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Stabilizing a class of time delay systems using the Hermite–Biehler theorem ☆

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Abstract

In this paper we use the Hermite–Biehler theorem to establish results for the design of fixed order controllers for a class of time delay systems. We extend results of the polynomial case to quasipolynomials using the property of interlacing in high frequencies of the class of time delay systems considered.

Keywords: Stability; Delay systems; Closed-loop control; Interlacing; Quasipolynomials

1. Introduction

The Hermite–Biehler theorem provides necessary and sufficient conditions for Hurwitz stability of real polynomials in terms of an interlacing property [1]. When a given real polynomial is not Hurwitz, the Hermite–Biehler theorem does not provide information on its roots distribution. A generalization of the Hermite–Biehler theorem on polynomials was first derived in a report by Ozguler and Koçan [2] where was

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given a formula for a signature of a polynomial not necessarily Hurwitz applicable to real polynomials with no imaginary axis roots except possibly a single root at the origin. This formula was used to solve the constant gain stabilization problem. Recently, in [3] a different formula applicable to arbitrary real polynomials but with no restrictions on root locations was derived and used in the problem of stabilizing proportional-integral-derivative (PID) controllers.

The dynamic behavior of many industrial plants may be mathematically described by a linear time invariant system with time delay. The problem of stability of linear time invariant systems with time delay involves finding the location of the roots of transcendental functions. An extension of the Hermite–Biehler theorem to cope with transcendental functions was first derived by Pontryagin [4]. In his paper are given necessary and sufficient conditions for the stability of transcendental functions of the form $H(s) = h(s, \mathrm{e}^s)$, where $h(s,t)$ is a polynomial in two variables. For the case $h(s,t)$ has a principal term he showed that the characteristics of the zeros of $H(s)$ are determined by the behavior of $H(\mathrm{j}\omega)$ with ω real. In [5] we can find a good review of stability methods for time delay systems. The classical method of stability verification is the Nyquist criterion which can be used to the class of time delay systems. However, for the analytical characterization of the stabilizing gains one may have to deal with non-linear inequalities.

In this paper we consider the problem of stabilizing a more general class of time delay systems using their property of interlacing in high frequencies. The well known fixed order controller structures such as the proportional and proportional integral controllers are considered to illustrate the applicability of the results presented.

1.1. Exponential polynomials and notation

In this section we study the zeros of exponential polynomials. Let $h(s,t)$ be a polynomial in s and t with constant, real or complex coefficients

$$h(s,t) = \sum_{m,n} a_{mn} s^m t^n. \tag{1}$$

The term $a_{pq}s^p t^q$ is called the principal term of the polynomial if $a_{pq} \neq 0$ where p and q the highest powers of s and t, respectively. For example $h(s,t) = s^3 t + st + s^2 + 1$ presents $p = 3$ and $q = 1$ and it has principal term $a_{pq} = 1$ but $h(s,t) = s^3 + s^2 t + t^2 + 1$ does not have the principal term.

Let $H(s) = h(s, \mathrm{e}^s)$. For $s = \mathrm{j}\omega$ with ω real, the function $H(\mathrm{j}\omega)$ can be written in terms of its real and imaginary parts, $H(\mathrm{j}\omega) = f_{\mathrm{r}}(\omega, \cos(\omega), \sin(\omega)) + \mathrm{j} f_{\mathrm{i}}(\omega, \cos(\omega), \sin(\omega))$, where $f_{\mathrm{r}}(\omega, u, v)$ and $f_{\mathrm{i}}(\omega, u, v)$ are polynomials with real and constant coefficients with respect to the variables ω, u and v.

Consider $f(s,u,v)$ a polynomial represented in the form

$$f(s,u,v) = \sum_{m,n} s^m \phi_m^{(n)}(u,v), \tag{2}$$

where $\phi_m^{(n)}(u, v)$ denotes the polynomial of degree n, which is homogeneous in u and v, that is, the sum of the exponents of u and v is n. Let $\phi_*^{(q)}(u, v)$ denote the coefficient of s^p in $f(s, u, v)$ so that

$$\phi_*^{(q)}(u, v) = \sum_{n \leqslant q} \phi_p^{(n)}(u, v) \tag{3}$$

and define $\Phi_*^{(q)}(s) = \phi_*^{(q)}(\cos s, \sin s)$. The notation given is clarified in the following example:

Example 1. Consider $H(s) = a\mathrm{e}^s + b - s\mathrm{e}^s$. For ω real we write $H(\mathrm{j}\omega)$ in terms of its real and imaginary parts $f_\mathrm{r}(\omega, u, v) = au + \omega v + b$ and $f_\mathrm{i}(\omega, u, v) = av - \omega u$, respectively, with $p = 1$, $q = 1$. Also, $f_\mathrm{r}(\omega, u, v)$ and $f_\mathrm{i}(\omega, u, v)$ have $\phi_*^{(1)}(u, v) = v$ and $\phi_*^{(1)}(u, v) = -u$, respectively.

