

Gravitational waves from the Hénon-Heiles system

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In this work we analyze the emission of gravitational waves from the Hénon-Heiles system. We show the qualitative differences among emission of the gravitational waves from regular and chaotic motions. [S0556-2821(97)01722-0]

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I. INTRODUCTION

Gravitational waves have been known to exist since the early days of general relativity. Einstein himself showed that in the weak field limit we get solutions that obey the wave equation. Despite its importance, at the present time there is only indirect evidence of the reality of gravitational waves (such as the analyzed binary pulsar which shows a decrease in the orbital period due to the emission of gravitational waves [1]). It is expected that with the new generation of gravitational wave detectors under construction, more systems will be observed. In this way, several analyses of the possible sources of gravitational waves have been worked out in recent years in the belief that new detectors may be testing models and theories of gravity, stellar models, galactic dynamics, and so on [2].

In the linear regime of general theory of relativity an important feature of a gravitational wave's emitted power is its dependence on the third-order derivative of the quadrupolar moment. In this case, any system which changes its configuration with time is expected to be a good candidate for a gravitational wave source. Systems suffering catastrophic events are among the best candidates to be observed with gravitational wave detectors and a flurry of work searching for the best candidates to be observed is being done by several groups in the world (see [3] and references therein). However, between these systems, which suffer changes in their configurations, there is one that has not been analyzed yet: a chaotic system. An important characteristic of this system is its high sensitivity to initial conditions and a complicated nonperiodic behavior. In gravitational systems, chaotic behavior was studied with care since the pioneering works of Poincaré which analyzed the stability of the Solar System. The three-body problem [4], the tumbling of Saturn's satellite hyperon [5], Hénon-Heiles model of galactic dynamics [6], geodesic motion around black holes [7], Bianchi type-IX cosmological models [8], and others [9] are examples of the chaotic behavior in some gravitational systems. In this work, we consider the emission of gravitational waves from a simple system which shows chaotic behavior: the Hénon-Heiles system [6]. This system was used for the first time to

study the dynamics of galactic systems. We know that the emission of gravitational waves from galaxies is negligible to its dynamics. Given this restriction, our main concern in this paper is to show the qualitative differences between gravitational waves from chaotic and regular gravitational systems. In this way, because of its simplicity, the Hénon-Heiles system is well suited for our purpose.

II. HÉNON-HEILES SYSTEM

The Hénon-Heiles system is described by the potential

$$V(x,y) = m\omega^2 \left[\frac{x^2 + y^2}{2} + \frac{1}{a} \left(x^2 y - \frac{y^3}{3} \right) \right]. \quad (1)$$

We have three constants in the above potential: the mass m , the frequency ω , and the parameter a , which set the length scale of the system (note that it is more common to use the parameter $\lambda \equiv m\omega^2/a$ instead of a). We followed the usual approach, setting $a = \omega = m = 1$ in our calculations. An important characteristic of this system is that in the range $E = 0 - 1/10$ the dynamics is well behaved, whereas for greater energy the chaoticity of the system increases with energy. At $E = 1/6$ it is completely chaotic and with $E > 1/6$ the trajectories are unbounded.

We performed several numerical simulations—using the standard fourth-order Runge-Kutta method—with different energy and initial conditions. The power emitted as a gravitational wave was calculated using the quadrupole formula

$$-\frac{dE}{dt} = \frac{G}{45c^5} \ddot{Q}_{\alpha\beta}^2, \quad (2)$$

with $Q_{\alpha\beta} = \int \mu (3x_{\alpha\beta} - \delta_{\alpha\beta} r^2) d\vec{r}$. In our case, we considered a pointlike object, and thus $\mu = \delta(\vec{r} - \vec{r}_i)$. Then,

$$Q_{xx} = 2x^2 - y^2, \quad (3)$$

$$Q_{yy} = 2y^2 - x^2, \quad (4)$$

$$Q_{xy} = 3xy, \quad (5)$$

with x, y being solutions of the Hénon-Heiles system. For the sake of simplicity, we assumed that the decrease in the energy carried away as gravitational waves is negligible during the time period considered, thus keeping its energy constant

