The Fractional-Nonlinear Robotic Manipulator: Modeling and Dynamic Simulations

S. A. David [1], J. M. Balthazar [2], B.H.S.Julio[3],C.Oliveira[4]
[1] University of São Paulo at Pirassununga,Pirassununga, Brasil, sergiodavid@usp.br
[2] São Paulo State University at Rio Claro, Rio Claro, Brasil, jmbaltha@rc.unesp.br
[3]University of São Paulo at Lorena, Lorena, Brasil, bruno.julio@usp.br
[4] Federal University of Dourados, Dourados, Brasil, ClivaldoOliveira@ufgd.edu.br

Abstract. In this paper, we applied the Riemann-Liouville approach and the fractional Euler-Lagrange equations in order to obtain the fractional-order nonlinear dynamics equations of a two link robotic manipulator. The aformentioned equations have been simulated for several cases involving: integer and non-integer order analysis, with and without external forcing acting and some different initial conditions. The fractional nonlinear governing equations of motion are coupled and the time evolution of the angular positions and the phase diagrams have been plotted to visualize the effect of fractional order approach. The new contribution of this work arises from the fact that the dynamics equations of a two link robotic manipulator have been modeled with the fractional Euler-Lagrange dynamics approach. The results reveal that the fractional-nonlinear robotic manipulator can exhibit different and curious behavior from those obtained with the standard dynamical system and can be useful for a better understanding and control of such nonlinear systems.

Keywords: Fractional Calculus, Lagrangian Mechanics, Robotics, Nonlinear Dynamics
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INTRODUCTION

The fractional calculus is a natural extension of classical mathematics. Perhaps the subject would be more aptly called “integration and differentiation of arbitrary order.” The theory of fractional calculus dates back to the birth of the theory of differential calculus, but its inherent complexity delayed the application of its associated concepts. The basic aspects of the theory of fractional calculus are outlined in [7]. In fact, the fractional order calculus can represent systems with high-order dynamics and complex nonlinear phenomena using few coefficients, since the arbitrary order of the derivatives provides an additional degree of freedom to fit a specific behavior. Numerous mathematicians have contributed to the history of fractional calculus by attempting to solve a fundamental problem to the best of their understanding. Each researcher sought a definition and therefore different approaches, which has led to various definitions of differentiation and antidifferentiation of non-integer orders that are provenly equivalent. Although all these definitions may be equivalent, from one specific standpoint, i.e., for a specific application, some definitions seem more attractive. Advances in high capacity computer technology and non-linear dynamics turn the attention of scientists and engineers to fractional variational principles, due to their various applications in many fields. Among them we can mention fluid flow, viscoelastic damping, diffusion phenomena, signal processing, electromagnetic theory, electrical circuits, probability and others. Several researchers have been inspired to examine this new possibility also in other scientific areas. Some work has been carried out in the field of dynamical systems and control theory [1,6], but the proposed models and algorithms are still in the preliminary stage.

Fractional dynamics has gained increasing popularity by researchers over the years. In the case of oscillatory problems, what is generally done is that systems are modeled using classical integer-order approach, and then some of the integer-order differential terms are replaced by fractional-order differentials, instead of modeling the system directly and completely using fractional approach.

For instance, in [8] some terms of the Duffing equation is replaced by a fractional derivative and then the chaotic dynamics of the system has been examined. In the investigation about modified magneto-elastic dynamic equation in [3] two integer-order differential terms of the system were replaced by fractional-order differential terms.

In this paper, the dynamics of a planar two link robotic manipulator has been modeled using fractional dynamics approach. For the sake of

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comparison and to show the similarities and differences between the solution sets, both classical integer-order and fractional-order modeling approaches are applied to the manipulator. The dynamic modeling, in both cases, was carried out with and without external forcing acting and some different initial conditions. Effects of fractional order modeling on the dynamics of the manipulator are observed and presented visually through of the time evolution of the angular positions and the phase diagrams.