Assuming $u = \cos(s)$ and $v = \sin(s)$ in (2) we define $F(s) = f(s, \cos(s), \sin(s))$. The result on the zeros of the function $F(s)$ due to Pontryagin is stated as follows [4,6]:

Theorem 1. *Let $f(s, u, v)$ be a polynomial with principal term $s^p\phi_p^{(q)}(u, v)$. If ϵ is such that $\Phi_*^{(q)}(\epsilon + \mathrm{j}\omega)$ does not take the value zero for real ω, then in the strip $-2\ell\pi + \epsilon \leqslant x \leqslant 2\ell\pi + \epsilon$, $s = x + \mathrm{j}\omega$ the function $F(s) = f(s, \cos(s), \sin(s))$ will have, for all sufficiently large values of ℓ, exactly $4q\ell + p$ zeros. Thus, in order for the function $F(s)$ to have only real roots, it is necessary and sufficient that in the interval $-2\ell\pi + \epsilon \leqslant x \leqslant 2\ell\pi + \epsilon$ it has exactly $4\ell q + p$ real roots starting with sufficiently large ℓ.*

Consider $F(\omega) = f(\omega, \cos(\omega), \sin(\omega))$ a entire transcendental function in the real argument ω, which assumes real values. In order to find the number of zeros of $F(\omega)$ in an interval, Theorem 1 is stated in the form that follows:

Theorem 2. *Consider real transcendental functions $f_\mathrm{r}(\omega, \cos(\omega), \sin(\omega))$ and $f_\mathrm{i}(\omega, \cos(\omega), \sin(\omega))$ such that $H(\mathrm{j}\omega) = f_\mathrm{r}(\omega, \cos(\omega), \sin(\omega)) + \mathrm{j} f_\mathrm{i}(\omega, \cos(\omega), \sin(\omega))$. Assume that $f_\mathrm{r}(\omega, u, v)$ and $f_\mathrm{i}(\omega, u, v)$ are polynomials with principal terms of the form $\omega^p\phi_p^{(q)}(u, v)$. Let η be an appropriate constant such that $\phi_*^{(q)}(u, v)$ in $f_\mathrm{r}(\omega, u, v)$ and $f_\mathrm{i}(\omega, u, v)$ do not vanish at $\omega = \eta$. Then for the equations $F_\mathrm{r}(\omega) = 0$ or $F_\mathrm{i}(\omega) = 0$ to have only real roots it is necessary and sufficient that in the interval $-2\pi\ell + \eta \leqslant \omega \leqslant 2\pi\ell + \eta$, $f_\mathrm{r}(\omega, u, v)$ and $f_\mathrm{i}(\omega, u, v)$ has exactly $4q\ell + p$ real roots starting with a sufficiently large ℓ.*

Following we illustrate the application of Theorem 2.

Example 2. Consider the Example 1 given before for $a < 1$ and $a < -b < (a_1^2 + a^2)^{1/2}$ where a_1 is the root of $\omega = a\tan(\omega)$ such that $0 < \omega < \pi$. If $a = 0$ we take $a_1 = \pi/2$. We claim that $F_\mathrm{i}(\omega) = a\sin(\omega) - \omega\cos(\omega) = 0$ has all its roots real and simple. In fact, from $F_{\mathrm{i}}(\omega) = 0$ we may write $\tan(\omega) = \omega/a$. Using Theorem 2 we can choose $\eta = 0$ since $\Phi_*^{(1)}(\omega) = -\cos(\omega) \neq 0$ for $\omega = \eta$. It is easy to see graphically that as $a < 1$, $F_{\mathrm{i}}(\omega)$ has $4\ell + 1$ real roots in the interval $-2\ell\pi \leqslant \omega \leqslant 2\ell\pi$, $\ell = 1, 2, \ldots$ Therefore, it follows that $F_{\mathrm{i}}(\omega)$ has only real roots.

2. The Hermite–Biehler theorem for a class of time delay systems

A linear time invariant system with delays has its characteristic function described by an entire function of the form

$$f(s) = \sum_{j=1}^{n} \mathrm{e}^{s\lambda_j} P_j(s), \tag{4}$$

where $P_j(s)$ for $j = 1, \ldots, n$ is an arbitrary polynomial in the complex variable s and the λ_j's are real numbers which satisfy,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n, \quad |\lambda_1| < \lambda_n. \tag{5}$$

Function (4) is called a quasipolynomial [7]. There are transcendental real functions $f_{\mathrm{r}}(\omega)$ and $f_{\mathrm{i}}(\omega)$ in the real variable ω associated with quasipolynomial (4) such that $f(\mathrm{j}\omega) = f_{\mathrm{r}}(\omega) + \mathrm{j} f_{\mathrm{i}}(\omega)$. From the used terminology, the zeros of the two real functions $f_{\mathrm{r}}(\omega)$ and $f_{\mathrm{i}}(\omega)$ alternate along the ω-axis if each of the functions has no multiple zeros, if between every two zeros of these functions there exists at least one zero of the other, and if the functions are never simultaneously equal to zero.

Another extension of the Hermite–Biehler was developed to study the stability of quasipolynomial (4) which is stated as follows [1]:

Theorem 3. *Under the above assumptions, the entire function $f(s)$ in* (4) *has all its zeros in the open left-half plane if and only if*

(i) *$f_{\mathrm{r}}(\omega)$ and $f_{\mathrm{i}}(\omega)$ have only real roots and these roots interlace,*
(ii) *$f_{\mathrm{i}}'(\omega^*) f_{\mathrm{r}}(\omega^*) - f_{\mathrm{i}}(\omega^*) f_{\mathrm{r}}'(\omega^*) > 0$ for some $\omega^* \in (-\infty, \infty)$(increasing phase condition),*

where $f_{\mathrm{r}}'(\omega)$ and $f_{\mathrm{i}}'(\omega)$ denote the first derivative with respect to ω of $f_{\mathrm{r}}(\omega)$ and $f_{\mathrm{i}}(\omega)$, respectively.

