

Effective action for QED in 2 + 1 dimensions at finite temperature

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(Received 19 February 1999; published 12 August 1999)

Both the parity-breaking and parity-invariant parts of the effective action for the gauge field in QED₃ with massive fermions at finite temperature are obtained exactly. This is feasible because we use a particular configuration of the background gauge field, namely a constant magnetic field and a time-dependent time component of the background gauge field. Our results allow us to compute exactly physically interesting quantities such as the induced charge density and fermion condensate whose dependence on the temperature, fermion mass and gauge field is discussed. [S0556-2821(99)03416-5]

PACS number(s): 12.20.Ds, 11.10.Kk, 11.10.Wx, 11.30.Rd

Many aspects of quantum electrodynamics in 2 + 1 dimensions (QED₃) [1] in the presence of external fields have already been studied. The dynamical generation of a Chern-Simons (CS) term for the gauge field [2], the formation of a fermion condensate in the presence of an external magnetic field [3,4] and the breaking of Lorentz symmetry [5,4] are some of the many features exhibited by QED₃. Some of these aspects are desirable in 3+1 dimensions and also very important in the study of superconductivity [6–9] and the quantum Hall effect [10] in planar systems.

In order to carry the study of such phenomena further it is important to consider the effects of temperature; these might sometimes be completely different from one's expectations. Particularly, it has been shown that the development of the fermion condensate catalyzed by a magnetic field is unstable at finite temperature [11] and that the issue of gauge invariance at finite temperature is only consistently addressed if a re-summation of the graphs is carried out [12–14]. This question has also been extensively reported using different techniques in the context of QED₃ [15–20] as well as by means of an analogous quantum mechanical model which allows one to perform a comparative analysis of the perturbation series with exact results [21,22]—in fact this was the approach used in one of the seminal works on the subject [12].

Although the exact contribution of the time component of the gauge field has been considered only for the parity violating term of the effective action, it also contributes to the parity invariant one and this may result in extra effects on quantities such as the fermion condensate and the density of charge generated dynamically. To explore such effects is the main purpose of this article. By making use of the Fock-Schwinger proper-time technique [23] we compute the one-loop effective action at finite temperature for a specific configuration, namely a constant magnetic field and a time-dependent time-component of the gauge field. We also address the aspects of gauge invariance and temperature dependence of the thermal condensate and the charge density. With this program our results go beyond those obtained in [18], where the same technique was used.

In QED the one-loop effective action for the gauge field is obtained by integrating out the fermions

$$e^{iS_{eff}[A]} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left[i \int d^3x \bar{\psi} (i \not{\partial} + e \not{A} - m) \psi \right] \times \exp \left[i \int d^3x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \theta \epsilon_{\mu\nu\alpha} A^\mu \partial^\nu A^\alpha \right) \right], \quad (1)$$

which can be calculated exactly only for some configurations of the field. In this case the gauge field is understood as a classical background field and many important features can be analyzed from the resulting effective action at zero temperature [23–25] and finite density [26,27] or at finite temperature [28,29] and finite density [30,31]. Whenever the effective action cannot be obtained exactly, it is useful to adopt some approximation. The derivative expansion technique is such an approximation [32] which has been successfully employed to analyze different aspects of effective actions in 1 + 1 [33] and in 2 + 1 dimensions [34,35].

The CS term at finite temperature was obtained by means of the derivative expansion [36] as well as other techniques [37–43]. Although there was strong evidence that the CS coefficient for a non-Abelian gauge field should be quantized at finite temperature [39], it was found to depend smoothly on the temperature, leading to the conjecture that it would not depend on the temperature at all [39]. Later it was proved that the derivative expansion could only be used to obtain the CS term for some very special configurations of the gauge field due to the intrinsic non-locality of the CS coefficient at finite temperature, which could not be removed because of the essential non-analyticity present in the vacuum-polarization graph [44–46]. More recently the subject gained new impetus in Refs. [47,48] where it was argued on the basis of gauge invariance of the partition function that there was a contradiction between the quantization of the CS coefficient at finite temperature and the results obtained at the perturbative level.

