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Rodrigo Donizete Euzébio

Estudo de conjuntos minimais para sistemas descontínuos em dimensões 2 e 3

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, Área de Concentração - Geometria e Topologia, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Campus de São José do Rio Preto.

Orientador: Prof. Dr. Claudio Aguinaldo Buzzi

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Aos meus pais, Ourivaldo e Maria, dedico.

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"The measure of our intellectual capacity is the capacity to feel less and less satisfied with our answers to better and better problems."

C.W. Churchmann

RESUMO

Nesta tese são estudados conjuntos minimais de campos de vetores suaves e descontínuos em dimensões 2 e 3. Primeiramente, restringimos o estudos de conjuntos minimais a ciclos limite e respondemos questões sobre existência, distribuição e quantidade de tais objetos em campos de vetores suaves e descontínuos em dimensão 3. Posteriormente, abordamos a existência de conjuntos minimais não triviais e caos em dimensão 2 para campos de vetores descontínuos. Apresentamos exemplos de conjuntos minimais não triviais e verificamos a presença de caos não determinístico em alguns destes conjuntos. Finalmente, apresentamos uma versão do Teorema de Poincaré-Bendixson para campos de vetores descontínuos que não apresentam regiões de deslize e escape.

Palavras-chave: Campos de vetores descontínuos. Conjuntos minimais. Caos não determinístico. Teorema de Poincaré-Bendixson.

ABSTRACT

In this thesis minimal sets of smooth and non-smooth vector fields in dimension 2 and 3 are studied. First the study of minimal sets is restricted to limit cycles. Questions about existence, distribution and quantity of such objects in smooth and non-smooth vector fields in dimension 3 are answered. Later, the existence of non-trivial minimal sets and chaos in dimension 2 is treated for non-smooth vector fields. Some examples of non-trivial minimal sets are presented and the presence of non-deterministic chaos on some of these sets is verified. Finally, a version of the Poincaré-Bendixson Theorem for non-smooth vector fields presenting neither escaping nor sliding motion is presented.

Keywords: Non-smooth vector fields. Minimal sets. Non-deterministic chaos. Poincaré-Bendixson Theorem.

Lista de Figuras

Table of Contents

Introdução

Conjuntos minimais são importantes objetos da teoria clássica dos sistemas dinâmicos e tem sido estudados exaustivamente ao longo das últimas décadas. Em particular, dentro do contexto de fluxo de campos de vetores planares, conjuntos minimais são parte essencial dos conjuntos limite e são descritos pelo clássico Teorema de Poincaré-Bendixson. Em um contexto mais amplo, contudo, conjuntos minimais de sistemas suaves podem ser toros com fluxo irracional ou mesmo outros conjuntos mais abstratos caracterizados principalmente pela presença de recorrência não trivial. A existência de tais objetos é garantida pelo Lema de Zorn (veja, por exemplo, [53]), que permite concluir que todo conjunto compacto e invariante pelo fluxo de um dado sistema possui um (sub)conjunto minimal. Não obstante, para campos de vetores de classe $\mathcal{C}^2$ definidos sobre certas variedades, o Teorema de Denjoy-Schwartz caracteriza os conjuntos minimais: eles são pontos de equilíbrio, órbitas periódicas e toros, e são comumente chamados de *conjuntos minimais triviais*.

No contexto dos conjuntos minimais triviais, encontrar *ciclos limite*, isto é, orbitas periódicas isoladas do conjunto das órbitas periódicas do sistema considerado, é um dos tópicos mais abordados da teoria qualitativa dos sistemas dinâmicos. Ciclos limite são importantes não apenas como um significativo ente matemático dentro da teoria de sistemas dinâmicos senão também nas aplicações de caracter prático. De fato, ciclos limite hiperbólicos, principalmente aqueles que são estáveis, tem um papel muito importante em modelos reais no sentido que eles imprimem estabilidade em tais modelos. Não por acaso, encontrar tais objetos é o objetivo de um famoso problema dentro do universo matemático, a saber, o 16º Problema de Hilbert.

O 16º Problema de Hilbert, problema este apresentado em duas partes, foi elaborado pelo matemático alemão David Hilbert e apresentado junto a outros 22 problemas durante a *Conferência Internacional de Matemáticos de Paris*, ocorrida durante 1900. Embora existam conexões entre as duas partes do referido problema (mais especificamente, que trata dos ciclos limite algébricos), a segunda parte toca diretamente à teoria dos sistemas dinâmicos e versa sobre a cota superior e a posição relativa de ciclos limite em sistemas polinomiais planares de grau n (veja [36]), sendo um dos poucos problemas ainda em aberto da lista apresentada por Hilbert. De fato, embora exista uma extensa relação de trabalhos contendo muitos resultados parciais interessantes sobre o tema, até o momento nenhuma resposta completa foi alcançada, mesmo no caso $n = 2$ (veja, por exemplo, [45] e as referências contidas neste artigo), embora se saiba que existe uma quantidade finita de ciclos limite neste caso particular (veja [27] e [38]). Atualmente, para sistemas quadráticos conjectura-se que o número máximo de ciclos limite seja 4 (3 ciclos limite em torno de um foco e 1 em torno de outro foco; veja [69]), mas não existe qualquer prova para esta conjectura até o momento.

Não obstante, devido a dificuldade em resolver o 16º Problema de Hilbert, um novo problema foi proposto pelo matemático russo Vladimir Arnol'd, a saber, o estudo da cota superior para o número de ciclos limite que bifurcam de um equilíbrio do tipo centro, conhecido como *versão fraca do 16º Problema de Hilbert* (veja [1] e [2]). Com efeito, atualmente muitos trabalhos relacionados a localizar e quantificar ciclos limite são dedicados à versão proposta por Arnol'd e, para este caso particular, alguns resultados podem ser encontrados em [21] e [37]. Mais especificamente, o número máximo de ciclos limite que podem bifurcar de um contínuo de órbitas periódicas preenchendo um aberto do plano $\mathbb{R}^2$ coincide com o número de zeros da *integral abeliana* de primeira ordem associada ao sistema em questão, um método baseado na aplicação de primeiro retorno de Poincaré (veja [4] e [57] para integrais abelianas). Outro método baseado na aplicação de Poincaré é a função de bifurcação de Malkin (veja Seção 3 do Capítulo 1), que sob certas hipóteses coincide com o método do averaging, ferramenta muito utilizada na literatura para estudar a versão fraca do 16º Problema de Hilbert. Para outros resultados baseados na aplicação de Poincaré, ver os trabalhos de Bogoliubov, Fatou, Krylov, Malkin, Poincaré, Pontryagin e Roseau, entre outros, bem como o livro de Sanders, Verhulst e Murdock ([5], [6], [32], [51], [55], [56], [58] e [59]).

Por outro lado, conjuntos minimais não triviais ocorrem em ambientes onde o campo de vetores possui regularidade mais baixa que aquela exigida pelo Teorema de Denjoy-Schwartz, ou seja, campos de vetores de classe $\mathcal{C}^1$, suaves por partes (de classe $\mathcal{C}^0$) ou mesmo descontínuos (exemplos de conjuntos minimais não triviais de um campo de vetores de classe $\mathcal{C}^1$ podem ser encontrados em [23] e de campos descontínuos em [12] e [13]). Deveras, sistemas descontínuos tem recebido muita atenção recentemente, principalmente devido ao fato que tais sistemas podem apresentar fenômenos pouco usuais ou mesmo que não ocorrem em sistemas suaves, bem como uma dinâmica mais rica e elaborada.

O estudo de sistemas descontínuos é bastante recente e tem atraído a atenção pelo caracter prático que assume em áreas como física, engenharia elétrica e teoria do controle. Com efeito, o estudo de tais sistemas tem mostrado que os mesmos fornecem uma modelagem muito mais realista em aplicações do que aquelas governadas por sistemas suaves. Em resumo, em sistemas descontínuos supõe-se a existência de uma superfície Σ de codimensão 1, chamada de *superfície de descontinuidade*, separando o retrato de fases em duas ou mais partes disjuntas (com exceção de Σ), sendo que em cada uma das partes está definido um campo de vetores e sob a superfície Σ estão definidos os campos de vetores adjacentes. Portanto, sobre pontos de Σ temos definidos dois campos de vetores.

Embora existam diferentes maneiras de definir um campo de vetores descontínuos, tem-se destacado a chamada *convenção de Filippov*, que determina o algoritmo para definir um campo de vetores sobre Σ (veja [33]). Por vezes, a um sistema descontínuos dizemos *sistema de Filippov*. Mais detalhes podem ser encontrados na Seção 2 do Capítulo 1 deste texto e referências contidas nesta seção. Baseados nesta convenção, muitos autores tem contribuido para o estudo de sistemas descontínuos. Apesar disso, a teoria é relativamente nova e existe uma necessidade latente de estabelecer boas definições e traduzir resultados da teoria clássica para o universo descontínuo. Neste sentido, um dos pontos iniciais da teoria dos sistemas descontínuos data de 1977 com o trabalho de Teixeira (veja [63]) sobre sistemas suaves em variedades bi-dimensionais com bordo. Mais tarde, outros trabalhos como [8], [33] e [41] também deram contribuições significativas à teoria, bem como mais recentemente os trabalhos [3] e [67].

Atualmente, ademais, já se conhece a validade de alguns resultados provenientes da teoria clássica dentro do contexto dos sistemas descontínuos. Por exemplo, o Teorema de Existência e Unicidade não é válido, uma vez que não existe unicidade de soluções nas regiões onde está definido o campo de Filippov (para mais detalhes, ver Seção 1.2 do Capítulo 1). Por outro lado, sobre certas condições, é sabido que o Teorema de Peixoto e o Teorema de Poincaré-Bendixson possuem versões para sistemas descontínuos (ver [13] e [50], respectivamente). Outros aspectos de sistemas descontínuos ao qual muitos pesquisadores desta área tem se dedicado é a teoria das bifurcações, com relativo destaque aos trabalhos de Teixeira (veja, por exemplo, [64], [65] e [66]).

Conjuntos minimais também tem sido intensamente estudados em sistemas descontínuos, tendo, contudo, grande atenção voltada aos conjuntos minimais triviais, embora o conceito de trivial possa ser ligeiramente diferente neste contexto. De fato, muitos trabalhos tem sido dedicados a encontrar órbitas periódicas e ciclos limite (também chamados, no cenário descontínuo, de pseudo ciclos ou ciclos canard) em sistemas descontínuos, principalmente através do estudo das bifurcações que ocorrem nestes sistemas. Vale ressaltar que tem-se verificado a ocorrência de um número maior de ciclos limite em sistemas descontínuos do que em seus análogos suaves (veja, por exemplo, [17] ou o Capítulo 3 desta tese), o que evidencia a riqueza dinâmica dos sistemas descontínuos e também motiva a busca por conjuntos minimais. Para um exemplo específico, basta notar que enquanto sistemas lineares suaves não possuem ciclos limite, em sistemas lineares descontínuos (isto é, sistemas descontínuos compostos apenas por campos de vetores lineares) podemos verificar a co-existência de até 3 ciclos limite (veja, por exemplo, [7], [19] e [44]).

Uma maneira eficiente de buscar conjuntos minimais triviais em sistemas descontínuos é através de perturbações descontínuas de pontos de equilíbrio do tipo centro, um problema muito parecido com aquele proposto por Arnol'd para sistemas suaves. Em verdade, métodos clássicos como o método da função de bifurcação de Malkin e o método do averaging já possuem versões para sistemas com baixa regularidade do campo de vetores e permitem encontrar ciclos limite neste contexto (ver [10], [11] e [43]). Contudo, embora estes e muitos outros trabalhos abordem o tema dos conjuntos minimais triviais, tem sido praticamente inexplorado o tema dos conjuntos minimais não triviais em sistemas descontínuos.

Para aqueles sistemas planares que não apresentam movimento deslizante (ou seja, que não possuem um campo de vetores de Filippov definido), em [13] prova-se uma versão do Teorema de Poincaré-Bendixson que garante que para tais sistemas existem apenas conjuntos minimais triviais (ver Capítulo 6). Ainda em [13], são apresentados exemplos de conjuntos minimais não triviais em sistemas descontínuos (ver Capítulos 4 e 5 desta tese). Desde que sistemas suaves planares possuem apenas conjuntos minimais triviais, tais exemplos representam mais uma diferença entre sistemas suaves e descontínuos. Vale ressaltar que os exemplos apresentados em [13] e posteriormente em [12] possuem uma rica dinâmica e algumas de suas propriedades contradizem conceitos básicos da teoria de conjuntos minimais para sistemas suaves, como o fato de sistemas minimais estarem contidos nos conjuntos limites, dentre outras contradições, como a existência de órbitas não densas no conjunto minimal. Por outro lado, em $\mathbb{R}^3$, surgem evidências da existência de conjuntos minimais não triviais em [22], embora os autores não tenham provado tal fato. Em todo caso, acredita-se que os exemplos apresentados em [13] e [12] sejam os primeiros exemplos na literatura de conjuntos minimais não triviais em sistemas descontínuos.

Outro aspecto observado recentemente em sistemas descontínuos autônomos é a ocorrência de caos no caso planar e a relação entre conjuntos minimais não triviais e sistemas caóticos (veja [12]). De fato, em [12] mostra-se que campos de vetores planares descontínuos definidos sobre certos conjuntos minimais (chamados conjuntos minimais *orientados*; veja Capítulos 4 e 5) não triviais com medida positiva sempre apresentam comportamento caótico. Vale ressaltar que, em sistemas planares suaves, a ocorrência de caos não é permitida devido ao Teorema da Curva de Jordan. Entretanto, as primeiras evidências de sistemas descontínuos caóticos surgem nos trabalhos de Jeffrey em $\mathbb{R}^3$ (veja, por exemplo, [22] e [39]). Além disso, em [39] Jeffrey também apresenta um exemplo de um sistema descontínuo com simetria em $\mathbb{R}^2$ apresentando caos. Portanto, mesmo no contexto planar, sistemas descontínuos podem apresentar certos comportamentos não presentes em sistemas suaves, como são os exemplos de conjuntos minimais não triviais e caos.

Na sequência apresentamos uma breve descrição dos capítulos da presente tese e evidenciamos a contribuição desta para os temas introduzidos previamente. Paralelamente, apresentamos os principais resultados obtidos.

Descrição dos capítulos e dos resultados principais

Apresentamos aqui uma breve descrição dos capítulos e enunciamos os principais resultados presentes nesta tese. A enumeração dos resultados obedece aquela que aparece no texto, em sua versão traduzida do inglês.

O primeiro capítulo desta tese está divido em três seções e apresenta os principais conceitos, métodos e ferramentas utilizadas através do texto. Na primeira seção, introduzimos a definição formal de conjuntos minimais triviais e não triviais, enunciamos alguns resultados clássicos relacionados a este tema e discorremos brevemente sobre alguns pontos específicos no sentido de contextualizá-los na tese. Na sequência, apresentamos uma seção contendo um resumo das principais definições e conveções de sistemas descontínuos segundo Filippov, definindo alguns entes desta teoria que tocam ao objetivo da presente tese. Outros resultados de sistemas descontínuos, que não serão utilizados durante este texto, são deixados através das referências contidas nesta introdução e na referida seção. Finalmente, na última seção do Capítulo 1, apresentamos dois resultados de caracter técnico baseados na função de bifurcação de Malkin e que serão utilizados, respectivamente, nos Capítulos 2 e 3.

Na sequência, o segundo capítulo traz condições suficientes para a existência de conjuntos minimais em um sistema em três dimensões, a saber, o sistema de Vallis. Tal sistema possui uma grande similaridade com o sistema de Lorenz e portanto apresenta objetos em comum tais como atratores estranhos e conjuntos compactos invariantes, embora não tenha sido relatada a existência de conjuntos minimais não triviais em tal sistema (veja [40], [42], [62] e [68]). Vale dizer que tampouco a existência de conjuntos minimais triviais foi estudada neste sistema, de tal forma que apenas um trabalho relata a existência de uma órbita periódica proveniente de uma bifurcação de Hopf para uma versão autônoma do sistema de Vallis (veja [62]). No Capítulo 2, contudo, encontramos cinco órbitas periódicas distintas desta última a qual citamos. Os detalhes deste capítulo estão publicados no periódico *Discrete and Continuous Dynamical Systems - Series A*, veja [28].

Este capítulo apresenta, ademais, um método de reescalonamento que permite aplicar a teoria do averaging em sistemas não perturbados (outros artigos utilizando a mesma metodologia podem ser encontrados em [29] e [49]).

O sistema de Vallis é dado por

$$
\begin{aligned}
\frac{dx}{dt} &= -ax + by + ap(t), \\
\frac{dy}{dt} &= -y + xz, \\
\frac{dz}{dt} &= -z - xy + 1,
\end{aligned}
\tag{1}
$$

onde $p(t)$ é uma função T-periódica de classe C^1 e os parâmetros a e b são positivos. O sistema (1) é chamado *sistema de Vallis*.

Considere agora I dado pela integral

$$I = \int_0^T p(s)ds,$$

e a função

$$J(t) = \int_0^t p(s)ds,$$

satisfazendo $J(T) = I$.

Os resultados a seguir fornecem condições suficientes para a existência de 5 ciclos limite no sistema de Vallis em função dos parâmetros do sistema, da função $p(t)$ e do período T, bem como a suas localizações aproximadas e estabilidade. Observe que, desde que o sistema (1) não depende de ε, os ciclos limite também não dependem deste parâmetro. Por este motivo, utilizamos o símbolo "$\approx$", que significa que o ciclo limite está muito perto do ponto que possui aquelas coordenadas. Portanto, os ciclos limites descritos através dos próximos resultados são ciclos de pequena amplitude. Os resultados são os seguintes:

Teorema 2.1. *Para $I \neq 0$ e $a \neq b$ o sistema de Vallis* (1) *tem uma solução T-periódica* $(x(t), y(t), z(t))$ *tal que*

$$(x(t), y(t), z(t)) \approx \left(\frac{aI}{T(a-b)}, \frac{aI}{T(a-b)}, 1 \right),$$

Além disso esta solução periódica é estável se $a > b$ e instável se $a < b$.

Teorema 2.2. *Para $I \neq 0$ o sistema de Vallis* (1) *tem uma solução T-periódica* $(x(t), y(t), z(t))$ *tal que*

$$(x(t), y(t), z(t)) \approx \left(-\frac{aI}{Tb}, -\frac{aI}{Tb}, 1 \right),$$

Além disso, esta solução periódica é sempre instável.

Teorema 2.3. *Para $I \neq 0$ o sistema de Vallis* (1) *tem uma solução T-periódica* $(x(t), y(t), z(t))$ *tal que*

$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, \frac{I}{T}, 1 \right),$$

Além disso, esta solução periódica é sempre estável.

Teorema 2.4. *Para $I \neq 0$ o sistema de Vallis* (1) *tem uma solução T-*

periódica $(x(t), y(t), z(t))$ *tal que*

$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, 0, 1\right),$$

Além disso, esta solução periódica é sempre estável.

Teorema 2.5. *Considere* $I = 0$ *e* $J(t) \neq 0$ *quando* $0 < t < T$. *Então o sistema de Vallis* (1) *possui uma solução* T*-periódica* $(x(t), y(t), z(t))$ *tal que*

$$(x(t), y(t), z(t)) \approx \left(-\frac{a}{T}\int_0^T J(s)ds, 0, 1\right),$$

Além disso, esta solução periódica é sempre estável.

Ainda neste capítulo, é provado que, através do método utilizado para provar os resultados (veja Seção 1.3 do Capítulo 1) e usando os reescalonamentos

$$\begin{aligned} x &= \varepsilon^{m_1}X, \qquad y = \varepsilon^{m_2}Y, \qquad z = \varepsilon^{m_3}Z, \\ p(t) &= \varepsilon^{n_1}P(t), \qquad a = \varepsilon^{n_2}A, \qquad b = \varepsilon^{n_3}B, \end{aligned} \tag{2}$$

o sistema de Vallis não possui nenhum outro ciclo limite diferente dos 5 apresentadas previamente, quaisquer que sejam a, b, m_i, n_i, para $i = 1, 2, 3$.

No Capítulo 3 são estudados os conjuntos minimais que bifurcam de um sistema que possui um cilindro preenchido por órbitas periódicas quando este cilindro é perturbado por funções descontínuas. Em particular, este trabalho é um dos primeiros a lidar com perturbação de variedades preenchidas por órbitas periódicas, diferentes de equilíbrios do tipo centro, via funções descontínuas. Ressaltamos que mesmo considerando perturbações contínuas, existem poucos trabalhos na literatura levando-se em conta que o conjunto perturbado não é um equilíbrio do tipo centro, citamos [46], [47] e [48]. Em particular, neste capítulo fazemos uma generalização de [48] para uma classe maior de cilindros e considerando perturbações descontínuas em vez de contínuas. Os resultados obtidos relatam a existência e fornecem a localização de conjuntos minimais triviais de sistemas descontínuos localizados sobre uma variedade bi-dimensional diferente de um aberto do plano. Os detalhes deste capítulo foram submetidos a um periódico especializado e podem ser encontrados em [17].

Observamos que a situação descrita previamente configura um problema similar a aquele proposto por Arnol'd em um contexto mais geral, uma vez que leva-se em conta não apenas órbitas periódicas que bifurcam de um equilíbrio do tipo centro senão de qualquer superfície bi-dimensional.

Outra contribuição importante deste capítulo é apresentar em detalhes e, até onde sabemos, pela primeira vez, um caso concreto onde aplica-se o método apresentado em [11] e detalhado na Seção 1.3 do Capítulo 1. Este método, baseado na função de bifurcação de Malkin, tem a vantagem de permitir que a perturbação feita a uma superfície (qualquer) preenchida por órbitas periódicas seja de classe $\mathcal{C}^0$, o que encontra grande aplicação dentro da teoria dos sistemas descontínuos.

O sistema com o qual trabalhamos é apresentado na sequência.

$$\begin{aligned} \dot{x} &= -y + x(x^2 + y^2 - 1), \\ \dot{y} &= x + y(x^2 + y^2 - 1), \\ \dot{z} &= h(x, y). \end{aligned} \tag{3}$$

Observamos que o cilindro $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ é invariante para o sistema (3). Ainda, se a função h é escrita como

$$h(x, y) = \rho(x, y)(x\,\phi(x^2, y^2) + xy\,\chi(x^2, y^2) + y\,\psi(x^2, y^2)),$$

de tal forma que $\rho(x, y)$ satisfaz $\rho(r\cos\theta, r\sin\theta) = 1$ para $r = 1$ em coordenadas cilíndricas, então por cada ponto sobre o cilindro C passa uma órbita periódica, ou seja, C está preenchido por órbitas periódicas do sistema (3).

Devido a geometria das órbitas sobre o cilindro, dividiremos o espaço $\mathbb{R}^3$ em duas partes, Σ^+ e Σ^-, através de uma superfície de descontinuidade Σ, de tal forma que cada órbita periódica do cilindro contenha partes em Σ^+ e Σ^-. Portanto Σ deve ser qualquer plano que contenha o eixo z. No nosso caso, para simplificar os cálculos, escolheremos Σ como sendo o plano $y = 0$. Consequentemente, faremos uma perturbação no sistema (3) da seguinte maneira: considere agora as funções $g^{\pm} = (p^{\pm}, q^{\pm}, r^{\pm})$ dadas por

$$
\begin{aligned}
p^{\pm}(x,y,z) &= \sum_{i+j+k\leq m} a^{\pm}_{ijk} x^i y^j z^k,\\
q^{\pm}(x,y,z) &= \sum_{i+j+k\leq n} b^{\pm}_{ijk} x^i y^j z^k,\\
r^{\pm}(x,y,z) &= \sum_{i+j+k\leq p} c^{\pm}_{ijk} x^i y^j z^k,
\end{aligned}
\tag{4}
$$

com $i,j,k,m,n,p \in \mathbb{N}$ e $a_{ijk}, b_{ijk}, c_{ijk} \in \mathbb{R}$, $\forall i,j,k \in \mathbb{N}$. Além disso, considere a função

$$g(x,y,z) = \frac{1}{2}(g^+(x,y,z) + g^-(x,y,z)) + \frac{1}{2}\mathrm{sgn}(y)(g^+(x,y,z) - g^-(x,y,z)),$$

e observe que a expressão da função g é diferente dependendo do sinal de y. Portanto, a perturbação do sistema (3) através da função g gera um sistema descontínuos tendo $y=0$ como região de descontinuidade Σ.

Agora considere as funções

$$A_h(\theta) = \cos\theta \frac{\partial h}{\partial x}(\cos\theta, \sin\theta) + \sin\theta \frac{\partial h}{\partial y}(\cos\theta, \sin\theta) \tag{5}$$

e

$$
\begin{aligned}
M_\delta(z) = &\int_0^{2\pi} -\frac{1}{2}\big[h(\cos\theta,\sin\theta)(-\cos\theta(q^+(\varsigma)+q^-(\varsigma))\\
&\quad + \sin\theta(p^+(\varsigma)+p^-(\varsigma))) + (r^+(\varsigma)+r^-(\varsigma)) +\\
&\quad (h(\cos\theta,\sin\theta)(\cos\theta(-q^+(\varsigma)+q^-(\varsigma)) + \sin\theta(p^+(\varsigma)-\\
&\quad p^-(\varsigma))) + (r^+(\varsigma)-r^-(\varsigma)))\varphi_\delta(\sin\theta)\big]\, ds,
\end{aligned}
\tag{6}
$$

com $\varsigma = \left(\cos\theta, \sin\theta, z + \int_0^s h(\cos v, \sin v) dv\right)$ e z algum valor real. Então valem os seguintes resultados.

Teorema 3.2. *Assuma que $A_h(\theta) = 0$, $\forall\, \theta \in [0, 2\pi)$. Então, para $|\varepsilon|$ suficientemente pequeno e, para cada z_0 tal que $M_\delta(z_0) = 0$ e $M'_\delta(z_0) \neq 0$, o sistema descontínuo gerado pela ε-perturbação do sistema* (3) *através da* *função g possui um ciclo limite bifurcando do contínuo de soluções periódicas do cilindro C com* $\varepsilon = 0$*. Além disso, existem no máximo* $s = \max\{m, n, p\}$ *valores de z para os quais* $M_\delta(z) = 0$*.*

Teorema 3.3. *Assuma que* $A_h(\theta) = 0$*,* $\forall\, \theta \in [0, 2\pi)$*,* $g^+ = g^-$ *e considere a função*

$$\overline{M}_\delta(z) = \int_0^{2\pi} -\left[h(\cos\theta, \sin\theta)(-\cos\theta q^+(\varsigma) + \sin\theta p^+(\varsigma)) + r^+(\varsigma)\right] ds, \tag{7}$$

onde $\varsigma = \left(\cos\theta, \sin\theta, z + \int_0^s h(\cos v, \sin v)dv\right)$ *e z é algum valor real. Então, para* $|\varepsilon|$ *suficientemente pequeno e, para cada* z_0 *tal que* $\overline{M}_\delta(z_0) = 0$ *e* $\overline{M}'_\delta(z_0) \neq 0$*, o sistema suave gerado pela* ε*-perturbação do sistema* (3) *através da função g possui um ciclo limite bifurcando do contínuo de soluções periódicas do cilindro C com* $\varepsilon = 0$*. Além disso, existem no máximo* $s = \max\{m, n, p\}$ *valores de z para os quais* $M_\delta(z) = 0$*.*

Neste capítulo também apresentamos dois exemplos comparando os resultados sobre o número de ciclos limite que podem bifurcar de C através de perturbações suaves e descontínuas e mostramos que a quantidade destes objetos é maior no caso de perturbações descontínuas.

O Capítulo 4 aborda os conjuntos minimais não triviais em sistemas planares descontínuos. Algumas definições são traduzidas da teoria clássica dos sistemas dinâmicos para o contexto dos sistemas dinâmicos descontínuos. Em particular, exibimos exemplos que indicam que não podemos generalizar os teoremas de Poincaré-Bendixson e Denjoy-Schwartz sem hipóteses extras sobre o campo de vetores descontínuos. Depois, no Capítulo 6, fornecemos condições necessárias para a não ocorrência de conjuntos minimais não triviais e classificamos os conjuntos limites e minimais (triviais) em sistemas descontínuos sob as referidas condições. Suspeita-se que os exemplos fornecidos neste capítulo são os primeiros exemplos de conjuntos minimais não triviais em sistemas descontínuos.

Neste capítulo também introduzimos a ideia de conjuntos minimais orientados ou *minimalidade orientada*, isto é, conjuntos positivo-minimais e/ou negativo-minimais para um dado campo de vetores descontínuos Z. Tais conceitos permitem distinguir importantes objetos da teoria dos sistemas descontínuos que não são minimais mas que, no entanto, apresentam características como compacidade e invariância em certo sentido. Dentre tais objetos, por um lado estão os pseudo equilíbrios e os pseudo ciclos que possuem regiões de deslize e, por outro, estão certos conjuntos com medida de Lebesgue positiva. Em particular, mostramos que se um conjunto é positivo-minimal e negativo-minimal, então tal conjunto é minimal, e fornecemos exemplos que evidenciam que a recíproca deste resultado não é verdadeira. Mais tarde, no Capítulo 5, apresentamos um importante teorema que relaciona sistemas descontínuos caóticos com conjuntos que são ao mesmo tempo positivo-minimais e negativo-minimais.

O próximo teorema exibe um exemplo de conjunto minimal não trivial. Note que tal conjunto possui medida de Lebesgue positiva.

Teorema 4.1. *Considere o campo de vetores descontínuos* $Z = (X, Y)$ *com* $X(x, y) = (1, -2x)$ *e* $Y(x, y) = (-2, 4x^3 - 2x)$*, onde* $\{y = 0\}$ *é a região de descontinuidade. Então o conjunto*

$$\Lambda = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1 \ \ e \ \ x^4/2 - x^2/2 \leq y \leq 1 - x^2\}. \tag{8}$$

é um conjunto minimal não trivial para o sistema descontínuo $\dot{q} = Z(q)$*.*

Durante o Capítulo 4 veremos também que Λ é um conjunto positivo-minimal e negativo-minimal, simultaneamente. Na sequência, os próximos dois teoremas fornecem exemplos de conjuntos minimais e também os classifica como positivo-minimais e/ou negativo-minimais. Observe, além disso, que tais teoremas também fornecem conjuntos com medida de Lebesgue positiva. Conjuntos minimais com tal propriedade serão chamados de *não triviais* ao longo do capítulo 4.