2.1. The interlacing property in high frequencies

Consider a special case of quasipolynomial (4)

$$\delta(s) = \mathrm{e}^{sT_M} d(s) + \sum_{k=1}^{M} \mathrm{e}^{s(T_M - T_k)} n_k(s), \tag{6}$$

where $d(s)$ and $n_k(s)$ for $k = 1, \dots, M$, are polynomials with real coefficients of the form

$$n_k(s) = b_{M_k} s^{M_k} + b_{M_k-1} s^{M_k-1} + \cdots + b_{0k}; \quad k = 1, \dots, M,$$
$$d(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_0$$

and

$$0 < T_1 < T_2 < \cdots < T_M. \tag{7}$$

We denote $\delta_{\mathrm{r}}(\omega)$ and $\delta_{\mathrm{i}}(\omega)$ the real and imaginary parts of the transcendental function associated with quasipolynomial (4), respectively.

We make the following assumptions:

A1. $M_k < n$ for $k = 1, \dots, M$.
A2. $d(s)$ and $n_k(s)$ in (6) for $k = 1, \dots, M$ are coprime polynomials.

Lemma 1. *Consider quasipolynomial* (6). *Under the assumption* A1 *above there exists* $0 < \omega_0 < \infty$ *such that* $\delta_{\mathrm{r}}(\omega)$ *and* $\delta_{\mathrm{i}}(\omega)$ *have only simple real roots and these roots interlace for* $\omega > \omega_0$.

Proof. Initially, we check the first part. We can write quasipolynomial (6) for $|s|$ large as follows:

$$\delta(s) \simeq \mathrm{e}^{sT_M} s^n + \sum_{k=1}^{M} b_{M_k} \mathrm{e}^{s(T_M - T_k)} s^{M_k}. \tag{8}$$

In fact, we have

$$\delta(s) = \mathrm{e}^{sT_M} s^n [1 + \varepsilon_0(s)] + \sum_{k=1}^{M} b_{M_k} \mathrm{e}^{s(T_M - T_k)} s^{M_k} [1 + \varepsilon_k(s)],$$

where

$$\varepsilon_0(s) := \frac{a_{n-1}}{s} + \cdots + \frac{a_0}{s^n}$$

and

$$\varepsilon_k(s) := \frac{1}{b_{M_k}} \left(\frac{b_{M_k-1}}{s} + \cdots + \frac{b_{0_k}}{s^{M_k}} \right), \quad k = 1, \dots, M \ [6].$$

Thus, $\varepsilon_k(s) \to 0$ as $|s| \to +\infty$, $k = 0, 1, \dots, M$. We can hence suppose that the zeros of $\delta(s)$ and the zeros of $\mathrm{e}^{sT_M} s^n + \sum_{k=1}^{M} b_{M_k} \mathrm{e}^{s(T_M - T_k)} s^{M_k}$ are close together for $|s|$ large. From (8) we have $\delta(s) \simeq \mathrm{e}^{sT_M} s^n [1 + \varepsilon^*(s)]$ with $\varepsilon^*(s) = \sum_{k=1}^{M} (b_{M_k} / \mathrm{e}^{sT_k} s^{n-M_k})$.

Substituting $s = \mathrm{j}\omega$ and using the fact that $\varepsilon^*(\mathrm{j}\omega) \to 0$ as $w \to \infty$ we can write

$$\delta(\mathrm{j}\omega) = \mathrm{e}^{\mathrm{j}\omega T_M} (\mathrm{j}\omega)^n \quad \text{for } \omega > \omega_0;\ \omega_0 \text{ sufficiently large.} \tag{9}$$

Expanding the exponential term we can write

$$\delta(\mathrm{j}\omega) = (\cos(\omega T_M) + \mathrm{j}\sin(\omega T_M))(\mathrm{j}\omega)^n.$$

We now assume n even. Then

$$\delta_{\mathrm{r}}(\omega) = (-1)^{n/2}\cos(\omega T_M)\omega^n, \tag{10}$$

$$\delta_{\mathrm{i}}(\omega) = (-1)^{n/2}\sin(\omega T_M)\omega^n. \tag{11}$$

Condition (i) of Lemma 1, as in [8], can be checked by using the Theorem 2, that is, the solutions of $\delta_{\mathrm{r}}(\omega)$ and $\delta_{\mathrm{i}}(\omega)$ are real roots and interlace. In fact, replacing $\omega_1 = \omega T_M$ in (10) and (11) and defining $u = \cos(\omega_1)$ and $v = \sin(\omega_1)$ we can write

$$\delta_{\mathrm{r}}(\omega_1, u, v) = \frac{(-1)^{n/2}}{T_M^n} u\omega_1^n, \tag{12}$$

$$\delta_{\mathrm{i}}(\omega_1, u, v) = \frac{(-1)^{n/2}}{T_M^n} v\omega_1^n. \tag{13}$$