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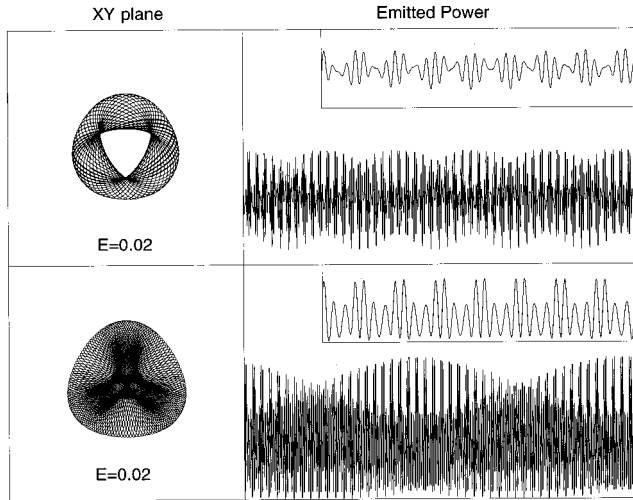


FIG. 1. The power emitted as gravitational waves by particle and its trajectories in the Hénon-Heiles potential with $E=0.020$, for two different initial conditions.

during the integration time. Indeed, assuming that the size of the system is nearly of typical galaxies, the energy carried away as gravitational waves would be very small. Later on in this work we will discuss qualitatively how the dynamics is changed when we consider the energy carried away as gravitational waves.

In Fig. 1 we show the results with $E=0.020$ and using two distinct initial conditions. With this energy, we know that the trajectories are regular. The figure shows the trajectories in the XY plane (left box) and the power emitted as a gravitational wave (right box). The small rectangle shows a small part of the whole spectrum. Note that the emitted power is periodic, reflecting the characteristics of the trajectory.

Because of the presence of chaotic motions in the Hénon-Heiles system, it is also important to consider how the spectrum of the emitted power changes in this case. For this purpose, we show in Fig. 2 the case with $E=0.165$ with two different initial conditions. In the bottom graph we have a regular motion and in the upper graph a chaotic motion. When the motion is regular, the emission of gravitational waves is nearly regular and periodic, similar to the previous case. However, in the chaotic case the emission is irregular and shows the presence of irregular peaks in the spectrum. These irregularities are due to its complicated nonperiodic trajectory.

In Table I, we show the statistics of the total energy emitted and the maximum power emitted with $E=0.020$ and $E=0.165$. In this last case a chaotic motion may occur, and thus also we show the same information separating regular motions from chaotic ones. The energy carried away as gravitational waves is greater in the regular case than in the chaotic one at the same energy. However, the maximum emitted power is greater in the chaotic case than the regular one. This difference is due to sudden changes in the trajectories, resulting in a high value of the derivative in its quadrupolar moment. Note that the chaotic case shows several nearly quiescent periods with a relatively small emission of gravitational waves. Because of these periods, the total en-

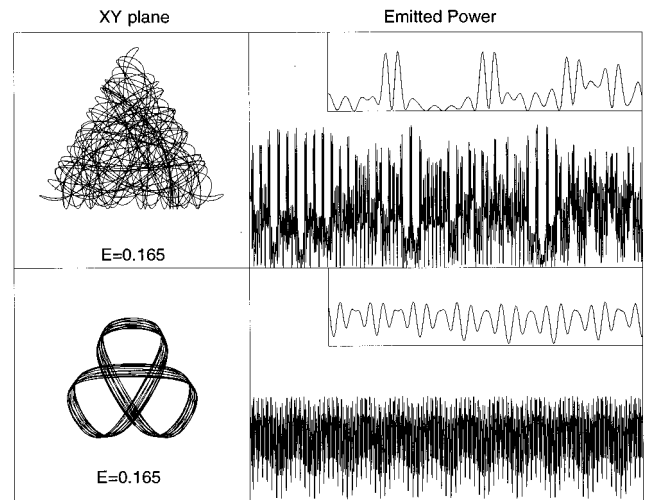


FIG. 2. The power emitted as gravitational waves by particles in the Hénon-Heiles system, with $E=0.165$, for two different initial conditions. We show also the trajectories in the XY plane.

ergy carried away as gravitational waves is lower than the regular case.

In Fig. 3 we show how the total energy emitted as gravitational waves varies with respect to the initial energy E . We also plot the energy emitted when the trajectories are circular, i.e., when the system consists of two bodies orbiting under the action of the Newtonian potential. In this case we know that

$$\left\langle \frac{dE}{dt} \right\rangle \propto E^5, \quad (6)$$

whereas in the Hénon-Heiles case, our numerical simulations resulted in

$$\left\langle \frac{dE}{dt} \right\rangle \propto E^{1.95}. \quad (7)$$

These graphs are normalized such that total energy emitted when $E=0.01$ is equal in both cases. Thus, the numerical values of $\langle dE/dt \rangle$ are meaningless, only its slope being important. In the Hénon-Heiles system we have an upper limit to the energy if we want a bound system, whereas in the Newtonian case there is not—of course, for very high energy it would be important to consider relativistic effects.