Teorema 4.2. *Considere o campo de vetores descontínuos* $Z_1 = (X, Y)$*, com* $X(x, y) = (1, -2x + 1)$ *e* $Y(x, y) = (-1, (-2 + x)(-22 + x(-7 + 4x)))$*, onde* $\{y = 0\}$ *é a região de descontinuidade. Então o conjunto*

$$\begin{aligned} \Lambda_1 = \ & \{(x, y) \in \mathbb{R}^2; -3 \leq x \leq 4 \ \ e \\ & (-4 + x)(-2 + x)^2(3 + x) \leq y \leq -(-4 + x)(3 + x)\}. \end{aligned} \tag{9}$$

é um conjunto minimal não trivial para o sistema descontínuo $\dot{q} = Z_1(q)$. No entanto, Λ_1 não é positivo-minimal nem negativo-minimal para este sistema descontínuo.

Teorema 4.3. *Considere o campo de vetores descontínuo Z_2 que possui o retrato de fases (restrito ao conjunto Λ_2 destacado) apresentado na Figura 1 onde $\{y = 0\}$ é a região de descontinuidade. Então o conjunto Λ_2 é um conjunto positivo-minimal e também minimal não trivial para o sistema descontínuo $\dot{q} = Z_2(q)$. No entanto, Λ_2 não é negativo-minimal para este sistema descontínuo.*

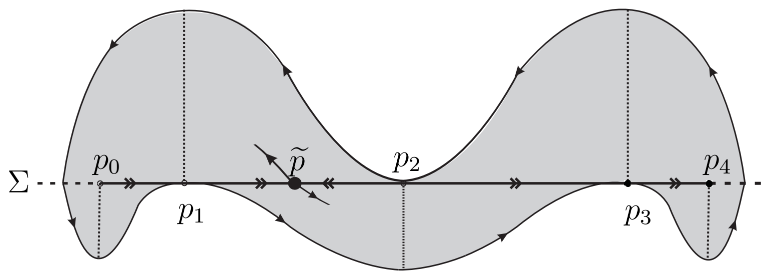

Figura 1: Conjunto minimal não trivial Λ_2.

Os detalhes deste capítulo fazem parte de dois artigos, um deles submetido para publicação a um periódico especializado e outro aceito para publicação no periódico *Ergodic Theory and Dynamical Systems* (veja [13] e [12], respectivamente).

No Capítulo 5 tratamos de sistemas planares descontínuos que apresentam comportamento caótico sobre conjuntos minimais não triviais. Introduzimos os conceitos relacionados a caos para sistemas descontínuos e relacionamos caos em tais sistemas com o conceito de minimalidade orientada introduzida no Capítulo 4. Suspeita-se que minimalidade orientada seja uma condição mais forte que caoticidade, uma vez que existem sistemas planares caóticos sobre certos conjuntos que não são minimais (veja exemplo em [39]), enquanto minimalidade orientada sob certas hipóteses garante condições suficientes para a existência de caos. Tais condições são explicitadas no Teorema 5.7 que apresentamos na sequência.

O conteúdo deste capítulo revela que, como comentado anteriormente, sistemas descontínuos podem apresentar caos mesmo em dimensão 2. Além disso, neste capítulo são apresentados exemplos de sistemas caóticos sem simetria ou presença de *pontos canard* (veja seção 1.2 do capítulo 1), diferente do exemplo de Jeffrey em [39].

A seguir apresentamos os principais resultados do Capítulo 5. O Teorema 5.4 nos diz que o campo de vetores descontínuos Z apresentado no Teorema 4.6 é caótico sobre o conjunto Λ.

Teorema 5.4. *Considere o campo de vetores descontínuo* $Z = (X, Y)$, *com* $X(x, y) = (1, -2x)$ *e* $Y(x, y) = (-2, 4x^3 - 2x)$, *onde* $\Sigma = \{y = 0\}$. *Então o sistema descontínuo* $\dot{q} = Z(q)$ *é caótico sobre o conjunto compacto invariante*

$$\Lambda = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1 \;\; and \;\; x^4/2 - x^2/2 \leq y \leq 1 - x^2\}.$$

Analogamente, o próximo teorema versa sobre a caoticidade do sistema descontínuo Z_2 apresentado no Teorema 4.8, sobre o conjunto Λ_2.

Teorema 5.6. *Considere o campo de vetores descontínuo* Z_2 *e o conjunto compacto invariante* Λ_2 *como apresentado no Teorema 4.8. Então* Z_2 *é caótico sobre* Λ_2.

Observamos que embora o conjunto Λ apresentado nos Teoremas 4.6 e 5.4 possua uma certa simetria e a coincidência de pontos de tangências visíveis e invisíveis (que nesta tese chamaremos estrutura canard), o conjunto Λ_2 apresentado previamente não possui qualquer destas características. Entendemos que este é o primeiro exemplo de um sistema planar caótico sobre um conjunto sem qualquer tipo de simetria.

Na sequência denotamos por $mes(\cdot)$ a medida de Lebesgue. O próximo resultado fornece uma condição suficiente para a existência de caos em sistemas planares descontínuos.

Teorema 5.7. *Seja* Z *um sistema planar descontínuo e* $\Lambda \subset \mathbb{R}^2$ *um conjunto compacto invariante. Se* Λ *é um conjunto positivo-minimal e negativo-minimal para* Z *satisfazendo* $mes(\Lambda) > 0$, *então* Z *é caótico sobre* Λ.

Salientamos que o exemplo apresentado por Jeffrey em [39] apresenta um exemplo de sistema planar descontínuo caótico que não é positivo/negativo-minimal ou mesmo minimal, ainda que tal conjunto tenha medida de Lebesgue positiva. Isto evidencia o fato de que a recíproca do Teorema 5.7 não é verdadeira.

Os resultados obtidos no Capítulo 5 estão aceitos para publicação no periódico *Ergodic Theory and Dynamical Systems*, veja [12].

Finalmente, no Capítulo 6, inspirados pelos capítulos anteriores sobre conjuntos minimais não triviais, apresentamos um resultado que caracteriza os conjuntos limites de sistemas planares descontínuos sobre certas hipóteses, bem como seus conjuntos minimais que, a saber, são todos triviais. Tal resultado na verdade configura uma extensão do Teorema de Poincaré-Bendixson para sistemas planares descontínuos sem movimento de deslize. Por movimento de deslize entendemos que está definido um campo de vetores de Filippov sobre a superfície de descontinuidade. Ressaltamos que os exemplos apresentados nos Capítulos 4 e 5 dizem que tal teorema não pode ser extendido para sistemas descontínuos com movimento deslizante sem a suposição de hipóteses extras, uma vez que definido um campo de Filippov, fenômenos como minimalidade não trivial e caos podem ocorrer, mesmo em dimensão 2.

Na sequência introduzimos os principais resultados do Capítulo 6. Eles lidam com os conjuntos limites de pontos e trajetórias de um campo de vetores planar descontínuo. Para uma definição precisa dos entes que aparecem nestes resultados, veja a Seção 1.2 do Capítulo 1 desta tese.

Teorema 6.2 [Poincaré-Bendixson para sistemas não suaves]. *Seja $Z = (X, Y)$ um campo de vetores planar descontínuo. Assuma que Z não possui movimento deslizante e que possua uma trajetória global $\Gamma_Z(t, p)$ cuja trajetória positiva $\Gamma_Z^+(t, p)$ esteja contida em um conjunto compacto K. Suponha também que X e Y têm um número finito de pontos de equilíbrio em K, nenhum deles sobre Σ, e um número finito de pontos de tangência com Σ. Então o conjunto ω-limite $\omega(\Gamma_Z(t, p))$ de $\Gamma_Z(t, p)$ é formado por um dos seguintes objetos:*

(i) um ponto de equilíbrio de X ou de Y;

(ii) uma órbita periódica de X ou de Y;

(iii) um gráfico de X ou de Y;

(iv) uma tangência s-singular de Z.

(v) um pseudo ciclo do tipo I de Z;

(vi) um pseudo gráfico de Z;

Observamos que as três primeiras possibilidade para o conjunto ω-limite de $\Gamma_Z(t,p)$ são conhecidas pelo teorema clássico de Poincaré-Bendixson. Os três últimos itens, entretanto, ocorrem devido a existência de região de descontinuidade. Observamos também que se a hipótese de não haver movimento deslizante for suprimida, então teremos exemplos como o conjunto Λ apresentado previamente, ou seja, teremos a existência de conjuntos limites com medida de Lebesgue positiva além de outras propriedades estranhas a teoria clássica dos conjuntos limites, como o fato de conjuntos limites de pontos em sistemas descontínuos serem desconexos (veja os detalhes através do Capítulo 6).

Salientamos que, desde que o Teorema de Existência e Unicidade não é válido para sistemas descontínuos, não observamos a unicidade das órbitas passando por um ponto, de tal forma que temos o seguinte corolário como consequência do Teorema 6.2.

Corolário 6.1. *Sobre as mesmas hipóteses do Teorema 6.2, o conjunto ω-limite $\omega(p)$ de um ponto p em um sistema planar descontínuo é um dos objetos descritos nos itens (i), (ii), (iii), (iv), (v) e (vi) ou a união de alguma (sub)coleção destes objetos.*

Observamos que o mesmo vale para o conjunto α-limite apenas mudando a orientação do tempo.

Outro corolário imediato do Teorema 6.2 toca ao tema dos conjuntos minimais. De fato, observe que, sob as mesmas hipóteses deste teorema, se existe um conjunto não vazio, compacto e invariante que não apresenta nenhum subconjunto próprio com tais características, então este conjunto é conjunto limite de alguma trajetória. Isso significa que os conjuntos minimais, como no caso clássico, estão contidos nos conjuntos limites.

Corolário 6.2. *Sobre as mesmas hipóteses do Teorema 6.2, os conjuntos minimais de um sistema descontínuo são todos triviais e dados por um dos seguintes objetos*

(i) um ponto de equilíbrio de X ou de Y;

(ii) uma órbita periódica de X ou de Y;

(iii) uma tangência s-singular de Z.

(iv) um pseudo ciclo do tipo I de Z;

Observamos que este Corolário, em certo sentido, é uma extensão do Teorema de Denjoy-Schwartz para sistemas descontínuos apresentando apenas regiões de costura e tangência. Além disso, note que os objetos listados no Corolário 6.2 são os mesmos do Teorema 6.2, com exceção dos gráficos e pseudo-gráficos, que não são conjuntos minimais uma vez que possuem subconjuntos próprios não vazios, compactos e invariantes, que são os pontos de equilíbrio.

Em resumo, nota-se que embora os resultados do capítulo 6 apresentem uma maior variedade dinâmica se comparada ao caso de sistemas suaves, o fato de não termos movimento deslizante permite fazer generalizações relativamente naturais do caso clássico. Os resultados obtidos no Capítulo 6 foram submetidos a um periódico especializado e podem ser encontrados em [13].

Na sequência descrevemos os capítulos em detalhes de acordo com aquilo que foi apreesentado previamente nesta introdução. Em acordo com a língua utilizada para redigir os artigos científicos que fundamentam esta tese, a saber, a língua inglesa, os capítulos foram redigidos também em inglês. Outras partes desta tese, como esta introdução e o título, seguem na língua nativa onde desenvolveu-se a maior parte da tese, em acordo com as normas estabelicidas pela UNESP.

Chapter 1

Preliminaries

In this chapter we present some concepts and results of smooth and non-smooth systems which will be important in order to develop the thesis. We start introducing some classical results and discussing some points about minimal sets for smooth systems. Then we briefly introduce the main concepts about non-smooth systems following the Filippov's convention in order to state the results of Chapters 3 to 6. Finally, we present two methods based on the Malkin's bifurcation function that achieve conditions for the existence of minimal sets which will be important in Chapters 2 and 3.

1.1 Minimal sets

Consider $E \subset \mathbb{R}^n$ an open set, x_0 an arbitrary point of E and $f \in \mathcal{C}^1(E)$ a vector field. Let $\phi_t(x_0) = \phi(t, x_0)$ be the flow of the system

$$\dot{x} = f(x), \tag{1.1}$$

with initial value $x(0) = x_0$ and defined in its maximal interval of existence $I(x_0)$. Understanding the behavior of ϕ_t depending on t and x_0 is the main aim of the continuous theory of dynamical system (as well as discrete and ergodic theory deal with diffeomorfisms and ergodic transformations, respectively). One should ask, for instance, about the invariance of the flow ϕ_t when the initial value x_0 varies in a compact set. Invariant and compact sets play a very important role in the theory of dynamical systems. Indeed, if ϕ_t is invariant on a compact set which possesses a proper compact subset on which ϕ_t is still invariant, we can reduce the dynamics for a more restrict set and so on. The "smallest" compact set (different from the empty set) achieving the invariance by the flow is *minimal* in the sense that it presents its proper dynamics. This is precisely the idea of minimal sets: they are fundamental dynamical systems. In the following we make clear the definition of a minimal set.

Definition 1.1. *Let $K \subset E$ be a nonempty set and ϕ_t the flow of system* (1.1)*. We say that K is a minimal set for system* (1.1) *if K is compact, invariant for ϕ_t and there exists no proper subset of K satisfying these properties. The minimal set K is trivial if it is an equilibrium point or a periodic orbit. Otherwise, K is called a non-trivial minimal set.*

We must note that when the flow is defined on a compact smooth boundaryless bi-dimensional manifold M and the minimal set K coincides with M, K is also called trivial. An important result concerning such kind of manifolds and trivial minimal sets is given by Denjoy and Schwartz. Indeed, they show that a minimal set of a $\mathcal{C}^2$ vector field defined on certain bi-dimensional compact manifolds is always trivial, i.e., it is an equilibrium point, a periodic orbit or the proper manifold, in this case, the torus (see [23] and [60]).

Theorem 1.2 (Theorem of Denjoy-Schwartz)**.** *A flow ϕ_t of system* (1.1) *of class $\mathcal{C}^2$ defined in a bi-dimensional compact connected boundaryless manifold M can not have a minimal set K different from an equilibrium point or a periodic orbit, unless $M = K$ is the torus.*

For flows associated to $\mathcal{C}^1$ vector fields defined on compact sets of $\mathbb{R}^2$, the occurrence of minimal sets is closed related with the limit sets of them. In fact, in such scenario, minimal sets are an essential part of the limit sets, as stated by the Poincaré-Bendixson Theorem. In what follows in such theorem, we call $\omega(\Gamma)$ the ω-limit of a trajectory Γ.

Theorem 1.3 (Poincaré-Bendixson's Theorem, see [54], pag. 245)**.** *Consider system* (1.1) *with* $f \in \mathcal{C}^1(E)$ *where* E *is an open subset of* $\mathbb{R}^2$ *and suppose that it has a trajectory* Γ *contained in a compact subset* F *on which system* (1.1) *has only a finite number of equilibrium points. Then it follows that* $\omega(\Gamma)$ *is either a equilibrium point, a periodic orbit or a graphic of system* (1.1)*.*

Another aspect of minimal sets concerns with the fact that the orbit $\phi_t(x_0)$ of a point x_0 of it is always recurrent. In fact, we know from the classical theory of dynamical systems that a compact set K is minimal if, and only if, the closure of the orbit of every point in K coincides with K. It means that the concept of minimal set is somehow related to recurrence. Indeed, it holds the following proposition (see [25]).

Proposition 1.1. *Let* Λ *be a minimal set of a vector field of class* $\mathcal{C}^1$ *defined on a compact manifold* $S \subseteq R^n$*. Then* Λ *is non-trivial if, and only if, the orbit* $\phi_t(p)$ *of each point* $p \in \Lambda$ *is non-trivial and recurrent.*

Finding minimal sets apart from the trivial ones is a hard task. Nevertheless the existence of such objects is assured by Zorn's Lemma. The next proposition states this fact (see Lemma 2.2 of [53]).

Proposition 1.2. *Let* $C \subset E$ *be a nonempty compact set which is invariant by the flow* ϕ_t *of system* (1.1)*. Then there exists a minimal set* $K \subset C$*.*

Although finding new examples of minimal sets is not easy, the Proposition 1.2 joint with the Denjoy-Schwartz Theorem say that it must exist other examples of non-trivial minimal sets by considering vector fields of classes $\mathcal{C}^1$ or less. Indeed, in [23] Denjoy presents a non-trivial minimal set for a vector field of class $\mathcal{C}^1$ defined on the torus T_2 but distinct from it. Recently, in [13] and [12] the authors present new examples of non-trivial minimal sets for non-smooth planar vector fields and also states conditions in order to have only trivial minimal sets in this scenario. These examples can be found in the Chapters 4, 5 and 6 or in the references [12] and [13].

In what follows we present the main statements about non-smooth systems due to Filippov. We stress out that we will not distinguish a non-smooth system from a discontinuous system, once usually, simply due to nomenclature's aspects, they are took as synonymous in the literature.

1.2 Basic theory about non-smooth systems

Non-smooth vector fields (NSVFs, for short) have become certainly one of the common frontiers between Mathematics and Physics or Engineering. Many authors have contributed to the study of NSVFs (see for instance the pioneering work [33] or the didactic works [3, 67], and references therein about details of these multivalued vector fields). In our approach Filippov's convention is considered. So, the vector field of the model is non-smooth across a *switching manifold* and it is possible for its trajectories to be confined onto the switching manifold itself. The occurrence of such behavior, known as sliding motion, has been reported in a wide range of applications. We can find important examples in electrical circuits having switches, in mechanical devices in which components collide into each other, in problems with friction, sliding or squealing, among others.

In order to state the main concepts of NSVFs, let V be an arbitrarily small neighborhood of $0 \in \mathbb{R}^n$. We consider a codimension one manifold Σ of $\mathbb{R}^n$ given by $\Sigma = f^{-1}(0)$, where $f : V \to \mathbb{R}$ is a $\mathcal{C}^1$ function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in V \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in V \mid f(q) \leq 0\}$. Note that we can assume, locally around the origin of $\mathbb{R}^n$, that $f(x, y) = y$.

Designate by χ the space of $\mathcal{C}^r$-vector fields on V, with $r \geq 1$ large enough for our purposes. Call Ω the space of vector fields $Z : V \to \mathbb{R}^n$ such that

$$Z(x,y) = \begin{cases} X(x,y), & \text{for} \quad (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for} \quad (x,y) \in \Sigma^-, \end{cases} \tag{1.2}$$

where $X = (X_1, X_2), Y = (Y_1, Y_2) \in \chi$. The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we accept it to be multivalued at points of Σ. The basic results of differential equations in this context were stated by Filippov in [33] and are summarized in what follows. For doing this, consider Lie derivatives

$$X.f(p) = \langle \nabla f(p), X(p) \rangle \qquad \text{and} \qquad X^i.f(p) = \left\langle \nabla X^{i-1}.f(p), X(p) \right\rangle, i \geq 2$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^n$. We distinguish the following regions on the discontinuity set Σ:

(i) $\Sigma^c \subseteq \Sigma$ is the *sewing region* if $(X.f)(Y.f) > 0$ on Σ^c .

(ii) $\Sigma^e \subseteq \Sigma$ is the *escaping region* if $(X.f) > 0$ and $(Y.f) < 0$ on Σ^e.

(iii) $\Sigma^s \subseteq \Sigma$ is the *sliding region* if $(X.f) < 0$ and $(Y.f) > 0$ on Σ^s.

The *sliding vector field* associated to $Z \in \Omega$ is the vector field Z^s tangent to Σ^s and defined at $q \in \Sigma^s$ by $Z^s(q) = m - q$ with m being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to Σ^s (see Figure 1.1). It is clear that if $q \in \Sigma^s$ then $q \in \Sigma^e$ for $-Z$ and then we can define the *escaping vector field* on Σ^e associated to Z by $Z^e = -(-Z)^s$. Next we use the notation Z^Σ for both cases. In our pictures we represent the dynamics of Z^Σ by double arrows.

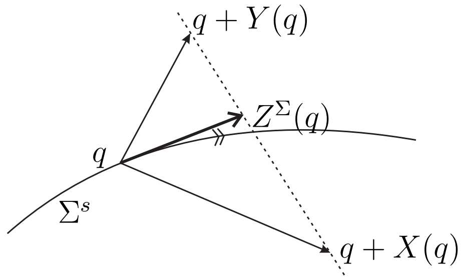

Figure 1.1: Filippov's convention.

We say that $q \in \Sigma$ is a Σ-*regular point* if

(i) $(X.f(q))(Y.f(q)) > 0$ or

(ii) $(X.f(q))(Y.f(q)) < 0$ and $Z^\Sigma(q) \neq 0$ (i.e., $q \in \Sigma^e \cup \Sigma^s$ and it is not an equilibrium point of Z^Σ).

The points of Σ which are not Σ-regular are called Σ-*singular*. We distinguish two subsets in the set of Σ-singular points: Σ^t and Σ^p. Any point $q \in \Sigma^p$ is called a *pseudo equilibrium of* Z and it is characterized by $Z^\Sigma(q) = 0$. Any point $q \in \Sigma^t$ is called a *tangential singularity* (or also *tangency point*) and it is characterized by $(X.f(q))(Y.f(q)) = 0$ (q is a tangent contact point between the trajectories of X and/or Y with Σ).

For a given $W \in \chi$, we say that r is the *contact order* of the trajectory Γ_W of W with Σ at p if $W^k f(p) = 0$, $\forall k = 0, \ldots, r-1$ and $W^r f(p) \neq 0$. For $W = X$ (respec. Y) we say that $p \in \Sigma$ is an*invisible tangency* if the contact order r of Γ_X (respec. Γ_Y) passing through p is even and $X^r f(p) < 0$ (respec. $Y^r f(p) > 0$). On the other hand, for $W = X$ (respec. Y) we say that $p \in \Sigma$ is a *visible tangency* if the contact order r of Γ_X (respec. Γ_Y) passing through p is odd or if it is even and $X^r f(p) > 0$ (respec. $Y^r f(p) < 0$).

A tangential singularity $p \in \Sigma^t$ is *singular* if p is a invisible tangency for both X and Y. On the other hand, a tangential singularity $p \in \Sigma^t$ is *regular* if it is not singular. Figures 1.2 and 1.3 illustrate all possible cases for regular and singular tangencies, respectively.

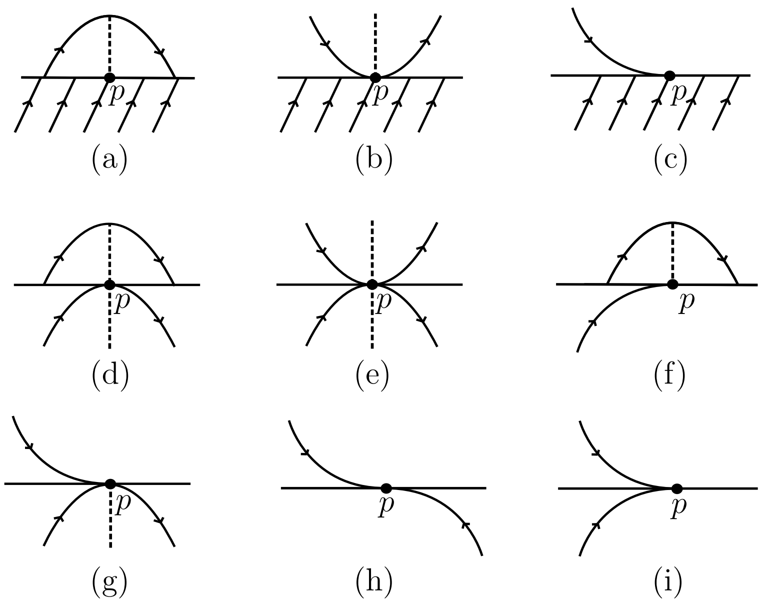

Figure 1.2: Cases where regular tangential singularities occur. The horizontal line represents the switching manifold. The dashed lines represent the curves where $Xf(p) = 0$ or $Yf(p) = 0$.

Remark 1.1. *Throughout this thesis we will say that a tangential singularity p has canard structure (or, equivalently, we will say that a canard*

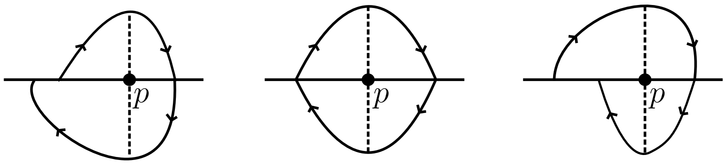

Figure 1.3: The particular cases where occur singular tangential singularities. The horizontal line represents the switching manifold.

phenomenon occurs) if p *belongs to* $\partial\Sigma^s \cap \partial\Sigma^e$. *In such case, note that any neighborhood of* p *in* Σ *presents sliding and escaping behavior. Also, any point in* Σ *which is a visible tangency for a vector field and an invisible tangency for the other one presents a canard structure. This shape will occur later on in this thesis (see Chapter 4). We must note that although we call such points as canards, we are not dealing with the concept of canard coming from the singular perturbation theory which, at first, has no relation to our context. The definition and details of canard phenomena in singular perturbation theory can be found in [26].*

Let $W \in \chi$. We denote its flow by $\phi_W(t,p)$. Thus,

$$\begin{cases} \dfrac{d}{dt}\phi_W(t,p) = W(\phi_W(t,p)), \\ \\ \phi_W(0,p) = p, \end{cases}$$

where $t \in I = I(p,W) \subset \mathbb{R}$, an interval depending on $p \in V$ and W.

The next two definitions state the concepts of *local* and *global trajectory* of a NSVF. The first one can be found in [34]. The second one is presented in [13].

Definition 1.4. *The* ***local trajectory (orbit)*** $\phi_Z(t,p)$ *of a NSVF given by* (1.2) *is defined as follows:*

- *For* $p \in \Sigma^+\backslash\Sigma$ *and* $p \in \Sigma^-\backslash\Sigma$ *the trajectory is given by* $\phi_Z(t,p) = \phi_X(t,p)$ *and* $\phi_Z(t,p) = \phi_Y(t,p)$ *respectively, where* $t \in I$.
- *For* $p \in \Sigma^c$ *such that* $X.f(p) > 0$, $Y.f(p) > 0$ *and taking the origin of the time at* p, *the trajectory is defined as* $\phi_Z(t,p) = \phi_Y(t,p)$ *for*

$t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p) = \phi_X(t,p)$ for $t \in I \cap \{t \geq 0\}$. For the case $X.f(p) < 0$ and $Y.f(p) < 0$ the definition is the same reversing the time.

- *For $p \in \Sigma^e$ and taking the origin of the time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_{Z^\Sigma}(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p)$ is either $\phi_X(t,p)$ or $\phi_Y(t,p)$ or $\phi_{Z^\Sigma}(t,p)$ for $t \in I \cap \{t \geq 0\}$. For the case $p \in \Sigma^s$ the definition is the same but reversing time.*

- *For p a regular tangency point and taking the origin of the time at p, the trajectory is defined as $\phi_Z(t,p) = \phi_1(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t,p) = \phi_2(t,p)$ for $t \in I \cap \{t \geq 0\}$, where each ϕ_1, ϕ_2 is either ϕ_X or ϕ_Y or ϕ_{Z^Σ}.*

- *For p a singular tangency point we have $\phi_Z(t,p) = p$ for all $t \in I$.*

Definition 1.5. *A **global trajectory (orbit)** $\Gamma_Z(t,p_0)$ of $Z \in \Omega$ passing through p_0 is a union*

$$\Gamma_Z(t,p_0) = \bigcup_{i \in \mathbb{Z}} \{\sigma_i(t,p_i); t_i \leq t \leq t_{i+1}\}$$

*of preserving-orientation local trajectories $\sigma_i(t,p_i)$ satisfying $\sigma_i(t_{i+1},p_i) = \sigma_{i+1}(t_{i+1},p_{i+1}) = p_{i+1}$ and $t_i \to \pm\infty$ as $i \to \pm\infty$. A global trajectory is a **positive** (respectively, **negative**) **global trajectory** if $i \in \mathbb{N}$ (respectively, $-i \in \mathbb{N}$) and $t_0 = 0$.*

Once in this thesis we are interested in minimal sets, it is very important to state the concept of periodic trajectory. Indeed, in [12] the authors introduce the notion of periodic trajectory for NSVFs, as follows in the next definition. Actually, it is analogous to the definition of periodic trajectory for smooth systems.

Definition 1.6. *Let $\Gamma_Z(t,q)$ a global trajectory of the NSVF* (1.2). *We say that Γ_Z is periodic if Γ_Z is periodic in the variable t, i.e., if there exist $T > 0$ such that $\Gamma_Z(t+T,q) = \Gamma_Z(t,q)$, for all $t \in \mathbb{R}$.*

More general than a global periodic trajectory is the concept of closed global trajectory. Indeed, every global periodic trajectory is closed but the converse may be not true. Nevertheless, two special kind of closed global orbits, not necessarily periodic, are the pseudo cycles and pseudo graphs. Such objects play an important role in the context of minimal and limit sets.

Definition 1.7. *Consider the non-smooth vector field* (1.2). *A closed global trajectory* Δ *of* Z *is a:*

(i) ***pseudo cycle*** *if* $\Delta \cap \Sigma \neq \emptyset$ *and it does not contain neither equilibrium nor pseudo equilibrium point (See Figure 1.4).*

(ii) ***pseudo graph*** *if* $\Delta \cap \Sigma \neq \emptyset$ *and it is a union of equilibria, pseudo equilibria and orbit-arcs of* Z *joining these points (See Figure 1.5).*

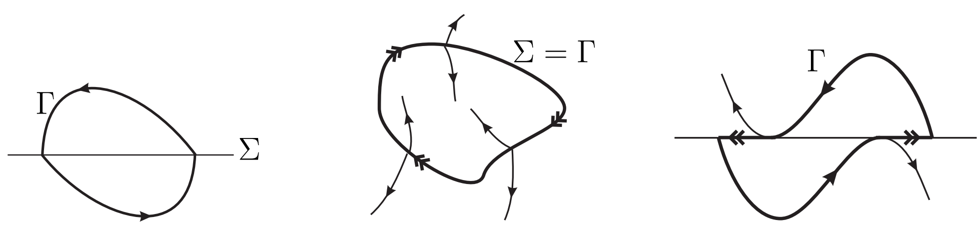

Figure 1.4: Possible kinds of pseudo cycles. From left to right: pseudo cycles of type I, II and III, respectively.