The solution to this puzzle was given in [12] where a (0 + 1)-dimensional model was used to demonstrate that the effective action at $T \neq 0$ is compatible with gauge invariance, if it is obtained exactly. Similar conclusions were reached in [13] after a zeta-function analysis of the fermion determinant in QED₃ at $T \neq 0$. More recently, in [14], the parity-breaking part of the effective action in QED₃ at $T \neq 0$ was calculated exactly thanks to the choice of a particular class of gauge transformations and provided that only the following re-

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stricted set of gauge configurations, $A_j=A_j(\vec{x})$, $A_3=A_3(\tau)$, is considered. The result shows that gauge invariance is compatible with the dynamically generated CS term.

We are going to adopt the same gauge configuration as given by $A_j=A_j(\vec{x})$, $A_3=A_3(\tau)$, but for a constant mag-

netic field $F_{12}=B$: $A_j=\frac{1}{2}F_{jk}x^k$, for which it is possible to obtain an exact result for the fermion propagator. This can be obtained from the covariant calculation [2] and made particular for our purpose. It is given in Minkowski space-time by

$$G(x,x')=i\frac{e^{3\pi i4}}{8\pi^{3/2}}\Phi(x,x')\int_0^\infty\frac{ds}{s^{3/2}}e^{-im^2s}\exp\left(i\left[eA_0(t-t')-\frac{(t-t')^2}{4s}+\frac{e|B|s}{4s}\cot(e|B|s)(\vec{x}-\vec{x}')^2\right]\right) \\ \times\left\{m-\gamma^0\frac{(t-t')}{2s}+\frac{e|B|s}{2s}\epsilon_{jk}\gamma^j(\vec{x}-\vec{x}')^k+\frac{e|B|s}{2s}\cot(e|B|s)\vec{\gamma}\cdot\vec{x}\right\}\frac{e|B|s}{\sin(e|B|s)}\left[\cos(e|B|s)-i\gamma^0\frac{*F_0}{|B|}\sin(e|B|s)\right], \quad (2)$$

where $\Phi(x,x')$ is the gauge dependent factor,

$$\Phi(x,x')=\exp\left(ie\int_{x'}^x d\eta^\mu A_\mu(\eta)\right), \quad (3)$$

and $*F_0=(1/2)\epsilon_{jk}F_{jk}$. One can note that a simple gauge transformation on this factor removes the $A_0=\text{const}$ taken into account explicitly in the propagator. However, we are going to see that at finite temperature there are factors dependent on the time component of the gauge field that cannot be removed by a gauge transformation.

To implement the finite temperature calculation in the imaginary-time formalism we recall that the propagator can also be rewritten as $G(x,x')=\Phi(x,x')\int[d^3k_E/(2\pi)^3]e^{-ik_E(x-x')}\tilde{G}(k_E)$, where

$$\tilde{G}(k_E)=-\int_0^\infty ds\exp\left(-is\left[m^2+(k_3+e\tilde{A}_3)^2+\tilde{k}^2\frac{\tan(e|B|s)}{e|B|s}\right]\right)\left\{m-\gamma_3(k_3-e\tilde{A}_3)-\vec{\gamma}\cdot\vec{k}+\epsilon_{ij}\gamma^i k^j\right. \\ \left.\times\tan(e|B|s)\right\}\left[1+\gamma_3\frac{*F_3}{|B|}\tan(e|B|s)\right] \quad (4)$$

is its Fourier transform in the Euclidean version ($\gamma_3=-i\gamma_0$) and \tilde{A}_3 is given by $A_3(\tau)\rightarrow\tilde{A}_3(\tau)=A_3(\tau)+\partial_\tau\Lambda(\tau)=\beta^{-1}\int_0^\beta d\tau' A_3(\tau')+2\pi k/e\beta$, where $\Lambda(\tau)$ is the gauge transformation mentioned in Eq. (7) of [14]. Once again we note that a simple redefinition of k_3 as a continuous variable renders $\tilde{G}(k_E)$ gauge independent, leaving the gauge dependence only on the factor $\Phi(x,x')$.