The principal terms of $\delta_{\mathrm{r}}(\omega_1)$ and $\delta_{\mathrm{i}}(\omega_1)$ are $\omega_1^n\phi_p^{(q)}(u, v) = \omega_1^n((-1)^{n/2}/T_M^n)u$ and $\omega_1^n((-1)^{n/2}/T_M^n)v$, respectively, and $p = n$ and $q = 1$ in both functions. Next, we choose $\eta = \pi/4$ since $\phi_*^{(1)}(\pi/4) \neq 0$. It is easy to see that $\phi_p^{(q)}(\omega_1)$ in (12) has four real roots in the interval $[-2\pi + \pi/4, 2\pi + \pi/4]$. Also, the root of ω_1^n is zero with multiplicity n. Hence, it follows that $\delta_{\mathrm{r}}(\omega_1)$ has $4 + n$ real roots in the interval $[-2\pi + \pi/4, 2\pi + \pi/4]$. Consequently, $\delta_{\mathrm{r}}(\omega_1)$ has $4\ell + n$ roots in the interval $[-2\pi\ell + \pi/4, 2\pi\ell + \pi/4]$ for $\ell = 1, 2, \ldots$ Using Theorem 2 we have that $\delta_{\mathrm{i}}(\omega_1)$ has only real roots. We can use the same reasoning to show that this result is valid for $\delta_{\mathrm{i}}(\omega_1)$. Observe that both $\delta_{\mathrm{r}}(\omega_1)$ and $\delta_{\mathrm{i}}(\omega_1)$ have only simple real roots except at the origin.

The solutions of $\delta_{\mathrm{i}}(\omega_1) = 0$ are given by $\omega = 0$ with multiplicity $n + 1$ and $\omega_1 = \ell\pi$, $\ell = 1, 2, \ldots$ and the solutions $\delta_{\mathrm{r}}(\omega_1) = 0$ are given by $\omega_1 = 0$ with multiplicity n and $\omega_1 = (2\ell + 1)\pi/2$, $\ell = 0, 1, 2, \ldots$ Therefore, we have that $\delta_{\mathrm{r}}(\omega_1)$ and $\delta_{\mathrm{i}}(\omega_1)$ interlace for $\omega_1 > \omega_0$ with ω_0 sufficiently large.

Similarly, the results can be verified for n odd. □

Remark 1. As a result of Lemma 1 we have that for $\omega > \omega_0$, ω_0 sufficiently large, under assumption A1 the quasipolynomial (6) interlaces being it stable or not.

3. Stabilization of a class of time delay systems

In this section, we first state a generalization of the Hermite–Biehler theorem for polynomials derived in [9] which is in the sequence used to establish results for the design of fixed order controllers for a class of time delay systems.

Let $P(\mathrm{j}\omega)$ be a given real polynomial with degree n and $P_f(\mathrm{j}\omega)$ the normalized plot by $1/f(\omega)$ where $f(\omega) = (1+\omega^2)^{n/2}$. Write

$$P_f(\mathrm{j}\omega) = p_f(\omega) + \mathrm{j}q_f(\omega), \tag{14}$$

where $p_f(\omega)$ and $q_f(\omega)$ denote the real and imaginary parts of $P_f(\omega)$.

Theorem 4. *Let $P(s)$ be a given real polynomial of degree n with no* jω *axis roots except for possibly one at the origin. Let $0 = \omega_{o_0} < \omega_{o_1} < \omega_{o_2} < \cdots < \omega_{o_{m-1}}$ be the real, non-negative, distinct finite zeros of $q_f(\omega)$ with odd multiplicities. Also define $\omega_m = \infty$. Then*

$$\sigma(P) = \begin{cases} \{\mathrm{sgn}[p_f(\omega_{o_0})] - 2\mathrm{sgn}[p_f(\omega_{o_1})] + 2\mathrm{sgn}[p_f(\omega_{o_2})] + \cdots + \\ (-1)^{m-1}2\mathrm{sgn}[p_f(\omega_{o_{m-1}})] + (-1)^m \mathrm{sgn}[p_f(\omega_{o_m})]\} \\ \times(-1)^{m-1}\mathrm{sgn}[q(\infty)] \quad \textit{if } n \textit{ is even} \\ \{\mathrm{sgn}[p_f(\omega_{o_0})] - 2\mathrm{sgn}[p_f(\omega_{o_1})] + 2\mathrm{sgn}[p_f(\omega_{o_2})] + \cdots + \\ (-1)^{m-1}2\mathrm{sgn}[p_f(\omega_{o_{m-1}})]\} \times (-1)^{m-1}\mathrm{sgn}[q(\infty)] \quad \textit{if } n \textit{ is odd}, \end{cases} \tag{15}$$

where $\sigma(P) :=$ number of open left half plane zeros of $P(s)$ $-$ number of open right half plane zeros of $P(s)$, $\sigma(P)$ denotes the signature of $P(s)$.

Remark 2. In Theorem 4 if the polynomial is Hurwitz stable $\sigma(P) = n$.

Let $\Delta_0^\infty\theta$ denote the net change in the argument $\theta(\omega) := \arctan[q(\omega)/p(\omega)]$ as ω increases from 0 to ∞. Then, we can state the following lemma by Gantmacher [10]:

Lemma 2. *Let $P(s)$ be a real polynomial with no imaginary axis roots. Then $\Delta_0^\infty\theta = (\pi/2)\sigma(P)$.*

We consider now the quasipolynomial (6) under assumptions A1 and A2. We shall analyze the roots of $\delta(\mathrm{j}\omega)$ in the frequency range determined by a sufficiently large ω_0 such that the roots of $\delta_\mathrm{r}(\omega)$ and $\delta_\mathrm{i}(\omega)$ interlace. Differing from the polynomials, the quasipolynomials have infinite roots. The results given in [9] deals with polynomials and make use of the number of roots to establish a procedure to design fixed order controllers. There, the number of roots of the polynomial is related to the real and finite zeros of $p_f(\omega)$ and $q_f(\omega)$. We now extend the results of [9] on the signature of polynomials for a class of quasipolynomials.