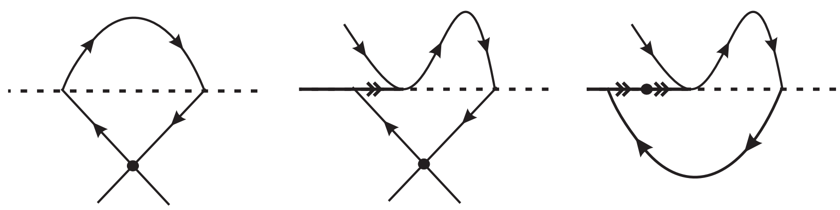

Figure 1.5: Examples of pseudo graphs. The horizontal line represents the switching manifold.

A very useful tool in order to study NSVFs is the regularization process introduced in [61] by Sotomayor and Teixeira. This method allows us to associate a NSVF to a smooth (or continuous) system and consequently it is possible to apply the major part of the methods from the classical theory of dynamical systems. Consequently, we can deduce certain aspects of the NSVFs from its regularization. In the next lines we briefly summarize it.

Indeed, a continuous function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a *transition function* if $\varphi(t) = -1$ for $t \leq -1$, $\varphi'(t) > 0$ for $t \in (-1, 1)$ and $\varphi(t) = 1$ for $t \geq 1$. So, for $\delta \in (0, 1]$ we say that the one-parameter family of continuous functions Z_δ given by

$$Z_\delta(t, q) = \frac{X(t, q) + Y(t, q)}{2} + \varphi_\delta(f(q))\frac{X(t, q) - Y(t, q)}{2},$$

is a φ-regularization of a non-smooth vector field $Z = (X, Y)$, where $q \in V$ and $\varphi_\delta(t) = \varphi(\frac{t}{\delta})$.

Observe that, for those points of Σ^+ whose $f(q) > \delta$, the regularized vector fields Z_δ coincides to X. Analogously, Z_δ coincides to Y for each point of Σ^- whose $f(q) < -\delta$. Some properties of the regularized systems can be found in [61] and [20], as well as in the references therein.

Other aspects of NSVFs can be found through the references contained in this section and in the Introduction. Moreover, some definitions pertinent to a special topic of NSVFs, as minimal and limit sets, can be found in the respective chapters which deal with such objects. More specifically, they can be found mainly throughout the Chapters 4, 5 and 6.

1.3 Methods inspired by the Malkin's bifurcation function

In this section we present two results that will be fundamental in the Chapters 2 and 3. We present them before the correspondent chapters because they possess technical and huge statements aside of the main purpose of the respective chapters. These results provide sufficient conditions for the existence of limit cycles after a perturbation of a continuum of periodic orbits filling up some Euclidean manifold.

The first one take into account $\mathcal{C}^2$ perturbations. In some texts, it is called *averaging method* (in Chapter 2 we will also call in this way), although both methods presented in this section are based on the classical result due to Malkin [51] and Roseau [58] (for additional information see [59]). In this thesis, we will apply such method in order to study the existence of limit cycles in a 3-dimensional nonlinear system. Indeed, consider the general problem of bifurcation of T-periodic solutions from the differential systems of the form

$$\dot{x} = f(t,x) + \varepsilon g(t,x) + \varepsilon^2 r(t,x,\varepsilon), \tag{1.3}$$

with $\varepsilon \neq 0$ sufficiently small. Here the functions $f, g : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $r : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$, T-periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{x} = f(t,x), \tag{1.4}$$

has a sub-manifold of periodic solutions.

Indeed let $x(t,z,\varepsilon)$ be the solution of system (1.4) such that $x(0,z,\varepsilon) = z$. We write the linearization of the unperturbed system along a periodic solution $x(t,z,0)$ as

$$\dot{y} = D_x f(t, x(t,z,0))y. \tag{1.5}$$

Then we have the following result (see [11]).

Theorem 1.8 (Buică/Llibre/Françoise)**.** *Assume there exists a k-dimensional sub-manifold $\mathcal{M}$ filled up with T-periodic solutions of system* (1.4)*. Let V be an open and bounded subset of $\mathbb{R}^k$ and let $\beta : Cl(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) $\mathcal{M} = \{z_\alpha = (\alpha, \beta(\alpha)); \alpha \in Cl(V)\}$ *and that for each $z_\alpha \in \mathcal{M}$ the solution $x(t, z_\alpha)$ of* (1.4) *is T-periodic;*

(ii) for each $z_\alpha \in \mathcal{M}$ there is a fundamental matrix U_{z_α} of system (1.5) *such that the matrix $U_{z_\alpha}^{-1}(0) - U_{z_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with* $\det \Delta_\alpha \neq 0$.

Define $M : Cl(V) \to \mathbb{R}^k$ *as*

$$M(\alpha) = \xi\left(\int_0^T U_{z_\alpha}^{-1}(t, z_\alpha) g(t, x(t, z_\alpha))dt\right),$$

where $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^k$ *is the projection of* $\mathbb{R}^n$ *onto its first* k *coordinates. Then the following statements hold.*

(a) If there exists $a \in V$ *with* $M(a) = 0$ *and* $\det((\partial M/\partial\alpha)(a)) \neq 0$, *then there exists a* T*-periodic solution* $x(t,\varepsilon)$ *of system* (1.3) *such that* $x(t,\varepsilon) \to z_\alpha$ *when* $\varepsilon \to 0$.

(b) The type of stability of the periodic solution $x(t,\varepsilon)$ *is given by the eigenvalues of the Jacobian matrix* $((\partial M/\partial\alpha)(a))$.

For a shorter proof of Theorem 1.8, item (a), see [9]. Other methods based on the Malkin's bifurcation function take into account different classes of differentiability on the functions f, g and r and the geometry of the solutions of the unperturbed part of system (1.3). Indeed, recently some of these methods have considered $\mathcal{C}^0$ and discontinuous functions. One of them is present in [43] and take into account discontinuous perturbations by considering $f \equiv 0$ and uses the regularization method introduced in Section 2 of the current chapter. In [10], however, a method based on the Brouwer theory can be found for $\mathcal{C}^0$ perturbations.

Besides of the methods cited previously, in [11] the authors present a method assuming $\mathcal{C}^0$ perturbations and without considering the extra hypothesis $f \equiv 0$. Although applying this method may be complicated due to technical hypotheses, it find place in several problems involving non-smooth systems, once the regularizations of such systems are at least of class $\mathcal{C}^0$. In the Chapter 3, it will be applied in order to find periodic orbits bifurcating from a cylinder filled up by periodic orbits. Next we present the method.

Theorem 1.9 (Buică/Llibre/Makarenkov)**.** *Consider the* T*-periodic differential system*

$$\dot{x} = f(t, x) + \varepsilon g(t, x, \varepsilon), \tag{1.6}$$

where $f \in \mathcal{C}^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$ are T-periodic in the first variable and g is locally uniformly Lipschitz with respect to its second variable. For $z \in \mathbb{R}^n$ denote by $x(\cdot, z, \varepsilon)$ the solution of (1.6) such that $x(0, z, \varepsilon) = z$. Assume that the unperturbed system

$$\dot{x} = f(t, x) \tag{1.7}$$

satisfies the following conditions:

i) There exists an open ball $U \subset \mathbb{R}^k$ with $k \leq n$ and a function $\xi \in \mathcal{C}^1(\overline{U}, \mathbb{R}^n)$ such that for $h \in \overline{U}$ the $n \times k$ matrix $D\xi(h)$ has rank k and $\xi(h)$ is the initial condition of a T-periodic solution of (1.7).

ii) For each $h \in \overline{U}$ the linear system

$$\dot{y} = D_x f(t, x(t, z, 0))y \tag{1.8}$$

with $z = \xi(h)$ has the Floquet multiplier $+1$ with the geometric multiplicity equal to k.

Let $u_1(\cdot, h)$, ..., $u_k(\cdot, h)$ be linearly independent T-periodic solutions of the adjoint linear system

$$\dot{u} = -(D_x f(t, x(t, \xi(h), 0)))^* u, \tag{1.9}$$

such that $u_1(0, h)$, ..., $u_k(0, h)$ are $\mathcal{C}^1$ with respect to h and define the function $M : \overline{U} \to \mathbb{R}^k$ (called the Malkin's bifurcation function) by

$$M(z) = \int_0^T \begin{pmatrix} \langle u_1(s, z)\ ,\ g(s, x(s, \xi(z), 0), 0)\rangle \\ \dots \\ \langle u_k(s, z)\ ,\ g(s, x(s, \xi(z), 0), 0)\rangle \end{pmatrix} ds.$$

Then the following statements hold.

1) For any sequences $(\varphi_m)_{m\geq 1}$ of $\mathcal{C}^0(\mathbb{R}, \mathbb{R}^n)$ and $(\varepsilon_m)_{m\geq 1}$ from $[0,1]$ such that $\varphi_m(0) \to \xi(z_0) \in \xi(\overline{U})$, $\varepsilon_m \to 0$ as $m \to \infty$ and φ_m is a T-periodic solution of (1.6) with $\varepsilon = \varepsilon_m$, we have that $M(z_0) = 0$.

2) If $M(z) \neq 0$ for any $z \in \partial U$ and the Brouwer degree d satisfies $d(M,U) \neq 0$, then there exists $\varepsilon_1 > 0$ sufficiently small such that for each $\varepsilon \in (0, \varepsilon_1]$ there is at least one T-periodic solution φ_ε of system (1.6) such that $\rho(\varphi_\varepsilon(0), \xi(\overline{U})) \to 0$ as $\varepsilon \to 0$, where $\rho(\varphi_\varepsilon(0), \xi(\overline{U})) = \min_{\zeta \in \xi(\overline{U})} \|\varphi_\varepsilon(0) - \zeta\|$ and $\|.\|$ is a norm in $\mathbb{R}^n$.

In addition we assume that there exists $z_0 \in U$ such that $M(z_0) = 0$, $M'(z) \neq 0$ for all $z \in \overline{U} \backslash \{z_0\}$ and the Brouwer degree d of M in U satisfies $d(M,U) \neq 0$. Moreover, calling $w_0 = \xi(z_0)$, we assume that:

iii) For $\delta > 0$ sufficiently small there exists $M_\delta \subset [0,T]$ Lebesgue measurable with $mes(M_\delta) = \tilde{o}(\delta)$ such that

$$\|g(t, w_1 + \zeta, \varepsilon) - g(t, w_1, 0) - g(t, w_2 + \zeta, \varepsilon) + g(t, w_2, 0)\| \\ \leq \tilde{o}(\delta) \|w_1 - w_2\|,$$

for all $t \in [0,T] \backslash M_\delta$ and for all w_1, $w_2 \in B_\delta(w_0)$, $\varepsilon \in [0, \delta]$ and $\zeta \in B_\delta(0)$.

iv) There exists $\delta_1 > 0$ and $L_M > 0$ such that

$$\|M(z_1) - M(z_2)\| \geq L_M \|z_1 - z_2\|, \text{ for all } z_1, z_2 \in B_{\delta_1}(z_0).$$

Then the following conclusion holds.

3) There exists $\delta_2 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1]$, φ_ε is the only T-periodic solution of (1.6) with initial condition in $B_{\delta_2}(w_0)$. Moreover $\varphi_\varepsilon(0) \to \xi(z_0)$ as $\varepsilon \to 0$.

Remark 1.2. *Since condition (iii) is rather technical, instead of use it, in this thesis we consider a simpler condition for the function g, as follows:*

v) For any $\lambda > 0$ sufficiently small there exists $M_\lambda \subset [0,T]$ Lebesgue measurable with $mes(M_\lambda) = o(\lambda)/\lambda$ and such that for every $t \in [0,T] \setminus M_\lambda$ and for all $w \in B_\delta(w_0)$, $\varepsilon \in [0, \lambda]$, $||D_w g(t, w, \varepsilon) - D_w g(t, w_0, 0)|| \leq o(\lambda)/\lambda$.

The condition v) is a sufficient one for iii). This fact follows from the Main Value Theorem.

Following we study the Vallis differential system using Theorem 1.8 present before.

Chapter 2

Minimal sets of a smooth 3-dimensional system by a rescaling method

In this chapter we study the existence of minimal sets for the Vallis differential system. By rescaling the variables, the parameters and the periodic function of the Vallis differential system we provide sufficient conditions for the existence of periodic solutions. We also determine their approximated location and characterize their kind of stability. In addition, we apply the averaging method based on the Malkin's bifurcation function without performing any perturbation in the Vallis system, as usual in the literature related to applications of such method.

2.1 Setting the problem

The Vallis system, introduced by Vallis [68] in 1988, is a periodic non-autonomous 3-dimensional system that models the atmosphere dynamics in the tropics over the Pacific Ocean, related to the yearly oscillations of precipitation, temperature and wind force. Denoting by x the wind force, by y the difference of near-surface water temperatures of the east and west parts of the Pacific Ocean, and by z the average near-surface water temperature, the Vallis system is

$$\begin{aligned}
\frac{dx}{dt} &= -ax + by + ap(t), \\
\frac{dy}{dt} &= -y + xz, \\
\frac{dz}{dt} &= -z - xy + 1,
\end{aligned} \tag{2.1}$$

where $p(t)$ is some $\mathcal{C}^1$ T-periodic function that describes the wind force under seasonal motions of air masses, and the parameters a and b are positive.

Although this model neglects some effects like Earth's rotation, pressure field and wave phenomena, it provides a correct description of the observed processes and recovers many of the observed properties of El Niño. The properties of El Niño phenomenon are studied analytically in [62] and [68]. More precisely, in [68] it is shown that taking $p \equiv 0$, it is possible to observe the presence of chaos by considering $a = 3$ and $b = 102$. Later on, in [62] it is proved that there exists a chaotic attractor for system (2.1) after a Hopf bifurcation. This chaotic motion can be fixed if we observe that system (2.1) and Lorenz system

$$\begin{aligned}
\frac{dx}{dt} &= -\sigma x - \sigma y, \\
\frac{dy}{dt} &= \rho x - y - xz, \\
\frac{dz}{dt} &= -\beta z + xy,
\end{aligned}$$

have the same phase portrait by taking $p(t) \equiv 0$, $a = b = \sigma$, $\rho = -1$, $\beta = 1$ and under the replacements of z by $z - 1$ and of x by $-x$.

In [42] the authors examine the localization problem of compact invariant sets of nonlinear autonomous systems and apply the results to the Vallis system (2.1). In [40] the localization method for invariant compact sets of the autonomous dynamical system studied in [42] is generalized to the case of a non-autonomous system, and the localization problem for system (2.1) is solved.

In this chapter we provide sufficient conditions in order that system (2.1) has periodic orbits, and additionally we characterize the stability of them. Other similar works can be found in [29], [30] and [31], the first one following the same lines of the current chapter. The study of existence of periodic orbits in the non-autonomous Vallis system has been poorly considered in the literature. Indeed, as far as we know, the only study in this direction is the Hopf bifurcation analyzed in [62] for a autonomous version of system (2.1).

We observe that the method used here for studying the periodic orbits can be applied to any periodic non-autonomous differential system. Indeed, as commented before, in [29] this method have been applied in order to prove the existence of periodic solutions in a periodic Duffing-Van der Pol oscillator. Besides, the same methodology is applied in [49] for study a FitzHugh-Nagumo system. Concerning this chapter, the results are published in the journal *Discrete and Continuous Dynamical Systems - Series A* and can be found in [28].

2.2 Main results

From now on unless we say the contrary we will call

$$I = \int_0^T p(s)ds.$$

Observe that, once system (2.1) does not depends on ε, the limit cycles obtained also does not depends on that parameter. For this reason, we use the symbol "$\approx$" in order to say that a given point approximates another one. Thus, the limit cycles described in the next results are *small amplitude limit cycles.*

Now we state the main results of the chapter.

Theorem 2.1. *For $I \neq 0$ and $a \neq b$ the Vallis system* (2.1) *has a T-periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{aI}{T(a-b)}, \frac{aI}{T(a-b)}, 1 \right),$$

Moreover this periodic orbit is stable if $a > b$ and unstable if $a < b$.

The stable periodic solution provided by Theorem 2.1 says that the Niño phenomenon exhibits a periodic behavior if the T-periodic function $p(t)$ and the parameters a and b of the system satisfy that $I \neq 0$ and $a > b$. Moreover, Theorem 2.1 states that this periodic solution lives near the point

$$(x, y, z) = \left(\frac{aI}{T(a-b)}, \frac{aI}{T(a-b)}, 1 \right).$$

Since the periodic solutions found in Theorems 2.3, 2.4 and 2.5 are also stable, we can provide a similar interpretation for them as we have done for the periodic solution of Theorem 1.

Theorem 2.2. *For $I \neq 0$ the Vallis system* (2.1) *has a T-periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(-\frac{aI}{Tb}, -\frac{aI}{Tb}, 1 \right),$$

Moreover this periodic orbit is always unstable.

Theorem 2.3. *For $I \neq 0$ the Vallis system* (2.1) *has a T-periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, \frac{I}{T}, 1 \right),$$

Moreover this periodic orbit is always stable.

Theorem 2.4. *For $I \neq 0$ the Vallis system* (2.1) *has a T-periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(\frac{I}{T}, 0, 1 \right),$$

Moreover this periodic orbit is always stable.

In what follows we consider the function

$$J(t) = \int_0^t p(s)ds,$$

and note that $J(T) = I$. So we have the following result.

Theorem 2.5. *Consider $I = 0$ and $J(t) \neq 0$ if $0 < t < T$. Then the Vallis system (2.1) has a T-periodic solution $(x(t), y(t), z(t))$ such that*

$$(x(t), y(t), z(t)) \approx \left(-\frac{a}{T} \int_0^T J(s)ds, 0, 1 \right).$$

Moreover this periodic orbit is always stable.

Now we perform a rescaling of the variables (x, y, z), of the function $p(t)$, and of the parameters a and b, as follows:

$$\begin{aligned} x = \varepsilon^{m_1} X, \qquad y = \varepsilon^{m_2} Y, \qquad z = \varepsilon^{m_3} Z, \\ p(t) = \varepsilon^{n_1} P(t), \qquad a = \varepsilon^{n_2} A, \qquad b = \varepsilon^{n_3} B, \end{aligned} \tag{2.2}$$

where ε is positive and sufficiently small, and m_i and n_j are non-negative integers, for all $i, j = 1, 2, 3$. The following proposition shows that using the rescaling (2.2), we can not detect other periodic solutions of system (2.1) using the averaging theory.

Proposition 2.1. *By using the averaging theory described in Theorem 1.8 joint with the rescalings (2.2) no other periodic solutions close to origin, except the ones presented in Theorems 2.1, 2.2, 2.3, 2.4 and 2.5, can be found in the Vallis system (2.1).*

Following we prove these results.

2.3 Proof of the results

The tool for proving the results is the averaging theory based on the Malkin's bifurcation function. This theory applies to periodic non-autonomous differential systems depending on a small parameter ε. Since the Vallis system already is a T-periodic non-autonomous differential system, in order to apply to it the averaging theory we need to introduce a small parameter in such system. This is reached doing convenient rescalings in the variables (x, y, z), in the parameters (a, b) and in the function $p(t)$, as performed in (2.2). Playing with these rescalings we shall obtain different result on the periodic solutions of the Vallis system. More precisely, using the rescaling (2.2), in the new variables (X, Y, Z) system (2.1) writes

$$
\begin{aligned}
\frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n_3} BY + \varepsilon^{-m_1+n_1+n_2} AP(t), \\
\frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2+m_3} XZ, \\
\frac{dZ}{dt} &= -Z - \varepsilon^{m_1+m_2-m_3} XY + \varepsilon^{-m_3}.
\end{aligned} \tag{2.3}
$$

Consequently, in order to have non-negative powers of ε we must impose the conditions

$$
m_3 = 0 \quad \text{and} \quad 0 \le m_2 \le m_1 \le L, \tag{2.4}
$$

where $L = \min\{m_2 + n_3, n_1 + n_2\}$. So system (2.3) becomes

$$
\begin{aligned}
\frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{-m_1+m_2+n_3} BY + \varepsilon^{-m_1+n_1+n_2} AP(t), \\
\frac{dY}{dt} &= -Y + \varepsilon^{m_1-m_2} XZ, \\
\frac{dZ}{dt} &= 1 - Z - \varepsilon^{m_1+m_2} XY.
\end{aligned} \tag{2.5}
$$

Our aim is to find periodic solutions of system (2.5) for some special values of m_i, n_j, $i, j = 1, 2, 3$, and after we go back through the rescaling (2.2) to guarantee the existence of periodic solutions in system (2.1). In what follows we consider the case where n_2 and n_3 are positives and $m_2 = m_1 < n_1 + n_2$. These conditions lead to the proofs of Theorems 2.1, 2.2 and 2.3. For this reason we present these proofs together in order to avoid repetitive arguments. Moreover, next we consider

$$
K = \int_0^T P(s)ds.
$$

Proofs of Theorems 2.1, 2.2 and 2.3: We start considering system (2.5) with n_2 and n_3 positive and $m_2 = m_1 < n_1 + n_2$. So we have

$$\begin{aligned}
\frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{n_3} BY + \varepsilon^{-m_1+n_1+n_2} AP(t), \\
\frac{dY}{dt} &= -Y + XZ, \\
\frac{dZ}{dt} &= 1 - Z - \varepsilon^{2m_1} XY.
\end{aligned} \tag{2.6}$$

Now we apply the averaging method to the differential system (2.6). In this chapter, different from the notation of Section 1.3 of Chapter 1, we will write $\mathbf{x}$ and $\mathbf{z}$ instead of "x" and "z", respectively, for indicate vectors. Then, calling $\mathbf{x} = (X, Y, Z)^T$ we get

$$f(t, \mathbf{x}) = \begin{pmatrix} 0 \\ -Y + XZ \\ 1 - Z \end{pmatrix}. \tag{2.7}$$

We start considering the system

$$\dot{\mathbf{x}} = f(t, \mathbf{x}). \tag{2.8}$$

Its solution $\mathbf{x}(t, \mathbf{z}, 0) = (X(t), Y(t), Z(t))$ such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = (X_0, Y_0, Z_0)$ is

$$\begin{aligned}
X(t) &= X_0, \\
Y(t) &= (1 - e^{-t}(1 + t))X_0 + e^{-t} Y_0 + e^{-t} t X_0 Z_0, \\
Z(t) &= 1 - e^{-t} + e^{-t} Z_0.
\end{aligned}$$

In order that $\mathbf{x}(t, \mathbf{z}, 0)$ is a periodic solution we must choose $Y_0 = X_0$ and $Z_0 = 1$. This implies that for every point of the straight line $X = Y$, $Z = 1$ passes a periodic orbit that lies in the phase space $(X, Y, Z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$. Here and in what follows $\mathbb{S}^1$ is the interval $[0, T]$ identifying T with 0. Consequently we have an one-dimensional manifold on which each point is periodic by considering $t \in \mathbb{S}^1$ in the phase space $(X, Y, Z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$.

Observe that, using the notation of Section 3 in Chapter 1, we have $n = 3$, $k = 1$, $\alpha = X_0$ and $\beta(X_0) = (X_0, 1)$, and consequently $\mathcal{M}$ is an one-dimensional manifold given by $\mathcal{M} = \{(X_0, X_0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$. The fundamental matrix $U_{\mathbf{z}}(t)$ of the linearization of system (2.8), satisfying that $U_{\mathbf{z}}(0)$ is the identity of $\mathbb{R}^3$, is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 - \cosh t + \sinh t & e^{-t} & e^{-t} t X_0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

and its inverse matrix $U_{\mathbf{z}}^{-1}(t)$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 - e^{t} & e^{t} & -e^{t} t X_0 \\ 0 & 0 & e^{t} \end{pmatrix}.$$

Since the matrix $U_{\mathbf{z}}^{-1}(0) - U_{\mathbf{z}}^{-1}(T)$ has an 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$\Delta = \begin{pmatrix} 1 - e^{T} & e^{T} T X_0 \\ 0 & 1 - e^{T} \end{pmatrix},$$

with $\det(\Delta) = (1 - e^{T})^2 \neq 0$ because $T \neq 0$, we can apply the averaging theory described in Section 1.3 of Chapter 1.

Let F be the vector field of system (2.6) minus f given in (2.7). Then the components of the function $U_{\mathbf{z}}^{-1}(t) g(t, \mathbf{x}(t, \mathbf{z}, 0))$ are

$$\begin{aligned} g_1(X_0, t) &= -\varepsilon^{n_2} A X_0 + \varepsilon^{n_3} B X_0 + \varepsilon^{-m_1+n_1+n_2} A P(t), \\ g_2(X_0, t) &= \varepsilon^{2m_1} e^{t} t X_0^3 + (1 - e^{t}) g_1(X_0, t), \\ g_3(X_0, t) &= -\varepsilon^{2m_1} e^{t} X_0^2. \end{aligned}$$

In order to apply averaging theory of first order we need to consider only terms up to order ε. Analyzing the expressions of g_1, g_2 and g_3 we note that these terms depend on the values of m_1 and n_j, for each $j = 1, 2, 3$. In fact, we just need to study the integral of g_1 because $k = 1$. Moreover studying the function g_1 we observe that the only possibility to obtain an isolated zero of the function

$$f_1(X_0) = \int_0^T g_1(X_0, t)dt$$

is assuming that $n_1+n_2-m_1 = 1$. Otherwise, the only solution of $f_1(X_0) = 0$ is $X_0 = 0$ which correspond to the equilibrium point $(X_0, Y_0, Z_0) = (0, 0, 1)$ of system (2.8). The same occurs if n_2 and n_3 are greater than 1 simultaneously. This analysis reduces to the existence of possible periodic solutions to the following cases:

(p_1) $n_2 = 1$ and $n_3 = 1$;

(p_2) $n_2 > 1$ and $n_3 = 1$;

(p_3) $n_2 = 1$ and $n_3 > 1$.

In the case (p_1) we have $U_{\mathbf{z}}^{-1}(t)g_1(t, \mathbf{x}(t, \mathbf{z}, 0)) = -AX_0 + BX_0 + AP(t)$, and then

$$f_1(X_0) = (-A + B)TX_0 + AK.$$

Consequently, if $A \neq B$, then $f_1(X_0) = 0$ implies

$$X_0 = \frac{AK}{T(A - B)}.$$

So, by using Theorem 1.8 of Chapter 1, we get that system (2.6) has a periodic solution $(X(t, \varepsilon), Y(t, \varepsilon), Z(t, \varepsilon))$ such that

$$(X(0, \varepsilon), Y(0, \varepsilon), Z(0, \varepsilon)) \to (X_0, Y_0, Z_0) = \left(\frac{AK}{T(A - B)}, \frac{AK}{T(A - B)}, 1\right)$$

when $\varepsilon \to 0$. Then, if we take, for instance, $n_1 = n_2 = n_3 = 1$ and going back through the rescaling (2.2) of the variables and parameters, we obtain that the periodic solution of system (2.6) becomes the periodic solution

$(x(t), y(t), z(t))$ of system (2.1) satisfying that

$$(x(t), y(t), z(t)) \approx \left(\frac{aI}{T(a-b)}, \frac{aI}{T(a-b)}, 1 \right).$$

Indeed, observe that

$$x_0 = \varepsilon X_0 = \varepsilon \frac{(a\varepsilon^{-1})(I\varepsilon^{-1})}{T\varepsilon^{-1}(a-b)} = \frac{aI}{T(a-b)}.$$

Moreover, we note that $f_1'(x_0) = \varepsilon f_1'(X_0) = -a + b \neq 0$, so the periodic orbit corresponding to x_0 is stable if $a > b$, and unstable otherwise. This completes the proof of Theorem 2.1.

Analogously the function f_1 in the cases (p_2) and (p_3) is

$$f_1(X_0) = TBX_0 + AK \qquad \text{and} \qquad f_1(X_0) = -TAX_0 + AK,$$

respectively. In the first case the condition $f_1(X_0) = 0$ implies

$$X_0 = -\frac{AK}{TB}.$$

Now we observe that we have $n_2 > 1$ and $n_3 = 1$. So, going back through the rescaling we obtain

$$x_0 = \varepsilon X_0 = \varepsilon \frac{(-a\varepsilon^{-n_2})(I\varepsilon^{-n_1})}{Tb\varepsilon^{-1}} = -\frac{aI}{Tb\varepsilon^{n_1+n_2-2}}.$$

and consequently, choosing $n_1 = 0$ and $n_2 = 2$, we get $x_0 = -aI/(Tb)$. Note also that $f_1'(x_0) = Tb > 0$, then the periodic orbit corresponding to x_0 is always unstable. Thus Theorem 2.2 is proved.

Finally, in the case (p_3), $f_1(X_0) = 0$ implies $X_0 = K/T$. So, taking $n_1 = 1$ and going back through the rescaling, we have $x_0 = \varepsilon X_0 = \varepsilon I/(T\varepsilon) = I/T$. Additionally, we have that $f_1'(x_0) = -Ta < 0$. Therefore the periodic solution that comes from x_0 is always stable. This proves Theorem 2.3. $\square$

Proof of Theorem 2.4: As in the proofs of Theorems 2.1, 2.2 and 2.3 we start considering a more general case in the powers of ε in (2.5) taking $n_2 > 0$ and

$m_2 < m_1 < L$. In this case the function $f(t, \mathbf{x})$ of system (1.4) is

$$f(t, \mathbf{x}) = \begin{pmatrix} 0 \\ Y \\ 1 - Z \end{pmatrix}. \tag{2.9}$$

Then the solution $\mathbf{x}(t, \mathbf{z}, 0)$ of system (1.4) satisfying $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z}$ is

$$(X(t), Y(t), Z(t)) = (X_0, e^{-t} Y_0, 1 - e^{-t} + e^{-t} Z_0).$$

This solution is periodic if $Y_0 = 0$ and $Z_0 = 1$. Then for every point of the straight line $Y = 0$, $Z = 1$ passes a periodic orbit that lies in the phase space $(X, Y, Z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$. We observe that using the notation of Section 3 in Chapter 1 we have $n = 3$, $k = 1$, $\alpha = X_0$ and $\beta(\alpha) = (0, 1)$. Consequently $\mathcal{M}$ is an one-dimensional manifold given by $\mathcal{M} = \{(X_0, 0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$.

