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On anomalous diffusion and the fractional generalized Langevin equation for a harmonic oscillator

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The fractional generalized Langevin equation (FGLE) is proposed to discuss the anomalous diffusive behavior of a harmonic oscillator driven by a two-parameter Mittag–Leffler noise. The solution of this FGLE is discussed by means of the Laplace transform methodology and the kernels are presented in terms of the three-parameter Mittag–Leffler functions. Recent results associated with a generalized Langevin equation are recovered. © *2009 American Institute of Physics.*
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I. INTRODUCTION

Anomalous diffusive processes associated with a physical[1–3] or a biological[4–6] process can be discussed in terms of the generalized Langevin equation (GLE).[7–9] In the literature we can find some recent papers on anomalous diffusion.[10–20]

There are many physical systems which are related to diffusion, the so-called diffusive processes, for example. The Brownian motion is a diffusive process. A simple classification of the diffusive processes can be associated with the mean square displacement (MSD) given by $\langle x^2(t)\rangle$, where $x(t)$ is the position of a particle in the time t. For the normal diffusion, the MSD grows linearly with time, $\langle x^2(t)\rangle \sim t$, while the other cases we have the so-called anomalous diffusion or, in general, anomalous diffusive processes, whose MSD grows as $\langle x^2(t)\rangle \sim t^{\nu}$, where $\nu \neq 1$, i.e., the MSD does not grow linearly with time. When $\nu > 1$ we say that we have superdiffusion, and for $\nu < 1$ that we have subdiffusion. In both cases the anomalous diffusion depends on a unique parameter.

The Mittag–Leffler function and its generalizations are useful in the study of anomalous diffusion. In Ref. 21 a type of Mittag–Leffler function was introduced in the study of kinetic equation, random walks, and anomalous diffusion.[8,9] It appears also in the calculation of some fractional Green's functions associated with the reaction-diffusion equations.[22] This Mittag–Leffler function also appears in Ref. 17 in the definition of a one-parameter correlation function.

Our objective in this paper is to discuss the fractional version of the GLE with the correlation function being given by a generalized Mittag–Leffler function involving two parameters, i.e, a two-parameter correlation function.

The paper is organized as follows. In Sec. II we present the classical Langevin equations, i.e., we recover the GLE and we introduce the fractional GLE (FGLE). Section III contains a formal solution of the FGLE in terms of the Laplace transform of the relaxation function. In Sec. IV, using a correlation function, given in terms of a two-parameter Mittag–Leffler function, we cal-

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culate the corresponding Laplace transform and present the kernels in terms of the three-parameter Mittag–Leffler functions. In Sec. V we present the temporal behavior of the relaxation functions, and, in Sec. VI, we consider some particular cases of our approach.

II. THE LANGEVIN EQUATIONS

The classical Langevin equation and GLE were revisited and a fractional treatment was proposed some years ago by Mainardi and Pironi.[23] In that paper the authors discussed the Langevin equation associated only with velocity. To discuss the Langevin equation associated with the displacement, one must calculate an integration with respect to time variable.

In this section, we consider the classical Langevin equation, as discussed by Viñales *et al.*,[24] and we introduce the FGLE which we will discuss in Sec. III.

A. The GLE

In Ref. 24 the authors discussed the anomalous diffusive behavior of a harmonic oscillator driven, with frequency ω, by a Mittag–Leffler noise, i.e., they discuss the so-called GLE,

$$\mathbf{D}_t^2 x(t) + \int_0^t \gamma(t-t')\mathbf{D}_{t'} x(t')\mathrm{d}t' + \omega^2 x(t) = \xi(t), \tag{1}$$

where $\mathbf{D} \equiv \mathrm{d}/\mathrm{d}t$ and $x(t)$ represents the position of a particle of mass $m=1$ at time t and $\gamma(t)$ is the frictional memory kernel and $\xi(t)$ is a random force obeying the fluctuation-dissipation theorem.

Using another recent result[16] the authors introduced a Mittag–Leffler noise, i.e., the memory kernel is given by

$$\gamma(t) = \frac{1}{k_B T}\frac{C_\lambda}{\tau^\lambda}E_\lambda[-(|t|/\tau)^\lambda], \tag{2}$$

where τ acts as a characteristic memory time and $0<\lambda<2$. The $E_\mu(\cdot)$ denotes the classical Mittag–Leffler function and C_λ is a coefficient dependent on the parameter λ but independent of time. Also, k_B is the Boltzmann constant and T is the absolute temperature of the environment.

The analytical relaxation functions for a Mittag–Leffler noise are presented in terms of the classical Mittag–Leffler function and the derivative of the two-parameter Mittag–Leffler function. The temporal behavior of the relaxation functions closes the paper.

B. The FGLE

In this section we discuss a generalization of the results obtained in Ref. 24 by means of the two-parameter Mittag–Leffler function, i.e., we permit the memory kernel given by a two-parameter Mittag–Leffler function, and the temporal behavior to be dependent of two parameters.

First, we introduce a generalized Mittag–Leffler noise by the following two-parameter correlation function:

$$C(t) = \frac{C_\lambda}{\tau^\lambda}t^{\nu-1}E_{\lambda,\nu}[-(|t|/\tau)^\lambda], \tag{3}$$

where C_λ is a constant and τ acts as a characteristic memory time, $0<\lambda\le\alpha+\nu-\beta$ (see below for definition of α and β) with $\lambda\neq\{1,2\}$, and $\nu>0$. We observe that this condition will follow from Eq. (26), and when $\alpha=2$, $\beta=1$, and $\nu=1$, we have the conditions $0<\lambda<1$ and $1<\lambda<2$ as in Ref. 24. We also note that two particular situations have been studied very recently, that is, (i) $\nu=1$ in Ref. 25 and (ii) $\alpha=2$ and $\beta=1$ in Ref. 26.