Write $\delta(\mathrm{j}\omega) = p(\omega) + \mathrm{j}q(\omega)$. Let $0 = \omega_{o_0} < \omega_{o_1} < \omega_{o_2} < \cdots < \omega_{o_m} < \cdots$ and $\omega_{e_1} < \omega_{e_2} < \cdots < \omega_{e_r} < \cdots$ be real, distinct finite zeros of $q(\omega)$ and $p(\omega)$, respectively.

Definition 1. Let $m+1$ and r be the number of zeros of $q(\omega)$ and $p(\omega)$, respectively. For $m+r$ even we define $\omega_0 = \omega_{o_m}$ otherwise we define $\omega_0 = \omega_{e_r}$.

Using Lemma 1 and Definition 1 for a sufficiently large ω_0, we have the following pattern for the zeros of $q(\omega)$ and $p(\omega)$ in the range defined by ω_0. If $m+r$ is even then ω_{o_m} belongs to the range $[0, \omega_0]$ while ω_{e_r} does not belong. Otherwise, if $m+r$ is odd than ω_{o_m} and ω_{e_r} belong to the range $[0, \omega_0]$. This is a consequency of the interlacing property. We are now ready to state the following definition for the signature of a quasipolynomial:

Definition 2. Let $\delta(s)$ be a given quasipolynomial with no jω axis roots except for possibly one at the origin. For a sufficiently large ω_0 as in Definition 1 let $0 = \omega_{o_0} < \omega_{o_1} < \omega_{o_2} < \cdots < \omega_{o_m} \leqslant \omega_0$ and $\omega_{e_1} < \omega_{e_2} < \cdots < \omega_{e_r} \leqslant \omega_0$ be real, distinct finite zeros of $q(\omega)$ and $p(\omega)$, respectively. Then, a signature for $\delta(s)$ which we denote as $\sigma_q(\delta)$ may be given by

$$\sigma_q(\delta) = \begin{cases} \{\operatorname{sgn}[p(\omega_{o_0})] - 2\operatorname{sgn}[p(\omega_{o_1})] + 2\operatorname{sgn}[p(\omega_{o_2})] + \cdots + \\ (-1)^{m-1} 2\operatorname{sgn}[p(\omega_{o_{m-1}})] + (-1)^m \operatorname{sgn}[p(\omega_{o_m})]\} \\ \times (-1)^{m-1} \operatorname{sgn}[q(\omega^+_{o_{m-1}})] \quad \text{if } m+r \text{ is even}, \\ \{\operatorname{sgn}[p(\omega_{o_0})] - 2\operatorname{sgn}[p(\omega_{o_1})] + 2\operatorname{sgn}[p(\omega_{o_2}) + \cdots + \\ (-1)^m 2\operatorname{sgn}[p(\omega_{o_m})]\} \times (-1)^m \operatorname{sgn}[q(\omega^+_{o_m})] \quad \text{if } m+r \text{ is odd}, \end{cases} \tag{16}$$

where $\sigma_q(\delta)$ is the counterpart of the signature of polynomials for the quasipolynomial case.

Remark 3. The frequency ω_0 defined as in Definition 1 is chosen such that there is interlacing after $\omega > \omega_0$.

Lemma 3. *Consider a stable Hurwitz quasipolynomial $\delta(s)$ under assumptions* A1 *and* A2. *Let m and r be as already defined. Then the frequency range signature for $\delta(s)$ is given by $\sigma_q(\delta) = m + r$.*

Proof. Using the interlacing property of Hurwitz quasipolynomial established in Theorem 3 the results follow by Definition 2. □

This signature $\sigma_q(\delta)$ shall be used in the establishment of a feasible string set for the stabilization problem. Let $q(\omega)$ and $p(\omega)$ be as before and $0 = \omega_{o_0} < \omega_{o_1} < \omega_{o_2} < \cdots < \omega_{o_m} \leqslant \omega_0$ be real, distinct finite zeros of $q(\omega)$. Let $\Delta_0^{\omega_0}\theta$ denote the net change in the argument $\theta(\omega)$ as ω increases from 0 to ω_0. Similarly to the polynomial case [3], we have the following:

1. If ω_{o_i} and $\omega_{o_{i+1}}$ are both zeros of $q(\omega)$ in the range $[0, \omega_0]$, then
$$\Delta_{\omega_{o_i}}^{\omega_{o_{i+1}}}\theta = \frac{\pi}{2}[\operatorname{sgn}[p(\omega_{o_i})] - \operatorname{sgn}[p(\omega_{o_{i+1}})]]\operatorname{sgn}[q(\omega^+_{o_i})] \tag{17}$$
2. If ω_{o_i} is a zero of $q(\omega)$ in the range $[0, \omega_0]$ while $\omega_{o_{i+1}}$ is not, then
$$\Delta_{\omega_{o_i}}^{\omega_{o_{i+1}}}\theta = \frac{\pi}{2}[\operatorname{sgn}[p(\omega_{o_i})]\ \operatorname{sgn}[q(\omega^+_{o_i})]. \tag{18}$$

3. For $i = 0, 1, 2, \ldots, m-1$ we have
$$\operatorname{sgn}[q(\omega^+_{o_{i+1}})] = -\operatorname{sgn}[q(\omega^+_{o_i})]. \tag{19}$$