From now on we consider only the gauge invariant propagator in order to compute gauge invariant quantities, for example the charge density which is given by $\langle j_3\rangle=ie\text{tr}[\gamma_3 G_E(x,x')]|_{x=x'}$. Performing the trace of the γ matrices and the integration over \vec{k} , we obtain the following expression for the charge density:

$$\langle j_3\rangle=(e^2/2\pi)\beta^{-1}\sum_{n=-\infty}^\infty\int_0^\infty ds\exp\{-is[m^2+(w_n+e\tilde{A}_3)^2]\}[m*F_3-|B|(w_n+e\tilde{A}_3)\coth(e|B|s)], \quad (5)$$

where $w_n=(2n+1)\pi/\beta$. We can see from the above expression that there are two contributions to the expectation value of the charge density, the parity violating (PV) one which is proportional to $*F_3$ and the parity invariant (PI) one which is proportional to $|B|$. Now we can obtain the parity violating charge density: namely,

$$\langle j_3\rangle^{PV}=-im\frac{e^2*F_3}{2\pi}\frac{1}{\beta}\sum_{n=-\infty}^\infty[m^2+(w_n+e\tilde{A}_3)^2]^{-1} \\ =-i\frac{m}{|m|}\frac{e^2*F_3}{4\pi}\tanh\left(\frac{|m|\beta}{2}\right) \\ \times\left[1+\tanh^2\left(\frac{|m|\beta}{2}\right)\tan^2\left(\frac{e\tilde{A}_3\beta}{2}\right)\right]^{-1} \\ \times\left[1+\tan^2\left(\frac{e\tilde{A}_3\beta}{2}\right)\right]. \quad (6)$$

We note that this expression is indeed gauge invariant under both small and large gauge transformations.

Since this term comes from the existence of a fermion mass, it is important to ask what happens to the charge density generated dynamically in the limit $m\rightarrow 0$. From the above expression we find that it vanishes independently of \tilde{A}_3 for any temperature, although it survives at zero temperature [2]. This phenomenon also happens to the fermion condensate in the reducible representation [11] and will also be seen to happen here with the irreducible representation. The melting of the charge condensation at finite temperature can clearly be seen by rewriting expression (6) as

$$\langle j_3\rangle^{PV}=-i(m/|m|)(e^2*F_3/4\pi) \\ \times[1-(e^{\beta(|m|+ie\tilde{A}_3)}+1)^{-1}-(e^{\beta(|m|-ie\tilde{A}_3)}+1)^{-1}]. \quad (7)$$

We can see that the $T=0$ contribution survives in the limit $m\rightarrow 0$ but it is canceled by the thermal fluctuations. In addition, Eq. (7) shows that there is a formal analogy between $e\tilde{A}_3$ and an imaginary chemical potential.

From Eq. (6) we can obtain the parity violating contribution to the effective action by recalling that $\langle j_3\rangle=-\delta S_{eff}/\delta A_3(\tau)$. Then we obtain

$$S_{eff}^{PV} = \frac{ie}{2\pi} \frac{m}{|m|} \int d^2x {}^*F_3 \arctan \left[\tanh \left(\frac{|m|\beta}{2} \right) \tan \left(\frac{e\tilde{A}_3\beta}{2} \right) \right]. \quad (8)$$

