Relating a Gluon Mass Scale to an Infrared Fixed Point in Pure Gauge QCD

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We show that in pure gauge QCD (or any pure non-Abelian gauge theory) the condition for the existence of a global minimum of energy with a gluon (gauge boson) mass scale also implies the existence of a fixed point of the β function. We argue that the frozen value of the coupling constant found in some solutions of the Schwinger-Dyson equations of QCD can be related to this fixed point. We also discuss how the inclusion of fermions modifies this property.

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Non-Abelian gauge theories have the property of asymptotic freedom [1]. For large momenta the coupling becomes small, and perturbation theory seems to be an appropriate computational tool. For small momenta the coupling grows large, and we have to rely on nonperturbative methods to study the infrared (IR) behavior of these theories. In general, it is easier to apply nonperturbative methods to pure gauge theories; i.e., the absence of fermions may simplify the calculations. One of these methods, in the case of pure gauge quantum chromodynamics, is the study of Schwinger-Dyson equations (SDE) for the gluon propagator [2]. Following this method it was found some years ago that the gluon propagator is highly singular in the IR, which could explain gluon confinement in a simple way [2]. This early calculation contained a series of approximations, and nowadays it is believed that the gluon propagator IR behavior is smoother.

The softer IR behavior of the gluon propagator indicates the existence of a gluon mass scale. This conclusion was reached by a large number of nonperturbative methods. Cornwall argued that the gluon acquires a dynamical mass solving a gauge invariant SDE [3]. Recent research using a similar method with different approximations also finds an IR finite propagator involving a gluon mass scale [4]. These calculations are consistent with lattice simulations of pure gauge QCD, where it is found that the gluon propagator is modified at some mass scale and is infrared finite [5]. A variational method approach to QCD is also compatible with dynamical gluon mass generation [6]. This gluon mass scale appears in some other nonperturbative methods [7], as well as is necessary in several phenomenological calculations [8].

At the same time that the dynamical gluon mass scale is generated, the theory develops a freezing of the IR coupling constant. This is a consequence that the coupling behavior is related to the renormalization of the theory propagators, and in this procedure the infrared behavior of the gluon is transmitted to the coupling. Actually the coupling constant found in Ref. [4], solving SDE, clearly shows the existence of an infrared fixed point of the QCD β function. Nevertheless, it is important to stress that the SDE solutions are always solved within some approximation and in general in one specific gauge and renormalization scheme. Therefore we expect that any relationship between the gluon mass scale and the infrared behavior of the coupling constant and, consequently, a fixed point of the β function could not be univocally determined. This fact is peculiar to our inability to deal with the strong interaction physics, because we expect that the absolute minimum of QCD vacuum energy will be compatible with a unique gluon mass scale (if this is the solution preferred by the vacuum).

In this work we show that the dynamical gluon mass scale generation implies the existence of a fixed point of the β function, although the presence of a fixed point does not necessarily imply dynamical mass generation. We start remembering that gauge theories without fundamental scalar bosons may generate dynamical masses through the phenomenon of dimensional transmutation [9]; i.e., we basically do not have arbitrary parameters once the gauge coupling constant (g) is specified at some renormalization point (μ). In these theories all the physical parameters will depend on this particular coupling.

Many years ago Cornwall and Norton [10] emphasized that the vacuum energy (Ω) in dynamically broken gauge theories could be defined as a function of the dynamical mass $m_g(p^2) \equiv m(g, \mu)$, where $m_g(p^2)$ in pure gauge QCD is related to the gluon polarization tensor. This mass is not necessarily the gluon mass as it appears in the Euclidean propagator determined in Ref. [3]; it may be any momentum dependent mass scale that induces an IR finite behavior for the gluon propagator as it appears in Ref. [4]. In the sequence $m(g, \mu)$ will be indicated just by m. Actually, the vacuum energy may also depend on the dynamical fermion and ghost masses. However, in that which concerns ghosts, there is no evidence for scalar fermion Goldstone excitations; i.e., it is rather unlikely that ghosts develop mass [11].

