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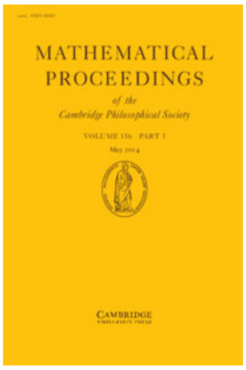

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Mathematical Proceedings of the Cambridge Philosophical Society / Volume 124 / Issue 02 / September 1998, pp 275 - 304
DOI: null, Published online: 08 September 2000

Link to this article: http://journals.cambridge.org/abstract_S0305004198002515

How to cite this article:
JACQUES-ELIE FURTER, ANGELA MARIA SITTA and IAN STEWART (1998). Algebraic path formulation for equivariant bifurcation problems. Mathematical Proceedings of the Cambridge Philosophical Society, 124, pp 275-304

Math. Proc. Camb. Phil. Soc. (1998), **124**, 275 275
Printed in the United Kingdom

Algebraic path formulation for equivariant bifurcation problems

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(*Received* 6 *November* 1996)

1. *Introduction*

In [**8**] we studied equivariant bifurcation problems with a symmetry group acting on parameters, from the point of view of singularity theory. We followed the now classical theory originated by Damon [**5**], using the ideas presented in [**5**, **13**, **14**]. We adapted general results about unfoldings, the algebraic characterization of finite determinacy, and the recognition problems, to multiparameter bifurcation problems $f(x, \lambda) = 0$ with 'diagonal' symmetry on both the state variables and on the bifurcation parameters. More precisely, such bifurcation problems satisfy the condition $f(\gamma x, \gamma\lambda) = \gamma f(x, \lambda)$ for all $\gamma \in \Gamma$, where Γ is a compact Lie group.

In this paper we attack the same problem from a different angle: the *path formulation*. This idea can be traced back to the first papers of Mather [**17**] and Martinet [**15**, **16**]. It was used explicitly in Golubitsky and Schaeffer [**12**] (see also their earlier paper [**11**]) as a way of relating bifurcation problems in one state variable without symmetry to a miniversal unfolding in the sense of catastrophe theory. At that time the techniques of singularity theory were not powerful enough to handle the full power of the idea efficiently – either in theory or in computational practice. This is why the path formulation was abandoned in favour of contact equivalence with distinguished parameters, as developed in Golubitsky and Schaeffer [**12**]. Considerable progress has been made since then; for example Montaldi and Mond [**19**] use the path formulation to apply the idea of $\mathscr{K}_V$-equivalence introduced by Damon [**6**] to equivariant bifurcation theory. Bridges and Furter [**3**] studied equivariant gradient bifurcation problems using the path formulation, and defined an equivalence relation in the space of paths and their unfoldings that respects contact equivalence of the gradients. Here we describe an algebraic approach to the path formulation that has the advantage of organizing the classification of normal forms. Moreover, it minimizes the calculation involved in obtaining the normal forms (compare with the classical framework in Furter *et al.* [**8**]). The geometric approach to the path

† Research supported by a grant from CAPES and CNPq.
‡ Research supported in part by a grant from the SERC (now EPSRC).

formulation using $\mathscr{K}_V$-equivalence is still open in the context of a symmetry group acting diagonally on parameters.

Let $\Sigma \subset \Gamma$ be the isotropy subgroup of λ, that is, $\Sigma = \{\sigma \in \Gamma \mid \sigma\lambda = \lambda,\ \forall\,\lambda\}$. For fixed λ the full equation is Σ-equivariant, but when $\bar{\lambda} \in \mathrm{Fix}\,(\Gamma)$, then the germ $f(x, \bar{\lambda})$ is Γ-equivariant. In the language of singularity theory, without additional constraints, we would consider the recognition problem for Γ-equivariant problems and unfold them in the Σ-theory. In our case we do not have a 'full' unfolding in the Σ-theory because Γ remains as a residue of symmetry when we enlarge the space to encompass the parameters.

In Furter *et al.* [**8**] we found normal forms for bifurcation problems with two state variables and two bifurcation parameters that are equivariant under an action of the dihedral group $\mathbf{D}_4$ on both state variables and parameters, see (1·1) below. This context was motivated by mathematical models describing the buckling of a square plate when forces act on its edges We used the classical framework to find the tangent spaces and higher order terms, from which we deduced the normal forms. We gave a corrected version of the generic normal form already obtained by Peters [**21**], and extended the classification to bifurcation problems of topological codimension one. We also described the bifurcation diagrams of the generic normal form.

We briefly put these results into a broader context. The study of equivariant bifurcation problems via singularity theory (Golubitsky *et al.* [**12**, **13**]) has mainly been concerned with models exhibiting *spontaneous* symmetry-breaking, where the equations maintain the same symmetry throughout the bifurcation, but the solutions lose symmetry as the parameters vary. Golubitsky and Schaeffer [**12**] and Golubitsky *et al.* [**13**] study one-parameter bifurcation problems where the symmetry groups acts only on the state variables. Peters [**21**] classified the bifurcation problems with a one-dimensional state variable and two bifurcation parameters, and extended the basic formalism to multiparameter bifurcation problems with diagonal symmetry on both state variables and bifurcation parameters. Simultaneously, in his Ph.D. thesis [**14**], Lari-Lavassani analysed multiparameter bifurcation problems with symmetry on the state variables.

However, there is another category of equivariant problems where the bifurcation equations satisfy less symmetry when some parameters are non-zero; this is called *forced* symmetry-breaking. There has been some some analysis of this situation using classical techniques in bifurcation theory (see Vanderbauwhede [**25**], Chillingworth [**4**], for instance). Although many of the results obtained so far are fairly general, they have mainly been applied to forced symmetry-breaking from a full orbit of solutions under a continuous Lie group – which arise for instance in periodic forcing of autonomous systems, or, in mechanics, for rigid body motion.

We now describe the structure of this paper. In Section 2 we recall the general theory derived in [**8**] in order to study Γ-equivariant multiparameter bifurcation problems via singularity theory, for a diagonal linear action of a compact Lie group Γ on the state variables x and on the multiparameter λ. We define an equivalence relation for such bifurcation problems using a change of coordinates (contact equivalence) that preserves the bifurcation structure (λ-slices) and the symmetry (Γ-action) of the problem. Two germs f, g representing bifurcation problems are said to be *equivalent* if there exist T, X, Λ such that

$$g(x,\lambda) = T(x,\lambda)\, f(X(x,\lambda), \Lambda(\lambda)) ,$$

where $T(x,\lambda)$ is an invertible matrix and $(x,\lambda) \mapsto X(x,\lambda), \Lambda(\lambda))$ is a local diffeomorphism. Both T and (X,Λ) must be symmetry- and orientation-preserving; that is, $T(\gamma x, \gamma\lambda)\,\gamma = \gamma\, T(x,\lambda)$, $X(\gamma x,\gamma\lambda) = \gamma\, X(x,\lambda)$, $\Lambda(\gamma\lambda) = \gamma\,\Lambda(\lambda)$, and $T(0,0)$, $X_x(0,0), \Lambda_\lambda(0)$ must be in the connected components of their respective identity operators.

In [**8**] we showed that this context fits into the classic framework of Damon [**5**]. Indeed we can either derive the main algebraic results – the finite determinacy and unfolding theorems – directly, or from the abstract formalism of Damon [**5**]. Finite codimension of the 'extended tangent space' of such f implies both that f is contact equivalent to a finite segment of its Taylor series (*finite determinacy*) and that any perturbation of f can be induced from a special perturbation F with $\operatorname{cod} f$ parameters (the *universal unfolding* of f).

In Section 3·1 we develop the idea of the organizing centre f_0 of a bifurcation problem f, $f_0(x) = f(x,0)$. Such an organizing center is still Γ-equivariant. Let Σ be the isotropy subgroup of the bifurcation multiparameter λ and suppose that f_0 has a Σ-universal unfolding $F(x,\alpha)$. Now consider f as a perturbation of f_0, and seek a germ $\tilde\alpha$ such that $f(x,\lambda) = F(x,\tilde\alpha(\lambda))$. We call such a germ $\tilde\alpha$ a *path*. (More accurately it is a path-germ.) Because f is Γ-equivariant and F is only Σ-equivariant in x, we define a Γ-action on the space of λ-paths in the parameter space of a *well-chosen* universal Σ-unfolding of the organizing centre, in a such way that the pullbacks $\tilde\alpha^* F$ of that Σ-unfolding by such paths become Γ-equivariant.

Our fundamental hypothesis (H0) is seemingly rather natural: we assume that

$$\operatorname{cod}^{\Sigma}(f_0) < \infty .$$

Indeed, in Section 3·2 we show that under (H0) the path formulation can always be introduced. Nevertheless, we also show in Section 3·1·2 that (H0) is not actually a necessary condition for f itself to be of finite codimension. The understanding of what happens there is an open question.

In Section 3·3 we define the tangent space and the unipotent tangent space of a Γ-equivariant path. The main result establishes an isomorphism between the normal space of a Γ-equivariant path and the normal space of the pullback of the Σ-unfolding of the organising center by this path. These results represent an *algebraic characterisation* of the path formulation, in the sense that we use only algebraic manipulations of the classical tangent spaces of Damon [**5**], as developed by Furter *et al.* [**8**], to construct the tangent spaces to a path. We rely upon the existence, from the start, of such a general theory. (The geometric approach, using the $\mathscr{K}_V$-equivalence defined in Damon [**6**] and applied in [**3**, **19**], with V being the relevant local bifurcation variety for F, is under investigation.) We finish in Section 3·4 with the proofs of the main results of Section 3·3.

In Section 4 we illustrate our theory by extending the classification of $\mathbf{D}_4$-equivariant bifurcation problems started in [**8**]. We consider problems with two state variables and two bifurcation parameters equivariant under the aforementioned action of the dihedral group $\mathbf{D}_4$. In complex notation, the effect of this action on a bifurcation problem $f(z,\lambda)$ is

$$f(\bar{z},\lambda) = \overline{f(z,\lambda)} \quad \text{and} \quad f(i\bar{z},\bar{\lambda}) = i\,\overline{f(z,\lambda)}\,. \tag{1·1}$$

This section ends with remarks on the use of our classification to tackle *gradient* $\mathbf{D}_4$-equivariant bifurcation problems. Some bifurcation problems, like those arising from models of the buckling of elastic shells, have a natural gradient structure. This acts as an additional constraint. Even if contact equivalence does not preserve the set of gradients $\vec{\mathscr{E}}^{\Gamma}_{\nabla,\lambda}$, it still induces an equivalence relation on $\vec{\mathscr{E}}^{\Gamma}_{\nabla,\lambda}$. Moreover, the perturbation (unfolding) theory extends to the gradient case: see Bridges and Furter [**3**] for general theoretical results on such questions.

In the multiparameter situation we must understand what structure is preserved by contact equivalence. In general, only the relative position of *open regions* in parameter space where the zero-set structure does not change in its principal characteristics is preserved. Without further information, one dimensional slices have in general no invariant meaning. In our situation, though, because of the symmetry on the parameters, the axis $\lambda_2 = 0$ is invariant under contact equivalence, so the structure in each half-plane is preserved. For two of our normal forms, $\mathbf{I}_0$ and $\mathbf{I}_6$, we can say more. They are also normal forms for the stronger contact equivalence that respects λ_1-slices for λ_2 = constant, that is, with

$$\Lambda(\lambda_1,\lambda_2) = (\Lambda_1(\lambda_1,\lambda_2), \Lambda(\lambda_2))\,.$$

Hence, in that case, the λ_2-sequence of those λ_1-slices has a perfectly good invariant meaning.

The proof of that fact is quite easy using the path formulation. The main part of the tangent space, which depends on F, is independent of the changes in the structure of the λ-space. So we need only change the second part of the tangent space, which depends on the λ-derivatives. When Γ acts nontrivially on the parameter λ we nevertheless may have to reconsider part of the general calculations, because we have to keep track of all the symmetries: see Sections 3 and 4).

Applications of the theory for $\mathbf{D}_4$-equivariant problems to forced symmetry breaking in four-cell rings are discussed in another paper currently in preparation, Furter [**7**], along with further $\mathbf{D}_4$-equivariant multiparameter bifurcation problems.

2. *Fundamentals of the general theory*

In this section we recall the fundamental concepts and principal results about unfoldings, finite determinacy and the recognition problem for multiparameter equivariant bifurcation problems, which we derived in [**8**] from the general abstract theory of Damon [**5**].

2·1. *Notation and definitions*

The *state variable* is $x = (x_1 \ldots x_n) \in \mathbb{R}^n$ and the *bifurcation parameter* is $\lambda = (\lambda_1 \ldots \lambda_\ell) \in \mathbb{R}^\ell$. Derivatives are denoted by subscripts, for example f_x for $\partial f/\partial x$, and the superscript ^{o} denotes the value of any function at the origin, so that $f^o = f(0)$, $f^o_x = f_x(0)$, and so on.

Let $\mathscr{E}_x$ be the ring of smooth germs $f\colon (\mathbb{R}^n,0) \to \mathbb{R}$ and $\mathscr{M}_x$ its maximal ideal. For $y \in \mathbb{R}^m$, $\mathscr{E}_{x,y}$ denotes the $\mathscr{E}_x$-module of smooth germs $g : (\mathbb{R}^n,0) \to \mathbb{R}^m$ and $\mathscr{M}_{x,y}$ its submodule of germs vanishing at the origin. When y is clear from the context we denote $\mathscr{E}_{x,y}$ by $\vec{\mathscr{E}}_x$ and $\mathscr{M}_{x,y}$ by $\vec{\mathscr{M}}_x$. In the path formulation, we also make use of the ring $\mathscr{O}_x$ and module $\vec{\mathscr{O}}_x$ of *real-analytic* germs.

Let $\mathrm{GL}(n)$ be the group of all invertible $n \times n$ real matrices and $\mathbf{O}(n)$ the n-dimensional orthogonal group. Let Γ be a compact Lie group acting on $\mathbb{R}^m$ and 'diagonally' on $\mathbb{R}^{n+\ell}$ via orthogonal representations $\rho_N: \Gamma \to \mathbf{O}(N)$, where $N = m, n, l$. (The abuse of notation involved here is intentional and useful.) We denote by γ_N the action on $\mathbb{R}^N$ induced by ρ_N, $N = n, l, m$, and identify γ_N with $\rho_N(\gamma)$ for all $\gamma \in \Gamma$. The connected component of the identity map in the subset of $\mathrm{GL}(n)$ consisting of all Γ-equivariant maps is denoted by $\mathscr{L}^o_\Gamma(n)$. The identity map in $\mathrm{GL}(n)$ is denoted by $\mathbf{I}_n$.

Let $\mathscr{E}^\Gamma_{(x,\lambda)} = \{h: (\mathbb{R}^{n+\ell}, 0) \to \mathbb{R} \mid h(\gamma_n x, \gamma_\ell \lambda) = h(x,\lambda),\ \forall\, \gamma \in \Gamma\}$ be the ring of smooth Γ-invariant germs and $\mathscr{M}^\Gamma_{(x,\lambda)}$ its maximal ideal. There exists a finite set of Γ-invariant polynomials $\{\bar{u}_i(x,\lambda)\}_{i=1}^r$ (see Schwarz [**23**]) such that any element $h \in \mathscr{E}^\Gamma_{(x,\lambda)}$ can be written as the pullback by $\bar{u} = (\bar{u}_1 \ \ldots \ \bar{u}_r)$ of a function of $u = (u_1 \ \ldots \ u_r)$; that is, $\mathscr{E}^\Gamma_{(x,\lambda)} = \bar{u}^* \mathscr{E}_u$. Similarly, taking $\mathscr{E}^\Gamma_\lambda = \{\eta: (\mathbb{R}^\ell, 0) \to \mathbb{R} \mid \eta(\gamma_\ell \lambda) = \eta(\lambda),\ \forall\, \gamma \in \Gamma\}$ and $\mathscr{M}^\Gamma_\lambda$ its maximal ideal, there also exist polynomials $\bar{v}(\lambda) = (\bar{v}_1(\lambda) \ \ldots \ \bar{v}_t(\lambda))$ with $\mathscr{E}^\Gamma_\lambda = \bar{v}^* \mathscr{E}_v$.

Let $\vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)} = \{f: (\mathbb{R}^{n+\ell}, 0) \to \mathbb{R}^m \mid f(\gamma_n x, \gamma_\ell \lambda) = \gamma_m f(x,\lambda),\ \forall\, \gamma \in \Gamma\}$ be the $\mathscr{E}^\Gamma_{(x,\lambda)}$-module of smooth Γ-equivariant germs. $\vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ is generated over $\mathscr{E}^\Gamma_{(x,\lambda)}$ by a finite set of Γ- equivariant polynomial maps $\{g_i\}_{i=1}^s$. Hence for any $f \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ there exist $\{h_j\}_{j=1}^s \in \mathscr{E}_u$ with

$$f = \bar{u}^*(h_1 g_1 + \cdots + h_s g_s)\,.$$

Thus we may identify $\vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ with $\bar{u}^* \mathscr{E}^s_u$ (in general that module is not free on $\vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$). Similarly, we represent $\vec{\mathscr{E}}^{\,\Gamma}_\lambda = \{\Lambda: (\mathbb{R}^\ell, 0) \to \mathbb{R}^\ell \mid \Lambda(\gamma_\ell \lambda) = \gamma_\ell \Lambda(\lambda),\ \forall\, \gamma \in \Gamma\}$ as $\bar{v}^* \mathscr{E}^{\hat{t}}_v$ for some $\hat{t}$. We denote by $\vec{\mathscr{M}}^{\,\Gamma}_{(x,\lambda)}$ ($\vec{\mathscr{M}}^{\,\Gamma}_\lambda$) the submodules of $\vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ ($\vec{\mathscr{E}}^{\,\Gamma}_\lambda$) of germs vanishing at the origin and, in general, $\vec{\mathscr{M}}^{\,\Gamma^k}_{(x,\lambda)} = (\mathscr{M}^k_{(x,\lambda)} \cdot \vec{\mathscr{E}}_{(x,\lambda)})^\Gamma$.

2·2. *Contact equivalence*

2·2·1. *$\mathscr{K}^\Gamma_\lambda$-equivalence*

Let

$$\mathbf{M}^\Gamma_{(x,\lambda)} = \{T: (\mathbb{R}^{n+\ell}, 0) \to M_m(\mathbb{R}) \mid T(\gamma_n x, \gamma_\ell \lambda)\,\gamma_m = \gamma_m\, T(x,\lambda),\ \forall\, \gamma \in \Gamma\}$$

be the $\mathscr{E}^\Gamma_{(x,\lambda)}$-module of Γ-commuting smooth matrix-valued maps. We also need the following $\mathscr{E}^\Gamma_{(x,\lambda)}$-module:

$$\vec{\Theta}^\Gamma_{(x,\lambda)} = \{X: (\mathbb{R}^{n+\ell}, 0) \to \mathbb{R}^n \mid X(\gamma_n x, \gamma_\ell \lambda) = \gamma_n X(x,\lambda),\ \forall\, \gamma \in \Gamma\}\,,$$

the following $\mathscr{E}^\Gamma_\lambda$-module:

$$\vec{\Theta}^\Gamma_\lambda = \{\Lambda: (\mathbb{R}^\ell, 0) \to \mathbb{R}^\ell \mid \Lambda(\gamma_\ell \lambda) = \gamma_\ell \Lambda(\lambda),\ \forall\, \gamma \in \Gamma\}$$

with their submodules

$$\vec{\Theta}^{\Gamma,o}_{(x,\lambda)} = \{X \in \vec{\Theta}^\Gamma_{(x,\lambda)} \mid X^o = 0\}$$

and

$$\vec{\Theta}^{\Gamma,o}_\lambda = \{\Lambda \in \vec{\Theta}^\Gamma_\lambda \mid \Lambda^o = 0\}.$$

The appropriate coordinate changes should preserve the zero-set, the special role of the bifurcation parameter, and the symmetry on both spaces. We therefore introduce the *contact group* $\mathscr{K}^{\Gamma}_{\lambda}$ defined by

$$\mathscr{K}^{\Gamma}_{\lambda} = \{(T, X, \Lambda) \in \mathbf{M}^{\Gamma}_{(x,\lambda)} \times \vec{\Theta}^{\Gamma,o}_{(x,\lambda)} \times \vec{\Theta}^{\Gamma,o}_{\lambda} \mid T^o \in \mathscr{L}^o_{\Gamma}(m),\ X^o_x \in \mathscr{L}^o_{\Gamma}(n),\ \Lambda^o_{\lambda} \in \mathscr{L}^o_{\Gamma}(\ell)\},$$

which acts in a natural way on $f \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ by

$$(T, X, \Lambda) \cdot f(x, \lambda) = T(x, \lambda)\, f(X(x, \lambda), \Lambda(\lambda)).$$

Two elements $f, g \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ are $\mathscr{K}^{\Gamma}_{\lambda}$-*equivalent* if they belong to the same $\mathscr{K}^{\Gamma}_{\lambda}$-orbit.

2·2·2. $\mathscr{K}^{\Gamma}_{\lambda,un}(k)$-*equivalence*

Let $\beta \in \mathbb{R}^k$, we extend in a straightforward manner the definitions of Section 2·2·1 to their β-parametrized versions, $\mathbf{M}^{\Gamma}_{(x,\lambda,\beta)}$, $\vec{\mathscr{E}}^{\Gamma}_{(x,\lambda,\beta)}$, $\vec{\Theta}^{\Gamma,o}_{(x,\lambda,\beta)}$, $\vec{\Theta}^{\Gamma,o}_{(\lambda,\beta)}$.

Perturbations of any $f \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ are described by *unfoldings* with k parameters of f, which are map germs $F \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda,\beta)}$, $\beta = (\beta_1 \ldots \beta_k)$, such that $F(x, \lambda, 0) = f(x, \lambda)$.