This paper is organized as follows. First, we presented the Fundamentals of fractional calculus. After, the classical integer-order and the fractional integer-order of the equations of motion to a planar two link robotic manipulator are derived and presented. Different sets of initial conditions and numerical values were considered. The simulations results are plotted and finally, a discussion is presented.

FRACTIONAL CALCULUS

Historically, fractional order calculus (FOC) has been unexplored or its applications delayed in engineering because of its inherent complexity, the apparent self-sufficiency of integer order calculus (IOC), and the fact that it lacks a fully acceptable geometric or physical interpretation. Clearly, numerous mathematicians such as Lacroix, Riemann, Liouville, Cauchy, Laurent, Caputo, among others, have contributed to the history of fractional calculus by attempting to solve a fundamental problem to the best of their understanding [2]. Each researcher sought a definition and therefore different approaches, which has led to various definitions of differentiation and antidifferentiation of non-integer orders that are provenly equivalent.

Fundamentals of Fractional Calculus

Based on the work published by Laurent and nowadays recognized as the definitive paper on the fundaments of fractional calculus, the Riemann-Liouville definition will be utilized in this paper. Briefly, the Riemann-Liouville approach can be understood from the definition of integration of arbitrary order $\upsilon > 0$:

$$ {}_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\,dt \tag{1}$$

The appropriate definition of differentiation of arbitrary order is to integrate it up to a point from which the desired result can be obtained by conventional differentiation.

Let $\nu = m - \rho$ where, for convenience, m is considered the smallest integer larger than υ and $0 < \rho \leq 1$.

Observe that,

$$ {}_cD_x^{\nu} f(x) = {}_cD_x^{m-\rho} f(x) \tag{2}$$

Thus,

$$ {}_cD_x^{m-\rho} f(x) = \frac{d^m}{dx^m}\left[{}_cD_x^{-\rho} f(x) \right] \tag{3}$$

and consequently,

$$ \frac{d^m}{dx^m}\left[{}_cD_x^{-\rho} f(x) \right] = \\ = \frac{d^m}{dx^m}\left[\frac{1}{\Gamma(\rho)} \int_c^x (x-t)^{\rho-1} f(t)\,dt \right] \tag{4}$$

Or similarly,

$$ {}_cD_x^{\nu} f(x) = \\ = \frac{d^m}{dx^m}\left[\frac{1}{\Gamma(m-\nu)} \int_c^x (x-t)^{m-\nu-1} f(t)\,dt \right] \tag{5}$$

Considering the Riemann-Liouville approach, one can write the derivative of arbitrary order (α) on the left side, i.e.,

$$ {}_aD_t^{\alpha} f(t) = \\ = \left(\frac{d}{dt}\right)^n \left[\frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau \right] \tag{6}$$

for $t > a$, and on right side, i.e.,

$$ {}_tD_b^{\alpha} f(t) = \\ = \left(-\frac{d}{dt}\right)^n \left[\frac{1}{\Gamma(n-\alpha)} \int_t^b (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau \right] \tag{7}$$

for $t < b$,
and $n-1 \leq \alpha < n$.

Fractional Euler-Lagrange Equations

Taking into account the Riemann-Liouville approach and considering an action function

$$S=\frac{1}{\Gamma(\alpha)}\int_a^b L(t,\ {}_aD_t^{\beta}\, q,\ {}_tD_b^{\gamma}\, q)(t-\tau)^{\alpha-1}\, d\tau \tag{8}$$

with,

$0 \le \beta \le 1$, $0 < \gamma < 1$, $0 \le \alpha \le 1$.