Consider the case $m + r$ is even. Using Definition 1 for a sufficiently large ω_0, we can write
$$\Delta_0^{\omega_0}\theta = \sum_{i=0}^{m-1} \Delta_{\omega_{o_i}}^{\omega_{o_{i+1}}}\theta, \tag{20}$$
where $\omega_0 = \omega_{o_m}$. Hence, substituting (17) into (20) and using (19) we obtain
$$\begin{aligned}\sum_{i=0}^{m-1} \Delta_{\omega_{o_i}}^{\omega_{o_{i+1}}}\theta = \sum_{i=0}^{m-1} &\frac{\pi}{2}\operatorname{sgn}[p(\omega_{o_i}) - \operatorname{sgn}[p(\omega_{o_{i+1}})]] \\ &\times (-1)^{m-1-i}\operatorname{sgn}[q(\omega^+_{o_{m-1}})].\end{aligned} \tag{21}$$

Consider now the case in which $m + r$ is odd. Again using Definition 1 for a sufficiently large ω_0, we can write
$$\Delta_0^{\omega_0}\theta = \sum_{i=0}^{m-1} \Delta_{\omega_{o_i}}^{\omega_{o_{i+1}}}\theta + \Delta_{\omega_{om}}^{\omega_0}\theta, \tag{22}$$
where $\omega_0 = \omega_{e_r}$. Substituting (18) in (22) and again using (19) we obtain
$$\begin{aligned}\Delta_0^{\omega_0}\theta = &\sum_{i=0}^{m-1} \frac{\pi}{2}\operatorname{sgn}[p(\omega_{o_i}) - \operatorname{sgn}[p(\omega_{o_{i+1}})]](-1)^{m-i}\operatorname{sgn}[q(\omega^+_{o_m})] \\ &+ \frac{\pi}{2}\operatorname{sgn}[p(\omega_{o_m})]\operatorname{sgn}[q(\omega^+_{o_m})].\end{aligned} \tag{23}$$

Now using Definition 2 we obtain
$$\Delta_0^{\omega_0}\theta = \frac{\pi}{2}\sigma_q(\delta). \tag{24}$$

In what follows we consider the problem of obtaining a family of stabilizing constant gain controllers for time delay systems with a single input delay. The feedback system characteristic function is given by
$$\delta^*(s, K) = d(s) + K\mathrm{e}^{-sT}n_1(s), \tag{25}$$
where K is a scalar, $d(s)$ and $n_1(s)$ are as already defined. Assuming $T > 0$ we obtain a quasipolynomial of the form of (6) with the same zeros of (25)
$$\delta(s, K) = \mathrm{e}^{sT}d(s) + Kn_1(s).$$
In the sequence we drop the subscript of $n_1(s)$ and again we consider $\delta(s, K)$ under assumptions A1 and A2. In the stabilization problem occurs the special class of quasipolynomial which is of the form $\delta(s, K)n(-s)$. Lemma 4 below gives a

frequency range signature for this product which is used to establish Theorem 5 on the stabilization problem. For a stabilizing $K = \overline{K}$ we can associate to m and r the number of zeros of $\delta(s, K)$ in the frequency range determined by frequency ω_0, using the Hermite–Biehler theorem. Thus, the product $\delta(s, K)n(-s)$ introduces a finite number of zeros in the frequency range considered. We denote m' the degree of $n(-s)$.

Lemma 4. *Let m and r define the real, distinct and finite zeros of the imaginary and real part of $\delta(\mathrm{j}\omega, \overline{K})$, respectively, for a stabilizing $\overline{K}$ and a sufficiently large frequency ω_0 defined as before. Then $\delta(s, K)$ is Hurwitz stable if and only if the signature for $\delta(s, K)n(-s)$ determined by the frequency ω_0 is given by $m + r - \sigma(n)$.*

Proof. The net change in the argument of $\delta(\mathrm{j}\omega, \overline{K})n(-\mathrm{j}\omega)$ is given by $\Delta_0^{\omega_0}\theta_{\delta n} = \Delta_0^{\omega_0}\theta_\delta + \Delta_0^{\omega_0}\theta_n$. Using (24) and (2) it yields $\Delta_0^{\omega_0}\theta_{\delta n} = (\pi/2)[\sigma_q(\delta) - \sigma(n)]$. Since $\delta(s, \overline{K})$ is Hurwitz stable by Lemma 3 we find that in the frequency range determined by ω_0, the frequency range signature of $\delta(s, K)n(-s)$ is $m + r - \sigma(n)$. It is easy to show the converse; that is, if the signature for $\delta(s, K)n(-s)$ is given by $m + r - \sigma(n)$ then $\delta(s, K)$ is Hurwitz. To use contradiction, suppose there exists a K^* that instabilizes $\delta(s, K)$ but yields the same $m + r$ in the interval $[0, \omega_0]$ considered. This implies that we have the same net change $\Delta_0^{\omega_0}\theta_\delta$ for both stable and unstable case of the quasipolynomial, which is an absurd. $\square$

Based on the results given in [9] we introduce the following definition:

Definition 3. Let $0 = \omega_{o_0} < \omega_{o_1} < \omega_{o_2} < \cdots < \omega_{o_i} < \omega_0$ be real, distinct finite zeros of $q(\omega)$. Then the set of strings A_I in a frequency range determined by the frequency ω_0 is defined as

$$A_\mathrm{I} = \{z_0, z_1, \ldots, z_i\},$$

where $z_0 \in \{-1, 0, 1\}$ and $z_t \in \{-1, 1\}$ for $t \neq 0$.