We denote by $\mathscr{K}^{\Gamma}_{\lambda,un}(k)$ the group of Γ-equivalences for unfoldings with k parameters. It is a natural extension of $\mathscr{K}^{\Gamma}_{\lambda}$ in the following sense:

$$\begin{aligned}\mathscr{K}^{\Gamma}_{\lambda,un}(k) = \{\ & (T, X, \Lambda, \Phi) \in \mathbf{M}^{\Gamma}_{(x,\lambda,\beta)} \times \vec{\Theta}^{\Gamma,o}_{(x,\lambda,\beta)} \times \vec{\Theta}^{\Gamma,o}_{(\lambda,\beta)} \times \mathscr{M}_{\beta,\beta} \mid \\ & (T, X, \Lambda) \text{ is a } k \text{ parameter unfolding of an element of } \mathscr{K}^{\Gamma}_{\lambda} \\ & \text{and } \Phi \text{ is a diffeomorphism germ } \}.\end{aligned}$$

The action of $\mathscr{K}^{\Gamma}_{\lambda,un}(k)$ on $F \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda,\beta)}$ is defined by

$$(T, X, \Lambda, \Phi) \cdot F(x, \lambda, \beta) = T(x, \lambda, \beta)\, F(X(x, \lambda, \beta), \Lambda(\lambda, \beta), \Phi(\beta)).$$

We say that $F, G \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda,\beta)}$ are $\mathscr{K}^{\Gamma}_{\lambda,un}(k)$-*equivalent* if they belong to the same $\mathscr{K}^{\Gamma}_{\lambda,un}(k)$-orbit.

2·2·3. *Tangent spaces*

Associated with $\mathscr{K}^{\Gamma}_{\lambda}$ we can define different tangent spaces to $f \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$. The *extended tangent space* to f is

$$\mathscr{T}^{\Gamma}_{e}(f) = \{Tf + f_x\, X + f_{\lambda}\, \Lambda \mid T \in \mathbf{M}^{\Gamma}_{(x,\lambda)},\ X \in \vec{\Theta}^{\Gamma}_{(x,\lambda)},\ \Lambda \in \vec{\Theta}^{\Gamma}_{\lambda}\}.$$

Note that it has only the structure of a $\mathscr{E}^{\Gamma}_{\lambda}$-module. The *extended normal space* to f is defined by

$$\mathscr{N}^{\Gamma}_{e}(f) = \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)} / \mathscr{T}^{\Gamma}_{e}(f)$$

and the Γ-*codimension* of f, $\mathrm{cod}^{\Gamma}(f)$, is defined as $dim_{\mathbb{R}}\ \mathscr{N}^{\Gamma}_{e}(f)$.

2·3. *The unfolding theory*

Let $F \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda,\beta)}$ be an unfoldings of $f \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ with k parameters, and let $G \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda,\alpha)}$ be an unfolding of $f \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ with r parameters. We say that G *maps into* F or G *factors through* F if there exist $T \in \mathbf{M}^{\Gamma}_{(x,\lambda,\alpha)}$, $X \in \vec{\Theta}^{\Gamma}_{(x,\lambda,\alpha)}$, $\Lambda \in \vec{\Theta}^{\Gamma}_{(\lambda,\alpha)}$ and $A : (\mathbb{R}^r, 0) \to (\mathbb{R}^k, 0)$ satisfying $T(x, \lambda, 0) = \mathbf{I}_m$, $X(x, \lambda, 0) = x$ and $\Lambda(\lambda, 0) = \lambda$, such that

$$G(x, \lambda, \alpha) = T(x, \lambda, \alpha)\ F(X(x, \lambda, \alpha), \Lambda(\lambda, \alpha), A(\alpha)).$$

The unfolding F is called *versal* if any unfolding G of f maps into F. If F is versal and has minimal number of parameters, it is called *miniversal*. The usual results from unfolding theory hold, as follows:

THEOREM 2·3·1 (The unfolding theorem). *Let* $f \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ *and* $F \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda,\alpha)}$ *be an unfolding of* f *with* k *parameters,* $\alpha = (\alpha_1 \ldots \alpha_k)$. *Then*

(i) F *is versal if and only if* $\vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)} = \mathscr{T}^{\Gamma}_{e}(f) + \mathbb{R} \cdot < F_{\alpha_1}(.,.,0) \ldots F_{\alpha_k}(.,.,0) >$.

(ii) *Two versal unfoldings of a germ in* $\vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ *are equivalent as unfoldings if and only if they have the same number of unfolding parameters.*

(iii) *Let* $W \subset \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ *be a finite dimensional complement of* $\mathscr{N}^{\Gamma}_{e}(f)$ *as a vector space, that is,* $\vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)} = \mathscr{T}^{\Gamma}_{e}(f) \oplus W$. *Let* $\{p_i\}_{i=1}^{\mathrm{cod}^{\Gamma}(f)}$ *be a basis for* W. *Then a miniversal unfolding of* g *is*

$$F(x, \lambda, \alpha) = f(x, \lambda) + \sum_{j=1}^{\mathrm{cod}^{\Gamma}(f)} \alpha_j \, p_j(x, \lambda).$$

(iv) *If* f *and* $g \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ *are two* $\mathscr{K}^{\Gamma}_{\lambda}$- *equivalent germs of finite codimension and* F *and* $G \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda,\alpha)}$, *with* $\alpha = (\alpha_1 \ldots \alpha_k)$, *are two miniversal unfoldings of* f *and* g, *respectively, then* F *and* G *are* $\mathscr{K}^{\Gamma}_{\lambda,un}(k)$*-equivalent. We say that* F, G *are universal unfoldings.*

2·4. *Determinacy*

For any mapping f we denote by $j^k(f)$ the Taylor polynomial of order k (or k-*jet*) of f. A germ $f \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ is k-$\mathscr{K}^{\Gamma}_{\lambda}$*-determined* if every germ $g \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ with $j^k(g) = j^k(f)$ is $\mathscr{K}^{\Gamma}_{\lambda}$-equivalent to f. A germ is *finitely* $\mathscr{K}^{\Gamma}_{\lambda}$*-determined* if it is k-$\mathscr{K}^{\Gamma}_{\lambda}$-determined for some integer k. As usual, there is a close relationship between being finitely determined and being of finite codimension. The first theorem follows from the general theory.

THEOREM 2·4·1 (Finite determinacy theorem). *A germ* $f \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ *is finitely* $\mathscr{K}^{\Gamma}_{\lambda}$*-determined if and only if* $\mathrm{cod}^{\,\Gamma}(f)$ *is finite.*

2·5. *The recognition problem*

The *recognition problem* seeks conditions under which a germ $g \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ is $\mathscr{K}^{\Gamma}_{\lambda}$-equivalent to a given normal form. To solve a particular recognition problem means explicitly to characterise the $\mathscr{K}^{\Gamma}_{\lambda}$-equivalence class in terms of a finite number of polynomial equalities and inequalities to be satisfied by the Taylor coefficients of the elements of that class.

2·5·1. *Intrinsic submodules and higher order terms*

Let $\Phi = (T, X, \Lambda) \in \mathscr{K}^{\Gamma}_{\lambda}$ and consider the mapping $f \mapsto \Phi(f) = T \cdot f \circ (X, \Lambda)$. A submodule $M \subset \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ is *intrinsic* if $\Phi(f) \in M$ for all $f \in M$ and all $\Phi \in \mathscr{K}^{\Gamma}_{\lambda}$. If $V \subset \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ then the *intrinsic part* of V, denoted by $\mathrm{Itr}\, V$, is the largest intrinsic submodule of $\vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ contained in V.

Let $f \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$. The 'perturbation term' $p \in \vec{\mathscr{E}}^{\Gamma}_{(x,\lambda)}$ is of *higher order* with respect to f if $g + p$ is $\mathscr{K}^{\Gamma}_{\lambda}$-equivalent to f for every g that is $\mathscr{K}^{\Gamma}_{\lambda}$-equivalent to f. By definition, such a perturbation cannot enter into a solution of the recognition problem for f.

We denote by $\mathscr{P}(f)$ the set of all higher order terms of f, that is,

$$\mathscr{P}(f) = \left\{ \, p \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)} \mid \; g + p \sim f \, , \; \forall g \sim f \, \right\}$$

where $\sim$ denotes $\mathscr{K}^{\Gamma}_{\lambda}$-equivalence.

Proposition 2·5·1. *For each* $f \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ *the set* $\mathscr{P}(f)$ *is an intrinsic submodule of* $\vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$.

2·5·2. *Unipotent* $\mathscr{K}^{\Gamma}_{\lambda}$*-equivalences*

The final subgroup of $\mathscr{K}^{\Gamma}_{\lambda}$ that we need is that of unipotent equivalences. The kernel of the projection map π sending $(T, X, \Lambda) \in \mathscr{K}^{\Gamma}_{\lambda}$ onto $(T^o, X^o_x, \Lambda^o_\lambda)$ is given by

$$\mathscr{U}^{\Gamma}_{\lambda} = \left\{ \, (T, X, \Lambda) \in \mathscr{K}^{\Gamma}_{\lambda} \mid T^o = \mathbf{I}_m, \; X^o_x = \mathbf{I}_n, \; \Lambda^o_\lambda = \mathbf{I}_\ell \, \right\}.$$

It is a normal subgroup of $\mathscr{K}^{\Gamma}_{\lambda}$ consisting of unipotent diffeomorphisms, and is called the subgroup of *unipotent* Γ-equivalences. Its associated tangent space at $f \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ is

$$\mathscr{T}\mathscr{U}^{\Gamma}(f) = \{ \, Tf + f_x X + f_\lambda \Lambda \mid T \in \mathbf{M}^{\Gamma}_{(x,\lambda)}, \; X \in \vec{\mathbf{\Theta}}^{\,\Gamma,o}_{(x,\lambda)}, \; \Lambda \in \vec{\mathbf{\Theta}}^{\,\Gamma,o}_{\lambda}, \\ T^o = 0 \, , \; X^o_x = 0 \, , \; \Lambda^o_\lambda = 0 \, \}.$$

As a consequence of Theorem 1·17 ([**9**], p. 108) we have the following proposition:

Proposition 2·5·2. *Let* $f \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ *be of finite* Γ*-codimension. Then* $\mathscr{P}(f) \supset \operatorname{Itr} \mathscr{T}\mathscr{U}^{\Gamma}(f)$.

Corollary 2·5·3. *Let* $p \in \operatorname{Itr} \mathscr{T}U^{\Gamma}(f)$. *Then* $f + p$ *is* $\mathscr{K}^{\Gamma}_{\lambda}$*-equivalent to* f.

3. *The path formulation*

In this section we describe a general 'algebraic' path formulation theory for Γ-equivariant bifurcation problems with diagonal Γ-action on state and parameter spaces.

3·1. *Organizing centres and equivariant paths*

Recall that Γ is a compact Lie group acting diagonally on state and parameter spaces. Let Σ be the subgroup of Γ leaving the λ-coordinate fixed. Technically $\Sigma = \operatorname{Ker} \rho_\ell$, so Σ is a normal subgroup of Γ. Let $\mathscr{E}^{\Sigma}_x$ be the set of Σ-invariant germs, let $\vec{\mathscr{E}}^{\,\Sigma}_x$ be the set of Σ-equivariant germs, let $\vec{\mathbf{\Theta}}^{\Sigma}_x$ be the set of Σ-equivariant vector fields on $\mathbb{R}^n$, and let $\mathbf{M}^{\Sigma}_x$ be the set of x-dependent Σ-commuting matrices. For $h \in \vec{\mathscr{E}}^{\,\Sigma}_x$ we have the following results (see [**13**]):

(i) $\mathscr{T}^{\Sigma}_e(h) = \{ \, Th + h_x X \mid T \in \mathbf{M}^{\Sigma}_x, \; X \in \vec{\mathbf{\Theta}}^{\Sigma}_x \, \}$ is the Σ-*extended tangent space* to h.

(ii) $\mathscr{N}^{\Sigma}_e(h) = \vec{\mathscr{E}}^{\,\Sigma}_x / \mathscr{T}^{\Sigma}_e(f_0)$ is the Σ-*normal space* to h.

(iii) $\operatorname{cod}^{\Sigma}(h) = \dim_{\mathbb{R}} \mathscr{N}^{\Sigma}_e(h)$.

(iv) If $\dim_{\mathbb{R}} \mathscr{N}^{\Sigma}_e(h) = r < \infty$ and $\{h_i\}_{i=1}^r \subset \vec{\mathscr{E}}^{\,\Sigma}_x$ is a basis for $\mathscr{N}^{\Sigma}_e(h)$ then a Σ-miniversal unfolding of h is $H(x, \alpha) = h(x) + \sum_{i=1}^r \alpha_i h_i(x)$.

Let $f \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$. Then the *organizing centre* f_0 of f is defined as $f_0(x) = f(x, 0)$. Clearly, f_0 is not only Σ-equivariant but it is actually Γ-equivariant. We assume henceforth that f_0 is of finite Σ-codimension. To fix the ideas, f_0 satisfies

$$\operatorname{cod}^{\Sigma}(f_0) = r \, . \tag{H0}$$

This hypothesis is fundamental in our work. In Section 3·2 we show that under (H0) the path formulation is always feasible (see Theorem 3·2·1). However, we also show in Section 3·1·2 that (H0) is *not* always necessary, by giving an example of a bifurcation problem where $\mathrm{cod}^{\Gamma}(f)$ is finite but f_0 does not satisfy (H0).

Let F_0: $(\mathbb{R}^{n+r}, 0) \to \mathbb{R}^m$ be the Σ-miniversal unfolding of f_0 with r parameters $\alpha = (\alpha_1 \ldots \alpha_r)$, constructed from a basis $\{h_i\}_{i=1}^r$ of $\mathscr{N}_e^{\Sigma}(f_0)$, namely

$$F_0(x, \alpha) = f_0(x) + \sum_{i=1}^{r} \alpha_i\, h_i(x)\,.$$

We say that α: $(\mathbb{R}^{\ell}, 0) \to (\mathbb{R}^r, 0)$ is a *path* in the r-dimensional parameter space of the miniversal unfolding of f_0. The pullback $(\bar{\alpha}^{\,*}F_0)$: $(\mathbb{R}^{n+\ell}, 0) \to \mathbb{R}^m$ is given by $(\bar{\alpha}^{\,*}F_0)(x, \lambda) = F_0(x, \bar{\alpha}(\lambda))$. We can now state the fundamental result about the existence of a space of paths. With the above notation let

$$\mathscr{P} = \{\ \alpha \in \mathscr{P}_{\ell,r}\ \mid\ \alpha \text{ is } \Gamma\text{-equivariant.}\}\,, \tag{3·1}$$

That is, $\alpha \in \mathscr{P}$ if and only if $\alpha(\gamma_{\ell}\lambda) = \gamma_r\, \alpha(\lambda)$ for all $\gamma \in \Gamma$). We call $\mathscr{P}$ the *space of paths*.

THEOREM 3·1·1 (Space of Paths). *There exists a basis* $\{h_i\}_{i=1}^r$ *of* $\mathscr{N}_e^{\Sigma}(f_0)$ *and a* Γ-*action on* $\mathbb{R}^r$ (*see* (3·2) *below*) *such that*

$$(\alpha^* F_0)(x, \lambda) = f_0(x) + \sum_{i=1}^{r} \alpha_i(\lambda)\, h_i(x)$$

is Γ-*equivariant for* $\alpha \in \mathscr{P}$.

We call such as basis a *good* basis, see Lemma 3·1·3. The proof of this result is in the next subsection.

3·1·1. *Space of paths*

We now construct the space $\mathscr{P}$ of Γ-equivariant λ-paths through the parameter space of F_0. Consider the isomorphism θ: $\vec{\mathscr{E}}_x^{\,\Sigma} / \mathscr{T}_e^{\Sigma}(f_0) \to \mathbb{R}^r$ defined by

$$\theta([g]) = \theta\left(\sum_{i=1}^{r} \alpha_i h_i\right) = (\alpha_1\ \ldots\ \alpha_r)\ .$$

Let $\varphi : \Gamma \times \vec{\mathscr{E}}_x^{\,\Sigma} \to \vec{\mathscr{E}}_x^{\,\Sigma}$ be the action of Γ on $\vec{\mathscr{E}}_x^{\,\Sigma}$ defined by $\varphi(\gamma, g) = \gamma_m^t\,(g \circ \gamma_n)$.

LEMMA 3·1·2. φ *is well-defined and* $\mathscr{T}_e^{\Sigma}(f_0)$ *is a* φ-*invariant submodule of* $\vec{\mathscr{E}}_x^{\,\Sigma}$.

Proof. Since Σ is a normal subgroup of Γ, a simple verification shows that for $g \in \vec{\mathscr{E}}_x^{\,\Sigma}$ the equation $\varphi(\gamma, g)(\sigma_n x) = \sigma_m\, \varphi(\gamma, g)(x)$ holds. Moreover, let $h \in \mathscr{T}_e^{\Sigma}(f_0)$; that is, there are $T \in \mathbf{M}_x^{\Sigma}$ and $X \in \vec{\mathbf{\Theta}}_x^{\Sigma}$ such that $h = T f_0 + (f_0)_x X$. Then $\varphi(\gamma, h) \in \mathscr{T}_e^{\Sigma}(f_0)$, $\forall\, \gamma \in \Gamma$. $\square$

As a consequence, we can project φ down to the quotient to define an action $\bar{\varphi}$ on $\Gamma \times \vec{\mathscr{E}}_x^{\,\Sigma} / \mathscr{T}_e^{\Sigma}(f_0)$ by $\bar{\varphi}(\gamma, [g]) = [\,\varphi(\gamma, g)\,]$. Then $\bar{\varphi}$ is well-defined, since $\mathscr{T}_e^{\Sigma}(f_0)$ is a φ-invariant submodule of $\vec{\mathscr{E}}_x^{\,\Sigma}$, and it defines a representation $\rho_r : \Gamma \to \mathrm{GL}(r)$, $\rho(\gamma) = \gamma_r^t$

where γ_r is the matrix defined by

$$[\bar{\varphi}(\gamma, h_i)] = \sum_{j=1}^{r} (\gamma_r)_{ij} [h_j], \quad 1 \leqslant i \leqslant r .$$

As usual we identify the action with its image.

We now consider a particular choice of a *polynomial* basis $\{h_i\}_{i=1}^r$ of $\mathscr{N}_e^\Sigma(f_0)$ for which the previous equivariance is 'exact' and $\gamma_r \subset \mathbf{O}(r)$; that is, we want the relations

$$\gamma_m^t h_i(\gamma_n x) = \sum_{j=1}^{r} (\gamma_r)_{ij} h_j(x), \quad \forall \gamma \in \Gamma, \forall 1 \leqslant i \leqslant r , \tag{3·2}$$

to hold for the polynomials, not only for the classes, and we want ρ_r to be an orthogonal representation. When such a choice is possible, we call such a basis a *good basis*.

Lemma 3·1·3. *There exists a good basis*

Proof. Since (H0) holds, that is $\operatorname{cod}^\Sigma(f_0) = r$, there exists $k \in \mathbb{N}$ such that $(\mathscr{M}_x^k)^\Sigma \cdot \vec{\mathscr{E}}_x^{\,\Sigma} \subset \mathscr{T}_e^\Sigma(f_0)$. Denoting by P_{k-1}^Σ the vector space of Σ-equivariant polynomials of degree less or equal to $k-1$, it follows that $\vec{\mathscr{E}}_x^{\,\Sigma}/\mathscr{T}_e^\Sigma(f_0) \subset \mathrm{P}_{k-1}^\Sigma$. As $\mathscr{T}_e^\Sigma(f_0) \bigcap \mathrm{P}_{k-1}^\Sigma$ is Σ-invariant, the tangent space has some Σ-invariant complement $\mathscr{N}_e^\Sigma(f_0)$ in P_{k-1}^Σ. Choose a basis $\{h_i\}_{i=1}^r$ for $\mathscr{N}_e^\Sigma(f_0)$ so that

$$\bar{\varphi}(\gamma, h_i) = \gamma_m^t h_i \circ \gamma_n = \sum_{j=1}^{r} (\gamma_r)_{ij} h_j ,$$

and change the coordinates again to make of ρ_r an orthogonal action. □

Proof of Theorem 3·1·1. We have constructed a good basis in Lemma 3·1·3; Now we define the space of paths $\mathscr{P}$ more precisely. For a good basis $\{h_i\}_{i=1}^r$, let $[g] = \sum_{i=1}^r \alpha_i h_i$. Then $\bar{\varphi}$ is explicitly given by

$$\bar{\varphi}(\gamma, [g]) = \sum_{i=1}^{r} \alpha_i (\gamma_m^t h_i \circ \gamma_n) = \sum_{i=1}^{r} \alpha_i \left(\sum_{j=1}^{r} (\gamma_r)_{ij} h_j \right) = \sum_{i=1}^{r} (\gamma_r^t \alpha)_i h_i ,$$

where $(\gamma_r^t)_{ij} = (\gamma_r)_{ji}$, $1 \leqslant i, j \leqslant r$. By considering the isomorphism $\theta\colon \vec{\mathscr{E}}_x^{\,\Sigma}/\mathscr{T}_e^\Sigma(f_0) \to \mathbb{R}^r$ we have $\rho_r(\gamma)(\alpha) = \gamma_r^t \alpha$ for all $\alpha \in \mathbb{R}^r$.

