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Espaços Vetoriais e Topológicos de Intervalos Generalizados com Alguns Conceitos de Cálculo e Otimização Intervalar

Tese de Doutorado
Pós-Graduação em Matemática

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós-Graduação em Matemática, Área de Concentração - Matemática Aplicada, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Câmpus de São José do Rio Preto.

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Aos que lutaram
e aos que lutam
em prol da democratização
da Universidade pública.
Dedico.

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A estrada vai além do que se vê.

Los Hermanos

Resumo

Neste trabalho apresentamos um método para munir o conjunto intervalar generalizado $M = I(\mathbb{R}) \bigcup \overline{I(\mathbb{R})}$, sendo $I(\mathbb{R}) = \{[a_1, a_2] : a_1 \leq a_2 \text{ e } a_1, a_2 \in \mathbb{R}\}$ e $\overline{I(\mathbb{R})} = \{[a_1, a_2] : [a_2, a_1] \in I(\mathbb{R})\}$, com algumas diferentes estruturas, como algébrica, topológica e métrica. Também equipamos M com relações de ordem. Na verdade, fizemos isso em um contexto mais geral, pois trabalhamos em $M^n = M \times M \times \cdots \times M$ para $n \in \mathbb{N}$. Nós formulamos problemas de otimização intervalar e relacionamos esses problemas com clássicos problemas de otimização multiobjetivo. Além disso, apresentamos uma versão do Teorema minmax no contexto intervalar e também desenvolvemos conceitos do cálculo em espaços intervalar generalizado, os quais são usados para encontrar o conjunto dos estados atingíveis de um inclusão diferencial clássica sob algumas condições dadas.

Palavras-chave: Espaços Topológicos e Vetoriais Intervalar Generalizado, Otimização Intervalar, Conceitos de Cálculo Intervalar Generalizado.

Abstract

This work presents a method to endow the generalized interval set $M = I(\mathbb{R}) \bigcup \overline{I(\mathbb{R})}$, where $I(\mathbb{R}) = \{[a_1, a_2] : a_1 \leq a_2 \text{ and } a_1, a_2 \in \mathbb{R}\}$ and $\overline{I(\mathbb{R})} = \{[a_1, a_2] : [a_2, a_1] \in I(\mathbb{R})\}$, with some different structures, such as algebraic, topological, and metric. We also equip M with order relations. Actually, we did this in a more general context because we worked in $M^n = M \times M \times \cdots \times M$ for $n \in \mathbb{N}$. We formulated interval optimization problems and related them to classic multi-objective optimization problems. We presented a version of the mini-max Theorem in the interval context, and also developed concepts of calculus on the generalized interval space which are used to find the attainable state set of a classic differential inclusion under some given conditions.

Keywords: Generalized Interval Topological and Vector Spaces, Interval Optimization, Concepts of Interval Generalized Calculus.

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$I(\mathbb{R})$: proper interval set;

$\overline{I(\mathbb{R})}$: improper interval set;

$(I(\mathbb{R}))^n$: n-cartesian proper interval set;

M : generalized interval set;

M^n : n-cartesian generalized interval set;

$|\cdot|$: norm in $\mathbb{R}$;

$\|\cdot\|_{\mathbb{R}^{2n}}$: norm in $\mathbb{R}^{2n}$;

$\|\cdot\|_\varphi$: norm in M^n induced by φ;

$(M^n, +_\varphi, \cdot_\varphi)$: n-dimensional generalized interval vector space;

$(M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$: n-dimensional normed generalized interval vector space;

$(M^n, +_\varphi, \cdot_\varphi, \odot_\varphi \|\cdot\|_\varphi)$: n-dimensional normed generalized interval vector space equipped with matrix product $\odot_\varphi$;

Chapter 1

Introduction

The interval analysis is an important area of study from both theoretical and practical points of views. It was introduced with the goal to work with uncertainty in mathematical and computational models that may arise, for example, from measurements or incomplete information about the data that constitute theses models (see [7], [10], [11], [12], [17], [24], [25], [26]), like the limited numerical representation capability that have the computers for numbers of kind π and $\sqrt{2}$, that have infinite digits in its decimal places. In this case the computers work with rounded floating point numbers. Moreover, most rationales like $\frac{1}{3}$ have rounded representations. Thus, the algebraic operations on floating points numbers, in every step, create uncertainty about the result obtained and these operations may have accumulative errors that may be significant. For example in the Gulf War, after the launching of a missile against the U.S. military, a U.S. Patriot missile failed to intercept a missile attack owing to errors generated by approximations in numbers that were part of algorithm implemented in the Patriot. The result of this was that twenty eight people died and ninety eight were injured (see http://www.diale.org/patriot.html for more information).
Another natural application of interval analysis is classical optimization which was studied from the beginning of interval analysis (see [7], [11], [24], [25]) and continues to these days (see [5]). For numerical verification of bounds on global optimization, interval methods are essential. A more recent application of interval analysis is in solving optimization problems that have some inaccurate coefficients arising from rounded values and/or from incomplete information. It is very natural to use interval analysis to work with this kind of problem because we can transform it into an interval optimization problem.
To talk about interval analysis we, necessarily, must consider the arithmetic operations as addition, subtraction, multiplication, and division, that are used in the interval set, since that these arithmetic operations influence directly some concepts of interval calculus as limit and differentiability. There are three people who, in the decade of the 50^{tees},

independently developed interval arithmetic, Warmus 1956 [34], Sunaga 1958 [33], and Moore 1959 [23]. R.E. Moore, however, is credited with being the father of interval analysis whose book on interval analysis was published in 1966 [24]. His book was the first mathematically well structured text about interval analysis. Summarizing, while Warmus and Sunaga were the first to create interval arithmetic, the development of mathematical analysis on intervals is a result of the work of Moore and subsequent researches.
It is known that the real interval set, denoted by

$$I(\mathbb{R}) = \{[a_1, a_2] : a_1 \leq a_2 \text{ and } a_1, a_2 \in \mathbb{R}\},$$

endowed with the arithmetic developed by R.E. Moore, is not a vector space because not every element in this space has additive inverse (see [24]). Thus, some concepts of interval calculus are not natural extensions of classics concepts. Some authors like ([8], [16], [18], [19], [20]) have introduced different types of arithmetic operations in $I(\mathbb{R})$, however, there are authors who work in a "bigger" set than $I(\mathbb{R})$, in the sense of containment of $I(\mathbb{R})$, by introducing other new arithmetic operations. This "bigger" set can be constructed in the way it was done in [30], i.e.,

$$I(\mathbb{R})^* = I(\mathbb{R}) \cup \big\{[-\infty, r], r \in \mathbb{R}\big\} \cup \big\{[r, +\infty], r \in \mathbb{R}\big\} \cup [-\infty, +\infty],$$

where $-\infty, +\infty$ are points of $\overline{\mathbb{R}}$.
Another way of constructing the "bigger" set, that was used in ([9], [13], [21],[22], [28], [29]), is given by

$$M := I(\mathbb{R}) \bigcup \overline{I(\mathbb{R})},$$

where $\overline{I(\mathbb{R})} = \{[a_1, a_2] : [a_2, a_1] \in I(\mathbb{R})\}$. The set M is called **generalized interval set**.

All the approaches cited above have been used in order to avert the mishaps generated by the fact that $I(\mathbb{R})$ is not a vector space and enable the construction of an interval analysis with another perspective.

For this exposition, the elements in $I(\mathbb{R})$ are called *proper intervals* and the elements in $\overline{I(\mathbb{R})}$ are called *improper intervals*. Also we will work in the more general context by considering the set $M^n = M \times M \times \cdots \times M$ with $n-$factors, for $n \in \mathbb{N}$. The set M^n endowed with an algebraic structure of vector space is called *n-Dimensional Generalized Interval Vector Space*.

This work is structured in the following way:

In Chapter 2 we will present a method to obtain various distinct vector space structures in M^n, a class of orders that can be set in M^n, and some examples of specific ordered vector spaces that have been cited in some published papers . Else, we will analyze some properties in these ordered spaces. In the Chapter 3 we will present a class of optimization problems that involves functions with generalized interval images. We will show how to find solutions to these problems by means of multi-objective optimization methods and present some examples of specific optimization problems. In Chapter 4 we will present a class of topological structures for generalized interval spaces together with some topological results. Yet in the Chapter 4 we present an important result involving the Von Neumann's Theorem in generalized interval spaces. In the Chapter 5 will be presented concepts of limit, continuity, and Lipschitz for functions of type $F : U \subseteq \mathbb{R} \longrightarrow M^n$, which notation is

$$F(x) = \big([f_1(x), f_2(x)], [f_3(x), f_4(x)], \cdots , [f_{2n-1}(x), f_{2n}(x)]\big),$$

where $f_i : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, with $i \in \{1, 2, \cdots , 2n\}$.
It will also be presented results linking these concepts about the function F and the respective classics concepts about functions $f_1, f_2, \cdots , f_{2n} : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ that we call extremes functions of F. One of the topics of interval calculus that have been studied hardly is the **differentiability of interval-valued functions** ([3], [4], [5], [6], [19], [31], [32]) and we, also in Chapter 5, will realize a study about this concept for functions of type $F : U \subseteq \mathbb{R} \longrightarrow M^n$, we will present results linking the derivative of the function F and the derivatives of the extremes functions of F. Moreover, we will present a concept of generalized interval matrix. In the Chapter 6 we will present the concepts of interval differential equation in M^n and linear interval differential equation in M^n. Also, two definitions of solution for each of these concepts will be presented, one is called proper solution and the other is just called solution. The proper solution is an element in $\big(I(\mathbb{R})\big)^n = I(\mathbb{R}) \times I(\mathbb{R}) \cdots \times I(\mathbb{R})$. Still in the Chapter 6, it will be present a method to find a solution for this differential equations and finally, we will relate the set of attainable states of a classical differential inclusion and the proper solution of a LIDE in M^n like Plotinikova did in [27].

Chapter 2

$n-$Dimensional Generalized Interval Vector Spaces

In this chapter, we present a way to equip the set M^n with a vector space structure by considering the Euclidean vector space $(\mathbb{R}^{2n}, +, \cdot)$, equipped with the usual operations, and using a bijection between the sets M^n and $\mathbb{R}^{2n}$. We also consider an order relation on $\mathbb{R}^{2n}$ and induce an order relation in M^n by making use of a bijection between these sets. So we present a result that allows us to say when $\big(I(\mathbb{R})\big)^n$ is a convex cone in M^n. We also discuss some examples that have been used in other papers ([5], [12], [27], [32]) as special cases of our method to obtain the structure of vector space and order relation in the generalized interval set.

2.1 Interval vector spaces and order relations.

Definition 2.1.1. *Given the usual vector space $(\mathbb{R}^{2n}, +, \cdot)$, a bijection $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ and $\alpha \in \mathbb{R}$, we denote by $(M^n, +_\varphi, \cdot_\varphi)$ the space in which the operations $+_\varphi : M^n \times M^n \longrightarrow M^n$ and $\cdot_\varphi : \mathbb{R} \times M^n \longrightarrow M^n$ are given by*

$$
\begin{aligned}
&([a_1, a_2], [a_3, a_4], \cdots, [a_{2n-1}, a_{2n}]) +_\varphi ([b_1, b_2], [b_3, b_4], \cdots, [b_{2n-1}, b_{2n}]) \\
&:= \varphi^{-1}\Big(\varphi\Big([a_1, a_2], [a_3, a_4], \cdots, [a_{2n-1}, a_{2n}]\Big) + \varphi\Big([b_1, b_2], [b_3, b_4], \cdots, [b_{2n-1}, b_{2n}]\Big)\Big)
\end{aligned}
$$

and

$$\alpha \cdot_\varphi ([a_1, a_2], [a_3, a_4], \cdots, [a_{2n-1}, a_{2n}]) := \varphi^{-1}\Big(\alpha \cdot \varphi\Big([a_1, a_2], [a_3, a_4], \cdots, [a_{2n-1}, a_{2n}]\Big)\Big),$$

respectively.

Theorem 2.1.1. *The space $(M^n, +_\varphi, \cdot_\varphi)$ is a vector space.*

Proof. The proof follows directly from the fact of that $(\mathbb{R}^{2n}, +, \cdot)$ is a vector space and from definition of operations "$+_\varphi$ " and " $\cdot_\varphi$ " given by Definition 2.1.1. □

Corollary 2.1.1. *The vector space* $(M^n, +_\varphi, \cdot_\varphi)$ *is isomorphic to* $(\mathbb{R}^{2n}, +, \cdot)$.

Proof. Trivial. □

Example 2.1.1. Consider $n = 1$ and let $\phi_1, \phi_2, \phi_3 : M \longrightarrow \mathbb{R}^2$ be the specific bijections defined by

$$\phi_1([a_1, a_2]) = (a_1, a_2 - a_1), \quad \phi_2([a_1, a_2]) = (a_1, a_2) \text{ and } \phi_3([a_1, a_2]) = \left(\frac{a_1 + a_2}{2}, \frac{a_2 - a_1}{2}\right),$$

whose inverses are given by

$$\phi_1^{-1}(a_1, a_2) = [a_1, a_2 + a_1], \quad \phi_2^{-1}(a_1, a_2) = [a_1, a_2] \text{ and } \phi_3^{-1}(a_1, a_2) = [a_1 - a_2, a_1 + a_2],$$

respectively. Then, $(M, +_{\phi_1}, \cdot_{\phi_1})$, $(M, +_{\phi_2}, \cdot_{\phi_2})$, and $(M, +_{\phi_3}, \cdot_{\phi_3})$, are isomorphic vector spaces to $(\mathbb{R}^2, +, \cdot)$.

Remark 2.1.1. The function ϕ_1 restricted to the set $I(\mathbb{R})$, which was used in [27], analyzes the first points and the lengths of each intervals. The function ϕ_2 restricted to the set $I(\mathbb{R})$, which was used in [12], works like the identity function, and the function ϕ_3 restricted to the set $I(\mathbb{R})$, which was used in [32], analyzes the midpoints and the radius of each intervals.

Example 2.1.2. Let $\psi_i : M \longrightarrow \mathbb{R}^2$ be a bijection for each $i \in \{1, 2, \cdots, n\}$. We will consider the function $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ defined by $\varphi \equiv (\psi_1 \times \psi_2 \times \cdots \times \psi_n)$. That is,

$$\begin{aligned} \varphi([a_1, a_2], [a_3, a_4], \cdots, [a_{2n-1}, a_{2n}]) &= (\psi_1 \times \cdots \times \psi_n)([a_1, a_2], [a_3, a_4], \cdots, [a_{2n-1}, a_{2n}]) \\ &= (\psi_1([a_1, a_2]), \psi_2([a_3, a_4]), \cdots, \psi_n([a_{2n-1}, a_{2n}])) \end{aligned}$$

for all $([a_1, a_2], [a_3, a_4], \cdots, [a_{2n-1}, a_{2n}]) \in M^n$.

The function φ is well defined because, given $X = ([x_1, x_2], [x_3, a_4], \cdots, [x_{2n-1}, x_{2n}])$ and $Y = ([y_1, y_2], [y_3, y_4], \cdots, [y_{2n-1}, y_{2n}])$, if $X = Y$, it follows that $[x_i, x_{i+1}] = [y_i, y_{i+1}]$ for all $i \in \{1, \cdots, 2n - 1\}$. Since each ψ_j is a bijection, in particular, each ψ_j is well-defined, so that $\psi_j([x_i, x_{i+1}]) = \psi_j([y_i, y_{i+1}])$ for each $i \in \{1, \cdots, 2n - 1\}$ and for all $j \in \{1, \cdots, n\}$. This means that $\varphi(X) = \varphi(Y)$. Therefore, if $\varphi(X) \neq \varphi(Y)$, then $X \neq Y$, so that, φ is well-defined.

We can use a similar argument to prove that the function $\varphi^{-1} : \mathbb{R}^{2n} \longrightarrow M^n$ given by $\varphi^{-1} \equiv (\psi_1^{-1} \times \psi_2^{-1} \times \cdots \times \psi_n^{-1})$, this is,

$$\begin{aligned}\varphi^{-1}((a_1,a_2),\cdots,(a_{2n-1},a_{2n})) &= (\psi_1^{-1} \times \psi_2^{-1} \times \cdots \times \psi_n^{-1})\left((a_1,a_2),\cdots,(a_{2n-1},a_{2n})\right)\\ &= \big(\psi_1^{-1}(a_1,a_2), \psi_2^{-1}(a_3,a_4),\cdots,\psi_n^{-1}(a_{2n-1},a_{2n})\big)\end{aligned}$$

for each $((a_1,a_2),(a_3,a_4),\cdots,(a_{2n-1},a_{2n})) \in \mathbb{R}^{2n}$, is well-defined. Moreover, it is easy to see that φ^{-1} is the inverse of φ. Thus, $(M^n, +_\varphi, \cdot_\varphi)$ and $(\mathbb{R}^{2n}, +, \cdot)$ are isomorphic vector spaces to each other.

Example 2.1.3. Consider the bijection $\phi_2 : M \longrightarrow \mathbb{R}^2$ given by $\phi_2([a,b]) = (a,b)$ for all $[a,b] \in M$ and set $\psi_i \equiv \phi_2$ for all $i \in \{1,\cdots,n\}$. Thus the bijective function $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ defined by $\varphi \equiv (\psi_1 \times \psi_2 \times \cdots \times \psi_n)$ is such that

$$\varphi([a_1,a_2],[a_3,a_4],\cdots,[a_{2n-1},a_{2n}]) = ((a_1,a_2),(a_3,a_4),\cdots,(a_{2n-1},a_{2n}))\,,$$

with inverse given by

$$\varphi^{-1}\left((a_1,a_2),(a_3,a_4),\cdots,(a_{2n-1},a_{2n})\right) = ([a_1,a_2],[a_3,a_4],\cdots,[a_{2n-1},a_{2n}]).$$

From Theorem 2.1.1 and from Corollary 2.1.1, it follows that $(M^n, +_\varphi, \cdot_\varphi)$ is an isomorphic vector space to $(\mathbb{R}^{2n}, +, \cdot)$, where the operations "$+_\varphi$ " and " $\cdot_\varphi$ " are given, respectively, by

$$\begin{aligned}&([a_1,a_2],[a_3,a_4],\cdots,[a_{2n-1},a_{2n}]) +_\varphi ([b_1,b_2],[b_3,b_4],\cdots,[b_{2n-1},b_{2n}])\\ &= \Big([a_1+b_1, a_2+b_2],[a_3+b_3,a_4+b_4],\cdots,[a_{2n-1}+b_{2n-1}, a_{2n}+b_{2n}]\Big)\end{aligned}$$

and

$$\alpha \cdot_\varphi ([a_1,a_2],[a_3,a_4],\cdots,[a_{2n-1},a_{2n}]) = ([\alpha\cdot a_1, \alpha\cdot a_2],[\alpha\cdot a_3,\alpha\cdot a_4],\cdots,[\alpha\cdot a_{2n-1},\alpha\cdot a_{2n}]).$$

Theorem 2.1.2. *Let $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ be an isomorphism. Then $\big(I(\mathbb{R})\big)^n$ is a convex cone in M^n if and only if $\varphi\big(\big(I(\mathbb{R})\big)^n\big)$ is a convex cone in $\mathbb{R}^{2n}$.*

Proof. Suppose $\varphi\big(\big(I(\mathbb{R})\big)^n\big)$ is a convex cone in $\mathbb{R}^{2n}$. Given $\alpha \in \mathbb{R}$ with $\alpha \geq 0$ and $\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big) \in \big(I(\mathbb{R})\big)^n$, it follows that

$$\alpha\cdot\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big) \in \varphi\big(\big(I(\mathbb{R})\big)^n\big).$$

Then $\varphi^{-1}\big(\alpha\cdot\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)\big) = \alpha \cdot_\varphi \big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big) \in \big(I(\mathbb{R})\big)^n$. Thus, $\big(I(\mathbb{R})\big)^n$ is cone in M^n.

Given $\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big),\big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)\in\big(I(\mathbb{R})\big)^n$, it follows that $\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big),\varphi\big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)\in\varphi\Big(\big(I(\mathbb{R})\big)^n\Big)$. But, by hypothesis, $\varphi\Big(\big(I(\mathbb{R})\big)^n\Big)$ is a convex cone, so that,

$$\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)+\varphi\big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)\in\varphi\Big(\big(I(\mathbb{R})\big)^n\Big).$$

Then,

$$\begin{aligned}&\varphi^{-1}\bigg(\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)+\varphi\big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)\bigg)\\&=\bigg(\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)\ +_\varphi\ \big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)\bigg)\in\big(I(\mathbb{R})\big)^n.\end{aligned}$$

Therefore, $\big(I(\mathbb{R})\big)^n$ is a convex cone in M^n.