This two-parameter correlation function has a very interesting behavior depending on the values of λ and μ, as can be seen in Fig. 1, where it is shown the graphics of the function $F_{\lambda,\nu}(u)=u^{\nu-1}E_{\lambda,\nu}(-u^\lambda)$, that appears in the definition of the correlation function, for different values of λ and ν. We note that since $E_{\lambda,\nu}(-z)\approx z^{-1}/\Gamma(\nu-\lambda)$ for $|z|\gg 1$, its asymptotic behavior is

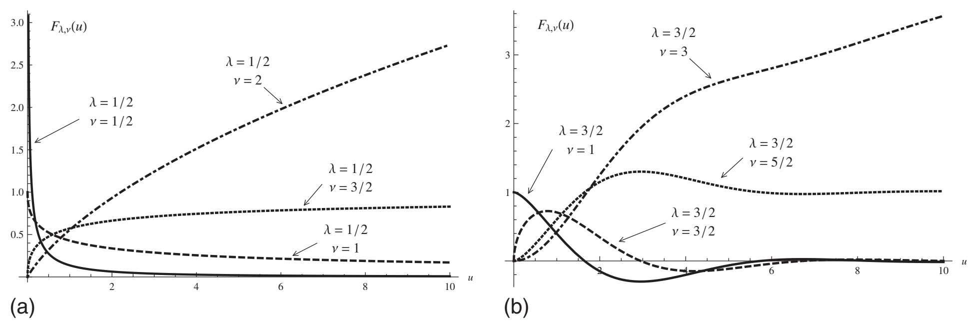

FIG. 1. The function $F_{\lambda,\nu}(u)=u^{\nu-1}E_{\lambda,\nu}(-u^{\lambda})$ for different values of λ and ν.

given by $F_{\lambda,\nu}(u)=u^{\nu-1-\lambda}/\Gamma(\nu-\lambda)$. Moreover, since $0<\alpha-\beta<2$ (see below), all functions displayed in Fig. 1 satisfy the above condition $0<\lambda\leq\alpha+\nu-\beta$ for some values of α and β.

We assume that the random force $F(t)$, which appears in a Langevin equation, obeys the fluctuation-dissipation theorem,[15] which is valid in the equilibrium state,

$$\langle F(t)F(\xi)\rangle = C(|t-\xi|) = k_B T\mu(|t-\xi|),$$

where $\mu(t)$ is the fractional memory kernel of the Langevin equation, k_B is the Boltzmann constant, and T is the absolute temperature of the environment.

Using equations above we can write for the so-called two-parameter frictional memory kernel,

$$\mu(t) \equiv \frac{\mu_\lambda}{\tau^\lambda} t^{\nu-1} E_{\lambda,\nu}[-(|t|/\tau)^\lambda], \tag{4}$$

where $\mu_\lambda=C_\lambda/k_B T$. The different notations for the memory kernel in Eqs. (2) and (4) is to call attention to the fact that in the first case the function involves one parameter while in the later it involves two parameters.

Thus, this paper deals with the dynamic of a fractional treatment of a harmonic oscillator with frequency ω under the influence of a random force modeled as Gaussian colored noise whose corresponding FGLE, associated with the displacement, can be written as

$$\mathbf{D}_t^\alpha x(t) + \int_0^t \mu(t-\xi)\mathbf{D}_\xi^\beta x(\xi)\mathrm{d}\xi + \omega^2 x(t) = F(t), \tag{5}$$

with $1<\alpha\leq 2$, $0<\beta\leq 1$, and $x(t)$ is the position of a particle of unitary mass at time t. Note that $0<\alpha-\beta<2$, as mentioned above.

The fractional derivative is to be considered in Caputo's sense,[7]

$$\mathbf{D}_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \mathrm{d}\xi \frac{\mathbf{D}_\xi^n x(\xi)}{(t-\xi)^{\alpha-n+1}},$$

with $n-1<\alpha\leq n$ and $\mathbf{D}_\xi^n \equiv \mathrm{d}^n/\mathrm{d}\xi^n$ being the ordinary derivative.[27] We note that the case $\alpha=2$ and $\beta=1$ recovers Eq. (1).

III. SOLUTION OF THE FGLE

This section presents the solution of Eq. (5) obtained by means of the Laplace transform methodology where the solution is presented in terms of the relaxation function, the derivative of the relaxation function, and the Laplace transform of the damping kernel. We consider Eq. (5) with the initial conditions $x(0)=x_0$ and $\dot{x}(0)=v_0$.

Introducing the Laplace transform in Eq. (5), we obtain a formal expression for the displacement, given by

$$x(t) = x_0[1-\omega^2 I(t)] + v_0\mathfrak{L}^{-1}[\hat{H}(s)s^{\alpha-2}] + \int_0^t H(t-\xi)F(\xi)\mathrm{d}\xi, \tag{6}$$

where the last term was obtained by means of the convolution theorem associated with the Laplace transform.

In Eq. (6) we have $\hat{H}(s)$, the Laplace transform of the relaxation function defined by

$$\hat{H}(s) \equiv \mathfrak{L}[H(t)] = \frac{1}{s^\alpha + s^\beta\hat{\mu}(s) + \omega^2},$$

where $\hat{\mu}(s)$ is the Laplace transform of the damping kernel, $\hat{\mu}(s)\equiv\mathfrak{L}[\mu(t)]$, and we have introduced the integration of the relaxation function,

$$I(t) = \int_0^t H(\xi)\mathrm{d}\xi.$$

Introducing the position mean value, defined by the relation,

$$\langle x(t)\rangle \equiv x_0[1-\omega^2 I(t)] + v_0\int_0^t H(t-\xi)\frac{\xi^{1-\alpha}}{\Gamma(2-\alpha)}\mathrm{d}\xi, \tag{7}$$

where we have used the convolution theorem in the expression for $\mathfrak{L}^{-1}$, we can write Eq. (6) in the following formal form:

$$x(t) = \langle x(t)\rangle + \int_0^t H(t-\xi)F(\xi)\mathrm{d}\xi. \tag{8}$$

On the other hand, to calculate the variances associated with a physical process, for example, one must introduce the derivative of the relaxation function, $h(t)=\dot{H}(t)$, and its respective integration as we will see in Sec. IV.