Let $\mathscr{P}_{\ell,r} = \{\alpha\colon (\mathbb{R}^\ell, 0) \to (\mathbb{R}^r, 0)\}$ be the set of paths in the unfolding parameter space. We define an action φ_p on $\mathscr{P}_{\ell,r}$ by $\varphi_p : \Gamma \times \mathscr{P}_{\ell,r} \to \mathscr{P}_{\ell,r}$, $(\gamma, \alpha) \mapsto \gamma_r^t (\alpha \circ \gamma_\ell)$. The fundamental space of paths we want to work with is the subspace of $\mathscr{P}_{\ell,r}$ defined as $\mathscr{P} = \operatorname{Fix} \varphi_p$, that is, $\alpha \in \mathscr{P}$ if and only if $\alpha(\gamma_\ell \lambda) = \gamma_r \alpha(\lambda)$, $\forall \gamma \in \Gamma$. The proof now follows from a straightforward calculation, carried out in detail in Sitta [**24**]. □

3·1·2. *Counterexample*

The following example shows that

$$\operatorname{cod}^\Gamma(f) < \infty \quad \text{does not imply that} \quad \operatorname{cod}^\Sigma(f_0) < \infty .$$

We consider $\mathbf{O}(2)$-equivariant bifurcation problems with 2 variables and 2 parameters. For convenience we shall use complex notation, that is, $(x_1, x_2, \lambda_1, \lambda_2) \in \mathbb{R}^4$ is identified with $(z,\lambda) = (x_1 + ix_2, \lambda_1 + i\lambda_2) \in \mathbb{C}^2$. Let $\mathbf{O}(2)$ act on $\mathbb{C}^2$ by $\theta \cdot (z,\lambda) = (e^{i\theta}z, e^{i\theta}\lambda)$ and $\kappa \cdot (z,\lambda) = (\bar{z},\bar{\lambda})$. Then as in [**24**]:

(i) $\mathscr{E}^{\mathbf{O}(2)}_{(z,\lambda)}$, the ring of $\mathbf{O}(2)$- invariant germs, is generated by $u = z\bar{z}$, $v = \lambda\bar{\lambda}$ and $\omega = z\bar{\lambda} + \bar{z}\lambda$.

(ii) $\vec{\mathscr{E}}^{\,\mathbf{O}(2)}_{(z,\lambda)}$, the $\mathscr{E}^{\mathbf{O}(2)}_{(z,\lambda)}$-module of $\mathbf{O}(2)$-equivariant map germs, is generated by z and λ, that is,

$$f(z,\lambda) = p(u,v,\omega)z + q(u,v,\omega)\lambda\,.$$

We also denote f by $[p,q]$.

(iii) Considering the action of $\mathbf{O}(2)$ on λ only, every $\mathbf{O}(2)$-equivariant mapping $\Lambda \in \vec{\mathscr{E}}^{\,\mathbf{O}(2)}_{\lambda}$ has the form $\Lambda(\lambda) = \xi(\lambda\bar{\lambda})\lambda = \xi(v)\lambda$, for some $\xi \in \mathscr{E}_v$.

(iv) $\mathbf{M}^{\mathbf{O}(2)}_{(z,\lambda)}$, the $\mathscr{E}^{\mathbf{O}(2)}_{(z,\lambda)}$-module of $\mathbf{O}(2)$-equivariant matrices, is generated by the following linear maps on $\mathbb{C}$:

$$\begin{aligned}
S_1(z,\lambda)\, w &= w,\\
S_2(z,\lambda)\, w &= (z\bar{\lambda} - \bar{z}\lambda)\, w,\\
S_3(z,\lambda)\, w &= z^2\bar{w},\\
S_4(z,\lambda)\, w &= z\lambda\bar{w},\\
S_5(z,\lambda)\, w &= \lambda^2\bar{w}.
\end{aligned}$$

(v) The *extended tangent space at* $f = [p,q] \in \vec{\mathscr{E}}^{\,\mathbf{O}(2)}_{(z,\lambda)}$ is defined by

$$\mathscr{T}^{\mathbf{O}(2)}_e(f) = \left\{\; Sf + f_x X + f_\lambda \Lambda \;\mid\; S \in \mathbf{M}^{\mathbf{O}(2)}_{(z,\lambda)},\ X \in \vec{\mathscr{E}}^{\,\mathbf{O}(2)}_{(z,\lambda)},\ \Lambda \in \vec{\mathscr{E}}^{\,\mathbf{O}(2)}_{\lambda} \;\right\}.$$

A calculation shows that

$$\mathscr{T}^{\mathbf{O}(2)}_e(f) = \mathscr{E}^{\mathbf{O}(2)}_{(z,\lambda)} \cdot < g_1 \ \ldots\ g_7 > + \mathscr{E}^{\mathbf{O}(2)}_{\lambda} \cdot < g_8 > \tag{3·3}$$

where

$$\begin{aligned}
g_1 &= [\, p,\ q\,]\,,\\
g_2 &= [\, \omega p + 2vq,\ -2up - \omega q\,]\,,\\
g_3 &= [\, up + \omega q,\ -uq\,]\,,\\
g_4 &= [\, vq,\ up\,]\,,\\
g_5 &= [\, -vp,\ \omega p + vq\,]\,,\\
g_6 &= [\, p + 2up_u + \omega p_\omega,\ 2uq_u + \omega q_\omega\,]\,,\\
g_7 &= [\, \omega p_u + 2vp_\omega,\ p + \omega q_u + 2vq_\omega\,]\,,\\
g_8 &= [\, 2vp_v + \omega p_\omega,\ q + 2vq_v + \omega q_\omega\,]\,.
\end{aligned}$$

The example is the generic bifurcation problem $f(z,\lambda) = \epsilon uz - \delta\lambda$, where $\epsilon^2 = \delta^2 = 1$. Using (3·3), $\mathscr{T}^{\mathbf{O}(2)}_e(f) = [\mathscr{M}_{(u,v,\omega)}, \mathscr{E}]$, so

$$\operatorname{cod}^{\mathbf{O}(2)}(f) = \dim_{\mathbb{R}} \vec{\mathscr{E}}^{\,\mathbf{O}(2)}_{(z,\lambda)} / \mathscr{T}^{\mathbf{O}(2)}_e(f) = 1.$$

The organizing centre of f is by $f_0(z) = uz$. The isotropy subgroup of λ is the trivial group, $\boldsymbol{\Sigma} = \mathbf{1}$. Thus, $\operatorname{cod}^{\boldsymbol{\Sigma}}(f_0) = \operatorname{cod}_{\mathscr{K}}(f_0)$. By Proposition 2·4 ([**26**], p. 494), any $\mathscr{K}$-finite germ is $\mathscr{C}$-finite, and using the geometric criterion for a germ to be $\mathscr{C}$-finite we find that f_0 has infinite $\boldsymbol{\Sigma}$-codimension since the complexification of f_0 has a non-isolated singularity.

That f is the generic bifurcation problem follows from the general theory of [**8**]. A straightforward calculation shows that $\mathscr{M}_{(u,v,\omega)}$ and $<v,\omega>$ are intrinsic ideals and that $[\mathscr{I},\mathscr{J}]$ is an intrinsic module if and only if $\mathscr{I}$ and $\mathscr{J}$ are intrinsic ideals and $<v,\omega>\cdot\mathscr{J}\subset\mathscr{I}\subset\mathscr{J}$. Therefore

$$\mathscr{P}(f)=[\mathscr{M}^2_{(u,v,\omega)}+\mathscr{M}_{(u,v,\omega)}\cdot<v,\omega>,\ \mathscr{M}_{(u,v,\omega)}]\,.$$

Hence if $p^o_u \neq 0$ and $q^o \neq 0$, simple rescalings show that $[p,q]$ is contact equivalent to f with $\epsilon = sg\,p^o_u$ and $\delta = sg\,q^o$.

3·2. *General path formulation*

Now we show that if (H0) holds then we can always define a path formulation for the bifurcation diagrams and their unfoldings.

Theorem 3·2·1 (path formulation). *Let $f\in\vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ with organizing centre $f_0\in\vec{\mathscr{E}}^{\,\Gamma}_x$ of finite $\mathscr{K}^{\Sigma}$-codimension r having miniversal unfolding $F_0(x,\alpha)=f_0(x)+\sum_{i=1}^r \alpha_i\,h_i(x)$. Then there exists a path $\hat{\alpha}\in\mathscr{P}$ such that f is $\mathscr{K}^{\Gamma}_{\lambda}$-contact equivalent to $\hat{\alpha}^*F_0$.*

Similarly, if F is an unfolding of f with parameters β, then there exists an unfolding A of $\hat{\alpha}$ with parameters β such that F is $\mathscr{K}^{\Gamma}_{(\lambda,\beta)}$-contact equivalent to A^*F_0.

Proof. We construct a $\mathscr{K}^{\Gamma}_{\lambda}$-trivial homotopy $G{:}[0,1]\to\vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ between f and $\bar{\alpha}(.,1)^*F_0$, defined by

$$G(x,\lambda,t)=(1-t)\,f(x,\lambda)+t\,F_0(x,\bar{\alpha}(\lambda,t))$$

for some (yet to be determined) $\bar{\alpha}(.,t)\in\mathscr{P}$ with $\bar{\alpha}(0,t)=0$, $\forall\,t\in[0,1]$. By 'trivial homotopy' we mean that G_t is $\mathscr{K}^{\Gamma}_{\lambda}$-contact equivalent to f, $\forall\,t\in[0,1]$, which would imply that $\hat{\alpha}=\bar{\alpha}(.,1)$.

To find $\bar{\alpha}$, define

$$H(x,\lambda,\alpha,t)=(1-t)\,f(x,\lambda)+t\,f_0(x)+\sum_{i=1}^r \alpha_i h_i(x).$$

Note that $G=(t\,\bar{\alpha}(\lambda,t))^*H$ and that H is Γ- equivariant. Moreover, F_0, G and H are all unfoldings of f_0.

The key ingredient is the following version of the Parametrized Preparation Theorem (the idea is to have germs in (x,λ,α) but not in $t\in[0,1]$).

Parametrised equivariant preparation theorem (see Arnold *et al.* [**1**]). *If*

$$\vec{\mathscr{E}}^{\,\Sigma}_x=\mathscr{T}\mathscr{K}^{\Sigma}_x(f_0)+\mathbb{R}\cdot<H_{\alpha_1}\,\ldots\,H_{\alpha_r}>$$

then

$$\vec{\mathscr{E}}^{\,\Sigma}_{(x,\lambda,\alpha,t)}=\mathscr{T}\mathscr{K}^{\Sigma}_{(x,\lambda,\alpha,t)}(H)+\mathscr{E}_{(\lambda,\alpha,t)}\cdot<H_{\alpha_1}\,\ldots\,H_{\alpha_r}>\qquad\forall\,t\in[0,1]\,. \tag{3·4}$$

Let $g(x,\lambda)=f(x,\lambda)-f_0(x)$. From the Hadamard Lemma there exist $\{M_j\}_{j=1}^k\subset\vec{\mathscr{E}}^{\,\Sigma}_x$ such that $g(x,\lambda)=\sum_{j=1}^k \lambda_j M_j(x,\lambda)$. We can use (3·4) to decompose each M_j, $1\leqslant j\leqslant k$. That is, there exist (S_j,Y_j,a_j) such that

$$M_j(x,\lambda)=S_j(x,\lambda,\alpha,t)\,H(x,\lambda,\alpha,t)+H_x(x,\lambda,\alpha,t)\,Y_j(x,\lambda,\alpha,t)+\sum_{i=1}^r (a_j)_i(\lambda,\alpha,t)\,h_i(x)\,.$$

Now define $S = \sum_{i=1}^{n} \lambda_i S_i$, $Y = \sum_{i=1}^{n} \lambda_i Y_i$ and $a = \sum_{i=1}^{r} \lambda_i a_i$. Then

$$f(x,\lambda) - f_0(x) = SH + H_x Y + \sum_{i=1}^{r} a_i(\lambda,\alpha,t)\, h_i(x)\,. \tag{3·5}$$

Moreover, $S(x,0,\alpha,t)$, $Y(x,0,\alpha)$ and $a(0,\alpha,t)$ are all identically 0 for all $t \in [0,1]$. Those two properties are preserved when we Γ-average (3·5) to get Γ-equivariant S, Y and a.

Now consider the following ODE for $t \in [0,1]$:

$$\frac{d}{dt}(t\,\bar{\alpha}(\lambda,t)) = a(\lambda,\bar{\alpha}(\lambda,t),t)\,.$$

For consistency at $t=0$ we need

$$\bar{\alpha}(\lambda,0) = a(\lambda,\bar{\alpha}(\lambda,0),0)\,. \tag{3·6}$$

We want to have $\bar{\alpha}(0,0) = 0$, and we know that $a(0,0,0) = a_\alpha(0,0,0) = 0$. We use the Implicit Function Theorem to find a unique solution $\bar{\alpha}(\lambda,0)$ of (3·6); we use that solution as an initial point for the ODE. Moreover, since $a(0,0,t) = 0$ we see that $\bar{\alpha}(0,t) = 0$ for all $t \in [0,1]$.

Once we have $\bar{\alpha}$ we can get the rest of the change of coordinates in the classical manner. We integrate $\dot{X} = Y(X,\lambda,t\bar{\alpha},t)$ to find $X(x,\lambda,t)$ such that $X(x,\lambda,0) = x$. As $Y(0,0,0,t) = 0$, $\forall\, t \in [0,1]$, we verify that $X(0,0,t) = 0$, $\forall\, t \in [0,1]$. Finally we integrate the matrix vectorfield

$$\dot{T}(x,\lambda,t) = T(x,\lambda,t)\, S(X(x,\lambda,t),\lambda,t\,\bar{\alpha}(\lambda,t),t)$$

to find T such that $T(x,\lambda,0) = \mathbf{I}_n$. Then

$$T^{-1}T_t\, G(X,\lambda,t) + G_x(X,\lambda,t)\, X_t + G_t(X,\lambda,t) \equiv 0$$

with

$$G_t(x,\lambda,t) = -(f(x,\lambda) - f_0(x)) + \sum_{i=1}^{r} \frac{d}{dt}(t\,\bar{\alpha}(\lambda,t))\, h_i(x).$$

We conclude that

$$\frac{d}{dt}\,(T\,G(X,\lambda,t)) \equiv 0\,,$$

so that $T\,G(X,\lambda,t)$ is a constant over time, equal to $G_0 = f$. $\square$

3·3. *Tangent spaces to a path*

Suppose that (H0) holds and let $\mathscr{N}_e^{\Sigma}(f_0)$ be generated by a good basis $\{h_i\}_{i=1}^{r}$. Let ρ_r be the orthogonal representation defined in Section 3·1·1. Consider the action of Γ on $\alpha \in \mathbb{R}^r$ given by $(\gamma,\alpha) \mapsto \gamma_r\alpha$. Note that the $\boldsymbol{\Sigma}$-miniversal unfolding of f_0, denoted as before by F_0, is Γ-*equivariant*. More precisely, $F_0(\gamma_n x, \gamma_r\alpha) = \gamma_m F_0(x,\alpha)$, $\forall\, \gamma \in \Gamma$.

In what follows we establish preliminary results needed to define the tangent space and the unipotent tangent space to a Γ- equivariant path. We have to keep track of the symmetry on $\lambda \in \mathbb{R}^\ell$. Because of that, we first enlarge the space of paths $\mathscr{P}$. Let $\hat{\mathscr{P}}$ be the set of Γ-equivariant paths defined by $\tilde{\beta}(\lambda) = (\tilde{\alpha}(\lambda),\, \lambda)$ for $\tilde{\alpha} \in \mathscr{P}$. Let $\pi_r\colon \mathbb{R}^{r+\ell} \to \mathbb{R}^r$ be the natural projection. For $\beta = (\alpha,\lambda) \in \mathbb{R}^{r+\ell}$, let $\hat{F}_o\colon (\mathbb{R}^{n+r+\ell},0) \to \mathbb{R}^m$ be defined as

$$\hat{F}_o(x,\beta) = F_o(x,\pi_r(\beta)) = (\mathbf{I}_n \times \pi_r)^* F_o(x,\beta)\,.$$

We also have the following β-parametrised spaces with the Γ-action on the β- space: $\vec{\mathscr{E}}^{\,\Gamma}_{(x,\beta)}$, $\mathbf{M}^{\Gamma}_{(x,\beta)}$, $\vec{\mathbf{\Theta}}^{\,\Gamma}_{(x,\beta)}$, $\vec{\mathscr{E}}^{\,\Gamma}_{\beta}$, $\mathscr{E}^{\Gamma}_{\beta}$ and $\vec{\mathbf{\Theta}}^{\,\Gamma,o}_{(x,\beta)}$. From a simple calculation it follows that $\hat{F}_o \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\beta)}$, $\tilde{\beta}^*\hat{F}_o = \tilde{\alpha}^*F_o$ and so $\tilde{\beta}^*\hat{F}_o \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$. We have already proved that $(\tilde{\alpha}^*F_o) \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ if $\tilde{\alpha} \in \mathscr{P}$ (Theorem 3·1·1).

The *tangent space at* $\hat{f} \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\beta)}$ is defined by

$$\mathscr{T}^{\Gamma}(\hat{f}) = \left\{ \hat{S}\hat{f} + \hat{f}_x\,\hat{\xi} \;\middle|\; \hat{S} \in \mathbf{M}^{\Gamma}_{(x,\beta)} \text{ and } \hat{\xi} \in \vec{\mathbf{\Theta}}^{\,\Gamma}_{(x,\beta)} \right\}$$

and the *unipotent tangent space at* $\hat{f} \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\beta)}$ is

$$\mathscr{T}\mathscr{U}^{\Gamma}(\hat{f}) = \{\hat{S}\hat{f} + \hat{f}_x\hat{\xi} \mid \hat{S} \in \mathbf{M}^{\Gamma}_{(x,\beta)},\, \hat{\xi} \in \vec{\mathbf{\Theta}}^{\,\Gamma,o}_{(x,\beta)} \quad \text{satisfying } \hat{S}^o = 0 \text{ and } \hat{\xi}^o_x = 0\}\,.$$

Because $\mathscr{O}^{\Gamma}_{\beta}$ is a noetherian ring (Montaldi [**20**]), the following intersections of $\mathscr{O}^{\Gamma}_{\beta}$-modules have a finite number of generators $\{h'_i\}^s_{i=1}$, $\{h''_i\}^t_{i=1}$, respectively, such that

$$\mathscr{T}^{\Gamma}(\hat{F}_o) \cap (\mathscr{O}_{\beta}\cdot\!<h_1\,\ldots\,h_r>)^{\Gamma} = \mathscr{O}^{\Gamma}_{\beta}\cdot\!<h'_1\,\ldots\,h'_s>, \tag{3·7}$$

and

$$\mathscr{T}\mathscr{U}^{\Gamma}(\hat{F}_o) \cap (\mathscr{O}_{\beta}\cdot\!<h_1\,\ldots\,h_r>)^{\Gamma} = \mathscr{O}^{\Gamma}_{\beta}\cdot\!<h''_1\,\ldots\,h''_t>\,. \tag{3·8}$$

Recall here that $\hat{F}_o$ is a polynomial. Note that $\mathscr{O}^{\Gamma}_{\beta}\cdot\!<h''_1\,\ldots\,h''_t>\subset \mathscr{O}^{\Gamma}_{\beta}\cdot\!<h'_1\,\ldots\,h'_s>$.

For any $1 \leqslant j \leqslant s$, we can decompose h'_j as $h'_j(x,\beta) = \sum_{i=1}^r (\eta_j)_i\,(\beta)\,h_i(x)$ where $\eta_j : (\mathbb{R}^{r+\ell},0) \to \mathbb{R}^r$, $\eta_j = (\eta_{j_1}\,\ldots\,\eta_{j_r})$ is Γ-equivariant, that is, $\eta_j\,(\gamma_{r+\ell}\beta) = \gamma_r\,\eta_j\,(\beta)$, for all $\gamma \in \Gamma$. We define

$$N = \mathscr{E}^{\Gamma}_{\beta}\cdot <\eta_1\,\ldots\,\eta_s>\,. \tag{3·9}$$

Similarly, for any $1 \leqslant j \leqslant t$, we can decompose h''_j as $h''_j(x,\beta) = \sum_{i=1}^r (\tilde{\eta}_j)_i\,(\beta)\,h_i(x)$ where $\tilde{\eta}_j : (\mathbb{R}^{r+\ell},0) \to \mathbb{R}^r$, $\tilde{\eta}_j = (\tilde{\eta}_{j_1}\,\ldots\,\tilde{\eta}_{j_r})$ is Γ-equivariant. We define

$$\tilde{N} = \mathscr{E}^{\Gamma}_{\beta}\cdot <\tilde{\eta}_1\,\ldots\,\tilde{\eta}_t>\,. \tag{3·10}$$

For $\tilde{\alpha} \in \mathscr{P}$, let $\omega_{\tilde{\alpha}} : \mathscr{P}_{\ell,r} \to \vec{\mathscr{E}}^{\,\Sigma}_{(x,\lambda)}$ be given by $\omega_{\tilde{\alpha}}(\xi) = \sum_{i=1}^r \xi_i\,h_i$. We define the *extended tangent space at the path* $\tilde{\alpha}$ by

$$\mathscr{T}_e(\tilde{\alpha}) = \tilde{\alpha}^*N + \tilde{\alpha}_{\lambda}\cdot \vec{\mathscr{E}}^{\,\Gamma}_{\lambda}$$

and the *unipotent tangent space at* $\tilde{\alpha}$ by

$$\mathscr{T}\mathscr{U}(\tilde{\alpha}) = \tilde{\alpha}^*\tilde{N} + \tilde{\alpha}_{\lambda}\cdot \vec{\mathscr{M}}^{\Gamma^2}_{\lambda}.$$

We denote by $\mathscr{P}(\tilde{\alpha})$ *the higher order terms* of $\tilde{\alpha} \in \mathscr{P}$ and define $\xi \in \mathscr{P}(\tilde{\alpha})$ if and only if $\omega_{\tilde{\alpha}}(\xi) \in \mathscr{P}(\tilde{\alpha}^*F_o)$.

Proposition 3·3·1. *Let* $\xi \in \mathscr{T}\mathscr{U}(\tilde{\alpha})$. *If* $\omega_{\tilde{\alpha}}(\xi) \in \mathrm{Itr}\,\mathscr{T}\mathscr{U}^{\Gamma}(\tilde{\alpha}^*F_o)$ *then* $\xi \in \mathscr{P}(\tilde{\alpha})$.