If ε denotes the variation of the function S, then

$$\Delta_\varepsilon S=\int_a^b L(q+\varepsilon\,\delta q,\ {}_aD_t^{\beta}\, q+$$
$$+\int_a^b \varepsilon\, {}_aD_t^{\beta}\,\delta q,\ {}_tD_b^{\gamma}\, q+\varepsilon\ {}_tD_b^{\gamma}\,\delta q)(t-\tau)^{\alpha-1}\, d\tau \tag{9}$$

The equation (9) can be rewritten

$$\Delta_\varepsilon S = \int_a^b \left(\frac{\partial L}{\partial q}(t-\tau)^{\alpha-1} + \right.$$
$$+\int_a^b \frac{\partial L}{\partial({}_aD_t^{\beta}\, q)}(t-\tau)^{\alpha-1}\, {}_aD_t^{\beta}\,\delta q +$$
$$\left.+\int_a^b \frac{\partial L}{\partial({}_tD_b^{\gamma}\, q)}(t-\tau)^{\alpha-1}\, {}_tD_b^{\gamma}\,\delta q\right)$$
$$\times\ \varepsilon\, d\tau+0(\varepsilon^2) \tag{10}$$

Or similarly,

$$\Delta_\varepsilon S=\int_a^b\left(\frac{\partial L}{\partial q}(t-\tau)^{\alpha-1}+\right.$$
$$+\int_a^b {}_aD_t^{\beta}\left[\frac{\partial L}{\partial({}_aD_t^{\beta}\, q)}(t-\tau)^{\alpha-1}\right]+$$
$$\left.\int_a^b {}_tD_b^{\gamma}\left[\frac{\partial L}{\partial({}_tD_b^{\gamma}\, q)}(t-\tau)^{\alpha-1}\right]\right)$$
$$\times\ \delta q\,\varepsilon\, d\tau+0(\varepsilon^2) \tag{11}$$

Thus, one can be the fractional Euler-Lagrange equations such as

$$\frac{\partial L}{\partial q}-\frac{1}{(t-\tau)^{\alpha-1}}\Big[{}_tD_b^{\beta}\Big(\frac{\partial L}{\partial({}_aD_t^{\beta}\, q)}(t-\tau)^{\alpha-1}\Big)+$$
$${}_aD_t^{\gamma}\Big(\frac{\partial L}{\partial({}_tD_b^{\gamma}\, q)}(t-\tau)^{\alpha-1}\Big)\Big]=0 \tag{12}$$

For β = γ = 1 and assuming that the Lagrangean depends only of the ${}_aD_t^{\beta}\, q$ or of the ${}_tD_b^{\gamma}\, q$, one can obtain

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q}\Big)-\frac{\partial L}{\partial q}+\frac{(\alpha-1)}{(t-\tau)}\frac{\partial L}{\partial \dot q}=0 \tag{13}$$

FRACTIONAL MODELING

Dynamic Equations of Motion

We have considered a planar two link robotic manipulator with two rotational joints (Figure 1).

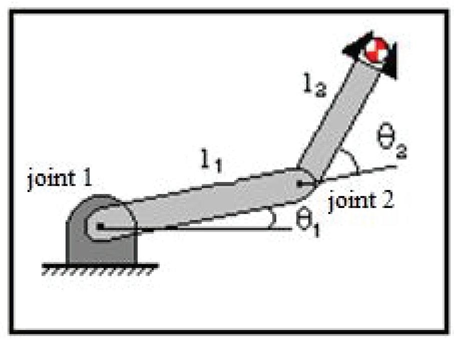

FIGURE 1. Two link planar manipulator

It well known that a classical approach, i.e, integer order calculus permit us to obtain the equations of motion [4,5] using

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q_i}\Big)-\frac{\partial L}{\partial q_i}=Q_i \tag{14}$$

where L = T – V is the Lagrangean. The generalized coordinates are θ_1 and θ_2. Thus, one can be write the Lagrangean as

$$L=\frac{1}{2}m_1 l_1^{\,2}\dot\theta_1^{\,2}+\frac{1}{2}m_2\,(l_1^2\,\dot\theta_1^2+l_2^2\,\dot\theta_2^2+$$
$$+2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_2-\theta_1))+(m_1+m_2)gl_1\cos\theta_1$$
$$+m_2gl_2\cos\theta_2 \tag{15}$$

where:

m_1 and m_2 are the masses of the links 1 and 2, respectively.
l_1 e l_2 are the lengths of the links 1 and 2, respectively;
g is the gravitational acceleration constant.