Finally, variances can be written as[8,24,25,28]

$$\sigma_{xx}(t) = k_B T[2I(t) - H^2(t) - \omega^2 I^2(t)],$$

$$\sigma_{vv}(t) = k_B T[1 - h^2(t) - \omega^2 H^2(t)],$$

$$\sigma_{xv}(t) = k_B T H(t)[1 - h(t) - \omega^2 I^2(t)]. \tag{9}$$

IV. A MITTAG–LEFFLER CORRELATION FUNCTION

Now we consider the two-parameter Mittag–Leffler correlation function. We have to calculate the Laplace transform of the memory kernel given by Eq. (4), i.e., the following integral:

$$\hat{\mu}(s) \equiv \mathfrak{L}[\mu(t)] = \mathfrak{L}\left\{\frac{\mu_\lambda}{\tau^\lambda}t^{\nu-1}E_{\lambda,\nu}[-(|t|/\tau)^\lambda]\right\}.$$

Using the result for $\rho=1$,

$$\mathfrak{L}[t^{\beta-1}E^\rho_{\alpha,\beta}(at^\alpha)] = \frac{s^{\alpha\rho-\beta}}{(s^\alpha - a)^\rho} \Leftrightarrow \mathfrak{L}^{-1}\left[\frac{s^{\alpha\rho-\beta}}{(s^\alpha-a)^\rho}\right] = t^{\beta-1}E^\rho_{\alpha,\beta}(at^\alpha), \tag{10}$$

where $\mathrm{Re}(s)>0$, $\mathrm{Re}(\beta)>0$, $a\in\mathbb{C}$, and $|as^{-\alpha}|<1$, we can write[29]

$$\hat{\mu}(s) = \mu_\lambda \frac{s^{\lambda-\nu}}{1+s^\lambda \tau^\lambda}, \tag{11}$$

which gives the corresponding Laplace transform of the relaxation function,

$$\hat{H}(s) = \hat{H}_0(s) + \hat{H}_1(s) \tag{12}$$

with

$$\hat{H}_1(s) = \tau^{-\lambda} s^{-\lambda} \hat{H}_0(s),$$

where $\hat{H}_0(s)$ is given by the following expression:

$$\hat{H}_0(s) = \frac{s^\lambda}{s^{\lambda+\alpha} + \frac{1}{\tau^\lambda} s^\alpha + \frac{\mu_\lambda}{\tau^\lambda} s^{\beta+\lambda-\nu} + \omega^2 s^\lambda + \frac{\omega^2}{\tau^\lambda}}. \tag{13}$$

The inverse Laplace transform of $\hat{H}_0(s)$ and $\hat{H}_1(s)$ have been calculated in the appendices, and the result is given by Eq. (A9), that is,

$$H_0(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^k \binom{k}{m} \left(\frac{\mu_\lambda}{\tau^\lambda}\right)^k \left(\frac{\tau^\lambda \omega^2}{\mu_\lambda}\right)^m t^{\xi_0^0 - 1} E_{\lambda,\xi_0^0}^{k-m+1}[-(t/\tau)^\lambda], \tag{14}$$

$$H_1(t) = \tau^{-\lambda} \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^k \binom{k}{m} \left(\frac{\mu_\lambda}{\tau^\lambda}\right)^k \left(\frac{\tau^\lambda \omega^2}{\mu_\lambda}\right)^m t^{\xi_1^0 - 1} E_{\lambda,\xi_1^0}^{k-m+1}[-(t/\tau)^\lambda], \tag{15}$$

where ξ_0^0 and ξ_1^0 are given by Eq. (A7), from which we can see that

$$\xi_1^0 = \xi_0^0 + \lambda, \tag{16}$$

with

$$\xi_0^0 = \alpha + (\alpha - \beta + \nu)k + (\beta - \nu)m. \tag{17}$$

Now we can calculate $H(t) = H_0(t) + H_1(t)$. Using Eqs. (14)–(16) we have

$$H(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^k \binom{k}{m} \left(\frac{\mu_\lambda}{\tau^\lambda}\right)^k \left(\frac{\tau^\lambda \omega^2}{\mu_\lambda}\right)^m t^{\xi_0^0 - 1} \cdot [E_{\lambda,\xi_0^0}^{k-m+1}[-(t/\tau)^\lambda] + (t/\tau)^\lambda E_{\lambda,\xi_0^0+\lambda}^{k-m+1}[-(t/\tau)^\lambda]]. \tag{18}$$

But the term between brackets can be calculated using Eq. (B4) in the appendices, which gives

$$H(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} (-1)^k \binom{k}{m} \left(\frac{\mu_\lambda}{\tau^\lambda}\right)^k \left(\frac{\tau^\lambda \omega^2}{\mu_\lambda}\right)^m t^{\xi_0^0 - 1} E_{\lambda,\xi_0^0}^{k-m}[-(t/\tau)^\lambda]. \tag{19}$$

Another useful expression can be written using Eq. (17) and switching the order of the sums,

$$H(t) = t^{\alpha-1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} \binom{m+n}{m} (\omega^2)^n \left(\frac{\mu_\lambda}{\tau^\lambda}\right)^m t^{\Delta_{m,n}} E_{\lambda,\alpha+\Delta_{m,n}}^m[-(t/\tau)^\lambda], \tag{20}$$

where

$$\Delta_{m,n} = (\alpha + \nu - \beta)m + \alpha n.$$

Expressions for $I(t)$ and $h(t)$ follow easily using Eqs. (B1) and (B2), which gives

$$I(t) = t^{\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\binom{m+n}{m}(\omega^2)^n\left(\frac{\mu_\lambda}{\tau^\lambda}\right)^m t^{\Delta_{m,n}}E^{m}_{\lambda,\alpha+\Delta_{m,n}+1}[-(t/\tau)^\lambda], \tag{21}$$

$$h(t) = t^{\alpha-2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\binom{m+n}{m}(\omega^2)^n\left(\frac{\mu_\lambda}{\tau^\lambda}\right)^m t^{\Delta_{m,n}}E^{m}_{\lambda,\alpha+\Delta_{m,n}-1}[-(t/\tau)^\lambda]. \tag{22}$$

V. TEMPORAL BEHAVIOR OF THE RELAXATION FUNCTIONS

In this section we study the short-time and the asymptotic behavior of the relaxation functions $H(t)$, $I(t)$, and $h(t)$.