Proof. Since $\xi \in \mathscr{T}\mathscr{U}(\tilde{\alpha})$, we deduce that $\omega_{\tilde{\alpha}}(\xi) \in \mathscr{T}\mathscr{U}^{\,\Gamma}(\tilde{\alpha}^*F_o)$ by Proposition 3·4·5. By hypothesis, $\omega_{\tilde{\alpha}}(\xi) \in \mathrm{Itr}\,\mathscr{T}\mathscr{U}^{\Gamma}(\tilde{\alpha}^*F_o)$ and so $\omega_{\tilde{\alpha}}(\xi) \in \mathscr{P}(\tilde{\alpha}^*F_o)$ by Proposition 2·5·2. By definition, $\xi \in \mathscr{P}(\tilde{\alpha})$. □

We define the *normal extended tangent space at* $\tilde{\alpha} \in \mathscr{P}$ by $\mathscr{N}_e(\tilde{\alpha}) = \mathscr{P}/\mathscr{T}_e(\tilde{\alpha})$, and define the *codimension* of $\tilde{\alpha} \in \mathscr{P}$ as $\mathrm{cod}^{\Gamma}(\tilde{\alpha}) = \dim_{\mathbb{R}}\mathscr{N}_e(\tilde{\alpha})$.

THEOREM 3·3·2. *The map $\omega_{\tilde{\alpha}}$ induces an isomorphism between $\mathcal{N}_e(\tilde{\alpha})$ and $\mathcal{N}_e^{\Gamma}(\tilde{\alpha}^*F_o)$ as real vector spaces. Moreover if $\{\phi_i\}_{i=1}^r \subset \mathscr{P}$ projects into a basis of $\mathcal{N}_e(\tilde{\alpha})$, then $\{\omega_{\tilde{\alpha}}(\phi_i)\}_{i=1}^r$ projects into a basis of $\mathcal{N}_e^{\Gamma}(\tilde{\alpha}^*F_o)$.*

COROLLARY 3·3·3. $\mathrm{cod}^{\Gamma}(\tilde{\alpha}^*F_o)$ *is finite if and only if* $\mathrm{cod}^{\Gamma}(\tilde{\alpha})$ *is finite.*

3·4. *Proofs.*

From the decomposition of h'_j, $1 \leqslant j \leqslant s$, define $\hat{\eta}_j(\beta) = (\eta_j(\beta), \lambda)$. Note that $\hat{\eta}_j \in \vec{\mathcal{O}}_{\beta}^{\Gamma}$, since $\hat{\eta}_j(\gamma_{r+\ell}\beta) = (\eta_j(\gamma_{r+\ell}\beta), \gamma_\ell\lambda) = (\gamma_r\eta_j(\beta), \gamma_\ell\lambda) = \gamma_{r+\ell}\,\hat{\eta}_j(\beta)$, $\forall\, \gamma \in \Gamma$.
Define

$$\hat{N} = \mathscr{E}_{\beta}^{\Gamma}\cdot\!<\hat{\eta}_1 \,\dots\, \hat{\eta}_s> . \tag{3·11}$$

Similarly, for $1 \leqslant j \leqslant t$, let $\hat{\tilde{\eta}}_j(\beta,\lambda) = (\tilde{\eta}_j(\beta),\lambda)$. Note that $\hat{\tilde{\eta}}(\beta) \in \vec{\mathcal{O}}_{\beta}^{\Gamma}$. Define

$$\hat{\tilde{N}} = \mathscr{E}_{\beta}^{\Gamma}\cdot\!<\hat{\tilde{\eta}}_1 \,\dots\, \hat{\tilde{\eta}}_t> . \tag{3·12}$$

Let $\omega_\beta : \vec{\mathscr{E}}_{\beta}^{\Gamma} \to \vec{\mathscr{E}}_{(x,\beta)}^{\Gamma}$ be defined as $\omega_\beta(\hat{\mu}) = \sum_{i=1}^r (\pi_r \circ \hat{\mu})_i\, h_i$. Clearly $\omega_\beta(\hat{\mu}) \in \vec{\mathscr{E}}_{(x,\beta)}^{\Gamma}$.

PROPOSITION 3·4·1. $\omega_\beta^{-1}(\mathscr{T}^{\Gamma}(\hat{F}_o)) = \hat{N}$ *and* $\omega_\beta^{-1}(\mathscr{T}\mathscr{U}^{\Gamma}(\hat{F}_o)) = \hat{\tilde{N}}$.

Proof. We show that $\omega_\beta^{-1}(\mathscr{T}^{\Gamma}(\hat{F}_o)) = \hat{N}$ in two steps.

(i) $\hat{N} \subset \omega_\beta^{-1}(\mathscr{T}^{\Gamma}(\hat{F}_o))$. Let $\hat{\eta} \in \hat{N}$, that is, $\hat{\eta} = \sum_{j=1}^s \mu_j\hat{\eta}_j$. Therefore

$$\begin{aligned}\omega_\beta(\hat{\eta}) &= \sum_{i=1}^r (\pi_r \circ \hat{\eta})_i h_i = \sum_{i=1}^r \left(\sum_{j=1}^s \mu_j(\pi_r \circ \hat{\eta}_j)\right)_i h_i \\ &= \sum_{j=1}^s \mu_j \left(\sum_{i=1}^r (\eta_j)_i h_i\right) = \sum_{j=1}^s \mu_j h'_j.\end{aligned}$$

It follows that $\omega_\beta(\hat{\eta}) \in \mathscr{E}_{\beta}^{\Gamma}\cdot\!<h'_1 \,\dots\, h'_s> \subset \mathscr{T}^{\Gamma}(\hat{F}_o)$.

(ii) $\omega_\beta^{-1}(\mathscr{T}^{\Gamma}(\hat{F}_o)) \subset \hat{N}$. Let $\hat{\mu} \in \vec{\mathscr{E}}_{\beta}^{\Gamma}$ with $\omega_\beta(\hat{\mu}) \in \mathscr{T}^{\Gamma}(\hat{F}_o)$. We have to show that $\hat{\mu} \in \hat{N}$. Therefore $\omega_\beta(\hat{\mu}) = \sum_{i=1}^r (\pi_r \circ \hat{\mu})_i h_i \in \mathscr{T}^{\Gamma}(\hat{F}_o)$ implies that $\omega_\beta(\hat{\mu}) \in \mathscr{E}_{\beta}^{\Gamma}\cdot$ $<h'_1 \,\dots\, h'_s>$. Hence,

$$\begin{aligned}\omega_\beta(\hat{\mu}) &= \sum_{j=1}^r \mu_j h'_j = \sum_{j=1}^s \mu_j \left(\sum_{i=1}^r (\eta_j)_i h_i\right) \\ &= \sum_{i=1}^r \left(\sum_{j=1}^s (\mu_j(\pi_r \circ \hat{\eta}_j))_i\right) h_i = \omega_\beta\left(\sum_{j=1}^s \mu_j\hat{\eta}_j\right).\end{aligned}$$

By uniqueness, $\hat{\mu} = \sum_{j=1}^s \mu_j\hat{\eta}_j$ and so $\hat{\mu} \in \hat{N}$.

The proposition follows then from (i) and (ii). The proof that $\omega_\beta^{-1}(\mathscr{T}\mathscr{U}^{\Gamma}(\hat{F}_o)) = \hat{\tilde{N}}$ is similar. □

PROPOSITION 3·4·2. $\omega_{\tilde{\alpha}}$ *is an* $\mathbb{R}$*-isomorphism between* $\mathscr{P}$ *and* $(\mathscr{E}_\lambda\cdot\!<h_1 \,\dots\, h_r>)^{\Gamma}$.

Proof. It is straightforward to show that $\omega_{\tilde{\alpha}}$ is a $\mathbb{R}$-linear map and that $\omega_{\tilde{\alpha}}$ is injective since $\{h_i\}_{i=1}^r$ is a basis of $\mathcal{N}_e^{\Sigma}(f_0)$. What remains to be shown is that $\omega_{\tilde{\alpha}}(\mathscr{P}) = (\mathscr{E}_\lambda\cdot <h_1 \,\dots\, h_r>)^{\Gamma}$. We have already proved that for $\xi \in \mathscr{P}$, $\omega_{\tilde{\alpha}}(\xi)$ is Γ-equivariant and so $\omega_{\tilde{\alpha}}(\mathscr{P}) \subset (\mathscr{E}_\lambda\cdot <h_1 \,\dots\, h_r>)^{\Gamma}$. It remains to be verified that

$(\mathscr{E}_\lambda\cdot < h_1 \ldots h_r >)^\Gamma \subset \omega_{\tilde{\alpha}}(\mathscr{P})$. Let $\zeta \in (\mathscr{E}_\lambda\cdot < h_1 \ldots h_r >)^\Gamma$, that is, $\zeta(x,\lambda) = \sum_{i=1}^r \nu_i(\lambda)\, h_i(x)$. Let $\nu(\lambda) = (\nu_1(\lambda) \ldots \nu_r(\lambda))$. We claim that $\nu \in \mathscr{P}$.

Define $\xi = \int_\Gamma \gamma_r^t\, \nu\, \gamma_\ell\, d\gamma$ where $d\gamma$ is the Haar measure on Γ. For $h(x) = (h_1(x) \ldots h_r(x))$, we have that $\xi \in \mathscr{P}$ and

$$\begin{aligned}(\omega_{\tilde{\alpha}}(\xi))(x,\lambda) &= <\xi(\lambda), h(x)> = <(\textstyle\int_\Gamma \gamma_r^t\, \nu\, \gamma_\ell)(\lambda), h(x)> \\ &= \textstyle\int_\Gamma <\nu(\gamma_\ell\lambda), \gamma_r^t\, h(x)> = \int_\Gamma <\nu(\gamma_\ell\lambda), (\mathbf{I}_r \otimes \gamma_m^{-1})\, h(\gamma_n x)> \\ &= \textstyle\int_\Gamma \gamma_m^{-1} <\nu(\gamma_\ell\lambda), h(\gamma_n x)> = \int_\Gamma \gamma_m^{-1} \zeta(\gamma_\ell\lambda, \gamma_n x) \\ &= \zeta(x,\lambda) = (\omega_{\tilde{\alpha}}(\nu))(x,\lambda)\,.\end{aligned}$$

Therefore, by injectivity, $\nu = \xi$ and so $\nu \in \mathscr{P}$. $\square$

For $\tilde{\beta} \in \hat{\mathscr{P}}$, we define $\omega_{\tilde{\beta}}\colon \hat{\mathscr{P}} \to \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$ by $\omega_{\tilde{\beta}}(\tilde{\xi}) = \omega_{\tilde{\alpha}}\,(\pi_r \circ \tilde{\xi})$. By definition, it follows that $\omega_{\tilde{\beta}}(\tilde{\xi}) = \sum_{i=1}^r \xi_i\, h_i$ with $\xi = \pi_r \circ \tilde{\xi} = (\xi_1 \ldots \xi_r)$ and $\omega_{\tilde{\beta}}$ is an $\mathbb{R}$-isomorphism between $\hat{\mathscr{P}}$ and $(\mathscr{E}_\lambda\cdot<h_1 \ldots h_r>)^\Gamma$. Recall that

(i) For $\tilde{\beta} \in \hat{\mathscr{P}}$, $(\tilde{\beta}^*\hat{F}_o) \in \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}$,
$\hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o) = \{T\,(\tilde{\beta}^*\hat{F}_o) + (\tilde{\beta}^*\hat{F}_o)_x\, X \mid T \in \mathbf{M}^\Gamma_{(x,\lambda)} \text{ and } X \in \vec{\mathbf{\Theta}}^{\,\Gamma}_{(x,\lambda)}\}$,
$\widehat{\mathscr{T}\mathscr{U}}^\Gamma(\tilde{\beta}^*\hat{F}_o) = \{T\,(\tilde{\beta}^*\hat{F}_o) + (\tilde{\beta}^*\hat{F}_o)_x X \mid T \in \mathbf{M}^\Gamma_{(x,\lambda)}, X \in \vec{\mathbf{\Theta}}^{\,\Gamma,o}_{(x,\lambda)} \text{ with } T^o = X^o_x = 0\}$.

(ii) From (3·9) and (3·10), $N = \mathscr{E}^\Gamma_\beta\cdot<\eta_1 \ldots \eta_s>$, $\eta_j\colon (\mathbb{R}^{r+\ell},0) \to \mathbb{R}^r$ is Γ-equivariant, $1 \leqslant j \leqslant s$, and $\tilde{N} = \mathscr{E}^\Gamma_\beta\cdot < \tilde{\eta}_1 \ldots \tilde{\eta}_t >$, $\tilde{\eta}_j\colon (\mathbb{R}^{r+\ell},0) \to \mathbb{R}^r$ is Γ- equivariant, $1 \leqslant j \leqslant t$.

Let $(\tilde{\beta}^*\eta_j)\colon (\mathbb{R}^\ell,0) \to \mathbb{R}^r$, $(\tilde{\beta}^*\eta_j)(\lambda) = \eta_j(\tilde{\alpha}(\lambda),\lambda)$, $1 \leqslant j \leqslant s$ and $(\tilde{\beta}^*\tilde{\eta}_j)\colon (\mathbb{R}^\ell,0) \to \mathbb{R}^r$ be given by $(\tilde{\beta}^*\tilde{\eta}_j)(\lambda) = \eta_j(\tilde{\alpha}(\lambda),\lambda)$, $1 \leqslant j \leqslant t$. It follows that $\tilde{\beta}^*\eta_j$ and $\tilde{\beta}^*\tilde{\eta}_j$ belong to $\mathscr{P}$ and we may write $\tilde{\beta}^*\eta_j = \tilde{\alpha}^*\eta_j$, $1 \leqslant j \leqslant s$ and $\tilde{\beta}^*\tilde{\eta}_j = \tilde{\alpha}^*\tilde{\eta}_j$, $1 \leqslant j \leqslant t$.

We define
$$\tilde{\alpha}^*N = \mathscr{E}^\Gamma_\lambda\cdot<\tilde{\alpha}^*\eta_1 \ldots \tilde{\alpha}^*\eta_s> \tag{3·13}$$
and
$$\tilde{\alpha}^*\tilde{N} = \mathscr{E}^\Gamma_\lambda\cdot<\tilde{\alpha}^*\tilde{\eta}_1 \ldots \tilde{\alpha}^*\tilde{\eta}_t>\,. \tag{3·14}$$

From (3·11) and (3·12), $\hat{N} = \mathscr{E}^\Gamma_\beta\cdot < \hat{\eta}_1 \ldots \hat{\eta}_s >$, $\hat{\eta}_j\colon (\mathbb{R}^{r+\ell},0) \to \mathbb{R}^r$ is Γ-equivariant, $1 \leqslant j \leqslant s$, and $\hat{\tilde{N}} = \mathscr{E}^\Gamma_\beta\cdot<\hat{\tilde{\eta}}_1 \ldots \hat{\tilde{\eta}}_t>$, $\hat{\tilde{\eta}}_j\colon (\mathbb{R}^{r+\ell},0) \to \mathbb{R}^r$ is Γ- equivariant, $1 \leqslant j \leqslant t$. By definition, $(\tilde{\beta}^*\hat{\eta}_j)(\lambda) = \hat{\eta}_j(\tilde{\alpha}(\lambda),\lambda) = (\eta_j(\tilde{\alpha}(\lambda),\lambda),\lambda)$, $1 \leqslant j \leqslant s$, and $(\tilde{\beta}^*\hat{\tilde{\eta}}_j)(\lambda) = \hat{\tilde{\eta}}_j(\tilde{\alpha}(\lambda),\lambda) = (\tilde{\eta}_j(\tilde{\alpha}(\lambda),\lambda),\lambda)$, $1 \leqslant j \leqslant t$. Hence $\tilde{\beta}^*\hat{\eta}_j$ and $\tilde{\beta}^*\hat{\tilde{\eta}}_j$ belong to $\hat{\mathscr{P}}$ for all $1 \leqslant j \leqslant s$, $1 \leqslant j \leqslant t$, respectively.

We define
$$\tilde{\beta}^*\hat{N} = \mathscr{E}^\Gamma_\beta\cdot<\tilde{\beta}^*\hat{\eta}_1 \ldots \tilde{\beta}^*\hat{\eta}_s> \tag{3·15}$$
and
$$\tilde{\beta}^*\hat{\tilde{N}} = \mathscr{E}^\Gamma_\beta\cdot<\tilde{\beta}^*\hat{\tilde{\eta}}_1 \ldots \tilde{\beta}^*\hat{\tilde{\eta}}_t>\,. \tag{3·16}$$

Proposition 3·4·3. *$\omega_{\tilde{\beta}}^{-1}(\hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o)) = \tilde{\beta}^*\hat{N}$ and $\omega_{\tilde{\beta}}^{-1}(\widehat{\mathscr{T}\mathscr{U}}^\Gamma(\tilde{\beta}^*\hat{F}_o)) = \tilde{\beta}^*\hat{\tilde{N}}$, where $\tilde{\beta}^*\hat{N}$ and $\tilde{\beta}^*\hat{\tilde{N}}$ are defined in* (3·15) *and* (3·16).

Proof. To prove $\omega_{\tilde{\beta}}^{-1}(\hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o)) = \tilde{\beta}^*\hat{N}$ we show first in two steps that $\omega_{\tilde{\beta}}^{-1}(\hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o)) \subset \tilde{\beta}^*\hat{N}$.

Step 1. By definition, $\tilde{\beta}$ is an immersion and so there exists $\mathbf{\Psi}$: $(\mathbb{R}^{r+\ell}, 0) \to (\mathbb{R}^{\ell}, 0)$ such that $\mathbf{\Psi} \circ \tilde{\beta} = \mathbf{I}_\ell$. It is then possible to exhibit a Γ-equivariant map $\tilde{\mathbf{\Psi}}$, that is, $\tilde{\mathbf{\Psi}} \circ \gamma_{r+\ell} = \gamma_\ell\, \tilde{\mathbf{\Psi}}$, such that $\tilde{\mathbf{\Psi}} \circ \tilde{\beta} = \mathbf{I}_\ell$.

We define $\tilde{\mathbf{\Psi}} = \int_\Gamma \gamma_\ell^t\, \mathbf{\Psi} \circ \gamma_{r+\ell}$. Using the properties of Haar integration, $\tilde{\mathbf{\Psi}}$: $(\mathbb{R}^{r+\ell}, 0) \to (\mathbb{R}^{\ell}, 0)$ and

$$\begin{aligned} \tilde{\mathbf{\Psi}} \circ \gamma'_{r+\ell} &= \int_\Gamma \gamma_\ell^t\, \mathbf{\Psi} \circ \gamma_{r+\ell}\gamma'_{r+\ell} = \int_\Gamma \gamma'_\ell\, \gamma_\ell'^{\,t}\, \gamma_\ell^t \circ \mathbf{\Psi} \circ \gamma_{r+\ell}\gamma'_{r+\ell} \\ &= \int_\Gamma \gamma'_\ell\, \nu^t\, \mathbf{\Psi} \circ \nu \;\; (\text{with } \nu = \gamma\gamma') \; = \gamma'_\ell \circ \tilde{\mathbf{\Psi}}\,. \end{aligned}$$

Further, $\tilde{\mathbf{\Psi}} \circ \tilde{\beta} = \int_\Gamma \gamma_\ell^t\, \mathbf{\Psi} \circ \gamma_{r+\ell} \circ \tilde{\beta} = \int_\Gamma \gamma_\ell^t\, \mathbf{\Psi} \circ \tilde{\beta} \circ \gamma_\ell = \int_\Gamma \gamma_\ell^t\, \gamma_\ell = \mathbf{I}_\ell$.

Via $\tilde{\mathbf{\Psi}}$,

(i) $\mathbf{M}^\Gamma_{(x,\lambda)} = \tilde{\beta}^*\mathbf{M}^\Gamma_{(x,\beta)}$ since $\tilde{\beta}^*\mathbf{M}^\Gamma_{(x,\beta)} \subset \mathbf{M}^\Gamma_{(x,\lambda)}$ and, for $T \in \mathbf{M}^\Gamma_{(x,\lambda)}$, we may define $\hat{S}(x,\beta) = (\tilde{\mathbf{\Psi}}^* T)(x,\beta) = T(x, \tilde{\mathbf{\Psi}}(\beta))$. Clearly, $\hat{S} \in \mathbf{M}^\Gamma_{(x,\beta)}$ and

$$(\tilde{\beta}^*\hat{S})(x,\lambda) = \hat{S}(x, \tilde{\beta}(\lambda)) = T(x, \tilde{\mathbf{\Psi}} \circ \tilde{\beta}(\lambda)) = T(x,\lambda)\,.$$

(ii) $\vec{\mathbf{\Theta}}^\Gamma_{(x,\lambda)} = \tilde{\beta}^*\vec{\mathbf{\Theta}}^\Gamma_{(x,\beta)}$ since $\tilde{\beta}^*\vec{\mathbf{\Theta}}^\Gamma_{(x,\beta)} \subset \vec{\mathbf{\Theta}}^\Gamma_{(x,\lambda)}$ and for $X \in \vec{\mathbf{\Theta}}^\Gamma_{(x,\lambda)}$, we may define $\hat{\xi}(x,\beta) = (\tilde{\mathbf{\Psi}}^* X)(x,\beta) = X(x, \tilde{\mathbf{\Psi}}(\beta))$. Clearly, $\hat{\xi} \in \vec{\mathbf{\Theta}}^\Gamma_{(x,\beta)}$, and

$$\tilde{\beta}^*\hat{\xi}(x,\lambda) = \hat{\xi}(x, \tilde{\beta}(\lambda)) = X(x, \tilde{\mathbf{\Psi}} \circ \tilde{\beta}(\lambda)) = X(x,\lambda)\,.$$

Step 2. Let $\tilde{\xi} \in \hat{\mathscr{P}}$ such that $\omega_{\tilde{\beta}}(\tilde{\xi}) \in \hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o)$. We have to show that $\tilde{\xi} \in \tilde{\beta}^*\hat{N}$. Because $\omega_{\tilde{\beta}}(\tilde{\xi}) = \omega_{\tilde{\alpha}}(\pi_r \circ \tilde{\xi}) = \omega_{\tilde{\alpha}}(\xi) = \sum_{i=1}^r \xi_i\, h_i \in \hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o)$ there exist $\mathrm{T} \in \mathbf{M}^\Gamma_{(x,\lambda)}$ and $X \in \vec{\mathbf{\Theta}}^\Gamma_{(x,\lambda)}$ such that

$$\omega_{\tilde{\beta}}(\tilde{\xi})(x,\lambda) = T(x,\lambda)\,(\tilde{\beta}^*\hat{F}_o)(x,\lambda) + (\tilde{\beta}^*\hat{F}_o)_x(x,\lambda)\,X(x,\lambda)\,.$$

From Step 1, $\omega_{\tilde{\beta}}(\tilde{\xi}) = \tilde{\beta}^*(\hat{S}\hat{F}_o + (\hat{F}_o)_x\,\hat{\xi}) \in \tilde{\beta}^*(\mathscr{T}^\Gamma(\hat{F}_o))$ with $\hat{S} \in \mathbf{M}^\Gamma_{(x,\beta)}$ and $\hat{\xi} \in \vec{\mathbf{\Theta}}^\Gamma_{(x,\beta)}$. By Proposition 3·4·1, $\omega_\beta^{-1}\,(\mathscr{T}^\Gamma(\hat{F}_o)) = \hat{N}$ and so we may write $\omega_{\tilde{\beta}}(\tilde{\xi}) \in \tilde{\beta}^*\,\omega_\beta(\hat{N})$; that is, there exists $\hat{\xi} \in \hat{N}$, $\hat{\xi} = \sum_{j=1}^s \mu_j\,\hat{\eta}_j$, such that

$$\begin{aligned} \omega_{\tilde{\beta}}(\tilde{\xi}) &= \tilde{\beta}^*\,\omega_\beta(\hat{\xi}) = \tilde{\beta}^*\,(\textstyle\sum_{i=1}^r (\pi_r \circ \hat{\xi})_i\, h_i) = \tilde{\beta}^*\left(\sum_{i=1}^r (\sum_{j=1}^s \mu_j\,(\pi_r \circ \hat{\eta}_j))_i\, h_i\right) \\ &= \textstyle\sum_{i=1}^r \tilde{\beta}^*\,(\pi_r \circ \sum_{j=1}^s \mu_j\,\hat{\eta}_j)_i\, h_i = \sum_{i=1}^r (\pi_r \circ (\tilde{\beta}^*\,\sum_{j=1}^s \mu_j\,\hat{\eta}_j))_i\, h_i = \omega_{\tilde{\beta}}\,(\tilde{\beta}^*\hat{\xi}). \end{aligned}$$

By uniqueness, $\tilde{\xi} = \tilde{\beta}^*\hat{\xi}$ with $\hat{\xi} \in \hat{N}$ and so $\tilde{\xi} \in \tilde{\beta}^*\hat{N}$. Therefore, $\omega_{\tilde{\beta}}^{-1}(\hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^* F_o)) \subset \tilde{\beta}^*\hat{N}$.

