Invariant conserved currents in gravity theories with local Lorentz and diffeomorphism symmetry

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We discuss conservation laws for gravity theories invariant under general coordinate and local Lorentz transformations. We demonstrate the possibility to formulate these conservation laws in many covariant and noncovariant(ly looking) ways. An interesting mathematical fact underlies such a diversity: there is a certain ambiguity in a definition of the (Lorentz-) covariant generalization of the usual Lie derivative. Using this freedom, we develop a general approach to the construction of *invariant* conserved currents generated by an arbitrary vector field on the spacetime. This is done in any dimension, for any Lagrangian of the gravitational field and of a (minimally or nonminimally) coupled matter field. A development of the ''regularization via relocalization'' scheme is used to obtain finite conserved quantities for asymptotically nonflat solutions. We illustrate how our formalism works by some explicit examples.

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I. INTRODUCTION

As is well known, in the Lagrangian approach every continuous symmetry of the action gives rise to a conservation law. This is the substance of the Noether theorem which provides a construction of the corresponding conserved currents that satisfy certain algebraic and differential identities. In particular, the energy and momentum of a physical system are described by the currents that are generated by the symmetry with respect to time and space translations. In a similar way, the angular momentum is described by a current related to the spacetime rotations.

In gravity theory, the definition of energy, momentum, and angular momentum is a nontrivial problem that has a long and rich history (see the reviews [1–3], for example). The difficulty is rooted deeply in the geometric nature of the gravitational theory and is related to the equivalence principle which identifies locally gravity and inertia. As a result, the conservation laws that arise from the general coordinate (diffeomorphism) invariance normally have the form of covariant, but not invariant, equations. The energy-momentum currents in general transform nontrivially under a change of the local coordinate system or a local frame. In a recent paper [4] we have discussed the covariance properties of conserved quantities in the framework of the tetrad approach to general relativity theory.

Along with the covariant conservation laws that are formulated in terms of the gravitational field variables independently of the additional geometric structures on a spacetime manifold, there exist a class of invariant conservation laws. The corresponding conserved currents are usually associated with a vector field that acts on spacetime as a generator of a certain symmetry of the gravitational configuration. However, no systematic and general method for the construction of such invariant conserved quantities was ever developed in the literature, at least not to our knowledge. For example, quite a long time ago Komar [5,6] proposed a very nice formula that gives reasonable values of the total energy (mass) and angular momentum for asymptotically flat configurations. However, although these formulas have proved their apparent feasibility, and despite certain achievements [7–21] in explaining their relevance to the concepts of the gravitational energy and angular momentum, many important issues remained unclear. In particular, the freedom in the choice of the gravitational Lagrangian, of the gravitational field variables, of the coupling to matter, of the matter Lagrangian, of the background geometry (is it needed at all?), of the class of relevant vector fields (should they be necessarily Killing vector fields? why?); all these questions did not have a satisfactory answer (in our opinion) in the existing literature. One of the motivations for the current study can be formulated as ''understanding Komar.''

Another motivation is related to the interesting work of Aros *et al.* [22,23] who have proposed a new conserved quantity for asymptotically anti-de Sitter (AdS) spacetimes. Actually, their result appeared both surprising and confusing to us. The reason is that their conservation law was derived from an invariant Lagrangian with the help of (apparently) *covariant* computations, however, the resulting conserved current (and the corresponding total charge) was *not a scalar* under local Lorentz transformations. How could this happen? Two immediate guesses naturally can

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be formulated. Either the construction [22,23] is fundamentally inconsistent, then it is necessary to find the source of the inconsistency. Or, if this result is nevertheless consistent, then perhaps one can improve it in such a way that from an invariant Lagrangian, by using covariant manipulations, one would be able to derive invariant conserved quantities. At the beginning, we considered the first guess to be most probably true. However, a careful analysis has demonstrated that the results of [22,23] are consistent, but can be improved.

In this paper, we present a detailed exposition of the corresponding analysis. At the same time, we show that indeed the results of Aros *et al.* can be improved along the lines mentioned above. Namely, one of the aims of our paper is to systematically investigate the derivation of covariant and invariant conservation laws for the local Lorentz and the diffeomorphism symmetry in gravity theories. As a result, we derive new explicitly invariant conservation laws for the currents that are true scalars under general coordinate and local Lorentz transformations. The Lagrangian approach seems to be most appropriate when one discusses generally covariant models. We will thus use the Lagrangian framework in this study. Nevertheless, it is worthwhile to recall that the Hamiltonian approach reveals many other important aspects, for example, the role of the conserved charges in defining the generators of the corresponding symmetry transformations [24]. These aspects are discussed in a review [3] (see also the references therein). For a covariant Hamiltonian formulation see, in particular, [25–28].

Since both the diffeomorphism and the local Lorentz symmetry are in the center of our attention, we naturally turn to the so-called Poincaré gauge approach to gravity [24,29–33] in which both symmetries are naturally realized. Einstein's general relativity arises as a particular (degenerate) case in this framework. In the gauge theory of gravity based on the Poincaré group (the semidirect product of the Lorentz group and the spacetime translations) mass (energy-momentum) and spin are treated on an equal footing as the sources of the gravitational field. The gravitational gauge potentials are the local coframe 1-form ϑ^α and the 1-form $\Gamma_\alpha{}^\beta$ of the metric-compatible connection. The absence of nonmetricity and use of orthonormal frames yield the skew symmetry of the connection, $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$. The spacetime manifold carries a Riemann-Cartan geometric structure with nontrivial curvature and torsion that arise as the corresponding gauge field strengths: $R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta + \Gamma_\gamma{}^\beta \wedge \Gamma_\alpha{}^\gamma$ and $T^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta$.

The structure of the paper is as follows. In Sec. II we consider the consequences of the invariance of a general Lagrangian of the gravitational field under local Lorentz transformations of the frames and under spacetime diffeomorphisms. The Noether identities corresponding to the Lorentz symmetry are derived. We then demonstrate that these identities underlie the possibility to recast the Lie equation for the diffeomorphism symmetry in an explicitly covariant form. However, such a "covariantization" is essentially nonunique, and it is determined by what we call a *generalized Lie derivative*. The latter can be introduced on a Riemann-Cartan manifold with the help of an arbitrary (1, 1) tensorial field that has a "connection"-like transformation law under the action of the Lorentz group. These observations are subsequently used in Sec. III A to derive several equivalent forms of the Noether identities for the diffeomorphism symmetry of the gravitational action. In Sec. IV we extend the Lagrange-Noether machinery to the Lagrangian of a matter field with an arbitrary coupling to gravity (i.e., we allow also for nonminimal coupling). Section V presents the main results of our paper: here we give the explicit construction of the invariant currents associated with an arbitrary vector field. The next Sec. VI provides a natural extension of the construction to the case of an arbitrary generalized Lie derivative. This, in particular, demonstrates the consistency of the original construction of Aros *et al.* By applying our general construction to the case of the Einstein-Cartan theory in Sec. VII we show that the Komar formula arises as a special case of our invariant current. Section VIII describes the relocalization of the currents that is induced by the change of the Lagrangian by a boundary term. In order to demonstrate how the general formalism works, in Sec. IX we apply our derivations to the computation of the conserved charges (total mass and angular momentum) for solutions without and with torsion. In the asymptotically nonflat cases, the conserved charges turn out to be divergent and we need to regularize them. We show that the "regularization via relocalization" method (which we recently used in [4]) successfully works also here. Finally, Sec. X contains a discussion of the results obtained and gives an outlook of further possible developments and of open problems.

Our general notations are as in [32]. In particular, we use the Latin indices $i, j, \ldots$ for local holonomic spacetime coordinates and the Greek indices $\alpha, \beta, \ldots$ label (co)frame components. Particular frame components are denoted by hats, $\hat{0}$, $\hat{1}$, etc. As usual, the exterior product is denoted by $\wedge$, while the interior product of a vector ξ and a p-form Ψ is denoted by $\xi \rfloor \Psi$. The vector basis dual to the frame 1-forms ϑ^α is denoted by e_α and they satisfy $e_\alpha \rfloor \vartheta^\beta = \delta_\alpha^\beta$. Using local coordinates x^i, we have $\vartheta^\alpha = h_i^\alpha dx^i$ and $e_\alpha = h^i_\alpha \partial_i$. We define the volume n-form by $\eta := \vartheta^{\hat{0}} \wedge \cdots \wedge \vartheta^{\hat{n}}$. Furthermore, with the help of the interior product we define $\eta_\alpha := e_\alpha \rfloor \eta$, $\eta_{\alpha\beta} := e_\beta \rfloor \eta_\alpha$, $\eta_{\alpha\beta\gamma} := e_\gamma \rfloor \eta_{\alpha\beta}$, etc., which are bases for $(n-1)$-, $(n-2)$-, and $(n-3)$-forms, etc., respectively. Finally, $\eta_{\alpha_1 \cdots \alpha_n} = e_{\alpha_n} \rfloor \eta_{\alpha_1 \cdots \alpha_{n-1}}$ is the Levi-Civita tensor density. The η-forms satisfy the useful identities:

$$\vartheta^\beta \wedge \eta_\alpha = \delta_\alpha^\beta \eta, \tag{1.1}$$

$$\vartheta^\beta \wedge \eta_{\mu\nu} = \delta^\beta_\nu \eta_\mu - \delta^\beta_\mu \eta_\nu, \tag{1.2}$$

$$\vartheta^\beta \wedge \eta_{\alpha\mu\nu} = \delta^\beta_\alpha \eta_{\mu\nu} + \delta^\beta_\mu \eta_{\nu\alpha} + \delta^\beta_\nu \eta_{\alpha\mu}, \tag{1.3}$$

$$\vartheta^\beta \wedge \eta_{\alpha\gamma\mu\nu} = \delta^\beta_\nu \eta_{\alpha\gamma\mu} - \delta^\beta_\mu \eta_{\alpha\gamma\nu} + \delta^\beta_\gamma \eta_{\alpha\mu\nu} - \delta^\beta_\alpha \eta_{\gamma\mu\nu}, \tag{1.4}$$

etc. The line element $ds^2 = g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$ is defined by the spacetime metric $g_{\alpha\beta}$ of signature $(+, -, \cdots, -)$.

II. GENERAL LAGRANGE-NOETHER MACHINERY

We work on a n-dimensional spacetime manifold. The Poincaré gauge potentials are the coframe ϑ^α and the Lorentz connection $\Gamma_\alpha{}^\beta$. We assume that the gravitational Lagrangian n-form V has the form

$$V = V(\vartheta^\alpha, T^\alpha, R_\alpha{}^\beta), \tag{2.1}$$

and that it is invariant under local Lorentz transformations. As usual, let us introduce, according to the canonical prescription, the following translational and rotational *gauge field momenta* $(n-2)$-forms [34]:

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha}, \qquad H^\alpha{}_\beta := -\frac{\partial V}{\partial R_\alpha{}^\beta}. \tag{2.2}$$

Moreover, we define the canonical energy-momentum and spin $(n-1)$-forms

$$E_\alpha := \frac{\partial V}{\partial \vartheta^\alpha}, \qquad E^{\alpha\beta} := -\vartheta^{[\alpha} \wedge H^{\beta]}, \tag{2.3}$$

for the gravitational gauge field.

A general variation of the gravitational Lagrangian then reads

$$\delta V = \delta\vartheta^\alpha \wedge \mathcal{E}_\alpha + \delta\Gamma_\alpha{}^\beta \wedge \mathcal{C}^\alpha{}_\beta - d(\delta\vartheta^\alpha \wedge H_\alpha + \delta\Gamma_\alpha{}^\beta \wedge H^\alpha{}_\beta), \tag{2.4}$$

where we have defined the variational derivatives with respect to (w.r.t.) the gravitational potentials:

$$\mathcal{E}_\alpha := \frac{\delta V}{\delta \vartheta^\alpha} = -DH_\alpha + E_\alpha, \tag{2.5}$$

$$\mathcal{C}^\alpha{}_\beta := \frac{\delta V}{\delta \Gamma_\alpha{}^\beta} = -DH^\alpha{}_\beta + E^\alpha{}_\beta. \tag{2.6}$$

For an infinitesimal Lorentz transformation, $\Lambda^\alpha{}_\beta = \delta^\alpha_\beta + \varepsilon^\alpha{}_\beta$, with $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, we have

$$\delta\vartheta^\alpha = \varepsilon^\alpha{}_\beta \vartheta^\beta, \qquad \delta\Gamma_\beta{}^\alpha = -D\varepsilon^\alpha{}_\beta. \tag{2.7}$$

Substituting this into (2.4), we find

$$\delta V = \varepsilon^\alpha{}_\beta(\vartheta^\beta \wedge \mathcal{E}_\alpha + D\mathcal{C}^\beta{}_\alpha). \tag{2.8}$$

Thus, the local Lorentz invariance of the gravitational Lagrangian, $\delta V = 0$, yields the Noether identity

$$D\mathcal{C}_{\alpha\beta} + \vartheta_{[\alpha} \wedge \mathcal{E}_{\beta]} \equiv 0. \tag{2.9}$$

Now, let us derive the consequences of the diffeomorphism invariance of V. Let f be an arbitrary local diffeomorphism on the spacetime manifold. It acts with the pullback map f^* on all the geometrical quantities, and the invariance of the theory means that [35]

$$V(f^*\vartheta^\alpha, f^*T^\alpha, f^*R_\alpha{}^\beta) = f^*(V(\vartheta^\alpha, T^\alpha, R_\alpha{}^\beta)). \tag{2.10}$$

Consider an arbitrary vector field ξ and the corresponding local 1-parameter group of diffeomorphisms f_t generated along this vector field. Then, using f_t in the above formula and differentiating w.r.t. the parameter t, we find the identity

$$(\ell_\xi\vartheta^\alpha) \wedge E_\alpha - (\ell_\xi T^\alpha) \wedge H_\alpha - (\ell_\xi R_\alpha{}^\beta) \wedge H^\alpha{}_\beta = \ell_\xi V. \tag{2.11}$$

Here the Lie derivative is given on exterior forms by

$$\ell_\xi = d\xi\rfloor + \xi\rfloor d. \tag{2.12}$$

At first sight, the left-hand side (l.h.s.) of (2.11) does not look to be invariant under local Lorentz transformations because of the usual (not covariant) derivatives in (2.12). However, it is invariant. In order to see this, let us recall the property of the Lie derivative $\ell_\xi(\varphi\omega) = \varphi\ell_\xi\omega + (\xi\rfloor d\varphi)\omega$ which is valid for any exterior form ω and any function (0-form) φ. Then, taking into account the local Lorentz transformations $\vartheta'^\alpha = \Lambda^\alpha{}_\beta\vartheta^\beta$, $E'_\alpha = (\Lambda^{-1})^\beta{}_\alpha E_\beta$, $T'^\alpha = \Lambda^\alpha{}_\beta T^\beta$, $H'_\alpha = (\Lambda^{-1})^\beta{}_\alpha H_\beta$, $R'_\alpha = \Lambda^\beta{}_\gamma(\Lambda^{-1})^\delta{}_\alpha R_\delta{}^\gamma$, and $H'^\alpha{}_\beta = \Lambda^\alpha{}_\gamma(\Lambda^{-1})^\delta{}_\beta H^\gamma{}_\delta$, we straightforwardly find that under the action of the local Lorentz transformation, the l.h.s. of (2.11) will be shifted by the term $(\Lambda^{-1})^\alpha{}_\gamma(\xi\rfloor d\Lambda^\gamma{}_\beta)(D\mathcal{C}^\beta{}_\alpha + \vartheta^\beta \wedge \mathcal{E}_\alpha)$. This is zero *in view of the Noether identity* (2.9).