The well known equations of motion for the classical (integer-order) calculus are

$$(m_1+m_2)l_1^2\ddot{\theta}_1+m_2l_1l_2\ddot{\theta}_2\cos(\theta_2-\theta_1) - m_2l_1l_2\dot{\theta}_2^2 \operatorname{sen}(\theta_2-\theta_1) + (m_1+m_2)gl_1 sen(\theta_1)=Q_1 \tag{16}$$

and

$$m_2l_2^2\ddot{\theta}_2+m_2l_1l_2\ddot{\theta}_1\cos(\theta_2-\theta_1) + m_2l_1l_2\dot{\theta}_1^2 \operatorname{sen}(\theta_2-\theta_1) + m_2gl_2sen(\theta_2)=Q_2 \tag{17}$$

where Q_1 and Q_2 are the time dependent generalized non-conservative forces (or torques) which value vanishes in the absent of external forces.

Considering, now, the Riemann-Liouville approach and fractional Euler-lagrange equations, one can be write

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} + \frac{(\alpha-1)}{(t-\tau)}\frac{\partial L}{\partial \dot{q}_i}=Q_i \tag{18}$$

One can combine with the equations (13) and (15) in order to obtain

$$\frac{(\alpha-1)}{(t-\tau)}\frac{\partial L}{\partial \dot{q}_1}=\frac{(\alpha-1)}{(t-\tau)}\frac{\partial L}{\partial \dot{\theta}_1}= =\frac{(\alpha-1)}{(t-\tau)}[(\mathrm{m}_1+m_2)l_1^2\dot{\theta}_1+ +m_2l_1l_2\dot{\theta}_2\cos(\theta_2-\theta_1)] \tag{19}$$

and,

$$\frac{(\alpha-1)}{(t-\tau)}\frac{\partial L}{\partial \dot{q}_2}=\frac{(\alpha-1)}{(t-\tau)}\frac{\partial L}{\partial \dot{\theta}_2}= =\frac{(\alpha-1)}{(t-\tau)}\left[m_2\,l_2^2\dot{\theta}_2+m_2l_1l_2\dot{\theta}_1\cos(\theta_2-\theta_1)\right] \tag{20}$$

Thus, it is possible to write the equations of motion using the Lagrange equations for fractional manipulator system as

$$(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2l_1l_2\ddot{\theta}_2\cos(\theta_2-\theta_1) - m_2l_1l_2\dot{\theta}_2^2 \operatorname{sen}(\theta_2-\theta_1) + (m_1 + m_2)gl_1sen(\theta_1) + \frac{(\alpha-1)}{(t-\tau)}\left[(\mathrm{m}_1+m_2)l_1^2\dot{\theta}_1+m_2l_1l_2\dot{\theta}_2\cos(\theta_2-\theta_1)\right] = Q_1 \tag{21}$$

$$m_2l_2^2\ddot{\theta}_2 + m_2l_1l_2\ddot{\theta}_1\cos(\theta_2-\theta_1) + m_2l_1l_2\dot{\theta}_1^2 \operatorname{sen}(\theta_2-\theta_1)+m_2gl_2sen(\theta_2) + \frac{(\alpha-1)}{(t-\tau)}\left[m_2\,l_2^2\dot{\theta}_2+m_2l_1l_2\dot{\theta}_1\cos(\theta_2-\theta_1)\right] = Q_2 \tag{22}$$

Left hand sides of the equations comprise of the terms in Equations (16) and (17). The terms containing α indicate the additional terms resulting from the fractional-order model, as given in Equations (21) and (22). Right hand sides denote the generalized force terms resulting from the forcing functions. There is a specific set of values for Q_1 and Q_2 for each case of forcing that will be shown in sequence.