Conversely, suppose $\big(I(\mathbb{R})\big)^n$ a convex cone in M^n. Let $\alpha\in\mathbb{R}$ with $\alpha\geq 0$ and let $\big((b_1,b_2),\cdots,(b_{2n-1},b_{2n})\big)\in\varphi\Big(\big(I(\mathbb{R})\big)^n\Big)$. There is $\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)\in\big(I(\mathbb{R})\big)^n$ such that $\big((b_1,b_2),\cdots,(b_{2n-1},b_{2n})\big)=\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)$. Then,

$$\begin{aligned}\varphi^{-1}\bigg(\alpha\big((b_1,b_2),\cdots,(b_{2n-1},b_{2n})\big)\bigg)&=\varphi^{-1}\bigg(\alpha\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)\bigg)\\&=\alpha\cdot_\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)\in\big(I(\mathbb{R})\big)^n.\end{aligned}$$

Consequently, $\alpha\big((b_1,b_2),\cdots,(b_{2n-1},b_{2n})\big)\in\varphi\Big(\big(I(\mathbb{R})\big)^n\Big)$. Furthermore, given

$$\big((b_1,b_2),\cdots,(b_{2n-1},b_{2n})\big),\big((d_1,d_2),\cdots,(d_{2n-1},d_{2n})\big)\in\varphi\Big(\big(I(\mathbb{R})\big)^n\Big),$$

there are

$$\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big),\big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)\in\big(I(\mathbb{R})\big)^n,$$

such that

$$\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)=\big((b_1,b_2),\cdots,(b_{2n-1},b_{2n})\big)$$

and

$$\varphi\big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)=\big((d_1,d_2),\cdots,(d_{2n-1},d_{2n})\big).$$

By hypothesis we have that

$$\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)\ +_\varphi\ \big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)\in\big(I(\mathbb{R})\big)^n.$$

Since φ is an isomorphism, it follows that

$$\begin{aligned}&\big((b_1,b_2),\cdots,(b_{2n-1},b_{2n})\big)+\big((d_1,d_2),\cdots,(d_{2n-1},d_{2n})\big)\\&=\varphi\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)+\varphi\big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)\\&=\varphi\bigg(\big([a_1,a_2],\cdots,[a_{2n-1},a_{2n}]\big)+_\varphi\big([c_1,c_2],\cdots,[c_{2n-1},c_{2n}]\big)\bigg)\in\varphi\Big(\big(I(\mathbb{R})\big)^n\Big).\end{aligned}$$

Therefore, $\varphi\Big(\big(I(\mathbb{R})\big)^n\Big)$ is a convex cone in $\mathbb{R}^{2n}$. □

Remark 2.1.2. Consider functions $\varphi : M \longrightarrow (\mathbb{R}^2, +, \cdot)$ and $\psi : (\mathbb{R}^2, +, \cdot) \longrightarrow M$ defined by

$$\varphi([a_1, a_2]) = (a_1 + 2, a_2) \text{ and } \psi(u, v) = [u - 2, v],$$

respectively. We have

$$(\psi \circ \varphi)([a_1, a_2]) = \psi(\varphi([a_1, a_2])) = \psi(a_1 + 2, a_2) = \big[(a_1 + 2) - 2, a_2\big] = [a_1, a_2]$$

and

$$(\varphi \circ \psi)(u, v) = \varphi(\psi(u, v)) = \varphi([u - 2, v]) = \big((u - 2) + 2, v\big) = (u, v).$$

Thus, $\psi = \varphi^{-1} : (\mathbb{R}^2, +, \cdot) \longrightarrow M$. Then, $\varphi : (M, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^2, +, \cdot)$ is an isomorphism, where "$+_\varphi$" and "$\cdot_\varphi$" are defined, respectively, by

$$\begin{aligned}
[a_1, a_2] +_\varphi [b_1, b_2] &= \varphi^{-1}\big(\varphi([a_1, a_2]) + \varphi([b_1, b_2])\big) \\
&= \varphi^{-1}\big((a_1 + 2, a_2) + \varphi(b_1 + 2, b_2)\big) = \varphi^{-1}(a_1 + b_1 + 2 + 2, a_2 + b_2) \\
&= \big[(a_1 + b_1 + 2 + 2) - 2, a_2 + b_2\big] = [a_1 + b_1 + 2, a_2 + b_2]
\end{aligned}$$

and

$$\begin{aligned}
\alpha \cdot_\varphi [a_1, a_2] &= \varphi^{-1}\big(\alpha \cdot \varphi([a_1, a_2])\big) = \varphi^{-1}\big(\alpha \cdot (a_1 + 2, a_2)\big) \\
&= \varphi^{-1}\big(\alpha \cdot (a_1 + 2), \alpha \cdot a_2\big) = \big[(\alpha \cdot (a_1 + 2)) - 2, \alpha \cdot a_2\big].
\end{aligned}$$

In this case, $I(\mathbb{R})$ is not a convex cone in $(M, +_\varphi, \cdot_\varphi)$, because, given $[1, 1], [4, 5] \in I(\mathbb{R})$ and $\alpha = 2 \geq 0$, we have

$$[1, 1] +_\varphi [4, 5] = [1 + 4 + 2, 1 + 5] = [7, 6] \notin I(\mathbb{R})$$

and

$$2 \cdot_\varphi [4, 5] = \big[(2(4 + 2)) - 2, 5\big] = [16 - 2, 5] = [14, 5] \notin I(\mathbb{R}).$$

Therefore, there exists isomorphism $\varphi : (M, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^2, +, \cdot)$, so that, $I(\mathbb{R})$ is not a convex cone in $(M, +_\varphi, \cdot_\varphi)$.

Consider the bijections $\phi_1, \phi_2, \phi_3 : M \longrightarrow \mathbb{R}^2$ given in Example 2.1.1. We have that $I(\mathbb{R})$ is a convex cone in the spaces $(M, +_{\phi_1}, \cdot_{\phi_1}), (M, +_{\phi_2}, \cdot_{\phi_2})$, and $(M, +_{\phi_3}, \cdot_{\phi_3})$. Indeed, given $[a_1, a_2], [b_1, b_2] \in M$ and $\alpha \in \mathbb{R}$, we have

$$[a_1, a_2] +_{\phi_i} [b_1, b_2] = [a_1 + b_1, a_2 + b_2] \text{ and } \alpha \cdot_{\phi_i} [a_1, a_2] = [\alpha a_1, \alpha a_2]$$

for all $i = 1, 2, 3$. Thus, given $[a_1, a_2], [b_1, b_2] \in I(\mathbb{R})$ and $\alpha \in \mathbb{R}$ with $\alpha \geq 0$, it follows that $a_1 + b_1 \leq a_2 + b_2$ and $\alpha a_1 \leq \alpha a_2$. Then,

$$[a_1, a_2] +_{\phi_i} [b_1, b_2] \in I(\mathbb{R}) \text{ and } \alpha \cdot_{\phi_i} [a_1, a_2] \in I(\mathbb{R}) \text{ for all } i = 1, 2, 3.$$

Corollary 2.1.2. *Consider the bijections $\phi_1, \phi_2, \phi_3 : M \longrightarrow \mathbb{R}^2$ given in Example 2.1.1. We have that $\phi_i(I(\mathbb{R}))$ is a convex cone in $(\mathbb{R}^2, +, \cdot)$ for all $i = 1, 2, 3$.*

Proof. From Remark 2.1.2, we have that $I(\mathbb{R})$ is a convex cone in the spaces $(M, +_{\phi_1}, \cdot_{\phi_1})$, $(M, +_{\phi_2}, \cdot_{\phi_2})$, and $(M, +_{\phi_3}, \cdot_{\phi_3})$. Then, from Theorem 2.1.2, we have that $\phi_i(I(\mathbb{R}))$ is a convex cone in $(\mathbb{R}^2, +, \cdot)$ for all $i = 1, 2, 3$. □

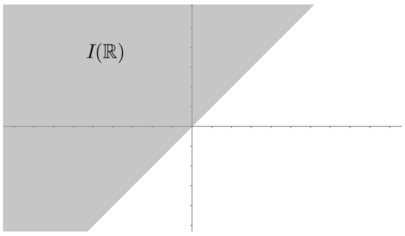

Figure 2.1: $I(\mathbb{R}) \subset M$

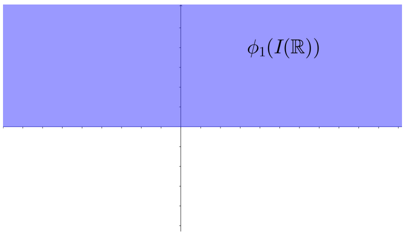

Figure 2.2: Convex Cone in $\mathbb{R}^2$ by ϕ_1.

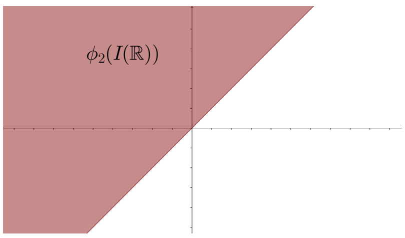

Figure 2.3: Convex Cone in $\mathbb{R}^2$ by ϕ_2.

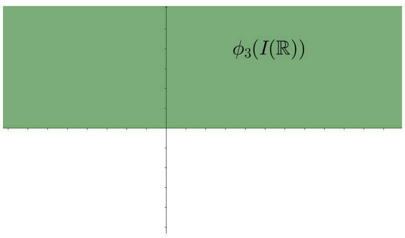

Figure 2.4: Convex Cone in $\mathbb{R}^2$ by ϕ_3.

Remark 2.1.3. A binary relation defined in $\mathbb{R}^{2n}$ is called non-strict partial order relation in $\mathbb{R}^{2n}$ if it satisfies the follows properties:

- reflexivity, antisymmetry, and transitivity. Moreover, if this binary relation also satisfies the property of dichotomy, then is a non-strict total order relation in $\mathbb{R}^{2n}$.

A binary relation defined in $\mathbb{R}^{2n}$ is called strict partial order relation in $\mathbb{R}^{2n}$, if it satisfies the follows properties:

- Anti Reflectivity, asymmetry, and transitivity. Moreover, if this binary relation also satisfies the property of trichotomy, then is a strict total order relation in $\mathbb{R}^{2n}$.

We will consider a (total) partial order relation $\leq_{\mathbb{R}^{2n}}$ in $\mathbb{R}^{2n}$.

We will denote by $\prec_{\mathbb{R}^{2n}}$, the binary relation in $\mathbb{R}^{2n}$, such that, given $u = (u_1, \cdots, u_{2n})$ and $v = (v_1, \cdots, v_{2n})$ in $\mathbb{R}^{2n}$, we have

$$u \prec_{\mathbb{R}^{2n}} v$$

if and only if $\quad u \leq_{\mathbb{R}^{2n}} v$ and $u \neq v$.

We will denote by $<_{\mathbb{R}^{2n}}$, the binary relation in $\mathbb{R}^{2n}$, such that, given $u = (u_1, \cdots, u_{2n})$ and $v = (v_1, \cdots, v_{2n})$ in $\mathbb{R}^{2n}$, we have

$$u <_{\mathbb{R}^{2n}} v$$

if and only if $\quad u \leq_{\mathbb{R}^{2n}} v$ and $u_i < v_i$ for all $i \in \{1, \cdots, 2n\}$.

Proposition 2.1.1. *Let $\leq_{\mathbb{R}^{2n}}$ be a (total) partial order relation in $\mathbb{R}^{2n}$, then*

(a) $\prec_{\mathbb{R}^{2n}}$ is also a strict (total) partial order relation in $\mathbb{R}^{2n}$.

(b) $<_{\mathbb{R}^{2n}}$ is also a strict partial order relation in $\mathbb{R}^{2n}$.

Proof. (a): Suppose that $\leq_{\mathbb{R}^{2n}}$ is a non-strict total order relation in $\mathbb{R}^{2n}$. Given the arbitrary elements $(a_1, a_2, \cdots, a_{2n-1}, a_{2n})$, $(b_1, b_2, \cdots, b_{2n-1}, b_{2n})$, and $(c_1, c_2, \cdots, c_{2n-1}, c_{2n})$ in $\mathbb{R}^{2n}$, we have:

1. Since $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) = (a_1, a_2, \cdots, a_{2n-1}, a_{2n})$, it follows that

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \not\prec_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n}).$$

2. If $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \prec_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n})$ it follows that

$$(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \not\prec_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n}).$$

Indeed, $\leq_{\mathbb{R}^{2n}}$ is a non-strict order relation and

$$(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \prec_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n}),$$

it follows that $(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \leq_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n})$ with

$$(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \neq (a_1, a_2, \cdots, a_{2n-1}, a_{2n}).$$

But, by hypothesis we have $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \leq_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n})$ with $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \neq (b_1, b_2, \cdots, b_{2n-1}, b_{2n})$. It follows that

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) = (b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \text{ and}$$

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \neq (b_1, b_2, \cdots, b_{2n-1}, b_{2n})$$

which is a contradiction.

3. Let

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \prec_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \text{ and}$$
$$(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \prec_{\mathbb{R}^{2n}} (c_1, c_2, \cdots, c_{2n-1}, c_{2n}).$$

Thus, we have

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \leq_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \text{ with}$$

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \neq (b_1, b_2, \cdots, b_{2n-1}, b_{2n}), \text{ and}$$

$$(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \leq_{\mathbb{R}^{2n}} (c_1, c_2, \cdots, c_{2n-1}, c_{2n}) \text{ with}$$
$$(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \neq (c_1, c_2, \cdots, c_{2n-1}, c_{2n}).$$

From item 2., it follows that $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \neq (c_1, c_2, \cdots, c_{2n-1}, c_{2n})$ and, since $\leq_{\mathbb{R}^{2n}}$ is a order relation in $\mathbb{R}^{2n}$, we have that

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \leq_{\mathbb{R}^{2n}} (c_1, c_2, \cdots, c_{2n-1}, c_{2n}).$$

Then, $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \prec_{\mathbb{R}^{2n}} (c_1, c_2, \cdots, c_{2n-1}, c_{2n})$.

4. Let $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \neq (b_1, b_2, \cdots, b_{2n-1}, b_{2n})$. Since $\leq_{\mathbb{R}^{2n}}$ is a total order relation in $\mathbb{R}^{2n}$, we have that

$$\text{either } (a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \leq_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n})$$

$$\text{or } (b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \leq_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n}).$$

Then, we have either $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \prec_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n})$ or $(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \prec_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n})$.

Therefore, $\prec_{\mathbb{R}^{2n}}$ is a strict total order relation in $\mathbb{R}^{2n}$.
We can use the similar argument to prove that, if $\leq_{\mathbb{R}^{2n}}$ is a non-strict partial order relation in $\mathbb{R}^{2n}$, then $\prec_{\mathbb{R}^{2n}}$ is a strict partial order relation in $\mathbb{R}^{2n}$.

(b): Suppose that $\leq_{\mathbb{R}^{2n}}$ is a total order relation in $\mathbb{R}^{2n}$. Given arbitrary elements $(a_1, a_2, \cdots, a_{2n-1}, a_{2n})$, $(b_1, b_2, \cdots, b_{2n-1}, b_{2n})$, and $(c_1, c_2, \cdots, c_{2n-1}, c_{2n})$ in $\mathbb{R}^{2n}$, we have:

1. Since $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) = (a_1, a_2, \cdots, a_{2n-1}, a_{2n})$, it follows that

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \not<_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n}).$$

2. If $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) <_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n})$ it follows that

$$(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \not<_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n}),$$

Indeed, if $\leq_{\mathbb{R}^{2n}}$ is a strict order relation, the the assertion is valid by means of the definition of $<_{\mathbb{R}^{2n}}$. If $\leq_{\mathbb{R}^{2n}}$ is a non-strict order relation and

$$(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) <_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n}),$$

it follows that $(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \leq_{\mathbb{R}^{2n}} (a_1, a_2, \cdots, a_{2n-1}, a_{2n})$ with $b_i < a_i$ for all $i \in \{1, \cdots, 2n\}$. On the other hand, by hypothesis we have that

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \leq_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \text{ with } a_i < b_i \text{ for all}$$

$i \in \{1, \cdots, 2n\}$. It follows that

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) = (b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \text{ and}$$

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \neq (b_1, b_2, \cdots, b_{2n-1}, b_{2n}).$$

which is a contradiction.

3. Let

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) <_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \text{ and}$$

$$(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) <_{\mathbb{R}^{2n}} (c_1, c_2, \cdots, c_{2n-1}, c_{2n}).$$

Thus, we have $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \leq_{\mathbb{R}^{2n}} (b_1, b_2, \cdots, b_{2n-1}, b_{2n})$ with $a_i < b_i$ for all $i \in \{1, \cdots, 2n\}$ and $(b_1, b_2, \cdots, b_{2n-1}, b_{2n}) \leq_{\mathbb{R}^{2n}} (c_1, c_2, \cdots, c_{2n-1}, c_{2n})$ with $b_i < c_i$ for all $i \in \{1, \cdots, 2n\}$. On other hand, $\leq_{\mathbb{R}^{2n}}$ is a order relation in $\mathbb{R}^{2n}$, thus $(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) \leq_{\mathbb{R}^{2n}} (c_1, c_2, \cdots, c_{2n-1}, c_{2n})$ and, since $a_i < b_i$ and $b_i < c_i$ for all $i \in \{1, \cdots, 2n\}$, it follows that $a_i < c_i$ for all $i \in \{1, \cdots, 2n\}$. Then,

$$(a_1, a_2, \cdots, a_{2n-1}, a_{2n}) <_{\mathbb{R}^{2n}} (c_1, c_2, \cdots, c_{2n-1}, c_{2n}).$$

Therefore, $<_{\mathbb{R}^{2n}}$ is a strict partial order relation in $\mathbb{R}^{2n}$.
We can use the similar argument to prove that, if $\leq_{\mathbb{R}^{2n}}$ is a partial order relation in $\mathbb{R}^{2n}$, then $<_{\mathbb{R}^{2n}}$ is a strict partial order relation in $\mathbb{R}^{2n}$.

□

Remark 2.1.4. The strict partial order relation $<_{\mathbb{R}^{2n}}$ given in the Proposition 2.1.1 is not a strict total order relation in $\mathbb{R}^2$ because, it does not satisfy the property of trichotomy. For example, given n=1 and let $\leq_{\mathbb{R}^2}$ be the specific total order relation in $\mathbb{R}^2$ given by

$$(a_1, a_2) \leq_{\mathbb{R}^2} (b_1, b_2) \Longleftrightarrow \begin{cases} a_1 < b_1 \text{ or} \\ a_1 = b_1 \text{ and } a_2 < b_2 \end{cases}.$$

Given $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$, respectively, by $(0, 4), (1, 2)$, we have that $(0, 4) \neq (1, 2)$ but, $(0, 4) \not<_{\mathbb{R}^2} (1, 2)$ because $(0, 4) \leq_{\mathbb{R}^2} (1, 2)$ and $a_2 > b_2$. Moreover, $(1, 2) \not<_{\mathbb{R}^2} (0, 4)$ because $(1, 2) \not\leq_{\mathbb{R}^2} (0, 4)$ and $a_1 > b_1$.

Definition 2.1.2. *Given a (total) partial order relation $\leq_{\mathbb{R}^{2n}}$ in $\mathbb{R}^{2n}$ and a bijection $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$, we denote by $\leq_\varphi$, $\prec_\varphi$, and $<_\varphi$, the binary relations in M^n defined, respectively, by*

$$A \leq_\varphi B \Longleftrightarrow \varphi(A) \leq_{\mathbb{R}^{2n}} \varphi(B);$$

$$A \prec_\varphi B \Longleftrightarrow \varphi(A) \prec_{\mathbb{R}^{2n}} \varphi(B);$$

$$A <_\varphi B \Longleftrightarrow \varphi(A) <_{\mathbb{R}^{2n}} \varphi(B);$$

for all $A, B \in M^n$.

Theorem 2.1.3. *Let $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ be a bijection. Given a (total) partial order relation $\leq_{\mathbb{R}^{2n}}$ in $\mathbb{R}^{2n}$, the binary relations $\leq_\varphi$ and $\prec_\varphi$, given in the Definition 2.1.2, are (total) partial order relations in M^n, and $<_\varphi$ is a partial order relation in M^n.*

Proof. This proof follows directly from the fact that $\leq_{\mathbb{R}^{2n}}$ is a partial (total) order in $\mathbb{R}^{2n}$, from Remark 2.1.3, and from Definition 2.1.2. $\square$

Example 2.1.4. Let $\leq_{\mathbb{R}^{2n}}$ be the specific partial order relation in $\mathbb{R}^{2n}$ given by

$$u \leq_{\mathbb{R}^{2n}} v \Longleftrightarrow u_i \leq v_i \quad \forall i \in \{1, ..., 2n\}$$

for all $u = (u_1, \cdots, u_{2n})$, $v = (v_1, \cdots, v_{2n}) \in \mathbb{R}^{2n}$. Consider the function $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ denoted by

$$\begin{aligned} & \varphi\left([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\right) \\ = & \Big(\varphi_1\left([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\right), \cdots, \varphi_{2n}\left([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\right)\Big). \end{aligned}$$

It follows that, $\leq_\varphi$ is an partial order in the set M^n, which is given by

$$\left([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\right) \leq_\varphi \left([b_1, b_2], \cdots, [b_{2n-1}, b_{2n}]\right)$$

if and only if

$$\varphi_i\left([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\right) \leq \varphi_i\left([b_1, b_2], \cdots, [b_{2n-1}, b_{2n}]\right)$$

for all $i \in \{1, \cdots, 2n\}$ and for all

$$\left([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\right), \left([b_1, b_2], \cdots, [b_{2n-1}, b_{2n}]\right) \in M^n.$$

In particular, for $n = 1$, if we consider the bijection $\phi_1 : M \longrightarrow \mathbb{R}^2$ given in the Example 2.1.1, it follows that the partial order relations $\leq_{\mathbb{R}^2}$ in $\mathbb{R}^2$ and $\leq_{\phi_1}$ in M are given, respectively, by

$$(a_1, a_2) \leq_{\mathbb{R}^2} (b_1, b_2) \Longleftrightarrow a_1 \leq b_1 \ \ and \ \ a_2 \leq b_2, \tag{2.1}$$

and

$$[a_1, a_2] \leq_{\phi_1} [b_1, b_2] \Longleftrightarrow a_1 \leq b_1 \ \ and \ \ a_2 - a_1 \leq b_2 - b_1. \tag{2.2}$$

If we consider ϕ_1 restricted to the set $I(\mathbb{R})$, the partial order (2.2) induces in $I(\mathbb{R})$ the partial order relation that was used in [5]. This order compares both the right extreme points of each par of intervals and the lengths of each par of intervals.

If we consider the bijection $\phi_2 : M \longrightarrow \mathbb{R}^2$ defined in the Example 2.1.1, we have the following partial order relation in M

$$[a_1, a_2] \leq_{\phi_2} [b_1, b_2] \Longleftrightarrow a_1 \leq b_1 \ \ and \ \ a_2 \leq b_2. \tag{2.3}$$

If we consider ϕ_2 restrict to the set $I(\mathbb{R})$, the partial order (2.3), induces in $I(\mathbb{R})$ the partial order relation that was used in [35]. This order compares the right extreme points of each par of intervals as well as the left extreme points of each par of intervals.

If we consider the bijection $\phi_3 : M \longrightarrow \mathbb{R}^2$ defined in the Example 2.1.1, we obtain the following partial order relation in M

$$[a_1, a_2] \leq_{\phi_3} [b_1, b_2] \iff \frac{a_1 + a_2}{2} \leq \frac{b_1 + b_2}{2} \quad and \quad \frac{a_2 - a_1}{2} \leq \frac{b_2 - b_1}{2}. \tag{2.4}$$

This order compares both midpoints and the radius of each par of intervals.

Example 2.1.5. Let $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ be the bijection denoted by

$$\begin{aligned}&\varphi\big([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\big)\\ =\ &\Big(\varphi_1\big([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\big), \cdots, \varphi_{2n}\left([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\right)\Big).\end{aligned}$$

Consider the total order in $\mathbb{R}^{2n}$ given by

$$u \prec_{\mathbb{R}^{2n}} v \iff \begin{cases} u_1 < v_1 \text{ or} \\ \exists i \in \{1, \cdots\cdots, 2n-1\} \text{ such that } u_1 = v_1, \ \ldots, \ u_i = v_i \text{ and } u_{i+1} < v_{i+1} \end{cases}$$

for all $u = (u_1, u_2, \cdots, u_{2n}), v = (v_1, v_2, \cdots, v_{2n}) \in \mathbb{R}^{2n}$. From Definition 2.1.2, it follows that $\prec_\varphi$ is a totally ordered relation in the set M^n, given by

$$\big([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\big) \prec_\varphi \big([b_1, b_2], \cdots, [b_{2n-1}, b_{2n}]\big)$$

if and only if

$$\varphi_1\big([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\big) < \varphi_1\big([b_1, b_2], \cdots, [b_{2n-1}, b_{2n}]\big)$$

or there is $i \in \{1, \cdots, 2n-1\}$ such that,

$$\varphi_j\big([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\big) = \varphi_j\big([b_1, b_2], \cdots, [b_{2n-1}, b_{2n}]\big)$$

for all $j \in \{1, \cdots, i\}$ and

$$\varphi_{i+1}\left([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\right) < \varphi_{i+1}\left([b_1, b_2], \cdots, [b_{2n-1}, b_{2n}]\right)$$

for all $\big([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\big), \big([b_1, b_2], \cdots, [b_{2n-1}, b_{2n}]\big) \in M^n$.