The vacuum energy $\Omega = \Omega(g, \mu)$, defined ahead, is a finite function of its arguments, because the perturbative contribution has been subtracted [10,12]. Ω must satisfy a homogeneous renormalization group equation [13]

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$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g}\right)\Omega = 0. \tag{1}$$

On the other hand, the dynamically generated masses can be written as $m = \mu f(g)$ [13], from what follows that $\mu(\partial m/\partial \mu) = m$ and, consequently,

$$m \frac{\partial \Omega}{\partial m} = -\beta(g) \frac{\partial \Omega}{\partial g}. \tag{2}$$

This last and simple equation will be central to our argument, because it relates the stationary condition for the vacuum energy ($\partial\Omega/\partial m = 0$) [3,12] to the condition of zeros of the β function, and we expect that the massive solution indeed minimizes the energy [6]. Therefore in a gauge theory with dynamically generated masses, the condition for an extremum of the vacuum energy

$$\beta(g) \frac{\partial \Omega}{\partial g} \bigg|_{\partial\Omega/\partial m = 0} = 0 \tag{3}$$

always implies $\beta(g) = 0$. Of course, this is true only if $\partial\Omega/\partial g \neq 0$ when $m \neq 0$. Note that only at the global minimum is the vacuum energy a gauge independent and meaningful quantity. Exactly at this point we expect that the mass scale, the coupling constant, and its β function are uniquely determined.

We note that the coupling constant in the IR has no unique determination, and it has been enough to match its functional form with its ultraviolet behavior. This diversity at the IR has the inconvenience that depending on the choice we make, we have to face very different scenarios, for instance, the singular behavior of the coupling [14] or its freezing at low energies [3,4]. In this sense, it would be appropriate to clarify what coupling constant and β function we are referring to, since Eq. (2) was written down without any specification of their functional form and the renormalization scale where they are to be computed. The point here is that Eq. (2) precedes any *a priori* definition of the coupling constant and its associated β function (at some renormalization scale), allowing us to obtain very general properties of these functions if we have some extra ingredient at hand. As discussed by Coleman and Weinberg many years ago [9], there is a unique way of linking g and μ, and this can be achieved by the vacuum energy at its minimum. In other words, whatever the definition of g and β we choose, the minimum of energy provides us with further information, demanding them to conform to the existence of an IR fixed point when there is dynamical mass generation.

To show that $\partial\Omega/\partial g \neq 0$ we must refer to the vacuum energy for composite operators [12], since the theory will admit only condensation of composite operators as, for instance, $\langle \frac{\alpha_s}{\pi} G^{\mu\nu} G_{\mu\nu} \rangle$ in the pure gauge theory and $\langle -\bar{\psi}\psi \rangle$ when we add fermions. In order to do so we introduce a bilocal field source $J(x, y)$, and Ω will be calculated after a series of steps starting from the generating functional $Z(J)$ [12]:

$$\begin{aligned} Z(J) &= \exp[\iota W(J)] \\ &= \int d\phi \exp\bigg[\iota\bigg(\int d^4x \mathcal{L}(x) \\ &\qquad + \int d^4x d^4y \phi(x) J(x, y) \phi(y)\bigg)\bigg], \end{aligned} \tag{4}$$

where ϕ can be a gauge boson or fermion field. From the generating functional we determine the effective action $\Gamma(G)$ which is a Legendre transform of $W(J)$ and is given by $\Gamma(G) = W(J) - \int d^4x d^4y G(x, y) J(x, y)$ (where G is a complete propagator) leading to $\delta\Gamma/\delta G(x, y) = -J(x, y)$. The physical solutions will correspond to $J(x, y) = 0$, which will reproduce the SDE of the theory [12].

In general, if J is the source of the operator $\mathcal{O}$, we have [15]

$$\frac{\delta\Gamma}{\delta J} \bigg|_{J=0} = \langle 0|\mathcal{O}|0\rangle. \tag{5}$$

For translationally invariant (ti) field configurations we can work with the effective potential given by $V(G) \int d^4x = -\Gamma(G)|_{\rm ti}$. Finally, from the above equations we can define the vacuum energy as [12]

$$\Omega = V(G) - V_{\rm pert}(G), \tag{6}$$

where we are subtracting from $V(G)$ its perturbative counterpart, and Ω is computed as a function of the nonperturbative propagators G. These propagators depend on the gauge boson, fermion, and ghost self-energies. We will not consider fermions and, as long as the ghost self-energy does not show any nontrivial pole, its direct contribution is washed out from the vacuum energy. It should be noted, however, that the ghosts can still interfere through its effect on the gluon propagator [4]. Ω is a function of the dynamical masses of the theory and is zero in the absence of mass generation [12]. We shall comment later on the actual Ω calculation.