Forces and Initial Conditions

Generalized forces can be obtained using the relation

$$Q_k=\sum_{i=1}^{n}F_i\,\frac{\partial r_i}{\partial q_k} \tag{23}$$

where Q_k is the generalized force associated with the k^{th} equation of motion, F_i is the applied force, $r_i$is the position of the point of application and q_k is the generalized coordinate. It should be noticed that the Q_k term vanishes in unforced systems, i.e, without external force acting.

The generalized force terms constitute the right hand sides of the equations of motion of the manipulator. The Q_1 term represents the right hand side of the first equation of motion, which had been derived by differentiating the Lagangean (15) with respect to $\theta_1$and its derivative, and likewise, Q_2 term represents the right hand side of the second equation of motion, which had been derived by differentiating the Lagrangean with respect to θ_2 and its derivative. Both, Q_1 and Q_2 will be equal to zero when there is no forcing function.

In this study, we presented two different cases: without force acting and with force acting. Specifically, the following cases we have investigated.

TABLE 1. Cases selected

Cases	Modeling	Order (α)	Initial conditions
1	Integer	1	Set: 1, 2
1	Fractional	0.4; 0.6; 0.9; 1.1; 1.2	Set: 1, 2
2	Integer	1	Set: 1, 2
2	Fractional	0.4; 0.6; 0.9; 1.1; 1.2	Set: 1,2
3	Integer	1	Set: 1,2
3	Fractional	0.4; 0.6; 0.9; 1.1; 1.2	Set: 1,2

Case 1: $Q_1 = Q_2 = 0$;

Case 2: $Q_1 = A\cos(wt)\, l_1 \sin(\theta_2 - \theta_1)$, $Q_2 = 0$.

Case 3: $Q_1 = -A\cos(wt)\, l_1 \sin(\theta_1)$,
$Q_2 = -A\cos(wt)\, l_2 \sin(\theta_2)$

Each equation of motion was solved for each set of these initial conditions, which are:

set 1:

$$\theta_1(0) = 1,\ \theta_2(0) = 2\ ,\ \dot{\theta}_1(0) = 0\ ,\dot{\theta}_2(0) =\ 0$$

set 2:

$$\theta_1(0) = 1, \theta_2(0) = -1, \dot{\theta}_1(0) = 0\ ,\dot{\theta}_2(0) =\ 0$$

where the angles are measured in radians. The sets 1 and 2 correspond to the case where the manipulator has an initial displacement relative to a reference but no initial velocity.

In this study, we have considered the following parameters:

$m_1 = m_2 = 1Kg$;

$l_1 = l_2 = 1m$;

α = 0.4 ; α = 0.6 ; α = 0.9 ; α = 1.0 ; α = 1.1 ; α = 1.2

g = 9.81m/s^2.

Some selected results of simulations will be presented in the following section.

RESULTS

An inspection of the Table 1 shows all the cases investigated in this work. The manipulator has been modeled and simulated with and without external forcing, using both integer and fractional order modeling. Equations modeling the dynamics of all the cases treated here have been solved and the time evolution of the angular positions and the phase diagrams have been plotted to visualize the effect of fractional order approach. The effect of fractional variable α is observed. As α approaches 1, the results approach those obtained by integer-order modeling.

For the sake of limit pages in this paper, we will show just the following situations simulated:

CASE 1 with SET 1;

CASE 2 with SET 2;

CASE 3 with SET 1;

CASE 1 – SET 1

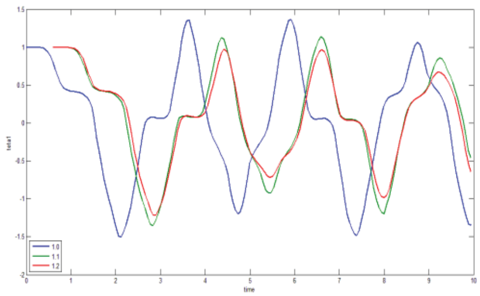

Figure 2 - θ_1 of α = [1, 1.1, 1.2] x time.