For the short-time behavior, we use

$$E^{\rho}_{\alpha,\beta}(-z) \approx \frac{1}{\Gamma(\beta)} - \frac{\rho z}{\Gamma(\alpha+\beta)} + \cdots \quad (|z| \ll 1),$$

and from Eqs. (20)–(22) we obtain for $t \ll 1$ that

$$I(t) \approx t^{\alpha}\left[\frac{1}{\Gamma(\alpha+1)} - \frac{\omega^2 t^\alpha}{\Gamma(2\alpha+1)} - \frac{\mu_\lambda t^{(\alpha+\nu-\beta)}}{\tau^\lambda\Gamma(\alpha+(\alpha+\nu-\beta)+1)} + \cdots\right], \tag{23}$$

$$H(t) \approx t^{\alpha-1}\left[\frac{1}{\Gamma(\alpha)} - \frac{\omega^2 t^\alpha}{\Gamma(2\alpha)} - \frac{\mu_\lambda t^{(\alpha+\nu-\beta)}}{\tau^\lambda\Gamma(\alpha+(\alpha+\nu-\beta))} + \cdots\right], \tag{24}$$

$$h(t) \approx t^{\alpha-2}\left[\frac{1}{\Gamma(\alpha-1)} - \frac{\omega^2 t^\alpha}{\Gamma(2\alpha-1)} - \frac{\mu_\lambda t^{(\alpha+\nu-\beta)}}{\tau^\lambda\Gamma(\alpha+(\alpha+\nu-\beta)-1)} + \cdots\right]. \tag{25}$$

Since $(\alpha+\nu-\beta)>0$, the first terms on the right hand side of the above expressions are indeed the zeroth-order approximation in the short-time regime for the given functions. However, the next term in the approximation depends on the value of $\nu-\beta$. Since $\nu>0$ and $0<\beta\leq 1$, the quantity $\nu-\beta$ can be positive, negative, or zero. If $\nu-\beta$ is positive, then the term in $t^{(\alpha+\nu-\beta)}$ is a second-order term, but if it is negative or zero, it is a first-order term.

Now we turn to the asymptotic expressions of $I(t)$, $H(t)$, and $h(t)$. In this case we use Eq. (B5),[30]

$$E^{\rho}_{\alpha,\beta}(-z) \approx \frac{z^{-\rho}}{\Gamma(\beta-\alpha\rho)} - \frac{\rho z^{-(\rho+1)}}{\Gamma(\beta-\alpha(\rho+1))} + \cdots \quad (|z| \gg 1).$$

Using it in Eq. (21) we obtain for $t \gg \tau$ that

$$I(t) \approx t^{\alpha}\sum_{m=0}^{\infty}(-1)^m(\mu_\lambda)^m\Bigg(t^{(\alpha-\lambda+\nu-\beta)m}\Bigg[E^{m+1}_{\alpha,(\alpha-\lambda+\nu-\beta)m+\alpha+1}(-\omega^2 t^\alpha) - m\left(\frac{t}{\tau}\right)^{-\lambda}E^{m+1}_{\alpha,(\alpha-\lambda+\nu-\beta)m+\alpha+1-\lambda}(-\omega^2 t^\alpha) + \cdots\Bigg]\Bigg), \tag{26}$$

where $\alpha-\lambda+\nu-\beta \geq 0$, and using the asymptotic expansion again, for $t \gg 1$,

$$I(t) \approx \frac{1}{\omega^2}\sum_{m=0}^{\infty}\frac{(-\mu_\lambda\omega^{-2}t^{\nu-\beta-\lambda})^m}{\Gamma[(\nu-\beta-\lambda)m+1]} - \frac{t^{-\alpha}}{\omega^4}\sum_{m=0}^{\infty}\frac{(m+1)(-\mu_\lambda\omega^{-2}t^{\nu-\beta-\lambda})^m}{\Gamma[(\nu-\beta-\lambda)m+1-\alpha]} + \frac{\mu_\lambda t^{\nu-\beta-\lambda}}{\omega^4}\left(\frac{t}{\tau}\right)^{-\lambda}\sum_{m=0}^{\infty}\frac{(m+1)(-\mu_\lambda\omega^{-2}t^{\nu-\beta-\lambda})^m}{\Gamma[(\nu-\beta-\lambda)(m+1)+1-\lambda]} + \cdots. \tag{27}$$

In the same way, we obtain for $H(t)$ for $t \gg 1$ that

$$H(t) \approx \frac{t^{-1}}{\omega^2}\sum_{m=1}^{\infty}\frac{(-\mu_\lambda \omega^{-2}t^{\nu-\beta-\lambda})^m}{\Gamma[(\nu-\beta-\lambda)m]} - \frac{t^{-\alpha-1}}{\omega^4}\sum_{m=0}^{\infty}\frac{(m+1)(-\mu_\lambda \omega^{-2}t^{\nu-\beta-\lambda})^m}{\Gamma[(\nu-\beta-\lambda)m-\alpha]} + \frac{\mu_\lambda t^{\nu-\beta-\lambda-1}}{\omega^4}\left(\frac{t}{\tau}\right)^{-\lambda}\sum_{m=0}^{\infty}\frac{(m+1)(-\mu_\lambda \omega^{-2}t^{\nu-\beta-\lambda})^m}{\Gamma[(\nu-\beta-\lambda)(m+1)-\lambda]} + \cdots, \quad (28)$$

and for $h(t)$,

$$h(t) \approx \frac{t^{-2}}{\omega^2}\sum_{m=1}^{\infty}\frac{(-\mu_\lambda \omega^{-2}t^{\nu-\beta-\lambda})^m}{\Gamma[(\nu-\beta-\lambda)m-1]} - \frac{t^{-\alpha-2}}{\omega^4}\sum_{m=0}^{\infty}\frac{(m+1)(-\mu_\lambda \omega^{-2}t^{\nu-\beta-\lambda})^m}{\Gamma[(\nu-\beta-\lambda)m-\alpha-1]} + \frac{\mu_\lambda t^{\nu-\beta-\lambda-2}}{\omega^4}\left(\frac{t}{\tau}\right)^{-\lambda}\sum_{m=0}^{\infty}\frac{(m+1)(-\mu_\lambda \omega^{-2}t^{\nu-\beta-\lambda})^m}{\Gamma[(\nu-\beta-\lambda)(m+1)-\lambda-1]} + \cdots. \quad (29)$$

At this point we must distinguish three cases.