We can now write Eq. (3) in the following form:

$$-\beta(g)\left[\frac{\partial\Omega}{\partial J}\frac{\partial J}{\partial g}\right]_{J=0} = 0. \tag{7}$$

Of course, we assume that the conditions for a global minimum of the vacuum energy $\partial\Omega/\partial m = 0$ and $J = 0$ are equivalent. However, $\partial\Omega/\partial J = -\partial\Gamma/\partial J$, and as a consequence of Eq. (5) we have

$$\beta(g)\langle 0|\mathcal{O}|0\rangle \frac{\partial J}{\partial g} \bigg|_{J=0} = 0. \tag{8}$$

Using the inversion method devised by Fukuda [16] it is possible to show that $\frac{\partial J}{\partial g} \neq 0$ when there is condensation, i.e., $\langle 0|\mathcal{O}|0\rangle \equiv \vartheta \neq 0$. In Ref. [16] it was verified that to compute a nonperturbative quantity like ϑ the usual procedure is to introduce a source J and to calculate the series

$$\vartheta = \sum_{n=0}^{\infty} g^n h_n(J). \qquad (9)$$

In practice we have to truncate Eq. (9) at some finite order, which gives us only the perturbative solution $\vartheta = 0$ when we set $J = 0$. The right-hand side of Eq. (9) should be double valued at $J = 0$ for another solution to exist, which is not the present case. The alternative method is to invert Eq. (9), solving it in favor of J and regarding ϑ as a quantity of the order of unity. One obtains the following series:

$$J = \sum_{n=0}^{\infty} g^n k_n(\vartheta), \qquad (10)$$

where the k_n's satisfying $n \leq m$ (m being some finite integer) are calculable from h_n, also satisfying $n \leq m$. One can find a nonperturbative solution of ϑ by setting $J = 0$ through a truncated version of Eq. (10). The important point for us is that by construction of Eq. (10) we verify that when $J = 0$ and $\vartheta \neq 0$ the same value of ϑ that satisfies Eq. (10) leads trivially to

$$\partial J/\partial g|_{J=0} \neq 0. \qquad (11)$$

To make this point clear, observe that Eq. (10) allows us to look at J as a function of g and ϑ; hence we can imagine a surface in the space spanned by J, g, and ϑ. Nevertheless, this surface has physical meaning only for $J = 0$, resulting in a curve in the (g, ϑ) plane where the derivative of Eq. (11) is calculated. Therefore, the two terms, $\partial J/\partial g$ and $\langle 0|\mathcal{O}|0\rangle$, of Eq. (8) are different from zero in the condensed phase.

According to the above discussion and looking at Eq. (8), the only possibility to obtain $\partial\Omega/\partial m = 0$ is when we have a fixed point $[\beta(g) = 0]$, from which comes our main assertion that the condition for the existence of a gluon (gauge boson) mass scale at the global minimum of the vacuum energy also implies the existence of a fixed point of the β function. The reverse is not necessarily true, since the theory may have a fixed point consistent with the absence of any dynamical mass.

It should be remembered that there are more than one SDE solution consistent with a dynamical mass scale for the gluon. These solutions, as discussed previously, depend on the different approximations used to solve the equations and gauge choice, and they necessarily do not lead to a global minimum of energy. It is reasonable to expect that only the true solution, massive or not, will give the absolute minimum of energy and if it has a gluon mass scale it will be related to a unique fixed point.

We can demonstrate the connection between the gauge boson mass scale and the existence of the fixed point in a different way if we particularize the problem to pure gauge QCD. Its Lagrangian is given by $\mathcal{L} = \frac{1}{2}G_{\mu\nu}^2$ and $\phi = A_\mu$ in Eq. (4). Following an argumentation presented by Cornwall [3] we can now rescale the fields $A_\mu^a = g^{-1}\hat{A}_\mu^a$, $G_{\mu\nu}^a = g^{-1}\hat{G}_{\mu\nu}^a$, and regularize the vacuum energy (and the potential) setting its perturbative part equal to zero in order to obtain