Let us now notice that the 1-form $(\Lambda^{-1})^\alpha{}_\gamma d\Lambda^\gamma{}_\beta$ is a flat Lorentz connection. Then the above observation can be used as follows. We can take an *arbitrary Lorentz-valued 0-form* $B^{\alpha\beta}(\xi) = -B^{\beta\alpha}(\xi)$ and add a *zero term*,

$$B_\beta{}^\alpha(D\mathcal{C}^\beta{}_\alpha + \vartheta^\beta \wedge \mathcal{E}_\alpha), \tag{2.13}$$

to the l.h.s. of (2.11). As is easily verified, this addition is *equivalent* to the replacement of the usual Lie derivative (2.12) with a *generalized Lie derivative* $L_\xi := \ell_\xi + B_\beta{}^\alpha\rho^\beta{}_\alpha$ when applied to coframe, torsion, and curvature. This generalized derivative will be covariant provided $B_\alpha{}^\beta$ transforms according to (A2). Here $\rho^\beta{}_\alpha$ denote the corresponding Lorentz generators for the object on which the derivative acts.

There are many options for the choice of the generalized covariant Lie derivative. One family of generalized Lie derivatives corresponds to the case in which $B_\beta{}^\alpha = \xi\rfloor A_\beta{}^\alpha$, where $A_\beta{}^\alpha$ is an *arbitrary Lorentz connection*. For this family, one can verify that $\ell_\xi + \xi\rfloor A_\beta{}^\alpha\rho^\beta{}_\alpha = \xi\rfloor\overset{A}{D} + \overset{A}{D}\xi\rfloor$, where $\overset{A}{D} := d + A_\beta{}^\alpha\rho^\beta{}_\alpha$ denotes the covariant derivative defined by the connection A. We will consider the following two possibilities:

$$Ł_\xi := \ell_\xi + \xi\rfloor\Gamma_\beta{}^\alpha\rho^\beta{}_\alpha = \xi\rfloor D + D\xi\rfloor, \tag{2.14}$$

$$\overset{\{\}}{Ł}_\xi := \ell_\xi + \xi\rfloor\overset{\{\}}{\Gamma}_\beta{}^\alpha\rho^\beta{}_\alpha = \xi\rfloor\overset{\{\}}{D} + \overset{\{\}}{D}\xi\rfloor. \tag{2.15}$$

Here $\Gamma_\alpha{}^\beta$ is the dynamical Lorentz connection of the theory, whereas $\overset{\{\}}{\Gamma}_\alpha{}^\beta$ is the Riemannian connection (i.e., the anholonomic form of the Christoffel symbols). We refer the reader to Appendix A for further details.

Yet another possibility satisfying the condition (A2), that is not directly related to a connection, is given by

$$\mathcal{L}_\xi := \ell_\xi - \Theta_\beta{}^\alpha\rho^\beta{}_\alpha, \tag{2.16}$$

i.e., $B_\alpha{}^\beta := -\Theta_\alpha{}^\beta$, where

$$\Theta^{\alpha\beta} := e^{[\alpha}\rfloor\ell_\xi\vartheta^{\beta]}. \tag{2.17}$$

This, our third choice of generalized covariant Lie derivative, is very different from the other two, as its existence is specific for the models with a coframe (tetrad) field (and a metric). We will call (2.16) a *Yano derivative* since its counterpart for the linear group was first introduced in [36]. See also [37] for similar, but different definitions.

III. GRAVITATIONAL NOETHER IDENTITIES FOR DIFFEOMORPHISM SYMMETRY

All the above three choices of the covariant Lie derivative are useful. The most common is the first option (2.14) which directly yields covariant Noether identities [32]. (For a general discussion of the Noether theorems in models with local symmetries see [38]).

A. Covariant Noether identities

Using the covariant Lie derivative $Ł_\xi$ we recast (2.11) into

$$(Ł_\xi\vartheta^\alpha)\wedge E_\alpha - (Ł_\xi T^\alpha)\wedge H_\alpha - (Ł_\xi R_\alpha{}^\beta)\wedge H^\alpha{}_\beta - Ł_\xi V = A + dB = 0, \tag{3.1}$$

from where (with $\xi^\alpha := \xi\rfloor\vartheta^\alpha$ denoting the components of the vector field ξ), we find

$$A = \xi^\alpha(-D\mathcal{E}_\alpha + e_\alpha\rfloor T^\beta\wedge\mathcal{E}_\beta + e_\alpha\rfloor R_\gamma{}^\beta\wedge\mathcal{C}^\gamma{}_\beta), \tag{3.2}$$

$$B = \xi^\alpha(E_\alpha - e_\alpha\rfloor T^\beta\wedge H_\beta - e_\alpha\rfloor R_\gamma{}^\beta\wedge H^\gamma{}_\beta - e_\alpha\rfloor V). \tag{3.3}$$

Since the diffeomorphism invariance holds for *arbitrary* vector fields ξ, A and B must necessarily vanish and thus we find the familiar Noether identities:

$$D\mathcal{E}_\alpha \equiv e_\alpha\rfloor T^\beta\wedge\mathcal{E}_\beta + e_\alpha\rfloor R_\gamma{}^\beta\wedge\mathcal{C}^\gamma{}_\beta, \tag{3.4}$$

$$E_\alpha \equiv e_\alpha\rfloor V + e_\alpha\rfloor T^\beta\wedge H_\beta + e_\alpha\rfloor R_\gamma{}^\beta\wedge H^\gamma{}_\beta. \tag{3.5}$$

B. Covariant Noether identities: another face

The second choice (2.15) brings (2.11) into a similar form:

$$(\overset{\{\}}{Ł}_\xi\vartheta^\alpha)\wedge E_\alpha - (\overset{\{\}}{Ł}_\xi T^\alpha)\wedge H_\alpha - (\overset{\{\}}{Ł}_\xi R_\alpha{}^\beta)\wedge H^\alpha{}_\beta - \overset{\{\}}{Ł}_\xi V = A' + dB = 0. \tag{3.6}$$

For B we still find (3.3), whereas A' is different, acquiring an additional piece proportional to the rotational Noether identity, namely

$$A' = \xi^\alpha[-D\mathcal{E}_\alpha + e_\alpha\rfloor T^\beta\wedge\mathcal{E}_\beta + e_\alpha\rfloor R_\gamma{}^\beta\wedge\mathcal{C}^\gamma{}_\beta + (e_\alpha\rfloor K_\gamma{}^\beta)(\vartheta^\gamma\wedge\mathcal{E}_\beta + D\mathcal{C}^\gamma{}_\beta)]. \tag{3.7}$$

In accordance with the above general analysis, the difference of (3.1) and (3.6) is proportional to the difference of the connections

$$K_\alpha{}^\beta := \overset{\{\}}{\Gamma}_\alpha{}^\beta - \Gamma_\alpha{}^\beta. \tag{3.8}$$

This quantity is known as contortion 1-form. In particular, the torsion is recovered from it as $T^\alpha = K^\alpha{}_\beta\wedge\vartheta^\beta$. The corresponding curvature 2-forms are related via

$$R_\alpha{}^\beta = \overset{\{\}}{R}_\alpha{}^\beta - \overset{\{\}}{D}K_\alpha{}^\beta + K_\gamma{}^\beta\wedge K_\alpha{}^\gamma. \tag{3.9}$$

Combining together the terms with $\mathcal{E}_\alpha$, with the help of (3.8) and (3.9) we can recast (3.7) into

$$A' = \xi^\alpha[-\overset{\{\}}{D}(\mathcal{E}_\alpha - \mathcal{C}^\gamma{}_\beta e_\alpha\rfloor K_\gamma{}^\beta) + \mathcal{C}^\gamma{}_\beta\wedge(e_\alpha\rfloor\overset{\{\}}{R}_\gamma{}^\beta - \overset{\{\}}{Ł}_\alpha K_\gamma{}^\beta)]. \tag{3.10}$$

Again, since the diffeomorphism invariance holds for an arbitrary vector field ξ, we recover the corresponding Noether identity in the alternative form

$$\overset{\{\}}{D}(\mathcal{E}_\alpha - \mathcal{C}^\gamma{}_\beta e_\alpha\rfloor K_\gamma{}^\beta) \equiv (e_\alpha\rfloor\overset{\{\}}{R}_\gamma{}^\beta - \overset{\{\}}{Ł}_\alpha K_\gamma{}^\beta)\wedge\mathcal{C}^\gamma{}_\beta. \tag{3.11}$$

The identities (3.4) and (3.11) are equivalent, but in contrast to the usual (3.4), the alternative form (3.11) is less known. It was derived previously in [33] using a different method.

C. "Noncovariant" Noether identities

Besides the above choices, it is still possible to work with the noncovariant ordinary Lie derivative (2.12). Then we start directly with the identity (2.11) which we recast into

$$(\ell_\xi \vartheta^\alpha) \wedge E_\alpha - (\ell_\xi T^\alpha) \wedge H_\alpha - (\ell_\xi R_\alpha{}^\beta) \wedge H^\alpha{}_\beta - \ell_\xi V = \mathcal{A} + dB = 0. \tag{3.12}$$

Again B is given by (3.3), but A changes into

$$\begin{aligned}\mathcal{A} &= \xi^\alpha[-D\mathcal{E}_\alpha + e_\alpha\rfloor T^\beta \wedge \mathcal{E}_\beta + e_\alpha \rfloor R_\gamma{}^\beta \wedge \mathcal{C}^\gamma{}_\beta \\ &\quad - (e_\alpha\rfloor \Gamma_\gamma{}^\beta)(\vartheta^\gamma \wedge \mathcal{E}_\beta + D\mathcal{C}^\gamma{}_\beta)] \\ &= \xi^\alpha[-d\mathcal{E}_\alpha + (e_\alpha\rfloor d\vartheta^\beta) \wedge \mathcal{E}_\beta \\ &\quad + (e_\alpha\rfloor d\Gamma_\gamma{}^\beta) \wedge \mathcal{C}^\gamma{}_\beta - (e_\alpha\rfloor\Gamma_\gamma{}^\beta) d\mathcal{C}^\gamma{}_\beta].\end{aligned} \tag{3.13}$$

As a result of the diffeomorphism invariance for an arbitrary ξ we find yet another form of the Noether identity (3.4), namely

$$d(\mathcal{E}_\alpha + \mathcal{C}^\gamma{}_\beta e_\alpha \rfloor \Gamma_\gamma{}^\beta) \equiv (\ell_{e_\alpha}\vartheta^\beta) \wedge \mathcal{E}_\beta + (\ell_{e_\alpha}\Gamma_\gamma{}^\beta) \wedge \mathcal{C}^\gamma{}_\beta. \tag{3.14}$$

One may call this form of the Noether identity "noncovariant" since it explicitly involves the noncovariant gravitational field potentials $\Gamma_\alpha{}^\beta$ (and not the corresponding field strengths) and the ordinary (noncovariant) Lie derivative. However, just like the noncovariantly looking expression (2.11), the identity (3.14) is in fact covariant. Moreover, all the three forms (3.4), (3.11), and (3.14) are completely equivalent.

IV. MATTER: DYNAMICS AND NOETHER IDENTITIES

Matter fields can be represented by scalar-, tensor-, or spinor-valued forms of some rank, and we will denote all of them collectively as Ψ^A, where the superscript A indicates the appropriate index (tensor and/or spinor) structure. We will assume that the matter fields Ψ^A belong to the space of some (reducible, in general) representation of the Lorentz group. For an infinitesimal Lorentz transformation (2.7) the matter fields transform as

$$\Psi'^A = \Psi^A + \delta\Psi^A, \qquad \delta\Psi^A = \varepsilon^\beta{}_\alpha(\rho^\alpha{}_\beta)^A{}_B\Psi^B. \tag{4.1}$$

Here $(\rho^\alpha{}_\beta)^A{}_B$ denote the corresponding matrices of generators of the Lorentz group. For definiteness, we consider the case when the matter field Ψ^A is a 0-form on the spacetime manifold. This includes scalar and spinor fields of any rank [39].

A. Lagrangian and field equations

We assume that the matter Lagrangian n-form L depends most generally on Ψ^A, $d\Psi^A$ and the gravitational potentials ϑ^α, $\Gamma_\alpha{}^\beta$. According to the *minimal coupling* prescription, derivatives of the gravitational potentials are not permitted. We usually adhere to this principle. However, Pauli-type terms and Jordan-Brans-Dicke-type terms may occur in phenomenological models or in the context of a symmetry breaking mechanism. Also the Gordon decomposition of the matter currents and the discussion of the gravitational moments necessarily requires the inclusion of Pauli-type terms, see [40,41]. Therefore, we develop our Lagrangian formalism in sufficient generality in order to cope with such models by including in the Lagrangian also the derivatives $d\vartheta^\alpha$, and $d\Gamma_\alpha{}^\beta$ of the gravitational potentials. In view of the assumed invariance of L under local Lorentz transformations, the derivatives can enter only in a covariant form, namely

$$L = L(\Psi^A, D\Psi^A, \vartheta^\alpha, T^\alpha, R_\alpha{}^\beta). \tag{4.2}$$

That is, the derivatives of the matter fields appear only in a covariant combination, $D\Psi^A = d\Psi^A + \Gamma_\alpha{}^\beta \wedge (\rho^\alpha{}_\beta)^A{}_B\Psi^B$, whereas the derivatives of the coframe and connection can only appear via the torsion and the curvature 2-forms.