In particular, if we consider $n = 1$ and the bijection $\phi_1 : M \longrightarrow \mathbb{R}^2$ given in the Example 2.1.1, we have that the order relation $\prec_{\phi_1}$ in M is given by

$$[a_1, a_2] \prec_{\phi_1} [b_1, b_2] \iff \begin{cases} a_1 < b_1 \text{ or} \\ a_1 = b_1, \text{ and } a_2 - a_1 < b_2 - b_1. \end{cases} \tag{2.5}$$

If we consider $n = 1$ and the bijection $\phi_2 : M \longrightarrow \mathbb{R}^2$ given in the Example 2.1.1, we have that the order relation $\prec_{\phi_2}$ in M is given by

$$[a_1, a_2] \prec_{\phi_2} [b_1, b_2] \Longleftrightarrow \begin{cases} a_1 < b_1 \text{ or} \\ a_1 = b_1, \text{ and } a_2 < b_2. \end{cases} \tag{2.6}$$

If we consider $n = 1$ and the bijection $\phi_3 : M \longrightarrow \mathbb{R}^2$ given in the Example 2.1.1, we have that the order relation $\prec_{\phi_3}$ in M is given by

$$[a_1, a_2] \prec_{\phi_3} [b_1, b_2] \Longleftrightarrow \begin{cases} \frac{a_1+a_2}{2} < \frac{b_1+b_2}{2} \text{ or} \\ \frac{a_1+a_2}{2} = \frac{b_1+b_2}{2}, \text{ and } \frac{a_2-a_1}{2} < \frac{b_2-b_1}{2}. \end{cases} \tag{2.7}$$

Given a set of intervals, it can express either quantitative or qualitative information. Then it is important to consider different kinds of order relations in this set to allows us to compare the information in either case. For example, suppose $[a_1, a_2]$ and $[b_1, b_2]$ express information about the income of people that live in two different cities, called A and B, respectively. If we wish compare the quality of life in these cities, suppose that the first criterion of evaluation is the middle income of people that live in the same city and the second criterion of evaluation is the income inequality of the people that live in the same city, it follows that the order in (2.7), allow us to say if to live in $[a_1, a_2]$ is better than to live in $[b_1, b_2]$ or to live in $[b_1, b_2]$ is better than to live in $[a_1, a_2]$. In this case, we have qualitative analysis.

On the other hand, if $[a_1, a_2]$ and $[b_1, b_2]$ express information about the ages of people that live in two different cities, called A and B, respectively. Suppose a_1 and a_2 are the smallest and the biggest age, respectively, in A and, suppose b_1 and b_2 are the smallest and the biggest age, respectively, in B. If we wish to know in what city live the person with the smallest age, then the order in (2.6) allows us obtain the answer. In this case, we have quantitative analysis.

Final remarks

Since we do not know an algebraic structure that makes $I(\mathbb{R})$ a vector space, it follows that we can not use the well known tools of linear algebra and functional analysis. Then the construction of a vector structure in the set M by means of the bijection between M and the usual vector space $(\mathbb{R}^2, +, \cdot)$, allows us to obtain concepts in $I(\mathbb{R})$ via the isomorphism between the vector spaces M and $(\mathbb{R}^2, +, \cdot)$, like the concepts of cone and convexity. Moreover, by means of the concepts of order in $\mathbb{R}^2$, we equipped M with the same concepts and, consequently, these same concepts can be defined in $I(\mathbb{R})$ and they provide tools to work with optimization problems in $I(\mathbb{R})$. Actually, our analysis is more general because the set that we have dealt with is M^n, thus M is a particular case and our arguments holds for $(I(\mathbb{R}))^n$ for all $n \in \mathbb{N}$.

Chapter 3

Optimization in n-Dimensional Generalized Interval Vector Spaces

In this chapter we present the structure of an interval optimization problem and a multi-objective optimization problem. We show that these two problems are closely related, the solution of one problem is also a solution of the other and vice versa. Actually, we show that solving the interval optimization problem proposed is equivalent to solving a multi-objective optimization problem. In addition, some examples are provided.

3.1 Optimization in interval spaces

Let $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ be a bijection and let $\leq_{\mathbb{R}^{2n}}$ be an order in $\mathbb{R}^{2n}$. Given a non-empty subspace X in $(\mathbb{R}^m, \|\cdot\|_{\mathbb{R}^m})$, where $\|\cdot\|_{\mathbb{R}^m}$ is a norm in $\mathbb{R}^m$, and $F : X \longrightarrow \big(I(\mathbb{R})\big)^n$ a well-defined function, we consider the following optimization problem:

$$\min_{x \in X} F(x). \tag{3.1}$$

Definition 3.1.1. *$\bar{x} \in X$ is a **Pareto solution** of (3.1) if and only if there does not exist any $x \in X\backslash\{\bar{x}\}$ such that*

$$F(x) \prec_{\varphi} F(\bar{x}).$$

Definition 3.1.2. *$\bar{x} \in X$ is a **locally Pareto solution** of (3.1) if and only if there is $\delta > 0$ such that, there does not exist any $x \in X \cap N(\bar{x}, \delta)$, where $N(x, \delta) = \{y \in X : \|y - x\|_{\mathbb{R}^m} < \delta\}$, that satisfies*

$$F(x) \prec_{\varphi} F(\bar{x}).$$

Definition 3.1.3. *$\bar{x} \in X$ is a **weak Pareto solution** of (3.1) if and only if there does not exist any $x \in X\backslash\{\bar{x}\}$ such that*

$$F(x) <_{\varphi} F(\bar{x}).$$

Definition 3.1.4. *$\bar{x} \in X$ is called* ***weak locally Pareto solution*** *of (3.1) if and only if there is $\delta > 0$ such that, there does not exist any $x \in X \cap N(\bar{x}, \delta)$, where $N(x, \delta) = \{y \in X : \|y - x\|_{\mathbb{R}^m} < \delta\}$, that satisfies*

$$F(x) <_{\varphi} F(\bar{x}).$$

Since $\varphi : (M^n, +_{\varphi}, \cdot_{\varphi}) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ is an isomorphism, $(I(\mathbb{R}))^n \subset M^n$, and $F : X \subseteq \mathbb{R} \longrightarrow (I(\mathbb{R}))^n$, then we can consider the following optimization problem

$$\min_{x \in X} \varphi\left(F(x)\right). \tag{3.2}$$

However, since $\varphi \circ F : X \longrightarrow \mathbb{R}^{2n}$, there exists $(\varphi \circ F)_i : X \longrightarrow \mathbb{R}$, with $i \in \{1, \cdots, 2n\}$, such that, $\varphi(F(x)) = ((\varphi \circ F)_1(x), (\varphi \circ F)_2(x), \cdots, (\varphi \circ F)_{2n}(x))$ for all $x \in X$. Thus, (3.2) can be written as the following multi-objective optimization problem

$$\min_{x \in X} \left((\varphi \circ F)_1(x), (\varphi \circ F)_2(x), \cdots, (\varphi \circ F)_{2n}(x)\right). \tag{3.3}$$

Definition 3.1.5. *$\bar{x} \in X$ is a* ***Pareto solution*** *of (3.3) if and only if there does not exist any $x \in X \backslash \{\bar{x}\}$ such that*

$$\varphi\left(F(x)\right) \prec_{\mathbb{R}^{2n}} \varphi\left(F(\bar{x})\right).$$

Definition 3.1.6. *$\bar{x} \in X$ is a* ***locally Pareto solution*** *of (3.3) if and only if there is $\delta > 0$ such that, there does not exist any $x \in X \cap N(\bar{x}, \delta)$, where $N(x, \delta) = \{y \in X : \|y - x\|_{\mathbb{R}^m} < \delta\}$, that satisfies*

$$\varphi\left(F(x)\right) \prec_{\mathbb{R}^{2n}} \varphi\left(F(\bar{x})\right).$$

Definition 3.1.7. *$\bar{x} \in X$ is a* ***weak Pareto solution*** *of (3.3) if and only if there does not exist any $x \in X \backslash \{\bar{x}\}$ such that*

$$\varphi\left(F(x)\right) <_{\mathbb{R}^{2n}} \varphi\left(F(\bar{x})\right).$$

Definition 3.1.8. *$\bar{x} \in X$ is a* ***weak locally Pareto solution*** *of (3.3) if and only if there is $\delta > 0$ such that, there does not exist any $x \in X \cap N(\bar{x}, \delta)$, where $N(x, \delta) = \{y \in X : \|y - x\|_{\mathbb{R}^m} < \delta\}$, that satisfies*

$$\varphi\left(F(x)\right) <_{\mathbb{R}^{2n}} \varphi\left(F(\bar{x})\right).$$

Theorem 3.1.1. *$\bar{x} \in X$ is a (locally) Pareto solution of (3.1) if and only if $\bar{x} \in X$ is a (locally) Pareto solution of (3.3). Moreover, $\bar{x} \in X$ is a weak (locally) Pareto solution of (3.1) if and only if $\bar{x} \in X$ is a weak (locally) Pareto solution of (3.3).*

Proof. From Definition 2.1.2 we have

$$F(x) \prec_{\varphi} F(\bar{x}) \text{ if and only if } \varphi(F(x)) \prec_{\mathbb{R}^{2n}} \varphi(F(\bar{x})).$$

Thus, there does not exist any $x \in X\backslash\{\bar{x}\}$ such that, $F(x) \prec_{\varphi} F(\bar{x})$ if and only if, there does not exist any $x \in X\backslash\{\bar{x}\}$ such that, $\varphi(F(x)) \prec_{\varphi} \varphi(F(\bar{x}))$. Therefore, $\bar{x} \in X$ is a Pareto solution of (3.1) if and only if $\bar{x} \in X$ is a Pareto solution of (3.3).
Moreover, there does not exist any $x \in X \cap N(\bar{x}, \delta)$ that satisfies $F(x) \prec_{\varphi} F(\bar{x})$ for some $\delta > 0$, if and only if, there does not exist any $x \in X \cap N(\bar{x}, \delta)$ that satisfies $\varphi(F(x)) \prec_{\mathbb{R}^{2n}} \varphi(F(\bar{x}))$ for the same $\delta > 0$. Therefore, $\bar{x} \in X$ is a locally Pareto solution of (3.1) if and only if $\bar{x} \in X$ is a locally Pareto solution of (3.3).

We can use a similar arguments to prove that $\bar{x} \in X$ is a weak (locally) Pareto solution of (3.1) if and only if $\bar{x} \in X$ is a weak (locally) Pareto solution of (3.3). □

Corollary 3.1.1. *If $\bar{x} \in X$ is a (locally) Pareto solution of (3.1), then $\bar{x} \in X$ is a weak (locally) Pareto solution of (3.1),*

Proof. If $\bar{x} \in X$ is a Pareto solution of (3.1), from Theorem 3.1.1 we have that $\bar{x} \in X$ is a Pareto solution of (3.3). Thus, there does not exist any $x \in X\backslash\{\bar{x}\}$, such that, $\varphi(F(x)) \prec_{\mathbb{R}^{2n}} \varphi(F(\bar{x}))$, that is, there does not exist any $x \in X\backslash\{\bar{x}\}$ such that $\varphi(F(x)) \leq_{\mathbb{R}^{2n}} \varphi(F(\bar{x}))$ and $\varphi(F(x)) \neq \varphi(F(\bar{x}))$. Then, there does not exist any $x \in X\backslash\{\bar{x}\}$ such that $\varphi(F(x)) \leq_{\mathbb{R}^{2n}} \varphi(F(\bar{x}))$ and $(\varphi \circ F)_i(x) <_{\mathbb{R}^{2n}} (\varphi \circ F)_i(\bar{x})$ for all $i \in \{1, \cdots, 2n\}$ because, otherwise, there would be $x \in X\backslash\{\bar{x}\}$ such that, either $\varphi(F(x)) \leq_{\mathbb{R}^{2n}} \varphi(F(\bar{x}))$ or $(\varphi \circ F)_i(x) <_{\mathbb{R}^{2n}} (\varphi \circ F)_i(\bar{x})$ for all $i \in \{1, \cdots, 2n\}$. But, by hypothesis, there does not exist any $x \in X\backslash\{\bar{x}\}$ such that $\varphi(F(x)) \leq_{\mathbb{R}^{2n}} \varphi(F(\bar{x}))$. Thus, there would be $x \in X\backslash\{\bar{x}\}$ such that $(\varphi \circ F)_i(x) <_{\mathbb{R}^{2n}} (\varphi \circ F)_i(\bar{x})$ for all $i \in \{1, \cdots, 2n\}$, it follows that there would be $x \in X\backslash\{\bar{x}\}$ such that $\varphi(F(x)) \neq \varphi(F(\bar{x}))$, which contradicts the hypothesis.

Therefore, $\bar{x} \in X$ is a weak Pareto solution of (3.3). But, by Theorem 3.1.1 we have that $\bar{x} \in X$ is a Pareto solution of (3.1).

We can use a similar argument to prove that, if $\bar{x} \in X$ is a locally Pareto solution of (3.1), then $\bar{x} \in X$ is a weak locally Pareto solution of (3.1). □

Remark 3.1.1. The next example will show that the reciprocal of Corollary 3.1.1 is not true.

Example 3.1.1. Given $m = n = 1$, $X = \mathbb{R}$, the specific bijection $\phi_2 : M \longrightarrow \mathbb{R}^2$ given in the Example 2.1.1, and the order in $\mathbb{R}^2$ defined by

$$(a_1, a_2) \leq_{\mathbb{R}^2} (b_1, b_2) \text{ if and only if } a_1 \leq_{\mathbb{R}} b_1 \text{ and } a_2 \leq_{\mathbb{R}} b_2.$$

Consider the function F defined by

$$F(x) = \begin{cases} [0, x] & \textit{if } x \geq 0 \\ [x, 0] & \textit{if } x \leq 0. \end{cases}$$

Then, $\bar{x} = 0$ is a weak Pareto solution of problem

$$\min_{x \in \mathbb{R}} F(x).$$

Indeed, suppose that there is an $x \in \mathbb{R}$ with $x \neq 0$, such that $F(x) <_{\phi_2} F(0)$, that is, $\phi_2(F(x)) <_{\mathbb{R}^2} \phi_2(F(0))$.

1. If $0 < x$, so $\phi_2(F(x)) = (0, x) <_{\mathbb{R}^2} \phi_2(F(0)) = (0, 0)$ implies $(0, x) \leq_{\mathbb{R}^2} (0, 0)$ with $0 < 0$ and $x < 0$. But, it is a contradiction.

2. If $x < 0$, so $\phi_2(F(x)) = (x, 0) <_{\mathbb{R}^2} \phi_2(F(0)) = (0, 0)$ implies $(x, 0) \leq_{\mathbb{R}^2} (0, 0)$ with $0 < 0$ and $x < 0$. But, it is a contradiction.

Thus, there does not exist any $x \neq 0$ such that $\phi_2(F(x)) <_{\mathbb{R}^2} \phi_2(F(0))$. Therefore, by Theorem 3.1.1, $\bar{x} = 0$ is a weak Pareto solution of problem.

On other hand, given $x < 0$ we have that $\phi_2(F(x)) \prec_{\mathbb{R}^2} \phi_2(F(0))$, that is, $\phi_2(F(x)) \leq_{\mathbb{R}^2} \phi_2(F(0))$ and $\phi_2(F(x)) \neq \phi_2(F(0))$. Indeed, $(x, 0) \leq_{\mathbb{R}^2} (0, 0)$ because $x \leq 0$ and $0 \leq 0$, moreover $\phi_2(F(x)) \neq \phi_2(F(0))$ because, $x < 0$, it follows that $(x, 0) \neq (0, 0)$. Therefore, $\bar{x} = 0$ is not a Pareto solution of problem.

Final remarks

In this Chapter we presented optimization problems over the space of proper interval-valued functions. These problems can be found, for example, in mathematical models in which the coefficients were obtained through numerical approximation. In order to have a control on the error generated through these numerical approximations, this model can be converted into an interval mathematics model. Since interval analysis has its limitation, then this chapter is important because, the construction of M^n allowed us to identify optimization problems of proper interval-valued functions with multi-objective optimization problems, which have a well established theory. Thus, solve an optimization problem with proper interval-valued functions is equivalent to solve a multi-objective optimization problem

Chapter 4

The Von Neumann's Theorem in Complete Linearly Ordered Generalized Interval Spaces

4.1 Complete Linearly Ordered Spaces

Here, we equip the set M^n with topological structures and then discuss some concepts and results on these topological spaces. We are particularly interested in spaces equipped with the order topology because in these spaces we can define the concept of linear order, which makes it possible to work with optimization problems of Von Neumann type, for example. Optimization problems of Von Neumann type will be dealt with in the next section.

Let $(\mathbb{R}^{2n}, \tau)$ be a topological space and let $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ be a bijection. We define the subset $\Im$ of M^n by

$$\Im := \{\varphi^{-1}(A) : A \in \tau\}.$$

Theorem 4.1.1. *The set $\Im$ is a topology in M^n.*

Proof. First we show that $\emptyset, M^n \in \Im$. Indeed, $\emptyset$ and $\mathbb{R}^{2n}$ are in the domain of φ^{-1} because τ is a topology. Since φ is a bijection $\varphi^{-1}(\emptyset) = \emptyset$ and $\varphi^{-1}(\mathbb{R}^{2n}) = M^n$. Thus, $\emptyset, M^n \in \Im$.

We now show that $\varphi^{-1}(A) \cap \varphi^{-1}(B) \in \Im$ for all $\varphi^{-1}(A), \varphi^{-1}(B) \in \Im$.
Given $\varphi^{-1}(A), \varphi^{-1}(B) \in \Im$, it follows that $A, B \in \tau$. Since τ is a topology in $\mathbb{R}^{2n}$, then $A \cap B \in \tau$ and, consequently , $\varphi^{-1}(A \cap B) \in \Im$.

On the other hand, $\varphi^{-1}(A \cap B) = \varphi^{-1}(A) \cap \varphi^{-1}(B)$. Therefore, $\varphi^{-1}(A) \cap \varphi^{-1}(B) \in \Im$.

It remains to be shown that $\Im$ is closed for arbitrary union of elements. Given a collection $\left(\varphi^{-1}(A_i)\right)_{i \in I} \in \Im$, where I is an arbitrary non-empty indexing set, it follows that $(A_i)_{i\in I} \in \tau$. Since τ is a topology in $\mathbb{R}^{2n}$, then $\left(\bigcup_{i\in I} A_i\right) \in \tau$. Thus, $\varphi^{-1}\left(\bigcup_{i\in I} A_i\right) \in \Im$. But,

$$\varphi^{-1}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \varphi^{-1}(A_i),$$

from which follows that $\bigcup_{i\in I} \varphi^{-1}(A_i) \in \Im$. Therefore, $\Im$ is a topology in M^n. □

Theorem 4.1.2. *Let $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ be a bijection.*

(a) If $(\mathbb{R}^{2n}, \tau)$ is a Hausdorff space, then $(M^n, \Im)$ is a Hausdorff space.

(b) If $(\mathbb{R}^{2n}, \tau)$ is a locally compact Hausdorff topological space, then $(M^n, \Im)$ is a locally compact Hausdorff topological space.

Proof. We start by proving (a). Given $x, y \in M^n$ with $x \neq y$, since φ is a bijection, it follows that, $\varphi(x) \neq \varphi(y)$. $(\mathbb{R}^{2n}, \tau)$ is a Hausdorff space, thus there exist $A, B \in \tau$ with $A \cap B = \emptyset$, such that $\varphi(x) \in A$ and $\varphi(y) \in B$. Since $A \cap B = \emptyset$, then $\emptyset = \varphi^{-1}(A \cap B) = \varphi^{-1}(A) \cap \varphi^{-1}(B)$. Furthermore, $x \in \varphi^{-1}(A)$ and $y \in \varphi^{-1}(B)$. Therefore, there exist $\varphi^{-1}(A), \varphi^{-1}(B) \in \Im$, with $\varphi^{-1}(A) \cap \varphi^{-1}(B) = \emptyset$ such that $x \in \varphi^{-1}(A)$ and $y \in \varphi^{-1}(B)$. Therefore, $(M^n, \Im)$ is a Hausdorff space.

We now prove (b). Given $x \in (M^n, \Im)$ arbitrarily, it follows that $\varphi(x) \in (\mathbb{R}^{2n}, \tau)$. By hypothesis, $(\mathbb{R}^{2n}, \tau)$ is a locally compact Hausdorff topological space, thus there exists $A \subset \mathbb{R}^{2n}$ compact which is a neighborhood of $\varphi(x)$. This means that $\varphi^{-1}(A) \subset M^n$ is a compact neighborhood of x in M^n, because given a collection of arbitrary open sets $(C_i)_{i\in I}$ in M^n, such that

$$\varphi^{-1}(A) \subset \bigcup_{i\in I} C_i,$$

it follows that, for each $i \in I$, $C_i = \varphi^{-1}(B_i)$ with $B_i \in \tau$. Thus,

$$\varphi^{-1}(A) \subset \bigcup_{i\in I} \varphi^{-1}(B_i).$$

Since φ is a bijection,

$$A = \varphi\left(\varphi^{-1}(A)\right) \subset \bigcup_{i\in I} \varphi\left(\varphi^{-1}(B_i)\right) = \bigcup_{i\in I} B_i.$$

Since A is compact in $(\mathbb{R}^{2n}, \tau)$, there exists $B_1, B_2, \cdots, B_k \in (B_i)_{i \in I}$ such that $A \subset \bigcup_{i=1}^{k} B_i$.
But,

$$\varphi^{-1}(A) \subset \varphi^{-1}\left(\bigcup_{i=1}^{k} B_i\right) = \bigcup_{i=1}^{k} \varphi^{-1}(B_i) = \bigcup_{i=1}^{k} C_i.$$

Thus, $\varphi^{-1}(A)$ is compact in $(M^n, \Im)$. By hypothesis $(\mathbb{R}^{2n}, \tau)$ is a Hausdorff space, thus it follows from part (a) that $(M^n, \Im)$ is also a Hausdorff space and for all $x \in M^n$ there exists a compact neighborhood $\varphi^{-1}(A)$. Therefore $(M^n, \Im)$ is a locally compact Hausdorff topological space. $\square$

Remark 4.1.1. Given a bijection $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ and a locally compact Hausdorff topological space $(\mathbb{R}^{2n}, \tau)$, it follows from Theorem 4.1.2 that $(M^n, \Im)$ is a locally compact Hausdorff topological space. Thus, if we consider the set $\overline{M^n} = M^n \cup \{p\}$, where p is an abstract point that does not belong to M^n, it follows from Alexandroff's Compactification Theorem (see e.g. [15]) that the subset $\widehat{\Im} \subset \overline{M^n}$ defined by

$$\widehat{\Im} := \Im \bigcup \{(M^n \backslash C) \cup \{p\} : C \textit{ is compact in } M^n\}$$

satisfies the following:

1. $\widehat{\Im}$ is a topology in $\overline{M^n}$
2. $\left(\overline{M^n}, \widehat{\Im}\right)$ is a compact Hausdorff topological space.
3. *If* M^n itself is not compact, then M^n is dense in $\left(\overline{M^n}, \widehat{\Im}\right)$.