Now we wish to show the converse: $\omega_{\tilde{\beta}}(\tilde{\beta}^*\hat{N}) \subset \hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o)$. Let $\hat{\xi} \in \hat{N}$, that is, $\hat{\xi} = \sum_{j=1}^s \mu_j\,\hat{\eta}_j$. By Proposition 3·4·1, $\omega_\beta(\hat{\xi}) \in \mathscr{T}^\Gamma(\hat{F}_o)$ and so there exist some $\hat{S} \in \mathbf{M}^\Gamma_{(x,\beta)}$ and $\hat{\xi}' \in \vec{\mathbf{\Theta}}^\Gamma_{(x,\beta)}$ with $\omega_\beta(\hat{\xi}) = \hat{S}\hat{F}_o + (\hat{F}_o)_x\,\hat{\xi}'$. Now, from Step 1,

$$\begin{aligned} \omega_{\tilde{\beta}}\,(\tilde{\beta}^*\hat{\xi}) &= \textstyle\sum_{i=1}^r (\pi_r \circ \tilde{\beta}^*\hat{\xi})_i\, h_i = \sum_{i=1}^r \tilde{\beta}^*(\pi_r \circ \hat{\xi})_i\, h_i = \tilde{\beta}^* \sum_{i=1}^r (\pi_r \circ \hat{\xi})_i\, h_i = \tilde{\beta}^*\omega_\beta(\hat{\xi}) \\ &= \tilde{\beta}^*\,(\hat{S}\hat{F}_o + (\hat{F}_o)_x\,\hat{\xi}') = (\tilde{\beta}^*\,\hat{S})\,(\tilde{\beta}^*\hat{F}_o) + (\tilde{\beta}^* F_o)_x\,(\tilde{\beta}^*\,\hat{\xi}') \\ &= \mathrm{T}\,(\tilde{\beta}^*\hat{F}_o) + (\tilde{\beta}^* F_o)_x\, X \end{aligned}$$

for some $T \in \mathbf{M}^\Gamma_{(x,\lambda)}$ and $X \in \vec{\mathbf{\Theta}}^\Gamma_{(x,\lambda)}$. Hence, $\omega_{\tilde{\beta}}\,(\tilde{\beta}^*\hat{\xi}) \in \hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o)$ and $\omega_{\tilde{\beta}}\,(\tilde{\beta}^*\hat{N}) \subset \hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o)$.

We conclude that $\omega_{\tilde{\beta}}^{-1}(\hat{\mathscr{T}}^\Gamma_e(\tilde{\beta}^*\hat{F}_o)) = \tilde{\beta}^*\hat{N}$.

It remains to show that $\omega_{\tilde\beta}^{-1}(\widehat{\mathscr{T}\mathscr{U}}^{\Gamma}(\tilde\beta^*\hat F_o)) = \tilde\beta^*\hat{\tilde N}$. This is similar to the proof of the previous result, so we omit the details of the calculations. We use Proposition 3·4·1, the definition of $\widehat{\mathscr{T}\mathscr{U}}^{\Gamma}(\hat F_o)$ and write, via $\tilde{\mathbf{\Psi}}$, $(\tilde\beta^*\hat S)(x,\lambda) = T(x,\lambda)$ with $\hat S \in \mathbf{M}^{\Gamma}_{(x,\beta)}$, $T \in \mathbf{M}^{\Gamma}_{(x,\lambda)}$, and $(\tilde\beta^*\hat\xi)(x,\lambda) = X(x,\lambda)$ with $\hat\xi \in \vec{\mathbf{\Theta}}^{\Gamma,o}_{(x,\beta)}$, $X \in \vec{\mathbf{\Theta}}^{\Gamma,o}_{(x,\lambda)}$. Note that $(\tilde\beta^*\hat S)^o = 0$ if and only if $T^o = 0$ and $(\tilde\beta^*\hat\xi)^o_x = 0$ if and only if $X^o_x = 0$. Therefore the proposition holds. □

Proposition 3·4·4. *Let* $\tilde\alpha \in \mathscr{P}$. *Then,* $\omega_{\tilde\alpha}^{-1}(\hat{\mathscr{T}}^{\Gamma}_e(\tilde\alpha^*F_o)) = \tilde\alpha^*N$ *and* $\omega_{\tilde\alpha}^{-1}(\widehat{\mathscr{T}\mathscr{U}}^{\Gamma}(\tilde\alpha^*F_o)) = \tilde\alpha^*\tilde N$.

Proof. This follows from Proposition 3·4·3 since $\tilde\alpha^*F_o = \tilde\beta^*\hat F_o$, $\omega_{\tilde\beta}(\tilde\xi) = \omega_{\tilde\alpha}(\pi_r \circ \tilde\xi)$ and $\tilde\alpha^*N = \tilde\beta^*(\pi_r \circ \hat N)$, $\tilde\alpha^*\tilde N = \tilde\beta^*(\pi_r \circ \hat{\tilde N})$. Here $\hat N$, $\hat{\tilde N}$ are given by (3·11), (3·12), and $\pi_r \circ \hat N$, $\pi_r \circ \hat{\tilde N}$ denote the $\mathscr{E}^{\Gamma}_{\beta}$-modules $\pi_r \circ \hat N = \mathscr{E}^{\Gamma}_{\beta}\cdot\!<\pi_r\circ\hat\eta_1 \,\ldots\, \pi_r\circ\hat\eta_s>$ and $\pi_r \circ \hat{\tilde N} = \mathscr{E}^{\Gamma}_{\beta}\cdot\!<\pi_r\circ\hat{\tilde\eta}_1 \,\ldots\, \pi_r\circ\hat{\tilde\eta}_t>$, respectively. □

Proposition 3·4·5. $\omega_{\tilde\alpha}^{-1}(\mathscr{T}^{\Gamma}_e(\tilde\alpha^*F_o)) = \mathscr{T}_e(\tilde\alpha)$ *and* $\omega_{\tilde\alpha}^{-1}(\mathscr{T}\mathscr{U}^{\Gamma}(\tilde\alpha^*F_o)) = \mathscr{T}\mathscr{U}(\tilde\alpha)$.

Proof. We show the first part in two steps:

(i) $\mathscr{T}_e(\tilde\alpha) \subset \omega_{\tilde\alpha}^{-1}(\mathscr{T}^{\Gamma}_e(\tilde\alpha^*F_o))$. We can split $\tilde\xi \in \mathscr{T}_e(\tilde\alpha)$ as $\tilde\xi = \xi_1 + \xi_2$ with $\xi_1 \in \tilde\alpha^*N$ and $\xi_2 \in \tilde\alpha_\lambda \cdot \vec{\mathscr{E}}^{\Gamma}_{\lambda}$. From Proposition 3·4·4, $\omega_{\tilde\alpha}(\xi_1) \in \hat{\mathscr{T}}^{\Gamma}_e(\tilde\alpha^*F_o)$. Let $\xi_2 = (L_1 \,\ldots\, L_r)$ where $L_i = \sum_{j=1}^{\ell}(\tilde\alpha_i)_{\lambda_j}\Lambda_j$. Then

$$\omega_{\tilde\alpha}(\xi_2)(x,\lambda) = \sum_{i=1}^{r} L_i(\lambda)\, h_i(x) = (F_o)_\alpha(x,\tilde\alpha(\lambda))\,\tilde\alpha_\lambda(\lambda)\,\Lambda(\lambda) = (\tilde\alpha^*F_o)_\lambda(x,\lambda)\,\Lambda(\lambda)\,.$$

Hence, $\omega_{\tilde\alpha}(\xi_2) = (\tilde\alpha^*F_o)_\lambda \circ \Lambda$ for some $\Lambda \in \vec{\mathscr{E}}^{\Gamma}_{\lambda}$. From the linearity of $\omega_{\tilde\alpha}$, $\omega_{\tilde\alpha}(\tilde\xi) \in \mathscr{T}^{\Gamma}_e(\tilde\alpha^*F_o)$. Therefore (i) holds.

(ii) Let $\tilde\xi \in \mathscr{P}$ and suppose that $\omega_{\tilde\alpha}(\tilde\xi) = \zeta \in \mathscr{T}^{\Gamma}_e(\tilde\alpha^*F_o)$. We have to show that $\tilde\xi \in \mathscr{T}(\tilde\alpha)$. We can split $\mathscr{T}^{\Gamma}_e(\tilde\alpha^*F_o) = \hat{\mathscr{T}}^{\Gamma}_e(\tilde\alpha^*F_o) + \left\{(\tilde\alpha^*F_o)_\lambda \circ \Lambda \;\middle|\; \Lambda \in \vec{\mathscr{E}}^{\Gamma}_{\lambda}\right\}$ and $\zeta = \zeta_1 + \zeta_2$. We claim that there exists $\xi_2 \in \tilde\alpha_\lambda \cdot \vec{\mathscr{E}}^{\Gamma}_{\lambda}$ such that $\omega_{\tilde\alpha}(\xi_2) = \zeta_2$. As a matter of fact, $\zeta_2 = (\tilde\alpha^*F_o)_\lambda \circ \Lambda$ for some $\Lambda \in \vec{\mathscr{E}}^{\Gamma}_{\lambda}$ and so

$$\zeta_2 = (\tilde\alpha^*F_o)_\lambda \circ \Lambda = \sum_{i=1}^{r}\left(\sum_{j=1}^{\ell}(\tilde\alpha_i)_{\lambda_j}\Lambda_j\right) h_i = \omega_{\tilde\alpha}(\tilde\alpha_\lambda \circ \Lambda)\,.$$

Therefore there exists ξ_2 satisfying the claim.

From the linearity of $\omega_{\tilde\alpha}$, $\zeta_1 = \omega_{\tilde\alpha}(\tilde\xi - \xi_2)$ and by Proposition 3·4·4, $\tilde\xi - \xi_2 \in \tilde\alpha^*N$. Hence, $\omega_{\tilde\alpha}^{-1}(\mathscr{T}^{\Gamma}_e(\tilde\alpha^*F_o)) \subset \mathscr{T}_e(\tilde\alpha)$. From (i), $\omega_{\tilde\alpha}^{-1}(\mathscr{T}^{\Gamma}_e(\tilde\alpha^*F_o)) = \mathscr{T}_e(\tilde\alpha)$.

The proof of the second part is analogous to what we have done for the first part: now we use Proposition 3·4·4 and the definitions of the unipotent tangent spaces at $\tilde\alpha^*F_o$ and $\tilde\alpha$. □

Proof of Theorem 3·3·2. Consider the following diagram:

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\ \omega_{\tilde\alpha}\ } & \vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)} \\ \Big\downarrow \pi_{\tilde\alpha} & & \Big\downarrow \pi_{\tilde\alpha} \\ \dfrac{\mathscr{P}}{\mathscr{T}_e(\tilde\alpha)} & \xrightarrow{\ \mathbf{\Omega}_{\tilde\alpha}\ } & \dfrac{\vec{\mathscr{E}}^{\,\Gamma}_{(x,\lambda)}}{\mathscr{T}^{\Gamma}_e(\tilde\alpha^{\,*}F_o)} \end{array}$$

Define $\mathbf{\Omega}_{\tilde\alpha}$: $\mathscr{N}_e(\tilde\alpha) \to \mathscr{N}^{\Gamma}_e(\tilde\alpha^{\,*}F_o)$ by $\mathbf{\Omega}_{\tilde\alpha}([v]) = [\omega_{\tilde\alpha}(v)]$.

(i) $\mathbf{\Omega}_{\tilde\alpha}$ is well-defined, that is, $[v_1] = [v_2]$ implies that $\mathbf{\Omega}_{\tilde\alpha}([v_1]) = \mathbf{\Omega}_{\tilde\alpha}([v_2])$. Note that if $[v_1] = [v_2]$ then $v_1 - v_2 \in \mathscr{T}_e(\tilde\alpha)$ and so, $\omega_{\tilde\alpha}(v_1 - v_2) \in \mathscr{T}^{\Gamma}_e(\tilde\alpha^{\,*}F_o)$ by Proposition 3·4·5. Hence, $[\omega_{\tilde\alpha}(v_1)] = [\omega_{\tilde\alpha}(v_2)]$.

(ii) $\mathbf{\Omega}_{\tilde\alpha}$ is $\mathbb{R}$-linear since $\omega_{\tilde\alpha}$ and the projections are $\mathbb{R}$-linear.

(iii) $\mathbf{\Omega}_{\tilde\alpha}$ is injective since if $\mathbf{\Omega}_{\tilde\alpha}([v_1]) = \mathbf{\Omega}_{\tilde\alpha}([v_2])$ then $\omega_{\tilde\alpha}(v_1) - \omega_{\tilde\alpha}(v_2) \in \mathscr{T}^{\Gamma}_e(\tilde\alpha^{\,*}F_o)$. From Proposition 3·4·5, $(v_1 - v_2) \in \omega_{\tilde\alpha}^{-1}(\mathscr{T}^{\Gamma}_e(\tilde\alpha^{\,*}F_o)) = \mathscr{T}_e(\tilde\alpha)$ and so, $[v_1] = [v_2]$.

(iv) $\mathbf{\Omega}_{\tilde\alpha}$ is surjective.

$$\operatorname{Im}\mathbf{\Omega}_{\tilde\alpha} = \left\{[f] \in \mathscr{N}^{\Gamma}_e(\tilde\alpha^{\,*}F_o) \mid [f] = \mathbf{\Omega}_{\tilde\alpha}([v]) \text{ for some } [v] \in \mathscr{N}_e(\tilde\alpha)\right\}.$$

Since $\operatorname{Im}\mathbf{\Omega}_{\tilde\alpha} \subset \mathscr{N}^{\Gamma}_e(\tilde\alpha^{\,*}F_o)$, it remains to show that $\mathscr{N}^{\Gamma}_e(\tilde\alpha^{\,*}F_o) \subset \operatorname{Im}\mathbf{\Omega}_{\tilde\alpha}$. Let $[f] \in \mathscr{N}^{\Gamma}_e(\tilde\alpha^{\,*}F_o)$. By the definition of $\mathbf{\Omega}_{\tilde\alpha}$, $[f] = \pi_{\tilde\alpha}(\omega_{\tilde\alpha}(\xi)) = (\mathbf{\Omega}_{\tilde\alpha} \circ \pi_{\tilde\alpha})(\xi) = \mathbf{\Omega}_{\tilde\alpha}([\xi])$ for some $\xi \in \mathscr{P}$, that is, $[f] \in \operatorname{Im}\mathbf{\Omega}_{\tilde\alpha}$. Therefore $\mathbf{\Omega}_{\tilde\alpha}$ is an isomorphism between $\mathscr{N}_e(\tilde\alpha)$ and $\mathscr{N}^{\Gamma}_e(\tilde\alpha^{\,*}F_o)$. From Proposition 3·4·5, the other claim of the theorem is straightforward, since $\omega_{\tilde\alpha}^{-1}(\mathscr{T}^{\Gamma}_e(\tilde\alpha^{\,*}F_o)) = \mathscr{T}_e(\tilde\alpha)$.

4. *Classification of* $\mathbf{D}_4$*-equivariant bifurcation problems using the path formulation*

In this section we confirm and extend the classification obtained in [**8**].

4·1. *Organizing centres*

The $\mathbf{D}_4$-action on $\mathbb{R}^2\times\mathbb{R}^2$ that we are studying is defined by

$$\begin{aligned} \hat\kappa\cdot(x_1,x_2,\lambda_1,\lambda_2) &= (\kappa\cdot(x_1,x_2),\lambda_1,\lambda_2) = (x_1,-x_2,\lambda_1,\lambda_2),\\ \hat\mu\cdot(x_1,x_2,\lambda_1,\lambda_2) &= (\mu\cdot(x_1,x_2),\kappa\cdot(\lambda_1,\lambda_2)) = (x_2,x_1,\lambda_1,-\lambda_2). \end{aligned} \tag{4·1}$$

The isotropy subgroup of λ is $\mathbf{\Sigma} \stackrel{\text{def}}{=} <1,\hat\kappa,\hat\mu\hat\kappa\hat\mu,(\hat\kappa\hat\mu)^2> \simeq \mathbf{Z}_2\oplus\mathbf{Z}_2$.

As defined before, (3·2) induces an action on the $\mathbf{\Sigma}$-unfolding parameters $\alpha \in \mathbb{R}^r$. So we define the action of $\mathbf{D}_4$ on (z,α,λ) where $z = (x_1,x_2)$, $\alpha = (\alpha_1\ldots\alpha_r)$ and $\lambda = (\lambda_1,\lambda_2)$ by

$$(\kappa\cdot x,\gamma^{\kappa}_r\cdot\alpha,\lambda_1,\lambda_2) \quad\text{and}\quad (\mu\cdot x,\gamma^{\mu}_r\cdot\alpha,\lambda_1,-\lambda_2).$$

When it is clear from the context, we denote this action by $(\gamma_2\,x,\gamma_r\,\alpha,\gamma_\ell\,\lambda)$.

Let $N = z\bar z$, $\delta = -\frac{1}{2}(z^2+\bar z^2)$ and $u_4 = \lambda_2^2$. The ring $\mathscr{E}^{\mathbf{\Sigma}}_z$ of $\mathbf{\Sigma}$- invariant germs is generated by N and δ. The $\mathscr{E}^{\mathbf{\Sigma}}_z$-module of $\mathbf{\Sigma}$-equivariant germs $\vec{\mathscr{E}}^{\,\mathbf{\Sigma}}_z$ is generated by z and $\bar z$, and we identify h and $[p,q]$. The set $\mathbf{M}^{\mathbf{\Sigma}}_z$ of $\mathbf{\Sigma}$-commuting matrices is the $\mathscr{E}^{\mathbf{\Sigma}}_z$-module generated by

$$S_1(z)\,w = w\,,\ S_2(z)\,w = \bar{w}\,,\ S_3(z)\,w = \tfrac{i}{2}\omega\,(w+\bar{w})\,,\ S_4(z)\,w = \tfrac{i}{2}\omega\,(\bar{w}-w)\,,$$

where $\omega = \frac{i}{4}(\bar{z}^2 - z^2)$.

The extended tangent space is $\mathscr{T}^{\Sigma}_{e}(h) = \{Sh + h_z X \mid S \in \mathbf{M}^{\Sigma}_{z},\ X \in \vec{\mathscr{E}}^{\,\Sigma}_{z}\}$ and a calculation shows that $\mathscr{T}^{\Sigma}_{e}(h) = \mathscr{E}^{\Sigma}_{z}\cdot\!<h_1\ \ldots\ h_6>$ where

$$\begin{aligned} h_1 &= [p,q]\,, & h_3 &= (p+q)[N-\delta,\ \delta-N]\,, & h_5 &= [Np_N + \delta p_\delta,\ Nq_N + \delta q_\delta]\,,\\ h_2 &= [q,p]\,, & h_4 &= (p-q)[N+\delta,\ N+\delta]\,, & h_6 &= [\delta p_N + Np_\delta,\ \delta q_N + Nq_\delta]\,. \end{aligned}$$

The general form for a $\mathbf{D}_4$-bifurcation problem is

$$f(z,\lambda) = p(u)\,z + q(u)\,\delta\bar{z} + r(u)\,\lambda_2\bar{z} + s(u)\,\lambda_2\delta z$$

where $p,q,r,s \in \mathscr{E}_u$, $u = (N,\ \Delta,\ \lambda_1,\ u_4)$, and $p^o = 0$.

The organizing centre of f is $f_0(z) = f(z,0) = p(N,\Delta)\,z + q(N,\Delta)\,\delta\bar{z}$.

We classify those organising centres using the $\mathbf{D}_4$-theory. If we want to stop at topological codimension 2 problems (with two parameters), we need to consider the following cases. We denote by $\Delta_{x,y}(p,q)$ the expression $p^o_x q^o_y - p^o_y q^o_x$.