For the total variation of the matter Lagrangian with respect to the material and the gravitational fields, we find

$$\delta L = \delta\vartheta^\alpha \wedge \frac{\partial L}{\partial\vartheta^\alpha} + \delta T^\alpha \wedge \frac{\partial L}{\partial T^\alpha} + \delta R_\alpha{}^\beta \wedge \frac{\partial L}{\partial R_\alpha{}^\beta} + \delta\Psi^A \wedge \frac{\partial L}{\partial \Psi^A} + \delta(D\Psi^A) \wedge \frac{\partial L}{\partial D\Psi^A} \tag{4.3}$$

$$= \delta\vartheta^\alpha \wedge \Sigma_\alpha + \delta\Gamma_\alpha{}^\beta \wedge \tau^\alpha{}_\beta + \delta\Psi^A \wedge \frac{\delta L}{\delta\Psi^A} + d\left[\delta\vartheta^\alpha \wedge \frac{\partial L}{\partial T^\alpha} + \delta\Gamma_\alpha{}^\beta \wedge \frac{\partial L}{\partial R_\alpha{}^\beta} + \delta\Psi^A \wedge \frac{\partial L}{\partial D\Psi^A}\right]. \tag{4.4}$$

Here, as usual, we denote the covariant *variational derivative* of L with respect to the matter field Ψ^A as

$$\frac{\delta L}{\delta\Psi^A} := \frac{\partial L}{\partial\Psi^A} - D\frac{\partial L}{\partial(D\Psi^A)}, \tag{4.5}$$

and the *canonical currents* of energy-momentum and spin are defined, respectively, by [42]

$$\Sigma_\alpha := \frac{\delta L}{\delta\vartheta^\alpha} = \frac{\partial L}{\partial\vartheta^\alpha} + D\frac{\partial L}{\partial T^\alpha}, \tag{4.6}$$

$$\tau^\alpha{}_\beta := \frac{\delta L}{\delta\Gamma_\alpha{}^\beta} = (\rho^\alpha{}_\beta)^A{}_B\Psi^B \wedge \frac{\partial L}{\partial(D\Psi^A)} + \vartheta^{[\alpha} \wedge \frac{\partial L}{\partial T^{\beta]}} + D\frac{\partial L}{\partial R_\alpha{}^\beta}. \tag{4.7}$$

The principle of stationary action $\delta\int V^{\rm tot} = 0$, $V^{\rm tot} := V + L$, for the coupled system of the gravitational and material fields, see (2.4) and (4.4), then yields the dynamical equations:

$$\frac{\delta L}{\delta\Psi^A} = 0, \tag{4.8}$$

$$\mathcal{E}_\alpha + \Sigma_\alpha = 0, \tag{4.9}$$

$$\mathcal{C}^\alpha{}_\beta + \tau^\alpha{}_\beta = 0. \tag{4.10}$$

B. Lorentz symmetry

It is straightforward to derive the Noether identities following from the invariance of L under the local Lorentz group and under diffeomorphisms. Using the infinitesimal transformations (2.7) and (4.1), we find for the *Lorentz symmetry*:

$$\delta L = \varepsilon^\beta{}_\alpha\left(\vartheta^\alpha \wedge \Sigma_\beta + D\tau^\alpha{}_\beta + (\rho^\alpha{}_\beta)^A{}_B\Psi^B \wedge \frac{\delta L}{\delta\Psi^A}\right) + d\left[\varepsilon^\beta{}_\alpha\left(-\tau^\alpha{}_\beta + (\rho^\alpha{}_\beta)^A{}_B\Psi^B \wedge \frac{\partial L}{\partial D\Psi^A} + \vartheta^\alpha \wedge \frac{\partial L}{\partial T^\beta} + D\frac{\partial L}{\partial R_\alpha{}^\beta}\right)\right] = 0. \tag{4.11}$$

The term in the total derivative vanishes identically in view of the definition (4.7) of the spin current $\tau^\alpha{}_\beta$. Then, from the arbitrariness of $\varepsilon^\alpha{}_\beta$, we find the corresponding Noether identity:

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha} \wedge \Sigma_{\beta]} \equiv -(\rho_{\alpha\beta})^A{}_B\Psi^B \wedge \frac{\delta L}{\delta\Psi^A} \cong 0. \tag{4.12}$$

With $\cong$ we denote "weak identities," i.e., an identity valid assuming that the *matter* field equations are satisfied.

C. Diffeomorphism symmetry

For a *diffeomorphism* generated by an arbitrary vector field ξ, we find directly from (4.3):

$$\ell_\xi L = (\ell_\xi\vartheta^\alpha) \wedge \frac{\partial L}{\partial\vartheta^\alpha} + (\ell_\xi T^\alpha) \wedge \frac{\partial L}{\partial T^\alpha} + (\ell_\xi R_\alpha{}^\beta) \wedge \frac{\partial L}{\partial R_\alpha{}^\beta} + (\ell_\xi\Psi^A) \wedge \frac{\partial L}{\partial\Psi^A} + (\ell_\xi D\Psi^A) \wedge \frac{\partial L}{\partial D\Psi^A}. \tag{4.13}$$

In complete analogy to the purely gravitational case, we have a wide choice of covariant Lie derivatives which we can use in the above identity instead of the ordinary Lie derivative ℓ_ξ. However, we will not repeat once again the detailed analysis of all possibilities, taking into account the equivalence of the final results. Instead, here we give details for the covariant option (2.14) only. We then find

$$-Ł_\xi L + (Ł_\xi\vartheta^\alpha) \wedge \frac{\partial L}{\partial\vartheta^\alpha} + (Ł_\xi T^\alpha) \wedge \frac{\partial L}{\partial T^\alpha} + (Ł_\xi R_\alpha{}^\beta) \wedge \frac{\partial L}{\partial R_\alpha{}^\beta} + (Ł_\xi\Psi^A) \wedge \frac{\partial L}{\partial\Psi^A} + (Ł_\xi D\Psi^A) \wedge \frac{\partial L}{\partial D\Psi^A} = A + dB = 0, \tag{4.14}$$

where, after some algebra, we find

$$A = -(\xi\rfloor\vartheta^\alpha)D\frac{\delta L}{\delta\vartheta^\alpha} + (\xi\rfloor T^\alpha) \wedge \frac{\delta L}{\delta\vartheta^\alpha} + (\xi\rfloor R_\beta{}^\gamma) \wedge \frac{\delta L}{\delta\Gamma_\beta{}^\gamma} + (\xi\rfloor D\Psi^A) \wedge \frac{\delta L}{\delta\Psi^A}, \tag{4.15}$$

$$B = (\xi\rfloor\vartheta^\alpha)\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor T^\alpha) \wedge \frac{\partial L}{\partial T^\alpha} + (\xi\rfloor R_\alpha{}^\beta) \wedge \frac{\partial L}{\partial R_\alpha{}^\beta} + (\xi\rfloor D\Psi^A) \wedge \frac{\partial L}{\partial D\Psi^A} - \xi\rfloor L. \tag{4.16}$$

Using the fact that ξ is a pointwise arbitrary vector field, we derive the following Noether identities from $B = 0$ and $A = 0$, respectively:

$$\Sigma_\alpha \equiv e_\alpha\rfloor L - (e_\alpha\rfloor D\Psi^A) \wedge \frac{\partial L}{\partial D\Psi^A} + D\frac{\partial L}{\partial T^\alpha} - (e_\alpha\rfloor T^\beta) \wedge \frac{\partial L}{\partial T^\beta} - (e_\alpha\rfloor R_\beta{}^\gamma) \wedge \frac{\partial L}{\partial R_\beta{}^\gamma}, \tag{4.17}$$

and

$$D\Sigma_\alpha \equiv (e_\alpha\rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha\rfloor R_\beta{}^\gamma) \wedge \tau^\beta{}_\gamma + w_\alpha \cong (e_\alpha\rfloor T^\beta) \wedge \Sigma_\beta + (e_\alpha\rfloor R_\beta{}^\gamma) \wedge \tau^\beta{}_\gamma, \tag{4.18}$$

where

$$w_\alpha := (e_\alpha\rfloor D\Psi^A)\frac{\delta L}{\delta\Psi^A} \cong 0. \tag{4.19}$$

Equation (4.17) yields the explicit form of the *canonical energy-momentum of matter*, with the first line representing the result known in the context of the *special*-relativistic classical field theory. The second line in (4.17) accounts for the possible Pauli terms as well as for Lagrange multiplier terms in the variations with constraints and it is absent for the case of minimal coupling. The first line in (4.18) is given in the *strong* form, without using the field equations for matter (4.8).

In complete analogy to Sec. III B and III C it is possible to derive the "Riemannian" and the "noncovariant" versions of the Noether identity (4.18). By using instead (2.14) the covariant Lie derivative (2.15) we then obtain

$$\overset{\{\}}{D}(\Sigma_\alpha - \tau^\gamma{}_\beta e_\alpha\rfloor K_\gamma{}^\beta) \cong (e_\alpha\rfloor\overset{\{\}}{R}_\gamma{}^\beta - \overset{\{\}}{Ł}_\alpha K_\gamma{}^\beta) \wedge \tau^\gamma{}_\beta, \tag{4.20}$$

whereas replacement of $Ł_\xi$ by the ordinary Lie derivative (2.12) yields

$$d(\Sigma_\alpha + \tau^\gamma{}_\beta e_\alpha\rfloor\Gamma_\gamma{}^\beta) \cong (\ell_{e_\alpha}\vartheta^\beta) \wedge \Sigma_\beta + (\ell_{e_\alpha}\Gamma_\gamma{}^\beta) \wedge \tau^\gamma{}_\beta. \tag{4.21}$$

The three forms of the Noether identity (4.18), (4.20), and (4.21) are equivalent. In [43] the metric-affine counterpart of (4.20) was used to demonstrate that structureless test particles always move along Riemannian geodesics.

V. CURRENTS ASSOCIATED WITH A VECTOR FIELD

In the two previous sections we have analyzed the Noether identities which follow from the diffeomorphism symmetry generated by any vector field ξ. The corresponding covariant currents do not depend on the latter. However, there exists a class of invariant conserved currents which are associated with a given (though arbitrary) vector field. A notable example is the well-known Komar construction [5,6]. As another example, take a symmetric energy-momentum tensor $T_j{}^i$ (which is covariantly conserved in diffeomorphism-invariant theories) and a *Killing* field $\xi = \xi^i \partial_i$ (that generates an isometry of the spacetime). Then $j^i := \xi^j T_j{}^i$ is a conserved current, and a conserved charge is defined as the integral $\int_S j^i \partial_i \rfloor \eta$ over a $(n-1)$-hypersurface S. Moreover, it is possible to construct a conserved current $(n-1)$-form for any solution of a diffeomorphism-invariant model even when ξ is not a Killing field [19]. Such a current and the corresponding charge are scalars under general coordinate transformations.

In this section, we derive globally conserved currents associated with a vector field by making use of the *third* covariant Lie derivative, namely, the Yano derivative (2.16). Note that it is defined without any connection.

A. Gravitational current

We start from the identity (2.11) for the gravitational Lagrangian, and replace the ordinary Lie derivative ℓ_ξ with the covariant Yano derivative $\mathcal{L}_\xi$. Then using the properties (A18) and (A19), we find

$$(\mathcal{L}_\xi \vartheta^\alpha) \wedge E_\alpha - (\mathcal{L}_\xi T^\alpha) \wedge H_\alpha - (\mathcal{L}_\xi R_\alpha{}^\beta) \wedge H^\alpha{}_\beta - \mathcal{L}_\xi V = \mathcal{L}_\xi \vartheta^\alpha \wedge \mathcal{E}_\alpha + \mathcal{L}_\xi \Gamma_\alpha{}^\beta \wedge \mathcal{C}^\alpha{}_\beta - d\mathcal{J}^{\mathrm{grav}}[\xi] = 0, \tag{5.1}$$

where we introduced the scalar $(n-1)$-form

$$\mathcal{J}^{\mathrm{grav}}[\xi] := \xi \rfloor V + \mathcal{L}_\xi \vartheta^\alpha \wedge H_\alpha + \mathcal{L}_\xi \Gamma_\alpha{}^\beta \wedge H^\alpha{}_\beta. \tag{5.2}$$

By making use of the definitions of the Yano derivative (A12) and (A13), and taking into account the Noether identity (3.5), we recast this current into the equivalent form

$$\mathcal{J}^{\mathrm{grav}}[\xi] = d(\xi^\alpha H_\alpha + \Xi_\alpha{}^\beta H^\alpha{}_\beta) + \xi^\alpha \mathcal{E}_\alpha + \Xi_\alpha{}^\beta \mathcal{C}^\alpha{}_\beta. \tag{5.3}$$

B. Matter current

In complete analogy with the previous subsection, starting now from the identity (4.13) for the *material* Lagrangian, and replacing the ordinary Lie derivative ℓ_ξ with the covariant Yano derivative $\mathcal{L}_\xi$, we obtain

$$\mathcal{L}_\xi \vartheta^\alpha \wedge \frac{\partial L}{\partial \vartheta^\alpha} + \mathcal{L}_\xi T^\alpha \wedge \frac{\partial L}{\partial T^\alpha} + \mathcal{L}_\xi R_\alpha{}^\beta \wedge \frac{\partial L}{\partial R_\alpha{}^\beta} + \mathcal{L}_\xi \Psi^A \wedge \frac{\partial L}{\partial \Psi^A} + \mathcal{L}_\xi D\Psi^A \wedge \frac{\partial L}{\partial D\Psi^A} - \mathcal{L}_\xi L = \mathcal{L}_\xi \vartheta^\alpha \wedge \Sigma_\alpha + \mathcal{L}_\xi \Gamma_\alpha{}^\beta \wedge \tau^\alpha{}_\beta + \mathcal{L}_\xi \Psi^A \wedge \frac{\delta L}{\delta \Psi^A} - d\mathcal{J}^{\mathrm{mat}}[\xi] = 0. \tag{5.4}$$

Here the $(n-1)$-form of the matter current is defined by

$$\mathcal{J}^{\mathrm{mat}}[\xi] := \xi \rfloor L - \mathcal{L}_\xi \vartheta^\alpha \wedge \frac{\partial L}{\partial T^\alpha} - \mathcal{L}_\xi \Gamma_\alpha{}^\beta \wedge \frac{\partial L}{\partial R_\alpha{}^\beta} - \mathcal{L}_\xi \Psi^A \wedge \frac{\partial L}{\partial D\Psi^A}. \tag{5.5}$$

Again using the definitions of the Yano derivative (A12) and (A13), together with the Noether identity (4.17), we find that

$$\mathcal{J}^{\mathrm{mat}}[\xi] = -d\left(\xi^\alpha \frac{\partial L}{\partial T^\alpha} + \Xi_\alpha{}^\beta \frac{\partial L}{\partial R_\alpha{}^\beta}\right) + \xi^\alpha \Sigma_\alpha + \Xi_\alpha{}^\beta \tau^\alpha{}_\beta. \tag{5.6}$$

C. Total current

Finally, for the coupled system of gravitational and matter fields described by the total Lagrangian $V^{\mathrm{tot}} = V + L$, the diffeomorphism invariance of V^{tot} gives rise to the total current $(n-1)$-form

$$\mathcal{J}[\xi] := \mathcal{J}^{\mathrm{grav}}[\xi] + \mathcal{J}^{\mathrm{mat}}[\xi] = \xi \rfloor (V + L) - \mathcal{L}_\xi \Psi^A \wedge \frac{\partial L}{\partial D\Psi^A} + \mathcal{L}_\xi \vartheta^\alpha \wedge \left(H_\alpha - \frac{\partial L}{\partial T^\alpha}\right) + \mathcal{L}_\xi \Gamma_\alpha{}^\beta \wedge \left(H^\alpha{}_\beta - \frac{\partial L}{\partial R_\alpha{}^\beta}\right). \tag{5.7}$$

By combining (5.1) and (5.4), we verify that the exterior derivative of this current is

$$d\mathcal{J}[\xi] = \mathcal{L}_\xi \vartheta^\alpha \wedge (\mathcal{E}_\alpha + \Sigma_\alpha) + \mathcal{L}_\xi \Gamma_\alpha{}^\beta \wedge (\mathcal{C}^\alpha{}_\beta + \tau^\alpha{}_\beta) + \mathcal{L}_\xi \Psi^A \wedge \frac{\delta L}{\delta \Psi^A}. \tag{5.8}$$

Finally, from (5.3) and (5.6), we find

$$\mathcal{J}[\xi] = d\Big[(\xi\rfloor\vartheta^\alpha)\Big(H_\alpha - \frac{\partial L}{\partial T^\alpha}\Big) + \Xi_\alpha{}^\beta\Big(H^\alpha{}_\beta - \frac{\partial L}{\partial R_\alpha{}^\beta}\Big)\Big] + (\xi\rfloor\vartheta^\alpha)(\mathcal{E}_\alpha + \Sigma_\alpha) + \Xi_\alpha{}^\beta(\mathcal{C}^\alpha{}_\beta + \tau^\alpha{}_\beta). \quad (5.9)$$

As follows from (5.8), for solutions of the coupled system of gravitational plus matter field equations (4.8), (4.9), and (4.10), the total current $(n-1)$-form (5.7) is conserved: $d\mathcal{J}[\xi] = 0$ *for any* ξ. Hence, it is possible to define a corresponding conserved charge $\mathcal{Q}[\xi]$ by integrating $\mathcal{J}[\xi]$ over a $(n-1)$-dimensional spatial hypersurface S. Moreover, as we see from the Eq. (5.9), on the solutions of the field equations (4.8), (4.9), and (4.10), this current is expressed in terms of a superpotential $(n-2)$-form. As a result, the corresponding charge can be computed as an integral over the spatial boundary ∂S:

$$\mathcal{Q}[\xi] := \int_S \mathcal{J}[\xi] = \int_{\partial S}\Big[(\xi\rfloor\vartheta^\alpha)\Big(H_\alpha - \frac{\partial L}{\partial T^\alpha}\Big) + \Xi_\alpha{}^\beta\Big(H^\alpha{}_\beta - \frac{\partial L}{\partial R_\alpha{}^\beta}\Big)\Big]. \quad (5.10)$$

For the usual case of *minimally coupled* matter $\partial L/\partial T^\alpha = 0$ and $\partial L/\partial R_\alpha{}^\beta = 0$, and this expression then simplifies considerably.