This is the result found in Refs. [13,14]. It can also be rewritten in a more suitable form to check the consequences of a gauge transformation,

$$S_{eff}^{PV} = \frac{e}{4\pi} \frac{m}{|m|} \int d^2x {}^*F_3 \left\{ ie \int_0^\beta d\tau A_3(\tau) + 2\pi ik \right. \\ \left. - \ln[1 + e^{-\beta(|m| - ie\tilde{A}_3)}] + \ln[1 + e^{-\beta(|m| + ie\tilde{A}_3)}] \right\}. \quad (9)$$

We point out that the superscript PV has been used for the effective action only for making clear that it is associated with the parity violating charge density. It is the effective Lagrangian which possesses the parity violating property and not the effective action. The latter is invariant under small gauge transformation ($k=0$) and changes by $iek\Phi/2$ under large gauge transformation, where Φ is the magnetic flux. The change under large gauge transformation comes from the zero-temperature contribution and, if the magnetic flux is quantized in units of $4\pi/e$, the partition function for the gauge field is invariant under large gauge transformation. Furthermore, one can also note that this is not the CS term at finite temperature; this is hidden in the factors with logarithmic functions which are not extensive quantities.

The parity invariant contribution to the charge density is given by

$$\langle j_3 \rangle^{PI} = -\frac{e^2|B|}{2\pi} \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int_0^\infty ds e^{-is[m^2 + (w_n + e\tilde{A}_3)^2]} \\ \times (w_n + e\tilde{A}_3) \cot(e|B|s) \\ = -i \frac{e^2|B|}{4\pi} \sum_{n=0}^{\infty} \sum_{s=1}^2 [(e^{\beta(E_{n,s} + ie\tilde{A}_3)} + 1)^{-1} \\ - (e^{\beta(E_{n,s} - ie\tilde{A}_3)} + 1)^{-1}], \quad (10)$$

where $E_{n,s} = \sqrt{m^2 + 2e|B|(n+s-1)}$ is the energy of the Landau levels.

We note that, at $T=0$, the parity invariant charge density vanishes, as expected [2,23]. More interestingly, it vanishes in the limit $\tilde{A}_3=0$, as well. Comparing Eqs. (7) and (10) we see that the parity violating contribution to the charge density is proportional to *F_3 and only the lowest Landau level contributes to it; on the other hand the parity invariant part is proportional to $|B|$ and all the Landau levels contribute. It is also important to remember that the parity violating contribution is a peculiar feature of QED in an odd number of space-time dimensions with fermions in the irreducible representation; on the other hand the parity invariant one exists in both the irreducible and the reducible representation of QED₃. Actually, it is the only contribution in the reducible representation, as is expected, since QED₃ in the reducible representation is very similar to QED₄.

Using $\langle j_3 \rangle = -\delta S_{eff}/\delta A_3(\tau)$ and Eq. (10) we can obtain the parity invariant effective Lagrangian, which is given by

$$\mathcal{L}_{eff}^{PI} = \frac{e|B|}{4\pi} \sum_{n=0}^{\infty} \sum_{s=1}^2 \{ E_{n,s} + \beta^{-1} \ln[1 + e^{-\beta(E_{n,s} + ie\tilde{A}_3)}] \\ + \beta^{-1} \ln[1 + e^{-\beta(E_{n,s} - ie\tilde{A}_3)}] \}. \quad (11)$$

The first term in the curly brackets is the zero temperature contribution and can be rewritten as

$$\mathcal{L}_{eff}^{PI}(T=0) = \frac{e^{i3\pi/4}}{8\pi^{3/2}} \int_0^\infty \frac{ds}{s^{5/2}} e^{-im^2s} e|B|s \cot(e|B|s). \quad (12)$$

Finally, we note that this effective Lagrangian has to be properly renormalized. The result can be read off directly from Refs. [24,30]:

$$\mathcal{L}_{eff}^{PI}(T=0) = \frac{e^{i3\pi/4}}{8\pi^{3/2}} \int_0^\infty \frac{ds}{s^{5/2}} e^{-im^2s} \\ \times [e|B|s \cot(e|B|s) - 1 - \frac{1}{3}(es|B|)^2], \quad (13)$$

and the whole effective Lagrangian is given by $\mathcal{L}_{eff} = -\frac{1}{2} {}^*F_3^2 - \theta {}^*F_3 A_3 + \mathcal{L}_{eff}^{PV} + \mathcal{L}_{eff}^{PI}$.