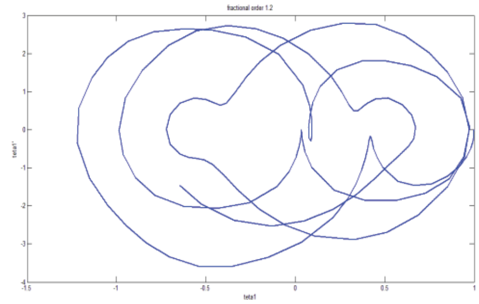

Figure 3- Phase plane of θ_1. (α = 1.2)

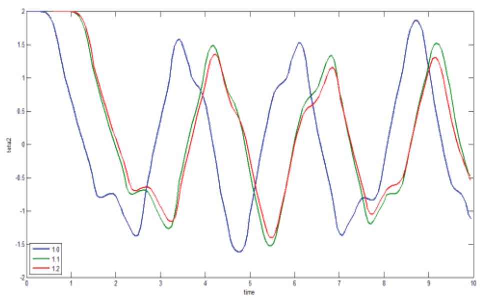

Figure 4 - θ_2 of α = [1, 1.1, 1.2] x time.

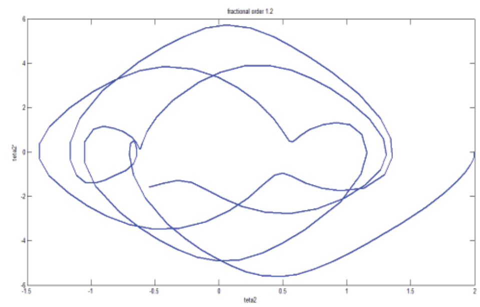

Figure 5- Phase plane of θ_2. (α = 1.2)

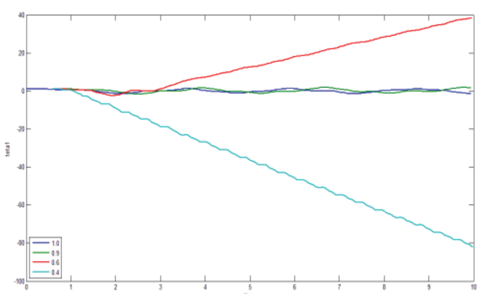

Figure 6 - θ_1 of α = [1, 0.9, 0.6, 0.4] x time.

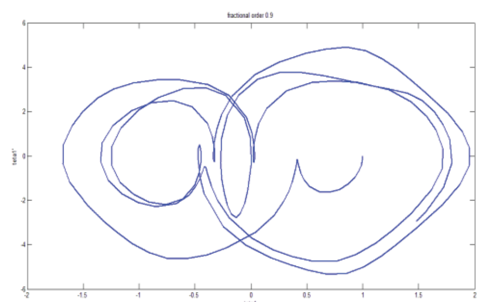

Figure 7- Phase plane of θ_1. (α = 0.9)

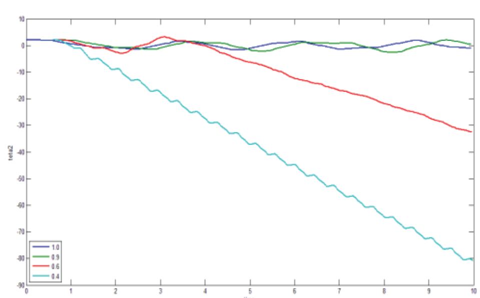

Figure 8 – θ_2 of α = [1, 0.9, 0.6, 0.4] x time.