Theorem 5. *Consider $q(\omega)$ and $p(\omega)$ as real and imaginary parts of $\delta(\mathrm{j}\omega, K)n(-\mathrm{j}\omega)$ and the quasipolynomial $\delta(s, K)$ as already defined. Suppose there is a stabilizing $\overline{K}$. Let $0 = \omega_{o_0} < \omega_{o_1} < \omega_{o_2} < \cdots < \omega_{o_i} \leqslant \omega_0$ be real, distinct finite zeros of $q(\omega)$ in a frequency range. Also, choose ω_0 as in Definition 1 associated to $\delta(s, \overline{K})$. Then the set of all K such that $\delta(s, K)$ is Hurwitz may be obtained using the following expression for the signature of $\delta(s, K)n^*(s)$:*

$$\sigma_q(\delta(s, K)n^*(s)) = \begin{cases} \{z_0 - 2z_1 + 2z_2 + \cdots + (-1)^{i-1}2z_{i-1} + (-1)^i z_i\} \\ \times(-1)^{i-1}\operatorname{sgn}[q(\omega_{o_{i-1}}^+)] \quad \textit{if } m + r + m' \textit{ is even} \\ \{z_0 - 2z_1 + 2z_2 + \cdots + (-1)^i 2z_i\} \\ \times(-1)^i\operatorname{sgn}[q(\omega_{o_i}^+)] \quad \textit{if } m + r + m' \textit{ is odd} \end{cases}$$

and

$$K = \cup K_r,$$

where $\{z_0, z_1, \ldots\} \in A_{\mathrm{I}}$ *such that*

$$\max_{z_t \in A_{\mathrm{I}}, z_t.\mathrm{sgn}[p_2(\omega_t)]=1} \left[-\frac{1}{G(\mathrm{j}\omega_t)} \right] < \min_{z_t \in A_{\mathrm{I}}, z_t.\mathrm{sgn}[p_2(\omega_t)]=-1} \left[-\frac{1}{G(\mathrm{j}\omega_t)} \right],$$

$$K_r = \left(\max_{z_t \in A_{\mathrm{I}}, z_t.\mathrm{sgn}[p_2(\omega_t)]=1} \left[-\frac{1}{G(\mathrm{j}\omega_t)} \right], \min_{z_t \in A_{\mathrm{I}}, z_t.\mathrm{sgn}[p_2(\omega_t)]=-1} \left[-\frac{1}{G(\mathrm{j}\omega_t)} \right] \right)$$

with r *the number of feasible strings*, $\delta(\mathrm{j}\omega, K)n^*(s) = p_1(\omega) + Kp_2(\omega) + \mathrm{j}q(\omega)$, $n^*(s) = n(-s)$, m' *is the degree of* $n(s)$ *with* $\sigma_q(\delta(s, K)n(-s))$ *given by* $m + r - m'$.

Proof. Considering the frequency range determined by ω_0 and using Lemma 4, the proof follows the same lines as for the polynomial case given in [9]. □

We now present an example to illustrate the application of Theorem 5.

Example 3. Consider the stabilization of a given time delay system using a proportional controller.

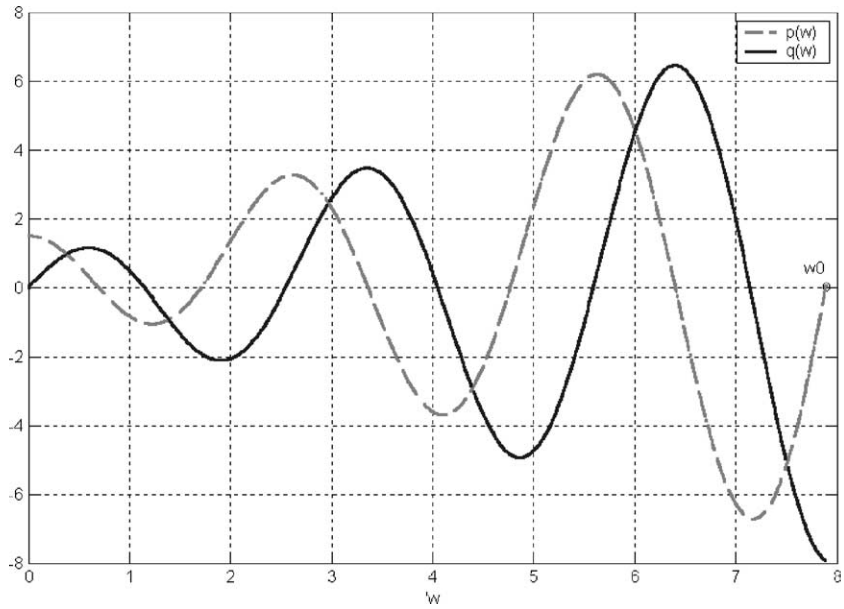

Fig. 1. Plots of $p(\omega)$ and $q(\omega)$ for $\delta(s, \overline{K})$ with $\overline{K} = 1/2$.

Case 1: The system is a first order plus a single time delay in the input as in [8]

$$G(s) = \frac{\mathrm{e}^{-2s}}{s+1}.$$

In this case $n(s) = 1$, $d(s) = s + 1$ and $T = 2$. First we encounter a value $\overline{K}$ which stabilizes the feedback system. Using Nyquist criterion we choose $\overline{K} = 1/2$ and write the system characteristic function as $\delta(\mathrm{j}\omega, \overline{K}) = p(\omega) + \mathrm{j}q(\omega)$, where $p(\omega) = 1/2 + \cos(2\omega) - \omega \sin(2\omega)$ and $q(\omega) = \omega \cos(2\omega) + \sin(2\omega)$. The plot $p(\omega)$ and $q(\omega)$ of $\delta(\mathrm{j}\omega, \overline{K})$ is shown in Fig. 1.