As a historic remark, let us mention that similar constructions for the *matter current* were described in the literature in the framework of the Einstein-Cartan theory [7,44,45], and also in the context of metric-affine gravity [46]. However, the conservation of those currents was derived assuming a (Killing) symmetry of the gravitational configuration. We can recover the latter result as follows. If we assume minimal coupling of matter to gravity and choose ξ as a generalized Killing vector which satisfies $\mathcal{L}_\xi\vartheta^\alpha = 0$ (cf. Eq. (A21)) and $\mathcal{L}_\xi\Gamma_\alpha{}^\beta = 0$, then (5.6) yields the matter current $\mathcal{J}^{\rm mat}[\xi] = \xi^\alpha\Sigma_\alpha + \Xi_\alpha{}^\beta\tau^\alpha{}_\beta$, which is conserved in view of (5.4) on the solutions of the matter field equations. For spinless matter, i.e. $\tau^\alpha{}_\beta = 0$, it is sufficient for ξ to satisfy the usual Killing equation, $\mathcal{L}_\xi\vartheta^\alpha = 0$.

In contrast to that, it is worthwhile to stress that in our derivations we never made any assumptions concerning the vector field ξ. The conservation of the current $\mathcal{J}[\xi]$ and the existence of the corresponding charge $\mathcal{Q}[\xi]$ do not depend on whether ξ is a usual/generalized Killing vector or not. Note also that we did not specify either the dimension of spacetime or the form of the gravitational Lagrangian.

VI. GENERALIZED LIE DERIVATIVES AND COVARIANT CURRENTS

In the previous section, we have applied the Lie derivative $\mathcal{L}_\xi$ in the sense of Yano to obtain invariant conserved currents associated with a vector field. This is one of our main results. However, the existence of arbitrary generalized Lie derivatives (see Appendix A) opens additional possibilities. Let us recall that in Sec. II, in particular, see Eq. (2.13), we discovered that all generalized Lie derivatives (i.e., for any (1, 1) field $B_\alpha{}^\beta$) are admissible for the analysis of the conservation laws related to the diffeomorphism symmetry. The specific examples for the corresponding Noether identities were considered in Sec. III.

So, a natural question arises: Can we find other conserved currents for the generalized Lie derivatives (for an arbitrary $B_\alpha{}^\beta$)? The answer is positive. Without repeating the computations, we just formulate the result.

Given a generalized Lie derivative L_ξ defined by some $B_\alpha{}^\beta$, the diffeomorphism invariance of the total Lagrangian $V^{\rm tot}$ of the coupled gravitational and matter fields associates to a vector field ξ (that generates a diffeomorphism) a current $(n-1)$-form

$$J_L[\xi] := \xi\rfloor(V+L) - L_\xi\Psi^A\wedge\frac{\partial L}{\partial D\Psi^A} + L_\xi\vartheta^\alpha\wedge\Big(H_\alpha - \frac{\partial L}{\partial T^\alpha}\Big) + L_\xi\Gamma_\alpha{}^\beta\wedge\Big(H^\alpha{}_\beta - \frac{\partial L}{\partial R_\alpha{}^\beta}\Big). \quad (6.1)$$

The subscript $_L$ reflects the fact that this current is defined with the help of the generalized Lie derivative L_ξ. Moreover, this current has two basic properties: (i) it satisfies

$$d(J_L[\xi]) = L_\xi\vartheta^\alpha\wedge(\mathcal{E}_\alpha + \Sigma_\alpha) + L_\xi\Gamma_\alpha{}^\beta\wedge(\mathcal{C}^\alpha{}_\beta + \tau^\alpha{}_\beta) + L_\xi\Psi^A\wedge\frac{\delta L}{\delta\Psi^A}. \quad (6.2)$$

(ii) it admits the representation

$$J_L[\xi] = d\Big[(\xi\rfloor\vartheta^\alpha)\Big(H_\alpha - \frac{\partial L}{\partial T^\alpha}\Big) + (\xi\rfloor\Gamma_\alpha{}^\beta - B_\alpha{}^\beta)\times\Big(H^\alpha{}_\beta - \frac{\partial L}{\partial R_\alpha{}^\beta}\Big)\Big] + (\xi\rfloor\vartheta^\alpha)(\mathcal{E}_\alpha + \Sigma_\alpha) + (\xi\rfloor\Gamma_\alpha{}^\beta - B_\alpha{}^\beta)(\mathcal{C}^\alpha{}_\beta + \tau^\alpha{}_\beta). \quad (6.3)$$

Consequently, for solutions of the field equations (4.8), (4.9), and (4.10), the new currents are conserved, $d(J_L[\xi]) = 0$, and the corresponding charges can be computed as an integral over the spatial boundary:

$$Q_L[\xi] := \int_S J_L[\xi] = \int_{\partial S}\Big[(\xi\rfloor\vartheta^\alpha)\Big(H_\alpha - \frac{\partial L}{\partial T^\alpha}\Big) + (\xi\rfloor\Gamma_\alpha{}^\beta - B_\alpha{}^\beta)\Big(H^\alpha{}_\beta - \frac{\partial L}{\partial R_\alpha{}^\beta}\Big)\Big]. \quad (6.4)$$

When $B_\alpha{}^\beta = -\Theta_\alpha{}^\beta$, we recover (5.7), (5.8), (5.9), and (5.10) in view of (A15). Below we describe a number of additional particular cases.

A. Covariant current for natural covariant Lie derivative

The covariant Lie derivative $\text{Ł}_\xi = D\xi\rfloor + \xi\rfloor D$ is defined by the dynamical connection $\Gamma_\alpha{}^\beta$ of the gravitational theory (hence the name natural). It corresponds to the choice $B_\alpha{}^\beta = \xi\rfloor\Gamma_\alpha{}^\beta$. As a result, the conserved current $J_\Gamma[\xi]$ (which is given by (6.1) with L_ξ replaced with Ł_ξ) gives rise to the charge

$$Q_\Gamma[\xi] := \int_S J_\Gamma[\xi] = \int_{\partial S}\left[(\xi\rfloor\vartheta^\alpha)\left(H_\alpha - \frac{\partial L}{\partial T^\alpha}\right)\right]. \tag{6.5}$$

This quantity is nontrivial only when the gravitational Lagrangian V depends on the torsion, and/or for nonminimal coupling of matter. For all other cases this charge is identically zero. For example, for the usual Einstein-Cartan theory we have $H_\alpha = 0$, and for the minimal coupling the derivative $\partial L/\partial T^\alpha = 0$ also vanishes, hence the above charge is trivial.

B. Covariant current for Riemannian covariant Lie derivative

The Riemannian covariant Lie derivative $\overset{\{\}}{\text{Ł}}_\xi = \overset{\{\}}{D}\xi\rfloor + \xi\rfloor\overset{\{\}}{D}$ is defined by the Christoffel connection $\overset{\{\}}{\Gamma}_\alpha{}^\beta$ (hence the notation with the subscript $_{\{\}}$). It corresponds to the choice $B_\alpha{}^\beta = \xi\rfloor\overset{\{\}}{\Gamma}_\alpha{}^\beta$. The corresponding current $J_{\{\}}[\xi]$ is obtained by substituting L_ξ with $\overset{\{\}}{\text{Ł}}_\xi$ in (6.1). It yields the charge

$$\begin{aligned} Q_{\{\}}[\xi] &:= \int_S J_{\{\}}[\xi] \\ &= \int_{\partial S}\left[(\xi\rfloor\vartheta^\alpha)\left(H_\alpha - \frac{\partial L}{\partial T^\alpha}\right)\right. \\ &\quad \left. - \xi\rfloor K_\alpha{}^\beta\left(H^\alpha{}_\beta - \frac{\partial L}{\partial R_\alpha{}^\beta}\right)\right]. \end{aligned} \tag{6.6}$$

Here $K_\alpha{}^\beta$ is the contortion, see (3.8). As a result, this charge reduces to (6.5) for all torsion-free solutions, as is the case, for example, in the standard general relativity theory.

C. Covariant current for background covariant Lie derivative

Yet another option arises when we introduce a nondynamical background connection $\bar{\Gamma}_\alpha{}^\beta$ (that is different from both $\Gamma_\alpha{}^\beta$ and $\overset{\{\}}{\Gamma}_\alpha{}^\beta$) and define a generalized Lie derivative with the help of $B_\alpha{}^\beta = \xi\rfloor\bar{\Gamma}_\alpha{}^\beta$. Then the key combination is $\xi\rfloor\Gamma_\alpha{}^\beta - B_\alpha{}^\beta = \xi\rfloor\Delta\Gamma_\alpha{}^\beta$, and the corresponding conserved charge reduces to

$$\begin{aligned} Q_\Delta[\xi] &:= \int_S J_\Delta[\xi] \\ &= \int_{\partial S}\left[(\xi\rfloor\vartheta^\alpha)\left(H_\alpha - \frac{\partial L}{\partial T^\alpha}\right)\right. \\ &\quad \left. + \xi\rfloor\Delta\Gamma_\alpha{}^\beta\left(H^\alpha{}_\beta - \frac{\partial L}{\partial R_\alpha{}^\beta}\right)\right]. \end{aligned} \tag{6.7}$$

The difference $\Delta\Gamma_\alpha{}^\beta := \Gamma_\alpha{}^\beta - \bar{\Gamma}_\alpha{}^\beta$ normally should be chosen so that it provides a finite value for integral over the spatial boundary. Similar constructions for the computations of conserved quantities in the gauge theories of gravity were used in [4,25–28,47,48].

D. Noncovariant current for ordinary Lie derivative

Finally, one can also consider the case when $B_\alpha{}^\beta = 0$. Then the generalized Lie derivative reduces to the ordinary one, $L_\xi = \ell_\xi$. As a result, we find the following charge

$$\begin{aligned} Q_0[\xi] &:= \int_S J_0[\xi] \\ &= \int_{\partial S}\left[(\xi\rfloor\vartheta^\alpha)\left(H_\alpha - \frac{\partial L}{\partial T^\alpha}\right)\right. \\ &\quad \left. + \xi\rfloor\Gamma_\alpha{}^\beta\left(H^\alpha{}_\beta - \frac{\partial L}{\partial R_\alpha{}^\beta}\right)\right]. \end{aligned} \tag{6.8}$$

Because of the noncovariant character of ℓ_ξ, this quantity is not invariant under local Lorentz transformations, i.e. $Q_0[\xi]$ depends in general on the choice of frame on the spatial boundary ∂S. The general expression (6.8) reduces to the result found by Aros *et al.* for the specific Lagrangian V considered in [22], see also [23,49].

VII. EINSTEIN(-CARTAN) GRAVITY

Komar [5] gave a formula for the computation of the gravitational energy in Einstein's general relativity theory which proved to give correct (i.e., physically reasonable) results for asymptotically flat configurations. Later [6] it was recognized that this formula also yields the angular momentum for rotating configurations. Yet, it remained unclear (for us, at least) how this nice formula fits into a general Noether scheme. In the few relevant studies, the explanations were based on an assumption that ξ is a Killing vector either of the physical spacetime geometry or of a background geometry [7–12]. One of the motivations for the current work was to clarify the status of the Komar construction.

Let us consider the Einstein-Cartan theory that is described by the Hilbert-Einstein Lagrangian plus, in general, a cosmological term:

$$V = -\frac{1}{2\kappa}(R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\lambda\eta). \tag{7.1}$$

Here κ is the gravitational coupling constant, and λ is the cosmological constant (with a dimension of the inverse length square). Making use of (2.2), (2.3), and (3.5), we find explicitly

$$H_\alpha = 0, \qquad E_\alpha = -\frac{1}{2\kappa}(R^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma} - 2\lambda\eta_\alpha), \quad (7.2)$$

$$H_{\alpha\beta} = \frac{1}{2\kappa}\eta_{\alpha\beta}, \qquad E_{\alpha\beta} = 0. \quad (7.3)$$

We assume minimal coupling of matter, so that $\partial L/\partial T^\alpha = 0$ and $\partial L/\partial R_\alpha{}^\beta = 0$. The covariant current (5.7) then reduces to

$$\mathcal{J}[\xi] = \xi\rfloor(V+L) - \mathcal{L}_\xi\Psi^A\wedge\frac{\partial L}{\partial D\Psi^A} + \frac{1}{2\kappa}\mathcal{L}_\xi\Gamma^{\alpha\beta}\wedge\eta_{\alpha\beta}. \quad (7.4)$$

This current is conserved, $d\mathcal{J}[\xi]=0$, on the solutions of the field equations (4.8), (4.9), and (4.10), which now read

$$\tfrac{1}{2}R^{\beta\gamma}\wedge\eta_{\alpha\beta\gamma} - \lambda\eta_\alpha = \kappa\Sigma_\alpha, \quad (7.5)$$

$$\tfrac{1}{2}T^\gamma\wedge\eta_{\alpha\beta\gamma} = \kappa\tau_{\alpha\beta}, \quad (7.6)$$

$$\frac{\delta L}{\delta\Psi^A} = 0. \quad (7.7)$$

On the other hand, "on shell" from (5.9) we read off

$$\mathcal{J}[\xi] = \frac{1}{2\kappa}d\{{}^*[dk + \xi\rfloor(\vartheta^\lambda\wedge T_\lambda)]\}, \quad (7.8)$$

where we used (7.2), (7.3), and (A17) (recall that $k := \xi_\alpha\vartheta^\alpha$ is the 1-form dual to the vector ξ). For the solutions of (7.6) for $\tau^\alpha{}_\beta = 0$, i.e. for spinless matter or in vacuum, the torsion vanishes, $T^\alpha = 0$, and hence the total charge (5.10) finally reduces to

$$\mathcal{Q}[\xi] = \frac{1}{2\kappa}\int_{\partial S}{}^*dk. \quad (7.9)$$

This invariant conserved quantity $\mathcal{Q}[\xi]$ is precisely the (n-dimensional generalization of the) Komar formula.