The fundamental matrix $U_{\mathbf{z}}(t)$ of the linearization of system (1.4) with f given by (2.9) satisfying $U_{\mathbf{z}}(0) = Id_3$ and its inverse $U_{\mathbf{z}}^{-1}(t)$ are given by

$$U_{\mathbf{z}}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad U_{\mathbf{z}}^{-1}(t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}.$$

Since the matrix $U_{\mathbf{z}}^{-1}(0) - U_{\mathbf{z}}^{-1}(T)$ has an 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$\Delta = \begin{pmatrix} 1 - e^T & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta) = (1 - e^T)^2 \neq 0$, we can apply the averaging theory. Again using the notations introduced in the proofs of Theorems 2.1, 2.2 and 2.3, since $k = 1$ we will look only to the integral of the first coordinate of $M = (f_1, f_2, f_3)$. In this case we have

$$g_1(X_0, Y_0, Z_0, t) = -\varepsilon^{n_2} A X_0 + \varepsilon^{-m_1 + n_1 + n_2} A P(t).$$

Comparing this function g_1 with the same function obtained in the proof of Theorems 2.1, 2.2 and 2.3, it is easy to see that this case correspond to the case (p_3) of the mentioned theorems. Then, in order to have periodic solutions, we need to choose $n_2 = 1$ and $n_1 + n_2 - m_1 = 1$. So, following the steps of the proof of case (p_3) by choosing $n_1 = 1$ and coming back through the rescaling (2.2) to system (2.1), Theorem 2.4 is proved. $\square$

Proof of Theorem 5*:* Now we start considering system (2.5) with $n_3 = 2$, $n_2 > 0$, $m_1 = n_1 + n_2$ and $m_2 < m_1 < m_2 + n_3$. With these conditions system (2.5) becomes

$$\begin{aligned} \frac{dX}{dt} &= -\varepsilon^{n_2} AX + \varepsilon^{m_2 - n_1 - n_2 + n_3} BY + AP(t), \\ \frac{dY}{dt} &= -Y + \varepsilon^{-m_2 + n_1 + n_2} XZ, \\ \frac{dZ}{dt} &= 1 - Z - \varepsilon^{m_2 + n_1 + n_2} XY. \end{aligned} \tag{2.10}$$

Once we are using the averaging method, by considering $\mathbf{x} = (X, Y, Z)^T$ and following the notation of Chapter 1 we obtain

$$f(t, \mathbf{x}) = \begin{pmatrix} AP(t) \\ -Y \\ 1 - Z \end{pmatrix}. \tag{2.11}$$

Now we note that the solution $\mathbf{x}(t, \mathbf{z}, 0) = (X(t), Y(t), Z(t))$ such that $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z} = (X_0, Y_0, Z_0)$ of the system

$$\dot{\mathbf{x}} = f(t, \mathbf{x}) \tag{2.12}$$

is

$$X(t) = X_0 + \int_0^t AP(s)ds, \quad Y(t) = e^{-t} Y_0, \quad Z(t) = 1 - e^{-t} + e^{-t} Z_0.$$

Since $I = 0$ and $J(t) \neq 0$ for $0 < t < T$, in order that $\mathbf{x}(t, \mathbf{z}, 0)$ is a periodic solution we need to fix $Y_0 = 0$ and $Z_0 = 1$. This implies that for every point in a neighbourhood of X_0 in the straight line $Y = 0$, $Z = 1$ passes a periodic orbit that lies in the phase space $(X, Y, Z, t) \in \mathbb{R}^3 \times \mathbb{S}^1$.

Following the notation of Chapter 1, we have $n = 3$, $k = 1$, $\alpha = X_0$ and $\beta(X_0) = (0, 1)$. Hence $\mathcal{M}$ is a one-dimensional manifold $\mathcal{M} = \{(X_0, 0, 1) \in \mathbb{R}^3 : X_0 \in \mathbb{R}\}$ and the fundamental matrix $U_{\mathbf{z}}(t)$ of the linearization of system (2.12) satisfying that $U_{\mathbf{z}}(0)$ is the identity of $\mathbb{R}^3$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}.$$

It is easy to see that the matrix $U_{\mathbf{z}}^{-1}(0) - U_{\mathbf{z}}^{-1}(T)$ has a 1×2 zero matrix in the upper right corner and a 2×2 lower right corner matrix

$$\Delta = \begin{pmatrix} 1 - e^T & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

with $\det(\Delta) = (1 - e^T)^2 \neq 0$. Then the hypotheses of Theorem 1.8 of the previous chapter are satisfied. Now the components of the function $U_{\mathbf{z}}^{-1}(t) g(t, \mathbf{x}(t, \mathbf{z}, \mathbf{0}))$ are

$$\begin{aligned} g_1(X_0, t) &= -\varepsilon^{n_2} A \left(X_0 + \int_0^t AP(s) ds \right) + AP(t), \\ g_2(X_0, t) &= \varepsilon^{-m_2 + n_1 + n_2} \left(X_0 + \int_0^t AP(s) ds \right) e^t, \\ g_3(X_0, t) &= 0. \end{aligned}$$

Taking n_1 and n_2 equal to one and observing that $k = 1$ and $n = 3$, we are interested only in the first component of the function g. Indeed, applying the averaging theory we must study the zeros of the first component of the function

$$M(X_0) = (f_1(X_0), f_2(X_0), f_3(X_0)) = \int_0^T U_{\mathbf{z}}^{-1}(t, \mathbf{z}) g(t, \mathbf{x}(t, \mathbf{z})) dt.$$

Since such component writes

$$-A\left(X_0 + \int_0^t AP(s)ds\right),$$

we have

$$\begin{aligned} f_1(X_0) &= \int_0^T -A\left(X_0 + \int_0^t AP(s)ds\right) dt \\ &= -ATX_0 - A^2\int_0^T\left(\int_0^t P(s)ds\right) ds. \end{aligned}$$

Therefore, from $f_1(X_0) = 0$ we obtain

$$X_0 = -\frac{A}{T}\int_0^T\left(\int_0^t P(s)ds\right) ds \neq 0.$$

So, using rescaling (2.2) we get

$$x_0 = \varepsilon^2 X_0 = -\varepsilon^2\frac{a\varepsilon^{-1}}{\varepsilon T}\int_0^T J(s)ds = -\frac{a}{T}\int_0^T J(s)ds.$$

Moreover, since $f_1'(x_0) = -a/T < 0$, because a and ε are positive, the T-periodic orbit detected by the averaging theory is always stable. This ends the proof. □

In order to prove Proposition 2.1 we will study all possible powers of the different $\varepsilon' s$ in system (2.5). Indeed, we consider the set $P = \{n_2, -m_1+m_2+n_3, -m_1+n_1+n_2, m_1-m_2\}$ of the relevant powers of ε in this system (see (2.4) and (2.5)), and observe that each integer of P must be non-negative. Therefore, we will study each one of the 16 possible combinations of values of the elements of P taking into account conditions (2.4). We start considering $n_2 > 0$. Then we have the following eight cases:

Case 1: $n_2 > 0$, $m_1 = m_2$, $n_3 = 0$ and $m_1 = n_1 + n_2$,

Case 2: $n_2 > 0$, $m_1 = m_2$, $n_3 = 0$ and $m_1 < n_1 + n_2$,

Case 3: $n_2 > 0$, $m_1 = m_2$, $n_3 > 0$ and $m_1 = n_1 + n_2$, Case 4: $n_2 > 0$, $m_1 = m_2$, $n_3 > 0$ and $m_1 < n_1 + n_2$,

Case 5: $n_2 > 0$, $m_1 > m_2$, $n_3 > 0$ and $m_1 = n_1 + n_2$,

Case 6: $n_2 > 0$, $m_1 > m_2$, $n_3 > 0$ and $m_1 < n_1 + n_2$,

Case 7: $n_2 > 0$, $m_1 > m_2$, $m_1 < m_2 + n_3$ and $m_1 = n_1 + n_2$,

Case 8: $n_2 > 0$, $m_1 > m_2$, $m_1 < m_2 + n_3$ and $m_1 < n_1 + n_2$.

The remainder cases from 9 to 16 are the same than the cases from 1 to 8, respectively, taking $n_2 = 0$ instead of $n_2 > 0$. We stress out that other rescaling may lead to different cases, so it can exist other periodic orbits close to the origin, apart from those ones presented in the previous theorems. Also, by applying hight order averaging, one may obtain more periodic orbits. In this chapter we do not do this due to the large number of cases by considering other possibilities for the power of ε.

We observe that the case 4 was studied in Theorems 2.1, 2.2 and 2.3. Additionally, Theorem 2.4 concerns to case 8, and Theorem 2.5 deals to case 7. Thus we will eliminate these cases in the proof of Proposition 2.1. In the other cases we will prove that some hypotheses of the averaging method do not hold.

Proof of Proposition 2.1*:* First we prove the proposition using system (2.5) in case 2. Indeed, considering $\mathbf{x} = (X, Y, Z)$, in case 2 system (1.4) becomes

$$\dot{\mathbf{x}} = f(t, \mathbf{x}) = (BY, -Y + XZ, 1 - Z)^T. \tag{2.13}$$

This last differential equation is uncoupled and its solution $Z(t)$ is $Z(t) = 1 - e^{-t} + e^{-t} Z_0$. It is easy to see that $Z_0 = 1$ is the only value of Z_0 for which $Z(t)$ is periodic. Now substituting the solution $Z(t)$ in the second differential equation of (2.13) and solving the system of differential equations $\dot{X} = BY$, $\dot{Y} = -Y + X$ we get

$$\begin{aligned} X(t) &= \frac{1}{2C}\left(C_1 e^{\frac{1}{2}(-C-1)t} + C_2 e^{\frac{1}{2}(C-1)t}\right), \\ Y(t) &= \frac{1}{2C}\left(C_3 e^{\frac{1}{2}(-C-1)t} + C_4 e^{\frac{1}{2}(C-1)t}\right), \end{aligned}$$

where $C = \sqrt{1+4B} > 1$, $C_1 = (C-1)X_0 - 2BY_0$, $C_2 = (C+1)X_0 + 2BY_0$, $C_3 = -2X_0 + (C+1)Y_0$ and $C_4 = 2X_0 + (C-1)Y_0$.

Without loss of generality we will study the conditions that turn the solution $X(t)$ into a periodic function. In order to do this, we need to choose C_1 and C_2 equal to zero because $C > 1$. Fixing $C_1 = 0$ we obtain

$$X_0 = \frac{-2BY_0}{C-1}.$$

Replacing this value into C_2 we get $(1+4B+C)Y_0$ which is positive unless $Y_0 = 0$. On the other hand the value $Y_0 = 0$ implies $X_0 = 0$, and since $Z_0 = 1$ we have the equilibrium point $(0,0,1)$ of system (2.13). This implies that system (2.13) has no periodic solutions, and then the averaging method cannot be applied in this case. Moreover, there is no loss of generality when we study only the solution $X(t)$ because if one of the functions $X(t)$, $Y(t)$ or $Z(t)$ of (1.4) is not periodic, system (2.13) cannot have a periodic solution. We will use this fact to end the proof of Proposition 2.1 in some other cases below.

In what follows we prove Proposition 2.1 for system (2.5) in case 10. Indeed observe that system (1.4) now writes

$$\dot{\mathbf{x}} = (-AX + BY, -Y + XZ, 1 - Z)^T. \tag{2.14}$$

As before we take the solution $Z(t) = 1 - e^{-t} + e^{-t}Z_0$ of $\dot{Z} = 1 - Z$ and we replace this solution with $Z_0 = 1$ in the other differential equations of (2.14). Therefore the solution $X(t)$ is

$$X(t) = \frac{1}{2D}\left(D_1 e^{\frac{1}{2}(-A-1-D)t} + C_2 e^{\frac{1}{2}(-A-1+D)t}\right)$$

where $D = \sqrt{(A-1)^2 + 4B} > 0$, $D_1 = (A-1+D)X_0 - 2BY_0$ and $D_2 = (-A+1+D)X_0 + 2BY_0$. We note that this expression is very similar to the expression of the solution $X(t)$ of system (2.5) in case 2 just taking A as zero. Moreover, it is possible to show that the same arguments used in case 2 are also true in this case, and consequently the averaging method does not apply to system (2.5) in case 10.

In case 12 we have

$$\dot{\mathbf{x}} = (-AX, -Y + XZ, 1 - Z)^T. \tag{2.15}$$

So the solutions $X(t)$ and $Z(t)$ for this system are, respectively, $X(t) = e^{-At}X_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. Then choosing $X_0 = 0$ and $Z_0 = 1$ in order that $X(t)$ and $Z(t)$ be periodic, the solution $Y(t)$ becomes $Y(t) = e^{-t}Y_0$, and consequently we have to take $Y_0 = 0$. However, the point $(X_0, Y_0, Z_0) = (0, 0, 1)$ is the equilibrium point of system (2.15), and therefore system (2.15) has no periodic solutions. Thus, in case 12 again we cannot apply the averaging theory.

In case 14 system (1.4) is

$$\dot{\mathbf{x}} = (-AX + BY, -Y, 1 - Z)^T, \tag{2.16}$$

whose solutions $Y(t)$ and $Z(t)$ starting at Y_0 and Z_0 are, respectively, $Y(t) = e^{-t}Y_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. These solutions are periodic if $Y_0 = 0$ and $Z_0 = 1$, and with these values the solution $X(t)$ writes $X(t) = e^{-At}X_0$. So, since $A \neq 0$, we need to take $X_0 = 0$ for having $X(t)$ periodic. The conclusion of Proposition 2.1 in this case follows as in case 12.

For proving Proposition 2.1 in case 16 we observe that the solutions $X(t)$, $Y(t)$ and $Z(t)$ of system (1.4) given by

$$\dot{\mathbf{x}} = (-AX, -Y, 1 - Z)^T, \tag{2.17}$$

are $X(t) = e^{-At}X_0$, $Y(t) = e^{-t}Y_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. The values X_0, Y_0 and Z_0 for which these solutions are periodic are $(X_0, Y_0, Z_0) = (0, 0, 1)$. So as before we cannot apply the averaging theory.

Now we prove that the averaging method does not work in system (2.5) in case 5. In fact, in this case the solutions $X(t)$, $Y(t)$ and $Z(t)$ of system

(1.4) starting at (X_0, Y_0, Z_0) are

$$(X(t), Y(t), Z(t)) \;=\; \left(X_0 + \int_0^t AP(s)ds, e^{-t}Y_0, 1 - e^{-t} + e^{-t}Z_0\right),$$

where now we suppose that $\int_0^T p(s)ds = 0$ in order that $x(t)$ be a T-periodic solution. Taking into account to the expressions of $Y(t)$ and $Z(t)$, it is easy to see that $Y_0 = 0$ and $Z_0 = 1$ are the only values of Y_0 and Z_0 for which $Y(t)$ and $Z(t)$ are periodic. We observe that using the notation of Section 1.3 in Chapter 1, we have $k = 1$, $n = 3$ and the fundamental matrix $U_{\mathbf{z}}(t)$ is

$$\begin{pmatrix} 1 & B(1-e^{-x}) & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}.$$

Its inverse matrix is

$$U_{\mathbf{z}}^{-1}(t) = \begin{pmatrix} 1 & B(1-e^{t}) & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}.$$

So the matrix $U_{\mathbf{z}}^{-1}(0) - U_{\mathbf{z}}^{-1}(T)$ does not have a 1×2 zero matrix in the upper right corner, because

$$U_{\mathbf{z}}^{-1}(0) - U_{\mathbf{z}}^{-1}(T) = \begin{pmatrix} 0 & B(e^{T}-1) & 0 \\ 0 & 1-e^{T} & 0 \\ 0 & 0 & 1-e^{T} \end{pmatrix}.$$

Indeed, since B is positive, we cannot apply averaging method in case 5.

Case 6 is similar to case 5. In fact, the solution $X(t)$, $Y(t)$ and $Z(t)$ of system (1.4) starting at (X_0, Y_0, Z_0) eliminating the non-periodic terms is

$$(X(t), Y(t), Z(t)) = (X_0, 0, 1),$$

and following the steps of Section 3 in Chapter 1 we obtain the same matrix $U_{\mathbf{z}}^{-1}(0) - U_{\mathbf{z}}^{-1}(T)$ of case 5. Hence we cannot apply the averaging method in this case.

Next we prove Proposition 2.1 in case 3. The solutions $X(t)$, $Y(t)$ and $Z(t)$ of system (1.4) are

$$
\begin{aligned}
X(t) &= X_0 + \int_0^t AP(s)ds, \\
Y(t) &= X_0 - e^{-t}X_0 + e^{-t}Y_0 + \int_0^t AP(s)ds - e^{-t}\int_0^t Ae^sP(s)ds, \\
Z(t) &= 1 - e^{-t} + e^{-t}Z_0,
\end{aligned}
$$

where we suppose that $I = 0$ in order that $X(t)$ be a T-periodic solution. Observe that if $I \neq 0$, $X(t)$ is not periodic and then we cannot apply the averaging method. The expression of $Z(t)$ implies that $Z_0 = 1$ is the only value of Z_0 for which $Z(t)$ is periodic. Moreover, we take $X_0 = Y_0 + W$ such that the solutions $X(t)$, $Y(t)$ and $Z(t)$ become

$$
\begin{aligned}
X(t) &= Y_0 + W + A\int_0^t P(s)ds, \\
Y(t) &= Y_0 + W + A\int_0^t P(s)ds - e^{-t}\left(W - A\int_0^t e^sP(s)ds\right), \\
Z(t) &= 1,
\end{aligned}
$$

where $\int_0^t P(s)ds$ is periodic because $I = 0$ and we suppose that $e^{-t}(W - A\int_0^t e^sP(s)ds)$ is periodic. Note that if a such W does not exists, then $Y(t)$ is non-periodic and the averaging theory does not apply. Hence, we assume that a such W exists and the solution $Y(t)$ is T-periodic.

We note that if $P(t) = \cos t$ and $W = A/2$ the solutions $X(t)$, $Y(t)$ and

$Z(t)$ are periodic, because with these considerations we have

$$X(t) = Y_0 + (A/2) + A \sin t,$$
$$Y(t) = (1/2)(A + 2Y_0 - A \cos t + A \sin t),$$
$$Z(t) = 1,$$

which are periodic functions. However we want to consider the general case instead of this particular case $P(t) = \cos t$ and $W = A/2$. Hence using the notation of Section 1.3 in Chapter 1, we have $k = 1$, $n = 3$ and the fundamental matrix $U_{\mathbf{z}}(t)$ is

$$\begin{pmatrix} e^t & 1 - e^t & e^t E(A, W, Y_0, t) \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{pmatrix},$$

where $E(A, W, Y_0, t) = \displaystyle\int_0^t e^{-2s} \left(Y_0 + W + A \int_0^s P(w)dw \right) ds$. Its inverse matrix $U_{\mathbf{z}}^{-1}(t)$ is

$$\begin{pmatrix} e^{-t} & 1 - e^{-t} & -e^t E(A, W, Y_0, t) \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{pmatrix}.$$

Then the matrix $U_{\mathbf{z}}^{-1}(0) - U_{\mathbf{z}}^{-1}(T)$ has a 2×2 lower right matrix

$$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & 1 - e^T \end{pmatrix},$$

whose determinant is zero. Then we cannot apply the averaging method in case 3.

We study system (2.5) in case 1. Now system (1.4) is

$$(\dot{X}, \dot{Y}, \dot{Z}) = (BY + AP(t), -Y + XZ, 1 - Z)^T. \tag{2.18}$$

This differential equation is uncoupled and its solution $Z(t)$ is $Z(t) = 1-e^{-t}+e^{-t}Z_0$. As before if $Z_0 = 1$ then $Z(t)$ is a periodic solution. Now substituting the solution $Z(t)$ in the second differential equation of (2.18) with $Z_0 = 1$ and solving the system of differential equations $\dot{X} = BY + AP(t)$, $\dot{Y} = -Y + X$ we get solutions very similar to the ones obtained in case 2. In fact, denoting by $X_2(t)$ and $Y_2(t)$ the solutions of case 2, for case 1 the solutions $X(t)$ can be written as

$$X(t) = X_2(t) + (1/2C)g_2(A, B, t)e^{\frac{1}{2}(-C-1)t},$$

where g_2 is

$$\frac{A}{2C}(-1 + C + Ce^{Ct}) \int_0^t e^{-\frac{1}{2}(-1+C)s} \left(1 + C + (-1 + C)e^{Cs}\right) P(s)ds+$$
$$\frac{2AB}{C}(1 - e^{Ct}) \int_0^t e^{-\frac{1}{2}(-1+C)s} \left(-1 + e^{Cs}\right) P(s)ds.$$

We observe that g_2 does not depend neither of X_0 nor of Y_0. For this reason we cannot eliminate the non-periodic terms of $X_2(t)$ through the expression $(1/2C)g_2(A, B, t)e^{\frac{1}{2}(-C-1)t}$, whatever be the function $g_2(A, B, t)$ chosen. So as we see in case 2 we must choose $(X_0, Y_0) = (0, 0)$ in order that $X_2(t)$ be periodic. Since $Z_0 = 1$, system (2.18) has no sub-manifold of periodic solutions as needs the averaging theory.

Case 9 is similar to case 1 in the sense that there is no choice of X_0, Y_0 and Z_0 in such way that the solution of the system

$$\dot{\mathbf{x}} = (-AX + BY + AP(t), -Y + XZ, 1 - Z)^T. \tag{2.19}$$

corresponding to system (2.5) in case 9 has a sub-manifold of periodic solutions. As before, $Z(t) = 1 - e^{-t} + e^{-t}Z_0$ is the solution of the last differential equation of system (2.19), and the value Z_0 for which this solution is periodic is $Z_0 = 1$. Substituting this solution into system (2.19) and solving it, we obtain a solution similar to the solution of system (2.14) in case 10 denoted here by $X_{10}(t)$. We have

$$X(t) = X_{10}(t) + g_{10}(A, B, t),$$

where g_{10} is

$$\frac{A}{2D}(-1 + A\left(1 - e^{Dt}\right) + D + (1 + D)e^{Dt}) \int_0^t e^{\frac{1}{2}(1+A-D)s} \left(1 + D + (-1 + D)e^{Ds} + A\left(-1 + e^{Ds}\right)\right) P(s)ds + \frac{2AB}{D}(1 - e^{Dt}) \int_0^t e^{\frac{1}{2}(1+A-D)s} \left(-1 + e^{Ds}\right) P(s)ds.$$

Note that g_{10} does not depend neither of X_0 nor of Y_0. The conclusion of this case follows from the fact that X_{10} is non-periodic unless $(X_0, Y_0) = (0, 0)$, and using the same arguments of the proof of case 1.

Now we consider system (2.5) in case 11

$$\dot{\mathbf{x}} = (-AX + AP(t), -Y + XZ, 1 - Z)^T. \tag{2.20}$$

Considering $A \neq 1$, as before we have $Z(t) = 1 - e^{-t} + e^{-t}Z_0$ and choose $Z_0 = 1$ because $Z(t)$ must be periodic. The solution $X(t)$ is

$$X(t) = e^{-At}X_0 + A\int_0^t e^{As}P(s)ds.$$

This means that we must take $X_0 = 0$ to have $X(t)$ periodic. Substituting $X_0 = 0$ and $Z(t) = 1$ in $Y(t)$, it becomes

$$Y(t) = Y_0e^{-t} + \frac{1}{A-1}e^{-(A+1)t}h_{11}(A, t),$$

where now h_{11} does not depends on Y_0 and writes

$$A(e^{At} - e^t)\int_0^t e^{As}P(s)ds + Ae^{At}\int_0^t \left(e^s - e^{As}\right)P(s)ds.$$

Since $A \neq 1$ and h_{11} does not depend on Y_0 we cannot eliminate the non-periodic term $Y_0 e^{-t}$ of $Y(t)$ unless we take $Y_0 = 0$. Consequently, as in cases 1 and 9 the averaging theory does not work out in case 11.

Moreover, if $A = 1$, we have the same solutions $X(t)$ and $Z(t)$. So, considering again $X_0 = 0$ and $Z_0 = 1$ the solution $Y(t)$ becomes $Y(t) = e^{-t}Y_0 + h(t)$, where h does not depend on Y_0. Hence, considering $A = 1$ again we cannot eliminate the non-periodic term $Y_0 e^{-t}$ of $Y(t)$ unless $Y_0 = 0$, and therefore the averaging cannot be applied.

Cases 13 and 15 can be proved in a similar way. More precisely, systems (1.4) corresponding to system (2.5) in cases 13 and 15 are

$$\dot{\mathbf{x}} = (-AX + BY + AP(t), -Y, 1 - Z)^T, \tag{2.21}$$

and

$$\dot{\mathbf{x}} = (-AX + AP(t), -Y, 1 - Z)^T, \tag{2.22}$$

respectively. In both cases solutions $Y(t)$ and $Z(t)$ are $Y(t) = e^{-t}Y_0$ and $Z(t) = 1 - e^{-t} + e^{-t}Z_0$. So in order to $Y(t)$ and $Z(t)$ be periodic we take $Y_0 = 0$ and $Z_0 = 1$. Then the solution $X(t)$ becomes

$$X(t) = e^{-At}X_0 + A\int_0^t e^s P(s)ds.$$

Again once $g(t) = \displaystyle\int_0^t e^s P(s)ds$ does not depend on X_0 it is not possible to eliminate the non-periodic term $e^{-At}X_0$ from $X(t)$ unless we take $X_0 = 0$. Therefore both systems (2.21) and (2.22) do not have a sub-manifold $\mathcal{M}$ filled with periodic solutions. Hence the averaging theory cannot be applied in cases 13 and 15. $\square$

2.4 Discussions and conclusions

In this chapter we have achieved sufficient conditions for the existence, location and stability of 5 periodic orbits for the Vallis differential system (2.1). These periodic orbits are distinct from the periodic orbit verified by a Hopf bifurcation in previous works on the Vallis systems. The non-autonomous version of the Vallis systems have been considered instead of the autonomous one. The method of averaging coming from the Malkin's bifurcation function, which is applicable usually in perturbed system, is applied for a system that has neither perturbation term nor equilibrium points.

The methodology used in order to reach the results consist in performing different rescaling in the variables, parameters and function of system (2.1). Additionally, each rescaling into the form (2.2) is considered and all cases are treated. In particular, we have shown that by using such rescaling and the method presented in Section 1.3 of Chapter 1, there is no periodic orbits different from those ones presented throughout this chapter.

Chapter 3

Bifurcation of limit cycles from a non-smooth perturbation of a two-dimensional isochronous cylinder

Detect the birth of limit cycles in non-smooth vector fields is a very important matter into the recent theory of dynamical systems and applied sciences. The goal of this chapter is to study the bifurcation of limit cycles from a continuum of periodic orbits filling up a two-dimensional isochronous cylinder of a vector field in $\mathbb{R}^3$. The approach involves the regularization process of non-smooth vector fields and a method based on the Malkin's bifurcation function for $\mathcal{C}^0$ perturbations. The results provide sufficient conditions in order to obtain limit cycles emerging from the considered cylinder through smooth and non-smooth perturbations of it. To the best of our knowledge they also illustrate the implementation by the first time of a new method based on the Malkin's bifurcation function. In addition, some points concerning the number of limit cycles bifurcating from non-smooth perturbations compared with smooth ones are studied. In summary the results yield a better knowledge about limit cycles in non-smooth vector fields in $\mathbb{R}^3$ and explicit a manner to obtain them by performing non-smooth perturbations in codimension one

Euclidean manifolds. The content of this chapter can be found in [17].

3.1 Setting the problem

This chapter concerns with the existence of limit cycles emerging from a continuum of periodic solutions filling up a two dimensional cylinder via a non-smooth perturbation. Such kind of problems are closed related to the weakest version of the famous 16th Hilbert's problem proposed by Arnol'd (see [1] and [2]). As commented in the introduction, Arnol'd asked about the number of limit cycles bifurcating from the perturbation of a center and up to now many authors have contributed with this subject. However, the problems of perturbation of a sub-manifold filled up by periodic solutions which appears in the literature are usually restricted to the plane. In our opinion the perturbation of other kind of two-dimensional manifolds has been poorly treated in the literature.

Recently in [48] the authors investigated the problem of perturbation of a two-dimensional cylinder filled up by periodic solutions in $\mathbb{R}^3$ by a smooth function. In their paper, the authors illustrated the implementation of a method based on the averaging theory for computing the limit cycles bifurcating from a continuum of periodic solutions occupying a cylinder. Other papers with similar approaches can be found in [46] and [47].