We will denote the point p by $\{-\infty\}$.

Definition 4.1.1. *Let $(X, \preceq)$ be an ordered set. An element $b \in X$ is an upper (lower) limit of a subset $A \subset X$ with respect to the order $\preceq$ if and only if $a \preceq b$ $(b \preceq a)$ for all $a \in A$.*

Definition 4.1.2. *Let $(X, \preceq)$ be a linearly (or totally) ordered set. We will consider the total order relation $\preceq$ in X defined by*

$$a \prec b \Longleftrightarrow a \preceq b \textit{ and } a \neq b \textit{ with } a, b \in X.$$

Given the following subsets of X

$$]x, y[= \{z \in X : x \prec z \prec y, \textit{ with } x, y \in X\},$$

$$L_x = \{z \in X : z \prec x\} \quad \textit{and}$$

$$R_y = \{z \in X : y \prec z\},$$

we know that $\mathfrak{B} = \{L_u, R_v,]x, y[: u, v, x, y \in X\}$ *is a base for the order topology*

$$\Im_{\prec} = \{A \subset X : \forall a \in A, \exists B \in \mathfrak{B} \text{ such that } a \in B \subset A\}.$$

The space $(X, \preceq, \Im_{\prec})$ *is called a* ***linearly ordered space****.*

We say that $]x, y[$ *is an open interval in* X *and the set* $[x, y]$*, given by*

$$[x, y] = \{z \in X : x \preceq z \preceq y, \text{ with } x, y \in X\},$$

is a closed interval in X*.*

Example 4.1.1. Let $(M^n, \leq_\varphi)$ be the pair where $\leq_\varphi$ is the order induced in M^n by the bijection $\varphi : M^n \longrightarrow \mathbb{R}^{2n}$ and by the lexicographic order in $\mathbb{R}^{2n}$. This means that $(M^n, \leq_\varphi)$ is a totaly ordered set. Then, given the order topology $\Im_{<_\varphi}$, it follows that $(M^n, \leq_\varphi, \Im_{<_\varphi})$ is a linearly ordered space.

Definition 4.1.3. *Let* $(X, \preceq, \Im_{\prec})$ *be a linearly ordered space. This space is called* ***complete space*** *if and only if all non-empty subsets of* X *have at least one lower limit.*

Example 4.1.2. Consider $\overline{M^n} = M^n \cup \{-\infty\}$, where $-\infty$ is an abstract point that does not belong to M^n, and let $\leq_{\mathbb{R}^{2n}}$ be a total order relation in $\mathbb{R}^{2n}$. Then, it follows from Proposition 2.1.1 that $<_{\mathbb{R}^{2n}}$ is a total order relation in in $\mathbb{R}^{2n}$ and from Theorem 2.1.3 that $<_\varphi$ is a total order relation in M^n.

If we consider in $\overline{M^n}$ the following binary relation

$$a \leq b \Longleftrightarrow \begin{cases} a = b \text{ or} \\ a <_\varphi b \text{ if } a, b \in M^n \text{ or} \\ b \in M^n \text{ and } a = -\infty, \end{cases}$$

then it follows that $(\overline{M^n}, \leq, \Im_<)$ is a complete linearly ordered space.

4.2 The Von Neumann's Theorem

This section is devoted to presenting a new version of the Von Neumann's Theorem for functions taking values in complete linearly ordered generalized interval spaces. Our version of the Von Neumann's Theorem can be seen as Corollary of the Von Neumann's Theorem in complete linearly ordered spaces given in [14]. In order to present our version of the Von Neumann's Theorem we first recall some concepts and results.

To simplify the notation, we will denote a topological space (X, ζ) only by X and we will consider the topological space X which are endowed with a function $G : X \times X \longrightarrow \{\text{connected subsets of } X\}$ such that $x, y \in G(x, y) = G(y, x)$ for all $x, y \in X$.

Definition 4.2.1. *([14]) A subset $K \subseteq X$ is convex if and only if for all $x, y \in K$ we have $G(x, y) \subset K$.*

Let Z be a complete linearly ordered space and $F : X \longrightarrow Z$ be a well-defined function.

Recall that a function $F : X \longrightarrow Z$ is **quasiconvex** (**quasiconcave**) (see[14]) if and only if the sets

$$\{x \in X : F(x) \preceq z\} \quad (\text{ resp. } \{x \in X : z \preceq F(x)\})$$

are convex for all $z \in Z$. Furthermore, F is called **upper semicontinuous** if the sets

$$\{x \in X : z \preceq F(x)\}, \text{ for all } z \in Z,$$

are closed in X.

Proposition 4.2.1. *(a) If X is compact and $F : X \longrightarrow Z$ is upper semicontinuous, then there is $\bar{x} \in X$ such that, $F(x) \preceq F(\bar{x})$ for all $x \in X$. That is,*

$$F(\bar{x}) = \max_{x \in X} F(x).$$

(b) Let $(F_i)_{i \in I}$ be a family of upper semicontinuous functions such that $F_i : X \longrightarrow Z$. Then, the function $x \mapsto \inf_{i \in I} F_i(x)$, which we denoted by F, is upper semicontinuous.

Proof. (a): Let $F(X) = A$. Thus, given $a \in A$, there is $x_a \in X$, such that, $F(x_a) = a$. Then, the set $F_a = \{x \in X : a \preceq F(x)\}$ is non-empty for all $a \in A$. We also have that F_a is compact for all $a \in A$, because F_a is closed in X and X is compact. Moreover, given $a_n, a_{n+1} \in A$ with $a_n \prec a_{n+1}$, it follows that $F_{a_{n+1}} \subset F_{a_n}$.

Given a family $(F_a)_{a \in A}$, we take an arbitrary sequence $(F_{a_i})_{i \in \mathbb{N}}$ such that $a_i \prec a_{i+1}$ (it is possible because Z is a linearly ordered space). Since each F_{a_i} is non-empty, there is $x_i \in F_{a_i}$ for all $i \in \mathbb{N}$. But, as $(x_i)_{i \in \mathbb{N}}$ is a sequence in F_{a_1}, it follows that there is a subsequence $(x_{i_j})_{j \in \mathbb{N}}$ of $(x_i)_{i \in \mathbb{N}}$ such that $x_{i_j} \to \widehat{x} \in F_{a_1}$, when $j \to \infty$. However, $(x_{i_j})_{j \geq k}$

is a sequence in F_{a_k} for all $k \in \mathbb{N}$. Moreover, each F_{a_k} is closed, thus $\widehat{x} \in F_{a_k}$ for all $k \in \mathbb{N}$. Therefore, there is $\widehat{x} \in \bigcap_{n=1}^{\infty} F_{a_n}$. In particular, the family $(F_a)_{a\in A}$, has the finite intersection propriety. But X is a compact, then by classic theorem of topology that says "S is compact if only if every family of closed sets in S with the finite intersection propriety has non-empty intersection" we have that $\bigcap_{a\in A} F_a \neq \emptyset$. Let $\bar{x} \in \bigcap_{a\in A} F_a$, then $a \preceq F(\bar{x})$ for all $a \in A$, but it is equivalent to $F(x) \preceq F(\bar{x})$ for all $x \in X$. Since $\bar{x} \in X$, it follow that $F(\bar{x}) = \max_{x\in X} F(x)$.

(b) Given $z \in Z$ arbitrary, we have that $\{x \in X : z \preceq F_i(x)\}$ is closed in X for all $i \in I$. Moreover, we have

$$\{x \in X : z \preceq F(x)\} = \bigcap_{i\in I}\{x \in X : z \preceq F_i(x)\}.$$

Thus, the set $\{x \in X : z \preceq F(x)\}$ is closed in X for all $z \in Z$, which implies that F is an upper semicontinuous function. □

We now state and prove the main result of this section.

Theorem 4.2.1 (The Von Neumann's Theorem)**.** *Given* $(\mathbb{R}^m, \tau_m)$ *and* $(\mathbb{R}^p, \tau_p)$ *both Euclidean topological spaces, let* $X \subset \mathbb{R}^m$ *be a convex compact topological subspace,* $Y \subset \mathbb{R}^p$ *a convex topological subspace, and* $(\overline{M^n}, \leq, \Im_<)$ *the complete linearly ordered space that was given in the Example 4.1.2. Finally, let* $F : X \times Y \longrightarrow \overline{M^n}$ *be a function having the following properties:*

(a) *The functions* $F(\cdot, y)$ *are quasiconcave on* X *and upper semicontinuous on* X *for all* $y \in Y$ *fixed.*

(b) *The functions* $F(x, \cdot)$ *are quasiconvex on* Y *and upper semicontinuous on* Y *for all* $x \in X$ *fixed.*

Then,

$$\max_{x\in X} \inf_{y\in Y} F(x, y) = \inf_{y\in Y} \max_{x\in X} F(x, y).$$

Proof. In order to prove this theorem, the expressions $\max_{x\in X} \inf_{y\in Y} F(x, y)$ and $\inf_{y\in Y} \max_{x\in X} F(x, y)$ must exist.

The expression $\max_{x\in X} \inf_{y\in Y} F(x, y)$ exist, because for each $y \in Y$ fixed, we have that the function $F_y : X \longrightarrow Z$ defined by $F_y(x) = F(x, y)$ is an upper semicontinuous function. Then, from item (b) of Proposition 4.2.1, it follows that the function $x \mapsto \inf_{y\in Y} F_y(x)$ is an upper semicontinuous function. But, by hypothesis, X is compact and, from item (a) of Proposition 4.2.1, we obtain that, there exists $\max_{x\in X} \inf_{y\in Y} F_y(x) = \max_{x\in X} \inf_{y\in Y} F(x, y)$.

Therefore, there exists the expression $\max_{x\in X}\inf_{y\in Y} F(x,y)$.

Now, we analyze the existence of the expression $\inf_{y\in Y}\max_{x\in X} F(x,y)$. For each $y \in Y$ fixed, we have that the function $F_y : X \longrightarrow Z$ defined by $F_y(x) = F(x,y)$ is upper semicontinuous. By hypothesis, X is compact and, from item (a) of Proposition 4.2.1 the maximum $\max_{x\in X} F_y(x)$ is achieved. On other hand, $\overline{M^n}$ is complete linearly ordered space, thus there is $z* \in \overline{M^n}$ such that $z^* = \inf_{y\in Y}\max_{x\in X} F(x,y)$. Therefore, the expression $\inf_{y\in Y}\max_{x\in X} F(x,y)$ exists too.

We will prove that the topological spaces X and Y are endowed with the functions

$$G_X : X \times X \longrightarrow \{\text{connected subsets of } X\}$$

and

$$G_Y : Y \times Y \longrightarrow \{\text{connected subsets of } Y\},$$

respectively, such that $x_1, x_2 \in G_X(x_1,x_2)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in G_Y(y_1,y_2)$ for all $y_1, y_2 \in Y$. To this end we will show that all convex topological subspaces U of an Euclidean topological space $(\mathbb{R}^k, \tau_k)$, with $k \in \mathbb{N}$, are spaces endowed with a function

$$G_U : U \times U \longrightarrow \{\text{connected subsets of } U\},$$

such that $u_1, u_2 \in G_U(u_1,u_2)$ for all $u_1, u_2 \in U$.

We consider the function $G_U : U \times U \longrightarrow \{\text{convex subsets of } U\}$ defined by

$$G_U(u_1,u_2) = \Big\{(1-t)u_1 + tu_2 : t \in [0,1]\Big\} \text{ for all } u_1, u_2 \in U.$$

Then, for all $u_1, u_2 \in U$, it follows that $u_1, u_2 \in G_U(u_1,u_2)$. Furthermore, since $G_U(u_1,u_2)$ is a convex subset in the induced topological space (U, τ_U), for all $u_1, u_2 \in U$ and, (U,τ_U) is a topological subspace induced by the Euclidean topological space $(\mathbb{R}^k, \tau_k)$, it follows that $G_U(u_1,u_2)$ is a connected subset of U.

We will prove that

$$\max_{x\in X}\inf_{y\in Y} F(x,y) \leq \inf_{y\in Y}\max_{x\in X} F(x,y).$$

Given $\bar{x} \in X$ and $y \in Y$ arbitrary. Then,

$$F(\bar{x},y) \leq \max_{x\in X} F(x,y)$$

for all $y \in Y$. Consequently,

$$\inf_{y\in Y} F(\bar{x},y) \leq \inf_{y\in Y}\max_{x\in X} F(x,y) \quad \forall \bar{x} \in X. \tag{4.1}$$

Since the function $x \mapsto \inf_{y\in Y} F(x,y)$ is upper semincontinuos and X is compact, it follows that there is $x_0 \in X$ such that

$$\inf_{y\in Y} F(x_0,y) = \max_{x\in X}\inf_{y\in Y} F(x,y).$$

On other hand, (4.1) is valid for all $x \in X$, in particular we have that

$$\max_{x\in X}\inf_{y\in Y} F(x,y) \leq \inf_{y\in Y}\max_{x\in X} F(x,y).$$

In order to prove the other inequality, we will show that $\bigcap_{\widehat{y}\in Y} K(\widehat{y}) \neq \emptyset$, where

$$K(\widehat{y}) = \Big\{\widehat{x} \in X : \inf_{y\in Y}\max_{x\in X} F(x,y) \leq F(\widehat{x},\widehat{y})\Big\}.$$

Thus there would be $\widehat{x} \in \bigcap_{\widehat{y}\in Y} K(\widehat{y})$ which implies that $\inf_{y\in Y}\max_{x\in X} F(x,y) \leq F(\widehat{x},\widehat{y}) \quad \forall \widehat{y} \in Y$. In particular,

$$\inf_{y\in Y}\max_{x\in X} F(x,y) \leq \inf_{y\in Y} F(\widehat{x},y). \tag{4.2}$$

Since $x \mapsto \inf_{y\in Y} F(x,y)$ is an upper semicontinuous function and X is compact, it follows that there is $x_0 \in X$ such that

$$\inf_{y\in Y} F(\widehat{x},y) \leq \inf_{y\in Y} F(x_0,y) = \max_{x\in X}\inf_{y\in Y} F(x,y).$$

Thus, from (4.2) we have

$$\inf_{y\in Y}\max_{x\in X} F(x,y) \leq \max_{x\in X}\inf_{y\in Y} F(x,y).$$

In order to show that $\bigcap_{\widehat{y}\in Y} K(\widehat{y}) \neq \emptyset$ we will prove the following affirmations:

Affirmation: $K(y)$ is a non-empty convex set for all $y \in Y$. (4.3)

Indeed, given $\bar{y} \in Y$, consider $K(\bar{y})$. Since $\overline{M^n}$ is a complete linearly ordered space, it follows that there is $z^* \in \overline{M^n}$ such that $z^* = \inf_{y\in Y}\max_{x\in X} F(x,y) \leq \max_{x\in X} F(x,\bar{y})$. On other hand, $x \mapsto F(x,\bar{y})$ is an upper semicontinuous function and X is compact, it follows that there is $\bar{x} \in X$ such that $\max_{x\in X} F(x,\bar{y}) = F(\bar{x},\bar{y})$. Thus,

$$z^* \leq \max_{x\in X} F(x,\bar{y}) = F(\bar{x},\bar{y}).$$

It follows that $\bar{x} \in \Big\{x \in X : \inf_{y\in Y}\max_{x\in X} F(x,y) \leq F(\bar{x},\bar{y})\Big\} = K(\bar{y})$. Therefore $K(\bar{y}) \neq \emptyset$ for all $\bar{y} \in Y$. Moreover, by definition of quasiconcave function and by the fact that $\inf_{y\in Y}\max_{x\in X} F(x,y) \in \overline{M^n}$ we have that $K(\bar{y})$ is convex.

Affirmation: $K(y)$ is compact for all $y \in Y$. (4.4)

Indeed, given $\bar{y} \in Y$, since $x \mapsto F(x,\bar{y})$ is an upper semicontinuous, it follows that $K(\bar{y})$ is closed and, by the fact that $K(\bar{y}) \subset X$ with X compact, it follows that $K(\bar{y})$ is compact.

Affirmation: $K(y) \subset K(y_1) \cup K(y_2)$ whenever $y_1, y_2 \in Y$ and $y \in G_Y(y_1,y_2)$. (4.5)

Indeed, suppose there is $\bar{y} \in G_Y(y_1, y_2)$ such that $K(\bar{y}) \not\subset K(y_1) \cup K(y_2)$. It follows that there is $\bar{x} \in K(\bar{y})$, so that, $\bar{x} \notin K(y_1)$ and $\bar{x} \notin K(y_2)$. That is,

$$\begin{aligned} \inf_{y\in Y}\max_{x\in X} F(x,y) &\leq F(\bar{x},\bar{y}) \\ F(\bar{x},y_1) &< \inf_{y\in Y}\max_{x\in X} F(x,y) \qquad (4.6) \\ F(\bar{x},y_2) &< \inf_{y\in Y}\max_{x\in X} F(x,y) \qquad (4.7) \end{aligned}$$

Since $\overline{M^n}$ is a linearly ordered space and $F(\bar{x}, y_1)$, $F(\bar{x}, y_2) \in \overline{M^n}$, it follows that either $F(\bar{x}, y_1) \leq F(\bar{x}, y_2)$ or $F(\bar{x}, y_2) \leq F(\bar{x}, y_1)$.

If $F(\bar{x}, y_1) \leq F(\bar{x}, y_2)$, set $B = \{y \in Y : F(\bar{x}, y) \leq F(\bar{x}, y_2)\}$. We have that $B \neq \emptyset$ because $y_1, y_2 \in B$. Moreover, by hypothesis the function $y \mapsto F(\bar{x}, y)$ is quasiconvex, consequently, B is convex and, it implies that $G_Y(y_1, y_2) \subset B$. By our supposition, $\bar{y} \in G_Y(y_1, y_2)$, then, $F(\bar{x}, \bar{y}) \leq F(\bar{x}, y_2)$. However $\bar{x} \in K(\bar{y})$, it follows that

$$\inf_{y\in Y}\max_{x\in X} F(x,y) \leq F(\bar{x},\bar{y}) \leq F(\bar{x},y_2),$$

which contradicts (4.7).

If $F(\bar{x}, y_2) \leq F(\bar{x}, y_1)$, set $C = \{y \in Y : F(\bar{x}, y) \leq F(\bar{x}, y_1)\}$. We have that $C \neq \emptyset$ because $y_1, y_2 \in C$. Moreover, by hypothesis the function $y \mapsto F(\bar{x}, y)$ is quasiconvex, consequently, C is convex and, it implies that $G_Y(y_1, y_2) \subset C$. By our supposition, $\bar{y} \in G_Y(y_1, y_2)$, then, $F(\bar{x}, \bar{y}) \leq F(\bar{x}, y_1)$. However $\bar{x} \in K(\bar{y})$ which implies that

$$\inf_{y\in Y}\max_{x\in X} F(x,y) \leq F(\bar{x},\bar{y}) \leq F(\bar{x},y_1),$$

which contradicts (4.6).

Therefore, there does not exist any $y \in G_Y(y_1, y_2)$ such that $K(\bar{y}) \not\subset K(y_1) \cup K(y_2)$. That is, $K(y) \subset K(y_1) \cup K(y_2)$ whenever $y_1, y_2 \in Y$ and $y \in G_Y(y_1, y_2)$.

Affirmation: If $\lim_{i\in I} y_i = \bar{y}$, $\lim_{i\in I} x_i = \bar{x}$, and $x_i \in K(y_i)$, then $\bar{x} \in K(\bar{y})$. (4.8)

Indeed, since $x \mapsto F(x, y)$ is an upper semicontinuous for each $y \in Y$ and $y \mapsto F(x, y)$ is an upper semicontinuous for each $x \in X$, then $(x, y) \mapsto F(x, y)$ is an upper semicontinuous, that is,

$$\{(x,y) \in X \times Y : z \leq F(x,y)\}$$

is closed for all $z \in \overline{M^n}$.

We have that that $\limsup_{i\in I} F(x_i, y_i) \leq F(\bar{x}, \bar{y})$, otherwise, since $(\overline{M^n}, \leq, \mathfrak{I}_<)$ is complete linearly ordered, it follows that there is $z \in \overline{M^n}$ such that

$$F(\bar{x},\bar{y}) < z < \limsup_{i\in I} F(x_i,y_i).$$

On other hand, $z < \limsup_{i\in I} F(x_i, y_i)$ it implies that there is a subsequence $(x_{i_j}, y_{i_j})_{j\in I}$ of $(x_i, y_i)_{j\in I}$ such that $z < F(x_{i_j}, y_{i_j})$. Thus, $(x_{i_j}, y_{i_j})_{j\in I}$ is a sequence in the set $\{(x,y) \in X : z \leq F(x,y)\}$ such that $\lim_{j\in I}(x_{i_j}, y_{i_j}) = (\bar{x}, \bar{y})$ and $(\bar{x}, \bar{y}) \notin \{(x,y) \in X : z \leq F(x,y)\}$, which contradicts the fact that $\{(x,y) \in X \times Y : z \leq F(x,y)\}$ is closed.

Thus, we have that $\limsup_{i\in I} F(x_i, y_i) \leq F(\bar{x}, \bar{y})$. Moreover,

$$\lim_{j\in I} F(x_{i_j}, y_{i_j}) \leq \limsup_{i\in I} F(x_i, y_i)$$

for all convergente subsequence $\big(F(x_{i_j}, y_{i_j})\big)_{j\in I}$ of the sequence $\big(F(x_i, y_i)\big)_{i\in I}$. Since $x_{i_j} \in K\big(y_{i_j}\big)$, it follows that $\inf_{y\in Y}\max_{x\in X} F(x,y) \leq F(x_{i_j}, y_{i_j})$ for all $j \in I$. Consequently,

$$\inf_{y\in Y}\max_{x\in X} F(x,y) \leq \lim_{j\in I} F(x_{i_j}, y_{i_j}) \leq \limsup_{i\in I} F(x_i, y_i) \leq F(\bar{x}, \bar{y}).$$

Then $\bar{x} \in K(\bar{y})$.

To prove that $\bigcap_{\widehat{y}\in Y} K(\widehat{y}) \neq \emptyset$, it is enough to prove that the family $\big(K(\widehat{y})\big)_{\widehat{y}\in Y}$ has the finite intersection property and to this end we will use (4.3)-(4.8).