Theorem 4·1·1. *Let $f \in \vec{\mathscr{E}}^{\,\mathbf{D}_4}_{(x,\lambda)}$ of topological codimension less or equal to 2, then its organizing centre belongs to the following list:*

(i) $f^1_0(z) = mNz + \epsilon_5\delta\bar{z}$, *($f^1_0$ is the generic organizing centre), the nondegeneracy conditions are $m \neq \pm 1, 0$ and $q^o \neq 0$,*

(ii) *(when $m = 0$)* $f^2_0(z) = \epsilon_2 N^2 z + \epsilon_5\delta\bar{z}$,

(iii) *(when $m = -1$)* $f^3_0(z) = -\epsilon_5 Nz + \epsilon_5\,\delta\bar{z}$,

(iv) *(when $m = 1$)* $f^4_0(z) = \epsilon_1 Nz + \epsilon_3 N^2 z + \epsilon_1\delta\bar{z}$,

(v) *(when $q^o = 0$)* $f^5_0(z) = \epsilon_1 Nz + \epsilon_6\Delta\,\delta\bar{z}$.

Here $m = p^o_N/|\,q^o|$, $\epsilon_1 = sg\,p^o_N$, $\epsilon_2 = sg\,p^o_{NN}$, $\epsilon_3 = sg\,(p^o_{NN} + 2p^o_\Delta - 2q^o_N)$, $\epsilon_5 = sg\,q^o$ and $\epsilon_6 = \epsilon_1\,sg\,\Delta_{N,\Delta}(p,q)$.

Proof. We first rule out many organizing centres via the following remarks.

Suppose that the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-codimension of an organizing centre f_0 is k. Observe that the modal parameters of f_0 are also modal parameters for the path $\tilde{\alpha}$ and all component of $\tilde{\alpha}$ are 0 at the origin unless they correspond to a modal parameter of f_0. Hence, at constant and first order in the invariants λ_1, u_4, the tangent space $\mathscr{T}_e(\tilde{\alpha})$ has dimension less than or equal to $k+m$ (the vectorfields component) + 6 (the λ_1, u_4-derivatives part). Thus we require $3k - (k+m+6) \geqslant 3+m$, hence $k \geqslant \frac{9}{2} + m$; that is, the germ must be of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-topological codimension at least 5.

The centres listed in the Theorem are of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-topological codimension less or equal to 3. The next layers can be found in [**3**]. Of those, the only centre remaining under consideration is

$$\epsilon N^3 z + \epsilon_0\delta\bar{z}$$

because it is of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-topological codimension 4. Its $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-universal unfolding is

$$(\epsilon N^3 + \alpha_1 + \alpha_2 N + \alpha_3 N^2)\,z + (\epsilon_0\delta + \alpha_4)\,\bar{z}$$

with the $\mathbf{Z}_2$-action given by $-1 \mapsto (\alpha_1, \alpha_2, \alpha_3, -\alpha_4)$. An explicit analysis in this case show that a general path has codimension at least 3. □

Note that f^3_0 is not distinguished in the $\mathbf{D}_4$-theory for organizing centres. It belongs to the *same* $\mathscr{K}^{\mathbf{D}_4}$- class as f^1_0, but its $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-universal unfolding is different, and

Table 1. *Normal forms. Here top-cod denotes the topological* $\mathbf{D}_4$*-codimension and cod the differentiable* $\mathbf{D}_4$*-codimension*

Case	Normal form	Top-cod	Cod
I_0	$[mN + \epsilon_0\lambda_1,\ \epsilon_5,\ 1,\ 0]$	0	1
I_1	$[mN + \epsilon_3 N\lambda_1 + \epsilon_4\lambda_1^2 + \alpha,\ \epsilon_5,\ 1,\ 0]$	1	2
I_2	$[mN + nNu_4 + \epsilon_0\lambda_1,\ \epsilon_5,\ \lambda_1 + \epsilon_8 u_4 + \alpha,\ 0]$	1	3
I_3	$[mN + \epsilon_3 N\lambda_1 + \epsilon_{11}\lambda_1^3 + \alpha + \beta\lambda_1,\ \epsilon_5,\ 1,\ 0]$	2	3
I_4	$[mN + \epsilon_7 N\lambda_1^2 + \epsilon_4\lambda_1^2 + \alpha + \beta N\lambda_1,\ \epsilon_5,\ 1,\ 0]$	2	3
I_5	$[mN + \epsilon_3 N\lambda_1 + \epsilon_4\lambda_1^2 + n_1 u_4 + \alpha,\ \epsilon_5,\ \lambda_1 + \beta,\ 0]$	2	4
I_6	$[mN + \epsilon_0\lambda_1 + \epsilon_9 Nu_4,\ \epsilon_5,\ u_4 + \alpha + \beta\lambda_1,\ 0]$	2	3
I_7	$[mN + \epsilon_0\lambda_1 + \alpha + \beta Nu_4,\ \epsilon_5,\ \lambda_1 + \epsilon_8 u_4,\ 0]$	2	3
I_8	$[mN + \epsilon_0\lambda_1 + nNu_4 + \alpha u_4,\ \epsilon_5,\ \lambda_1 + \epsilon_{10}u_4^2 + \beta,\ 0]$	2	4

so we have to consider as an additional class for our classification. With only one parameter and no symmetry it was again not necessary to make that distinction (cf. [**10**]).

4·2. *Classification of* $\mathbf{D}_4$*-equivariant problems with organizing centre I*

The following theorem gives the classification up to topological codimension 2 of $\mathbf{D}_4$-problems with organizing centre $f_0^1(z) = mNz + \epsilon_5\,\delta\bar{z}$, $m \neq \pm 1, 0$.

THEOREM 4·2·1 (Recognition Theorem for f_0^1). *Let* $f = [p, q, r, s]$ *be a* $\mathbf{D}_4$-*bifurcation problem with organizing centre* f_0^1. *Then* f *is of topological codimension 0,1 or 2 if and only if it belongs to the following list. Moreover,* f *is* $\mathscr{K}_\lambda^{\mathbf{D}_4}$*-equivalent to the given normal form in each case below if and only if it satisfies the correspondent sets of defining and nondegeneracy conditions listed below in Section* 4·2·1. *In all cases* $p^o = 0$. *The parameters* α, β *are the unfolding parameters, as* m, m_1, m_2, n, n_1 *are moduli.*

4·2·1. *Additional information*

Case I_0: Nondegeneracy conditions

$$p_N^o \cdot p_{\lambda_1}^o \cdot r^o \cdot ({p_N^o}^2 - q^{o2}) \neq 0.$$

Case I_1: Defining condition

$$p_{\lambda_1}^o = 0.$$

Nondegeneracy conditions

$$p_N^o \cdot p_{\lambda_1\lambda_1}^o \cdot q^o \cdot r^o \cdot ({p_N^o}^2 - q^{o2}) \cdot (p_{N\lambda_1}^o q^o - q_{\lambda_1}^o p_N^o) \neq 0.$$

Case I_2: Defining condition

$$r^o = 0.$$

Nondegeneracy conditions

$$p_N^o \cdot p_{\lambda_1}^o \cdot q^o \cdot ({p_N^o}^2 - q^{o2}) \cdot \Delta_{\lambda_1,u_4}(p, r) \cdot \xi_3 \cdot \xi_4 \neq 0.$$

Case I_3: Defining conditions

$$p_{\lambda_1}^o = p_{\lambda_1\lambda_1}^o = 0.$$

Nondegeneracy conditions

$$p_N^o \cdot p_{\lambda_1\lambda_1\lambda_1}^o \cdot r^o \cdot ({p_N^o}^2 - q^{o2}) \cdot (p_{N\lambda_1}^o q^o - q_{\lambda_1}^o p_N^o) \neq 0.$$

Case I_4: Defining conditions

$$p_{\lambda_1}^o = p_{N\lambda_1}^o q^o - q_{\lambda_1}^o p_N^o = 0.$$

Nondegeneracy conditions

$$p_N^o \cdot p_{\lambda_1\lambda_1}^o \cdot r^o \cdot ({p_N^o}^2 - q^{o2}) \cdot \xi_7 \neq 0.$$

Case I_5: Defining conditions

$$r^o = p^o_{\lambda_1} = 0.$$

Nondegeneracy conditions

$$p^o_N \cdot q^o \cdot ({p^o_N}^2 - q^{o2}) \cdot \xi_8 \neq 0.$$

Case I_6: Defining conditions

$$r^o = \xi_4 = 0.$$

Nondegeneracy conditions: $p^o_N \cdot p^o_{\lambda_1} \cdot ({p^o_N}^2 - q^{o2}) \cdot \xi_3 \cdot \xi_5 \neq 0.$

Case I_7: Defining conditions

$$r^o = \xi_3 = 0.$$

Nondegeneracy conditions

$$p^o_N \cdot p^o_{\lambda_1} \cdot ({p^o_N}^2 - q^{o2}) \cdot \Delta_{\lambda_1,u_4}(p,r) \cdot \xi_4 \neq 0.$$

Case I_8: Defining conditions

$$r^o = \Delta_{\lambda_1,u_4}(p,r) = 0.$$

Nondegeneracy conditions

$$p^o_N \cdot p^o_{\lambda_1} \cdot ({p^o_N}^2 - q^{o2}) \cdot \xi_3 \cdot \xi_4 \cdot \xi_6 \neq 0.$$

Coefficients: $\epsilon_0 = sg\, p^o_{\lambda_1}$, $\epsilon_1 = sg\, p^o_N$, $\epsilon_2 = sg\, p^o_{NN}$, $\epsilon_3 = \epsilon_5\, sg\,(p^o_{N\lambda_1} q^o - q^o_{\lambda_1} p^o_N)$, $\epsilon_4 = sg\, p^o_{\lambda_1\lambda_1}$, $\epsilon_5 = sg\, q^o$, $\epsilon_7 = \epsilon_1\, \epsilon_5\, sg\, \xi_7$, $\epsilon_9 = \epsilon_0, \epsilon_5\, sg\,({p^o_N}^2 - q^{o2})\, \xi_3$, $\epsilon_{10} = (p^o_{\lambda_1} \cdot ({p^o_N}^2 - q^{o2}) \cdot sg\, \xi_6/\xi_4$,

$$\epsilon_8 = \epsilon_0\, sg\, \frac{(p^{o^2}_N - q^{o^2}) \cdot \Delta_{\lambda_1,u_4}(p,r)}{p^o_{\lambda_1} \cdot (q^o s^o - r^o_N p^o_N) + r^o_{\lambda_1} \cdot (p^{o^2}_N - q^{o^2})},\ \epsilon_{11} = sg\, p^o_{\lambda_1\lambda_1\lambda_1}.$$

$$\xi_2 = p^o_{\lambda_1} \cdot (r^o_N - s^o) + r^o \cdot (q^o_{\lambda_1} - p^o_{N\lambda_1}),$$

$$\begin{aligned}\xi_3 = {} & p^o_N \cdot (p^{o^2}_N - q^{o^2}) \cdot \Delta_{\lambda_1,u_4}(p,q) + q^o \cdot (p^{o^2}_N - q^{o^2}) \cdot (p^o_{N\lambda_1} p^o_{u_4} - p^o_{Nu_4} p^o_{\lambda_1}) \\ & + p^o_N \cdot p^o_{\lambda_1} \cdot q^o \cdot (r^{o^2}_N + s^{o^2}) - p^o_{\lambda_1} \cdot r^o_N \cdot s^o \cdot (p^{o^2}_N + q^{o^2}),\end{aligned}$$

$$\xi_4 = p^o_{\lambda_1} \cdot (q^o s^o - r^o_N p^o_N) + r^o_{\lambda_1} \cdot ({p^o_N}^2 - q^{o2}),$$

$$\xi_5 = p^o_{u_4} \cdot (q^o s^o - p^o_N r^o_N) + r^o_{u_4} \cdot ({p^o_N}^2 - q^{o2}),$$

$$\xi_6 = p^o_{u_4} \cdot r^o_{\lambda_1} \cdot (p^o_{\lambda_1 u_4} p^o_{\lambda_1} - p^o_{\lambda_1\lambda_1} p^o_{u_4}) + p^o_{\lambda_1} \cdot p^o_{u_4} \cdot (r^o_{\lambda_1\lambda_1} p^o_{u_4} - r^o_{\lambda_1 u_4} p^o_{\lambda_1}) - p^o_{\lambda_1} \cdot \Delta_{u_4 u_4,\lambda_1}(p,r),$$

$$\begin{aligned}\xi_7 = {} & p^o_{\lambda_1\lambda_1} \cdot p^o_{u_4} \cdot q^o \cdot ({p^o_N}^2 - q^{o2}) + p^o_N \cdot r^2_o \cdot (p^o_{N\lambda_1\lambda_1} q^o - p^o_N q^o_{\lambda_1\lambda_1}) \\ & + p^o_{\lambda_1\lambda_1} \cdot q^o \cdot r^o \cdot (s^o q^o - p^o_N r^o_N) + p^o_{\lambda_1\lambda_1} \cdot r^{o2} \cdot (p^o_N q^o_N - p^o_\Delta q^o - \tfrac{1}{2} p^o_{NN} q^o),\end{aligned}$$

$$\xi_8 = p^o_{\lambda_1\lambda_1} \cdot p^o_{u_4} \cdot r^o_{\lambda_1} \cdot (p^o_{N\lambda_1} q^o - q^o_{\lambda_1} p^o_N),$$

$$m = \frac{p^o_N}{|q^o|}, \quad n = \frac{\epsilon_5\, p^o_{\lambda_1} \cdot (q^{o^2} - p^{o^2}_N)}{q^{o^2} \cdot [p^o_{\lambda_1} \cdot (s^o q^o - p^o_N r^o_N) + r^o_{\lambda_1} \cdot (p^{o^2}_N - q^{o^2})]^2}\ \xi_3, \quad n_1 = \frac{p^o_{u_4} \cdot |\, p^o_{\lambda_1\lambda_1}\,|}{2\, {r^o_{\lambda_1}}^2}.$$

4·2·2. *Variational problems*

Another criterion affecting the choice of a normal form is the gradient structure of some bifurcation problems. For instance, the first example in [**8**] of buckling of elastic shells is usually given a variational formulation. We refer to Bridges and Furter [**3**] for a theory of the contact equivalence classification of gradient bifurcation problems. The difficulty is that $\mathscr{K}^{\mathbf{D}_4}_\lambda$- equivalence does not in general preserve the gradient structure of the problem, although it defines an equivalence relation on $\vec{\mathscr{E}}^{\,\mathbf{D}_4}_{\nabla,\lambda}$. Bridges and Furter [**3**] show that it is enough to look for normal forms that are gradients, and that the basis of the vector space $\vec{\mathscr{E}}^{\,\mathbf{D}_4}_{\nabla,\lambda}/(\vec{\mathscr{E}}^{\,\mathbf{D}_4}_{\nabla,\lambda} \cap \mathscr{T}_e(f))$ provides the generators needed for the gradient universal unfolding of f. In effect we are looking for normal forms and universal unfolding terms that are the gradients of suitable equivariant functionals.

In our problem, a routine calculation shows that

$$f_0(z) = p(N,\Delta)\, z + q(N,\Delta)\, \delta \bar{z}$$

is a gradient if and only if $q_N + 2p_\Delta \equiv 0$. By inspection all of the organising centers classified in Theorem 4.1.1 satisfy that condition. For their universal unfoldings, that condition together with the additional one $r_N + s + 2\Delta s_\Delta = 0$, is also readily checked. Thus we have the following result:

THEOREM 4·2·2. *The list of gradient bifurcation problems and the universal unfoldings in $\vec{\mathscr{E}}^{\mathbf{D}_4}_{\nabla,\lambda}$ of organizing centres f_0^1 of topological codimension up to 2 is the same as the list given in Theorem* 4·2·1.

This is again a situation where symmetry puts enough constraints on the diagrams so that the difference between gradient systems and the rest is negligible. This already happens for $\mathbf{D}_n$-equivariant ($n \geqslant 3$) bifurcation problems with one parameter, see Bridges and Furter [**3**].

4·2·3. *Hierarchy of parameters*

So far we have not considered any hierarchical structure involving parameters, such as $\lambda_1 \gg \lambda_2$ or $\lambda_2 \gg \lambda_1$, but retaining the same symmetry constraints. Such hierarchies require us to consider changes of coordinates in $\mathscr{K}^{\mathbf{D}_4}_\lambda$ with Λ satisfying

$$\Lambda(\lambda_1, \lambda_2) = (\Lambda_1(\lambda_1, \lambda_2), \Lambda_2(\lambda_2)) \tag{4·2}$$

or

$$\Lambda(\lambda_1, \lambda_2) = (\Lambda_1(\lambda_1), \Lambda_2(\lambda_1, \lambda_2)).$$

The advantage of such more restricted changes of coordinates is that they respect the order in λ_2 (respectively λ_1) of the λ_1-slices (respectively λ_2-slices), instead of simply respecting open regions in parameter space. We ask if any of the normal forms in Theorem 4·2·1 is also a normal form for this more restrictive equivalence, with the same codimension. The path formulation is particularly well-adapted to answer such a question, because the vectorfields generating N and $\tilde{N}$ are independent of such considerations. Only the part $\tilde{\alpha}_\lambda$ will change. Hence the new tangent spaces for the equivalence corresponding to (4·2) are given by

$$\mathscr{T}_e(\tilde{\alpha}) = \tilde{\alpha}^* N + \mathscr{E}^{\mathbf{D}_4}_\lambda \cdot \tilde{h}_4 + \mathscr{E}^{\mathbf{D}_4}_{\lambda_2} \cdot \tilde{h}_5$$

and

$$\mathscr{T}\mathscr{U}(\tilde{\alpha}) = \tilde{\alpha}^* \tilde{N} + \mathscr{M}^{\mathbf{D}_4^2}_\lambda \cdot \tilde{h}_4 + \mathscr{M}^{\mathbf{D}_4}_{\lambda_2} \cdot \tilde{h}_5,$$

where $\tilde{\alpha}^* N$ and $\tilde{\alpha}^* \tilde{N}$ are given in Theorem 4·3·1.

Note that the residual $\mathbf{Z}_2$-symmetry on λ_2 already imposes some restrictions. In particular, a simple inspection shows that the second assumption on Λ is too strong – none of our normal forms persists with the same codimension. It is then a straightforward verification to see that only $\mathbf{I}_0$ and $\mathbf{I}_6$ remain as normal forms with the same universal unfoldings for the more restrictive change of coordinates in (4·2).

4·3. *Proofs*

For Case I, $f_0^1(z) = mNz + \epsilon_5\,\delta\bar{z}$, $m \neq \pm 1, 0$. Using the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-description, we write $f_0^1 = [mN, \epsilon_5\delta]$. A calculation shows that

$$\mathscr{T}^\Sigma_e(f_0^1) = \mathscr{E}^\Sigma_{N,\delta} \cdot < [0, N],\ [\delta, 0],\ [0, \delta^2],\ [N^2, 0],\ [mN, \epsilon_5\delta] > .$$

The $\mathbf{\Sigma}$-normal space $\mathscr{N}_e^{\Sigma}(f_0^1)$ is generated by $[1,0]$, $[N,0]$ and $[0,1]$ as a real vector space. The $\mathbf{\Sigma}$-unfolding of f_0^1 is given by $F_0^1(z,\alpha) = \hat{m}\, Nz + \alpha_1\, z + (\epsilon_5\delta + \alpha_3)\,\bar{z}$, where $\hat{m}$ represents $(m+\alpha_2)$. Because $\mathbf{D}_4$ acts on z as $\bar{z}$ and $i\bar{z}$, it follows that $\{\ [1,0],\ [N,0],\ [0,1]\ \}$ is a good basis; the action on the α-space is generated by $\hat{\kappa}$ acting as $\mathbf{I}_3$ and $\hat{\mu}$ as -1 on the last component, but as the identity on the first two.