In this chapter the goal is to generalize the study presented in [48] for a bigger class of cylinders and also take into account non-smooth perturbations. We stress out that this is not the situation considered in the paper [48]. We consider the differential system

$$\begin{aligned} \dot{x} &= -y + x(x^2 + y^2 - 1), \\ \dot{y} &= x + y(x^2 + y^2 - 1), \\ \dot{z} &= h(x, y) \end{aligned} \tag{3.1}$$

with $h \in \mathcal{C}^2$. Observe that the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is an invariant set for system (3.1), once in cylindrical coordinates we get

$\dot{r} = r(r^2-1)$, where r is the radius. The solution passing through the point $(\cos\theta_0, \sin\theta_0, z_0) \in C$ at time $t=0$ is $x(t) = \cos(t+\theta_0)$, $y(t) = \sin(t+\theta_0)$ and

$$z(t) = z_0 + \int_0^t h(\cos(s+\theta_0), \sin(s+\theta_0))ds. \tag{3.2}$$

Consequently the solutions on the cylinder C are periodic if the last integral is periodic. In order to verify such property about this integral, we must impose some conditions on the function h. Otherwise, the cylinder is invariant but not filled up with periodic orbits. Indeed, we will consider the functions h which can be written into the form $h(x,y) = \rho(x^2+y^2)\overline{h}(x,y)$, where $\overline{h}(x,y) = \sum_{i+j\geq 1} a_{ij}x^iy^j$, $a_{ij} \in \mathbb{R}$. Then we will achieve conditions on the natural values i and j for which the integral

$$\int_0^t h(\cos s, \sin s)ds = \rho(1) \cdot \sum_{i+j\geq 1} a_{ij} \int_0^t \cos^i s\, \sin^j s\, ds, \tag{3.3}$$

is periodic, when now we take $\theta_0 = 0$ in order to simplify the expressions. Note that we are evaluating the integral in cylindrical coordinates taking $r=1$ (in order to address the cylinder). The expression into the integral takes the following form

$$\cos^i s\, \sin^j s = \sum_{m=0}^{\left[\frac{i+j}{2}\right]} c_m \cos((i+j-2m)s),$$

or

$$\cos^i s\, \sin^j s = \sum_{m=0}^{\left[\frac{i+j}{2}\right]} d_m \sin((i+j-2m)s),$$

if $i+j$ is even or odd, respectively (see [18]). Using the formulae below and a table of integrals one can see that in both cases the integral are periodic unless $i+j-2m=0$ when $i+j$ is even. Indeed, in such case the cosine of the first expression provide a constant term which is not periodic after integration. However, the condition $i+j-2m=0$ when j is even implies that i is also even. Then in order to live the last integral of equality (3.3) periodic we must impose that i and j can not be even simultaneously. Moreover, by playing with the series expansion of the function $\overline{h}(x,y)$, it can be put into the following form

$$\overline{h}(x,y) = h_1(x^2,y^2) + x\,h_2(x^2,y^2) + xy\,h_3(x^2,y^2) + y\,h_4(x^2,y^2).$$

Hence, since the power of x and y can not be even simultaneously, we are interested in the class of functions presenting the form $\tilde{h}(x,y) = x\,\phi(x^2,y^2) + xy\,\chi(x^2,y^2) + y\,\psi(x^2,y^2)$. Therefore, since the periodic orbits live on the cylinder C, we will take into account that the functions $h(x,y) = \rho(x^2 + y^2)\tilde{h}(x,y)$ once $\rho(r^2\cos^2\theta + r^2\sin^2\theta) = \rho(1)$ for $r = 1$ in polar coordinates.

In this chapter we perform a non-smooth perturbation in system (3.1). It means that we consider two special perturbations of system (3.1) depending on the region of $\mathbb{R}^3$, which lead us to a non-smooth system. The results are obtained by using the Malkin's bifurcation function (see [11]) after the performing of a regularization of such non-smooth system. We stress out that apart from the results presented in this chapter, it has an especial importance because we exhibit a thoroughly implementation of the method presented in Section 1.3 of Chapter 1. As far as we know, there is no other examples of implementation of such method in the literature.

Indeed, we will consider a plane separating the cylinder C into two parts in order to perturb each one into two different functions. Nevertheless, due to the arrangement of C which is around the z-axis, we should take the plane containing this axis, so every periodic orbit on the cylinder is divided by the plane. Here, in order to simplify the calculations, we will take $y = 0$ as this plane. As we commented, observe that each orbit on the cylinder intersects Σ transversally in two distinct points. It is clear that the switching manifold in this case is given by $\Sigma = F^{-1}(0)$ where $F(x,y,z) = y$. Note that the intersection of the cylinder C with Σ are the straight lines $x = \pm 1$; note also that Σ separates C in two connected components (see Figure 3.1).

Now we perturb system (3.1). Taking into account the geometry of Σ, we

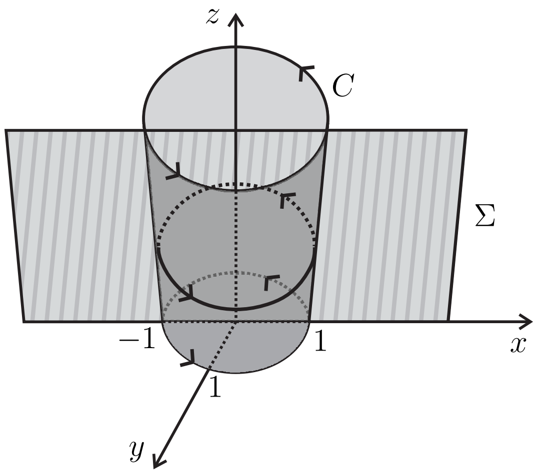

Figure 3.1: The intersection between Σ and C.

will consider the polynomials $g^{\pm} = (p^{\pm}, q^{\pm}, r^{\pm})$ given by

$$\begin{aligned} p^{\pm}(x,y,z) &= \sum_{i+j+k\leq m} a^{\pm}_{ijk} x^i y^j z^k, \\ q^{\pm}(x,y,z) &= \sum_{i+j+k\leq n} b^{\pm}_{ijk} x^i y^j z^k, \\ r^{\pm}(x,y,z) &= \sum_{i+j+k\leq p} c^{\pm}_{ijk} x^i y^j z^k, \end{aligned} \tag{3.4}$$

with $i, j, k \in \mathbb{N}$ and $a_{ijk}, b_{ijk}, c_{ijk} \in \mathbb{R}$, $\forall i, j, k \in \mathbb{N}$. Moreover, consider the function

$$g(x,y,z) = \frac{1}{2}(g^+(x,y,z) + g^-(x,y,z)) + \frac{\operatorname{sgn}(y)}{2}(g^+(x,y,z) - g^-(x,y,z)),$$

and observe that the expression of the function g take different forms depending on the signal of y, i.e., $g(\mathbf{x}) = g^+(\mathbf{x})$ if $\mathbf{x} \in \Sigma^+ = \{y \geq 0\}$ and $g(\mathbf{x}) = g^-(\mathbf{x})$ if $\mathbf{x} \in \Sigma^- = \{y \leq 0\}$ for each $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} = (x, y, z)$. Then, by performing a perturbation in system (3.1) through the non-smooth function g we obtain the non-smooth differential system

$$\dot{\mathbf{x}}_{\varepsilon} = f(\mathbf{x}) + \varepsilon g(\mathbf{x}). \tag{3.5}$$

where $f(\mathbf{x})$ is the vector field of system (3.1) and ε is a small parameter. We will call X the vector field defined in Σ^+ and Y the vector field defined in Σ^-.

Following the Filippov's convention, we have $XF(x,y,z) = YF(x,y,z) = y(x^2+y^2-1)+x$ when $\varepsilon = 0$ and then $XF(\pm 1, 0, z) \cdot YF(\pm 1, 0, z) = 1$ Therefore $C \cap \Sigma \subset \Sigma^c$. Also, if $|\varepsilon| \neq 0$ is sufficiently small, the intersection $C \cap \Sigma$ still occurs in sewing points since the transversality of the solutions passing through sewing points is stable.

A powerful tool for study the perturbation of a continuum of periodic solutions as system (3.5) is the averaging theory. Despite, in [43] the authors exhibits a result based on the averaging theory where it is possible to consider non-smooth vector fields into the *standard form*, i.e., when $f(t,X) \equiv 0$. However, system (3.5) is not in the standard form, then we can not apply the results obtained in [43]. In fact, once function g in system (3.5) is non-smooth, as far as we know there is no perturbation method in the literature that works out in this system. Nevertheless, in those cases where g is $\mathcal{C}^0$, we can apply the result based on the Malkin's bifurcation function presented in [11], even if the considered system is not in the standard form. This method is summarized in Theorem 1.9 in the Section 1.3 of Chapter 1. In fact, in this chapter we apply such method in order to achieve the results.

Taking into account such points, to achieve our results we choose to work with a regularization of system (3.5) since its perturbed part is non-smooth instead of $\mathcal{C}^0$. We obtain the results firstly for the regularized system Z_δ of system (3.5) via Theorem 1.9 and then we adapt such results by doing $\delta \to 0$. Indeed, following the notation of Chapter 1, first we identify $q = X = (x,y,z)$, $V = \mathbb{R}^3$ and $F(X) = y$. If we consider the $\mathcal{C}^0$ transition function

$$\varphi_\delta(\tau) = \begin{cases} -1, & \text{if } \tau \leq -\delta, \\ \dfrac{\tau}{\delta}, & \text{if } -\delta < \tau < \delta, \\ 1, & \text{if } \tau \geq \delta, \end{cases} \tag{3.6}$$

then a $\mathcal{C}^0$ φ_δ-regularization of system (3.5) writes

$$\begin{aligned}\dot{\mathbf{x}}_\delta^\varepsilon &= f_\delta^\varepsilon(\mathbf{x},\varepsilon)\\ &= \frac{f^+(\mathbf{x},\varepsilon)+f^-(\mathbf{x},\varepsilon)}{2}+\varphi_\delta(y)\frac{f^+(\mathbf{x},\varepsilon)-f^-(\mathbf{x},\varepsilon)}{2}\\ &= f(\mathbf{x})+\varepsilon g_\delta(\mathbf{x},\varepsilon),\end{aligned} \tag{3.7}$$

where

$$g_\delta(\mathbf{x},\varepsilon)=\begin{cases} g^-(\mathbf{x}), & y\le -\delta,\\ \dfrac{g^+(\mathbf{x})+g^-(\mathbf{x})}{2}+\dfrac{y}{\delta}\left(\dfrac{g^+(\mathbf{x})-g^-(\mathbf{x})}{2}\right), & |y|<\delta,\\ g^+(\mathbf{x}), & y\ge\delta.\end{cases} \tag{3.8}$$

We must stress that the vector field of system (3.7) is $\mathcal{C}^0$ and it has the same unperturbed part of system (3.5), i.e., system (3.7) also possesses the cylinder C filled by periodic solutions when ε is zero and for all $\delta>0$. In addition, taking $\delta\to 0$ in system (3.7) we obtain the non-smooth system (3.5). As we said before, in this chapter we will apply Theorem 1.9 and then

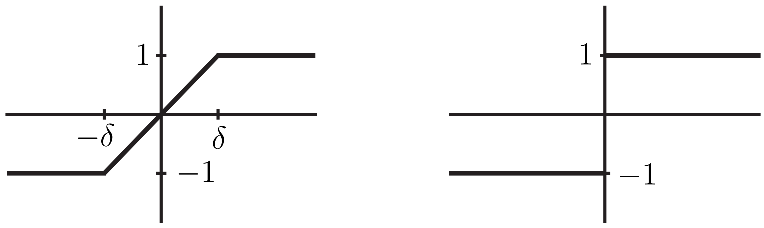

Figure 3.2: The graph of φ_δ for $\delta>0$ (left) and for $\delta\to 0$ (right).

we take $\delta\to 0$ in order to extend the results for system (3.5).

Now we introduce two functions depending on the function h (which determines the shape of the periodic solutions on the cylinder C) and the perturbations $p^\pm$, $q^\pm$ and $r^\pm$ which are very important to the results. Indeed, first consider the function

$$A_h(\theta) = \cos\theta \frac{\partial h}{\partial x}(\cos\theta, \sin\theta) + \sin\theta \frac{\partial h}{\partial y}(\cos\theta, \sin\theta). \tag{3.9}$$

Observe that $A_h(\theta)$ is rather technical but it plays an important role in the implementation of Theorem 1.9 for system (3.7). Note also that it depends only on function h.

Next, let $M_\delta : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by

$$\begin{aligned} M_\delta(z) = \quad & \int_0^{2\pi} -\frac{1}{2}\big[h(\cos\theta, \sin\theta)(-\cos\theta(q^+(\varsigma) + q^-(\varsigma)) \\ & + \sin\theta(p^+(\varsigma) + p^-(\varsigma))) + (r^+(\varsigma) + r^-(\varsigma)) + \\ & (h(\cos\theta, \sin\theta)(\cos\theta(-q^+(\varsigma) + q^-(\varsigma)) + \sin\theta(p^+(\varsigma) - \\ & p^-(\varsigma))) + (r^+(\varsigma) - r^-(\varsigma)))\varphi_\delta(\sin\theta)\big]\, ds, \end{aligned} \tag{3.10}$$

where $\varsigma = \big(\cos\theta, \sin\theta, z + \int_0^s h(\cos v, \sin v)dv\big)$ and z is some real value. We will see in Subsection 3.2 that under suitable assumptions, the simple zeros of function M_δ provide limit cycles bifurcating from the continuum of periodic solutions on the cylinder C. Observe that, up this step, the development of the calculations does not depend on the transition function φ_δ. Actually, since the relevant variable is this particular case is z and we have φ_δ evaluated in $\sin\theta$, the results will not changing by replacing the transition function. It means that we can obtain the same bounds for the number os limit cycles bifurcating from C independently if the expression of φ_δ.

Now we establish the main results of the chapter.

3.2 Main Results

Lemma 3.1. *Suppose that $A_h(\theta) = 0$, $\forall\theta \in [0, 2\pi)$. Then, for $|\varepsilon|$ sufficiently small and for every z_0 such that $M_\delta(z_0) = 0$ and $M'_\delta(z_0) \neq 0$, the smooth system* (3.7) *has a limit cycle bifurcating from the continuum of periodic solutions of the cylinder C with $\varepsilon = 0$. Moreover, there exist at most $s =$*

$\max\{m,n,p\}$ *values of* z *for which* $M_\delta(z)=0$.

Remark 3.1. *We note that the condition* $A_h(\theta)=0$ *is not empty. For instance, it is not difficult to verify that for each* $c_i\in\mathbb{R}$ *the functions*

$$\hat{h}(x,y)=\sum_{i=0}^{\infty}\frac{c_i}{(x^2+y^2)^{\frac{2i+1}{2}}}x^{2i+1}$$

satisfy such property. Moreover, it holds that the function $\rho(x,y)=1/(x^2+y^2)^{\frac{2i+1}{2}}$ *satisfies* $\rho(r\cos\theta,r\sin\theta)=1$ *and the power of* x *in the polynomials* c_ix^{2i+1} *is always odd, i.e., the cylinders defined by the function* $\hat{h}$ *presented previously is filled by periodic orbits of system* (3.1). *Actually, this facts says that the results take into account infinitely many different cylinders.*

The next theorem is the main result of this chapter and says that the limit cycles that we find for the regularized system (3.7) are preserved for the non-smooth system (3.5) when $\delta\to 0$.

Theorem 3.1. *Assume that* $A_h(\theta)=0$, $\forall\theta\in[0,2\pi)$. *Then, for* $|\varepsilon|$ *sufficiently small and for each* z_0 *such that* $M_\delta(z_0)=0$ *and* $M_\delta'(z_0)\neq 0$, *the non-smooth system* (3.5) *has a limit cycle bifurcating from the continuum of periodic solutions of the cylinder* C *with* $\varepsilon=0$. *Moreover, there exist at most* $s=\max\{m,n,p\}$ *values of* z *for which* $M_\delta(z)=0$.

A particular case of perturbations of system (3.1) is to consider $g^+=g^-$ in (3.4), i.e., perform the same perturbation in Σ^+ and Σ^-. In this particular case we obtain a smooth perturbation of system (3.1). In such case system (3.5) becomes smooth and it coincides with its regularized system (3.7). The next theorem states the results in this situation.

Theorem 3.2. *Assume that* $A_h(\theta)=0$, $\forall\theta\in[0,2\pi)$, $g^+=g^-$ *and consider the function*

$$\overline{M}_\delta(z)=\int_0^{2\pi}-\left[h(\cos\theta,\sin\theta)(-\cos\theta q^+(\varsigma)+\sin\theta p^+(\varsigma))+r^+(\varsigma)\right]ds, \tag{3.11}$$

where $\varsigma=\left(\cos\theta,\sin\theta,z+\int_0^s h(\cos v,\sin v)dv\right)$ *and* z *is some real value. Then, for* $|\varepsilon|$ *sufficiently small and for each* z_0 *such that* $\overline{M}_\delta(z_0)=0$ *and*

$\overline{M}'_\delta(z_0) \neq 0$*, the smooth system* (3.5) *has a limit cycle bifurcating from the continuum of periodic solutions of the cylinder* C *with* $\varepsilon = 0$*. Moreover, there exist at most* $s = \max\{m, n, p\}$ *values of* z *for which* $\overline{M}_\delta(z) = 0$*.*

We observe that although Theorems 3.1 and 3.2 provide the same upper bound for the number of limit cycles bifurcating from C by performing non-smooth and smooth functions, respectively, for concrete examples we may reach different upper bounds in each case. In fact, in similar situations usually non-smooth systems present more limit cycles than smooth ones. In Subsection 3.4 we will discuss this topic in more details through a specific example. Before that, in what follows we present the proof of the results.

3.3 Proofs of the main results

Now we apply the methods and tools described in Chapter 1 in order to prove the results presented in Subsection 3.2. One should note that the method used for proving the results deals with perturbations up to first order in the parameter ε. It means that eventually we may obtain more limit cycles than those bounds indicated through the results of the previous section by considering higher order perturbations in ε. We start proving Lemma 3.1.

Proof of Lemma 3.1: Consider system (3.7) and assume that for this system we verify $A_h(\theta) = 0$ for all $\theta \in [0, 2\pi)$. Since the periodic solutions of system (3.1), that we are perturbing, live on the cylinder C, we will perform a cylindrical change of coordinates in system (3.7) by introducing the new variables (z, r, θ) given implicitly by $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$. In the new variables (z, r, θ) system (3.7) writes

$$
\begin{aligned}
\dot{z} = \;& h(r\cos\theta, r\sin\theta) + \varepsilon\frac{1}{2}\left[r^+(\vartheta) + r^-(\vartheta) + (r^+(\vartheta) - r^-(\vartheta))\right.\\
& \left.\varphi_\delta(r\sin\theta)\right],\\
\dot{r} = \;& -r + r^3 + \varepsilon\frac{1}{2}\left[\cos\theta(p^+(\vartheta) + p^-(\vartheta)) + \sin\theta(q^+(\vartheta) + q^-(\vartheta)) + \right.\\
& \left.\cos\theta(p^+(\vartheta) - p^-(\vartheta)) + \sin\theta(q^+(\vartheta) - q^-(\vartheta))\varphi_\delta(r\sin\theta)\right],\\
\dot{\theta} = \;& 1 + \varepsilon\frac{1}{2r}\left[\cos\theta(q^+(\vartheta) + q^-(\vartheta)) - \sin\theta(p^+(\vartheta) + p^-(\vartheta)) + \right.\\
& \left.\cos\theta(q^+(\vartheta) - q^-(\vartheta)) + \sin\theta(-p^+(\vartheta) + p^-(\vartheta))\varphi_\delta(r\sin\theta)\right],
\end{aligned}
\tag{3.12}
$$

where $\vartheta = (r\cos\theta, r\sin\theta, z)$.

Now we change the independent variable t of system (3.12) to the new variable θ and obtain the following equivalent system

$$
\begin{aligned}
\frac{dz}{d\theta} &= h(r\cos\theta, r\sin\theta) + \varepsilon\frac{1}{2r}\left[h(r\cos\theta, r\sin\theta)(-\cos\theta(q^+(\vartheta) + \right.\\
&\quad q^-(\vartheta)) + \sin\theta(p^+(\vartheta) + p^-(\vartheta))) + r(r^+(\vartheta) + r^-(\vartheta)) + \\
&\quad (h(r\cos\theta, r\sin\theta)(\cos\theta(-q^+(\vartheta) + q^-(\vartheta)) + \sin\theta(p^+(\vartheta) - \\
&\quad \left. p^-(\vartheta))) + r(r^+(\vartheta) - r^-(\vartheta)))\varphi_\delta(r\sin\theta)\right] + O_2\\
&= h(r\cos\theta, r\sin\theta) + \varepsilon G_\delta^1(\theta, z, r, \varepsilon),\\
\frac{dr}{d\theta} &= -r + r^3 + \varepsilon\frac{1}{2}\left[-(q^+(\vartheta) + q^-(\vartheta))((r^2-1)\cos\theta - \sin\theta) + \right.\\
&\quad (p^+(\vartheta) + p^-(\vartheta))((r^2-1)\sin\theta + \cos\theta) + (-(q^+(\vartheta) + q^-(\vartheta))\\
&\quad ((r^2-1)\cos\theta - \sin\theta) + (p^+(\vartheta) + p^-(\vartheta))((r^2-1)\sin\theta + \\
&\quad \left. \cos\theta))\varphi_\delta(r\sin\theta)\right] + O_2\\
&= -r + r^3 + \varepsilon G_\delta^2(\theta, z, r, \varepsilon),
\end{aligned}
\tag{3.13}
$$

where again $\vartheta = (r\cos\theta, r\sin\theta, z)$ and $O_2 = O(\varepsilon^2)$. Observe that the vector field of system (3.13) is 2π-periodic. Additionally, in order to see that its perturbed part is locally uniformly Lipschitz in the variables $(z, r) \in \mathbb{R}^2$, consider the function $G_\delta : \mathbb{R} \times \mathbb{R}^2 \times [0,1] \longrightarrow \mathbb{R}^2$ as $G_\delta(\theta, w_1, w_2, \varepsilon) = (G^1_\delta(\theta, w_1, w_2, \varepsilon), G^2_\delta(\theta, w_1, w_2, \varepsilon))$. Consider also the sets

$$
\begin{aligned}
R_1 &= \{(\theta, w_1, w_2, \varepsilon) \in \mathbb{R} \times \mathbb{R}^2 \times [0,1]; w_2 \sin\theta \leq -\delta\}, \\
R_2 &= \{(\theta, w_1, w_2, \varepsilon) \in \mathbb{R} \times \mathbb{R}^2 \times [0,1]; -\delta \leq w_2 \sin\theta \leq \delta\}, \\
R_3 &= \{(\theta, w_1, w_2, \varepsilon) \in \mathbb{R} \times \mathbb{R}^2 \times [0,1]; w_2 \sin\theta \geq \delta\},
\end{aligned}
$$

and let $K \subset \mathbb{R} \times \mathbb{R}^2 \times [0,1] = \bigcup_{i=1,2,3} R_i$ be a compact set. In order to see that G_δ is Lipschitz on K, it is sufficient to show that G_δ is Lipschitz on the convex hull $\overline{K}_C$ of K, once $K \subseteq \overline{K}_C$. Indeed, let x and y be two arbitrary points of $\mathbb{R}^2$ in $\overline{K}_C$. Now consider S the segment connecting x and y and $S_i = S \cap R_i$, $i = 1,2,3$. This intersection consists of a finite number of closed segments contained in S, since the boundaries between R_1 and R_2 and between R_2 and R_3 are codimension one manifolds of $\mathbb{R} \times \mathbb{R}^2 \times [0,1]$. The restrictions $G_\delta|_{S_i}$ are polynomial in the variables w_1 and w_2 for each $i = 1,2,3$ and consequently they are also $\mathcal{C}^\infty$. It means that each restriction $G_\delta|_{S_i}$ is locally L_i-Lipschitz on the compact set S_i, for each $i = 1,2,3$, which is equivalent to be L_i-Lipschitz. Then for all $(\theta, x, \varepsilon), (\theta, y, \varepsilon) \in \overline{K}_C$, there exists $L = \max\{L_1, L_2, L_3\}$ such that

$$
\begin{aligned}
&||G_\delta(\theta, x, \varepsilon) - G_\delta(\theta, y, \varepsilon)|| \\
&\quad \leq L_1 \sum_{p^j \in S_1} ||p_j - p_{j+1}|| + L_2 \sum_{p^k \in S_2} ||p_k - p_{k+1}|| + L_3 \sum_{p^l \in S_3} ||p_l - p_{l+1}|| \\
&\qquad \leq L \left(\sum_{p^j \in S_1} ||p_j - p_{j+1}|| + \sum_{p^k \in S_2} ||p_k - p_{k+1}|| + \sum_{p^l \in S_3} ||p_l - p_{l+1}|| \right),
\end{aligned}
$$

where p^s is the segment with ends in p_s and p_{s+1} for $s \in \{j,k,l\}$, $x = p_s$ and $y = p_r$, for some $s, r \in \mathbb{N}$. Consequently, if n is the number of intersections of S with the boundaries of each R_i, $i = 1,2,3$, then once S is a segment we obtain

$$
\begin{aligned}
||G_\delta(\theta, x, \varepsilon) - G_\delta(\theta, y, \varepsilon)|| &\leq L\left(||x - p_1|| + \ldots + ||p_n - y||\right) \\
&\leq L||x - y||.
\end{aligned}
$$

Hence G_δ is locally uniformly Lipschitz in the variables $(z, r) \in \mathbb{R}^2$.

Now we call $X = (z, r)$ and consider system (3.13) with $\varepsilon = 0$. Then we obtain

$$\frac{dX}{d\theta} = f(\theta, X), \tag{3.14}$$

where $f(\theta, X) = (h(r\cos\theta, r\sin\theta), -r+r^3)$. By hypothesis $f \in \mathcal{C}^2$ and in the zr-plane, the straight line $r = 1$ is invariant. Hence the solution $X(\theta, X_0, 0)$ with initial condition $X_0 = (z_0, 1)$ is

$$X(\theta, X_0, 0) = (z(\theta, z_0), r(\theta, z_0)) = \left(z_0 + \int_0^\theta h(\cos s, \sin s)ds, 1\right).$$

Since $\int_0^\theta h(\cos s, \sin s)ds$ is periodic, it follows that $z(\theta, z_0)$ is 2π-periodic in the variable θ and for each point in a neighborhood of $z = z_0$ on the straight line $r = 1$ passes a 2π-periodic solution that lies in the phase space $(z, r, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$. Consequently system (3.14) has a family $\mathcal{M} = \{(z, r) \in \mathbb{R}^2 : r = 1\}$ of 2π-periodic solutions.

Now consider $R_0 > 0$. There exists an open ball $U \subset \mathbb{R}$, $R_0 \in \mathbb{R}$, $U = \{z_0 \in (-R_0, R_0)\}$ and a function $\xi \in \mathcal{C}^1(\overline{U}, \mathbb{R}^2)$,

$$\xi(z) = \left(z + \int_0^\theta h(\cos s, \sin s)ds, 1\right),$$

which is a parametrization of each periodic solution on $\mathcal{M}$ satisfying that for any $z \in \overline{U}$, we have $D\xi(z) = (1, 0)^T$, whose rank is 1. Note that $\xi(z)$ is the initial condition of a 2π-periodic solution of (3.14).

Now we linearize system (3.14) along its periodic solutions $X(\theta, \xi(z), 0)$. We get

$$\frac{dY}{d\theta} = D_X f(\theta, X(\theta, \xi(z), 0))Y. \tag{3.15}$$

The matrix $D_X f(\theta, X(\theta, \xi(z), 0))$ writes

$$\begin{pmatrix} \dfrac{\partial u}{\partial z}(\xi(z), 1) & \dfrac{\partial u}{\partial r}(\xi(z), 1) \\ \dfrac{\partial(-r + r^3)}{\partial z}(\xi(z), 1) & \dfrac{\partial(-r + r^3)}{\partial r}(\xi(z), 1) \end{pmatrix},$$

with $u(r,\theta) = h(x(r,\theta), y(r,\theta))$, $x(r,\theta) = r\cos\theta$ and $y(r,\theta) = r\sin\theta$.

Now, if we observe that $\frac{\partial u}{\partial r}(\xi(z), 1) = A_h(\theta)$, then it is easy to check that system (3.15) writes

$$\frac{dY}{d\theta} = \begin{pmatrix} 0 & A_h(\theta) \\ 0 & 2 \end{pmatrix} Y, \tag{3.16}$$

and has the fundamental matrix $N_Y(\theta)$ given by

$$N_Y(\theta) = \begin{pmatrix} 1 & \displaystyle\int_0^\theta e^{2s} A_h(s)ds \\ 0 & e^{2\theta} \end{pmatrix}. \tag{3.17}$$

We note that $N_Y(0) = I_2$. Thus, since $A_h(\theta) = 0$ by hypothesis, the monodromy matrix $C = N_Y^{-1}(0)N_Y(2\pi)$ is

$$C = \begin{pmatrix} 1 & 0 \\ 0 & e^{4\pi} \end{pmatrix}, \tag{3.18}$$

and consequently system (3.15) has the Floquet multiplier $+1$ with the geometric multiplicity equal to 1.

Next we take the adjoint linear system

$$\frac{dU}{d\theta} = -(D_X f(\theta, X(\theta, \xi(z), 0)))^* U. \tag{3.19}$$

Since system (3.19) is the adjoint of system (3.15), its fundamental matrix $N_U(\theta)$ is $N_U(\theta) = -(N_Y(\theta))^*$ and consequently a linearly independent solution is $u_1(\theta, z) = (-1, 0)^T$. Observe that $u_1(0, z)$ is $\mathcal{C}^1$ with respect to z. Therefore the Malkin's bifurcation function $M_\delta(z)$ takes the form

$$\begin{aligned} M_\delta(z) &= \int_0^{2\pi} < u_1(s,z), G_\delta(s, X(s, \xi(z), 0)) > ds \\ &= \int_0^{2\pi} -G_\delta^1(s, X(s, \xi(z), 0)) ds. \end{aligned} \tag{3.20}$$

In other words, we obtain the formula

$$\begin{aligned} M_\delta(z) = \int_0^{2\pi} & -\frac{1}{2}\left[h(\cos\theta,\sin\theta)(-\cos\theta(q^+(\varsigma)+q^-(\varsigma))\right. \\ & +\sin\theta(p^+(\varsigma)+p^-(\varsigma)))+(r^+(\varsigma)+r^-(\varsigma))+ \\ & (h(\cos\theta,\sin\theta)(\cos\theta(-q^+(\varsigma)+q^-(\varsigma))+\sin\theta(p^+(\varsigma)- \\ & \left. p^-(\varsigma)))+(r^+(\varsigma)-r^-(\varsigma)))\varphi_\delta(\sin\theta)\right] ds, \end{aligned} \tag{3.21}$$

where now $\varsigma = \big(\cos\theta, \sin\theta, z + \int_0^s h(\cos v, \sin v)dv\big)$.