Suppose that $\bigcap_{i}^{n} K(y_i) \neq \emptyset$ for every choice of $y_1, \cdots, y_n \in Y$, but $\bigcap_{i=1}^{n+1} K(y_i^*) \neq \emptyset$ for some $y_1^*, \cdots, y_{n+1}^* \in Y$. Set $K^*(y) = \bigcap_{i=3}^{n+1} K(y_i^*) \cap K(y)$ for all $y \in Y$, then

$$K^*(y_1^*) \cap K^*(y_2^*) = \emptyset. \tag{4.9}$$

The set $K^*(y)$ is convex and compact for all $y \in Y$. (4.10)

Indeed, for all $y \in Y$ and all $i \in \{3, 4, \cdots, n+1\}$ we have, by (4.3) and by (4.4), that $K(y)$ and $K(y_i^*)$ are convex and compact sets. It implies that $K^*(y)$ is convex and compact for all $y \in Y$.

We have that $K^*(y) \subset K^*(y_1^*) \cup K^*(y_2^*)$ whenever $y_1^*, y_2^* \in Y$ and $y \in G_Y(y_1^*, y_2^*)$. (4.11)

Indeed, from (4.5) we have that $K(y) \subset K(y_1^*) \cup K(y_2^*)$ whenever $y_1^*, y_2^* \in Y$ and $y \in G_Y(y_1^*, y_2^*)$. Thus, given $y \in G_Y(y_1^*, y_2^*)$ and $x \in K^*(y)$ arbitrary, it follows that

$$\begin{aligned} x &\in \bigcap_{i=3}^{n+1} K(y_i^*) \cap K(y) \\ &\subset \big(K(y_n^*) \cap \cdots K(y_{n+1}^*)\big) \cap \big(K(y_1^*) \cup K(y_2^*)\big) = K^*(y_1^*) \cup K^*(y_2^*). \end{aligned}$$

Since was chosen $x \in K^*(y)$ arbitrary, then $K^*(y) \subset K^*(y_1^*) \cup K^*(y_2^*)$. Actually, either

$$K^*(y) \subset K^*(y_1^*) \text{ or } K^*(y) \subset K^*(y_2^*) \text{ whenever } y_1^*, y_2^* \in Y \text{ and } y \in G_Y(y_1^*, y_2^*). \tag{4.12}$$

Indeed, from (4.9) it follows that $K^*(y) \not\subset K^*(y_1^*) \cap K^*(y_2^*)$. Suppose there is $x_1, x_2 \in K^*(y)$ such that, $x_1 \in K^*(y_1^*)$ and $x_2 \in K^*(y_2^*)$. Since $K^*(y)$ is convex, it implies that $G_X(x_1, x_2) \subset K^*(y)$, thus the convex set $G_X(x_1, x_2)$ can be written the following way:

$$G_X(x_1, x_2) = \big(G_X(x_1, x_2) \cap K^*(y_1^*)\big) \cup \big(G_X(x_1, x_2) \cap K^*(y_2^*)\big).$$

On other hand, $G_X(x_1, x_2)$ is closed for all $x_1, x_2 \in X$ because, given a sequence $(u_i)_{i \in I}$ convergent to $\bar{u} \in X$ we have that for each $i \in I$, there is $\lambda_i \in [0, 1]$, such that $u_i = (1 - \lambda_i)x_1 + \lambda_i x_2$. Since the $(\lambda_i)_{i \in I}$ is a sequence in $[0, 1]$, it implies that there is a subsequence $(\lambda_{i_j})_{j \in I}$ such that $\lim\limits_{j \in I} \lambda_{i_j} = \bar{\lambda} \in [0, 1]$, thus $\bar{u} = (1 - \bar{\lambda})x_1 + \bar{\lambda} x_2 \in G_X(x_1, x_2)$. Furthermore, from (4.9) $\big(G_X(x_1, x_2) \cap K^*(y_1^*)\big)$ and $\big(G_X(x_1, x_2) \cap K^*(y_2^*)\big)$ are disjoint. In this way, $G_X(x_1, x_2)$ is an union of tow non-empty, closed and disjoint sets. But, it contradicts the connectedness of $G_X(x_1, x_2)$. Thus, either $K^*(y) \subset K^*(y_1^*)$ or $K^*(y) \subset K^*(y_2^*)$.

We set

$$V = \big\{y \in G_Y(y_1^*, y_2^*) : K^*(y) \subset K^*(y_1^*)\big\} \text{ and } W = \big\{y \in G_Y(y_1^*, y_2^*) : K^*(y) \subset K^*(y_2^*)\big\}.$$

We have that $G_Y(y_1^*, y_2^*) = V \cup W$ with V and W convex disjoint. Indeed, given $y \in G_Y(y_1^*, y_2^*)$, from (4.11) it follows that $K^*(y) \subset K^*(y_1^*) \cup K^*(y_2^*)$ consequently, $y \in V \cup W$. Thus, $G_Y(y_1^*, y_2^*) \subset V \cup W$. On other hand, given $y \in V \cup W$, by definition of V and W, it follows that $y \in G_Y(y_1^*, y_2^*)$. Thus $V \cup W \subset G_Y(y_1^*, y_2^*)$. Then, $G_Y(y_1^*, y_2^*) = V \cup W$. Moreover, from (4.9) we have that $V \cap W = \emptyset$ and, given $v_1, v_2 \in V$ arbitrary, by the definition of V, it follows that $K^*(v_1) \subset K^*(y_1^*)$ and $K^*(v_2) \subset K^*(y_1^*)$. Given $v \in G_Y(v_1, v_2)$, it follows from (4.11) that $K^*(v) \subset K^*(v_1) \cup K^*(v_2) \subset K^*(y_1^*)$, and consequently, $v \in V$. Thus, $G_Y(v_1, v_2) \subset V$ for all $v_1, v_2 \in V$, that is, V is a convex set.

We can use a similar argument to prove that W is a convex set.

Since $G_Y(y_1^*, y_2^*)$ is an union of V with W, disjoints convex sets, there is $y_0 \in G_Y(y_1^*, y_2^*)$ such that

$$K^*(y) \subset K^*(y_1^*) \text{ for all } y \in G_Y(y_1^*, y_0) \backslash \{y_0\} \tag{4.13}$$

and

$$K^*(y) \subset K^*(y_2^*) \text{ for all } y \in G_Y(y_0, y_2^*) \backslash \{y_0\}. \tag{4.14}$$

Since $y_0 \in G_Y(y_1^*, y_2^*)$, from (4.12) we have that either $K^*(y_0) \subset K^*(y_1^*)$ or $K^*(y_0) \subset K^*(y_2^*)$.

$$\text{We have that, if } K^*(y_0) \subset K^*(y_1^*) \text{ it follows that } \bigcap_{y \in G_Y(y_0, y_2^*) \backslash \{y_0\}} K^*(y) \neq \emptyset. \tag{4.15}$$

Indeed, since $K^*(y)$ is compact for all $y \in Y$, it follows that $\big(K^*(y)\big)_{y \in G_Y(y_0, y_2^*) \backslash \{y_0\}}$ is a family of compact set, then to prove (4.15) we can show that $\big(K^*(y)\big)_{y \in G_Y(y_0, y_2^*) \backslash \{y_0\}}$ has finite intersection property, for this propose, it is enough to prove that

$$K^*(\widehat{y}) \subset K^*(\bar{y}) \quad \text{whenever} \quad \widehat{y} \in G_Y(y_0, \bar{y})\backslash\{y_0\} \quad \text{and} \quad \bar{y} \in G_Y(y_0, y_2^*)\backslash\{y_0\}.$$

Given $\widehat{y} \in G_Y(y_0, \bar{y})\backslash\{y_0\}$ and $\bar{y} \in G_Y(y_0, y_2^*)\backslash\{y_0\}$ arbitrary. Thus, $\widehat{y} \in G_Y(y_0, \bar{y})\backslash\{y_0\} \subset G_Y(y_0, \bar{y})$ and from (4.11), it follows that

$$K^*(\widehat{y}) \subset K^*(y_0) \cup K^*(\bar{y}).$$

Moreover, $\widehat{y} \in G_Y(y_0, \bar{y})\backslash\{y_0\} \subset G_Y(y_0, y_2^*)\backslash\{y_0\}$ and from (4.14) we have that $K^*(\widehat{y}) \subset K^*(y_2^*)$. Thus,

$$\begin{aligned}
K^*(\widehat{y}) &\subset \big(K^*(y_0) \cup K^*(\bar{y})\big) \cap K^*(y_2^*) \\
&\subset \big(K^*(y_1^*) \cup K^*(\bar{y})\big) \cap K^*(y_2^*) \\
&= \big(K^*(y_1^*) \cap K^*(y_2^*)\big) \cup \big(K^*(\bar{y}) \cap K^*(y_2^*)\big) \\
&= \emptyset \cup \big(K^*(\bar{y}) \cap K^*(y_2^*)\big) \\
&= \big(K^*(\bar{y}) \cap K^*(y_2^*)\big) = K^*(\bar{y}),
\end{aligned}$$

consequently, $\displaystyle\bigcap_{y \in G_Y(y_0, y_2^*)\backslash\{y_0\}} K^*(y) \neq \emptyset$.

Given $x_0 \in \left(\displaystyle\bigcap_{y \in G_Y(y_0, y_2^*)\backslash\{y_0\}} K^*(y)\right)$, from definition of $K^*(y)$, it follows that

$$\inf_{y \in Y} \max_{x \in X} F(x, y) \leq F(x_0, \bar{y}) \quad \text{for all} \quad \bar{y} \in G_Y(y_0, y_2^*)\backslash\{y_0\}. \tag{4.16}$$

Thus, we make $\bar{y} \to y_0$, since $y \mapsto F(x_0, y)$ is an upper semicontinuous function, we have that

$$\limsup_{\bar{y} \to y_0} F(x_0, \bar{y}) \leq F(x_0, y_0). \tag{4.17}$$

Then, given a sequence $(y_i)_{i \in I}$ in $G_Y(y_0, y_2^*)\backslash\{y_0\}$ with $\lim\limits_{i \in I} y_i = y_0$, it follows from (4.16) and (4.17) that

$$\inf_{y \in Y} \max_{x \in X} F(x, y) \leq \limsup_{y_i \to y_0} F(x_0, y_i) \leq \limsup_{y \to y_0} F(x_0, y) \leq F(x_0, y_0). \tag{4.18}$$

On other hand, $y_2^* \in G_Y(y_0, y_2^*)\backslash\{y_0\}$, it follows that $x_0 \in K^*(y_2^*)$. But, by our supposition $K^*(y_0) \subset K^*(y_1^*)$, then by (4.9) we have that $x_0 \notin K^*(y_0)$ and by definition of $K^*(y_0)$, it follows that $x_0 \notin K(y_0)$, which contradicts (4.18).

Supposing $K^*(y_0) \subset K^*(y_2^*)$ we can use a similar arguments to prove that is supposition is an absurd. Thus, there is not any $y_0 \in G_Y(y_1^*, y_2^*)$ such that $K^*(y_0) \subset K^*(y_1^*)$ or $K^*(y_0) \subset K^*(y_2^*)$.

Then it is absurd the supposition that $\bigcap_{i=1}^{n} K(y_i) \neq \emptyset$ for all choice of $y_1, \cdots, y_n \in Y$ and $\bigcap_{i=1}^{n+1} K(y_i^*) = \emptyset$ for some $y_1^*, \cdots, y_{n+1}^* \in Y$. Thus, $\bigcap_{\widehat{y} \in Y} K(\widehat{y}) \neq \emptyset$.

Therefore,

$$\max_{x \in X} \inf_{y \in Y} F(x, y) = \inf_{y \in Y} \max_{x \in X} F(x, y).$$

□

Final remarks

In this chapter we provided a topological structure for the set M^n, which was constructed through a bijection between M^n and $\mathbb{R}^{2n}$. The topological space M^n with this construction inherited the topological properties of $\mathbb{R}^{2n}$ and allowed us to approach a very important topic of game theory and economics, which is known as Von Neumann's Theorem, on the interval space.

In the proof of our version of Von Neumann's Theorem, we used the ideas in the proof of version of Von Neumann's Theorem given in [14].

Chapter 5

Some Concepts of Calculus of Generalized Interval-valued Functions

Here we present a way to equip the set M^n with a normed space structure by considering the set $\mathbb{R}^{2n}$ equipped with some norm. Also, we present some concepts of calculus of generalized interval-valued functions of real variables. The concepts are limit, continuity, Lipschitz condition, and differentiability, which together with the results we develop here, allows us to analyze interval differential equations and relate them to the classical differential equations in the next chapter.

5.1 Calculus on generalized interval space

Definition 5.1.1. *Given an isomorphism $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$, consider the metric space $(\mathbb{R}^{2n}, || \cdot ||_{\mathbb{R}^{2n}})$, where $|| \cdot ||_{\mathbb{R}^{2n}}$ is an arbitrary norm. Define the following function $|| \cdot ||_\varphi : M^n \longrightarrow \mathbb{R}$ by*

$$\left\|([a_1, a_2], [a_3, a_4], \cdots, [a_{2n-1}, a_{2n}])\right\|_\varphi := \|\varphi([a_1, a_2], [a_3, a_4], \cdots, [a_{2n-1}, a_{2n}])\|_{\mathbb{R}^{2n}} .$$

Theorem 5.1.1. *The space $(M^n, +_\varphi, \cdot_\varphi, || \cdot ||_\varphi)$ is a normed space.*

Proof. This proof follows directly from Definition 5.1.1. □

Given the functions $F, G : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi)$ and $\lambda \in \mathbb{R}$, we will use the following notation:

$$(F + G)(t) := F(t) +_\varphi G(t),$$

$$(F - G)(t) := F(t) -_\varphi G(t) = F(t) +_\varphi (-1) \cdot_\varphi G(t),$$

$$(\lambda \cdot F)(t) := \lambda \cdot_\varphi F(t).$$

Definition 5.1.2. *Given a function $F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \| \cdot \|_\varphi)$ and $t_0 \in U$, we say that $L_+ \in M^n$ ($L_- \in M^n$) is **right hand limit** (**left hand limit**) of F as $t \to t_0$ if and only if for each $\epsilon > 0$, there exist $\delta > 0$ such that for all $t \in U$ with $0 < t - t_0 < \delta$ ($0 < t_0 - t < \delta$), it follows that $\|F(t) -_\varphi L\|_\varphi < \epsilon$. We denote the right hand limit L_+ by $\lim\limits_{t \to t_0^+} F(t)$ and the left hand limit L_- by $\lim\limits_{t \to t_0^-} F(t)$. If $L_- = L_+ = L \in M^n$ we say that L is the limit of of F as $t \to t_0$ and we denote L by $\lim\limits_{t \to t_0} F(t)$.*

Theorem 5.1.2. *Let $F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \| \cdot \|_\varphi)$, $t_0 \in U$, and $L \in M^n$ be given. Then L is the limit of F as $t \to t_0$ if and only if $L = \varphi^{-1}\left(\lim\limits_{t \to t_0} \varphi(F(t)) \right)$.*

Proof. Let $L = \lim\limits_{t \to t_0} F(t)$ and let $\| \cdot \|_{\mathbb{R}^{2n}}$ be a norm in $\mathbb{R}^{2n}$ associated with the norm $\| \cdot \|_\varphi$ in M^n. From the definition of limit we have for each $\epsilon > 0$ given, that there exists $\delta > 0$ such that for all $t \in U$ with $0 < |t - t_0| < \delta$ it follows that $\|F(t) -_\varphi L\|_\varphi < \epsilon$. But this is equivalent to: for all $\epsilon > 0$ given, there exists $\delta > 0$ such that for all $t \in U$ with $0 < |t - t_0| < \delta$ it follows that $\|\varphi(F(t)) - \varphi(L)\|_{\mathbb{R}^{2n}} < \epsilon$. That means $\lim\limits_{t \to t_0} \varphi(F(t)) = \varphi(L)$. But this is equivalent to $\varphi^{-1}\left(\lim\limits_{t \to t_0} \varphi(F(t)) \right) = L$. □

Remark 5.1.1. *Given an isomorphism $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ and a function $F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \| \cdot \|_\varphi)$, we obtain a function $H : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{2n}$ given by the composition $H = \varphi \circ F$, illustrated in the following diagram:*

$$\begin{tikzcd} U \subseteq \mathbb{R} \arrow[r, "F"] \arrow[d, "H"] & M^n \arrow[dl, "\varphi"] \\ \mathbb{R}^{2n} & \end{tikzcd}.$$

Corollary 5.1.1. *Let $F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \| \cdot \|_\varphi)$, $t_0 \in U$, and $L \in M^n$. So, $L = \lim\limits_{t \to t_0} F(t)$ if and only if the function $H : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{2n}$ given by $H = \varphi \circ F$ is such that $\lim\limits_{t \to t_0} H(t) = \varphi(L)$.*

Proof. From Remark 5.1.1 follows the existence of the function $H : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{2n}$ given by $H = \varphi \circ F$. Thus

$$\lim_{t \to t_0} H(t) = \lim_{t \to t_0} (\varphi \circ F)(t) = \lim_{t \to t_0} \varphi(F(t)). \tag{5.1}$$

By the proof of Theorem 5.1.2 we have $L = \lim\limits_{t \to t_0} F(t)$ if and only if $\lim\limits_{t \to t_0} \varphi(F(t)) = \varphi(L)$. Then, from (5.1) we have $L = \lim\limits_{t \to t_0} F(t)$ if and only if $\lim\limits_{t \to t_0} H(t) = \varphi(L)$. □

Corollary 5.1.2. *Let $\psi_i : (M, +_{\psi_i}, \cdot_{\psi_i}) \longrightarrow (\mathbb{R}^2, +, \cdot)$ be an isomorphism for all $i \in \{1, \cdots, n\}$. Given the specific isomorphism, $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ defined by*

$$\begin{aligned} \varphi([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) &= (\psi_1 \times \cdots \times \psi_n)([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) \\ &= (\psi_1([a_1, a_2]), \cdots, \psi_n([a_{2n-1}, a_{2n}])) \end{aligned}$$

for all $([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) \in M^n$. *Let* $F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$ *be a function denoted by* $F(t) = \big(F_1(t), \cdots, F_n(t)\big)$, *where* $F_i : U \subseteq \mathbb{R} \longrightarrow (M, +_{\psi_i}, \cdot_{\psi_i}, \|\cdot\|_{\psi_i})$ *with* $F_i(t) = [f_{2i-1}(t), f_{2i}(t)]$ *for all* $i \in \{1, \cdots, n\}$. *Then,* $L = (L_1, L_2, \cdots, L_n) \in M^n$ *is the limit of* F *as* $t \to t_0$ *if and only if* L_i *is the limit of* F_i *as* $t \to t_0$ *for all* $i \in \{1, \cdots, n\}$.

Proof. It follows from Corollary 5.1.1 that $L \in M^n$ is the limit of F as $t \to t_0$ if and only if the function $H = \varphi \circ F : U \subseteq \mathbb{R} \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ satisfies

$$\lim_{t \to t_0} H(t) = \varphi(L). \tag{5.2}$$

However, by the definitions of φ and H we have that there is $H_i \equiv \psi_i \circ F_i : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ for all $i \in \{1, \cdots, n\}$, such that $H(t) = \big(H_1(t), \cdots, H_n(t)\big)$. Moreover, $\varphi(L) = \varphi(L_1, \cdots, L_n) = \big(\psi_1(L_1), \cdots, \psi_n(L_n)\big)$. Thus (5.2) is equivalent to $\lim\limits_{t \to t_0} H_i(t) = \psi_i(L_i)$ with $H_i \equiv \psi_i \circ F_i : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ for all $i \in \{1, \cdots, n\}$ and by Corollary 5.1.1, this is equivalent to L_i be the limit of F_i as $t \to t_0$ for all $i \in \{1, \cdots, n\}$.

Therefore, $L = (L_1, \cdots, L_n) \in M^n$ is the limit of F as $t \to t_0$ if and only if L_i is the limit of F_i as $t \to t_0$ for all $i \in \{1, \cdots, n\}$. □

Theorem 5.1.3. *Given* $F, G : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$, $t_0 \in U$, $\lambda \in \mathbb{R}$, *and* $L_1, L_2 \in M^n$ *such that*

$$\lim_{t \to t_0} F(t) = L_1 \quad \textit{and} \quad \lim_{t \to t_0} G(t) = L_2,$$

the following properties are valid:

1. $\lim\limits_{t \to t_0} (F + G)(t) = L_1 +_\varphi L_2$.
2. $\lim\limits_{t \to t_0} (F - G)(t) = L_1 -_\varphi L_2$.
3. $\lim\limits_{t \to t_0} (\lambda \cdot F)(t) = \lambda \cdot_\varphi L_1$.

Proof. We just prove item 1. The proofs of the other two items are similar. Since $\lim\limits_{t \to t_0} F(t) = L_1$ and $\lim\limits_{t \to t_0} G(t) = L_2$, by Corollary 5.1.1 there are the functions $H_1, H_2 : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{2n}$ given by $H_1 = \varphi \circ F$ and $H_2 = \varphi \circ G$, such that $\lim\limits_{t \to t_0} H_1(t) = \varphi(L_1)$ and $\lim\limits_{t \to t_0} H_2(t) = \varphi(L_2)$. On the other hand, φ is a isomorphism, thus given $H = H_1 + H_2 : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{2n}$ we have that $\lim\limits_{t \to t_0} H(t) = \varphi(L_1 +_\varphi L_2)$, where $H = \varphi \circ (F + G)$. Then, by Corollary 5.1.1 we have $\lim\limits_{t \to t_0} (F + G)(t) = L_1 +_\varphi L_2$. □

Definition 5.1.3. *Given a function* $F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$ *and* $t_0 \in U$, *we say that* F *is a continuous function at* t_0 *if and only if* $\lim\limits_{t \to t_0} F(t) = F(t_0)$. *If* F *is continuous at all points* $t \in U$, *then we say that* F *is continuous.*

Theorem 5.1.4. *Consider $F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$. Then, F is continuous if and only if $H = \varphi \circ F : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{2n}$ is continuous.*

Proof. Since φ is a isomorphism and $(M^n, +_\varphi, \cdot_\varphi)$ has finite dimension, then φ and φ^{-1} are continuous. On other hand, $H = \varphi \circ F$, consequently, $F = \varphi^{-1} \circ H$. Therefore, F is continuous if and only if H is continuous. □

Remark 5.1.2. It is known that each space $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ is constructed through of the bijection φ between M^n and $\mathbb{R}^{2n}$, Thus, given a specific bijection between M^n and $\mathbb{R}^{2n}$, we can provide, through of Theorem 5.1.4, a condition that allow us to say when a function $F : U \subseteq \mathbb{R} :\longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$ is continuous. For example, let $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ be the specific isomorphism defined by

$$\begin{aligned}\varphi([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) &= (\psi\big([a_1, a_2]\big), \cdots, \psi\big([a_{2n-1}, a_{2n}]\big)\\ &= ((a_1, a_2 - a_1), \cdots, (a_{2n-1}, a_{2n} - a_{2n-1})),\end{aligned}$$

where $\psi : (M, +_\psi, \cdot_\psi) \longrightarrow (\mathbb{R}^2, +, \cdot)$ is the specific isomorphism $\psi([u, v]) = (u, v - u)$. Let $F : U \subseteq \mathbb{R} :\longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$ be a function such that

$$F(t) = ([f_1(t), f_2(t)], [f_3(t), f_4(t)], \cdots, [f_{2n-1}(t), f_{2n}(t)]),$$

where $f_i : U \longrightarrow \mathbb{R}$, $i \in \{1, \cdots, 2n\}$. F is continuous if and only if for each $i \in \{1, \cdots, 2n\}$, f_i is also continuous.