(i) $\lambda+\beta-\nu>0$. In this case $t^{\nu-\beta-\lambda} \ll 1$ for $t \gg 1$ and the asymptotic expressions for the relaxation functions become

$$I(t) \approx \frac{1}{\omega^2} - \frac{\mu_\lambda}{\omega^4}\frac{\sin \pi(\lambda+\beta-\nu)}{\pi}\frac{\Gamma(\lambda+\beta-\nu)}{t^{\lambda+\beta-\nu}} - \frac{1}{\omega^4}\frac{\sin \pi\alpha}{\pi}\frac{\Gamma(\alpha)}{t^{\alpha}} + \cdots, \quad (30)$$

$$H(t) \approx \frac{\mu_\lambda}{\omega^4}\frac{\sin \pi(\lambda+\beta-\nu)}{\pi}\frac{\Gamma(\lambda+\beta-\nu+1)}{t^{\lambda+\beta-\nu+1}} + \frac{1}{\omega^4}\frac{\sin \pi\alpha}{\pi}\frac{\Gamma(\alpha+1)}{t^{\alpha+1}} + \cdots, \quad (31)$$

$$h(t) \approx -\frac{\mu_\lambda}{\omega^4}\frac{\sin \pi(\lambda+\beta-\nu)}{\pi}\frac{\Gamma(\lambda+\beta-\nu+2)}{t^{\lambda+\beta-\nu+2}} - \frac{1}{\omega^4}\frac{\sin \pi\alpha}{\pi}\frac{\Gamma(\alpha+2)}{t^{\alpha+2}} + \cdots. \quad (32)$$

We note, from Eqs. (7) and (9), that in the limit $t\to\infty$ we have $\langle x(\infty)\rangle=0$ and $\sigma_{xx}(\infty)=k_BT/\omega^2$.

(ii) $\lambda+\beta-\nu=0$. When this condition is satisfied we can calculate the sums in Eqs. (27)–(29) and obtain that

$$I(t) \approx \frac{1}{\omega^2+\mu_\lambda} - \frac{1}{(\omega^2+\mu_\lambda)^2}\frac{\sin \pi\alpha}{\pi}\frac{\Gamma(\alpha)}{t^{\alpha}} + \frac{\mu_\lambda \tau^\lambda}{(\omega^2+\mu_\lambda)^2}\frac{\sin \pi\lambda}{\pi}\frac{\Gamma(\lambda)}{t^{\lambda}} + \cdots, \quad (33)$$

$$H(t) \approx \frac{1}{(\omega^2+\mu_\lambda)^2}\frac{\sin \pi\alpha}{\pi}\frac{\Gamma(\alpha+1)}{t^{\alpha+1}} - \frac{\mu_\lambda \tau^\lambda}{(\omega^2+\mu_\lambda)^2}\frac{\sin \pi\lambda}{\pi}\frac{\Gamma(\lambda+1)}{t^{\lambda+1}} + \cdots, \quad (34)$$

$$h(t) \approx -\frac{1}{(\omega^2+\mu_\lambda)^2}\frac{\sin \pi\alpha}{\pi}\frac{\Gamma(\alpha+2)}{t^{\alpha+2}} + \frac{\mu_\lambda \tau^\lambda}{(\omega^2+\mu_\lambda)^2}\frac{\sin \pi\lambda}{\pi}\frac{\Gamma(\lambda+2)}{t^{\lambda+2}} + \cdots. \quad (35)$$

In this case, the limit $t\to\infty$ gives $\langle x(\infty)\rangle=x_0\mu_\lambda/(\omega^2+\mu_\lambda)$ and $\sigma_{xx}(\infty)=k_BT/\Omega^2$, where $\Omega=(\omega^2+\mu_\lambda)/\sqrt{\omega^2+2\mu_\lambda}$.

(iii) $\lambda+\beta-\nu<0$. In this case the first sum on the right hand side of Eq. (27) can be identified with $E_{\nu-\beta-\lambda,1}(-z)$ for $z=\mu_\lambda\omega^{-2}t^{\nu-\beta-\lambda}$. There is, however, a problem with identifying the second and third sums with $E^2_{\nu-\beta-\lambda,1-\alpha}(-z)$ and $E^2_{\nu-\beta-\lambda,1-\lambda}(-z)$, respectively. The problem is that $1-\alpha<0$ and $1-\alpha-\lambda<0$. However, when it comes to considering the asymptotic behavior, those functions behave like Mittag–Leffler functions. In fact, let us write

$$\sum_{m=0}^{\infty}\frac{(m+1)(-z)^m}{\Gamma(am-b)}=\sum_{m=0}^{N-1}\frac{(m+1)(-z)^m}{\Gamma(am-b)}+(-z)^N[E^2_{a,Na-b}(-z)+NE_{a,Na-b}(-z)],$$

where N is the smallest integer such that $Na-b>0$. Then, if we use Eq. (B5) for $E^2_{a,Na-b}(-z)$ and $E_{a,Na-b}(-z)$, we obtain, after some cancellations,

$$\sum_{m=0}^{\infty}\frac{(m+1)(-z)^m}{\Gamma(am-b)}=\sum_{k=0}^{\infty}(-1)^k k\frac{(-z)^{-(k+2)}}{\Gamma[-b-a(k+2)]},\quad |z|>1,$$

which is exactly the asymptotic expansion given by Eq. (B5) for $E^2_{a,-b}(-z)$.

Now, with these observations in mind, we obtain in this case

$$I(t)\approx\frac{1}{\mu_\lambda}\frac{\sin\pi(\nu-\beta-\lambda)}{\pi}\frac{\Gamma(\nu-\beta-\lambda)}{t^{\nu-\beta-\lambda}}+\cdots, \tag{36}$$

$$H(t)\approx-\frac{1}{\mu_\lambda}\frac{\sin\pi(\nu-\beta-\lambda)}{\pi}\frac{\Gamma(\nu-\beta-\lambda+1)}{t^{\nu-\beta-\lambda+1}}+\cdots, \tag{37}$$

$$h(t)\approx\frac{1}{\mu_\lambda}\frac{\sin\pi(\nu-\beta-\lambda)}{\pi}\frac{\Gamma(\nu-\beta-\lambda+2)}{t^{\nu-\beta-\lambda+2}}+\cdots. \tag{38}$$

The limit $t\to\infty$ gives in this case $\langle x(\infty)\rangle=x_0$ and $\sigma_{xx}(\infty)=0$.

It is interesting to note that when $\beta=1$ this case corresponds to $\nu-\lambda>1$ and then the asymptotic behavior of the frictional memory kernel $\mu(t)\approx(\mu_\lambda/\Gamma(\nu-\lambda))t^{\lambda-\nu-1}$ shows that it diverges when $t\to\infty$.

VI. PARTICULAR CASES

There is one very interesting case of the relaxation function. We can easily see in Eqs. (20)–(22) that the parameters ν and β appear in those equations in the form of the combination $\nu-\beta$. Therefore, when $\nu=\beta$, the relaxation functions do **not** depend on the order of the fractional derivative, β, or on the second parameter of the two-parameter Mittag–Leffler function ν. In this case, for the relaxation functions we have $\Delta_{m,n}=\alpha(m+n)$ in Eqs. (20)–(22).