In order to use Theorem 1.9 of Chapter 1 to assure the existence of limit cycles for system (3.7), we observe that for each $z_0 \in U$ such that $M_\delta(z_0) = 0$ and $M_\delta'(z_0) \neq 0$, the Implicit Function Theorem says that $M_\delta'(z) \neq 0$ for all $z \in \overline{U}$, and then we get $d(M_\delta, U) \neq 0$ since M_δ is continuous. In addition, we must verify condition *iii*) of Section 2. However, taking into account Remark 1.2, we will verify condition *v*) instead of condition *iii*). Indeed, let λ be a positive number, $\varepsilon \in [0,\lambda]$, $w_0 = \xi(z_0)$ and consider the values $p_+ = \arcsin(\delta/r)$ and $p_- = \arcsin(-\delta/r)$. Observe that function φ_δ is continuous except in the points $\theta = p_\pm$. In addition, consider the sets $L_\lambda^\pm = \{\omega \in [0,2\pi]; |\omega - p_\pm| < 2\delta\}$ and $L_\lambda = L_\lambda^- \cup L_\lambda^+$. Thus $mes(L_\lambda) = 8\lambda = o(\lambda)/\lambda$.

Now we observe that for all $\theta \in [0,2\pi] \setminus L_\lambda$ and for all $w \in B_\lambda(w_0)$ the function G_δ is $\mathcal{C}^\infty$. Indeed, G_δ does not switch from one region R_i to another R_j when $i \neq j$ and w varies on $B_\lambda(w_0)$, for $i,j = 1,2,3$. It holds once $\theta \in [0,2\pi] \setminus L_\lambda$ and the radius of the ball $B_\lambda(w_0)$ is smaller than the radius of each neighborhood $(p_\pm - \lambda, p_\pm + \lambda)$ of $p_\pm$. Then, from the Mean Value Theorem we obtain

$$||D_wG_\delta(\theta,w,\varepsilon) - D_wG_\delta(\theta,w_0,0)|| \leq \sup_{s\in\overline{S}} ||D_w^2G_\delta(s)|| \cdot |w - w_0|,$$

where $S = B_\lambda(w_0)$ and $D_w^2G_\delta$ denotes the second derivative of the function D_wG_δ. Thus, once $D_w^2G_\delta$ is $\mathcal{C}^1$ on the compact set V, it holds

$$||D_wG_\delta(\theta,w,\varepsilon) - D_wG_\delta(\theta,w_0,0)|| \leq K|w - w_0| \leq K\lambda = \frac{K\lambda^2}{\lambda},$$

and then condition $v)$ holds once $K\lambda^2 = o(\lambda)$.

In order to prove condition $iv)$, first observe that M_δ is a real-valued function whose domain is $\mathbb{R}$. Now, given $z_0 \in U$ satisfying $M_\delta(z_0) = 0$ with $M'_\delta(z) \neq 0$ for all $z \in \overline{U}$, consider $\delta_1 > 0$ an arbitrary value such that $V = (z_0 - \delta_1, z_0 + \delta_1) \subset U$. Thus, by the Mean Value Theorem, there exists $c \in V$ such that

$$|M_\delta(b) - M_\delta(a)| = |M'_\delta(c)| \cdot |b - a|,$$

for all $a, b \in V$ and $c \in (a, b)$. The proof of condition $iv)$ follows taking $L_{M_\delta} = \inf\{|M'_\delta(z)|; z \in V\} > 0$ and observing that $|M'_\delta(c)| \geq L_{M_\delta}$, i.e., $|M_\delta(b) - M_\delta(a)| \geq L_{M_\delta}|b - a|$ for all $a, b \in V$.

Therefore Theorem 1.9 assures that there exist $\varepsilon_1 > 0$ sufficiently small and $\delta_2 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1)$, there exists a unique 2π-periodic solution (consequently a limit cycle) $\varphi_\delta^\varepsilon \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^2)$ of the regularized system (3.13) with condition in $B_{\delta_2}(\xi(z_0))$ satisfying $\varphi_\delta^\varepsilon(0) \to \xi(z_0)$ when $\varepsilon \to 0$. Consequently the equivalent systems (3.12) and (3.7) also possess the limit cycle $\varphi_\delta^\varepsilon(t)$ satisfying such properties.

Finally, replacing the expressions of $p^\pm$, $q^\pm$ and $r^\pm$ given in (3.4) into the expression (3.10), we obtain the polynomial

$$M_\delta(z) = I_s(\delta)z^s + \ldots + I_1(\delta)z + I_0(\delta), \tag{3.22}$$

where $s = \max\{m, n, p\}$ and

$$I_j(\delta) = \int_0^{2\pi} \phi_j(\theta, \delta)d\theta \in \mathbb{R},$$

for some ϕ_j depending on θ and δ with $j = 0, 1, \ldots, s$.

Therefore, since $M_\delta(z)$ is a polynomial in z possessing at most s zeros, then $s = \max\{m, n, p\}$ is a upper bound for the number of zeros of M_δ. But consequently, by using Theorem 1.9, s is also the upper bound for the number of limit cycles that can bifurcate from the cylinder of system (3.13) by using the bifurcation Malkin's function up to first order. Then it follows that the same holds for the equivalent system (3.7). This finishes the proof of Lemma 3.1. $\square$

Observe that in the particular case treated in this chapter, the regularized system (3.7) with $\varepsilon = 0$ does not depend on δ. Thus, neither the initial condition $\xi(z)$ and consequently nor $\varphi_\delta^\varepsilon$ depends on δ in the last proof. Moreover, once $\xi(z_0) = (z_0 + \int_0^\theta h(\cos s, \sin s)ds, 1)$ has the second component equal to one (what means $r = 1$, in the cylindrical coordinates), the limit cycle $\varphi_\delta^\varepsilon(t)$ lives on the cylinder C, i.e., $\varphi_\delta^\varepsilon(t)$ bifurcates from the continuum of periodic solutions on C.

In what follows we prove Theorem 3.1.

Proof of Theorem 3.1. Suppose that $A_h(\theta) = 0$ for all $\theta \in [0, 2\pi)$ and that for $|\varepsilon|$ sufficiently small we have a value z_0 such that $M_\delta(z_0) = 0$ and $M_\delta'(z_0) \neq 0$, where M_δ is given in (3.10). Then, by Lemma 3.1, there exists a limit cycle $\varphi_\delta^\varepsilon(t)$ for system (3.7) satisfying $\varphi_\delta^\varepsilon(0) \to \xi(z_0)$ when $\varepsilon \to 0$, as described in the proof of Lemma 3.1. Now consider Σ^{z_0} a transversal section of $\varphi_\delta^\varepsilon$ contained in the cylinder C for the Poincaré map $P_\delta^\varepsilon : \Sigma^{z_0} \longrightarrow \Sigma^{z_0}$, where $P_\delta^\varepsilon(z) = X_\delta(2\pi, \xi(z), \varepsilon)$, $\varphi_\delta^\varepsilon(0) \in \Sigma^{z_0}$ and $X_\delta(t, X, \varepsilon)$ is a solution of the regularized system (3.7). Then it follows that $P_\delta^\varepsilon(\varphi_\delta^\varepsilon(0)) = \varphi_\delta^\varepsilon(0)$.

Consider also $P^\varepsilon : \Sigma^{z_0} \longrightarrow \Sigma^{z_0}$ the Poincaré map of the non-smooth system (3.5) with $P^\varepsilon(z) = X(2\pi, \xi(z), \varepsilon)$. Observe that by taking ε sufficiently small the Poincaré map P^ε is a composition of Poincaré maps of the regularized system and it is well defined and continuous for every $z \in \Sigma^{z_0}$. Moreover, each fixed point of P^ε corresponds to a periodic solution of the non-smooth system (3.5). Then it holds that $\lim\limits_{\delta \to 0} P_\delta^\varepsilon(z) = P^\varepsilon(z)$, i.e., P^ε is the pointwise limit of P_δ^ε.

Therefore the point $\varphi^\varepsilon(0) = \lim_{\delta \to 0} \varphi_\delta^\varepsilon(0)$ is a fixed point of the Poincaré map $P^\varepsilon(z)$ and consequently the non-smooth system (3.5) has a limit cycle $\varphi^\varepsilon(t)$ such that $\varphi^\varepsilon(0) \to (\xi(z_0), 1)$ when $\varepsilon \to 0$. $\square$

Finally we prove Theorem 3.2. It is an immediate consequence of Lemma 3.1.

Proof of Theorem 3.2: Since $g^+ = g^-$, we obtain $p^+ = p^-$, $q^+ = q^-$ and $r^+ = r^-$. Thus, $A_h(\theta) = 0$ for all $\theta \in [0, 2\pi)$, from formula (3.10) we get

$$\overline{M}_\delta(z) = \int_0^{2\pi} - \left[h(\cos\theta, \sin\theta)(-\cos\theta q^+(\varsigma) + \sin\theta p^+(\varsigma)) + r^+(\varsigma)\right] ds, \tag{3.23}$$

where $\varsigma = \left(\cos\theta, \sin\theta, z + \int_0^s h(\cos v, \sin v)dv\right)$. In addition, replacing the expressions of p^+, q^+ and r^+ given in (3.4), we obtain a polynomial

$$\overline{M}_\delta(z) = \overline{I}_s(\delta)z_0^s + \ldots + \overline{I}_1(\delta)z_0 + \overline{I}_0(\delta), \tag{3.24}$$

where again $s = \max\{m, n, p\}$ and

$$\overline{I}_j(\delta) = \int_0^{2\pi} \overline{\phi}_j(\theta, \delta)d\theta \in \mathbb{R},$$

for some $\overline{\phi}_j$ depending on θ and δ with $j = 0, 1, \ldots, s$.

The proof follows straightforward from Lemma 3.1. $\square$

We should mention that the formula obtained in (3.23) does not coincides precisely to the one presented in [48] due to a subtle technical mistake performed in that paper. However, it is important to note that such misunderstanding does not affects the content of that paper since the goal of the authors was to present the methodology for computing limit cycles that bifurcate from a continuum of periodic orbits forming a subset of $\mathbb{R}^n$.

Next we present two concrete examples and some particular perturbations of it in order to discuss some points about the results. More specifically, we compare the results of Theorems 3.1 and 3.2.

3.4 Examples and Conclusion

In this section we present some considerations about the number of periodic solutions that can bifurcate from a special cylinder (more specifically, we fix a function h) taking into account continuous and discontinuous perturbations. We must note that obtaining a global result about the achievement of the number of periodic solutions from formula (3.10) in terms of $h(x,y)$ and the values m, n and p is a hard task. Besides, we show that usually it is not possible neither reach the bound presented in Theorems 3.1 and 3.2 nor make the respective bounds coincide.

First consider $h(x,y) = x/(\sqrt{x^2+y^2})$. Thus system (3.1) is defined in $\mathbb{R}^3 \setminus \{(0,0,z); z \in \mathbb{R}\}$ and satisfies the conditions imposed in the beginning of the chapter. We stress out that this particular case of function h was studied in [48] by considering continuous perturbations. Observe that the function h is $\mathcal{C}^2$ and its expression in cylindrical coordinates is $h(\theta) = \cos\theta$. Moreover, it satisfies $\int_0^{2\pi} h(\theta)d\theta = 0$ and it is not difficult to see that $A_h(\theta) = 0$ for all $\theta \in [0, 2\pi)$. Now we consider the perturbations in (3.4) with $m = 2$, $n = p = 0$ and $a^{\pm}_{ijk} = 0$ for all $i,j,k \in \mathbb{N}$ satisfying $i+j+k \leq 1$. We choose this particular perturbation in order to simplify the calculations, once those ones with large expressions may lead to hard computations. Namely, the perturbations are

$$\begin{aligned}
p^{\pm}(x,y,z) &= a^{\pm}_{200}x^2 + a^{\pm}_{020}y^2 + a^{\pm}_{002}z^2 + a^{\pm}_{110}xy + a^{\pm}_{101}xz + a^{\pm}_{011}yz, \\
q^{\pm}(x,y,z) &= b^{\pm}_{000}, \\
r^{\pm}(x,y,z) &= c^{\pm}_{000}.
\end{aligned} \tag{3.25}$$

By using formula (3.11) we obtain

$$\overline{M}_\delta(z) = \frac{\pi}{4}(a^{+}_{101} + a^{+}_{110} + 4b^{+}_{000} + 8c^{+}_{000}).$$

Then $\overline{M}_\delta$ has no zeros if $a^{+}_{101}+a^{+}_{110}+4b^{+}_{000}+8c^{+}_{000} \neq 0$ and a continuum of zeros otherwise, and consequently Theorem 3.2 does not provide any limit cycle bifurcating from the cylinder C for these particular cases of perturbations and function h. On the other hand, now we use formula (3.10), which provides the periodic solutions bifurcating from C via discontinuous perturbations. Nevertheless, note that formula (3.10) depends on the function $\varphi_\delta(\sin\theta)$, then we need to apply a careful approach. Indeed, we will study this case in two steps. First, assume that $\delta \geq 1$ and observe that in this situation we obtain $|\sin\theta|/\delta \leq 1$. Then $\varphi_\delta(\sin\theta) = \sin\theta/\delta$ and from formula (3.10) we get

$$
\begin{aligned}
M_\delta(z) = \; & \frac{\pi}{8}(a^+_{101} + a^+_{110} + 4b^+_{000} + 8c^+_{000} + a^-_{101} + a^-_{110} + 4b^-_{000} + 8c^-_{000}) \\
& + \frac{\pi}{8\delta}(a^+_{101} - a^-_{101})z.
\end{aligned}
$$

Observe that considering $a^+_{101} - a^-_{101} \neq 0$, function M_δ has exactly one zero z_0, namely,

$$
z_0 = -\frac{(a^+_{101} + a^+_{110} + 4b^+_{000} + 8c^+_{000} + a^-_{101} + a^-_{110} + 4b^-_{000} + 8c^-_{000})\delta}{a^+_{101} - a^-_{101}},
$$

and z_0 satisfies $M'_\delta(z_0) = \frac{\pi}{8\delta}(a^+_{101} - a^-_{101}) \neq 0$.

Now suppose that $\delta < 1$ and consider $\theta_\delta \in (0, \pi/2)$ such that $\sin\theta_\delta = \delta$, i.e., $\theta_\delta = \arcsin\delta$. Note that in order to use formula (3.10), we must split the limit of integration of the integral in smaller pieces, taking into account the expression of $\varphi_\delta(\sin\theta)$ as follows.

$$
\begin{aligned}
&\varphi_\delta(\sin\theta) = 1, && \text{for} \quad \delta \leq \sin\theta \leq 1, \\
&\varphi_\delta(\sin\theta) = \sin\theta/\delta, && \text{for} \quad -\delta < \sin\theta < \delta, \\
&\varphi_\delta(\sin\theta) = -1, && \text{for} \quad -1 \leq \sin\theta \leq -\delta.
\end{aligned}
$$

Hence, the expression of M_δ is obtained by performing integral (3.10) with θ ranging in the partition $\{0, \theta_\delta, \pi - \theta_\delta, \pi + \theta_\delta, 2\pi - \theta_\delta, 2\pi\}$ of the interval $[0, 2\pi]$ (see Figure 3.3).

Therefore, for $\delta < 1$ we obtain the formula

$$
\begin{aligned}
M_\delta(z) = \; & \frac{\pi}{8}(a^+_{101} + a^+_{110} + 4b^+_{000} + 8c^+_{000} + a^-_{101} + a^-_{110} + 4b^-_{000} + 8c^-_{000}) \\
& - \frac{\delta\sqrt{1-\delta^2}(-5 + 2\delta^2) - 3\arcsin(\delta)}{12\delta}(a^+_{101} - a^-_{101})z,
\end{aligned}
$$

where we assume that $a^+_{101} - a^-_{101} \neq 0$. In addition, one should note that the function $\Lambda(\delta) = \delta\sqrt{1-\delta^2}(-5 + 2\delta^2) - 3\arcsin(\delta)$ satisfies $\Lambda(0) = 0$ and $\Lambda'(\delta) = -8(1-\delta^2)^{3/2} < 0$ for all $0 < \delta < 1$, i.e., $\Lambda(\delta) \neq 0$ when $0 < \delta < 1$.

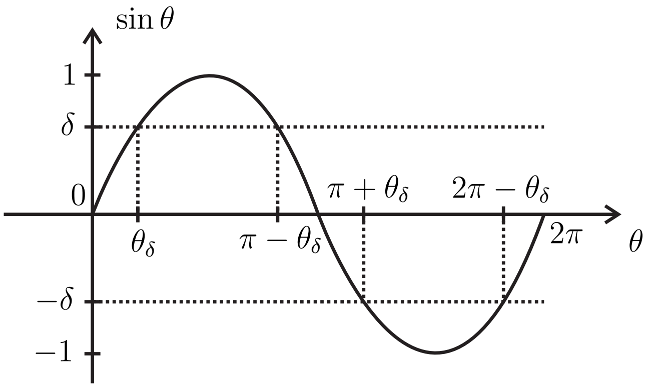

Figure 3.3: Different intervals of integration of formula (3.24).

Then function M_δ possesses the zero

$$z_0 = \frac{3\pi(a^+_{101} + a^+_{110} + 4b^+_{000} + 8c^+_{000} + a^-_{101} + a^-_{110} + 4b^-_{000} + 8c^-_{000})\delta}{2(\delta\sqrt{1-\delta^2}(-5+2\delta^2) - 3\arcsin(\delta))(a^+_{101} - a^-_{101})}.$$

Moreover, it is easy to see that

$$M'_\delta(z_0) = -\frac{\delta\sqrt{1-\delta^2}(-5+2\delta^2) - 3\arcsin(\delta)}{12\delta}(a^+_{101} - a^-_{101}) \neq 0,$$

and consequently Lemma 3.1 assures the existence of a limit cycle for system (3.7) considering the function $h(x,y) = x/\left(\sqrt{x^2+y^2}\right)$, perturbation (3.25) and ε sufficiently small. However, by Theorem 3.1, this periodic solution remains when δ tends to zero, in such sense that system (3.5) has also a periodic solution. Indeed, when $\delta \to 0$ we achieve $\delta < 1$ and then we get

$$\begin{aligned} M_\delta(z) = \; & \frac{\pi}{8}(a^+_{101} + a^+_{110} + 4b^+_{000} + 8c^+_{000} + a^-_{101} + a^-_{110} + 4b^-_{000} + 8c^-_{000}) \\ & + \frac{2}{3}(a^+_{101} - a^-_{101})z. \end{aligned}$$

Thus the periodic solution that emerges from the cylinder C for system (3.5) converges to the periodic solution with initial condition $(z_0, 1) \in C$ when ε

is sufficiently small, where

$$z_0 = -\frac{3\pi(a_{101}^{+} + a_{110}^{+} + 4b_{000}^{+} + 8c_{000}^{+} + a_{101}^{-} + a_{110}^{-} + 4b_{000}^{-} + 8c_{000}^{-})}{16(a_{101}^{+} - a_{101}^{-})}.$$

It is easy to check that when $\delta = 1$ both expressions of M_δ and z_0 coincides.

In short, in the case where $h(x, y) = x/\sqrt{x^2 + y^2}$ and the perturbations of system (3.1) are given by (3.25), we have one limit cycle by considering discontinuous perturbations and no one when we consider continuous ones. Also, it shows that although Theorems 3.1 and 3.2 provide the same upper bound for the number of limit cycles by using the Malkin's bifurcation function, the achievement of the number of periodic solutions in each case may be different. Finally, observe that in both cases, the upper bound $s = 2 = \max\{2, 0, 0\}$ is not reach.

It is not arduous to exhibit other examples where the number of limit cycles by considering polynomial non-smooth perturbations is greater than when we consider smooth ones, but the expressions of M_δ and mainly the zeros z_0 may become huge and we will not present them here. Despite of it, we exhibit some tables indicating the upper bound for the number of limit cycles that can bifurcate from smooth and non-smooth perturbations for the case where $m \geq n, p$ and $a_{ijk}^{\pm} = 0$ for all $i, j, k \leq \max\{m, n, p\} - 1$, with $m = 1, 2, 3$. This calculations were performed with the help of the algebraic manipulator Wolfram Mathematica.

Finally, we stress out that the same analysis can be performed by considering different expressions of the function $h(x, y)$, i.e., changing the arrangement of the periodic solutions on the cylinder C.

Indeed, by considering $h(x, y) = xy/(x^2 + y^2)$, we achieve all the necessary suppositions about such function. In addition, considering the same perturbations of the previous discussion we obtain the following tables.

Comparing the tables for both expressions of $h(x, y)$ we can see that the bifurcation of periodic orbits depends on the shape of the periodic orbits on C. Nevertheless, again the upper bound for the number of periodic orbits

Table 1: Case $m = 1$		
$n \backslash p$	**0**	**1**
0	0	1
1	1	1

Table 2: Case $m = 2$			
$n \backslash p$	**0**	**1**	**2**
0	0 (1)*	1	2
1	1	1	2
2	2	2	2

Table 3: Case $m = 3$				
$n \backslash p$	**0**	**1**	**2**	**3**
0	1 (2)	1 (2)	2	3
1	1 (2)	1 (2)	2	3
2	2	2	2	3
3	3	3	3	3

Table 3.1: Upper bound for the number of limit cycles for particular values of m, n and p when $h(x,y) = x/\sqrt{x^2+y^2}$. The number between brackets indicates the upper bound for the non-smooth case, when it is different from the continuous one. The * indicates the case studied previously.

when we perform discontinuous perturbations is greater than considering continuous perturbations.

Table 4: Case $m = 1$		
$n \backslash p$	**0**	**1**
0	0	1
1	0 (1)	1

Table 5: Case $m = 2$			
$n \backslash p$	**0**	**1**	**2**
0	1	1	2
1	1	1	2
2	1 (2)	1 (2)	2

Table 6: Case $m = 3$				
$n \backslash p$	**0**	**1**	**2**	**3**
0	2	2	2	3
1	2	2	2	3
2	2	2	2	3
3	2 (3)	2 (3)	2 (3)	3

Table 3.2: Upper bound for the number of limit cycles for particular values of m, n and p when $h(x, y) = xy/(x^2 + y^2)$. The upper bound for the number of periodic solutions and the dependence of them in terms of m, n and p change according to function h.

Chapter 4

Non-trivial minimal sets in planar non-smooth systems

In this chapter we treat some aspects of non-trivial minimal sets in planar non-smooth vector fields. Once non-smooth vector fields present no uniqueness of trajectories in the presence of sliding motion, the concept of invariance is introduced depending strongly on the orientation of the trajectories. Indeed, such dependence allows us to state two distinct definitions of minimal sets for non-smooth vector fields. In order to do this, we distinguish those sets that are invariant only in a sense of time, or both sense simultaneously. We also present some examples of such minimal sets in each case. We will see that in these examples it can be observed the occurrence of non-trivial recurrence. Moreover, the minimal sets presented in this chapter have non-empty interior and they are not predicted neither in classical Poincaré-Bendixson Theorem nor in the Denjoy-Schwartz Theorem. Additionally, we compare some properties of minimal sets in both smooth and non-smooth scenario and stress out some relations between the two definitions of minimal sets.

4.1 Setting the problem

For smooth vector fields there is a very developed theory nowadays, mainly in the planar case. In such environment, questions about minimality, for instance, are entirely answered. Indeed, for planar systems the Poincaré-Bendixson theorem says that for a given flow the minimal sets are only equilibria or limit cycles while in higher dimension the minimal sets are described by the Denjoy-Schwartz theorem (under some suitable hypothesis − see [35]). Therefore minimal sets are always limit sets of trajectories running for future or past, so they are invariant compact connected sets.

A very interesting subject is to study the occurrence of minimal sets in the non-smooth scenario and ask about what kind of properties they satisfy and how these objects look like. Besides, non-trivial minimality have been little studied in the literature of the non-smooth systems. Nevertheless, the specific topic addressed in this chapter concerns with minimal sets in non-smooth systems. It also deals with what we have called orientable minimality, i.e., minimality depending on the orientation of the time. For smooth vector fields this is a very important subject because minimal sets are an essential part of limit sets, as we commented before. However, different from the smooth case, it may not happen in the non-smooth context main due to the strong dependence of limit sets on the orientation of the trajectories. This fact has inspired us to introduce the concept of orientable minimality by distinguishing invariance for positive and negative global trajectories. The advantage by taking into account such approach is to differ some interesting sets that are not properly minimal but also present compactness and invariance in some sense. Later in chapter 6 we will discuss about the validity of the Poincaré-Bendixson theorem for non-smooth systems and state that an analogous theorem can be achieved by supposing that there is no sliding motion. This theorem will guarantee that sliding motion is a necessary condition for the existence of non-trivial minimal sets.

4.2 Definitions and first statements

We start introducing the definition of invariance and minimal sets for non-smooth systems. They are natural generalizations of such concepts which appear in the smooth theory.

Definition 4.1. *Consider $Z \in \Omega$. A set $A \subset \mathbb{R}^2$ is **invariant** for Z if, for each $p \in A$ and for all global trajectory $\Gamma_Z(t,p)$ of Z passing through p, it holds $\Gamma_Z(t,p) \subset A$.*

The next example clarify the dependence of the invariance of a set in terms of the orientation of the trajectories.

Example 4.1. *Consider the pseudo cycle Γ, of kind III, presented in Figure 4.1. Note that it is the α-limit set of all global trajectories on a neighbor-*

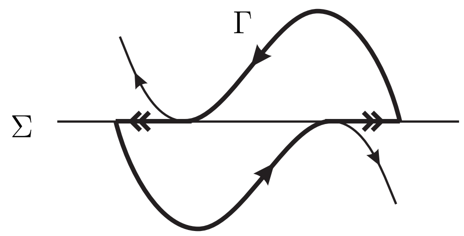

Figure 4.1: Pseudo cycle of kind III.

hood of it. However, since the trajectory runs out from Γ trough the escaping regions of Γ, this set is not invariant according to Definition 4.1. This phenomenon points out a distinct aspect of limit sets which are not predicted for the classical theory of smooth vector fields, once the α and ω-limit sets are invariant sets in the last context.

Definition 4.2. *Consider $Z \in \Omega$. A set $M \subset \mathbb{R}^2$ is **minimal** for Z if*

(i) $M \neq \emptyset$;

(ii) M is compact;

(iii) M is invariant for Z;

(iv) M does not contain proper subset satisfying (i), (ii) and (iii).

Observe that this definition of minimal sets is very similar to that one presented in Section 1.2 of Chapter 1 for smooth systems. Indeed, if there is only trajectory $\Gamma_Z(t,p)$ passing through each point of M, then the definition of invariance in both smooth and non-smooth scenario coincides. Consequently, the definition of minimal sets also coincides in such contexts.

Additionally, one may ask when a minimal set is trivial or not in the non-smooth case. Although this is a natural question that arises by studying minimal sets, it is not established until the present moment for non-smooth systems. Then, inspired in the definition of trivial minimal set of smooth systems, in what follows we propose a definition for these objects in the non-smooth context.

Definition 4.3. *A minimal set $K \subset \mathbb{R}^2$ for the non-smooth vector field Z is called non-trivial if its Lebesgue measure is positive. On the other hand, K is called a trivial minimal set if its Lebesgue measure is zero.*

Example 4.2. *One of the most explored objects of the non-smooth theory of dynamical systems are the pseudo-cycles of type I, which are the same as periodic orbits. (see Figure 4.2). Indeed, analogously to the smooth case,*

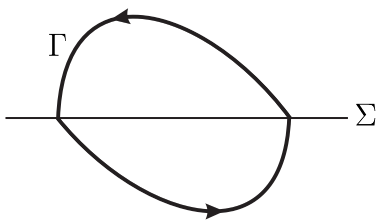

Figure 4.2: Pseudo cycle of kind I.

these objects are trivial minimal sets, since they have measure zero in $\mathbb{R}^2$ and the intersection between them and the switching manifold consist only of sewing points, i.e., they keep the invariance for both future and past times.

Remark 4.1. *We stress out that in our Definition 4.3 we take into account that the non-smooth vector field Z is defined on an open set $V \subset \mathbb{R}^2$, i.e., $K \subset V$. In those cases where the non-smooth vector field is defined on a compact bi-dimensional manifold M, this definition may be lightly different by considering K also a trivial minimal set when $K = M$, as we commented in the Section 1.2 of Chapter 1. We also observe that by using this definition the minimal sets of smooth systems, i.e., periodic orbits and equilibrium points, are still trivial.*

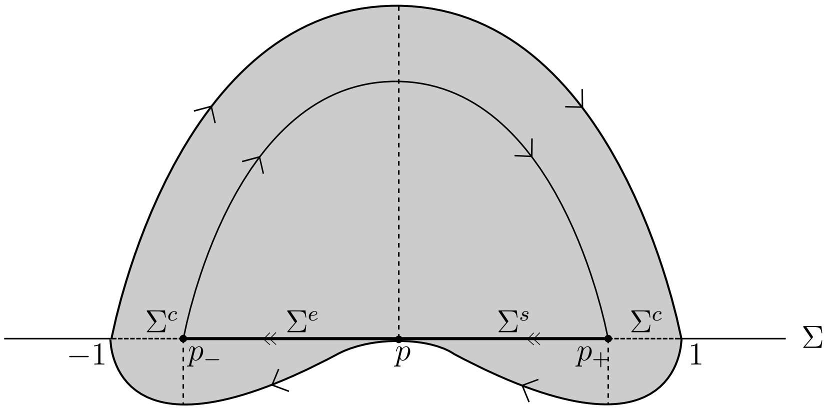

Figure 4.3: The non-trivial minimal set Λ.

Finding minimal sets of vector fields is one of the most important tasks of the qualitative theory of dynamical systems. However, the minimal sets presented in the literature are always trivial (see, for instance, [15, 16, 34]). Next we present a non-trivial minimal set in the scenario of non-smooth systems. As far as we know, this is the first example of a non-trivial minimal set in a non-smooth vector field.