Let $\mathscr{P}_{2,3} = \{\tilde{\alpha}\colon (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)\}$ be the set of paths defined from the bifurcation parameter space to the unfolding parameter space. The action ψ of $\mathbf{D}_4$ on $\tilde{\alpha} \in \mathscr{P}_{2,3}$ is given by

$$\gamma \cdot \tilde{\alpha} = \gamma_3^t \cdot (\tilde{\alpha} \circ \gamma_\ell)\,, \quad \forall\, \gamma \in \mathbf{D}_4.$$

$\mathscr{P}$ denotes the set of $\mathbf{D}_4$-equivariant paths, that is,

$$\mathscr{P} = \mathscr{P}_{2,3}|_{\mathrm{Fix}\,\psi} = \{\tilde{\alpha} \in \mathscr{P} \ \mid\ \tilde{\alpha}(\gamma_\ell\lambda) = \gamma_3\,\tilde{\alpha}(\lambda)\}.$$

A calculation shows that $\tilde{\alpha} \in \mathscr{P}$ if and only if

$$\tilde{\alpha}(\lambda_1,\lambda_2) = (\alpha_1(\lambda_1,u_4),\ \alpha_2(\lambda_1,u_4),\ \lambda_2\alpha_3(\lambda_1,u_4)).$$

By Theorem 3·2·1, f is $\mathscr{K}_\lambda^{\mathbf{D}_4}$-equivalent to $\tilde{\alpha}^* F_0^1$ where

$$(\tilde{\alpha}^* F_0^1)(z,\lambda) = [(m+\alpha_2(\lambda_1,u_4))\,N + \alpha_1(\lambda_1,u_4),\ \epsilon_5,\ \alpha_3(\lambda_1,u_4),\ 0].$$

Theorem 4·3·1. *Let $\tilde{\alpha} \in \mathscr{P}$. Then the tangent spaces at $\tilde{\alpha}$ are*

$$\mathscr{T}_e(\tilde{\alpha}) = \mathscr{E}_\lambda^{\mathbf{D}_4} \cdot <\tilde{h}_1\ \ldots\ \tilde{h}_5> \tag{4·3}$$

and

$$\mathscr{T}\mathscr{U}(\tilde{\alpha}) = \mathscr{E}_\lambda^{\mathbf{D}_4} \cdot <\tilde{h}_1,\ \tilde{h}_2,\ \alpha_1\tilde{h}_3,\ \lambda_1\tilde{h}_3,\ u_4\tilde{h}_3,\ \lambda_1^2\tilde{h}_4,\ u_4\tilde{h}_4,\ \lambda_1\tilde{h}_5,\ u_4\tilde{h}_5> \tag{4·4}$$

where

$$\left.\begin{aligned}
\tilde{h}_1 &= (\hat{m}u_4\alpha_3^2, (1-\hat{m}^2)\alpha_1,\ \hat{m}\alpha_1\alpha_3\lambda_2),\\
\tilde{h}_2 &= (u_4\alpha_1\alpha_3, \hat{m}(1-\hat{m}^2)u_4\alpha_3,\ \alpha_1^2\lambda_2),\\
\tilde{h}_3 &= (\alpha_1, 0, \alpha_3\lambda_2),\\
\tilde{h}_4 &= ((\alpha_1)_{\lambda_1},\ (\alpha_2)_{\lambda_1},\ (\alpha_3)_{\lambda_1}\lambda_2),\\
\tilde{h}_5 &= (2u_4(\alpha_1)_{u_4},\ 2u_4(\alpha_2)_{u_4},\ \alpha_3\lambda_2 + 2u_4(\alpha_3)_{u_4}\lambda_2),
\end{aligned}\right\} \tag{4·5}$$

To find the tangent space at a path $\tilde{\alpha} \in \mathscr{P}$, we follow § 3·2. We start by calculating the tangent space at a germ $\hat{f} \in \vec{\mathscr{E}}_{(z,\beta)}^{\mathbf{D}_4}$. We denoted (α,λ) by β and define

$$\vec{\mathscr{E}}_{(z,\beta)}^{\mathbf{D}_4} = \left\{\hat{f}\colon (\mathbb{C}\ \times\mathbb{R}^3\times\mathbb{R}^2,0) \to \mathbb{C}\ \mid\ \hat{f}(\gamma_2 z, \gamma_{3+\ell}\beta) = \gamma_2\hat{f}(z,\beta),\ \forall\,\gamma \in \mathbf{D}_4\right\},$$

$$\vec{\mathbf{\Theta}}_{(z,\beta)}^{\mathbf{D}_4,o} = \{\hat{\xi} \in \vec{\mathbf{\Theta}}_{(z,\beta)}^{\mathbf{D}_4}\ \mid\ \hat{\xi}^o = 0\},$$

$$\mathbf{M}_{(z,\beta)}^{\mathbf{D}_4} = \{\hat{S}\colon (\mathbb{C}\ \times\mathbb{R}^3\times\mathbb{R}^2,0) \to \mathrm{GL}(2)\ \mid\ \hat{S}(\gamma_2 z,\gamma_{3+\ell}\beta)\,\gamma_2 = \gamma_2\,\hat{S}(z,\beta),\ \forall\,\gamma \in \mathbf{D}_4\ \}.$$

The action of $\mathbf{D}_4$ on $\mathbb{C}\ \times\mathbb{R}^3\times\mathbb{R}^2$ is given by

$$\begin{aligned}
\tilde{\kappa}\cdot(z,\alpha_1,\alpha_2,\alpha_3,\lambda_1,\lambda_2) &= (\bar{z},\alpha_1,\alpha_2,\alpha_3,\lambda_1,\lambda_2),\\
\tilde{\mu}\cdot(z,\alpha_1,\alpha_2,\alpha_3,\lambda_1,\lambda_2) &= (i\bar{z},\alpha_1,\alpha_2,-\alpha_3,\lambda_1,-\lambda_2).
\end{aligned}$$

The $\mathbf{D}_4$-invariants for this action are

$$\left.\begin{aligned}
&\hat{u}_1 = N = z\bar{z}, \quad \hat{u}_2 = \Delta = \delta^2 = \tfrac{1}{4}(z^2+\bar{z}^2)^2, \quad \hat{u}_3 = \lambda_1, \quad \hat{u}_4 = \lambda_2^2 = u_4,\\
&\hat{u}_5 = \alpha_1, \quad \hat{u}_6 = \alpha_2, \quad \hat{u}_7 = \alpha_3^2, \quad \hat{u}_8 = \delta\alpha_3, \quad \hat{u}_9 = \delta\lambda_2, \quad \hat{u}_{10} = \alpha_3\lambda_2\,.
\end{aligned}\right\} \tag{4·6}$$

They satisfy the following relations: $\hat{u}_8^2 = \hat{u}_2\hat{u}_7$, $\hat{u}_9^2 = \hat{u}_2\hat{u}_4$ and $\hat{u}_{10}^2 = \hat{u}_7\hat{u}_4$. Let $\bar{u} = (\hat{u}_1 \dots \hat{u}_7)$ where the $\hat{u}_i$, $1 \leqslant i \leqslant 7$, are stated in (4·6).

PROPOSITION 4·3·2. (i) *$\vec{\mathscr{E}}^{\mathbf{D}_4}_{(z,\beta)}$ is freely generated by z, $\delta\bar{z}$, $\lambda_2\bar{z}$, $\alpha_3\bar{z}$, $\delta\lambda_2 z$, $\delta\alpha_3 z$, $\alpha_3\lambda_2 z$ and $\delta\alpha_3\lambda_2\bar{z}$ as an $\mathscr{E}_{\bar{u}}$-module.*

(ii) *$\mathbf{M}^{\mathbf{D}_4}_{(z,\beta)}$ is the $\mathscr{E}_{\bar{u}}$- module freely generated by the following linear maps on $\mathbb{C}$:*

$$\begin{aligned}
&\hat{S}_1(z,\beta)\,w = w\,, \quad \hat{S}_2(z,\beta)\,w = \lambda_2\bar{w}\,, \quad \hat{S}_3(z,\beta)\,w = \delta\bar{w}\,,\\
&\hat{S}_4(z,\beta)\,w = \alpha_3\bar{w}\,, \quad \hat{S}_5(z,\beta)\,w = \delta\lambda_2 w\,, \quad \hat{S}_6(z,\beta)\,w = \delta\alpha_3 w\,,\\
&\hat{S}_7(z,\beta)\,w = \alpha_3\lambda_2 w\,, \quad \hat{S}_8(z,\beta)\,w = \delta\alpha_3\lambda_2\bar{w}\,, \quad \hat{S}_9(z,\beta)\,w = i\omega\bar{w}\,,\\
&\hat{S}_{10}(z,\beta)\,w = -i\lambda_2\omega w\,, \quad \hat{S}_{11}(z,\beta)\,w = -i\delta\omega w\,, \quad \hat{S}_{12}(z,\beta)\,w = -i\alpha_3\omega w\,,\\
&\hat{S}_{13}(z,\beta)\,w = i\delta\lambda_2\omega\bar{w}\,, \quad \hat{S}_{14}(z,\beta)\,w = i\delta\alpha_3\omega\bar{w}\,,\\
&\hat{S}_{15}(z,\beta)\,w = i\alpha_3\lambda_2\omega\bar{w}\,, \quad \hat{S}_{16}(z,\beta)\,w = -i\delta\alpha_3\lambda_2\omega w\,,
\end{aligned}$$

where $\omega = \dfrac{i}{4}(\bar{z}^2 - z^2)$.

From Proposition 4·3·2, every $\hat{f} \in \vec{\mathscr{E}}^{\mathbf{D}_4}_{(z,\beta)}$ is written as

$$\begin{aligned}
\hat{f}(z,\alpha,\lambda) \quad = \quad & p_1(\bar{u})\,z + p_2(\bar{u})\,\delta\bar{z} + p_3(\bar{u})\,\lambda_2\bar{z} + p_4(\bar{u})\,\alpha_3\bar{z}\\
& + p_5(\bar{u})\,\delta\lambda_2 z + p_6(\bar{u})\,\delta\alpha_3 z + p_7(\bar{u})\,\alpha_3\lambda_2 z + p_8(\bar{u})\,\delta\alpha_3\lambda_2\bar{z}
\end{aligned}$$

and we identify $\hat{f}$ and $[p_1 \dots p_8]$.

The tangent space at $\hat{f} \in \vec{\mathscr{E}}^{\mathbf{D}_4}_{(z,\beta)}$ is given by

$$\mathscr{T}_e^{\mathbf{D}_4}(\hat{f}) = \left\{ \hat{S}\hat{f} + \hat{f}_z\,\hat{X} \;\middle|\; \hat{S} \in \mathbf{M}^{\mathbf{D}_4}_{(z,\beta)},\ \hat{X} \in \vec{\mathscr{E}}^{\mathbf{D}_4}_{(z,\beta)} \right\}.$$

PROPOSITION 4·3·3. *$\mathscr{T}_e^{\mathbf{D}_4}(\hat{f}) = \mathscr{E}_{\bar{u}}\cdot\langle g_1 \dots g_{24}\rangle$ where*

$$\begin{aligned}
g_1 &= [p_1,\, p_2,\, p_3,\, p_4,\, p_5,\, p_6,\, p_7,\, p_8]\,,\\
g_2 &= [u_4p_3,\, u_4p_5,\, p_1,\, u_4p_7,\, p_2,\, u_4p_8,\, p_4,\, p_6]\,,\\
g_3 &= [\Delta p_2,\, p_1,\, \Delta p_5,\, \Delta p_6,\, p_3,\, p_4,\, \Delta p_8,\, p_7]\,,\\
g_4 &= [\alpha_3^2p_4,\, \alpha_3^2p_6,\, \alpha_3^2p_7,\, p_1,\, \alpha_3^2p_8,\, p_2,\, p_3,\, p_5]\\
g_5 &= [\Delta u_4p_5,\, u_4p_3,\, \Delta p_2,\, \Delta u_4p_8,\, p_1,\, u_4p_7,\, \Delta p_6,\, p_4],\\
g_6 &= [\Delta\alpha_3^2p_6,\, \alpha_3^2p_4,\, \Delta\alpha_3^2p_8,\, \Delta p_2,\, \alpha_3^2p_7,\, p_1,\, \Delta p_5,\, p_3],\\
g_7 &= [\alpha_3^2u_4p_7,\, \alpha_3^2u_4p_8,\, \alpha_3^2p_4,\, u_4p_3,\, \alpha_3^2p_6,\, u_4p_5,\, p_1,\, p_2],\\
g_8 &= [\Delta\alpha_3^2u_4p_8,\, \alpha_3^2u_4p_7,\, \Delta\alpha_3^2p_6,\, \Delta u_4p_5,\, \alpha_3^2p_4,\, u_4p_3,\, \Delta p_2,\, p_1],\\
g_9 &= [Np_1 - \Delta p_2,\, p_1 - Np_2,\, \Delta p_5 - Np_3,\, \Delta p_6 - Np_4,\, Np_5 - p_3,\, Np_6 - p_4,\\
&\qquad Np_7 - \Delta p_8,\, p_7 - Np_8],\\
g_{10} &= [\Delta u_4p_5 - Nu_4p_3,\, Nu_4p_5 - u_4p_3,\, Np_1 - \Delta p_2,\, Nu_4p_7 - \Delta u_4p_8,\\
&\qquad p_1 - Np_2,\, u_4p_7 - Nu_4p_8, \Delta p_6 - Np_4, Np_6 - p_4],\\
g_{11} &= [\Delta p_1 - N\Delta p_2,\, Np_1 - \Delta p_2,\, N\Delta p_5 - \Delta p_3,\, N\Delta p_6 - \Delta p_4,\, \Delta p_5 - Np_3,\\
&\qquad \Delta p_6 - Np_4,\, \Delta p_7 - N\Delta p_8,\, Np_7 - \Delta p_8],\\
g_{12} &= [\Delta\alpha_3^2p_6 - N\alpha_3^2p_4,\, N\alpha_3^2p_6 - \alpha_3^2p_4,\, N\alpha_3^2p_7 - \Delta\alpha_3^2p_8,\, Np_1 - \Delta p_2,\, \alpha_3^2p_7 - N\alpha_3^2p_8,\\
&\qquad p_1 - Np_2, \Delta p_5 - Np_3,\, Np_5 - p_3],\\
g_{13} &= [N\Delta u_4p_5 - \Delta u_4p_3,\, \Delta u_4p_5 - Nu_4p_3,\, \Delta p_1 - N\Delta p_2,\, \Delta u_4p_7 - N\Delta u_4p_8,\\
&\qquad Np_1 - \Delta p_2,\, Nu_4p_7 - \Delta u_4p_8, N\Delta p_6 - \Delta p_4, \Delta p_6 - Np_4],\\
g_{14} &= [N\Delta\alpha_3^2p_6 - \Delta\alpha_3^2p_4,\, \Delta\alpha_3^2p_6 - N\alpha_3^2p_4,\, \Delta\alpha_3^2p_7 - N\Delta\alpha_3^2p_8,\, \Delta p_1 - N\Delta p_2,\\
&\qquad N\alpha_3^2p_7 - \Delta\alpha_3^2p_8,\, Np_1 - \Delta p_2, N\Delta p_5 - \Delta p_3,\, \Delta p_5 - Np_3],\\
g_{15} &= [Nu_4\alpha_3^2p_7 - u_4\alpha_3^2\Delta p_8,\, u_4\alpha_3^2p_7 - Nu_4\alpha_3^2p_8,\, \Delta\alpha_3^2p_6 - N\alpha_3^2p_4,\, \Delta u_4p_5 - Nu_4p_3,\\
&\qquad N\alpha_3^2p_6 - \alpha_3^2p_4,\, Nu_4p_5 - u_4p_3, Np_1 - \Delta p_2,\, p_1 - Np_2],
\end{aligned}$$

$$
\begin{aligned}
g_{16} = [&\Delta\alpha_3^2 u_4 p_7 - N\Delta\alpha_3^2 u_4 p_8,\ N\alpha_3^2 u_4 p_7 - \Delta\alpha_3^2 u_4 p_8,\ N\Delta\alpha_3^2 p_6 - \Delta\alpha_3^2 p_4,\\
&N\Delta u_4 p_5 - \Delta u_4 p_3,\ \alpha_3^2\Delta p_6 - N\alpha_3^2 p_4,\ \Delta u_4 p_5 - N u_4 p_3, \Delta p_1 - N\Delta p_2,\ N p_1 - \Delta p_2],\\
g_{17} = [&2Np_{1N} + 4\Delta p_{1\Delta} + p_1,\ 2Np_{2N} + 4\Delta p_{2\Delta} + 3p_2,\ 2Np_{3N} + 4\Delta p_{3\Delta} + p_3,\\
&2Np_{4N} + 4\Delta p_{4\Delta} + p_4,\ 2Np_{5N} + 4\Delta p_{5\Delta} + 3p_5,\ 2Np_{6N} + 4\Delta p_{6\Delta} + 3p_6,\\
&2Np_{7N} + 4\Delta p_{7\Delta} + p_7,\ 2Np_{8N} + 4\Delta p_{8\Delta} + 3p_8],\\
g_{18} = [&-2\Delta p_{1N} - 4N\Delta p_{1\Delta} + \Delta p_2,\ -2\Delta p_{2N} - 4N\Delta p_{2\Delta} - 2Np_2 + p_1,\\
&-2\Delta p_{3N} - 4N\Delta p_{3\Delta} + \Delta p_5,\ -2\Delta p_{4N} - 4N\Delta p_{4\Delta} + \Delta p_6,\\
&-2\Delta p_{5N} - 4N\Delta p_{5\Delta} - 2Np_5 + p_3,\ -2\Delta p_{6N} - 4N\Delta p_{6\Delta} - 2Np_6 + p_4,\\
&-2\Delta p_{7N} - 4N\Delta p_{7\Delta} + \Delta p_8,\ -2\Delta p_{8N} - 4N\Delta p_{8\Delta} - 2Np_8 + p_7],\\
g_{19} = [&-2\Delta u_4 p_{5N} - 4N\Delta u_4 p_{5\Delta} - 2Nu_4 p_5 + u_4 p_3,\ -2u_4 p_{3N} - 4Nu_4 p_{3\Delta} + u_4 p_5,\\
&-2\Delta p_{2N} - 4N\Delta p_{2\Delta} - 2Np_2 + p_1,\ -2\Delta u_4 p_{8N} - 4N\Delta u_4 p_{8\Delta} - 2Nu_4 p_8 + u_4 p_7,\\
&-2p_{1N} - 4Np_{1\Delta} + p_2,\ -2u_4 p_{7N} - 4Nu_4 p_{7\Delta} + u_4 p_8,\\
&-2\Delta p_{6N} - 4N\Delta p_{6\Delta} - 2Np_6 + p_4,\ -2p_{4N} - 4Np_{4\Delta} + p_6],\\
g_{20} = [&\alpha_3^2 p_4 - 2N\alpha_3^2 p_6 - 2\Delta\alpha_3^2 p_{6N} - 4N\Delta\alpha_3^2 p_{6\Delta},\ \alpha_3^2 p_6 - 2\alpha_3^2 p_{4N} - 4N\alpha_3^2 p_{4\Delta},\\
&\alpha_3^2 p_7 - 2\Delta\alpha_3^2 p_{8N} - 4N\Delta\alpha_3^2 p_{8\Delta} - 2N\alpha_3^2 p_8,\ p_1 - 2\Delta p_{2N} - 4N\Delta p_{2\Delta} - 2Np_2,\\
&-2\alpha_3^2 p_{7N} - 4\alpha_3^2 N p_{7\Delta} + \alpha_3^2 p_8,\ -2p_{1N} - 4Np_{1\Delta} + p_2,\\
&p_3 - 2\Delta p_{5N} - 4N\Delta p_{5\Delta} - 2Np_5, p_5 - 2p_{3N} - 4Np_{3\Delta}],\\
g_{21} = [&2N\Delta u_4 p_{5N} + 4\Delta^2 u_4 p_{5\Delta} + 3\Delta u_4 p_5,\ 2Nu_4 p_{3N} + 4\Delta u_4 p_{3\Delta} + u_4 p_3,\\
&2N\Delta p_{2N} + 4\Delta^2 p_{2\Delta} + 3\Delta p_2,\ 2N\Delta u_4 p_{8N} + 4\Delta^2 u_4 p_{8\Delta} + 3\Delta u_4 p_8,\\
&2Np_{1N} + 4\Delta p_{1\Delta} + p_1,\ 2Nu_4 p_{7N} + 4\Delta u_4 p_{7\Delta} + u_4 p_7,\\
&2N\Delta p_{6N} + 4\Delta^2 p_{6\Delta} + 3\Delta p_6, 2Np_{4N} + 4\Delta p_{4\Delta} + p_4],\\
g_{22} = [&2N\Delta\alpha_3^2 p_{6N} + 4\Delta^2\alpha_3^2 p_{6\Delta} + 3\Delta\alpha_3^2 p_6,\ 2N\alpha_3^2 p_{4N} + 4\Delta\alpha_3^2 p_{4\Delta} + \alpha_3^2 p_4,\\
&2N\Delta\alpha_3^2 p_{8N} + 4\Delta^2\alpha_3^2 p_{8\Delta} + 3\Delta\alpha_3^2 p_8,\ 2N\Delta p_{2N} + 4\Delta^2 p_{2\Delta} + 3\Delta p_2,\\
&2N\alpha_3^2 p_{7N} + 4\Delta\alpha_3^2 p_{7\Delta} + \alpha_3^2 p_7,\ 2Np_{1N} + 4\Delta p_{1\Delta} + p_1,\\
&2N\Delta p_{5N} + 4\Delta^2 p_{5\Delta} + 3\Delta p_5, 2Np_{3N} + 4\Delta p_{3\Delta} + p_3],\\
g_{23} = [&2Nu_4\alpha_3^2 p_{7N} + 4\Delta\alpha_3^2 u_4 p_{7\Delta} + u_4\alpha_3^2 p_7,\ 2Nu_4\alpha_3^2 p_{8N} + 4\Delta u_4\alpha_3^2 p_{8\Delta} + 3\alpha_3^2 u_4 p_8,\\
&2N\alpha_3^2 p_{4N} + 4\Delta\alpha_3^2 p_{4\Delta} + \alpha_3^2 p_4,\ 2Nu_4 p_{3N} + 4\Delta u_4 p_{3\Delta} + u_4 p_3,\\
&2N\alpha_3^2 p_{6N} + 4\Delta\alpha_3^2 p_{6\Delta} + 3\alpha_3^2 p_6,\ 2Nu_4 p_{5N} + 4\Delta u_4 p_{5\Delta} + 3u_4 p_5,\\
&2Np_{1N} + 4\Delta p_{1\Delta} + p_1,\ 2Np_{2N} + 4\Delta p_{2\Delta} + 3p_2],\\
g_{24} = [&-2\Delta\alpha_3^2 u_4 p_{7N} - 4N\Delta\alpha_3^2 u_4 p_{7\Delta} + \Delta\alpha_3^2 u_4 p_8,\\
&-2\Delta\alpha_3^2 u_4 p_{8N} - 4N\Delta\alpha_3^2 u_4 p_{8\Delta} - 2Nu_4\alpha_3^2 p_8 + u_4\alpha_3^2 p_7,\\
&-2\Delta\alpha_3^2 p_{4N} - 4N\Delta\alpha_3^2 p_{4\Delta} + \Delta\alpha_3^2 p_6,\ -2\Delta u_4 p_{3N} - 4N\Delta u_4 p_{3\Delta} + u_4\Delta p_5,\\
&-2\Delta\alpha_3^2 p_{6N} - 4N\Delta\alpha_3^2 p_{6\Delta} - 2N\alpha_3^2 p_6 + \alpha_3^2 p_4,\\
&-2\Delta u_4 p_{5N} - 4N\Delta u_4 p_{5\Delta} - 2Nu_4 p_5 + u_4 p_3,\\
&-2\Delta p_{1N} - 4N\Delta p_{1\Delta} + \Delta p_2, -2\Delta p_{2N} - 4N\Delta p_{2\Delta} - 2Np_2 + p_1].
\end{aligned}
$$

Proposition 4·3·4. *The unipotent tangent space at* $\hat{f} \in \vec{\mathscr{E}}^{\mathbf{D}_4}_{(z,\beta)}$ *is*

$$
\begin{aligned}
\mathscr{T}\mathscr{U}^{\mathbf{D}_4}(\hat{f}) \ = \ \mathscr{E}_{\tilde{u}}\cdot\!<\!&Ng_1,\ \Delta g_1,\ \alpha_1 g_1,\ \alpha_2 g_1,\ \alpha_3^2 g_1,\ \lambda_1 g_1,\ u_4 g_1,\ g_2\ \ldots\ g_{16},\\
&Ng_{17},\ \Delta g_{17},\ \alpha_1 g_{17},\ \alpha_2 g_{17},\ \alpha_3^2 g_{17},\ \lambda_1 g_{17},\ u_4 g_{17},\ g_{18}\ \ldots\ g_{24}\!>
\end{aligned}
$$

where $g_1\ \ldots\ g_{24}$ *are stated in Proposition* 4·3·3.