TABLE I. Total energy emitted ΔE and maximum power P_{\max} when energy is $E=0.165$ and $E=0.020$, with $\sigma=G/(45c^5)$. In the first case, trajectories may be chaotic. In the table we show the results when considering both chaotic and regular motions, only regular motions, and only chaotic motions.

Energy	$\Delta E/\sigma$	P_{\max}/σ	Type of motion
0.165	$(2.92 \pm 0.57) \times 10^3$	14.16 ± 1.77	chaotic+regular
	$(3.55 \pm 0.09) \times 10^3$	11.97 ± 0.38	regular
	$(2.56 \pm 0.37) \times 10^3$	15.48 ± 0.19	chaotic
0.020	$(4.10 \pm 0.66) \times 10$	0.166 ± 0.008	regular

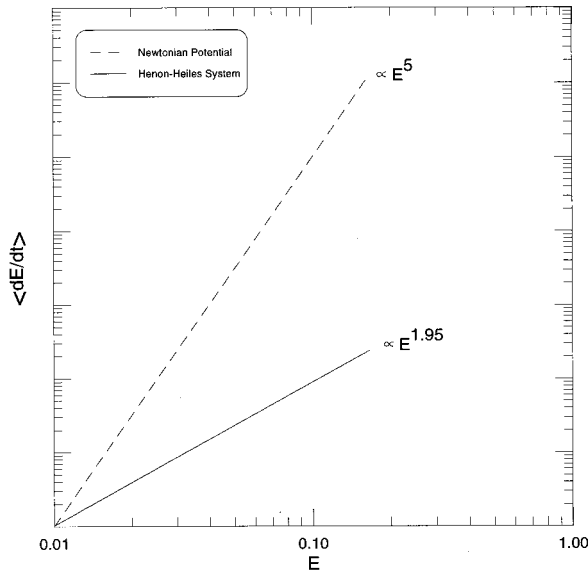


FIG. 3. Mean power emitted as a function of the initial energy with solid line representing the Hénon-Heiles case and dotted line the two bodies in circular orbit in a Newtonian potential. The results are normalized such that at energy $E=0.01$ we set dE/dt equal in both cases.

As we have said before, in a more realistic situation it would be important to consider the energy loss due to the emission of gravitational waves. For this purpose, consider a system initially with $E > 1/10$. Then its energy decreases with time at a rate which depends on the characteristics of the system (its mass m , frequency ω , and length scale a). However, in all cases after a suitable time interval, the system will lose sufficient energy, reaching a regime with $E < 1/10$ and becoming a regular one. It is important to remember that chaotic systems are very sensitive to the variations of their initial conditions, and thus during the passage from the $E > 1/10$ to $E < 1/10$ state the system may wander from chaotic to regular, resulting in a very complicated dynamics. Given these complications, the final state after a suitable time interval will be a regular state.

III. CONCLUDING REMARKS

In this work we have shown that the presence of chaotic motion results in an irregular emission of gravitational waves. However it is important to remember that our model—the Hénon-Heiles potential—is highly idealized, in such a way that our results are not applicable to a realistic astrophysical system. (However, in a recent work of Vieira and Letelier, they show that a black hole system with an external halo has a gravitational potential with terms similar to the Hénon-Heiles system [10].) Given such restrictions, the fact that gravitational waves from chaotic systems show different features, a detailed analysis of a more realistic system will be interesting. Several possible extensions of gravitational waves from chaotic systems may be performed. Note that a highly irregular and peaked spectrum may be a signature of a chaotic motion. However, in some cases these signals may be confused with stochastic noise due to several other sources in the sky. In this way it would be desirable to have adequate data analysis tools to distinguish a stochastic from a chaotic origin of the detected signal. Also, a detailed search of possible realistic astrophysical systems which display chaotic motions with detectable emission of gravitational waves would be desirable.

However, since gravitational waves carry away energy from the system, an initially chaotic system may change its dynamics from this regime to a regular one during its evolution. In this way, it may be important to consider the emission of gravitational waves in the study of long period behavior of a chaotic system, even when gravitational wave emission is negligible. Following this line of reasoning, we are now analyzing the behavior of a restricted three-body problem, which is a paradigm of a gravitational system suffering chaotic motions.

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