VIII. RELOCALIZATION OF THE CURRENTS

All our constructions are invariant under coordinate and local Lorentz transformations. However, besides the local coordinate and the local Lorentz freedom, there is another ambiguity in the definition of the conserved quantities. Namely, the field equations always allow for a *relocalization* of the gravitational field momenta. As a result, the conserved currents and the values of the total charges can be changed by means of the relocalization of a translational and rotational momenta.

More specifically, we consider here the case when a relocalization is produced by the change of the gravitational field Lagrangian by a total derivative:

$$V' = V + d\Phi, \qquad \Phi = \Phi(\vartheta^\alpha, \Gamma_\alpha{}^\beta, T^\alpha, R_\alpha{}^\beta). \quad (8.1)$$

The term $d\Phi$ changes only the boundary part of the action, leaving the field equations unchanged. We will assume a boundary $(n-1)$-form Φ whose general variation can be written as

$$\delta\Phi = \delta\vartheta^\alpha\wedge\frac{\partial\Phi}{\partial\vartheta^\alpha} + \delta\Gamma_\alpha{}^\beta\wedge\frac{\partial\Phi}{\partial\Gamma_\alpha{}^\beta} + \delta T^\alpha\wedge\frac{\partial\Phi}{\partial T^\alpha} + \delta R_\alpha{}^\beta\wedge\frac{\partial\Phi}{\partial R_\alpha{}^\beta}. \quad (8.2)$$

Then, taking the exterior derivative of (8.2) and expressing the variations of $d\vartheta^\alpha$, dT^α, $d\Gamma_\alpha{}^\beta$, and $dR_\alpha{}^\beta$ in terms of the variations of ϑ^α, T^α, $\Gamma_\alpha{}^\beta$, and $R_\alpha{}^\beta$, we find

$$\frac{\partial d\Phi}{\partial\vartheta^\alpha} = -d\frac{\partial\Phi}{\partial\vartheta^\alpha} + \Gamma_\alpha{}^\beta\wedge\frac{\partial\Phi}{\partial\vartheta^\beta} + R_\alpha{}^\beta\wedge\frac{\partial\Phi}{\partial T^\beta}, \quad (8.3)$$

$$\frac{\partial d\Phi}{\partial T^\alpha} = d\frac{\partial\Phi}{\partial T^\alpha} - \Gamma_\alpha{}^\beta\wedge\frac{\partial\Phi}{\partial T^\beta} + \frac{\partial\Phi}{\partial\vartheta^\alpha}, \quad (8.4)$$

$$\frac{\partial d\Phi}{\partial R_\alpha{}^\beta} = d\frac{\partial\Phi}{\partial R_\alpha{}^\beta} + \Gamma_\lambda{}^\alpha\wedge\frac{\partial\Phi}{\partial R_\lambda{}^\beta} - \Gamma_\beta{}^\lambda\wedge\frac{\partial\Phi}{\partial R_\alpha{}^\lambda} + \frac{\partial\Phi}{\partial\Gamma_\alpha{}^\beta} + \vartheta^{[\alpha}\wedge\frac{\partial\Phi}{\partial T^{\beta]}}. \quad (8.5)$$

We assume that, just like the original Lagrangian V, the boundary term $d\Phi$ is invariant under Lorentz transformations. Then (making use of the Lagrange-Noether machinery outlined above) one can verify that

$$\frac{\partial d\Phi}{\partial\Gamma_\alpha{}^\beta} = -d\frac{\partial\Phi}{\partial\Gamma_\alpha{}^\beta} - \Gamma_\lambda{}^\alpha\wedge\frac{\partial\Phi}{\partial\Gamma_\lambda{}^\beta} + \Gamma_\beta{}^\lambda\wedge\frac{\partial\Phi}{\partial\Gamma_\alpha{}^\lambda} - \vartheta^{[\alpha}\wedge\frac{\partial\Phi}{\partial\vartheta^{\beta]}} - T^{[\alpha}\wedge\frac{\partial\Phi}{\partial T^{\beta]}} - R_\lambda{}^\alpha\wedge\frac{\partial\Phi}{\partial R_\lambda{}^\beta} + R_\beta{}^\lambda\wedge\frac{\partial\Phi}{\partial R_\alpha{}^\lambda} \equiv 0. \quad (8.6)$$

Note that unlike the n-form $d\Phi$, the $(n-1)$-form Φ itself does not necessarily need to be a scalar under local Lorentz transformations. Using (8.3), (8.4), and (8.5) in the general definitions (2.2) and (2.3), we then find the relocalized momenta and gravitational currents:

$$H'_\alpha = H_\alpha - d\frac{\partial\Phi}{\partial T^\alpha} + \Gamma_\alpha{}^\beta\wedge\frac{\partial\Phi}{\partial T^\beta} - \frac{\partial\Phi}{\partial\vartheta^\beta}, \quad (8.7)$$

$$H'^\alpha{}_\beta = H^\alpha{}_\beta - d\frac{\partial\Phi}{\partial R_\alpha{}^\beta} - \Gamma_\lambda{}^\alpha\wedge\frac{\partial\Phi}{\partial R_\lambda{}^\beta} + \Gamma_\beta{}^\lambda\wedge\frac{\partial\Phi}{\partial R_\alpha{}^\lambda} - \frac{\partial\Phi}{\partial\Gamma_\alpha{}^\beta} - \vartheta^{[\alpha}\wedge\frac{\partial\Phi}{\partial T^{\beta]}}, \quad (8.8)$$

$$E'_\alpha = E_\alpha - d\frac{\partial\Phi}{\partial\vartheta^\alpha} + \Gamma_\alpha{}^\beta\wedge\frac{\partial\Phi}{\partial\vartheta^\beta} + R_\alpha{}^\beta\wedge\frac{\partial\Phi}{\partial T^\beta}, \quad (8.9)$$

$$E'_{\alpha\beta} = E_{\alpha\beta} + \vartheta_{[\alpha} \wedge d\frac{\partial\Phi}{\partial T^{\beta]}} - \vartheta_{[\alpha} \wedge \Gamma_{\beta]}{}^{\lambda} \wedge \frac{\partial\Phi}{\partial T^{\lambda}} - \vartheta_{[\alpha} \wedge \frac{\partial\Phi}{\partial\vartheta^{\beta]}}. \tag{8.10}$$

Accordingly, the relocalized translational and rotational momenta (8.7) and (8.8) determine the relocalized conserved currents and charges when they are substituted into the corresponding formulas (5.7), (5.8), (5.9), and (5.10).

The choice of a boundary term is fairly arbitrary. For example, a "universal" relocalization (in the sense that it is available in all spacetime dimensions) is defined with the help of a background connection $\bar{\Gamma}_{\alpha}{}^{\beta}$. The latter can be introduced as the limit of the dynamical connection $\Gamma_{\alpha}{}^{\beta}$ at spatial infinity ∂S, for instance, or fixed from other arguments. Given the background connection, one can always (for any n) add to a Lagrangian V the covariant boundary term $\alpha_0 d\Phi_0$ with a constant α_0 and the $(n-1)$-form

$$\Phi_0 := \eta_{\alpha\beta} \wedge \Delta\Gamma^{\alpha\beta}, \qquad \Delta\Gamma^{\alpha\beta} := \Gamma^{\alpha\beta} - \bar{\Gamma}^{\alpha\beta}. \tag{8.11}$$

Then the field momenta are relocalized as

$$\begin{aligned} H'_{\alpha} &= H_{\alpha} - \alpha_0 \eta_{\alpha\beta\lambda} \wedge \Delta\Gamma^{\beta\lambda}, \\ H'_{\alpha\beta} &= H_{\alpha\beta} - (-1)^n \alpha_0 \eta_{\alpha\beta}. \end{aligned} \tag{8.12}$$

Alternatively, there is a possibility to make the arbitrariness in the choice of the boundary term more narrow by restricting one's attention to the topological invariants which always can be added to the action. This option, however, depends on the spacetime dimension. For instance, in four dimensions ($n = 4$) we can consider a 3-parameter family of boundary forms

$$\Phi := \alpha_1\Phi_1 + \alpha_2\Phi_2 + \alpha_3\Phi_3, \tag{8.13}$$

$$\Phi_1 := T^{\alpha} \wedge \vartheta_{\alpha}, \tag{8.14}$$

$$\Phi_2 := \Gamma_{\alpha}{}^{\beta} \wedge (R_{\beta}{}^{\alpha} + \tfrac{1}{3}\Gamma_{\beta}{}^{\lambda} \wedge \Gamma_{\lambda}{}^{\alpha}), \tag{8.15}$$

$$\Phi_3 := \eta_{\alpha\beta\mu\nu}\Gamma^{\alpha\beta} \wedge (R^{\mu\nu} + \tfrac{1}{3}\Gamma^{\mu\lambda} \wedge \Gamma_{\lambda}{}^{\nu}). \tag{8.16}$$

These 3-forms correspond to the Nieh-Yan [50,51], the Pontryagin, and the Euler topological invariants, respectively. They represent the so-called gravitational (translational and rotational) Chern-Simons 3-forms, see [52] for more details. Substituting this into (8.7), (8.8), (8.9), and (8.10) we then obtain a particular ("topological") relocalization:

$$H'_{\alpha} = H_{\alpha} - 2\alpha_1 T_{\alpha}, \tag{8.17}$$

$$H'_{\alpha\beta} = H_{\alpha\beta} - \alpha_1\vartheta_{\alpha} \wedge \vartheta_{\beta} + 2\alpha_2 R_{\alpha\beta} - 2\alpha_3\eta_{\alpha\beta\mu\nu}R^{\mu\nu}, \tag{8.18}$$

$$E'_{\alpha} = E_{\alpha} - 2\alpha_1 DT_{\alpha}, \tag{8.19}$$

$$E'_{\alpha\beta} = E_{\alpha\beta} + 2\alpha_1\vartheta_{[\alpha} \wedge T_{\beta]}. \tag{8.20}$$

We will use this specific relocalization in the subsequent discussion of the regularization of invariant conserved quantities.

XI. EXAMPLES

In order to illustrate how our general formalism works, in this section we specialize to the case of four-dimensional theories: $n = 4$, $\kappa = 8\pi G/c^3$. At first, we study the purely Riemannian (without torsion) solutions of the Einstein-Cartan theory which have asymptotic AdS behavior. After that, we consider similar configurations with torsion that arise as solutions in the quadratic Poincaré gauge gravity theory.

A. Kerr-AdS solution in Einstein-Cartan theory

When the cosmological constant λ is nontrivial, the Einstein-Cartan field equations (7.5) and (7.6) admit in vacuum the generalized Kerr solution with AdS asymptotics ($\lambda < 0$). We use a spherical local coordinate system (t, r, θ, φ), and choose the coframe as

$$\vartheta^{\hat{0}} = \sqrt{\frac{\Delta}{\Sigma}}[cdt - a\Omega\sin^2\theta d\varphi], \tag{9.1}$$

$$\vartheta^{\hat{1}} = \sqrt{\frac{\Sigma}{\Delta}}dr, \tag{9.2}$$

$$\vartheta^{\hat{2}} = \sqrt{\frac{\Sigma}{f}}d\theta, \tag{9.3}$$

$$\vartheta^{\hat{3}} = \sqrt{\frac{f}{\Sigma}}\sin\theta[-acdt + \Omega(r^2 + a^2)d\varphi]. \tag{9.4}$$

Here the functions and constants are defined by

$$\Delta := (r^2 + a^2)\left(1 - \frac{\lambda}{3}r^2\right) - 2mr, \tag{9.5}$$

$$\Sigma := r^2 + a^2\cos^2\theta, \tag{9.6}$$

$$f := 1 + \frac{\lambda}{3}a^2\cos^2\theta, \tag{9.7}$$

$$m := \frac{GM}{c^2}, \qquad \Omega := \frac{1}{1 + \frac{\lambda}{3}a^2}, \tag{9.8}$$

and $0 < t < \infty$, $0 < r < \infty$, $0 < \theta < \pi$, and $0 < \varphi < 2\pi$. For this solution the curvature has two nonvanishing irreducible pieces: $R^{\alpha\beta} = {}^{(1)}R^{\alpha\beta} + {}^{(6)}R^{\alpha\beta}$ (see Appendix B for definitions of the irreducible parts of the curvature). The curvature scalar is $R = -4\lambda$, and the Weyl 2-form (B14) has a very special structure. Namely, it is expressed only in terms of a certain scalar 2-form $w = \frac{1}{2}w_{\alpha\beta}\vartheta^\alpha \wedge \vartheta^\beta$ as follows:

$$W^{\alpha\beta} = \tilde{w}^{\alpha\beta *}w - w^{\alpha\beta}w - \tfrac{1}{3}{}^*(w \wedge {}^*w)\vartheta^\alpha \wedge \vartheta^\beta - \tfrac{1}{3}{}^*(w \wedge w)\eta^{\alpha\beta}. \tag{9.9}$$

Here $\tilde{w}^{\alpha\beta} := \frac{1}{2}\eta^{\alpha\beta\mu\nu}w_{\mu\nu}$. The 2-form w is given by a simple formula:

$$w = u\vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} + v\vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}, \tag{9.10}$$

where its two nontrivial components $u := w_{\hat{0}\hat{1}}$ and $v := w_{\hat{2}\hat{3}}$ are functions of r and θ. Their explicit form is not of interest since we will need only the two invariants

$$^*(w \wedge {}^*w) = \frac{3mr(r^2 - 3a^2\cos^2\theta)}{\Sigma^3}, \tag{9.11}$$

$$^*(w \wedge w) = \frac{3ma\cos\theta(3r^2 - a^2\cos^2)}{\Sigma^3}. \tag{9.12}$$

One can straightforwardly find u and v from these two equations, but as we said, we do not need these functions explicitly.

For a vector field $\xi = \xi^i\partial_i$ with constant holonomic components, ξ^i, in the coordinate system used in (9.1), (9.2), (9.3), and (9.4), the computation of conserved charges is fairly straightforward. Substituting (C1) into (7.9), and using (C2)–(C4), we find in the asymptotically AdS case ($\lambda < 0$):

$$\mathcal{Q}[\xi] = \xi^0\left[\frac{\Omega M c^2}{2} - \frac{4\pi\Omega c\lambda}{3\kappa} r_\infty(r_\infty^2 + a^2)\right] - \xi^3\Omega^2 Mca. \tag{9.13}$$

Here r_∞ is the radius of the spatial boundary sphere ∂S. It is worthwhile to note that $\mathcal{Q}[\partial_r] = 0$ and $\mathcal{Q}[\partial_\theta] = 0$.

When $\lambda = 0$, we recover the usual Komar result with $\mathcal{Q}[\partial_t] = Mc^2/2$ and $\mathcal{Q}[\partial_\varphi] = -Mca$. For a nontrivial negative cosmological constant, the conserved charge $\mathcal{Q}[\partial_\varphi] = -\Omega^2 Mca$ is finite, but $\mathcal{Q}[\partial_t]$ diverges as $r_\infty \to \infty$. Hence, a regularization is needed.