Proof of Theorem 4·3·1. We calculate the tangent space $\mathscr{T}_e(\tilde{\alpha})$ at $\tilde{\alpha}$ in two steps. The space $\mathscr{T}\mathscr{U}(\tilde{\alpha})$ is computed following the same lines.

Step 1. $\hat{F}^1_0 : (\mathbb{C} \times\mathbb{R}^3\times\mathbb{R}^2, 0) \to \mathbb{C}$ is defined by

$$\hat{F}^1_0(z, \alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2) = F^1_0(z, \alpha_1, \alpha_2, \alpha_3) = (\hat{m}N + \alpha_1)\, z + \epsilon_5\, \delta\bar{z} + \alpha_3\, \bar{z}$$

where $\hat{m} = m + \alpha_2$.

Using Proposition 4·1·6, a calculation shows that $\mathscr{T}_e^{\mathbf{D}_4}(\hat{F}_0^1) = \mathscr{E}_{\bar{u}}\cdot < g_1 \ldots g_{15} >$, where

$$
\begin{aligned}
&g_1 = \epsilon_5\delta\bar{z} + \hat{m}Nz\,, \quad g_2 = (\hat{m}^2-1)N\lambda_2\bar{z} + \hat{m}\alpha_1\lambda_2\bar{z} + \hat{m}\alpha_3\lambda_2 z\,,\\
&g_3 = \hat{m}\Delta z + \alpha_1 Nz\,, \quad g_4 = (\hat{m}^2-1)\Delta\lambda_2\bar{z} - \alpha_1\alpha_3\lambda_2 z - \alpha_1^2\lambda_2\bar{z}\,,\\
&g_5 = \epsilon_5(1-\hat{m}^2)\delta\lambda_2 z + \alpha_1\lambda_2\bar{z} + \alpha_3\lambda_2 z,\\
&g_6 = \epsilon_5(\hat{m}^2-1)\delta\alpha_3\lambda_2\bar{z} + {\alpha_1}^2\lambda_2\bar{z} + \alpha_1\alpha_3\lambda_2 z,\\
&g_7 = \hat{m}^2N\delta\alpha_3 z + \epsilon_5\alpha_1^2 Nz\,, \quad g_8 = (1-\hat{m}^2)N^2\lambda_2\bar{z} + \alpha_1\alpha_3\lambda_2 z + \alpha_1^2\lambda_2\bar{z}\,,\\
&g_9 = \hat{m}N^2 z + \alpha_1 Nz\,, \quad g_{10} = \alpha_1 z + \alpha_3\bar{z}\,, \quad g_{11} = \alpha_1\alpha_3\lambda_2 z + \alpha_3^2\lambda_2\bar{z}\,,\\
&g_{12} = \hat{m}\alpha_3^2 z + (1-\hat{m}^2)\alpha_1 Nz + \hat{m}\alpha_1\alpha_3\bar{z}\,,\\
&g_{13} = \alpha_1\alpha_3\lambda_2 z + \hat{m}(1-\hat{m}^2)N\alpha_3\lambda_2 z + \alpha_1^2\lambda_2\bar{z}\,,\\
&g_{14} = \alpha_1\alpha_3^2 z + \hat{m}(1-\hat{m}^2)\alpha_3^2 Nz + \alpha_1^2\alpha_3\bar{z}\,,\\
&g_{15} = \hat{m}\alpha_3^2\alpha_3\lambda_2 z + (1-\hat{m}^2)\alpha_1 N\alpha_3\lambda_2 z + \hat{m}\alpha_1\alpha_3^2\lambda_2\bar{z}\,.
\end{aligned}
$$

It follows that $\mathscr{T}_e^{\mathbf{D}_4}(\hat{F}_0^1) \bigcap (\mathscr{E}_\beta\cdot\!<z, Nz, \bar{z}>)^{\mathbf{D}_4} = \mathscr{E}_\beta^{\mathbf{D}_4}\cdot\!<h_1', h_2', h_3', h_4', h_5', h_6'>$ where

$$
\begin{aligned}
h_1' &= \hat{m}\alpha_3^2 z + (1-\hat{m}^2)\alpha_1 Nz + \hat{m}\alpha_1\alpha_3\bar{z},\\
h_2' &= \alpha_1\alpha_3\lambda_2 z + \hat{m}(1-\hat{m}^2)N\alpha_3\lambda_2 z + \alpha_1^2\lambda_2\bar{z},\\
h_3' &= \alpha_1{\alpha_3}^2 z + \hat{m}(1-\hat{m}^2)\alpha_3^2 Nz + \alpha_1^2\alpha_3\bar{z},\\
h_4' &= \hat{m}\alpha_3^2\alpha_3\lambda_2 z + (1-\hat{m}^2)\alpha_1 N\alpha_3\lambda_2 z + \hat{m}\alpha_1\alpha_3^2\lambda_2\bar{z},\\
h_5' &= \alpha_1 z + \alpha_3\bar{z},\\
h_6' &= \alpha_1\alpha_3\lambda_2 z + \alpha_3^2\lambda_2\bar{z}\,.
\end{aligned}
$$

Let $\eta_j\colon (\mathbb{R}^3\times\mathbb{R}^2, 0) \to \mathbb{R}^3$, $1 \leqslant j \leqslant 6$, be given by

$$
\begin{aligned}
\eta_1(\alpha,\lambda) &= (\hat{m}\alpha_3^2, (1-\hat{m}^2)\alpha_1, \hat{m}\alpha_1\alpha_3),\\
\eta_2(\alpha,\lambda) &= (\alpha_1\alpha_3\lambda_2,\ \hat{m}(1-\hat{m}^2)\alpha_3\lambda_2,\ \alpha_1^2\lambda_2),\\
\eta_3(\alpha,\lambda) &= (\alpha_1\alpha_3^2,\ \hat{m}(1-\hat{m}^2)\alpha_3^2,\ \alpha_1^2\alpha_3),\\
\eta_4(\alpha,\lambda) &= (\hat{m}\alpha_3^3\lambda_2,\ (1-\hat{m}^2)\alpha_1\alpha_3\lambda_2,\ \hat{m}\alpha_1\alpha_3^2\lambda_2),\\
\eta_5(\alpha,\lambda) &= (\alpha_1, 0, \alpha_3),\\
\eta_6(\alpha,\lambda) &= (\alpha_1\alpha_3\lambda_2, 0, \alpha_3^2\lambda_2).
\end{aligned}
$$

The η_j are $\mathbf{D}_4$-equivariant since

$$\eta_j(\alpha_1,\alpha_2,-\alpha_3,\lambda_1,-\lambda_2) = \gamma_3\cdot\eta_j(\alpha_1,\alpha_2,\alpha_3,\lambda_1,\lambda_2)$$

with $\gamma_3\cdot\eta_j = (\eta_{j_1}, \eta_{j_2}, -\eta_{j_3})$, $1 \leqslant j \leqslant 6$.

Let $N = \mathscr{E}_\beta^{\mathbf{D}_4}\cdot < \eta_1 \ldots \eta_6 >$. For $\tilde{\alpha} \in \mathscr{P}$, we define $\tilde{\alpha}^*N = \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot < \tilde{\alpha}^*\eta_1 \ldots \tilde{\alpha}^*\eta_6 >$, where

$$
\begin{aligned}
\tilde{\alpha}^*\eta_1 &= (\hat{m}u_4\alpha_3^2, (1-\hat{m}^2)\alpha_1,\ \hat{m}\alpha_1\alpha_3\lambda_2),\\
\tilde{\alpha}^*\eta_2 &= (u_4\alpha_1\alpha_3, \hat{m}(1-\hat{m}^2)u_4\alpha_3, \alpha_1^2\lambda_2),\\
\tilde{\alpha}^*\eta_3 &= (u_4\alpha_1\alpha_3^2, \hat{m}(1-\hat{m}^2)u_4\alpha_3^2, \alpha_1^2\alpha_3\lambda_2),\\
\tilde{\alpha}^*\eta_4 &= (\hat{m}\alpha_3^3u_4^2, (1-\hat{m}^2)\alpha_1\alpha_3 u_4,\ \hat{m}\alpha_1 u_4\alpha_3^2\lambda_2),\\
\tilde{\alpha}^*\eta_5 &= (\alpha_1, 0, \alpha_3\lambda_2),\\
\tilde{\alpha}^*\eta_6 &= (\alpha_1 u_4\alpha_3,\ 0,\ u_4\alpha_3^2\lambda_2).
\end{aligned}
$$

Since $\alpha_i \in \mathscr{E}_\lambda^{\mathbf{D}_4}$, we have that $\tilde{\alpha}^*N = \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\tilde{h}_1, \tilde{h}_2, \tilde{h}_3>$ with

$$
\left.
\begin{aligned}
\tilde{h}_1 &= (\hat{m}\,u_4\alpha_3^2, (1-\hat{m}^2)\alpha_1,\ \hat{m}\,\alpha_1\alpha_3\lambda_2),\\
\tilde{h}_2 &= (u_4\alpha_1\alpha_3, \hat{m}\,(1-\hat{m}^2)u_4\alpha_3, \alpha_1^2\lambda_2),\\
\tilde{h}_3 &= (\alpha_1, 0, \alpha_3\lambda_2).
\end{aligned}
\right\} \tag{4·7}
$$

Step 2. Clearly $\vec{\mathscr{E}}_\lambda^{\mathbf{D}_4}$ is generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}$ as an $\mathscr{E}_\lambda^{\mathbf{D}_4}$-module. Then, as

$$\tilde{\alpha}_\lambda = \begin{pmatrix} (\alpha_1)_{\lambda_1} & 2\lambda_2(\alpha_1)_{u_4} \\ (\alpha_2)_{\lambda_1} & 2\lambda_2(\alpha_2)_{u_4} \\ \lambda_2(\alpha_3)_{\lambda_1} & \alpha_3 + 2u_4(\alpha_3)_{u_4} \end{pmatrix},$$

$$\tilde{h}_4 = \tilde{\alpha}_\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} = ((\alpha_1)_{\lambda_1},\ (\alpha_2)_{\lambda_1},\ (\alpha_3)_{\lambda_1}\lambda_2)$$

$$\tilde{h}_5 = \tilde{\alpha}_\lambda \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix} = (2u_4\,(\alpha_1)_{u_4},\ 2u_4(\alpha_2)_{u_4},\ \alpha_3\,\lambda_2 + 2(\alpha_3)_{u_4}u_4\,\lambda_2)\,.$$

In a similar way we get $\tilde{N}$ and

$$\mathscr{T}\mathscr{U}(\tilde{\alpha}) = \tilde{\alpha}^*\tilde{N} + \tilde{\alpha}_\lambda \vec{\mathscr{M}}_\lambda^{\mathbf{D}_4^2}. \quad \square$$

Proof of Theorem 4·2·1. We use (4·3) and (4·4) to find, respectively, the tangent space and the unipotent tangent space of the path associated with each normal form, denoted hereafter by g.

The first part of the theorem follows from Propositions 3·3·1 and 3·4·5. To obtain the normal form and the non-degeneracy conditions we change of coordinates modulo an intrinsic submodule contained in the intrinsic part of $\mathscr{T}\mathscr{U}^{\mathbf{D}_4}(g)$, which in turn is contained in $\mathscr{P}(g)$ by Proposition 2·5·2. The miniversal unfolding follows from Theorems 2·3·1 and 3·3·2. To conclude the proof, we now state the basic data for each case.

Case I_0. Associated path: $\tilde{\alpha}(\lambda) = (\epsilon_0\lambda_1,\ 0,\ \lambda_2)$.

$$\begin{aligned}
\mathscr{T}_e(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot<(1,0,0),\ (0,\lambda_1,0),\ (0,u_4,0),\ (0,0,\lambda_2)>\,.\\
\mathscr{N}_e(\tilde{\alpha}) &=<(0,1,0)>;\ m \text{ is a modal parameter.}\\
\mathscr{T}\mathscr{U}(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot<(\lambda_1^2,0,0),\ (u_4,0,0),\ (0,\lambda_1,0), (0,u_4,0),\ (0,0,\lambda_1\lambda_2),\ (0,0,u_4\lambda_2)>\,.
\end{aligned}$$

Case I_1. Associated path: $\tilde{\alpha}(\lambda) = (\epsilon_4\lambda_1^2, \epsilon_3\lambda_1, \lambda_2)$.

$$\begin{aligned}
\mathscr{T}_e(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot<(u_4,0,0),\ (0,\lambda_1,0),\ (0,u_4,0),\ (2\epsilon_4\lambda_1,\epsilon_3,0),\ (0,0,\lambda_2)>\,.\\
\mathscr{N}_e(\tilde{\alpha}) &=<(1,0,0),\ (0,1,0)>;\ m \text{ is a modal parameter.}\\
\mathscr{T}\mathscr{U}(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot<(u_4,0,0),\ (\lambda_1^3,0,0),\ (0,\lambda_1^2,0),\ (0,u_4,0),\ (0,0,\lambda_1\lambda_2),\ (0,0,u_4\lambda_2)>\,.
\end{aligned}$$

Case I_2. Associated path: $\tilde{\alpha}(\lambda) = (\epsilon_0\lambda_1, nu_4, (\lambda_1+\epsilon_8u_4)\lambda_2)$.

$$\begin{aligned}
\mathscr{T}_e(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot<(-2m\lambda_1^2, \epsilon_0(1-m^2)\lambda_1, 0),\ (u_4,0,0),\ (\epsilon_0,0,\lambda_2),\\
&\qquad (-\epsilon_0\lambda_1, 2nu_4, 0),\ (\epsilon_0 m(1-m^2)\lambda_1^2, 0, 2n\lambda_1^2\lambda_2)>\,.\\
\mathscr{N}_e(\tilde{\alpha}) &= <(0,1,0),\ (0,u_4,0),\ (0,0,\lambda_2)>;\ m \text{ and n are modal parameters.}\\
\mathscr{T}\mathscr{U}(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot<(\lambda_1^2,0,0),\ (u_4^2,0,0),\ (0,\lambda_1,0),\ (0,u_4^2,0),\ (0,0,u_4\lambda_1\lambda_2),\\
&\qquad (\epsilon_0u_4, 0, u_4\lambda_2),\ (0,0,\lambda_1^2\lambda_2)>\,.
\end{aligned}$$

Case I_3. Associated path: $\tilde{\alpha}(\lambda) = (\epsilon_{11}\lambda_1^3, \epsilon_3\lambda_1, \lambda_2)$.

$$\begin{aligned}
\mathscr{T}_e(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot<(mu_4+\epsilon_3u_4\lambda_1,0,0),\ (0,u_4,0),\ (\lambda_1^3,0,0),\ (3\epsilon_{11}\lambda_1^2,\epsilon_3,0),\ (0,0,\lambda_2)>\,.\\
\mathscr{N}_e(\tilde{\alpha}) &=<(0,1,0),\ (1,0,0),\ (\lambda_1,0,0)>;\ m \text{ is a modal parameter.}\\
\mathscr{T}\mathscr{U}(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot<(u_4,0,0),\ (\lambda_1^4,0,0),\ (0,u_4,0),\ (0,\lambda_1^2,0),\ (0,0,\lambda_1\lambda_2),\ (0,0,u_4\lambda_2)>\,.
\end{aligned}$$

Case I_4. Associated path: $\tilde{\alpha}(\lambda) = (\epsilon_4\lambda_1^2, \epsilon_7\lambda_1^2, \lambda_2)$.

$$\begin{aligned}\mathscr{T}_e(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(u_4,0,0),\ (\lambda_1^2,0,0),\ (\epsilon_4\lambda_1,\epsilon_7\lambda_1,0),\ (0,u_4,0),\ (0,0,\lambda_2)\!>.\\ \mathscr{N}_e(\tilde{\alpha}) &= \,<\!(1,0,0),\ (0,1,0),\ (0,\lambda_1,0)\!>;\ m \text{ is a modal parameter.}\\ \mathscr{T}\mathscr{U}(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(m\,u_4,\epsilon_4(1-m^2)\lambda_1^2,0),\ (0,u_4,0),\ (\lambda_1^3,0,0),\ (0,\lambda_1^3,0),\\ &\qquad (0,0,\lambda_1\lambda_2),\ (0,0,u_4\lambda_2)\!>.\end{aligned}$$

Case I_5. Associated path: $\tilde{\alpha}(\lambda) = (\epsilon_4\lambda_1^2 + n_1u_4, \epsilon_3\lambda_1, \lambda_1\lambda_2)$.

$$\begin{aligned}\mathscr{T}_e(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(mu_4\lambda_1^2,(1-m^2)n_1u_4,m\lambda_1\lambda_2),\ (4n_1u_4\lambda_1^3,0,\epsilon_4\lambda_1^2\lambda_2),\\ &\qquad (2n_1u_4,0,\lambda_1\lambda_2),\ (2\epsilon_4\lambda_1,\epsilon_3,\lambda_2),\ (0,\lambda_1,0)\!>.\\ \mathscr{N}_e(\tilde{\alpha}) &= \,<\!(1,0,0),\ (u_4,0,0),\ (0,1,0),\ (0,0,1)\!>;\ m \text{ and } n_1 \text{ are modal parameters.}\\ \mathscr{T}\mathscr{U}(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(\epsilon_4m\lambda_1^4,(1-m^2)n_1^2u_4,2\epsilon_4n_1m\lambda_1^4\lambda_2),\ (u_4\lambda_1^3,0,0),\ (0,\lambda_1^2,0),\\ &\qquad (0,u_4\lambda_1,0),\ (2n_1u_4\lambda_1,0,\lambda_1^2\lambda_2),\ (2\epsilon_4u_4\lambda_1,\epsilon_3u_4,u_4\lambda_2),\\ &\qquad (n_1u_4\lambda_1-\epsilon_4\lambda_1^3,0,0),\ (n_1u_4^2-\epsilon_4u_4\lambda_1^2,0,0)\!>.\end{aligned}$$

Case I_6. Associated path: $\tilde{\alpha}(\lambda) = (\epsilon_0\lambda_1, \epsilon_9u_4, u_4\lambda_2)$.

$$\begin{aligned}\mathscr{T}_e(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(1,0,0),\ (0,\lambda_1,0),\ (0,u_4,0),\ (0,0,\lambda_1^2\lambda_2),\ (0,0,u_4\lambda_2)\!>.\\ \mathscr{N}_e(\tilde{\alpha}) &= \,<\!(0,1,0),\ (0,0,\lambda_2),\ (0,0,\lambda_1\lambda_2)\!>;\ m \text{ is a modal parameter.}\\ \mathscr{T}\mathscr{U}(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(u_4,0,0),\ (\lambda_1^2,0,0),\ (0,\lambda_1,0),\ (0,u_4^2,0),\ (0,0,\lambda_1^2\lambda_2),\\ &\qquad (0,0,u_4\lambda_1\lambda_2),\ (0,0,u_4^2\lambda_2)\!>.\end{aligned}$$

Case I_7. Associated path: $\tilde{\alpha}(\lambda) = (\epsilon_0\lambda_1, 0, (\lambda_1 + \epsilon_8u_4)\lambda_2)$.

$$\begin{aligned}\mathscr{T}_e(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(0,\lambda_1,0),\ (0,u_4^2,0),\ (0,0,u_4\lambda_2),\ (\epsilon_0,0,\lambda_2),\ (\epsilon_0,0,\lambda_2),\ (\lambda_1,0,0)\!>.\\ \mathscr{N}_e(\tilde{\alpha}) &= \,<\!(1,0,0),\ (0,1,0),\ (0,u_4,0)\!>;\ m \text{ is a modal parameter.}\\ \mathscr{T}\mathscr{U}(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(0,\lambda_1,0),\ (0,u_4^2,0),\ (0,0,u_4\lambda_1\lambda_2),\ (0,0,u_4^2\lambda_2),\ (\lambda_1^2,0,0),\\ &\qquad (\epsilon_0u_4,0,u_4\lambda_2),\ (0,0,\lambda_1^2\lambda_2)\!>.\end{aligned}$$

Case I_8. Associated path: $\tilde{\alpha}(\lambda) = (\epsilon_0\lambda_1, nu_4, (\lambda_1 + \epsilon_{10}u_4^2)\lambda_2)$.

$$\begin{aligned}\mathscr{T}_e(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(\epsilon_0,0,\lambda_2),\ (\epsilon_0\lambda_1,0,u_4^2\lambda_2),\ (0,2nu_4,\lambda_1\lambda_2),\ (0,\lambda_1,0)\!>.\\ \mathscr{N}_e(\tilde{\alpha}) &= \,<\!(u_4,0,0),\ (0,u_4,0),\ (0,1,0),\ (0,0,\lambda_2)\!>;\ m \text{ and } n \text{ are modal parameters.}\\ \mathscr{T}\mathscr{U}(\tilde{\alpha}) &= \mathscr{E}_\lambda^{\mathbf{D}_4}\cdot\!<\!(\lambda_1^2,0,0),\ (\epsilon_0u_4,0,u_4\lambda_2),\ (0,\lambda_1,0),\ (0,2nu_4^2,u_4\lambda_1\lambda_2),\\ &\qquad (0,0,u_4^2\lambda_1\lambda_2),\ (0,0,u_4^3\lambda_2),\ (0,0,\lambda_1^2\lambda_2)\!>.\end{aligned}$$

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