Indeed, let $H = \varphi \circ F : U \subseteq \mathbb{R} :\longrightarrow (\mathbb{R}^{2n}, +, \cdot)$, from Theorem 5.1.4 we have that, F is continuous if and only if H is continuous. But,

$$H(t) = \Big(\big(f_1(t), f_2(t) - f_1(t)\big), \big(f_3(t), f_4(t) - f_3(t)\big), \cdots, \big(f_{2n-1}(t), f_{2n}(t) - f_{2n-1}(t)\big)\Big),$$

thus, H is continuous if and only if for each $i \in \{1, \cdots, 2n - 1\}$ f_i and $f_{i+1} - f_i$ are continuous. On the other hand, for each $i \in \{1, \cdots, 2n-1\}$, f_i and $f_{i+1} - f_i$ are continuous if and only if for each $i \in \{1, \cdots, 2n\}$, f_i is also continuous.

Therefore, F is continuous if and only if for each $i \in \{1, \cdots, 2n\}$, f_i is also continuous.

Theorem 5.1.5. *Given the functions $F, G : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$, $t_0 \in U$, and $\lambda \in \mathbb{R}$, if F and G are continuous at t_0, then the functions*

$$F + G,\ \ F - G,\ \ \lambda \cdot F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$$

are also continuous at t_0.

Proof. The proof of this theorem is a direct consequence of Theorem 5.1.3. □

Definition 5.1.4. *Given a function $F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$, we say that F is Lipschitz if and only if there is $k > 0$ such that*

$$\|F(t_0) -_\varphi F(t_1)\|_\varphi \leq k|t_0 - t_1| \quad \forall t_0, t_1 \in U.$$

Remark 5.1.3. Consider a specific isomorphism $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ and let $F : U \subseteq \mathbb{R} :\longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$. By making use of a similar argument to that used in the Remark 5.1.2 and Theorem 5.1.4, we can provide conditions under which F is Lipschitz when its extreme functions are. For example, consider the specific isomorphism $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ defined by

$$\begin{aligned}\varphi([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) &= (\psi\big([a_1, a_2]\big), \cdots, \psi\big([a_{2n-1}, a_{2n}]\big)\\ &= ((a_1, a_2 - a_1), \cdots, (a_{2n-1}, a_{2n} - a_{2n-1})),\end{aligned}$$

where $\psi : (M, +_\psi, \cdot_\psi) \longrightarrow (\mathbb{R}^2, +, \cdot)$ is the specific isomorphism $\psi([u, v]) = (u, v - u)$.
Let $F : U \subseteq \mathbb{R} :\longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$ be a function such that

$$F(t) = ([f_1(t), f_2(t)], \cdots, [f_{2n-1}(t), f_{2n}(t)]),$$

where $f_i : U \subseteq \mathbb{R} :\longrightarrow \mathbb{R}$, $i \in \{1, \cdots, 2n\}$. F is Lipschitz if and only if, for each $i \in \{1, \cdots, 2n\}$, f_i is also Lipschitz.
Indeed, if F is Lipschitz, it follows that, there is $k > 0$ such that for all $t_0, t_1 \in U$ we have $\|F(t_0) -_\varphi F(t_1)\|_\varphi \leq k|t_0 - t_1|$. That is, $\|\varphi(F(t_0)) - \varphi(F(t_1))\|_{\mathbb{R}^{2n}} \leq k|t_0 - t_1| \quad \forall t_0, t_1 \in U$. Thus,

$$\begin{aligned}&\left\|\Big(f_1(t_0) - f_1(t_1), \cdots, f_{2n-1}(t_0) - f_{2n-1}(t_1), f_{2n}(t_0) - f_{2n}(t_1) - \big(f_{2n-1}(t_0) - f_{2n-1}(t_1)\big)\Big)\right\|_{\mathbb{R}^{2n}}\\ &\qquad \leq k|t_0 - t_1| \quad \textit{for all } t_0, t_1 \in U.\end{aligned}$$

Since that in $\mathbb{R}^{2n}$ all norms are equivalent, we can consider the max norm. Then, for all $i \in \{1, \cdots, 2n-1\}$, it follows that $|f_i(t_0) - f_i(t_1)| \leq k|t_0 - t_1|$ for all $t_0, t_1 \in U$. Moreover, for all $t_0, t_1 \in U$, $\quad |f_{2n}(t_0) - f_{2n}(t_1) - (f_{2n-1}(t_0) - f_{2n-1}(t_1))| \leq k|t_0 - t_1|$. Thus,

$$|f_{2n}(t_0) - f_{2n}(t_1)| - |(f_{2n-1}(t_0) - f_{2n-1}(t_1))| \leq k|t_0 - t_1|.$$

By setting $C = 2k$, we obtain

$$\begin{aligned}|f_{2n}(t_0) - f_{2n}(t_1)| &\leq k|t_0 - t_1| + |(f_{2n-1}(t_0) - f_{2n-1}(t_1))|\\ &\leq 2k|t_0 - t_1| \text{ for all } t_0, t_1 \in U,\end{aligned}$$

which implies that $|f_i(t_0) - f_i(t_1)| \leq C|t_0 - t_1|$ for all $t_0, t_1 \in U$ and for all $i \in \{1, \cdots, 2n\}$.
Therefore, if F is Lipschitz, it follows that for each $i \in \{1, \cdots, 2n\}$, f_i is also Lipschitz.

Reciprocally, let f_i be Lipschitz for each $i \in \{1, \cdots, 2n\}$. Thus, there are $C_1, C_2, \cdots, C_{2n} > 0$ such that $|f_i(t_0) - f_i(t_1)| \leq C_i|t_0 - t_1|$ for all $t_0, t_1 \in U$.

Let $C = C_1 + C_2 + \cdots + C_{2n}$. Then, for all $i \in \{1, \cdots, 2n\}$ we have

$$|f_i(t_0) - f_i(t_1)| \leq C|t_0 - t_1| \text{ for all } t_0, t_1 \in U$$

and for all $i \in \{1, \cdots, 2n-1\}$ we have

$$|f_{i+1}(t_0) - f_{i+1}(t_1) - (f_i(t_0) - f_i(t_1))\,| \leq C|t_0 - t_1| \text{ for all } t_0, t_1 \in U.$$

Since we consider the max norm in $\mathbb{R}^{2n}$, we may conclude that

$$\left\| \Big(f_1(t_0) - f_1(t_1), \cdots, f_{2n-1}(t_0) - f_{2n-1}(t_1), f_{2n}(t_0) - f_{2n}(t_1) - \big(f_{2n-1}(t_0) - f_{2n-1}(t_1)\big) \Big) \right\|_{\mathbb{R}^{2n}} \leq C|t_0 - t_1|, \quad \text{for all } t_0, t_1 \in U,$$

that is, $\|F(t_0) -_\varphi F(t_1)\|_\varphi \leq C|t_0 - t_1|$ for all $t_0, t_1 \in U$. Therefore, if for each $i \in \{1, \cdots, 2n\}$, f_i is Lipschitz, then F is Lipschitz.

Definition 5.1.5. *Given an open set $U \subseteq \mathbb{R}$ and a function $F : U \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$, we say that F is **right differentiable** (**left differentiable**) at $t \in U$ if and only if there exists the right hand limit (the left hand limit)*

$$\lim_{h \to 0^+} \big(F(t+h) -_\varphi F(t)\big) \cdot_\varphi \frac{1}{h} \quad \left(\lim_{h \to 0^-} \big(F(t+h) -_\varphi F(h)\big) \cdot_\varphi \frac{1}{h} \right).$$

*We denote $\lim_{h \to 0^+} \big(F(t+h) -_\varphi F(h)\big) \cdot_\varphi \frac{1}{h}$ and $\lim_{h \to 0^-} \big(F(t+h) -_\varphi F(h)\big) \cdot_\varphi \frac{1}{h}$ by $F'_+(t)$ and $F'_-(t)$, respectively. $F'_+(t)$ and $F'_-(t)$ are called **right derivative** and **left derivative**, respectively, of F at t. If $F'_+(t) = F'_-(t)$ we say that F is **differentiable** at t and we denote by $F'(t)$ the derivative of F at t.*

Theorem 5.1.6. *Given an open set $U \subseteq \mathbb{R}$ and $F : U \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$, F differentiable at $t \in U$ if and only if $H = \varphi \circ F : U \longrightarrow \mathbb{R}^{2n}$ is differentiable at t. Moreover, if F is differentiable at $t \in U$, then $F'(t) = \varphi^{-1}\big(H'(t)\big)$.*

Proof. F is differentiable at t if and only if, there is $F'(t) \in M^n$ such that $F'(t) = \lim_{h \to 0} \big(F(t+h) -_\varphi F(h)\big) \cdot_\varphi \frac{1}{h}$. However, by the isomorphism φ, this is equivalent to existence of

$$\varphi\big(F'(t)\big) = \varphi\left(\lim_{h \to 0} \big(F(t+h) -_\varphi F(h)\big) \cdot_\varphi \frac{1}{h} \right).$$

On other hand,

$$\begin{aligned}
\varphi\big(F'(t)\big) &= \varphi\left(\lim_{h \to 0} \big(F(t+h) -_\varphi F(h)\big) \cdot_\varphi \frac{1}{h} \right) \\
&= \lim_{h \to 0} \Big(\varphi\big(F(t+h)\big) - \varphi\big(F(t)\big) \Big) \cdot \frac{1}{h} \\
&= \lim_{h \to 0} \big(H(t+h) - H(t)\big) \cdot \frac{1}{h}.
\end{aligned}$$

Thus $\varphi\big(F'(t)\big)$ exists if and only if H is differentiable at t. Therefore, F is differentiable at $t \in U$ if and only if H is differentiable at t. Moreover, if F is differentiable at $t \in U$, it follows from above equality that $F'(t) = \varphi^{-1}\big(H'(t)\big)$. □

Corollary 5.1.3. *Let $\psi_i : (M, +_{\psi_i}, \cdot_{\psi_i}) \longrightarrow (\mathbb{R}^2, +, \cdot)$ be an isomorphism for each $i \in \{1, \cdots, n\}$. Given the specific isomorphism, $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ defined by*

$$\begin{aligned}\varphi([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) &= (\psi_1 \times \cdots \times \psi_n)\,([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) \\ &= (\psi_1([a_1, a_2]), \cdots, \psi_n([a_{2n-1}, a_{2n}]))\end{aligned}$$

for all $([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) \in M^n$ and an open set and $U \subseteq \mathbb{R}$. Let $F : U \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$ be a function denoted by

$$F(t) = \big(F_1(t), \cdots, F_n(t)\big),$$

where $F_i : U \subseteq \mathbb{R} \longrightarrow (M, +_{\psi_i}, \cdot_{\psi_i}, \|\cdot\|_{\psi_i})$ with $F_i(t) = [f_{2i-1}(t), f_{2i}(t)]$ for all $i \in \{1, \cdots, n\}$. Then F is differentiable at $t \in U$ if and only if F_i is differentiable at t for all $i \in \{1, \cdots, n\}$. Moreover, if F is differentiable at t, then $F'(t) = (F_1'(t), \cdots, F_n'(t))$

Proof. From Theorem 5.1.6 F is differentiable at $t \in U$ if and only if the function $H = \varphi \circ F : U \subseteq \mathbb{R} \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ is differentiable at $t \in U$. However, by the definition of φ and by the definition of H we have that there is $H_i \equiv \psi_i \circ F_i : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ for all $i \in \{1, \cdots, n\}$, such that $H(t) = \big(H_1(t), \cdots, H_n(t)\big)$. Thus, H is differentiable at $t \in U$ if and only if H_i is differentiable at $t \in U$ for all $i \in \{1, \cdots, n\}$ and by Theorem 5.1.6, this is equivalent to F_i is differentiable at $t \in U$ for all $i \in \{1, \cdots, n\}$. Thus, F is differentiable at $t \in U$ if and only if F_i is differentiable at $t \in U$ for all $i \in \{1, \cdots, n\}$. Moreover, if F is differentiable at $t \in U$ from Theorem 5.1.6 and by definition of the φ we have that

$$\begin{aligned}F'(t) &= \varphi^{-1}(H'(t)) = \varphi^{-1}(H_1'(t), \cdots, H_n'(t)) \\ &= \big(\psi_1^{-1}(H_1'(t)), \cdots, \psi_n^{-1}(H_n'(t))\big) = (F_1'(t), \cdots, F_n'(t)),\end{aligned}$$

that is, $F'(t) = (F_1'(t), \cdots, F_n'(t))$. □

Theorem 5.1.7. *Given an open set $U \subseteq \mathbb{R}$, let $F, G : U \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$ be differentiable functions at $t \in U$. Then*

1. $(F + G)'(t) = F'(t) +_\varphi G'(t)$;
2. $(F - G)'(t) = F'(t) -_\varphi G'(t)$;
3. $(\lambda \cdot F)'(t) = \lambda \cdot_\varphi F'(t)$.

Proof. It is enough to prove item 1., because the proofs of the other items are similar. Since F and G are differentiable at t, there are functions $H_1, H_2 : U \longrightarrow \mathbb{R}^{2n}$ given by $H_1 = \varphi \circ F$ and $H_2 = \varphi \circ G$, such that $F'(t) = \varphi^{-1}\big(H_1'(t)\big)$ and $G'(t) = \varphi^{-1}\big(H_2'(t)\big)$. Given $H = H_1 + H_2 : U \longrightarrow \mathbb{R}^{2n}$, we have $H'(t) = H_1'(t) + H_2'(t)$ and $H = \varphi \circ (F + G)$. Thus,

$$\begin{aligned} (F+G)'(t) &= \varphi^{-1}\big(H'(t)\big) = \varphi^{-1}\big(H_1'(t) + H_2'(t)\big) \\ &= \varphi^{-1}\big(H_1'(t)\big) +_{\varphi} \varphi^{-1}\big(H_2'(t)\big) = F'(t) +_{\varphi} G'(t). \end{aligned}$$

Therefore, $(F+G)'(t) = F'(t) +_{\varphi} G'(t)$. $\square$

Theorem 5.1.8. *Given an open set $U \subseteq \mathbb{R}$, let function $F : U \longrightarrow (M^n, +_{\varphi}, \cdot_{\varphi}, \| \cdot \|_{\varphi})$ be differentiable at $t_0 \in U$. Then function F is continuous at t_0.*

Proof. Since F is differentiable at t_0, it follows that $H = \varphi \circ F : U \longrightarrow \mathbb{R}^{2n}$ is differentiable at t_0. In particular, H is continuous at t_0. On the other hand,

$$\begin{aligned} \lim_{t \to t_0} F(t) &= \varphi^{-1}\left(\lim_{t \to t_0} \varphi(F(t))\right) = \varphi^{-1}\left(\lim_{t \to t_0} \varphi((\varphi^{-1} \circ H)(t))\right) \\ &= \varphi^{-1}\left(\lim_{t \to t_0} (H(t))\right) = \varphi^{-1}(H(t_0)) = \Big(\varphi^{-1} \circ H\Big)(t_0) = F(t_0). \end{aligned}$$

Therefore, F is continuous at t_0. $\square$

Remark 5.1.4. If we consider a specific isomorphism $\varphi : (M^n, +_{\varphi}, \cdot_{\varphi}) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$, let $F : U :\longrightarrow (M^n, +_{\varphi}, \cdot_{\varphi}, \| \cdot \|_{\varphi})$, with $U \subseteq \mathbb{R}$ open, through of Theorem 5.1.6 we can characterize the derivative of F. For example if we consider the specific isomorphism $\varphi : (M^n, +_{\varphi}, \cdot_{\varphi}) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ defined by

$$\begin{aligned} \varphi([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) &= \big(\psi\big([a_1, a_2]\big), \cdots, \psi\big([a_{2n-1}, a_{2n}]\big) \\ &= ((a_1, a_2 - a_1), \cdots, (a_{2n-1}, a_{2n} - a_{2n-1})), \end{aligned}$$

given $U \subseteq \mathbb{R}$ open, let $F : U \longrightarrow (M^n, +_{\varphi}, \cdot_{\varphi}, \| \cdot \|_{\varphi})$ be such that

$$F(t) = \big([f_1(t), f_2(t)], \cdots, [f_{2n-1}(t), f_{2n}(t)]\big).$$

Then, F is differentiable at $t_0 \in U$ if and only if for each $i \in \{1, \cdots, 2n\}$, f_i is differentiable at t_0. Moreover, if F is differentiable at t_0 then,

$$F'(t_0) = \big([f_1'(t_0), f_2'(t_0)], \cdots, [f_{2n-1}'(t_0), f_{2n}'(t_0)]\big).$$

For all $t \in U$ we have

$$\begin{aligned} H(t) &= (\varphi \circ F)(t) = \varphi(F(t)) \\ &= \Big(\big(f_1(t), f_2(t) - f_1(t)\big), \cdots, \big(f_{2n-1}(t), f_{2n}(t) - f_{2n-1}(t)\big)\Big). \quad (*) \end{aligned}$$

From Definition 5.1.5 we have that F is differentiable at t_0 if and only if H is differentiable at t_0. However, H is differentiable at t_0 if and only if f_i and $f_{i+1} - f_i$ are differentiable at t_0 for all $i \in \{1, \cdots, 2n-1\}$. But, f_i and $f_{i+1} - f_i$ are differentiable at t_0 for all $i \in \{1, ..., 2n-1\}$ if and only if f_i is differentiable at t_0 for all $i \in \{1, \cdots, 2n\}$. Therefore, F is differentiable at t_0 if and only if for each $i \in \{1, \cdots, 2n\}$, f_i is differentiable at t_0. From $(*)$ we see that $H'(t_0) = \Big(\big(f_1'(t_0), f_2'(t_0) - f_1'(t_0)\big), \cdots, \big(f_{2n-1}'(t_0), f_{2n}'(t_0) - f_{2n-1}'(t_0)\big)\Big)$. Then,

$$\begin{aligned} F'(t_0) &= \varphi^{-1}(H'(t_0)) \\ &= \varphi^{-1}\Big(\big(f_1'(t_0), f_2'(t_0) - f_1'(t_0)\big), \cdots, \big(f_{2n-1}'(t_0), f_{2n}'(t_0) - f_{2n-1}'(t_0)\big)\Big) \\ &= \Big(\big[f_1'(t_0), f_2'(t_0) - f_1'(t_0) + f_1'(t_0)\big], \cdots, \big[f_{2n-1}'(t_0), f_{2n}'(t_0) - f_{2n-1}'(t_0) + f_{2n-1}'(t_0)\big]\Big) \\ &= \Big(\big[f_1'(t), f_2'(t_0)\big], \big[f_3'(t_0), f_4'(t_0)\big], \cdots, \big[f_{2n-1}'(t_0), f_{2n}'(t_0)\big]\Big). \end{aligned}$$

Therefore, $F'(t_0) = \Big(\big[f_1'(t), f_2'(t_0)\big], \cdots, \big[f_{2n-1}'(t_0), f_{2n}'(t_0)\big]\Big)$.

Example 5.1.1. Given an open set $U \subseteq \mathbb{R}$, let $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ be the specific isomorphism defined by

$$\begin{aligned} \varphi([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) &= (\psi\big([a_1, a_2]\big), \cdots, \psi\big([a_{2n-1}, a_{2n}]\big) \\ &= ((a_1, a_2 - a_1), \cdots, (a_{2n-1}, a_{2n} - a_{2n-1})). \end{aligned}$$

We consider the function $F : U \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \| \cdot \|_\varphi)$ given by $F(t) = g(t) \cdot_\varphi C$, where $C = \Big([c_1, c_2], \cdots, [c_{2n-1}, c_{2n}]\Big) \in M^n$ and $g : U \longrightarrow \mathbb{R}$ is a real function. Then, F is differentiable at $t_0 \in U$ if and only if g is differentiable at t_0. Moreover, If g is differentiable at t_0, then

$$F'(t_0) = \Big(\big[c_1 \cdot g'(t_0), c_2 \cdot g'(t_0)\big], \cdots, \big[c_{2n-1} \cdot g'(t_0), c_{2n} \cdot g'(t_0)\big]\Big) = g'(t_0) \cdot_\varphi C.$$

Indeed, we have that

$$\begin{aligned} F(t) &= g(t) \cdot_\varphi C \\ &= \Big(\big[g(t) \cdot c_1, g(t) \cdot c_2\big], \cdots, \big[g(t) \cdot c_{2n-1}, g(t) \cdot c_{2n}\big]\Big) \\ &= \Big(\big[c_1 \cdot g(t), c_2 \cdot g(t)\big], \cdots, \big[c_{2n-1} \cdot g(t), c_{2n} \cdot g(t)\big]\Big) \\ &= \Big(\big[f_1(t), f_2(t)\big], \cdots, \big[f_{2n-1}(t), f_{2n}(t)\big]\Big), \end{aligned}$$

where $f_i \equiv c_i g$, $i \in \{1, \cdots, 2n\}$. From Remark 5.1.4, F is differentiable at t_0 if and only if for each $i \in \{1, \cdots, 2n\}$, f_i is differentiable at t_0. But, for each $i \in \{1, \cdots, 2n\}$, f_i is differentiable at t_0 if and only if g is differentiable at t_0. Thus, F is differentiable at t_0 if and only if g is differentiable at t_0. Moreover, if g is differentiable at t_0, from Theorem 5.1.4 we have

$$F'(t_0) \quad = \quad \Big(\big[c_1 \cdot g'(t_0), c_2 \cdot g'(t_0)\big], \cdots, \big[c_{2n-1} \cdot g'(t_0), c_{2n} \cdot g'(t_0)\big]\Big) = g'(t_0) \cdot_\varphi C.$$

5.2 Generalized Interval Matrix

In the next chapter we will present the concept and some results about differential equation. To this end, we will present in this section a concept of generalized interval matrix and the relation between the set of all generalized interval matrix with the generalized interval set.

Set

$$\mathfrak{M} = \left\{ \begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} : [a_{2i-1}, a_{2i}] \in M \textit{ for all } i = 1, 2, \cdots, n. \right\},$$

The set $\mathfrak{M}$ ***is called generalized interval matrix set***. Given $\begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix}$ and $\begin{pmatrix} [b_1, b_2] \\ \vdots \\ [b_{2n-1}, b_{2n}] \end{pmatrix}$ in $\mathfrak{M}$, we say that $\begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} = \begin{pmatrix} [b_1, b_2] \\ \vdots \\ [b_{2n-1}, b_{2n}] \end{pmatrix} \in \mathfrak{M}$ if and only if $[a_{2i-1}, a_{2i}] = [b_{2i-1}, b_{2i}]$ for all $i \in \{1, \cdots, n\}$.