B. Regularization via relocalization

In a recent paper [4], we have demonstrated that total conserved quantities can be regularized by means of a relocalization of the gravitational field momenta. In Sec. VIII, we discussed a specific relocalization generated by a boundary term in the action. Here we will use the same method to remove the divergence of the conserved charge (9.13) for the Kerr-AdS configuration.

A straightforward inspection shows that one can solve the regularization problem with the help of the relocalization (8.17), (8.18), (8.19), and (8.20) generated by the Chern-Simons boundary terms (8.14), (8.15), and (8.16). More exactly, we can verify that the translational and the rotational Chern-Simons forms (8.14) and (8.15) do not affect the total charge, whereas the Euler boundary term (8.16) does the job. Hence we put $\alpha_1 = \alpha_2 = 0$, and consider the relocalization $H_{\alpha\beta} \to H'_{\alpha\beta} = H_{\alpha\beta} - 2\alpha_3\eta_{\alpha\beta\mu\nu}R^{\mu\nu}$ that is generated by the change of the Lagrangian $V \to V' = V + \alpha_3 d\Phi_3$ by the boundary term (8.16). Using (7.3) and the irreducible decomposition of the curvature (B9), we find

$$H'_{\alpha\beta} = \left(\frac{1}{2\kappa} - \frac{4\alpha_3\lambda}{3}\right)\eta_{\alpha\beta} - 2\alpha_3\eta_{\alpha\beta\mu\nu}\bar{R}^{\mu\nu}, \tag{9.14}$$

with $\bar{R}^{\alpha\beta} := R^{\alpha\beta} - \frac{\lambda}{3}\vartheta^\alpha \wedge \vartheta^\beta$. The term $\eta_{\alpha\beta}$ contributes to $\mathcal{Q}'[\xi]$ with the usual Komar expression $^*dk$ which makes the conserved charge (9.13) infinite. Hence we choose $\alpha_3 = 3/8\kappa\lambda$ and eliminate the corresponding term completely. As a result, we end with the regularized invariant conserved charge

$$\mathcal{Q}'[\xi] = \int_{\partial S}\Xi^{\alpha\beta}H'_{\alpha\beta} = -\frac{3}{4\kappa\lambda}\int_{\partial S}\eta_{\alpha\beta\mu\nu}(e^\alpha\rfloor D\xi^\beta)\bar{R}^{\mu\nu}. \tag{9.15}$$

On the Kerr-AdS solution, the relocalized momentum (9.14) is constructed in terms of the Weyl 2-form

$$H'_{\alpha\beta} = -\frac{3}{4\kappa\lambda}\eta_{\alpha\beta\mu\nu}W^{\mu\nu}. \tag{9.16}$$

Making use of (A17) and (C1), and of the explicit form of the Weyl 2-form for the Kerr-AdS spacetime, c.f. Eq. (9.9), we derive

$$\eta_{\alpha\beta\mu\nu}\Xi^{\alpha\beta}W^{\mu\nu} = \tfrac{2}{3}[^*(w \wedge {}^*w)^*(2\omega - \chi) - {}^*(w \wedge w)(2\omega - \chi)]. \tag{9.17}$$

Substituting now (9.11), (9.12), and (C1)–(C4), we finally calculate the regularized charge:

$$\mathcal{Q}'[\xi] = \xi^0\Omega Mc^2 - \xi^3\Omega^2 Mca. \tag{9.18}$$

In other words, we found finite values of the covariant total charge $\mathcal{Q}$ for the Kerr-AdS solution, which read

$$\mathcal{Q}'[\partial_t] = \Omega Mc^2, \qquad \mathcal{Q}'[\partial_\varphi] = -\Omega^2 Mca, \qquad \mathcal{Q}'[\partial_r] = 0, \qquad \mathcal{Q}'[\partial_\theta] = 0. \tag{9.19}$$

Our results agree with those in [53,54]. It is worthwhile to note that the relocalized gravitational Lagrangian for $\alpha_3 = 3/8\kappa\lambda$ can be written as

$$V' = \frac{3}{8\kappa\lambda}\eta_{\alpha\beta\mu\nu}\bar{R}^{\alpha\beta} \wedge \bar{R}^{\mu\nu}. \tag{9.20}$$

This Lagrangian was studied extensively in the framework of various approaches to gravity on the basis of the de Sitter group, see [55–58], for example. The same action was also used in [22,23] for the derivation of the conserved current associated with a vector field.

Our results (9.19) agree with those in [22,23] for this particular case (note that the timelike Killing vector used in [22] corresponds to $\Omega\partial_\varphi$ in our notation). However, such an agreement appears to be a mere (although remarkable) coincidence since in [22] the authors consider the noncovariant charge (6.8), which depends in general on the choice of a frame at spatial infinity.

In order to illustrate this difference, let us take the Kerr-AdS coframe (9.1), (9.2), (9.3), and (9.4) and evaluate the noninvariant charge $Q_0[\partial_t]$ for a tetrad ϑ'^α that is obtained from the original one by means of the local Lorentz transformation

$$\vartheta'^{\hat 0} = \vartheta^{\hat 0}\cosh\zeta(x^i) + \vartheta^{\hat 1}\sinh\zeta(x^i), \tag{9.21}$$

$$\vartheta'^{\hat 1} = \vartheta^{\hat 0}\sinh\zeta(x^i) + \vartheta^{\hat 1}\cosh\zeta(x^i), \tag{9.22}$$

$$\vartheta'^{\hat 2} = \vartheta^{\hat 2}, \qquad \vartheta'^{\hat 3} = \vartheta^{\hat 3}. \tag{9.23}$$

Choosing the function as $\zeta(x^i) = \alpha_0 rt\sin\theta$ (α_0 is a constant) we find that Eq. (6.8) yields $Q'_0[\partial_t] = \Omega Mc^2 - \alpha_0 6\pi^2\Omega m/\kappa\lambda$. The value of the charge can thus be arbitrary, depending on the constant α_0. This charge can even be made divergent. For $\zeta = \alpha_0 r^2 t\sin\theta$, for example, we find $Q'_0[\partial_t] = \Omega Mc^2 - r_\infty\alpha_0 6\pi^2\Omega m/\kappa\lambda$ which diverges when the spatial boundary is $r_\infty \to \infty$. It is certainly true that the parameters of the Lorentz transformation above look quite exotic, so to say. However, this demonstrates, as a matter of principle, the fact that Q_0 depends on the choice of a frame.

In contrast, our invariant formula (5.10) yields the same finite value for all coordinates and all frames, which is much more advantageous.

C. Kerr-AdS solution with torsion

In vacuum, the Einstein-Cartan theory coincides with Einstein's general relativity theory, and all the solutions are characterized by a vanishing torsion. In order to test our approach for the configurations with torsion, we will consider a different model, namely, the quadratic Poincaré gauge theory with the Lagrangian [59–61]

$$V = -\frac{1}{2\kappa}\left[T^\alpha \wedge \vartheta^\beta \wedge {}^*(T_\beta \wedge \vartheta_\alpha) + \frac{3}{4\lambda}R^{\alpha\beta} \wedge {}^*R_{\alpha\beta}\right]. \tag{9.24}$$

This model was extensively studied [53,59–62] and it was demonstrated that it is a natural generalization of the Einstein-Cartan theory. In particular, it was shown that this model has a correct Einsteinian limit. Note that we use a slightly different notation for the coupling constants in (9.24), as compared to [59–61]. Here $\kappa = 8\pi G/c^3$ and λ has dimensions of (length)$^{-2}$.

The translational and the rotational gauge field momenta (2.2) now read:

$$H_\alpha = \frac{1}{\kappa}\vartheta^\beta \wedge {}^*(T_\beta \wedge \vartheta_\alpha), \tag{9.25}$$

$$H_{\alpha\beta} = \frac{3}{4\kappa\lambda}{}^*R_{\alpha\beta}. \tag{9.26}$$

Substituting these expressions into the vacuum field equations $\mathcal{E}_\alpha = 0$ and $\mathcal{C}^\alpha{}_\beta = 0$, one can verify that there is a generalized Kerr-AdS solution that is described as follows. The coframe is again given by the above formulas (9.1), (9.2), (9.3), and (9.4), whereas the components of the torsion 2-form read:

$$T^{\hat 0} = \sqrt{\frac{\Sigma}{\Delta}}\Big[-v_1\vartheta^{\hat 0}\wedge\vartheta^{\hat 1} - 2v_4\vartheta^{\hat 2}\wedge\vartheta^{\hat 3} + \sqrt{\frac{\Sigma}{\Delta}}\mathcal{T}\wedge(v_2\vartheta^{\hat 2} + v_3\vartheta^{\hat 3})\Big], \tag{9.27}$$

$$T^{\hat 1} = T^{\hat 0}, \tag{9.28}$$

$$T^{\hat 2} = \sqrt{\frac{\Sigma}{\Delta}}\mathcal{T}\wedge(v_5\vartheta^{\hat 2} + v_4\vartheta^{\hat 3}), \tag{9.29}$$

$$T^{\hat 3} = \sqrt{\frac{\Sigma}{\Delta}}\mathcal{T}\wedge(-v_4\vartheta^{\hat 2} + v_5\vartheta^{\hat 3}). \tag{9.30}$$

Here we have denoted the functions

$$v_1 = \frac{m(r^2 - a^2\cos^2\theta)}{\Sigma^2}, \qquad v_4 = -\frac{mra\cos\theta}{\Sigma^2}, \qquad v_5 = \frac{mr^2}{\Sigma^2}, \tag{9.31}$$

$$v_2 = -\sqrt{\frac{f}{\Sigma}}\frac{mra^2\sin\theta\cos\theta}{\Sigma^2}, \qquad v_3 = -\sqrt{\frac{f}{\Sigma}}\frac{mr^2 a\sin\theta}{\Sigma^2}, \tag{9.32}$$

and introduced the 1-form $\mathcal{T} := \vartheta^{\hat 0} - \vartheta^{\hat 1}$. One can verify that the axial torsion piece vanishes for this configuration, while the torsion trace is proportional to the above 1-form, that is:

$$\vartheta_\alpha \wedge T^\alpha = 0, \qquad T = e_\alpha \rfloor T^\alpha = -\frac{m}{\sqrt{\Sigma\Delta}}\mathcal{T}. \tag{9.33}$$

The Riemann-Cartan curvature 2-form of this solution consists of only two irreducible parts (see the definitions in Appendix B),

$$R^{\alpha\beta} = {}^{(4)}R^{\alpha\beta} + {}^{(6)}R^{\alpha\beta}, \tag{9.34}$$

which read explicitly as follows

$${}^{(4)}R^{\alpha\beta} = \frac{\lambda mr}{3\Delta}{}^{(4)}\mathcal{R}^{\alpha\beta}, \qquad {}^{(6)}R^{\alpha\beta} = \frac{\lambda}{3}\vartheta^\alpha\wedge\vartheta^\beta. \tag{9.35}$$

Here the nonvanishing components of the fourth irreducible part are given by ${}^{(4)}\mathcal{R}^{\hat{0}\hat{2}} = -{}^{(4)}\mathcal{R}^{\hat{2}\hat{0}} = {}^{(4)}\mathcal{R}^{\hat{1}\hat{2}} = -{}^{(4)}\mathcal{R}^{\hat{2}\hat{1}} = \mathcal{T} \wedge \vartheta^{\hat{2}}$, and ${}^{(4)}\mathcal{R}^{\hat{0}\hat{3}} = -{}^{(4)}\mathcal{R}^{\hat{3}\hat{0}} = {}^{(4)}\mathcal{R}^{\hat{1}\hat{3}} = -{}^{(4)}\mathcal{R}^{\hat{3}\hat{1}} = \mathcal{T} \wedge \vartheta^{\hat{3}}$.

Substituting all this into (9.25) and (9.26), we explicitly find the translational momentum

$$H_{\hat{0}} = \frac{1}{\kappa}\sqrt{\frac{\Sigma}{\Delta}}\Big[-2v_4\vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} - 2v_5\vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} + \sqrt{\frac{\Sigma}{\Delta}}\mathcal{T} \wedge (v_3\vartheta^{\hat{2}} - v_2\vartheta^{\hat{3}})\Big], \tag{9.36}$$

$$H_{\hat{1}} = -H_{\hat{0}}, \tag{9.37}$$

$$H_{\hat{2}} = \frac{1}{\kappa}\sqrt{\frac{\Sigma}{\Delta}}\mathcal{T} \wedge [-v_4\vartheta^{\hat{2}} + (v_1 - v_5)\vartheta^{\hat{3}}], \tag{9.38}$$

$$H_{\hat{3}} = \frac{1}{\kappa}\sqrt{\frac{\Sigma}{\Delta}}\mathcal{T} \wedge [(v_5 - v_1)\vartheta^{\hat{2}} - v_4\vartheta^{\hat{3}}], \tag{9.39}$$

and the rotational field momentum

$$H^{\alpha\beta} = \frac{1}{4\kappa}\eta^{\alpha\beta} + \frac{mr}{4\kappa\Delta}{}^{*(4)}\mathcal{R}^{\alpha\beta}. \tag{9.40}$$

We consider again vector fields with constant components ξ^i in the coordinate system used above. Then, from (9.1)–(9.4) and (9.36)–(9.39), for the translational contribution described by the first term in (5.10) we obtain:

$$\int_{\partial S}(\xi\rfloor\vartheta^\alpha)H_\alpha = \xi^0(\Omega Mc^2) + \xi^3\Big(-\frac{2}{3}\Omega Mac\Big). \tag{9.41}$$

On the other hand, taking into account that the axial torsion vanishes (9.33), the Eq. (A17) again yields $\Xi_{\alpha\beta} = \frac{1}{2}e_\beta\rfloor e_\alpha\rfloor(dk)$. Using then (9.40), after some algebra we find the second (rotational contribution) term in the charge (5.10):

$$\int_{\partial S}\Xi^{\alpha\beta}H_{\alpha\beta} = \frac{1}{4\kappa}\int_{\partial S}\Xi^{\alpha\beta}\eta_{\alpha\beta} + \frac{m}{4\kappa} \times \int_{\partial S}\frac{r}{\Delta}\Xi^{\alpha\beta *(4)}\mathcal{R}_{\alpha\beta} \tag{9.42}$$

$$= \Big[\xi^0\Big(\frac{1}{4}\Omega Mc^2 - \frac{2\pi\Omega\lambda c}{3\kappa}r_\infty(r_\infty^2 + a^2)\Big) + \xi^3\Big(-\frac{1}{2}\Omega^2 Mac\Big)\Big] + \Big[\xi^3\Big(-\frac{1}{6}\Omega^2 Mac\Big)\Big]. \tag{9.43}$$

In (9.43) we have explicitly specified by the square brackets the contributions of each integral in (9.42). Combining (9.41) with (9.43), we obtain the invariant conserved charges

$$\mathcal{Q}[\partial_t] = \frac{5}{4}\Omega Mc^2 - \frac{2\pi\Omega\lambda c}{3\kappa}r_\infty(r_\infty^2 + a^2), \qquad \mathcal{Q}[\partial_\varphi] = -\frac{4}{3}\Omega^2 Mac, \tag{9.44}$$

whereas again $\mathcal{Q}[\partial_r] = \mathcal{Q}[\partial_\theta] = 0$. As in the Riemannian case discussed in Sec. IX A, we thus find a finite value for the angular momentum $\mathcal{Q}[\partial_\varphi]$, but a divergent energy $\mathcal{Q}[\partial_t]$.