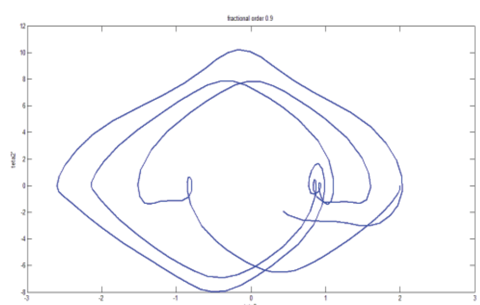

Figure 9- Phase plane of θ_2. (α = 0.9)

CASE 2 – SET 2

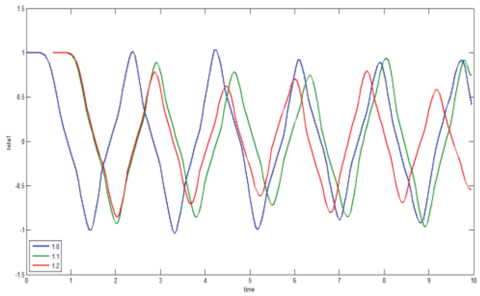

Figure 10 - θ_1 of α = [1, 1.1, 1.2] x time.

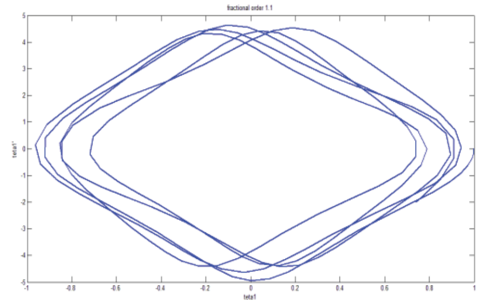

Figure 11- Phase plane of θ_1. (α = 1.1)

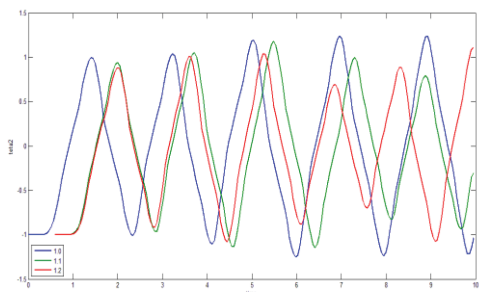

Figure 12 - θ_2 of α = [1, 1.1, 1.2] x time

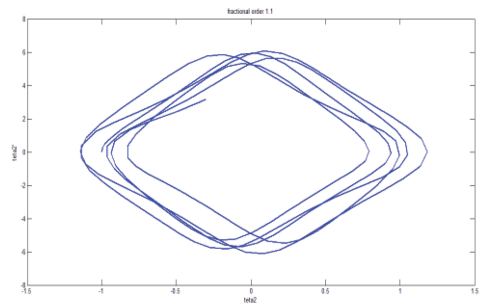

Figure 13- Phase plane of θ_2. (α = 1.1)

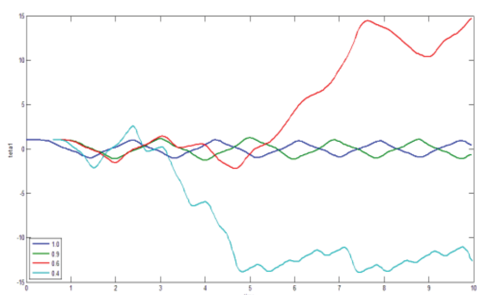

Figure 14- θ_1 of α = [1, 0.9, 0.6, 0.4] x time.

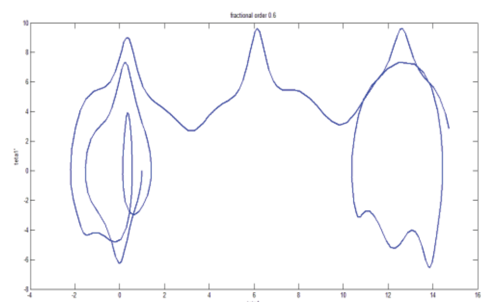

Figure 15- Phase plane of θ_1. (α = 0.6)

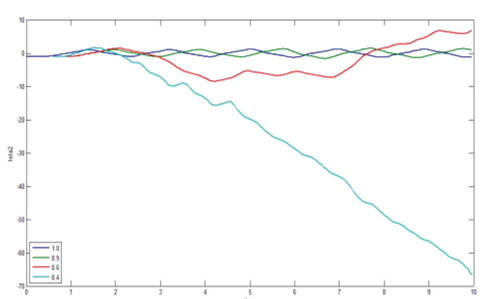

Figure 16 – θ_2 of α = [1, 0.9, 0.6, 0.4] x time.