Another quantity which is significant in the analysis of symmetry breaking is the fermion condensate. It can be shown to be given by $\langle \bar{\psi}\psi \rangle = i \text{tr}[G_E(x, x')]|_{x=x'}$. It also has two contributions which will be evaluated by means of the well-known formula $\langle \bar{\psi}\psi \rangle = -\partial \mathcal{L}_{eff}/\partial m$. The first one, which comes from the parity invariant effective Lagrangian, is also present in QED₃ with fermions in the reducible representation and is given by

$$\langle \bar{\psi}\psi \rangle^{PI} = -m \frac{e|B|}{4\pi} \sum_{n=0}^{\infty} \sum_{s=1}^2 \frac{1}{E_{n,s}} \\ \times [1 - (e^{\beta(E_{n,s} + ie\tilde{A}_3)} + 1)^{-1} - (e^{\beta(E_{n,s} - ie\tilde{A}_3)} + 1)^{-1}]. \quad (14)$$

This expression is the same found in [11] using real-time formalism, apart from a factor of 1/2 due to the trace of gamma matrices. Therefore the analysis carried out there can naturally be applied here. First we note that although this part of the condensate is non-vanishing in the limit $m \rightarrow 0$ at zero temperature [3], it melts at any finite temperature for $m \rightarrow 0$ independently of A_3 . In the reducible representation this is the only contribution to the order parameter. In [49] a B -dependent critical temperature was found for the case of small but non-vanishing mass. Above this critical temperature and in the regime $m^2 \ll T^2 \ll eB$ the condensate vanishes.

The second contribution to the condensate comes from the parity violating effective Lagrangian and reads

$$\langle \bar{\psi}\psi \rangle^{PV} = -i \frac{e {}^*F_3}{4\pi} \tan \left(\frac{e\tilde{A}_3\beta}{2} \right) \\ \times \left[1 + \tanh^2 \left(\frac{|m|\beta}{2} \right) \tan^2 \left(\frac{e\tilde{A}_3\beta}{2} \right) \right]^{-1} \\ \times \left[1 - \tanh^2 \left(\frac{|m|\beta}{2} \right) \right]$$

$$= -\frac{e^* F_3}{4\pi} \{ (e^{\beta(|m| - ie\tilde{A}_3)} + 1)^{-1} - (e^{\beta(|m| + ie\tilde{A}_3)} + 1)^{-1} \}. \quad (15)$$

We note that it vanishes if $T=0$ or $\tilde{A}_3=0$.

In conclusion, we have found the thermal effective action for a particular configuration of the gauge field, namely $A_j = \frac{1}{2} F_{jk} x^k$ with $F_{12}=B=\text{const}$ and $A_3=A_3(\tau)$. Because the presence of $A_3(\tau)$, we found that the associated partition function is gauge invariant. In the course of our derivation we also calculated the charge density and the fermion condensate, highlighting their origins in terms of the contribution of the Landau levels and the time component of the

gauge field. Although we found these quantities very unstable in the zero limit of the relevant parameters, the fermion mass and $A_3(\tau)$ analysis developed here can be useful whenever quantum fluctuations of the gauge field are taken into account. We also pointed out the similarities of our results with those obtained at finite density. More investigations of these similarities, their physical meaning and consequences are under study and will be reported elsewhere.

M.H. is supported by Fundação de Amparo a Pesquisa do Estado de São Paulo (FAPESP-Brazil). G.M. is grateful to PPARC (UK) for financial support. The authors wish to thank Professor I. J. R Aitchison, Dr. N. E. Mavromatos and Dr. A. Momen for valuable discussions.

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