The short-time regime of the relaxation functions in this case are

$$I(t)\approx\frac{t^\alpha}{\Gamma(\alpha+1)}-\frac{(\omega^2+\mu_\lambda/\tau^\lambda)t^{2\alpha}}{\Gamma(2\alpha+1)}\cdots, \tag{39}$$

$$H(t)\approx\frac{t^{\alpha-1}}{\Gamma(\alpha)}-\frac{(\omega^2+\mu_\lambda/\tau^\lambda)t^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots, \tag{40}$$

$$h(t)\approx\frac{t^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{(\omega^2+\mu_\lambda/\tau^\lambda)t^{2\alpha-2}}{\Gamma(2\alpha-1)}+\cdots. \tag{41}$$

For the asymptotic expansion, we note that since $\lambda>0$, the case $\nu=\beta$ must be a particular case of the situation where $\lambda+\beta-\nu>0$. Then we have

$$I(t)\approx\frac{1}{\omega^2}-\frac{\mu_\lambda}{\omega^4}\frac{\sin\pi\lambda}{\pi}\frac{\Gamma(\lambda)}{t^\lambda}-\frac{1}{\omega^4}\frac{\sin\pi\alpha}{\pi}\frac{\Gamma(\alpha)}{t^\alpha}+\cdots, \tag{42}$$

$$H(t) \approx \frac{\mu_\lambda}{\omega^4}\frac{\sin \pi\lambda}{\pi}\frac{\Gamma(\lambda+1)}{t^{\lambda+1}} + \frac{1}{\omega^4}\frac{\sin \pi\alpha}{\pi}\frac{\Gamma(\alpha+1)}{t^{\alpha+1}} + \cdots, \tag{43}$$

$$h(t) \approx -\frac{\mu_\lambda}{\omega^4}\frac{\sin \pi\lambda}{\pi}\frac{\Gamma(\lambda+2)}{t^{\lambda+2}} - \frac{1}{\omega^4}\frac{\sin \pi\alpha}{\pi}\frac{\Gamma(\alpha+2)}{t^{\alpha+2}} + \cdots. \tag{44}$$

Moreover, we note that when $\alpha=2$ the above expressions reduce to those ones obtained by Viñales *et al.*[24] We must observe, however, that in Ref. 24 the authors considered $\nu=\beta=1$.

Another interesting case is the limit $\tau\to 0$. In this case the correlation function given by Eq. (3) corresponds to the power law,

$$C_0(t) = C_\lambda \frac{t^{\nu-\lambda-1}}{\Gamma(\nu-\lambda)}, \quad (\tau \to 0),$$

and the expression for the relaxation functions,

$$I(t) = t^\alpha \sum_{n=0}^{\infty} (-\omega^2 t^\alpha)^n E^{n+1}_{\alpha-\lambda+\nu-\beta,\alpha(n+1)+1}[-\mu_\lambda t^{(\alpha-\lambda+\nu-\beta)}], \tag{45}$$

$$H(t) = t^{\alpha-1} \sum_{n=0}^{\infty} (-\omega^2 t^\alpha)^n E^{n+1}_{\alpha-\lambda+\nu-\beta,\alpha(n+1)}[-\mu_\lambda t^{(\alpha-\lambda+\nu-\beta)}], \tag{46}$$

$$h(t) = t^{\alpha-2} \sum_{n=0}^{\infty} (-\omega^2 t^\alpha)^n E^{n+1}_{\alpha-\lambda+\nu-\beta,\alpha(n+1)-1}[-\mu_\lambda t^{(\alpha-\lambda+\nu-\beta)}]. \tag{47}$$

These expressions generalize the ones given in Ref. 15.

Finally, let us consider $\omega\to 0$. In this limit we recover the results from Ref. 25,

$$I(t) = t^\alpha \sum_{m=0}^{\infty} \left(-\frac{\mu_\lambda}{\tau^\lambda} t^{\alpha+\nu-\beta}\right)^m E^m_{\lambda,\alpha+(\alpha+\nu-\beta)+1}[-(t/\tau)^\lambda], \tag{48}$$

$$H(t) = t^{\alpha-1} \sum_{m=0}^{\infty} \left(-\frac{\mu_\lambda}{\tau^\lambda} t^{\alpha+\nu-\beta}\right)^m E^m_{\lambda,\alpha+(\alpha+\nu-\beta)m}[-(t/\tau)^\lambda], \tag{49}$$

$$h(t) = t^{\alpha-2} \sum_{m=0}^{\infty} \left(-\frac{\mu_\lambda}{\tau^\lambda} t^{\alpha+\nu-\beta}\right)^m E^m_{\lambda,\alpha+(\alpha+\nu-\beta)m-1}[-(t/\tau)^\lambda], \tag{50}$$

and when $\omega\to 0$ and $\tau\to 0$ we generalize the results in Refs. 15 and 31–33,

$$I(t) = t^\alpha E_{\alpha-\lambda+\nu-\beta,\alpha+1}[-\mu_\lambda t^{(\alpha-\lambda+\nu-\beta)}], \tag{51}$$

$$H(t) = t^{\alpha-1} E_{\alpha-\lambda+\nu-\beta,\alpha}[-\mu_\lambda t^{(\alpha-\lambda+\nu-\beta)}], \tag{52}$$

$$h(t) = t^{\alpha-2} E_{\alpha-\lambda+\nu-\beta,\alpha-1}[-\mu_\lambda t^{(\alpha-\lambda+\nu-\beta)}]. \tag{53}$$

VII. CONCLUDING REMARKS

We conclude that Eqs. (20)–(22), our main results, determine the temporal evolution of the mean values (displacement and velocity) and the variances. We also note that in the limit $\omega \to 0$, our results recover the fractional free particle case[25] and for the integer case, i.e., integer derivatives, the results recover also a recent result obtained by Viñales *et al.*[24]

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APPENDIX A: CALCULATION OF RELAXATION FUNCTIONS

In Sec. IV, we have introduced a particular memory kernel given in terms of the two-parameter Mittag–Leffler function. To calculate the variances we need to know which is the corresponding relaxation function, for example. Here we show the calculation of the inverse Laplace transform associated with the functions $\hat{H}_0(s)$ and $\hat{H}_1(s)$ as given by Eq. (12).