Example 4.3. *Consider $Z = (X, Y) \in \Omega$, where $X(x, y) = (1, -2x)$, $Y(x, y) = (-2, 4x^3 - 2x)$ and $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$. The parametric equation for the integral curves of X and Y with initial conditions $(x(0), y(0)) = (0, k_+)$ and $(x(0), y(0)) = (0, k_-)$, respectively, are known. Indeed, its algebraic expressions are given by $y = -x^2 + k_+$ and $y = x^4/2 - x^2/2 + k_-$, respectively. It is easy to see that $p = (0, 0)$ is an invisible tangency point of X and a visible one of Y. It is also easy to note that the points $p_\pm = (\pm\sqrt{2}/2, 0)$ are both invisible tangency points of Y. Note that between p_- and p there exists an escaping region and between p and p_+ a sliding one. Further, every point between $(-1, 0)$ and p_- or between p_+ and $(1, 0)$ belongs to a sewing region. Consider now the particular trajectories of X and Y for the cases when $k_+ = 1$ and $k_- = 0$, respectively. These particular curves delimit a bounded region of plane that we call Λ, which is a non-trivial minimal set for Z (see Theorem 4.6 and its proof in Section 4.4). Figure 4.3 summarizes these facts.*

One may check that the minimal set Λ has nonempty interior, differently from the smooth context where we have the opposite situation. Consequently Λ is a non-trivial minimal set. We will also see, later in this chapter, that the future and the past of each point in Λ coincides with it.

Now consider the pseudo cycle of kind III, that we called Γ, presented in Figure 4.1. Observe that while Γ is invariant for the past, for the future each trajectory of Γ escapes from it through its escaping regions. It means that we must distinguish such objects since there exists a dependence in the orientation of the time. Indeed, motivated by these facts, following we present new definitions on invariance and minimality taking into account the previous considerations.

Definition 4.4. *A set $A \subset \mathbb{R}^2$ is **positive-invariant** (respectively, **negative-invariant**) if for each $p \in A$ and all positive global trajectory $\Gamma_Z^+(t,p)$ (respectively, negative global trajectory $\Gamma_Z^-(t,p)$) passing through p it holds $\Gamma_Z^+(t,p) \subset A$ (respectively, $\Gamma_Z^-(t,p) \subset A$).*

Remark 4.2. *It follows directly from Definition 4.4 that a given set is invariant if and only if it is positive-invariant and negative-invariant.*

In what follows we present the definition of what we have called orientable minimality. The definition is similar to the one of minimal sets and takes into account positive and negative invariance.

Definition 4.5. *Consider $Z \in \Omega$. A set $M \subset \mathbb{R}^2$ is **positive-minimal** (respectively, **negative-minimal**) if*

(i) $M \neq \emptyset$;

(ii) M is compact;

(iii) M is positive-invariant (respectively, negative-invariant) for Z;

(iv) M does not contain proper subset satisfying (i), (ii) and (iii).

The following lemma is a trivial consequence of Definition 4.5.

Lemma 4.1. *Consider $M \in \mathbb{R}^2$ and Z a non-smooth vector field. If M is positive-minimal and negative-minimal for Z, then M is minimal for Z.*

Proof of Lemma 4.1. In fact, since M is positive-minimal and negative-minimal, then M is a nonempty compact set and from Remark 4.2 M is invariant and does not contain a proper nonempty compact invariant subset. $\square$

Throughout this chapter we will see that the converse of Lemma 4.1 does not hold. We will also see that the minimal set presented in Example 4.3 is also positive-minimal and negative-minimal, simultaneously. This fact will be proved later in this chapter. Nevertheless, in the sequel we present some new examples of minimal sets and study when they are positive and/or negative minimal, or only minimal.

Example 4.4. *As we have observed before, the pseudo cycle Γ exhibited in Figure 4.1 is not positive-invariant but is negative-invariant. In fact, it is also easy to see that Γ is negative-minimal but it is not positive-minimal.*

The next example provides a pseudo cycle of kind II which is a negative-minimal set.

Example 4.5. *Consider the pseudo cycle of type II presented in Figure 4.4. Observe that it is a negative-minimal set once Γ is compact and each point*

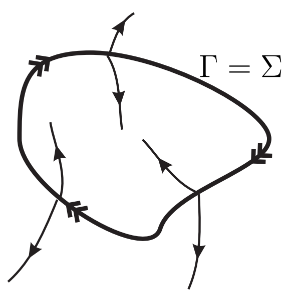

Figure 4.4: Pseudo cycle of kind II.

on it belongs to the escape region, i.e., Γ is negative-invariant. On the other hand, Γ is not positive-minimal. Indeed, for future times each point on Γ escape from it, so Γ is not positive-invariant.

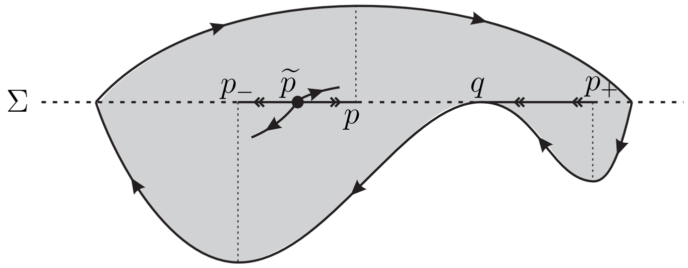

Figure 4.5: The minimal set Λ_1. Λ_1 is neither positive-minimal nor negative-minimal.

A consequence of Examples 4.1 and 4.5 is that pseudo cycles of types II and III are trivial positive-minimal or negative-minimal sets, but never both simultaneously. On the other hand, pseudo cycles of kind I (see Example 4.2) are positive-minimal and negative-minimal, simultaneously; consequently, by Lemma 4.1, pseudo cycles of type I are also minimal sets.

The next two examples provide others non-trivial minimal sets. Besides, observe that in Example 4.3 we verified the occurrence of canard phenomena (see Remark 1.1 of Chapter 1). However, in the next example of minimal set, such characteristic is not required. In addition, the next example exhibits a minimal set which is neither positive-minimal nor negative-minimal. This stress out that the converse of Lemma 4.1 does not hold.

Example 4.6. *Consider $Z_1 = (X, Y) \in \Omega$, where $X(x, y) = (1, -2x + 1)$, $Y(x, y) = (-1, (-2 + x)(-22 + x(-7 + 4x)))$ and $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$. The parametric equation for the integral curves of X and Y with initial conditions $(x(0), y(0)) = (0, k_+)$ and $(x(0), y(0)) = (0, k_-)$, respectively, are known and their algebraic expressions are given by $y = -(-4+x)(3+x)+k_+$ and $y = (-4+x)(-2+x)^2(3+x)+k_-$, respectively. It is easy to see that $p = (1/2, 0)$ is an invisible tangency point of X, $q = (2, 0)$ is a visible tangency point of Y and the points $p_\pm = ((7\pm\sqrt{401})/8, 0)$ are both invisible tangency points of Y. Note that, in Σ, between p_- and p there exists an escaping region with a repeller pseudo equilibrium $\widetilde{p}$ on it and between q and p_+ there exists a sliding region. Further, every point between $(-3, 0)$ and p_-, between p and q or between p_+ and $(4, 0)$ belongs to a sewing region. Con-*

sider now the particular trajectories of X and Y for the cases when $k_+ = 0$ and $k_- = 0$, respectively. These particular curves delimit a bounded region of plane that we call Λ_1, which is neither positive-minimal nor negative-minimal for Z_1 (see Theorem 4.7). Figure 4.5 summarizes these facts.

The next example is a small variation of the Example 4.3. It exhibits a non-trivial minimal set which is not negative-minimal but it is positive-minimal. It means, in particular, that there is no *symmetrical* properties involving positive-minimal sets and negative-minimal sets.

Example 4.7. *Consider Z_2 a non-smooth vector field presenting the phase portrait exhibited in Figure 4.6. Here, there exists a compact set Λ_2 bounded by trajectories of X and Y. As illustrated, p_1 and p_3 are invisible tangency points of X, p_2 is a visible tangency point of X, p_1 and p_3 are visible tangency points of Y and p_0, p_2 and p_4 are invisible tangency points of Y (note that p_1, p_2 and p_3 present canard structure). It is easy to see that Λ_2 is invariant for Z and that there is no proper subset of it which is compact and invariant. So, Λ_2 is minimal for Z_2 (see Theorem 4.8). Assume that there exists a pseudo equilibrium $\widetilde{p}$ between p_1 and p_2. Following the orientation of the trajectories in Figure 4.6 and the third bullet of Definition 1.4 we conclude that Λ_2 is not negative-minimal since $\{\widetilde{p}\}$ is a compact negative-invariant set for Z_2. Moreover, Λ_2 is positive-minimal since it is positive-invariant and it has no proper compact subset which is positive-invariant.*

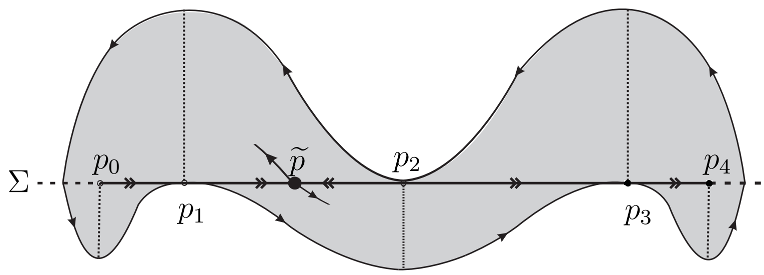

Figure 4.6: The minimal set Λ_2.

Observe that the previous examples emphasize that non-trivial minimality can occur even in systems presenting no symmetrical properties or having no canard points. Moreover, based on the previous examples, it seems that the existence of canard points is somehow related to sets which are positive-minimal and negative-minimal simultaneously. Indeed, in Example (4.3) we have a canard structure, and the set presented in such example is both positive-minimal and negative-minimal. The Example 4.6, on the other hand, presents a perturbation of the set presented in Example 4.3 which destroys the canard structure. Consequently, in this particular case, such perturbation create proper sets which are positive-invariant or negative-invariant.

Also, by observing Examples 4.3, 4.6 and 4.7 we note that the presence of sliding and escaping regions on Σ generates many different objects with very rich dynamics. The case where does not occur sliding or escape regions is present in Chapter 6. In such context, we will see that there exist only trivial minimal sets.

In this section we presented some examples of trivial and non-trivial minimal sets in planar non-smooth vector fields. We suspect that some of this examples are the first ones in the literature. In addition, they clarify the relation between ordinary minimality and orientable minimality. In the next section, we enunciate the claims stated previously. Next, in Chapter 5, we provide a motivation in order to study orientable minimality by making a connection between such sets and chaotic behavior.

4.3 Main Results

Now we state the main results of this chapter, discussed throughout Section 4.2. The Theorems 4.6, 4.7 and 4.8 correspond to Examples 4.3, 4.6 and 4.7, respectively.

Theorem 4.6. *Consider $Z = (X, Y) \in \Omega$, where $X(x, y) = (1, -2x)$, $Y(x, y) = (-2, 4x^3 - 2x)$ and $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$. The set*

$$\Lambda = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1 \ \ and \ \ x^4/2 - x^2/2 \leq y \leq 1 - x^2\}. \qquad (4.1)$$

is a minimal set for Z.

Theorem 4.7. *Consider* $Z_1 = (X, Y) \in \Omega$*, where* $X(x, y) = (1, -2x + 1)$*,* $Y(x, y) = (-1, (-2 + x)(-22 + x(-7 + 4x)))$ *and* $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$*. The set*

$$\begin{aligned} \Lambda_1 = \ & \{(x,y) \in \mathbb{R}^2; -3 \leq x \leq 4 \ \textit{and} \\ & (-4+x)(-2+x)^2(3+x) \leq y \leq -(-4+x)(3+x)\}. \end{aligned} \tag{4.2}$$

is minimal for Z_1 *but it is neither positive-minimal nor negative-minimal.*

Theorem 4.8. *Consider the notations of Example 4.7. The set* Λ_2 *is minimal and also positive-minimal for* Z_2*, but it is not negative-minimal for this non-smooth vector field.*

The next remark is an analogous of Theorem 4.8 by considering the opposite orientation of the time.

Remark 4.3. *Consider a non-smooth system presenting the phase portrait exhibited in Figure 4.6 with opposite orientation. Consider also the notation of Example 4.7. Then, following the same ideas of the last example, we obtain that* Λ_2 *is minimal and negative-minimal for* Z_2 *but it is not positive-minimal for this non-smooth system.*

Apart from the points indicated previously in this text, the following result points out another unusual aspect of the non-trivial minimal set Λ, not predicted for the classical theory.

Theorem 4.9. *Let* Λ *as presented in Theorem 4.7. If* $q \in \Lambda$ *then there exists a trajectory passing trough* q *that is not dense in* Λ*.*

We remember that, according to Definition 4.1, a global trajectory $\Gamma_Z(t, p)$ could not be unique provided that the uniqueness of solutions does not hold. This is actually the main reason for what a minimal set may possess a trajectory that is not dense.

Next we prove the main results.

4.4 Proof of the main results

In this section we prove the results presented in the last section. Indeed, the proofs are similar and take into account the richness of dynamics achieve by the sets Λ, Λ_1 and Λ_2. Moreover, the proofs strongly use the fact that these sets present sliding and escaping regions, once such regions preserve the invariance in one sense of time but produce a dense set of trajectories by iterating the points on them in the opposite sense.

Proof of Theorem 4.6. It is easy to see that Λ is compact and has nonempty interior. Moreover, by Definition 1.4, on $\partial\Lambda \setminus \{p\}$ we have uniqueness of trajectory (here ∂B means the boundary of the set B). Note that the global trajectory of any point in Λ meets p for some time t^*. Since p is a visible tangency point for Y and $p \in \overline{\partial\Sigma^e} \cap \overline{\partial\Sigma^s}$, according to the fourth bullet of Definition 1.4 any trajectory passing through p remain in Λ. Consequently Λ is invariant for Z. Moreover, given $p_1, p_2 \in \Lambda$ the positive global trajectory by p_1 reaches the sliding region between p and p_+ and slides to p. The negative global trajectory by p_2 reaches the escaping region between p and p_- and slides to p. So, there exists a global trajectory connecting p_1 and p_2. Now, let $\Lambda' \subset \Lambda$ be a invariant set. Given $q_1 \in \Lambda'$ and $q_2 \in \Lambda$ since there exists a global trajectory connecting them we conclude that $q_2 \in \Lambda'$. Therefore, $\Lambda' = \Lambda$ and Λ is a minimal set. $\square$

It is not difficult to produce another examples of non-trivial minimal set based on Λ. For instance, through a particular small perturbation of it we still have a non-trivial minimal set $\tilde{\Lambda}$ (see Figure 4.7).

Observe that in the proof of Theorem 4.6 we used the fact the any two arbitrary points can be connected by a trajectory going to the future or past. This is not the general case but sometimes such property can be achieve in only one sense of time. That is the case in the proof of the Theorem 4.7.

Proof of Theorem 4.7. Note that Λ_1 is compact and has nonempty interior. Moreover, the intersection of $\partial\Lambda_1$ with Σ occurs only in sewing and tangential points. Consequently, according to the Definition 1.4 the trajectory of any point starting in Λ_1 remain in this set. Consequently Λ_1 is invariant.

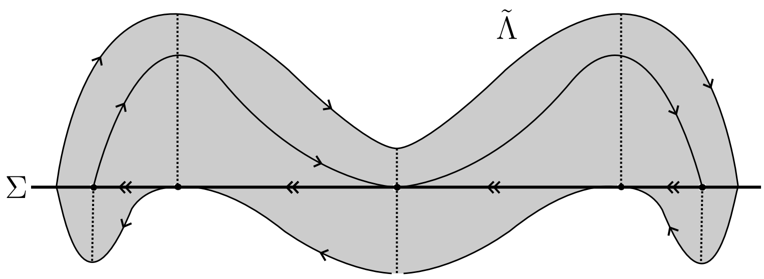

Figure 4.7: Non-trivial minimal set $\tilde{\Lambda}$ for $\tilde{Z}$ presenting nonempty interior.

Now observe that the future of each point in Λ_1 runs to the boundary of Λ_1 in a finite time and do not escape from it. The analogous situation occurs with $\tilde{p}$ when we consider the time running to the past, which means that Z_1 is neither positive-minimal nor negative-minimal. Nevertheless, let $\Lambda_1' \subset \Lambda_1$ be an invariant set. Then, by the invariance of Λ_1' and the previous comments it is clear that $\partial\Lambda_1 \subset \Lambda_1'$ and $\tilde{p} \in \Lambda_1'$. Now take a point $u \in \Lambda_1 \setminus \Lambda_1'$ and note that there exists a time $t_u > 0$ for which the positive trajectory through u, $\Gamma^+(t,u)$ satisfies $\Gamma^+(t_u,u) = v \in \partial\Lambda_1 \subset \Lambda_1'$. Then, by the invariance of Λ_1', the negative trajectory of v, $\Gamma^-(t,v)$, is contained in Λ_1'. In particular, the point $\Gamma^-(-t_u,v) = u$ belongs to Λ_1'. Therefore, $\Lambda_1' = \Lambda_1$ and then Λ_1 is minimal for Z_1. $\square$

The proof of of Theorem 4.8 is analogous to the proof of Theorem 4.7.

Proof of Theorem 4.8. As in the proofs of the previous Prepositions, Λ_2 is compact and has nonempty interior. Moreover, following the same idea of the proof of Preposition 4.7, one can see that Λ_2 is also invariant. Now consider an arbitrary point $p \in \Lambda_2$ and the set

$$\Gamma_p^+(t,p) = \bigcup_{\gamma_p^+ \in \Psi} \gamma_p^+(t,p),$$

where Ψ is the set of all positive trajectories $\gamma_p^+(t,p)$ satisfying $\gamma_p^+(0,p) = p$. Using definition 1.4 of local trajectories it is not hard to see that $\Gamma_p^+(t,p) = \Lambda_2$, once there are infinitively many trajectories reaching escaping regions. Consequently $\Gamma_p^+(t,p)$ is invariant and Λ_2 is positive-minimal for Z_2.

Now let $\Lambda_2' \subset \Lambda_2$ be an invariant set. By the last paragraph we get $\Lambda_2' \subset \Gamma_p^+(t, p)$. Then, as Λ_2' is invariant, it holds that $\Gamma_p^+(t, \Lambda_2') \subset \Lambda_2'$. The result follows by observing that $\Gamma_p^+(t, \Lambda_2') = \Lambda_2$, i.e., $\Lambda_2' = \Lambda_2$. In order to see that Λ_2 is not negative-minimal for Z_2, only note that $\emptyset \neq \{\tilde{p}\} \subset \Lambda_2$ is compact and negative-invariant. This ends the proof of Theorem 4.8. $\square$

Proof of Theorem 4.9. Observe Figure 4.3. By Definition 1.5, there exists a global trajectory Γ_0 of Z which coincides to the closed curve $\partial\Lambda$, the boundary of Λ. Moreover, as shown at the proof of Theorem 4.6, given an arbitrary point $q \in \Lambda$, each global orbit passing through q also reaches $p = (0, 0)$ in finite time. Let Γ_1 be an arc of trajectory of Z joining q and p. So, $\Gamma = \Gamma_0 \cup \Gamma_1$ is a non-dense trajectory of Z in Λ passing through $q \in \Lambda$. $\square$

4.5 Discussions and conclusions

In this chapter we highlighted some points about non-trivial minimal sets in non-smooth vector fields. We presented new definitions concerning minimal sets by taking into account not just invariance and compactness, but also the dependence on the orientation of the trajectories. Under the light of such new definitions, we introduced some examples of minimal sets, positive-minimal sets and negative-minimal sets. We must stress out that these examples configure the first ones dealing with non-trivial minimal sets in such sense that they present positive Lebesgue measure. We also discussed some properties of such sets and we compared them with the ones stated for smooth systems. Additionally, we call attention for the fact that such phenomena occurs due to the existence of sliding regions on the switching manifold. It fortifies the fact that non-smooth system present lots of interesting behaviors, even in simple contexts as the planar one treated here. We belief that the results presented in this chapter increase the knowledge about minimal set and limit sets in the non-smooth scenario. Indeed, the next chapters support some of these expectations by presenting, based on the current one, new results addressing chaos and a version of the Poincaré-Bendixson Theorem for planar non-smooth systems, respectively.

Chapter 5

Chaotic non-smooth systems possessing non-trivial minimal sets

In this chapter we introduce the idea of non-deterministic chaos in non-smooth systems. We observe the occurrence of such phenomenon in some systems which possess non-trivial minimal sets. We also investigate some relations between orientable minimality and chaos. Indeed, we verify that sets which are simultaneously positive-minimal and negative-minimal achieving non-trivial conditions present chaotic behavior.

We start making some remarks about chaos. Then, we translate some concepts concerning chaos from the classical theory to the non-smooth one. Finally we state the results and prove them. They address some examples of chaotic non-smooth systems on non-trivial minimal sets and highlight some properties concerning chaotic systems. Nevertheless, the main result give us a sufficient condition in order to get chaotic systems in terms of non-trivial minimal sets, once it provide infinitely many examples of such systems.

5.1 Setting the problem

The recent theory of non-smooth vector fields has shown that such systems usually present a richer dynamics than smooth ones. As we commented in the previous chapter, it happens basically due to the non-existence of a theorem that achieve the uniqueness of a trajectory crossing a switching manifold Σ through escaping and sliding regions. In fact, the trajectory passing by such regions on Σ can runs out from Σ to one of the adjacent vector fields or remains on it. Such behavior lead us to wonder a kind of non-determinism on Σ. Indeed, in this chapter we make this concept clear by introducing the notion of topological transitivity and sensitive dependence for non-smooth systems.

One should note that while chaos in smooth systems are massively studied in dimension three or higher, the Jordan's curve Theorem assures that there is no chaotic behavior in planar smooth systems. However, in this chapter we will see that this is not the case in non-smooth systems. Indeed, in this chapter we are concerned with the occurrence of chaos in planar non-smooth vector fields. More than that, we relate the concept of chaos with positive-minimal sets and negative-minimal ones by proving that these last objects provide a sufficient condition to the existence of chaos in non-smooth systems.

Examples of chaos in non-smooth systems have been presented by Jeffrey, some of them related with the *T-singularity* (see [22] and [39]). Although the major part of these examples occur in dimension 3, in [39] it is also presented an example in dimension 2. However, different from this example in dimension 2, in this chapter the sets presenting chaos are non-trivial minimal ones.

Since the dynamic on sliding and escaping regions are set-valued, following the previous nomenclature of [22] and [52], it is non-deterministic. In fact, the definition of non-deterministic chaos for non-smooth vector fields was first introduced in [22], where the authors adapt the classical definition of, for example [52], to this context. Of course, the definition must contemplate topological transitivity and sensitivity dependence to initial conditions. For this purpose, consider the following definitions:

Definition 5.1. *System* (1.2) *is topologically transitive on an invariant set* W *if for every pair of nonempty, open sets* U *and* V *in* W*, there exist* $q \in U$*,* $\Gamma_Z^+(t,q)$ *a positive global trajectory and* $t_0 > 0$ *such that* $\Gamma_Z^+(t_0,q) \in V$*.*

Definition 5.2. *System* (1.2) *exhibits sensitive dependence on a compact invariant set* W *if there is a fixed* $r > 0$ *satisfying* $r < diam(W)$ *such that for each* $x \in W$ *and* $\varepsilon > 0$ *there exist a* $y \in B_\varepsilon(x) \cap W$ *and positive global trajectories* Γ_x^+ *and* Γ_y^+ *passing through* x *and* y*, respectively, satisfying*

$$d_H(\Gamma_x^+, \Gamma_y^+) = \sup_{a \in \Gamma_x^+, b \in \Gamma_y^+} d(a,b) > r,$$

where $diam(W)$ *is the diameter of* W *and* d *is the Euclidean distance.*

As observed in [22], the two previous definitions coincide with those ones used for smooth systems when the flow is single-valued, making this a natural extension for a set-valued flow. Now we define a non-deterministic chaotic set:

Definition 5.3. *System* (1.2) *is chaotic on a compact invariant set* W *if it is topologically transitive and exhibits sensitive dependence on* W*.*

We observe that this definition do not ask about the density of the periodic orbits, as occurs in some classical texts of smooth dynamical systems (see, for instance, Devaney $-$ [24]). However, in the examples presented in the next section we also achieve such property.

Now we present the results.

5.2 Main results

This section present the results of the chapter. The following one will be necessary in order to prove Theorem 5.4 in what follows.

Lemma 5.1. *For any two points* x *and* y *in the set*

$$\Lambda = \{(x_1,x_2) \in \mathbb{R}^2; -1 \leq x_1 \leq 1 \text{ and } x_1^4/2 - x_1^2/2 \leq x_2 \leq 1 - x_1^2\},$$

there exist a positive global trajectory $\Gamma^+(t,x)$ passing through x and $t_0 > 0$ such that $\Gamma^+(t_0,x) = y$.

The previous lemma says that any two points in Λ can be connected by some positive global trajectory (note that Λ was introduced before, in Chapter 4, Figure 4.3). Its proof is straightforward if we observe that a global trajectory of any two points in Λ meets p for positive and negative times, as we saw in the last chapter. Such lemma will be fundamental in the proof of the following result. It presents a chaotic set coming from a non-trivial minimal set.

Theorem 5.4. *Consider $Z = (X,Y) \in \Omega$, where $X(x,y) = (1,-2x)$, $Y(x,y) = (-2,4x^3-2x)$ and $\Sigma = f^{-1}(0) = \{(x,y) \in \mathbb{R}^2; y = 0\}$ with $f(x,y) = y$. Then the planar non-smooth vector field Z is chaotic on the compact invariant set*

$$\Lambda = \{(x,y) \in \mathbb{R}^2; -1 \leq x \leq 1 \ \textit{and} \ x^4/2 - x^2/2 \leq y \leq 1 - x^2\} \tag{5.1}$$

shown in Figure 4.3.

As we commented before, apart of topologically transitive and sensitive dependence, the classical definition of chaos given by Devaney in [24] demands also that periodic trajectories of the considered system are dense in Λ. Nevertheless, the non-smooth system exhibited in Theorem 5.4 also present such property, as we see in the next result.

Theorem 5.5. *Consider Z and Λ as presented in Theorem 5.4. Then the periodic trajectories of Z are dense in Λ.*

The following result indicates the presence of chaos in another system studied in the last chapter.

Theorem 5.6. *Consider the non-smooth vector field Z_2 and the set Λ_2 as presented in Example 4.7 of Chapter 4. Then Z_2 is chaotic on Λ_2.*

We must note that the example exhibited in [39] of a chaotic planar system presents a symmetry. Moreover, such set is not minimal. On the other hand, Theorem 5.6 presents an example of a chaotic non-smooth systems which is minimal (non-trivial) and present no symmetry.

As we stress out before, it can be noted some relations between minimality and chaos. Indeed, Theorems 5.4 and 5.6 present examples of non-smooth vector fields which are chaotic on non-trivial minimal sets. This fact suggests such relation between chaoticity and minimality. We make this fact clear through the following result. From now on we denote by $mes(\cdot)$ the Lebesgue measure.

Theorem 5.7. *Let Z be a planar non-smooth vector field and $\Lambda \subset \mathbb{R}^2$ a compact invariant set. If Λ is simultaneously positive-minimal and negative-minimal satisfying $mes(\Lambda) > 0$, then Z is chaotic on Λ.*

Theorem 5.7 is a very interesting result because presents a connection between two important objects of non-smooth systems' theory, namely, the chaotic planar systems and the non-trivial minimal sets. However, the example of a chaotic system introduced in [39] says that the converse does not hold, since such example is no even minimal. Moreover, one should note that we can not change the hypotheses of Theorem 5.7 by considering minimal sets instead of simultaneous positive-minimal sets and negative-minimal sets. Indeed, consider the non-smooth vector field Z_1 and the set Λ_1 as presented in Proposition 4.7 of Chapter 4. As we proved, Λ_1 is minimal for Z_1. Nevertheless, Z_1 is not chaotic on Λ_1, since it is not topologically transitive on Λ_1.

In order to see that, consider a nonempty open set U located in Σ^+ just above the sliding segment S between q and p_+ in such way that all points of U reach S from Σ^+ to Σ and do not enter in the region $\Sigma^- \setminus \Sigma$. Consider also a nonempty open set V under the same conditions of U, however, located under S on Σ^-. Thus it is clear that all points of U and V reach S in a finite positive time and slides to $\partial\Lambda_1$ through the point q. However, since $\partial\Lambda_1$ is positive-minimal for Z_1, it follows that the trajectories of U and V do not escape from $\partial\Lambda_1$ for positive values of time. Consequently we can not connect points of U and V through a positive global trajectory and therefore Z_1 is not topologically transitive on Λ_1.

In order to prove Theorem 5.7 in the sequel, we introduce the next two lemmas. The first one is a generalization of Lemma 5.1.

Lemma 5.2. *Under the same hypotheses of Theorem 5.7, it holds that for any $x, y \in \Lambda$, there exists a global trajectory $\Gamma(t, y)$ passing through y and $t^* > 0$ such that $\Gamma^+(t^*, y) = x$.*

Lemma 5.3. *Under the same hypotheses of Theorem 5.7, if any two points of Λ can be connected by a global trajectory of Z, then Z is chaotic on Λ.*

In the next section we prove the main results.

5.3 Proof of the main results

Now we prove the main results of the present chapter. Observe that some of them have been proved in the last section.

We already know that the set Λ presented in Chapter 4 is a non-trivial minimal set. Now we prove that it also presents chaotic behavior, according to the definitions of the first section.

Proof of Theorem 5.4. In order to prove that the non-smooth vector field Z is topologically transitive on Λ, we observe that Λ is compact and invariant since it is minimal (see Proposition 1 of [13]). Now consider nonempty open sets U and V in Λ. Since U and V are nonempty, there exist at least an element λ_1 in U and another one λ_2 in V. By Lemma 5.1, there exist a positive global trajectory $\Gamma^+(t, \lambda_1)$ passing through λ_1 and $t_0 > 0$ such that $\Gamma^+(t_0, \lambda_1) = \lambda_2 \in V$. Consequently the non-smooth vector field Z is topologically transitive on the invariant set Λ.