$$Z = Z_p^{-1} \int d\hat{A}_\mu \exp\left[-g^{-2} \int d^4x \frac{1}{4} \left\langle \sum_a (\hat{G}_{\mu\nu}^a)^2 \right\rangle \right] = e^{-V\Omega}, \qquad (12)$$

where V is the volume of Euclidean space-time and Z_p is the perturbative functional. Differentiating with respect to g it follows that

$$\frac{\partial \ln Z}{\partial g} = \frac{1}{2g} \int d^4x \left\langle \sum_a (\hat{G}_{\mu\nu}^a)^2 \right\rangle_{\mathrm{reg}} = -\frac{V\partial\Omega}{\partial g}, \qquad (13)$$

where the subscript "reg" on the gluon condensate indicates that the regularization is by subtraction of the perturbative expectation value in the same way as indicated in Eq. (6). The factor V on the right-hand side is canceled with the one coming out from the x integration. As long as the condensate is different from zero for some $g > 0$, and there are indications that this happens for any $g > 0$ [17] (and the same would happen for the gluon mass scale [3,4]), $\partial\Omega/\partial g \neq 0$, since this quantity is proportional to the condensate. This argumentation is correct only in the light-cone gauge (or any ghost-free gauge) as discussed in Ref. [3], for which the derivation of Eq. (13) is valid. Furthermore, the condensate must be consistent with the deepest minimum of energy. According to Eqs. (2) and (3), this result constitutes an alternative proof of our statement that the theory has a nontrivial fixed point at the global minimum of energy, though restricted to a particular scheme.

It would be suitable to compute the vacuum energy Ω and show explicitly the connection between its minimum and the fixed point. However, to compute Ω we must know the full nonperturbative Green functions of the theory, which obviously is not an easy task. In general, this is accomplished using IR finite propagators within some rough approximations [3,6,18].

We can now discuss what happens if instead of a pure gauge theory we also have fermions. Actually part of the arguments presented here were already discussed by some of us when studying fermionic condensation and mass generation in the case of strong coupling QED [19], but the implications were not fully realized and only later it became clear to us [20] that the vacuum energy in QCD with massless fermions is basically dominated by the gluonic (gauge boson) condensation (or mass) rather than by the fermionic one. This fact can be observed if we recover some of the results of Ref. [20] in the following form:

$$\langle\Omega\rangle \propto -\frac{1}{16\pi^2}\left[\frac{3(N^2-1)}{2} a m^4 + N b \eta^4\right], \qquad (14)$$

where $\langle\Omega\rangle$ is the QCD vacuum energy at the extrema of energy in the case that we have a massless fermion,

$N = 3$ is the number of colors, a and b are constants determined by the theory and calculated in Ref. [20], m is the gluon mass, and η is the dynamical fermion mass, which are ultimately connected to the gluon and fermion condensates. There are several points to discuss about this expression. First, it was derived in Landau gauge and involves many approximations. We are far from a satisfactory determination of the full momentum dependence of the dynamical masses used as input to compute Eq. (14), but we believe that this equation can roughly describe the actual behavior. Second, currently assumed values for the gluon and fermion masses [8,18–20] indicate that the first term of the right-hand side dominates the other by at least 1 order of magnitude. Usual estimates of the dynamical masses give the ratio $m/\eta \approx 2$. Third, as is well known, the gluonic SDE are coupled to the fermionic and the ghost ones, i.e., the dynamical gauge boson mass is affected by the presence of fermions and ghosts and vice versa. However, in the case of fermions the effect is small, at least in what concerns the gluon mass [20,21]. Therefore, the global minimum of energy of QCD (or any other non-Abelian gauge theory) is dictated by the gauge bosons, and we can argue that any fixed point of the theory will be determined by the gauge boson sector. The fermions introduce only small changes in the position of the vacuum energy. It is also clear that if we increase the number of fermions too much we will change the values of the dynamical masses as well as the relative importance of each term in Eq. (14).

In conclusion, we have shown that the condition for the global minimum of the vacuum energy for a non-Abelian gauge theory with a dynamically generated gauge boson mass scale implies the existence of a nontrivial IR fixed point of the theory. This vacuum energy depends on the dynamical masses through the nonperturbative propagators of the theory. Our results show that the freezing of the QCD coupling constant observed in the calculations of Refs. [3,4] can be a natural consequence of the onset of a gluon mass scale, giving strong support to their claim.

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