The source of the divergence is easily detected: From (9.42) and (9.43) we can see that it is again the usual Komar term $\Xi^{\alpha\beta}\eta_{\alpha\beta} = {}^*(dk)$ that is responsible for all the infinite contributions to $\mathcal{Q}[\partial_t]$. Therefore, we can try to regularize the conserved charges (9.44) by relocalizing the field momenta in the same way as it was done for the Riemannian case in (9.14). Namely, we add an Euler boundary term, see (8.16), to the gravitational action (9.24). In this case the translational momentum does not change, $H'_\alpha = H_\alpha$, see (8.17), but the rotational momentum does change, according to (8.18). Making use of (9.34), (9.35), and (9.40), we then find the relocalized Lorentz momentum:

$$H'_{\alpha\beta} = \Big(\frac{1}{4\kappa} - \frac{4\alpha_3\lambda}{3}\Big)\eta_{\alpha\beta} + \frac{mr}{\Delta}\Big(\frac{1}{4\kappa}{}^{*(4)}\mathcal{R}_{\alpha\beta} - \frac{2\alpha_3\lambda}{3}\eta_{\alpha\beta\mu\nu}{}^{(4)}\mathcal{R}^{\mu\nu}\Big). \tag{9.45}$$

Clearly, the divergent contributions are canceled provided we fix the coefficient of the boundary term as $\alpha_3 = \frac{3}{16\kappa\lambda}$. In addition, we recall the double duality property of the fourth irreducible piece of the curvature, namely, ${}^{(4)}R_{\alpha\beta} \equiv -\frac{1}{2}\eta_{\alpha\beta\mu\nu}{}^{(4)}R^{\mu\nu}$, see Eq. (164) of [33], for example. The resulting relocalized quantity then reads

$$H'_{\alpha\beta} = \frac{mr}{2\kappa\Delta}{}^{*(4)}\mathcal{R}_{\alpha\beta}. \tag{9.46}$$

Using the explicit expression (9.35) for ${}^{(4)}\mathcal{R}_{\alpha\beta}$, together with (A17) and (9.33), we then find

$$\int_{\partial S}\Xi^{\alpha\beta}H'_{\alpha\beta} = \xi^3\Big(-\frac{1}{3}\Omega^2 Mac\Big). \tag{9.47}$$

Combining (9.41) and (9.47), we finally obtain the regularized covariant conserved charge:

$$\mathcal{Q}'[\xi] = \xi^0(\Omega Mc^2) + \xi^3(-\Omega^2 Mac). \tag{9.48}$$

It is satisfactory to see that the new current yields the "standard" values for the energy and the angular momentum, $\mathcal{Q}'[\partial_t] = \Omega Mc^2$, $\mathcal{Q}'[\partial_\varphi] = -\Omega^2 Mac$, as well as the trivial values $\mathcal{Q}'[\partial_r] = 0$ and $\mathcal{Q}'[\partial_\theta] = 0$.

It seems worthwhile to mention that although in both examples (without and with torsion) the resulting values of the regularized charge are the same, there is an important difference in the way how these values are actually composed. As compared to the Riemannian case discussed in

Sec. IX A, where the *rotational* term $\Xi^{\alpha\beta}H_{\alpha\beta}$ alone gave rise both to the total energy and to the total angular momentum, in the solution with torsion the whole energy comes only from the *translational* contribution $\xi^{\alpha}H_{\alpha}$, whereas the total angular momentum arises from the combined contributions of the translational and rotational terms.

X. DISCUSSION AND CONCLUSION

In this work, we have analyzed the problem of defining conserved currents in diffeomorphism- and local Lorentz-invariant theories. Einstein's general relativity theory, Einstein-Cartan theory, and general Poincaré gauge theories of gravity belong to this class of models. We have presented a systematic derivation of a general expression for conserved currents that are invariant under both coordinate and local Lorentz transformations. Such currents $\mathcal{J}[\xi]$ are associated with a given, but completely arbitrary, vector field ξ on the spacetime manifold. The conservation law, $d\mathcal{J}[\xi] = 0$, holds "on-shell," i.e. on every solution of the coupled system of the gravitational and matter field equations. Our results are valid in any spacetime dimension for the field-theoretic models with arbitrary Lagrangians. Since the latter are always defined up to a total derivative, we discussed the relocalization of the gravitational momenta, induced by boundary terms in the action, in order to find how such a relocalization affects the conserved currents and charges.

As we stressed, the total invariant current $\mathcal{J}[\xi]$ is conserved for any vector field ξ when the field equations are satisfied. In contrast, the separate gravitational $\mathcal{J}^{\rm grav}[\xi]$ and matter $\mathcal{J}^{\rm mat}[\xi]$ currents are not conserved in general, not even on-shell. However, if the vector field ξ generates a symmetry of the field configuration, i.e. if the generalized Killing equations are satisfied for the coframe and connection, $\mathcal{L}_\xi\vartheta^\alpha = 0$ and $\mathcal{L}_\xi\Gamma_\alpha{}^\beta = 0$, then (5.1) and (5.4) yield two independent conservation laws, $d\mathcal{J}^{\rm grav}[\xi] = 0$ and $d\mathcal{J}^{\rm mat}[\xi] = 0$. Moreover, in the case of spinless matter ($\tau^\alpha{}_\beta = 0$) or in vacuum, the separate conservation laws arise for the case when the vector field ξ is a usual isometry ($\mathcal{L}_\xi g_{ij} = 0$). In other words, separate invariant conservation laws exist for the matter and/or gravitational currents under stronger (Killing symmetry) conditions on the field configurations, as compared to the conservation of the total (gravitational plus matter) current that only requires the field equations to be satisfied.

As an immediate consequence of the general results, we have demonstrated that the Komar construction arises as a particular invariant current for the Hilbert-Einstein Lagrangian of the gravitational field. In this sense, the whole formalism can be viewed as a generalization of the Komar currents to arbitrary gravitational models with more complicated Lagrangians. As we verify, the usual Komar charges diverge for spacetimes which are not asymptotically flat (in particular, for the asymptotically anti-de Sitter spacetimes). We have shown that the general scheme of "regularization via relocalization" can be used in this case to obtain finite total conserved charges. In addition, we considered asymptotically anti-de Sitter solutions of the quadratic Poincaré gauge theory in order to test how our general formalism works for models with nontrivial torsion degrees of freedom. We found divergent total charges, which can again be regularized with the help of a relocalization induced by a suitable boundary term added to the action. Rather curiously, we found the same total final finite charges for the Riemannian and for the quadratic Poincaré model. This result appears to be quite satisfactory and rather nontrivial, since in the quadratic Poincaré model the total regularized charges arise from different sectors of the dynamical field degrees of freedom (translational versus rotational). Moreover, we have verified that $\mathcal{Q}[\partial_r] = \mathcal{Q}[\partial_\theta] = 0$ in both models.

It seems worthwhile to mention some specific mathematical results obtained in this study. Namely, we have discovered that the local Lorentz invariance of the theory allows for a certain freedom in the description of the consequences of the diffeomorphism invariance. This freedom is equivalent to a definition of a "generalized Lie derivative" that acts on the geometrical and matter fields. In other words, there is no unique or natural definition of the Lie derivative for the Lorentz-covariant fields (coframe, torsion, and Lorentz-covariant matter fields). For example, it is always possible to define the Lie derivative of Lorentz-covariant fields as the usual Lie derivative ℓ_ξ. However, the result will not be a Lorentz-covariant field and the conserved currents derived from this choice will not be invariant under local Lorentz transformations. We have found a variety of consistent covariant Lie derivatives, all leading to the definition of invariant conserved currents. Technically, the latter is a consequence of the commuting property of the exterior and generalized Lie derivative.

There is a number of interesting directions in which we can further develop the current formalism. In particular, the general approach can be naturally extended to include gravity interacting with gauge fields, thus allowing to obtain conserved quantities invariant also under gauge transformations in addition to the coordinate and local Lorentz invariance. Furthermore, it is possible to generalize the current framework to metric-affine gravity, so that to include gravitational models with local invariance under the general linear group. These developments will be discussed elsewhere.

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APPENDIX A: GENERALIZED LIE DERIVATIVES

In this paper, generalized Lie derivatives are heavily used. Here we collect the corresponding mathematical definitions and describe properties of the generalized Lie derivatives.

We call an operator L_ξ in the algebra of tensor fields over spacetime a generalized Lie derivative if it is a *derivation* of the tensor algebra (as defined in [63], e.g.) which satisfies the following properties (for any Lorentz-valued p-form ω^A, q-form φ^A, and constant λ):

(a) $L_{\lambda\xi}\omega^A = \lambda L_\xi \omega^A$ (linearity),

(b) $L_\xi(\omega^A \wedge \varphi^B) = (L_\xi\omega^A)\wedge\varphi^B + \omega^A\wedge L_\xi\varphi^B$ (Leibniz rule),

(c) $L_\xi\phi = \ell_\xi\phi = \xi\rfloor d\phi + d(\xi\rfloor\phi)$ (usual Lie derivative for any *scalar-valued* p-form ϕ).

In accordance with the general mathematical theory (see Proposition 3.3 of [63]), every derivation is constructed from the ordinary Lie derivative ℓ_ξ and a (1, 1) tensorial field. In this work, we explicitly define the generalized Lie derivative by the formula

$$L_\xi\omega^A := \ell_\xi\omega^A + B_\alpha{}^\beta(\rho^\alpha{}_\beta)^A{}_B\wedge\omega^B \tag{A1}$$

when it acts on any (Lorentz-)covariant p-form ω^A. The (1, 1) field $B_\alpha{}^\beta$ depends linearly on ξ in order to provide the above property (a). If, under the local Lorentz transformation $\vartheta^\alpha \to \vartheta'^\alpha = \Lambda^\alpha{}_\beta\vartheta^\beta$, this field transforms as

$$B'_\alpha = (\Lambda^{-1})^\rho{}_\alpha B_\rho{}^\gamma\Lambda^\beta{}_\gamma - (\Lambda^{-1})^\gamma{}_\alpha(\xi\rfloor d\Lambda^\beta{}_\gamma), \tag{A2}$$

then the generalized Lie derivative (A1) of a covariant object is again a covariant object with the same transformation properties. In addition, we would like to know how L_ξ acts on noncovariant objects, such as $d\omega^A$, for example. In order to find this, it suffices to define the Lie derivative of the connection $L_\xi\Gamma_\alpha{}^\beta$. Then we can compute $L_\xi(d\omega^A)$ by writing $d\omega^A = D\omega^A - \Gamma_\alpha{}^\beta(\rho^\alpha{}_\beta)^A{}_B\wedge\omega^B$, and by using the Leibniz rule and the definition (A1).

We define the generalized Lie derivative of the connection by

$$L_\xi\Gamma_\alpha{}^\beta := \ell_\xi\Gamma_\alpha{}^\beta - (dB_\alpha{}^\beta + \Gamma_\lambda{}^\beta B_\alpha{}^\lambda - \Gamma_\alpha{}^\lambda B_\lambda{}^\beta). \tag{A3}$$

If we recall the expression of the curvature 2-form in terms of the connection 1-form, we can recast the above definition into an equivalent form:

$$L_\xi\Gamma_\alpha{}^\beta = D(\xi\rfloor\Gamma_\alpha{}^\beta - B_\alpha{}^\beta) + \xi\rfloor R_\alpha{}^\beta. \tag{A4}$$

Without going into details, let us explain this important point. The ordinary Lie derivative commutes with the exterior differential, $[\ell_\xi, d] = 0$. Using this fact together with the Leibniz rule, we can easily compute the Lie derivative of the curvature 2-form: $\ell_\xi R_\alpha{}^\beta = \ell_\xi(d\Gamma_\alpha{}^\beta + \Gamma_\lambda{}^\beta\wedge\Gamma_\alpha{}^\lambda) = d(\ell_\xi\Gamma_\alpha{}^\beta) + (\ell_\xi\Gamma_\lambda{}^\beta)\wedge\Gamma_\alpha{}^\lambda + \Gamma_\lambda{}^\beta\wedge(\ell_\xi\Gamma_\alpha{}^\lambda) = D(\ell_\xi\Gamma_\alpha{}^\beta)$. Here we formally write the right-hand side as a covariant derivative, however in reality the resulting expression is *not* covariant. The generalized Lie derivative (A3) removes this deficiency. Indeed, the operation (A1) is well defined on the covariant curvature 2-form, and the result is the covariant 2-form $L_\xi R_\alpha{}^\beta$. Now, by applying the covariant differential D to (A3), or, equivalently, to (A4), we straightforwardly verify that $D(L_\xi\Gamma_\alpha{}^\beta) = L_\xi R_\alpha{}^\beta$. In other words, in the replacement of the noncovariant ℓ_ξ with a covariant L_ξ, the definition of the generalized Lie derivative of the connection (A3) provides the correct replacement of the formal noncovariant relation $\ell_\xi R_\alpha{}^\beta = D(\ell_\xi\Gamma_\alpha{}^\beta)$ with an appropriate covariant formula $L_\xi R_\alpha{}^\beta = D(L_\xi\Gamma_\alpha{}^\beta)$.

With these definitions, one can prove that the generalized Lie derivative commutes with the exterior derivative, i.e. $[L_\xi, d] = 0$. For an arbitrary Lorentz-valued form ω^A, we have

$$\begin{aligned} L_\xi(d\omega^A) &= L_\xi(D\omega^A - \Gamma_\alpha{}^\beta(\rho^\alpha{}_\beta)^A{}_B\wedge\omega^B) \\ &= d\ell_\xi\omega^A + (\ell_\xi\Gamma_\alpha{}^\beta - L_\xi\Gamma_\alpha{}^\beta)(\rho^\alpha{}_\beta)^A{}_B\wedge\omega^B \\ &\quad + B_\alpha{}^\beta(\rho^\alpha{}_\beta)^A{}_B d\omega^B \\ &\quad + B_\alpha{}^\beta\Gamma_\gamma{}^\delta[(\rho^\alpha{}_\beta)^A{}_C(\rho^\gamma{}_\delta)^C{}_B \\ &\quad - (\rho^\gamma{}_\delta)^A{}_C(\rho^\alpha{}_\beta)^C{}_B]\wedge\omega^C. \end{aligned} \tag{A5}$$

Using now (A3) and the commutation relation of the Lorentz generators,

$$\begin{aligned} [\rho_{\alpha\beta}, \rho_{\lambda\rho}]^A{}_B = \tfrac{1}{2}[&g_{\alpha\rho}\rho_{\lambda\beta} - g_{\lambda\beta}\rho_{\alpha\rho} + g_{\alpha\lambda}\rho_{\beta\rho} \\ &- g_{\beta\rho}\rho_{\lambda\alpha}]^A{}_B, \end{aligned} \tag{A6}$$

we ultimately find

$$\begin{aligned} L_\xi(d\omega^A) &= d\ell_\xi\omega^A + dB_\alpha{}^\beta(\rho^\alpha{}_\beta)^A{}_B\wedge\omega^B \\ &\quad + B_\alpha{}^\beta(\rho^\alpha{}_\beta)^A{}_B d\omega^B \\ &= d\ell_\xi\omega^A + d(B_\alpha{}^\beta(\rho^\alpha{}_\beta)^A{}_B\omega^B) = d(L_\xi\omega^A). \end{aligned} \tag{A7}$$

Similarly, we can verify that $L_\xi(d\Gamma_\alpha{}^\beta) = d(L_\xi\Gamma_\alpha{}^\beta)$. The vanishing of the commutator $[L_\xi, d]$ on all other geometric quantities follows then directly from these formulas and the Leibniz rule.