Now we prove that Z exhibits sensitive dependence on Λ. Indeed, take $m = diam(\Lambda)$ and consider $r = m/2 > 0$. Since $r < m$ then there exists two elements a and b in Λ such that $d(a, b) > r$. Now consider $x \in \Lambda$, $\varepsilon > 0$ and fix $y \in B_\varepsilon(x) \cap \Lambda$. By Lemma 5.1 there exist positive global trajectories $\Gamma^+_Z(t, x)$ of x and $\Gamma^+_Z(t, y)$ of y and numbers $t_1, t_2 > 0$ such that $\Gamma^+_Z(t_1, x) = a$ and $\Gamma^+_Z(t_2, y) = b$. Then $d_H(\Gamma^+_Z(t_1, x), \Gamma^+_Z(t_2, y)) = d(a, b) > r$ and consequently

Z exhibits sensitive dependence on Λ. Thus the planar non-smooth vector field Z is chaotic on the invariant compact set Λ. $\square$

Proof of Theorem 5.6. The proof of Theorem 5.6 follows the same lines of the proof of Theorem 5.4. Indeed, it is enough to note that any two points of Λ_2 can be connected by a positive global trajectory. $\square$

The next section present other property of the set Λ, usually related to chaos.

In this Subsection we prove Theorem 5.5. It assures that, apart from topological transitivity and sensitive dependence, the set Λ also satisfies a property concerning the density of its periodic orbits.

Proof of Theorem 5.5. The proof is completed if we show that for every point $x \in \Lambda$ passes a periodic trajectory of Z. In order to see that, consider σ_0 the closed arc connecting point x with itself ($\sigma_0 \neq \{x\}$). The existence of such arc is due to Lemma 5.1. Then the global trajectory

$$\Gamma_Z(t,x) = \bigcup_{i \in \mathbb{Z}} \{\sigma_i(t,x); t_i \leq t \leq t_{i+1}\}$$

satisfying $\sigma_i = \sigma_0$ for all $i \in \mathbb{Z}$ is t_1-periodic and passes through x. Observe that $\sigma_i(kt_1, x) = x$, for all $k \in \mathbb{Z}$ and for all $i \in \mathbb{Z}$. $\square$

Lemmas 5.2 and 5.3 are fundamental in the proof of Theorem 5.7. Their proofs are in the sequel.

Proof of Lemma 5.2. Since $mes(\Lambda) > 0$, by Poincaré-Bendixson Theorem for non-smooth systems presented in [13] (see Theorem 6.2 in the next chapter), there exists at least a set $A \subset \Sigma \cap (\Sigma^e \cup \Sigma^s)$. Otherwise, we have $\Sigma \cap \Lambda = \Sigma^c \cup \Sigma^t$ and then by the referred theorem we get $mes(\Lambda) = 0$, where Σ^t is the set of tangencies points of Z. For each $a \in A$, denote by Π_a^+ the set of all positive global trajectories passing through a and by Π_a^- its negative analogous. Now consider the sets

$$A_a^{\pm} = \bigcup_{\Gamma_a \in \Pi_a^{\pm}} \Gamma_a(t,a) \subset \Lambda.$$

Actually we have $A_a^{\pm} = \Lambda$, since $A_a^{\pm}$ is positive-invariant (respectively negative-invariant) contained in the positive-minimal (respectively negative-minimal) set Λ. In order to see that A_a^{+} is positive-invariant, let p be a point in A_a^{+} and $\Gamma_p(t,p)$ a positive global trajectory passing through p. Since $p \in A_a^{+}$, then there exists a positive global trajectory $\tilde{\Gamma}_a(t,a)$ passing through a and $t_0 > 0$ such that $\tilde{\Gamma}_a(t_0,a) = p$. Consequently $\Gamma_p(t,p)$ belongs to A_a^{+} once it is restrained to the positive global trajectory $\hat{\Gamma}_a(t,a) = \tilde{\Gamma}_a(t,a) \cup \Gamma_p(t,p) \subset A_a^{+}$. Analogously we can prove that A_a^{-} is negative-invariant.

Now consider $x, y \in \Lambda$ arbitrary points. Since $A_a^{-} = \Lambda = A_a^{+}$, there exists $\Gamma_a^{+}(t,a) \in A_a^{+}$ a positive global trajectory, $\Gamma_a^{-}(t,a) \in A_a^{-}$ a negative global trajectory and values $t_x > 0$, $t_y < 0$ such that $\Gamma_a^{+}(t_x,a) = x$ and $\Gamma_a^{+}(t_y,a) = y$. Consequently there exists a global trajectory $\Gamma(t,y)$ passing through y and $t^* = t_x + |t_y| > 0$ such that $\Gamma(t^*,y) = x$. □

Proof of Lemma 5.3. The proof of Lemma 5.3 is similar to the proof of Theorem 5.4 by using Lemma 5.2 instead of Lemma 5.1. □

Now we prove Theorem 5.7.

Proof of Theorem 5.7. The proof is straightforward from Lemmas 5.2 and 5.3. Indeed, Lemma 5.2 says that, under the hypotheses concerning Λ in Theorem 5.7, any two points of Λ can be connected. In such case, however, Lemma 5.3 assures the chaoticity of Λ, then it follows the result. □

5.4 Discussions and conclusions

In this chapter we contrast some aspects about non-deterministic chaos in planar non-smooth vector fields. We introduced the definitions of topological transitivity, sensitive dependence and chaos for non-smooth vector fields. Through these definitions, we verified the existence of non-smooth systems presenting non-deterministic chaos on some sets. In particular, in the examples presented throughout this chapter, the considered sets were non-trivial minimal sets. Moreover, one of these sets presented no symmetry and points without canard structure. As far as we know, this is the first time that non-smooth systems with such characteristic are observed in the planar case.

We also present a result connecting chaos with positive-minimal sets and negative-minimal ones. This theorem stress out that positive Lebesgue measure plays an important role in the scenario of chaotic non-smooth systems. Thus such relation suggest that non-trivial minimal sets (apparently presenting canard phenomena) are sufficient conditions to the presence of chaos. In addition, we present some lemmas which were important to prove the results but having also a properly interest. For instance, Lemma 5.2 highlights the fact that, on minimal sets presenting positive measure, we can always connect two arbitrary points. Nevertheless, through this chapter and the previous one, we could see that such property find place in both contexts of minimal sets and chaotic systems.

Chapter 6

Poincaré-Bendixson Theorem for non-smooth systems

In this chapter we are concerned with minimal sets and limit sets of non-smooth systems on the plane. For the classical theory it is well known the Poincaré-Bendixson Theorem which establishes that the limit set of a smooth vector field is either an equilibrium point or a periodic orbit or a graph. In the class of non-smooth systems, that do not present neither sliding nor escaping regions, a version of Poincaré-Bendixson Theorem is presented. In fact, in this case we add to the classical limit sets a s-singular tangency, a pseudo cycle and a pseudo graph. In addition, some examples illustrating the non-uniqueness of orbits and the non-connectedness of limit sets are presented. We also present a corollary of the Poincaré-Bendixson Theorem for non-smooth systems which is a result similar to the Denjoy-Schwartz Theorem. In this case, we classify the possible minimal sets that may occur and observe that all of them are trivial.

6.1 Setting the problem

In the previous two chapters we exhibited some examples of minimal sets having positive Lebesgue measure, which we have called *non-trivial* minimal sets. However, in each of these examples we observed the presence of sliding and/or escaping regions on the referred set. These facts lead us to some natural questions: how about the minimal sets of non-smooth vector fields presenting neither sliding nor escaping regions? Could they be non-trivial? Are they contained in the limit sets just as occurs in the smooth case? In this direction, inspired by the smooth classical results, in this chapter we goal to answer these questions for non-smooth systems presenting only sewing and tangencies points.

Indeed, for planar smooth vector fields there is a well developed theory nowadays. This theory is based on some important results. A now exhaustive list of such results include: *The Existence and Uniqueness Theorem*, *Hartman-Grobman Theorem*, *Poincaré-Bendixson Theorem* and *The Peixoto Theorem*, among others. A very interesting subject is to answer if these results are true or not at the non-smooth vector fields' scenario. It is already known that the first theorem is not true (see Example 6.1 and Figure 6.1 below) and the last one is true (under suitable conditions, see [50]). Another extension to non-smooth theory of classical results on planar smooth vector fields include the concept of *Poincaré Index* of a vector field in relation to a curve, as stated in [14].

The specific topic addressed in this chapter concerns with a version of the Poincaré-Bendixson Theorem for non-smooth vector fields. This theorem will provide not only the limit sets but also the minimal sets which may occur in planar non-smooth system presenting only sewing and tangential points. In particular, we will achieve the triviality of the minimal sets, different from some examples presented in Chapter 4 and 5.

In smooth vector fields, under relatively weak hypothesis, Poincaré-Bendixson Theorem provide all possible limit sets for a given orbit when it is restricted to a compact set. In particular, minimal sets in smooth vector fields are contained in the limit sets. In this chapter, however, we translate such theorem for the non-smooth context by considering that the switching manifold writes $\Sigma = \Sigma^c \cup \Sigma^t$. It means that we are assuming no sliding motion on Σ. We remember that new and unpredictable phenomena may happen by supposing that there exists sliding and escaping regions on Σ. This is actually expected once the presence of escaping and sliding regions on Σ necessarily destroys the uniqueness of global trajectories passing through such regions. In particular, this means that we can not generalize the Poincaré-Bendixson Theorem presented in this chapter without assuming extra hypothesis.

Next we introduce the concepts of α-limit and ω-limit of a global trajectory and a point for non-smooth systems. In order to do this, as made for defining invariance in Chapter 4, we must to consider each trajectory passing through a given point. This is necessary due to the non-uniqueness of trajectories. Observe, however, that for single-valued trajectories the next definition coincides to the classical one.

Definition 6.1. *Given $\Gamma_Z(t, p_0)$ a global trajectory, the set $\omega(\Gamma_Z(t, p_0)) = \{q \in V; \exists\,(t_n)$ satisfying $\Gamma_Z(t_n, p_0) \to q$ when $t_n \to +\infty\}$ (respectively $\alpha(\Gamma_Z(t, p_0)) = \{q \in V; \exists\,(t_n)$ satisfying $\Gamma_Z(t_n, p_0) \to q$ when $t_n \to -\infty\}$) is called ω-**limit** (respectively α-**limit**) set of $\Gamma_Z(t, p_0)$. The ω-limit (respectively α-limit) set **of a point** **p** is the union of the ω-limit (respectively α-limit) sets of all global trajectories passing through p.*

Next we present two examples in order to better understand the role that orbits and limit cycles play considering sliding motion or only sewing and tangential points. The first one makes clear the idea of non-uniqueness of trajectories. Observe that we can verify the existence of disconnected limit sets.

Example 6.1. *Consider Figure 6.1. We observe that the global orbit passing through $q \in \Sigma$ is not necessarily unique. In fact, according to the third bullet of Definition 1.4 of Chapter 1, the positive local trajectory by the point $q \in \Sigma$ can follow three distinct paths, namely, Γ_1, Γ_2 and Γ_3. In particular, this fact exemplifies that the Existence and Uniqueness Theorem is not true in the scenario of non-smooth vector fields. Moreover, the ω-limit set of Γ_i, $i = 1, 2, 3$ is, respectively, a focus, a pseudo-equilibrium and a limit cycle. Consequently, the ω-limit set of q, being the union of these objects, is not a connected set. However, the α-limit set of q is a connected set composed by the pseudo-equilibrium p.*

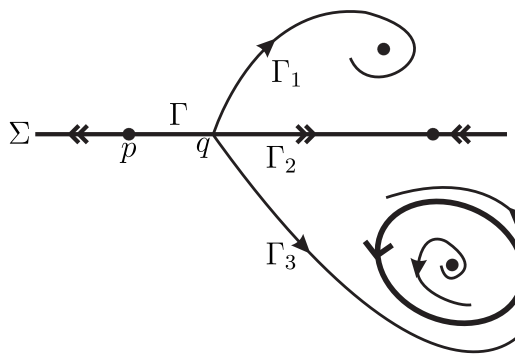

Figure 6.1: Disconnected limit sets and non-uniqueness of trajectories. The horizontal line represents the switching manifold.

Observe that non-connected limit sets can occur in smooth systems by supposing that the trajectories are not contained in a compact set. In the non-smooth context, however, non-connected limit sets may appear even in a compact set. Indeed, consider the compact invariant set Λ_1 presented in the previous chapters. It is not difficult to note that the α-limit set of the point $q \in \partial\Lambda_1$ is the union $\partial\Lambda_1 \cup \{\tilde{p}\}$, which is non-connected.

In what follows we present an example where we achieve the condition $\Sigma = \Sigma^c \cup \Sigma^t$. Again, we can observe disconnected limit sets.

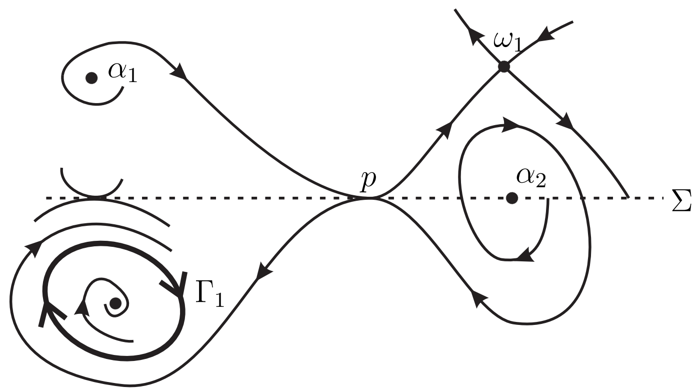

Figure 6.2: Both the α-limit set $\{\alpha_1, \alpha_2\}$ and the ω-limit set $\{\omega_1, \Gamma_1\}$ of the point p are disconnected. Sliding motion on Σ is not allowed.

Example 6.2. *Consider now Figure 6.2. Observe that there is neither sliding nor escaping regions on the switching manifold. Since the uniqueness of trajectories by* p *is not achieved (neither for positive nor for negative times) both the* α *and the* ω*-limit sets are disconnected sets. The* α*-limit set of* p *is composed by the focus* α_1 *and the s-singular tangency* α_2*. The* ω*-limit set of* p *is composed by the saddle* ω_1 *and the periodic orbit* Γ_1*. Consequently the limit sets for this particular examples are two equilibrium points, a limit cycle and a s-singular tangency. In particular, all of them are also trivial minimal sets.*

In the latter example we verify that the limit set of the point p can be a singular tangency. Of course, there is no such objects in the classical theory, but this fact stress out the importance of introducing new results from the theory of smooth dynamical systems to the non-smooth one. Take into account, however, that this case does not allow sliding motion. Nevertheless, as we will see in the next section, by considering $\Sigma = \Sigma^c \cup \Sigma^t$ we clearly maintain the limit sets provided by the classical Poincaré-Bendixson Theorem, but we also see the presence of other kind of objects. Beyond singular tangency, out result says that pseudo-cycles of kind I and pseudo-graphs may also be a limit set.

6.2 Main results

In the sequel we state the main results of this chapter. They deal with minimal sets and limit sets.

Theorem 6.2 [Poincaré-Bendixson for non-smooth systems]. *Let* $Z = (X, Y) \in \Omega$ *be a non-smooth vector field. Assume that* Z *does not have sliding motion and it has a global trajectory* $\Gamma_Z(t,p)$ *whose positive trajectory* $\Gamma_Z^+(t,p)$ *is contained in a compact subset* $K \subset V$*. Suppose also that* X *and* Y *have a finite number of critical points in* K*, no one of them in* Σ*, and a finite number of tangency points with* Σ*. Then, the* ω*-limit set* $\omega(\Gamma_Z(t,p))$ *of* $\Gamma_Z(t,p)$ *is one of the following objects:*

(i) an equilibrium of X *or* Y*;*

(ii) a periodic orbit of X or Y;

(iii) a graph of X or Y;

(iv) a singular tangency of Z;

(v) a pseudo cycle of kind I of Z;

(vi) a pseudo graph of Z.

We observe that the three first possibilities for the ω-limit set of $\Gamma_Z(t,p)$ in Theorem 6.2 are related with the classical Poincaré-Bendixson Theorem. Furthermore, the other possibilities appear due to the special type of discontinuous region Σ that we are considering. Note that they are the analogous of the ordinary equilibrium points, limit cycles and graphs. Moreover, any sequence $(\Gamma_Z(n,p))_n \subset \Gamma_Z(t,p) \subset K$ has a convergent subsequence with limit q. Obviously q belongs to K since it is compact and by construction $q \in \omega(\Gamma_Z(t,p))$. Thus we have $\omega(\Gamma_Z(t,p)) \neq \emptyset$.

One must see that, as we commented before, by supposing sliding and escaping regions on Σ, we may get invariant compact sets having positive Lebesgue measure (see Examples 4.3, 4.6 and 4.7). Such objects are not predicted by the classical theory and are still misunderstood. Consequently we can not generalize Theorem 6.2 without assuming extra hypothesis. However, we stress out that although Theorem 6.2 demands $\Sigma \subset \Sigma^c \cup \Sigma^t$, this condition may be softened by asking that K does not touch any escaping or sliding region and that $\partial K \cap \Sigma^t = \emptyset$. In summary, Theorem 6.2 still holds if we do not allow that the compact set K may be reached through some escaping or sliding regions.

As consequence of Theorem 6.2, since the uniqueness of trajectories passing through a point is not achieved, we have the following corollary:

Corollary 6.1. *Under the same hypothesis of Theorem 6.2 the ω-limit set $\omega(p)$ of a point $p \in V$ is one of the objects described in items (i), (ii), (iii), (iv), (v) and (vi) or a union of some (sub)collection of them.*

The same holds for the α-limit set, reversing time.

Now observe that we can also identify the minimal sets by taking $\Sigma = \Sigma^c \cup \Sigma^t$ directly from Theorem 6.2. Indeed, under this hypothesis it is easy to see that, if there exist a non-empty compact invariant set presenting no proper subset with such characteristics, then this set is a limit set. It means that, as occurs in smooth systems, minimal sets are necessarily limit sets. Of course, here we are strongly using the fact that $\Sigma = \Sigma^c \cup \Sigma^t$. These points summarized the proof of the next result.

Corollary 6.2. *Under the same hypothesis of Theorem 6.2, the minimal sets of a given non-smooth systems are trivial and given by one of the following objects:*

(i) *an equilibrium of X or Y;*

(ii) *a periodic orbit of X or Y;*

(iiii) *a s-singular tangency of Z;*

(iv) *a pseudo cycle of Z;*

Proof. The proof follows straightforward from the previous comments and the fact that each object (i), (ii), (iii) and (iv) presented in Lemma 6.2 have zero Lebesgue measure in $\mathbb{R}^2$. $\square$

Observe that Lemma 6.2 is a simple generalization of the Denjoy-Schwartz Theorem for non-smooth vector fields presenting only sewing and tangential points.

Following we prove the results stated in the present section.

6.3 Proof of the main results

Proof of Theorem 6.2. Consider $p \in V$ and $\Gamma_Z(t,p)$ a global trajectory of p satisfying $\Gamma_Z(0,p) = p$. If there exists a time $t_0 > 0$ such that $\Gamma_Z(t,p)$ does not collide with Σ for $t > t_0$ then we can apply the classical Poincaré-Bendixson Theorem in order to conclude that one of the three first cases (i),

(ii) or (iii) happens. Otherwise, there exists a sequence $(t_i) \subset \mathbb{R}$ of positive times, $t_i \to +\infty$, such that $p_i = \Gamma_Z(t_i, p) \in \Sigma$.

The hypothesis that we do not have sliding motion implies $Xf(p_i) \cdot Yf(p_i) \geq 0$ with $Xf(p_i) = 0 \Leftrightarrow Yf(p_i) = 0$. Indeed, for each $i \in \mathbb{N}$ we say that $p_i \in T(p)$ if one of the following cases happens: (i) $Xf(p_i) \cdot Yf(p_i) > 0$ or (ii) $Xf(p_i) = Yf(p_i) = 0$ and the local trajectory of p_i is unique. If $Xf(p_i) = Yf(p_i) = 0$ and p_i may follow two distinct paths of local trajectory then we say that $p_i \in N(p)$. Observe that, by hypothesis, $N(p)$ is a finite set. We separate the proof in two cases: $T(p)$ is finite and $T(p)$ is not finite.

Assume that $T(p)$ is a finite set. We denote by n_p and t_p the number of elements of the sets $N(p)$ and $T(p)$ respectively. According to Definition 1.4, a global trajectory of Z by $p_l \in N(p)$ can follows one of two distinct paths. Let us denote by Γ_m an arc of $\Gamma_Z(t, p)$ connecting two consecutive points p_i and p_{i+1}, $i \in \mathbb{N}$. In this case there exists at most $2^{n_p} + t_p$ arcs Γ_m of $\Gamma_Z(t, p)$. So, there exists a (sub)set $\Upsilon \subset \{1, 2, \ldots, 2^{n_p} + t_p\}$ such that $\Gamma = \bigcup_{j \in \Upsilon} \Gamma_j$ is a closed orbit intersecting Σ (i.e., a pseudo cycle) contained in $\Gamma_Z(t, p)$ and with the property that $\Gamma_Z(t, p)$ visit each arc Γ_j of Γ an infinite number of times. In what follows we prove that $\omega(\Gamma_Z(t, p)) = \Gamma$. In fact, as $\Gamma_Z(t, p)$ must visit each arc Γ_j of Γ an infinite number of times then $\Gamma \subset \omega(\Gamma_Z(t, p))$. On the other hand, if $x_0 \in \omega(\Gamma_Z(t, p))$ then there exists a sequence $(s_k) \subset \mathbb{R}$, $s_k \to +\infty$, such that $\Gamma_Z(s_k, p) = x_k \to x_0$. Moreover, since $\Gamma_Z(t, p)$ also is composed by a finite number of arcs Γ_m, $s_k \to +\infty$ and $\Gamma_Z(t, p)$ has no equilibria (otherwise it does not visit Σ infinitely many times), there exists a subsequence (x_{k_j}) of (x_k) that visits some arcs Γ_m infinitely many times. Since Γ includes all arcs Γ_j for which the global trajectory visit Γ_j for an infinite sequence of times, $x_{k_j} \in \Gamma$ a compact set, and consequently $x_0 \in \Gamma$.

Now assume that $T(p)$ is not a finite set. In this case, there exist a point $q \in \Sigma$ and a subsequence $(t_{i_j}) = (s_j)$ of (t_i) such that

$$\lim_{j \to \infty} \Gamma_Z(s_j, p) = q \tag{6.1}$$

since $\Gamma_Z^+(t, p) \subset K$, a compact set. Observe that $q \in \omega(\Gamma_Z(t, p)) \cap \Sigma \neq \emptyset$. As

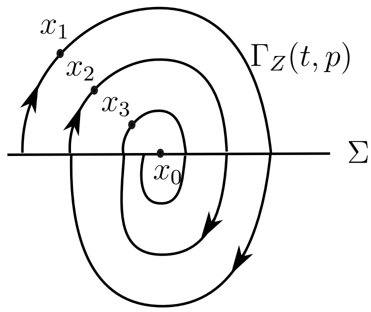

Figure 6.3: Case where there exists a s-singular tangency in $\omega(\Gamma_Z(t,p)) \cap \Sigma$.

we do not have sliding motion, for each $x \in \omega(\Gamma_Z(t,p)) \cap \Sigma$, we have only two options for it: either x is a s-singular tangency or x is a regular point.

If there exists $x_0 \in \omega(\Gamma_Z(t,p)) \cap \Sigma$ a s-singular tangency then $\omega(\Gamma_Z(t,p)) = \{x_0\}$. In fact, when both X and Y have an invisible tangency point at x_0 and there exists a sequence $(s_k) \subset \mathbb{R}$, $s_k \to +\infty$, such that $\Gamma_Z(s_k,p) = x_k \to x_0$ then there is a small neighborhood V_{x_0} of x_0 in V such that all trajectory of Z that starts at a point of V_{x_0} converges to x_0. See Figure 6.3. Therefore, $\omega(\Gamma_Z(t,p)) = \{x_0\}$ and $x_0 = q$.

Suppose now that all points in $\omega(\Gamma_Z(t,p)) \cap \Sigma$ are regular ones. Again we separate the analysis in two cases: either $\omega(\Gamma_Z(t,p))$ contains equilibria or contains no equilibria. Consider the case when $\omega(\Gamma_Z(t,p))$ contains no equilibria. Let q as in Equation (6.1). If $q \in T(q)$ then it is clear that the local trajectory passing through q is unique and $\Gamma_Z(\varepsilon,q) \in \omega(\Gamma_Z(t,p))$ for $\varepsilon > 0$ sufficiently small. If $q \in N(q)$ then q is a visible tangency for both X and Y. So, there are two possible choices for the positive local trajectory of Z passing through q and at least one of them is such that it is contained in $\omega(\Gamma_Z(t,p))$. By continuity, the global trajectory $\Gamma(t,q)$ of Z that passes through q, contained in $\omega(\Gamma_Z(t,p))$, must come back to a neighborhood V_q of q in Σ. The late affirmation is true, because if it does not come back then it remains in Σ^+ or in Σ^-. So, the set $\omega(\Gamma(t,q))$ is a periodic orbit of X or Y, because there are no equilibrium points in $\omega(\Gamma_Z(t,p))$. But it is a contradiction with the fact that the orbit $\Gamma_Z(t,p)$ must visit any neighborhood of q infinitely many times. Moreover, by the

Jordan Curve Theorem, $\Gamma(t,q) \cap V_q = \{q\}$, otherwise there exists a flow box not containing q for which $\Gamma(t,q)$ and, consequently, $\Gamma(t,p)$, do not depart it. This is a contradiction with the fact that the orbit $\Gamma_Z(t,p)$ must visit any neighborhood of q infinite many times. Therefore, $\Gamma_Z(t,q)$ is closed (i.e., is a pseudo cycle) and $\omega(\Gamma_Z(t,p)) = \Gamma_Z(t,q)$.

The remaining case is when $\omega(\Gamma_Z(t,p))$ has equilibria either of X or of Y. In this case for each regular point $q \in \omega(\Gamma_Z(t,p))$ consider the local orbit $\Gamma_Z(t,q)$ which is contained in $\omega(\Gamma_Z(t,p))$. The set $\omega(\Gamma_Z(t,q))$ can not be a periodic orbit or a graph contained in Σ^+ or in Σ^-, because the orbit $\Gamma_Z(t,p)$ must visit any neighborhood of q infinite many times. So, the unique option is that $\omega(\Gamma_Z(t,q)) = \{z_i\}$ where z_i is an equilibrium of X or of Y. Similarly, the α-limit set $\alpha(\Gamma_Z(t,q)) = \{z_j\}$ where z_j is an equilibrium of X or of Y. Thus, with an appropriate ordering of the equilibria z_k, $k = 1, 2 \ldots, m$, (which may not be distinct) and regular orbits $\Gamma_k \subset \omega(\Gamma_Z(t,p))$, $k = 1, 2 \ldots, m$, we have

$$\alpha(\Gamma_k) = z_k \quad \text{and} \quad \omega(\Gamma_k) = z_{k+1}$$

for $k = 1, \ldots, m$, where $z_{m+1} = z_1$. It follows that the global trajectory $\Gamma_Z(t,p)$ either spirals down to or out toward $\omega(\Gamma_Z(t,p))$ as $t \to +\infty$. It means that in this case $\omega(\Gamma_Z(t,p))$ is a pseudo graph composed by the equilibria z_k and the arcs Γ_k connecting them, $k = 1, \ldots, m$.

This concludes the proof of Theorem 6.2. □

Now we perform the proof of Corollary 6.1. Example 6.2 presented before illustrates its consequences.

Proof of Corollary 6.1. In fact, since by Definition 6.1 the ω-limit set of a point is the union of the ω-limit set of all global trajectories passing through it, the conclusion is obvious. □

6.4 Discussions and conclusions

In this chapter we concerned with limit sets and minimal sets for non-smooth systems presenting no sliding motion. We presented a new definition of limit sets of points and trajectories by considering the non-uniqueness of trajectories through some tangential points. Under these new definitions, we checked that limit sets may be lightly different from those ones which appear in smooth systems, in the sense that it can be a union of some objects and consequently disconnected.

In the highlight of the chapter, we present a version of the classical Poincaré-Bendixson Theorem for non-smooth systems considering that the switching manifold satisfies $\Sigma = \Sigma^c \cup \Sigma^t$. The result that we obtained provide us natural extensions of the limit sets occurring in smooth systems. Nevertheless, it suggest that considering non-smooth systems while avoiding sliding motion is a first safe step in order to build the statements of this new theory of dynamical systems concerned with non-smooth objects. Indeed, the overwork is deal with some points on the switching manifold which are visible tangencies of both systems, once in this (only) case we do not achieve uniqueness of trajectories.

In the second spotlight of the chapter we provide a trivial lemma of the Poincaré-Bendixson Theorem which concerned with minimal sets. We verified that as the smooth case, in the context of such theorem the minimal sets are an essential part of the limit sets. Indeed, it is a simplified extension of the Denjoy-Schwartz Theorem once extends the minimal sets to s-singular tangencies and pseudo cycles of kind I, apart from equilibrium points and periodic orbits. In particular, they are all trivial. In particular, it says that we can not obtain the examples of minimal sets of Chapters 4 and 5 achieving positive Lebesgue measure. Indeed, a corollary of this fact is that sliding motion is a necessary condition in order to obtain non-trivial minimal sets.

Finally, we stress out that this chapter is close related with Chapters 4 and 5. Indeed, in this thesis we only made a didactic separation in order to smooth the reading and contrast the different themes throughout them. Indeed, these three chapters are part of the two sequential coupled papers [13] and [12], the first one in its preprint version and the second one to appear in the journal Ergodic Theory and Dynamical Systems.

Index

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