Now, we choose $m = 6$ and $r = 5$ which yields $\omega_0 = \omega_{e5} = 7.89$ by Definition 1 and $\sigma_q(\delta) = 11$. The zeros of $q(\omega)$ are found as $\omega_{o_0} = 0$, $\omega_{o_1} = 1.13$, $\omega_{o_2} = 2.53$, $\omega_{o_3} = 4.05$, $\omega_{o_4} = 5.60$, $\omega_{o_5} = 7.10$. We now obtain the string $A_{\mathrm{I}} = \{1, -1, 1, -1, 1, -1\}$ as in Definition 3 which must satisfy $z_0 - 2z_1 + 2z_2 - 2z_3 + 2z_4 - 2z_5 = 11$. Hence from Theorem 5, we have the set of stabilizing gains $K \in [-1, 1.51]$.

Case 2: The system is a third order plus a single time delay in the input

$$G(s) = \frac{\mathrm{e}^{-s}(s+0.2)(s+4)}{(s+1)(s+30)(s+0.5)}.$$

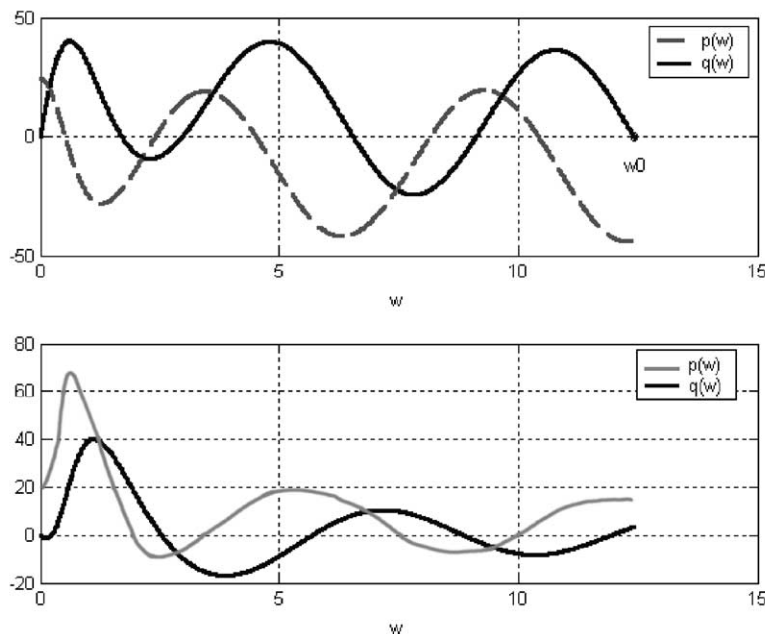

Fig. 2. The plots of $p(\omega)$ and $q(\omega)$ for $\delta(s, \overline{K})$ (upper) and $\delta(s, \overline{K})n(-s)$ (lower) with $\overline{K} = 12$.

In this case $n(s) = (s + 0.2)(s + 4)$, $d(s) = (s + 1)(s + 30)(s + 0.5)$ and $T = 1$. Again, we use the Nyquist criterion and choose $\overline{K} = 12$. Now, we choose $m = 5$ and $r = 5$ yielding $\sigma_q(\delta) = 10$ with $\omega_0 = \omega_{o5} = 12.44$. We obtain $m + r + \deg(n(s)) = 12$. Thus, we must have $\sigma_q(\delta(s, K)n(-s)) = m + r - \sigma(n) = 8$. Writing $\delta(\mathrm{j}\omega, \overline{K})n(-\mathrm{j}\omega) = p(\omega) + \mathrm{j}q(\omega)$ we find the zeros of $q(\omega)$ as $\omega_{o_0} = 0$, $\omega_{o_1} = 0.23$, $\omega_{o_2} = 2.53$, $\omega_{o_3} = 5.70$, $\omega_{o_4} = 8.84$, $\omega_{o_5} = 11.95$. Fig. 2 presents the plots $p(\omega)$ and $q(\omega)$ of both $\delta(s, \overline{K})$ and $\delta(s, \overline{K})n(-s)$. To find the zeros of $q(\omega)$ one can use the function *fzero* of the Matlab. We now find the string $A_{\mathrm{I}} = \{-1, -1, 1, -1, 1, -1\}$ which satisfies $z_0 - 2z_1 + 2z_2 - 2z_3 + 2z_4 - z_5 = 8$ and Theorem 5. Hence, we have the set of stabilizing gains as $K \in [-13.87, 17.58]$.

Both results in Example 3 can be checked using the Nyquist criterion via the Nyquist plot of $G(s)$. However, we give in this paper an analytical characterization of all the stabilizing constant gains and this is useful to the design of optimal solutions considering various performance criteria such as the H_2 and H_∞ norms of certain closed loop transfer functions.

The development of the PI and PID controllers follows the same lines given in [9] but with the new characterization of a signature for $\delta(s)$ as in (16) and will be addressed in a future publication.

4. Conclusion

In this paper we extend results of stabilization of linear time invariant systems to a class of time delay systems using the Hermite–Biehler Theorem. We derived a signature for the quasipolynomial case which was used in the problem of stabilizing constant gain controllers. The constant gain design presented gives an analytical characterization of all the stabilizing constant gains.

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