Then, in general, we calculate the following inverse Laplace transform:

$$H_i(t) \equiv \mathfrak{L}^{-1}[\hat{H}_i(s)] = \mathfrak{L}^{-1}\left[\frac{s^{\sigma_i}}{s^{\alpha} + as^{\beta} + bs^{\gamma} + cs^{\mu} + d}\right],$$

with the restriction $\alpha > \beta > \gamma > \mu$, and i can take the values 0 and 1, such that $\sigma_0 = \lambda$ and $\sigma_1 = 0$. In our particular case the real parameters are as below

Exponents	Coefficients
$\alpha \to \lambda + \alpha$	$a \to 1/\tau^{\lambda}$
$\beta \to \alpha$	$b \to \mu_{\lambda}/\tau^{\lambda}$
$\gamma \to \beta + \lambda - \nu$	$c \to \omega^2$
$\mu \to \lambda$	$d \to \omega^2/\tau^{\lambda}$

We also note that, in the case $\omega^2 = 0$ (frequency zero), i.e., the fractional free case is recovered[25] and the integer case $\alpha = 2$ and $\beta = 1 = \nu$, discussed by Vinãles *et al.*[24] is also recovered.

Using the geometric series we can write

$$\frac{s^{\sigma_i}}{s^{\alpha} + as^{\beta} + bs^{\gamma} + cs^{\mu} + d} = \sum_{k=0}^{\infty} (-1)^k s^{\sigma_i - \beta - \beta k} \frac{(d + cs^{\mu} + bs^{\gamma})^k}{(s^{\alpha-\beta} + a)^{k+1}},$$

with $a \neq 0$ and $|(d + cs^{\mu} + bs^{\gamma})/(s^{\alpha-\beta} + a)| < 1$.

The binomial series allow us to write, in a convenient way, the above equation as follows:

$$\frac{s^{\sigma_i}}{s^{\alpha} + as^{\beta} + bs^{\gamma} + cs^{\mu} + d} = \sum_{k=0}^{\infty} (-1)^k b^k k! \sum_{\ell=0}^{k} \frac{(d/b)^{\ell}}{\ell!} \sum_{j=0}^{k-\ell} \frac{(c/b)^j}{j!\,(k-\ell-j)!} \Lambda_{\sigma_i},$$

where

$$\Lambda_{\sigma_i} \equiv \Lambda_{\sigma_i}(k,\ell,j;\alpha,\beta,\gamma,\mu) = \frac{s^{\sigma_i - \beta(k+1) + \mu j + \gamma(k-\ell-j)}}{(s^{\alpha-\beta} + a)^{k+1}}.$$

Next we apply the inverse Laplace transform in both sides of the last equation,

$$\mathfrak{L}^{-1}\left[\frac{s^{\sigma_i}}{s^{\alpha}+as^{\beta}+bs^{\gamma}+cs^{\mu}+d}\right]=\sum_{k=0}^{\infty}(-1)^k b^k k!\sum_{\ell=0}^{k}\frac{(d/b)^{\ell}}{\ell!}\sum_{j=0}^{k-\ell}\frac{(c/b)^j}{j!\,(k-\ell-j)!}\cdot\mathfrak{L}^{-1}\left[\frac{s^{\sigma_i-\beta(k+1)+\mu j+\gamma(k-\ell-j)}}{(s^{\alpha-\beta}+a)^{k+1}}\right].$$

To calculate the inverse Laplace transform in the second member, we use Eq. (10) and we get

$$\mathfrak{L}^{-1}\left[\frac{s^{\sigma_i}}{s^{\alpha}+as^{\beta}+bs^{\gamma}+cs^{\mu}+d}\right]=\sum_{k=0}^{\infty}(-1)^k b^k k!\sum_{\ell=0}^{k}\frac{(d/b)^{\ell}}{\ell!}\sum_{j=0}^{k-\ell}\frac{(c/b)^j}{j!\,(k-\ell-j)!}\cdot t^{\xi_i-1}E_{\alpha-\beta,\xi_i}^{k+1}(-at^{\alpha-\beta}),$$

where $\xi_i=-\sigma_i+\alpha+(\alpha-\gamma)k+\gamma\ell-(\mu-\gamma)j$ with $i=0,1$ and $E_{\mu,\nu}^{\rho}(\cdot)$ is the three-parameter Mittag–Leffler function.

Using the notation introduced in Eq. (12), we have

$$H_i(t)\equiv(\tau^{-\lambda})^i\mathfrak{L}^{-1}\left[\frac{s^{\sigma_i}}{s^{\lambda+\alpha}+\frac{1}{\tau^{\lambda}}s^{\alpha}+\frac{\mu_\lambda}{\tau^{\lambda}}s^{\beta+\lambda-\nu}+\omega^2 s^{\lambda}+\frac{\omega^2}{\tau^{\lambda}}}\right]$$
$$=\sum_{k=0}^{\infty}(-1)^k k!\left(\frac{\mu_\lambda}{\tau^{\lambda}}\right)^k\sum_{\ell=0}^{k}\frac{(\omega^2/\mu_\lambda)^{\ell}}{\ell!}\sum_{j=0}^{k-\ell}\frac{(\omega^2\tau^{\lambda}/\mu_\lambda)^j}{j!\,(k-\ell-j)!}t^{\xi_i-1}E_{\lambda,\xi_i}^{k+1}[-(t/\tau)^{\lambda}], \quad \text{(A1)}$$

where $\xi_i=-\sigma_i+\lambda+\alpha+(\alpha-\beta+\nu)k+(\beta+\lambda-\nu)\ell+(\beta-\nu)j$ and $i=0,1$.

To simplify we introduce a function time independent defined by the following expression:

$$\mathbf{C}(k,\ell,j)\equiv(-1)^k k!\left(\frac{\mu_\lambda}{\tau^{\lambda}}\right)^k\frac{(\omega^2/\mu_\lambda)^{\ell}}{\ell!}\frac{(\omega^2\tau^{\lambda}/\mu_\lambda)^j}{j!\,(k-\ell-j)!}, \quad \text{(A2)}$$

and then Eq. (A1) can be written as

$$H_i(t)=(\tau^{-\lambda})^i\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{k-\ell}\mathbf{C}(k,\ell,j)t^{\xi_i-1}E_{\lambda,\xi_i}^{k+1}[-(t/\tau)^{\lambda}], \quad \text{(A3)}$$

where $\xi_i=-\sigma_i+\lambda+\alpha+(\alpha-\beta+\nu)k+(\beta+\lambda-\nu)\ell+(\beta-\nu)j$ with $i=0,1$ and $\sigma_i=\lambda\delta_{i,0}$.