Let $\xi : \mathfrak{M} \longrightarrow (M^n, +_\varphi, \cdot_\varphi)$ be the function defined by

$$\xi \begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} = \big([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\big).$$

We have that ξ is well-defined, because given

$$\begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix}, \begin{pmatrix} [b_1, b_2] \\ \vdots \\ [b_{2n-1}, b_{2n}] \end{pmatrix} \in \mathfrak{M}, \text{ if } \xi \begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} \neq \xi \begin{pmatrix} [b_1, b_2] \\ \vdots \\ [b_{2n-1}, b_{2n}] \end{pmatrix},$$

it implies that $\big([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]\big) \neq \big([b_1, b_2], \cdots, [b_{2n-1}, b_{2n}]\big)$. Thus, there is

$i \in \{1, \cdots, n\}$ such that $[a_{2i-1}, a_{2i}] \neq [b_{2i-1}, b_{2i}]$, consequently,

$$\begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} \neq \begin{pmatrix} [b_1, b_2] \\ \vdots \\ [b_{2n-1}, b_{2n}] \end{pmatrix}.$$

It is easy to see that ξ is a bijection. Thus, we can equip $\mathfrak{M}$ with a vector structure, defining the algebraic operation " $+_\xi$ " and " $\cdot_\xi$ " by

$$\begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} +_\xi \begin{pmatrix} [b_1, b_2] \\ \vdots \\ [b_{2n-1}, b_{2n}] \end{pmatrix} = \xi^{-1} \left(\xi \begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} +_\varphi \xi \begin{pmatrix} [b_1, b_2] \\ \vdots \\ [b_{2n-1}, b_{2n}] \end{pmatrix} \right)$$

and

$$\alpha \cdot_\xi \begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} = \xi^{-1} \left(\alpha \cdot_\alpha \xi \begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} \right)$$

for all $\begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix}, \begin{pmatrix} [b_1, b_2] \\ \vdots \\ [b_{2n-1}, b_{2n}] \end{pmatrix} \in \mathfrak{M}$ and for all $\alpha \in \mathbb{R}$.

Thus, we have that $\xi : (\mathfrak{M}, +_\xi, \cdot_\xi) \longrightarrow (M^n, +_\varphi, \cdot_\varphi)$ is an isomorphism and in the algebraic point of view the elements in $(M^n, +_\varphi, \cdot_\varphi)$ are the same as the elements in $(\mathfrak{M}, +_\xi, \cdot_\xi)$.

We denote by $([a_1, a_2] \cdots [a_{2n-1}, a_{2n}])$ the transposed matrix of $\begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix}$ *which is denoted by* $\begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix}^T$. *Thus,* $\begin{pmatrix} [a_1, a_2] \\ \vdots \\ [a_{2n-1}, a_{2n}] \end{pmatrix} = ([a_1, a_2] \cdots [a_{2n-1}, a_{2n}])^T$.

Final remarks

In this section we presented a new approach to deal with interval calculus. Since the classic calculus is well known and the vector space $(M^n, +_\varphi, \cdot_\varphi, \| \cdot \|_\varphi)$ is isomorphic to $(\mathbb{R}^{2n}, +, \cdot, \| \cdot \|_{\mathbb{R}^{2n}})$, we obtain all algebraic and metric properties existing in $(\mathbb{R}^{2n}, +, \cdot, \| \cdot \|_{\mathbb{R}^{2n}})$. With these concepts we will provide in the next chapter a way of obtaining proper derivative of a generalized interval-valued function and present a new version of a generalized interval problem with initial conditions, which is related to a classic differential inclusion.

Chapter 6

Interval Differential Equation in M^n

In this chapter we present the concept of generalized interval differential equation, in another words, interval differential equation in M^n endowed with a particular structure of vector space. We present a particular case of generalized interval differential Equation which we call linear generalized interval differential equation. That allows us to formulate an appropriate interval problem with initial value in which its proper solution is related to the attainable set of a classic differential inclusion under some conditions. Also, some results connecting a proper solution of a generalized interval differential equation to classic solution of systems of differential equations are provided.

6.1 Generalized Interval Differential Equation in M^n

Definition 6.1.1. Given an open set $U \subseteq \mathbb{R}$ and the functions $X : U \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$ and $F : U \times (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi) \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$. The equation

$$X'(t) = F(t, X(t)) \tag{6.1}$$

is called **interval differential equation in** M^n, where

$$X(t) = ([x_1(t), x_2(t)], [x_3(t), x_4(t)], ..., [x_{2n-1}(t), x_{2n}(t)])$$

and

$$F(t, X(t)) = \Big(\big[f_1(t, X(t)), f_2(t, X(t))\big], \cdots, \big[f_{2n-1}(t, X(t)), f_{2n}(t, X(t))\big] \Big).$$

Definition 6.1.2. *A function $\Psi : I \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$ is called a **solution of** (6.1) if and only if it satisfies:*

1. *$I \subseteq U$ is a (open or closed) interval in $\mathbb{R}$;*
2. *$\Psi'(t) = F(t, \Psi(t))$ for all $t \in I$. If t is a extreme point in I then $\Psi'(t)$ is a lateral derivative.*

Theorem 6.1.1. *Let $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ be an isomorphism. The function $\Psi : I \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \| \cdot \|_\varphi)$ is a solution of the generalized differential equation (6.1) if only if the function $\zeta \equiv \varphi \circ \Psi : I \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ is a solution of the classic systems of differential equations*

$$H'(t) = \hat{F}(t, H(t)), \tag{6.2}$$

where function $\hat{F} : U \times \mathbb{R}^{2n}$ is given by $\hat{F}(t, y) = \varphi\left(F\big(t, \varphi^{-1}(y)\big)\right)$

Proof. $\Psi : I \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \| \cdot \|_\varphi)$ is a solution of (6.1) if and only if $\Psi'(t) = F(t, \Psi(t))$. However, from Theorem 5.1.6, it is equivalent to existence of the function $\zeta \equiv \varphi \circ \Psi : I \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ such that $\varphi^{-1}\big(\zeta'(t)\big) = F\left(t, \varphi^{-1}\big(\zeta(t)\big)\right)$, that is

$$\zeta'(t) = \varphi\left(F\big(t, \varphi^{-1}(\zeta(t))\big)\right) = \hat{F}(t, \zeta(t)).$$

Therefore, $\Psi : I \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \| \cdot \|_\varphi)$ is a solution of (6.1) if and only if $\zeta : I \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ is a solution of (6.2). □

Remark 6.1.1. When resolving (6.1), the solution may not be a proper interval along the time evolution. In order to obtain a proper solution we need the following definition.

Definition 6.1.3. *Let $\Psi(t) = ([\Psi_1(t), \Psi_2(t)], [\Psi_3(t), \Psi_4(t)], \cdots, [\Psi_{2n-1}(t), \Psi_{2n}(t)])$ be a solution of (6.1). Then, the function $\widetilde{\Psi}(t) = (\widetilde{\Upsilon_1}(t), \widetilde{\Upsilon_2}(t), \cdots, \widetilde{\Upsilon_n}(t))$ such that for all $i \in \{1, 2, \cdots, n\}$,*

$$\widetilde{\Upsilon_i}(t) = \begin{cases} [\Psi_{2i-1}(t), \Psi_{2i}(t)], & \text{if } [\Psi_{2i-1}(t), \Psi_{2i}(t)] \in I(\mathbb{R}) \\ [\Psi_{2i}(t), \Psi_{2i-1}(t)], & \text{if } [\Psi_{2i-1}(t), \Psi_{2i}(t)] \in \overline{I(\mathbb{R})} \end{cases}$$

*is called a **proper solution** of (6.1).*

Remark 6.1.2. The definition of proper solution was inspired by the paper [27]. This definition allows us to link the next section to a particular classic linear differential inclusion.

6.2 Linear Interval differential Equation (LIDE) in M^n

Let $\psi_i : (M, +_{\psi_i}, \cdot_{\psi_i}) \longrightarrow (\mathbb{R}^2, +, \cdot)$ be an isomorphism for all $i \in \{1, \cdots, n\}$. In this section we will consider the specific isomorphism, $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ defined by

$$\begin{aligned} \varphi([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) &= (\psi_1 \times \cdots \times \psi_n)([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) \\ &= (\psi_1([a_1, a_2]), \cdots, \psi_n([a_{2n-1}, a_{2n}])) \end{aligned}$$

for all $([a_1, a_2], [a_3, a_4] \cdots, [a_{2n-1}, a_{2n}]) \in M^n$.

Definition 6.2.1. *We call*

$$X'(t) = A(t) \odot_\varphi X(t) +_\varphi F(t), \tag{6.3}$$

a ***LIDE in*** M^n*, where* $A : I \subseteq \mathbb{R} \longrightarrow \mathbb{M}_{n\times n}$*,* $X, F : I \subseteq \mathbb{R} \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \|\cdot\|_\varphi)$*,* I *is a (open or closed) interval in* $\mathbb{R}$*,and the operation* $\odot_\varphi : \mathbb{M}_{(n\times n)} \times M^n \longrightarrow M^n$ *is defined by*

$$\begin{aligned}
&\odot_\varphi \left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} ; ([m_1, m_2], [m_3, m_4], \cdots, [m_{2n-1}, m_{2n}]) \right) \\
&= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \odot_\varphi ([m_1, m_2], [m_3, m_4], \cdots, [m_{2n-1}, m_{2n}]) \\
&= \left(\sum_{j=1}^{n} a_{1j} \cdot_{\psi_j} [m_{2j-1}, m_{2j}], \sum_{j=1}^{n} a_{2j} \cdot_{\psi_j} [m_{2j-1}, m_{2j}], \cdots, \sum_{j=1}^{n} a_{nj} \cdot_{\psi_j} [m_{2j-1}, m_{2j}] \right).
\end{aligned}$$

Let $T > 0$, $A : [0, T] \longrightarrow \mathbb{M}_{(n\times n)}$, and $X, F : [0, T] \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \odot_\varphi, \|\cdot\|_\varphi)$, where A and F are continuous functions. Given the interval problem with initial value

$$\begin{cases} X'(t) = A(t) \odot_\varphi X(t) +_\varphi F(t), \\ X(0) = X_0, \end{cases} \tag{6.4}$$

we denote

$$\begin{aligned}
X(t) &= (X_1(t), X_2(t), \cdots, X_n(t)) \\
&= ([x_1(t), x_2(t)], [x_2(t), x_3(t)], \cdots, [x_{2n-1}(t), x_{2n}(t)])\,;
\end{aligned}$$

$$\begin{aligned}
F(t) &= (F_1(t), F_2(t), \cdots, F_n(t)) \\
&= ([f_1(t), f_2(t)], [f_3(t), f_4(t)], \cdots, [f_{2n-1}(t), f_{2n}(t)])\,;
\end{aligned}$$

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix},$$

where $a_{ij} : [0, T] \longrightarrow \mathbb{R}$ for each $i, j \in \{1, \cdots, n\}$.

From Corollary 5.1.3 we have that $X'(t) = A(t) \odot_\varphi X(t) +_\varphi F(t)$ can be written as

$$\begin{aligned}
X'(t) &= (X_1'(t), \cdots, X_n'(t)) \qquad (6.5) \\
&= A(t) \odot_\varphi (X_1(t), \cdots, X_n(t)) +_\varphi (F_1(t), \cdots, F_n(t))\,. \qquad (6.6)
\end{aligned}$$

On the other hand, if we apply φ and the definition of the algebraic operation $\odot_\varphi$ we can rewrite (6.5) as

$$\begin{aligned}\varphi(X'(t)) &= (\psi_1(X_1'(t)), \cdots, \psi_n(X_n'(t)))\\ &= \left(\sum_{i=1}^{n} a_{1i}(t)\psi_i(X_i(t)), \cdots, \sum_{i=1}^{n} a_{ni}(t)\psi_i(X_i(t))\right) +_\varphi (\psi_1(F_1(t)), \cdots, \psi_n(F_n(t))).\end{aligned}$$

But this is equivalent to classic systems of differential equations:

$$\begin{pmatrix} \psi\left(X_1'(t)\right) \\ \psi\left(X_2'(t)\right) \\ \vdots \\ \psi\left(X_n'(t)\right) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} a_{1i}(t)\psi_i(X_i(t)) + \psi_1(F_1(t)) \\ \sum_{i=1}^{n} a_{2i}(t)\psi_i(X_i(t)) + \psi_2(F_2(t)) \\ \vdots \\ \sum_{i=1}^{n} a_{ni}(t)\psi_i(X_i(t)) + \psi_n(F_n(t)) \end{pmatrix}. \tag{6.7}$$

Let $V_i, W_i : [0,T] \longrightarrow \mathbb{R}^2$ be the functions given by $V_i(t) = (\psi_i \circ X_i)(t)$ and $W_i(t) = (\psi_i \circ F_i)(t)$, respectively, for all $i \in \{1, \cdots, n\}$. Then $V_i'(t) = \psi_i(X_i'(t))$ for all $i \in \{1, \cdots, n\}$. Thus the System (6.7) can be written as following classic system of differential equations

$$\begin{pmatrix} V_1'(t) \\ V_2'(t) \\ \vdots \\ V_n'(t) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} a_{1i}(t)V_i(t) + W_1(t) \\ \sum_{i=1}^{n} a_{2i}(t)V_i(t) + W_2(t) \\ \vdots \\ \sum_{i=1}^{n} a_{ni}(t)V_i(t) + W_n(t) \end{pmatrix}, \tag{6.8}$$

which is equivalent to:

$$\begin{pmatrix} V_1'(t) \\ V_2'(t) \\ \vdots \\ V_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \cdot \begin{pmatrix} V_1(t) \\ V_2(t) \\ \vdots \\ V_n(t) \end{pmatrix} + \begin{pmatrix} W_1(t) \\ W_2(t) \\ \vdots \\ W_n(t) \end{pmatrix}. \tag{6.9}$$

Setting the functions $V, W : [0,T] \longrightarrow \mathbb{R}^{2n}$ by $V(t) = (V_1(t), \cdots, V_n(t))$ and $W(t) = (W_1(t), \cdots, W_n(t))$ we can rewrite (6.9) as

$$V'(t) = A(t) \cdot V(t) + W(t). \tag{6.10}$$

Thus we have proved the following theorem:

Theorem 6.2.1. *Let $T > 0$, $A : [0,T] \longrightarrow \mathbb{M}_{(n\times n)}$ be a matrix function and $X, F : [0,T] \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \odot_\varphi, \|\cdot\|_\varphi)$, where A and F are continuous functions. Then $\Psi : I \subseteq [0,1] \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \odot_\varphi, \|\cdot\|_\varphi)$ given by $\Psi(t) = (\Psi_1(t), \cdots, \Psi_n(t))$ is a solution of the interval problem with initial value*

$$\begin{cases} X'(t) = A(t) \odot_\varphi X(t) +_\varphi F(t), \\ X(0) = X_0, \end{cases}$$

if and only if $\big(\psi_1(\Psi_1(t)), \cdots, \psi_n(\Psi_n(t))\big)$ is a solution of classic problem with initial value

$$\begin{cases} V'(t) = A(t) \cdot V(t) + W(t), \\ V(0) = V_0. \end{cases}$$

□

Example 6.2.1. Consider the specific isomorphism, $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ defined by

$$\begin{aligned} \varphi([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) &= (\psi_1 \times \cdots \times \psi_n)([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) \\ &= (\psi_1([a_1, a_2]), \cdots, \psi_n([a_{2n-1}, a_{2n}])), \end{aligned}$$

where $\psi_i \equiv \psi : (M, +_\psi, \cdot_\psi) \longrightarrow (\mathbb{R}^2, +, \cdot)$ is given by $\psi(a_1, a_2) = (a_1, a_2 - a_1)$ for all $i \in \{1, \cdots, n\}$.

Let $T > 0$, $A : [0,T] \longrightarrow \mathbb{M}_{(n\times n)}$ be a matrix function and $X, F : [0,T] \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \odot_\varphi, \|\cdot\|_\varphi)$, where A and F are continuous functions. We will consider the interval problem with initial value

$$\begin{cases} X'(t) = A(t) \odot_\varphi X(t) + F(t), \\ X(0) = X_0. \end{cases}$$

From Theorem 6.2.1 we have that to solve this problem is equivalent to solve

$$\begin{cases} V'(t) = A(t) \cdot V(t) + W(t), \\ V(0) = V_0. \end{cases}$$

However, by definitions of V and W and by definition of φ, this problem is equivalent to:

$$\begin{pmatrix} (x_1'(t), \delta_1'(t)) \\ (x_3'(t), \delta_3'(t)) \\ \vdots \\ (x_{2n-1}'(t), \delta_{2n-1}'(t)) \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{i=1}^n a_{1i}(t)\,(x_{2i-1}(t), \delta_{2i-1}(t)) \\ \displaystyle\sum_{i=1}^n a_{2i}(t)\,(x_{2i-1}(t), \delta_{2i-1}(t)) \\ \vdots \\ \displaystyle\sum_{i=1}^n a_{ni}(t)\,(x_{2i-1}(t), \delta_{2i-1}(t)) \end{pmatrix} + \begin{pmatrix} (f_1(t), \gamma_2(t)) \\ (f_3(t), \gamma_4(t)) \\ \vdots \\ (f_{2n-1}(t), \gamma_{2n}(t)) \end{pmatrix},$$

where $\delta_{2i-1}(t) = x_{2i}(t) - x_{2i-1}(t)$ and $\gamma_{2i-1}(t) = f_{2i}(t) - f_{2i-1}(t)$ for all $i \in \{1, 2, \cdots, n\}$. Thus, this system can be written as:

$$\begin{pmatrix} x_1'(t) \\ x_3'(t) \\ \vdots \\ x_{2n-1}'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_3(t) \\ \vdots \\ x_{2n-1}(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_3(t) \\ \vdots \\ f_{2n-1}(t) \end{pmatrix}$$

$$\begin{pmatrix} \delta_1'(t) \\ \delta_3'(t) \\ \vdots \\ \delta_{2n-1}'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \cdot \begin{pmatrix} \delta_1(t) \\ \delta_3(t) \\ \vdots \\ \delta_{2n-1}(t) \end{pmatrix} + \begin{pmatrix} \gamma_1(t) \\ \gamma_3(t) \\ \vdots \\ \gamma_{2n-1}(t) \end{pmatrix}.$$

Therefore (6.4) is equivalent to following system

$$\begin{pmatrix} x_1'(t) \\ x_3'(t) \\ \vdots \\ x_{2n-1}'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_3(t) \\ \vdots \\ x_{2n-1}(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_3(t) \\ \vdots \\ f_{2n-1}(t) \end{pmatrix}$$

$$\begin{pmatrix} \delta_1'(t) \\ \delta_3'(t) \\ \vdots \\ \delta_{2n-1}'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \cdot \begin{pmatrix} \delta_1(t) \\ \delta_3(t) \\ \vdots \\ \delta_{2n-1}(t) \end{pmatrix} + \begin{pmatrix} \gamma_1(t) \\ \gamma_3(t) \\ \vdots \\ \gamma_{2n-1}(t) \end{pmatrix},$$

with initial condition $\underline{x(0)} = (x_1(0) \;\; x_3(0) \;\; \cdots \;\; x_{2n-1}(0))^T$ and $\delta(0) = (\delta_1(0) \;\; \delta_3(0) \;\; \cdots \;\; \delta_{2n-1}(0))^T$, whose solution is

$$\underline{x(t)} = \Phi(t,0)\underline{x(0)} + \int_0^t \Phi(t,s)\underline{f(s)}ds \tag{6.11}$$

$$\delta(t) = \Phi(t,0)\delta(0) + \int_0^t \Phi(t,s)\gamma(s)ds, \tag{6.12}$$

where $\underline{x(t)} = (x_1(t) \;\; x_3(t) \;\; \cdots x_{2n-1}(t))^T$, $\quad \delta(t) = (\delta_1(t) \;\; \delta_3(t) \;\; \cdots \;\; \delta_{2n-1}(t))^T$, $\underline{f(t)} = (f_1(t) \;\; f_3(t) \;\; \cdots \;\; f_{2n-1}(t))^T$ and $\Phi(t,s) = \exp \displaystyle\int_s^t A(\tau)d\tau$. Thus, we have

$$\begin{aligned} X(t) &= (X_1(t), ..., X_n(t)) \\ &= ([x_1(t), x_1(t) + \delta_1(t)], \cdots, [x_{2n-1}(t), x_{2n-1}(t) + \delta_{2n-1}(t)]) \\ &= ([x_1(t), x_1(t) + (x_2(t) - x_1(t))], \cdots, [x_{2n-1}(t), x_{2n-1}(t) + (x_{2n}(t) - x_{2n-1}(t))]) \\ &= ([x_1(t), x_2(t)], \cdots, [x_{2n-1}(t), x_{2n}(t)]) \end{aligned}$$

is a solution of (6.4), and $\widetilde{X}(t) = (\widetilde{X_1}(t), \widetilde{X_2}(t), ..., \widetilde{X_n}(t))$ is given by

$$\widetilde{X_i}(t) = \begin{cases} [x_{2i-1}(t), x_{2i}(t)], & \text{if } [x_{2i-1}(t), x_{2i}(t)] \in I(\mathbb{R}) \\ [x_{2i}(t), x_{2i-1}(t)], & \text{if } [x_{2i-1}(t), x_{2i}(t)] \in \overline{I(\mathbb{R})} \end{cases}$$
$$= \begin{cases} [x_{2i-1}(t), x_{2i-1}(t) + \delta_{2i-1}(t)], & \text{if } \delta_{2i-1}(t) \geq 0 \\ [x_{2i-1}(t) + \delta_{2i-1}(t), x_{2i-1}(t)], & \text{if } \delta_{2i-1}(t) < 0, \end{cases}$$

with $i \in \{1, \cdots, n\}$, is the proper solution of (6.4).

6.3 Relations Between LDI and LIDE in M^n with coefficient matrix

This section we will state and prove a theorem provideing a relation between a linear differential inclusion (LDI) and a LIDE in M^n. It is known that, if we consider the set of all convex subsets in $\mathbb{R}^n$ and the set of all compacts subsets in $\mathbb{R}^n$ denoted, respectively, by conv$(\mathbb{R}^n)$ and by comp$(\mathbb{R}^n)$, given $T > 0$, $t \in [0, T]$, $y \in \mathbb{R}^n$, $A : [0, T] \longrightarrow \mathbb{M}_{(n \times n)}$, and $F : [0, T] \longrightarrow \text{comp}(\mathbb{R}^n)$, with A and F continuous, then the linear differential inclusion

$$\dot{y} \in A(t)y + F(t), \quad y(0) \in X_0 \tag{6.13}$$

with initial condition $R(0) = X_0$, has the set of attainable states $R(t)$ given by

$$R(t) = \Phi(t, 0)X_0 + \int_0^t \Phi(t, s)F(s)ds, \tag{6.14}$$

where

$$\Phi(t, s) = \exp \int_s^t A(\tau),$$

and the integral of multifunction is the Aumann integral [2].