In conclusion, let us give the following useful identities for the basic gravitational field variables:

$$L_\xi\vartheta^\alpha = D\xi^\alpha + \xi\rfloor T^\alpha + (B_\beta{}^\alpha - \xi\rfloor\Gamma_\beta{}^\alpha)\wedge\vartheta^\beta, \tag{A8}$$

$$L_\xi T^\alpha = D(L_\xi\vartheta^\alpha) + (L_\xi\Gamma_\beta{}^\alpha)\wedge\vartheta^\beta, \tag{A9}$$

$$L_\xi R_\alpha{}^\beta = D(L_\xi\Gamma_\alpha{}^\beta), \tag{A10}$$

$$[L_\xi, D]\omega^A = (L_\xi\Gamma_\alpha{}^\beta)(\rho^\alpha{}_\beta)^A{}_B\wedge\omega^B. \tag{A.11}$$

1. Yano's derivative

The covariant Lie derivative in the sense of Yano is defined by (2.16) and (2.17). Accordingly, this turns out to be a particular realization of the generalized Lie derivative that corresponds to the choice $B^{\alpha\beta} = -\Theta^{\alpha\beta} = -e^{[\alpha}\rfloor\ell_\xi\vartheta^{\beta]}$.

This choice is special because in a certain sense it is "minimal." Let us explain this property. The usual Lie derivative ℓ_ξ (in which case $B_\beta{}^\alpha = 0$) of covariant geometrical and matter fields is not covariant under local Lorentz transformations. Specifically, let us consider the Lie derivative of the coframe. This is a 1-form and we can decompose it as follows: $\ell_\xi\vartheta^\alpha = (S_\beta{}^\alpha + A_\beta{}^\alpha)\vartheta^\beta$, where $S_\beta{}^\alpha$ is symmetric, $S_{[\alpha\beta]} = 0$, and $A_\beta{}^\alpha$ is antisymmetric, $A_{(\alpha\beta)} = 0$. Explicitly we find $S_{\alpha\beta} = e_{(\alpha}\rfloor\ell_\xi\vartheta_{\beta)} \equiv h^i_\alpha h^j_\beta \ell_\xi g_{ij}/2$ and $A_{\alpha\beta} = e_{[\alpha}\rfloor\ell_\xi\vartheta_{\beta]}$. We can verify that $S_\beta{}^\alpha$ is a tensor under local Lorentz transformations, but $A_\beta{}^\alpha$ is not. Therefore, the second piece is the source of the noncovariance of the usual Lie derivative $\ell_\xi\vartheta^\alpha$ of the coframe. There exists a unique $B_\beta{}^\alpha$ which introduces a covariant Lie derivative $L_\xi\vartheta^\alpha$ by just removing the second noncovariant term above, namely $B_{\alpha\beta} = -A_{\alpha\beta} = -e_{[\alpha}\rfloor\ell_\xi\vartheta_{\beta]} =: -\Theta_{\alpha\beta}$. This definition is minimal in the sense that it does not require the choice of any arbitrary constant. For instance, if $t_\beta{}^\alpha$ is a given *tensor field*, then the nonminimal choice $B'_\beta = -\Theta_\beta{}^\alpha + \alpha t_\beta{}^\alpha$ also leads to a covariant Lie derivative, but introduces an unknown scalar α. In [64–66] the choice $B_{\alpha\beta} = -\Theta_{\alpha\beta}$ is referred to as "Kosmann lift" and it was derived with the help of fairly *ad hoc* assumptions.

For this minimal choice we have, for vector-valued forms, connection, and for matter fields, respectively:

$$\mathcal{L}_\xi\omega^\alpha = \ell_\xi\omega^\alpha - \Theta_\beta{}^\alpha\omega^\beta, \tag{A12}$$

$$\mathcal{L}_\xi\Gamma_\alpha{}^\beta = D\Xi_\alpha{}^\beta + \xi\rfloor R_\alpha{}^\beta, \tag{A13}$$

$$\mathcal{L}_\xi\Psi^A = \ell_\xi\Psi^A - \Theta_\beta{}^\alpha(\rho^\beta{}_\alpha)^A{}_B\Psi^B. \tag{A14}$$

Here, in accordance with the general relation (A4), we have

$$\Xi_\alpha{}^\beta := \xi\rfloor\Gamma_\alpha{}^\beta + \Theta_\alpha{}^\beta. \tag{A15}$$

For a given vector field ξ, we can define a corresponding 1-form by $k := (\xi\rfloor\vartheta_\alpha)\vartheta^\alpha = \xi_\alpha\vartheta^\alpha$. Then a straightforward computation shows that

$$\Xi_{\alpha\beta} \equiv e_{[\alpha}\rfloor D\xi_{\beta]} - \xi\rfloor e_{[\alpha}\rfloor T_{\beta]} \tag{A16}$$

$$\equiv \tfrac{1}{2}e_\beta\rfloor e_\alpha\rfloor[dk + \xi\rfloor(\vartheta^\lambda\wedge T_\lambda)]. \tag{A17}$$

It is worthwhile to calculate explicitly the Yano derivative of the torsion, curvature, and of the coframe:

$$\mathcal{L}_\xi T^\alpha = D(\mathcal{L}_\xi\vartheta^\alpha) + (\mathcal{L}_\xi\Gamma_\beta{}^\alpha)\wedge\vartheta^\beta, \tag{A18}$$

$$\mathcal{L}_\xi R_\alpha{}^\beta = D(\mathcal{L}_\xi\Gamma_\alpha{}^\beta), \tag{A19}$$

$$\mathcal{L}_\xi\vartheta^\alpha = D\xi^\alpha + \xi\rfloor T^\alpha - \Xi_\beta{}^\alpha\vartheta^\beta \tag{A20}$$

$$= \vartheta_\beta\overset{\{\}}{D}{}^{(\alpha}\xi^{\beta)}. \tag{A21}$$

These formulas are quite useful for many practical computations, but with a word of caution: The two last equations are actually somewhat misleading since one might have a wrong impression that a (either non-Riemannian or Riemannian) connection is involved. However, the Yano derivative $\mathcal{L}_\xi$ is (like the ordinary Lie derivative ℓ_ξ) defined independently of any connection. This fact becomes more clear if we recall that the holonomic form of the second factor in (A21) reads

$$\overset{\{\}}{D}_{(\alpha}\xi_{\beta)} = \tfrac{1}{2}h^i_\alpha h^j_\beta(\xi^k\partial_k g_{ij} + g_{ik}\partial_j\xi^k + g_{kj}\partial_i\xi^k). \tag{A22}$$

As a bonus, from this observation we learn that the Yano derivative of the coframe vanishes if and only if ξ is a Killing vector field.

It is worthwhile to note that in the absence of torsion the Yano derivative of the usual spinor field reduces to the Lie derivative of Kosmann [67].

2. Covariant Lie derivative $Ł_\xi$

The covariant Lie derivative (2.14) corresponds to the choice $B_\alpha{}^\beta = \xi\rfloor\Gamma_\alpha{}^\beta$. Accordingly, the basic relations for this case read:

$$Ł_\xi\vartheta^\alpha = D\xi^\alpha + \xi\rfloor T^\alpha, \tag{A23}$$

$$Ł_\xi\omega^\alpha = \xi\rfloor D\omega^\alpha + D(\xi\rfloor\omega^\alpha), \tag{A24}$$

$$Ł_\xi\Psi^A = \xi\rfloor D\Psi^A + D(\xi\rfloor\Psi^A), \tag{A25}$$

$$Ł_\xi\Gamma_\alpha{}^\beta = \xi\rfloor R_\alpha{}^\beta. \tag{A26}$$

APPENDIX B: IRREDUCIBLE DECOMPOSITIONS

In a Riemann-Cartan spacetime, the torsion and the curvature can be decomposed into three and six irreducible parts, respectively.

Namely, the torsion 2-form is decomposed as $T^\alpha = {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha$, with

$${}^{(2)}T^\alpha = \tfrac{1}{3}\vartheta^\alpha\wedge(e_\nu\rfloor T^\nu), \tag{B1}$$

$${}^{(3)}T^\alpha = -\tfrac{1}{3}{}^*(\vartheta^\alpha\wedge{}^*(T^\nu\wedge\vartheta_\nu)) = \tfrac{1}{3}e^\alpha\rfloor(T^\nu\wedge\vartheta_\nu), \tag{B2}$$

$${}^{(1)}T^\alpha = T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha. \tag{B3}$$

The curvature 2-form is decomposed as $R^{\alpha\beta} = \sum_{I=1}^{6}{}^{(I)}R^{\alpha\beta}$, with

$$^{(2)}R^{\alpha\beta} = -{}^*(\vartheta^{[\alpha} \wedge \Psi^{\beta]}), \tag{B4}$$

$$^{(3)}R^{\alpha\beta} = -\tfrac{1}{12}{}^*(X\vartheta^\alpha \wedge \vartheta^\beta), \tag{B5}$$

$$^{(4)}R^{\alpha\beta} = -\vartheta^{[\alpha} \wedge \Phi^{\beta]}, \tag{B6}$$

$$^{(5)}R^{\alpha\beta} = -\tfrac{1}{2}\vartheta^{[\alpha} \wedge e^{\beta]}\rfloor(\vartheta^\gamma \wedge R_\gamma), \tag{B7}$$

$$^{(6)}R^{\alpha\beta} = -\tfrac{1}{12}R\vartheta^\alpha \wedge \vartheta^\beta, \tag{B8}$$

$$^{(1)}R^{\alpha\beta} = R^{\alpha\beta} - \sum_{I=2}^{6} {}^{(I)}R^{\alpha\beta}. \tag{B9}$$

Here

$$\begin{aligned} R^\alpha &:= e_\beta\rfloor R^{\alpha\beta}, \qquad & R &:= e_\alpha\rfloor R^\alpha,\\ X^\alpha &:= {}^*(R^{\beta\alpha} \wedge \vartheta_\beta), \qquad & X &:= e_\alpha\rfloor X^\alpha, \end{aligned} \tag{B10}$$

and

$$\Psi_\alpha := X_\alpha - \tfrac{1}{4}\vartheta_\alpha X - \tfrac{1}{2}e_\alpha\rfloor(\vartheta^\beta \wedge X_\beta), \tag{B11}$$

$$\Phi_\alpha := R_\alpha - \tfrac{1}{4}\vartheta_\alpha R - \tfrac{1}{2}e_\alpha\rfloor(\vartheta^\beta \wedge R_\beta). \tag{B12}$$

The components of the 2-form $R_\alpha{}^\beta = \frac{1}{2}R_{\mu\nu\alpha}{}^\beta\vartheta^\mu \wedge \vartheta^\nu$ are identified with the curvature tensor $R_{\mu\nu\alpha}{}^\beta$. Accordingly, the Ricci tensor is defined by the components of the 1-form $R_\alpha = \mathrm{Ric}_{\beta\alpha}\vartheta^\beta$, where explicitly we have $\mathrm{Ric}_{\alpha\beta} = R_{\gamma\alpha\beta}{}^\gamma$. This tensor is not symmetric, in general. The curvature scalar is, as usual, $R = g^{\alpha\beta}\mathrm{Ric}_{\alpha\beta}$. It determines the 6th irreducible part (B8) of the curvature. From (B12) we learn that the 4-th part of the curvature is given by the symmetric traceless Ricci tensor,

$$\Phi_\alpha = (\mathrm{Ric}_{(\alpha\beta)} - \tfrac{1}{4}Rg_{\alpha\beta})\vartheta^\beta. \tag{B13}$$

The first irreducible part (B9) introduces the generalized Weyl tensor $C_{\mu\nu\alpha}{}^\beta$ which is defined by the components of the Weyl 2-form

$$W_\alpha{}^\beta := {}^{(1)}R_\alpha{}^\beta = \tfrac{1}{2}C_{\mu\nu\alpha}{}^\beta\vartheta^\mu \wedge \vartheta^\nu. \tag{B14}$$

Accordingly, the 1st, 4th, and 6th curvature parts reproduce the well-known irreducible decomposition of the Riemannian curvature tensor into the Weyl, traceless Ricci, and curvature scalar parts. The 2nd, 3rd, and 5th curvature parts are purely non-Riemannian since they all arise from the nontrivial right-hand side of the first Bianchi identity $R_\alpha{}^\beta \wedge \vartheta^\alpha = DT^\beta$, see (B10) and (B11).

APPENDIX C: VECTOR FIELD IN KERR-ADS SPACETIME

Let us take an arbitrary vector field $\xi = \xi^i\partial_i$, the components of which are four constant parameters ξ^0, ξ^1, ξ^2, ξ^3. Then for the Kerr-AdS spacetime with the coframe given by (9.1), (9.2), (9.3), and (9.4), we find for the differential of the corresponding 1-form k:

$$dk = \omega + \chi, \quad \text{with } \omega = 2\mathcal{A}\vartheta^{\hat 0} \wedge \vartheta^{\hat 1} + 2\mathcal{B}\vartheta^{\hat 2} \wedge \vartheta^{\hat 3}. \tag{C1}$$

We have here explicitly the coefficients

$$\begin{aligned}\mathcal{A} = \xi^0 c\left[\frac{\lambda r}{3} - \frac{m(r^2 - a^2\cos^2\theta)}{\Sigma^2}\right] \\ + \xi^3 a\Omega\sin^2\theta\left[\frac{r}{\Sigma}\left(1 - \frac{\lambda r^2}{3}\right) + \frac{m(r^2 - a^2\cos^2\theta)}{\Sigma^2}\right],\end{aligned} \tag{C2}$$

$$\begin{aligned}\mathcal{B} = \xi^0 ca\cos\theta\left[\frac{\lambda}{3} + \frac{2mr}{\Sigma^2}\right] \\ - \xi^3\Omega\cos\theta\left[\frac{r^2 + a^2}{\Sigma}\left(1 + \frac{\lambda a^2\cos^2\theta}{3}\right) + \frac{2mra^2\sin^2\theta}{\Sigma^2}\right].\end{aligned} \tag{C3}$$

The 2-form χ in (C1) reads

$$\begin{aligned}\chi = \frac{2\sqrt{f\Delta}}{\Sigma}\Big[\xi^3\Omega\sin\theta(a\cos\theta\vartheta^{\hat 0} \wedge \vartheta^{\hat 2} + r\vartheta^{\hat 3} \wedge \vartheta^{\hat 1}) \\ - \left(\xi^1\frac{a^2\sin\theta\cos\theta}{\Delta} + \xi^2\frac{r}{f}\right)\vartheta^{\hat 1} \wedge \vartheta^{\hat 2}\Big].\end{aligned} \tag{C4}$$

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