If we switch the order of the two last sums and define $m=j+\ell$, the above expression can be written as

$$H_i(t)=(\tau^{-\lambda})^i\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{j=0}^{m}\mathbf{C}'(k,m,j)t^{\xi_i-1}E_{\lambda,\xi_i}^{k+1}[-(t/\tau)^{\lambda}], \quad \text{(A4)}$$

where

$$\mathbf{C}'(k,m,j)\equiv\frac{(-1)^k k!\left(\frac{\mu_\lambda}{\tau^{\lambda}}\right)^k(\tau^{\lambda}\omega^2/\mu_\lambda)^m(\tau^{\lambda})^{-j}}{j!\,(m-j)!\,(k-m)!}. \quad \text{(A5)}$$

Now, let us write

$$\xi_i=\xi_i^0+\lambda j, \quad \text{(A6)}$$

where

$$\xi_i^0=-\lambda\delta_{i,0}+\lambda+\alpha+(\alpha-\beta+\nu)k+(\beta-\nu)m. \quad \text{(A7)}$$

Using this we can write

$$H_i(t) = (\tau^{-\lambda})^i \sum_{k-0}^{\infty} \sum_{m=0}^{k} \frac{(-1)^k k!}{(k-m)!} \left(\frac{\mu_\lambda}{\tau^\lambda}\right)^k \left(\frac{\tau^\lambda \omega^2}{\mu_\lambda}\right)^m t^{\xi_i^0 - 1} \cdot \left[\sum_{j=0}^{m} \frac{1}{j\,!\,(m-j)!} ((t/\tau)^\lambda)^j E^{k+1}_{\lambda,\xi_i^0+\lambda j}[-(t/\tau)^\lambda] \right]. \tag{A8}$$

The sum between brackets is the one given by Eq. (B3), and then

$$H_i(t) = (\tau^{-\lambda})^i \sum_{k-0}^{\infty} \sum_{m=0}^{k} (-1)^k \binom{k}{m} \left(\frac{\mu_\lambda}{\tau^\lambda}\right)^k \left(\frac{\tau^\lambda \omega^2}{\mu_\lambda}\right)^m t^{\xi_i^0 - 1} E^{k-m+1}_{\lambda,\xi_i^0}[-(t/\tau)^\lambda]. \tag{A9}$$

APPENDIX B: THREE-PARAMETER MITTAG–LEFFLER FUNCTION

Our objective in this appendix is to prove some results that we have used. Let $E^{\rho}_{\alpha,\beta}(z)$ be the three-parameter Mittag–Leffler function defined by the series representation,

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!},$$

with $z \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\rho) > 0$, and $(\cdot)_k$ is the Pochhammer symbol. Then the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{t^{\xi-1} E^{k+1}_{\lambda,\xi}[-(t/\tau)^\lambda]\} = t^{\xi-2} E^{k+1}_{\lambda,\xi-1}[-(t/\tau)^\lambda], \tag{B1}$$

$$\int_0^t x^{\xi-1} E^{k+1}_{\lambda,\xi}[-(x/\tau)^\lambda] \mathrm{d}x = t^{\xi} E^{k+1}_{\lambda,\xi+1}[-(t/\tau)^\lambda], \tag{B2}$$

$$\sum_{j=0}^{m} \frac{z^j}{j\,!\,(m-j)!} E^{\rho}_{\lambda,\lambda j+\beta}(-z) = \frac{1}{m!} E^{\rho-m}_{\lambda,\beta}(-z), \tag{B3}$$

$$E^{\rho}_{\alpha,\beta}(z) - E^{\rho-1}_{\alpha,\beta}(z) = z E^{\rho}_{\alpha,\beta+\alpha}(z), \tag{B4}$$

$$E^{\rho}_{\alpha,\beta}(-z) = \sum_{k=0}^{\infty} (-1)^k \frac{(\rho)_k}{\Gamma[\beta - \alpha(k+\rho)]} \frac{z^{-(k+\gamma)}}{k!}, \quad |z| > 1. \tag{B5}$$

Proof: Equations (B1) and (B2) follow immediately from the series representation of $E^{\rho}_{\alpha,\beta}(z)$.

In order to prove Eq. (B3), we start using the series representation of $E^{\rho}_{\alpha,\beta}(z)$ on left hand side, which will be denoted by Λ,

$$\Lambda = \sum_{j=0}^{m} \sum_{k=0}^{\infty} \frac{z^j}{j\,!\,(m-j)!} \frac{(\rho)_k}{\Gamma(\lambda k + \lambda j + \beta)} \frac{(-z)^k}{k!}.$$

Denoting $k+j=i$ and switching the order of the sums we have that

$$\Lambda = \sum_{i=0}^{\infty} \sum_{j=0}^{m} \frac{(-z)^i}{\Gamma(\lambda i + \beta)} \frac{(-1)^j}{j\,!\,(m-j)!} \frac{(\rho)_{i-j}}{(i-j)!},$$

where we have used the fact that $(i-j)!=0$ for $j=i+1, i+2, \ldots$. But

$$\frac{(\rho)_{i-j}}{(i-j)!} = \binom{i-j+\rho-1}{i-j} = \binom{i-j+\rho-1}{\rho-1},$$

and using the identity

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j}\binom{a-j}{b} = \binom{a-m}{b-m},$$

we have that

$$\Lambda = \frac{1}{m!}\sum_{i=0}^{\infty} \frac{(-z)^i}{\Gamma(\lambda i+\beta)}\binom{i+\rho-1-m}{\rho-1+m} = \frac{1}{m!}\sum_{i=0}^{\infty} \frac{(\rho-m)_i}{\Gamma(\lambda i+\beta)}\frac{(-z)^i}{i!},$$

which gives the right hand side of Eq. (B3).

Equation (B4) can be proven starting from the expression on its left hand side and using

$$(\rho)_k - (\rho-1)_k = k(\rho)_{k-1},$$

which gives the result after the identification of the resulting series. We must observe that in principle this result holds only for $\mathrm{Re}(\rho-1) > 0$; however, if we define

$$E^0_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)}, \tag{B6}$$

then we can consider that Eq. (B4) holds for $\mathrm{Re}(\rho) \geq 1$, which is the case considered in the text.

Finally, the proof of Eq. (B5) can be found in the appendix of Ref. 30.

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