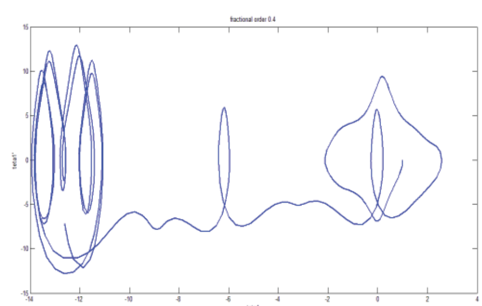

Figure 17- Phase plane of θ_1. (α = 0.4)

CASE 3 – SET 1

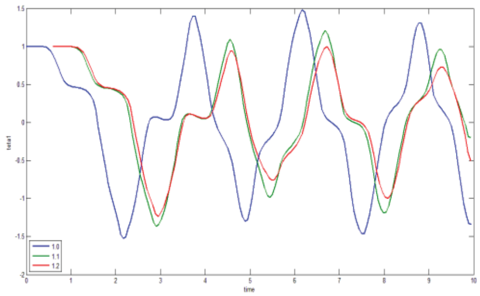

Figure 18 - θ_1 of α = [1, 1.1, 1.2] x time.

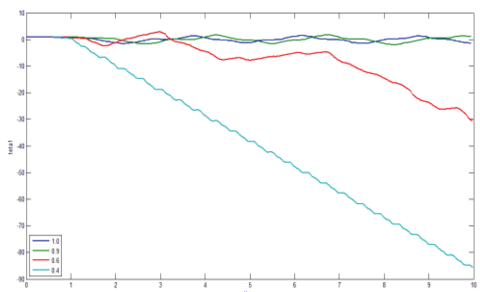

Figure 19 - θ_1 of α = [1, 0.9, 0.6, 0.4] x time.

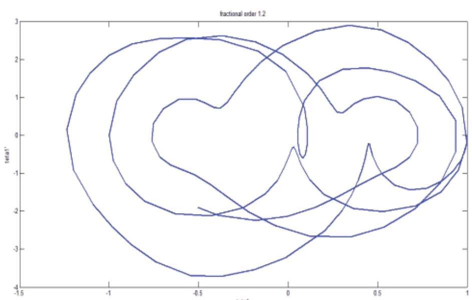

Figure 20- Phase plane of θ_1. (α = 1.2)

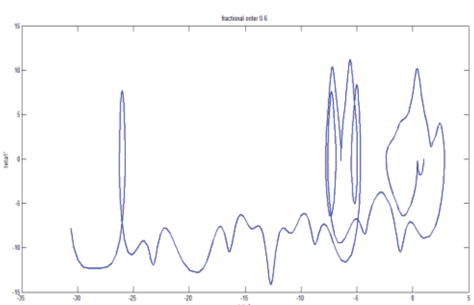

Figure 21- Phase plane of θ_1. (α = 0.6)

DISCUSSION AND CONCLUSIONS

The inspection of Table 1 reveals the cases examined in this study. Integer and fractional-order modeling of a planar two link robotic manipulator were done under various forcing conditions and three different sets of initial conditions. Five different values have been used for the fractional order α. It was observed that as α approaches 1, phase diagrams converge to those obtained by integer-order modeling. Physically possible initial conditions were applied in both modeling approaches and the results were presented graphically. However, experiments should be conducted to obtain more realistic results about wheter fractional-order models are more accurate than integer-order models. In addition, this study can be used as a source for a better understanding and control of such nonlinear systems.

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