Our next result is the theorem which justifies the name of this section.Hewever, to present this result and its prove we recall the **definition of support function**: let E be a non-empty subset in $\mathbb{R}^n$. The support function of E is the function $C(E, \cdot)$ with value in $\mathbb{R} \cup \{+\infty\}$ given by

$$C(E, \alpha) = \sup\{\langle x, \alpha \rangle : x \in E\},$$

where $\langle \cdot, \cdot \rangle$ is usual inner product in $\mathbb{R}^n$.

Theorem 6.3.1. *Given the linear differential inclusion*

$$\dot{y} \in A(t)y + F(t), \text{ with } y(0) \in X_0, \tag{6.15}$$

where $T > 0$, $y \in \mathbb{R}^n$, $A : [0,T] \longrightarrow \mathbb{M}_{(n\times n)}$ is continuous, $F : [0,T] \longrightarrow (I(\mathbb{R}))^n$ is continuous in the norm $\|\cdot\|_\varphi$ and denoted by

$$\begin{aligned} F(t) &= ([f_1(t), f_2(t)], \cdots, [f_{2n-1}(t), f_{2n}(t)]), \\ &= (F_1(t) \cdots, F_n(t)]), \end{aligned}$$

where $f_i : [0,T] \longrightarrow \mathbb{R}$ is a continuous function for all $i \in \{1, \cdots, 2n\}$ and $X : [0,T] \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \odot_\varphi, \|\cdot\|_\varphi)$ with $X_0 \in (I(\mathbb{R}))^n$. Then, to find the set of attainable states $R(t)$ of this problem is equivalent to find the proper solution, $\widetilde{X}(t)$, of (6.16)

$$\begin{cases} X'(t) = A(t) \odot_\varphi X(t) +_\varphi F(t), \\ X(0) = X_0 \end{cases} \tag{6.16}$$

whenever $C(R(t)\alpha) = C(\widetilde{X}(t), \alpha)$.

Proof. First of all, the problem (6.15) has set of attainable states. Indeed, since $f_1, \cdots, f_{2n} : [0,T] \longrightarrow \mathbb{R}$ are continuous functions, it follows that for all $t_0 \in [0,T]$ we have that $\lim\limits_{t\to t_0} f_i(t) = f_i(t_0)$ for all $i \in \{1, \cdots, 2n\}$. Then, by Aubin and Cellina in [1] we have that for all $i \in \{1, \cdots, n\}$ is valid

$$\lim_{t\to t_0} F_i(t) = \lim_{t\to t_0} [f_{2i-1}(t), f_{2i}(t)] = \left[\lim_{t\to t_0} f_{2i-1}(t), \lim_{t\to t_0} f_{2i}(t)\right] = [f_{2i-1}(t_0), f_{2i}(t_0)] = F_i(t_0),$$

where the limits of the sets F_i are given in the Pompeu-Hausdorff metric in $\mathbb{R}$ $\rho : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $\rho([c_1, c_2], [d_1, d_2]) = \max\{|c_1 - d_1|, |c_2 - d_2|\}$.

Given $D : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ the Pompeu-Hausdorff metric in $\mathbb{R}^n$ we have that

$$\begin{aligned} \lim_{t\to t_0} D(F(t), F(t_0)) &= \lim_{t\to t_0} \max\left\{\rho\big(F_i(t), F_i(t_0)\big) \quad i \in \{1, \cdots, n\}\right\} \\ &= \max\left\{\lim_{t\to t_0} \rho\big(F_i(t), F_i(t_0)\big) \quad i \in \{1, \cdots, n\}\right\} = 0 \end{aligned}$$

Thus, F viewed as a multifunction, is continuous in the Pompeu-Hausdorff metric.

Also, we have that the Problem (6.16) has a solution, because by hypothesis F is continuous in the norm $\|\cdot\|_\varphi$.

Since $R(t)$ and the proper solution $\widetilde{X}(t)$ of (6.4) are both convex and closed subsets of $\mathbb{R}^n$, we have that $R(t) = \widetilde{X}(t)$ whenever $C(R(t)\alpha) = C(\widetilde{X}(t), \alpha)$ for all $\alpha \in \mathbb{R}^n$. □

Example 6.3.1. Consider the specific isomorphism, $\varphi : (M^n, +_\varphi, \cdot_\varphi) \longrightarrow (\mathbb{R}^{2n}, +, \cdot)$ defined by

$$\begin{aligned} \varphi([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) &= (\psi_1 \times \cdots \times \psi_n)([a_1, a_2], \cdots, [a_{2n-1}, a_{2n}]) \\ &= (\psi_1([a_1, a_2]), \cdots, \psi_n([a_{2n-1}, a_{2n}])), \end{aligned}$$

where $\psi_i \equiv \psi : (M, +_\psi, \cdot_\psi) \longrightarrow (\mathbb{R}^2, +, \cdot)$ is given by $\psi(a_1, a_2) = (a_1, a_2 - a_1)$ for all $i \in \{1, \cdots, n\}$.

Consider the inclusion (6.15), where $T > 0$, $y \in \mathbb{R}^n$, $A : [0,T] \longrightarrow \mathbb{M}_{(n\times n)}$ is continuous, $F : [0,T] \longrightarrow (I(\mathbb{R}))^n$ is continuous in the norm $\|\cdot\|_\varphi$ denoted by $F(t) = ([f_1(t), f_2(t)], \cdots, [f_{2n-1}(t), f_{2n}(t)])$ and $X : [0,T] \longrightarrow (M^n, +_\varphi, \cdot_\varphi, \odot_\varphi, \|\cdot\|_\varphi)$ with $X_0 \in (I(\mathbb{R}))^n$ denoted by $X(t) = ([x_1(t), x_2(t)], \cdots, [x_{2n-1}(t), x_{2n}(t)])$.

Since F is continuous in the norm $\|\cdot\|_\varphi$, from (5.1.2) we have that $f_i : [0,T] \longrightarrow \mathbb{R}$ is a continuous function for all $i \in \{1, \cdots, 2n\}$ and by prove of Theorem 6.3.1 follows that F is continuous in the Pompeu-Hausdorff metric.

Then, to find the set of attainable states with condition initial X_0, this is equivalent to find the proper solution, $\widetilde{X}(t)$, of (6.16) whenever

$$\sum_{i=1}^{n}\left\{|\alpha_i|\cdot\left|\sum_{j=1}^{n}\lambda_{ij}(t,0)(x_{2j}(0)-x_{2j-1}(0)) + \int_0^t\left(\sum_{j=1}^{n}\lambda_{ij}(t,s)(f_{2j}(s)-f_{2j-1}(s))\right)ds\right|\right\} =$$

$$= \sum_{i=1}^{n}\left\{(x_{2j}(0)-x_{2j-1}(0))\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right| + \int_0^t\left((f_{2j}(s)-f_{2j-1}(s))\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right|\right)ds\right\},$$

where $\lambda_{ij}(t,s)$ are the entries of matrix $\Phi(t,s) = \exp\displaystyle\int_s^t A(\tau)d\tau$. In order to find this condition, we set $\delta_{2i-1}(t) = x_{2i}(t) - x_{2i-1}(t)$ and $\gamma_{2i-1}(t) = f_{2i}(t) - f_{2i-1}(t)$ for all $t \in [0,T]$ and for all $i \in \{1, 2, \cdots, n\}$.

Given $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n$ arbitrarily, the support function of $\widetilde{X}(t)$ is given by

$$C\left(\widetilde{X}(t),\alpha\right) = \sum_{i=1}^{n} C\left(\widetilde{X_i}(t),\alpha_i\right) = \sum_{i=1}^{n}\sup\left\{\langle x,\alpha_i\rangle : x \in \widetilde{X_i}(t)\right\}.$$

Since

$$\sup\left\{\langle x,\alpha_i\rangle : x\in\widetilde{X_i}(t)\right\}$$
$$= \begin{cases} \sup\left\{\langle x,\alpha_i\rangle : x \in \left[x_{2i-1}(t), x_{2i-1}(t)+\delta_{2i-1}(t)\right]\right\} & \textit{if } \delta_{2i-1}(t)\geq 0\\ \sup\left\{\langle x,\alpha_i\rangle : x \in \left[x_{2i-1}(t)+\delta_{2i-1}(t), x_{2i-1}(t)\right]\right\} & \textit{if } \delta_{2i-1}(t) < 0, \end{cases}$$

it follows that:

1. If $\delta_{2i-1}(t) \geq 0$ e $\alpha_i \geq 0$, then

$$\begin{aligned}\sup\left\{\langle x,\alpha_i\rangle : x\in\widetilde{X_i}(t)\right\} &= \sup\left\{\langle x,\alpha_i\rangle : x \in \left[x_{2i-1}(t), x_{2i-1}(t)+\delta_{2i-1}(t)\right]\right\}\\ &= (x_{2i-1}(t)+\delta_{2i-1}(t))\cdot\alpha_i\\ &= \left(x_{2i-1}(t)+\frac{\delta_{2i-1}(t)}{2}\right)\cdot\alpha_i + \left|\frac{\delta_{2i-1}(t)}{2}\right|\cdot|\alpha_i|.\end{aligned}$$

2. If $\delta_{2i-1}(t) < 0$ and $\alpha_i \geq 0$, thus

$$\begin{aligned}
\sup\left\{\langle x, \alpha_i\rangle : x \in \widetilde{X}_i(t)\right\} &= \sup\left\{ \langle x, \alpha_i\rangle : x \in \left[x_{2i-1}(t) + \delta_{2i-1}(t), x_{2i-1}(t)\right]\right\} \\
&= x_{2i-1}(t) \cdot \alpha_i \\
&= \left(x_{2i-1}(t) + \frac{\delta_{2i-1}(t)}{2}\right) \cdot \alpha_i + \frac{-\delta_{2i-1}(t)}{2} \cdot \alpha_i \\
&= \left(x_{2i-1}(t) + \frac{\delta_{2i-1}(t)}{2}\right) \cdot \alpha_i + \left|\frac{\delta_{2i-1}(t)}{2}\right| \cdot |\alpha_i|.
\end{aligned}$$

3. If $\delta_{2i-1}(t) < 0$ and $\alpha_i < 0$, thus

$$\begin{aligned}
\sup\left\{\langle x, \alpha_i\rangle : x \in \widetilde{X}_i(t)\right\} &= \sup\left\{ \langle x, \alpha_i\rangle : x \in \left[x_{2i-1}(t) + \delta_{2i-1}(t), x_{2i-1}(t)\right]\right\} \\
&= \left(x_{2i-1}(t) + \frac{\delta_{2i-1}(t)}{2}\right) \cdot \alpha_i + \frac{-\delta_{2i-1}(t)}{2} \cdot (-\alpha_i) \\
&= \left(x_{2i-1}(t) + \frac{\delta_{2i-1}(t)}{2}\right) \cdot \alpha_i + \left|\frac{\delta_{2i-1}(t)}{2}\right| \cdot |\alpha_i|.
\end{aligned}$$

4. If $\delta_{2i-1}(t) \geq 0$ and $\alpha_i < 0$, thus

$$\begin{aligned}
\sup\left\{\langle x, \alpha_i\rangle : x \in \widetilde{X}_i(t)\right\} &= \sup\left\{ \langle x, \alpha_i\rangle : x \in \left[x_{2i-1}(t), x_{2i-1}(t) + \delta_{2i-1}(t)\right]\right\} \\
&= \left(x_{2i-1}(t) + \frac{\delta_{2i-1}(t)}{2}\right) \cdot \alpha_i + \frac{\delta_{2i-1}(t)}{2} \cdot (-\alpha_i) \\
&= \left(x_{2i-1}(t) + \frac{\delta_{2i-1}(t)}{2}\right) \cdot \alpha_i + \left|\frac{\delta_{2i-1}(t)}{2}\right| \cdot |\alpha_i|.
\end{aligned}$$

Thus, from $1-4$ we have

$$\sum_{i=1}^{n} \sup\left\{\langle x, \alpha_i\rangle : x \in \widetilde{X}_i(t)\right\} = \sum_{i=1}^{n} \left(x_{2i-1}(t) + \frac{\delta_{2i-1}(t)}{2}\right) \cdot \alpha_i + \left|\frac{\delta_{2i-1}(t)}{2}\right| \cdot |\alpha_i|.$$

However, denoting the entries in $\Phi(t,s)$ as $\lambda_{ij}(t,s)$ with $i \in \{1, \cdots, n\}$ and $0 \leq s \leq t \leq T$, we can write $x_{2i-1}(t)$ and $\delta_{2i-1}(t)$ in terms of $x_{2i-1}(0)$ and $\delta_{2i-1}(0)$, respectively, because from (6.11) and from (6.12) we have

$$\begin{aligned}
\begin{pmatrix} x_1(t) \\ x_3(t) \\ \vdots \\ x_{2n-1}(t) \end{pmatrix} &= \begin{pmatrix} \lambda_{11}(t,0) & \lambda_{12}(t,0) & \cdots & \lambda_{1n}(t,0) \\ \lambda_{21}(t,0) & \lambda_{22}(t,0) & \cdots & \lambda_{2n}(t,0) \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{n1}(t,0) & \lambda_{n2}(t,0) & \cdots & \lambda_{nn}(t,0) \end{pmatrix} \cdot \begin{pmatrix} x_1(0) \\ x_3(0) \\ \vdots \\ x_{2n-1}(0) \end{pmatrix} + \\
&+ \int_0^t \begin{pmatrix} \lambda_{11}(t,s) & \lambda_{12}(t,s) & \cdots & \lambda_{1n}(t,s) \\ \lambda_{21}(t,s) & \lambda_{22}(t,s) & \cdots & \lambda_{2n}(t,s) \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{n1}(t,s) & \lambda_{n2}(t,s) & \cdots & \lambda_{nn}(t,s) \end{pmatrix} \cdot \begin{pmatrix} f_1(s) \\ f_3(s) \\ \vdots \\ f_{2n-1}(s) \end{pmatrix} ds
\end{aligned}$$

and

$$\begin{pmatrix} \delta_1(t) \\ \delta_3(t) \\ \vdots \\ \delta_{2n-1}(t) \end{pmatrix} = \begin{pmatrix} \lambda_{11}(t,0) & \lambda_{12}(t,0) & \cdots & \lambda_{1n}(t,0) \\ \lambda_{21}(t,0) & \lambda_{22}(t,0) & \cdots & \lambda_{2n}(t,0) \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{n1}(t,0) & \lambda_{n2}(t,0) & \cdots & \lambda_{nn}(t,0) \end{pmatrix} \cdot \begin{pmatrix} \delta_1(0) \\ \delta_3(0) \\ \vdots \\ \delta_{2n-1}(0) \end{pmatrix} +$$

$$+ \int_0^t \begin{pmatrix} \lambda_{11}(t,s) & \lambda_{12}(t,s) & \cdots & \lambda_{1n}(t,s) \\ \lambda_{21}(t,s) & \lambda_{22}(t,s) & \cdots & \lambda_{2n}(t,s) \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{n1}(t,s) & \lambda_{n2}(t,s) & \cdots & \lambda_{nn}(t,s) \end{pmatrix} \cdot \begin{pmatrix} \gamma_1(s) \\ \gamma_3(s) \\ \vdots \\ \gamma_{2n-1}(s) \end{pmatrix} ds.$$

Thus, for each $i \in \{1, 2, \cdots, n\}$, we have

$$x_{2i-1}(t) = \sum_{j=1}^{n} \lambda_{ij}(t,0) \cdot x_{2j-1}(0) + \int_0^t \sum_{j=1}^{n} \lambda_{ij}(t,s) f_{2j-1}(s) ds$$

and

$$\delta_{2i-1}(t) = \sum_{j=1}^{n} \lambda_{ij}(t,0) \cdot \delta_{2j-1}(0) + \int_0^t \sum_{j=1}^{n} \lambda_{ij}(t,s) \gamma_{2j-1}(s) ds.$$

This means that

$$\begin{aligned}
&\sum_{i=1}^{n} \sup \left\{ \langle x, \alpha_i \rangle : x \in \widetilde{X_i}(t) \right\} \\
&= \sum_{i=1}^{n} \left(x_{2i-1}(t) + \frac{\delta_{2i-1}(t)}{2} \right) \cdot \alpha_i + \left| \frac{\delta_{2i-1}(t)}{2} \right| \cdot |\alpha_i| \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \lambda_{ij}(t,0) \cdot x_{2j-1}(0) + \int_0^t \sum_{j=1}^{n} \lambda_{ij}(t,s) f_{2j-1}(s) ds \right) \alpha_i \\
&+ \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \lambda_{ij}(t,0) \cdot \frac{\delta_{2j-1}(0)}{2} + \int_0^t \sum_{j=1}^{n} \lambda_{ij}(t,s) \frac{\gamma_{2j-1}(s)}{2} ds \right) \alpha_i \\
&+ \sum_{i=1}^{n} \frac{|\alpha_i|}{2} \cdot \left| \sum_{j=1}^{n} \lambda_{ij}(t,0) \cdot \delta_{2j-1}(0) + \int_0^t \sum_{j=1}^{n} \lambda_{ij}(t,s) \gamma_{2j-1}(s) ds \right| \\
&= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} \lambda_{ij}(t,0) \left(x_{2j-1}(0) + \frac{\delta_{2j-1}(0)}{2} \right) \alpha_i + \int_0^t \sum_{j=1}^{n} \lambda_{ij}(t,s) \left(f_{2j-1}(s) + \frac{\gamma_{2j-1}(s)}{2} \right) \alpha_i ds \right\} \\
&+ \frac{1}{2} \cdot \left(\sum_{i=1}^{n} |\alpha_i| \cdot \left| \sum_{j=1}^{n} \lambda_{ij}(t,0) \delta_{2j-1}(0) + \int_0^t \left(\sum_{j=1}^{n} \lambda_{ij}(t,s) \gamma_{2j-1}(s) \right) ds \right| \right).
\end{aligned}$$

Thus,

$$C(\widetilde{X}(t),\alpha) =$$
$$= \sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\lambda_{ij}(t,0)\left(x_{2j-1}(0)+\frac{\delta_{2j-1}(0)}{2}\right)\alpha_i + \int_0^t\sum_{j=1}^{n}\lambda_{ij}(t,s)\left(f_{2j-1}(s)+\frac{\gamma_{2j-1}(s)}{2}\right)\alpha_i ds\right\}$$
$$+\frac{1}{2}\cdot\left(\sum_{i=1}^{n}|\alpha_i|\cdot\left|\sum_{j=1}^{n}\lambda_{ij}(t,0)\delta_{2j-1}(0)+\int_0^t\left(\sum_{j=1}^{n}\lambda_{ij}(t,s)\gamma_{2j-1}(s)\right)ds\right|\right).$$

On the other hand, consider

$$R(t) = \Phi(t,0)X_0 + \int_0^t \Phi(t,s)F(s)ds,$$

with $X_0 = (X_1^0, X_2^0, \cdots, X_n^0)$, where for each $i \in \{1,2,\cdots,n\}$ we have

$$X_i^0 = [x_{2i-1}(0), x_{2i-1}(0)+\delta_{2i-1}(0)], \text{ with } \delta_{2i-1}(0) = x_{2i}(0) - x_{2i-1}(0) \geq 0$$

and $F(s) = (F_1(s), F_2(s), \cdots, F_n(s))$, where for each $i \in \{1,2,\cdots,n\}$

$$F_i(s) = [f_{2i-1}(s), f_{2i-1}(s)+\gamma_{2i-1}(s)], \text{ with } \gamma_{2i-1}(s) = f_{2i}(s) - f_{2i-1}(s) \geq 0$$

for all $s \in [0,T]$.

The support function of $R(t)$ is given by

$$\begin{aligned} C(R(t),\alpha) &= C\left(\Phi(t,0)X_0 + \int_0^t \Phi(t,s)F(s)ds, \alpha\right) \\ &= C\left(\Phi(t,0)X_0,\alpha\right) + C\left(\int_0^t \Phi(t,s)F(s)ds,\alpha\right) \\ &= C\left(\Phi(t,0)X_0,\alpha\right) + \int_0^t C\left(\Phi(t,s)F(s),\alpha\right)ds \\ &= C\left(X_0, \Phi(t,0)^T\alpha\right) + \int_0^t C\left(F(s),\Phi(t,s)^T\alpha\right)ds. \end{aligned}$$

However, we have

$$\Phi(t,0)^T\alpha = \begin{pmatrix} \lambda_{11}(t,0) & \lambda_{21}(t,0) & \cdots & \lambda_{n1}(t,0) \\ \lambda_{12}(t,0) & \lambda_{22}(t,0) & \cdots & \lambda_{n2}(t,0) \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{1n}(t,0) & \lambda_{2n}(t,0) & \cdots & \lambda_{nn}(t,0) \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n}\lambda_{j1}(t,0)\alpha_j \\ \sum_{j=1}^{n}\lambda_{j2}(t,0)\alpha_j \\ \vdots \\ \sum_{j=1}^{n}\lambda_{jn}(t,0)\alpha_j \end{pmatrix}$$

and

$$\Phi(t,s)^T\alpha = \begin{pmatrix} \lambda_{11}(t,s) & \lambda_{21}(t,s) & \cdots & \lambda_{n1}(t,s) \\ \lambda_{12}(t,s) & \lambda_{22}(t,s) & \cdots & \lambda_{n2}(t,s) \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{1n}(t,s) & \lambda_{2n}(t,s) & \cdots & \lambda_{nn}(t,s) \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} \lambda_{j1}(t,s)\alpha_j \\ \sum_{j=1}^{n} \lambda_{j2}(t,s)\alpha_j \\ \vdots \\ \sum_{j=1}^{n} \lambda_{jn}(t,s)\alpha_j \end{pmatrix}.$$

Using the same arguments that we used to find $C(\widetilde{X}(t),\alpha)$, we have

$$\begin{aligned} &C\left(X_0, \Phi(t,0)^T\alpha\right) \\ &= \sum_{i=1}^{n}\left\{\left(x_{2i-1}(0)+\frac{\delta_{2i-1}(0)}{2}\right)\left(\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right)+\left|\frac{\delta_{2i-1}(0)}{2}\right|\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right|\right\} \end{aligned}$$

and

$$\begin{aligned} &C\left(F(s), \Phi(t,s)^T\alpha\right) \\ &= \sum_{i=1}^{n}\left\{\left(f_{2i-1}(s)+\frac{\gamma_{2i-1}(s)}{2}\right)\left(\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right)+\left|\frac{\gamma_{2i-1}(s)}{2}\right|\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right|\right\}. \end{aligned}$$

Since $\delta_{2i-1}(0) \geq 0$ and $\gamma_{2i-1}(s) \geq 0$, it follows that

$$\begin{aligned} &C(R(t),\alpha) = C\left(X_0, \Phi(t,0)^T\alpha\right) + \int_0^t C\left(F(s), \Phi(t,s)^T\alpha\right) \\ &= \sum_{i=1}^{n}\left\{\left(x_{2i-1}(0)+\frac{\delta_{2i-1}(0)}{2}\right)\left(\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right)+\frac{\delta_{2i-1}(0)}{2}\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right|\right\} \\ &+ \int_0^t \sum_{i=1}^{n}\left\{\left(f_{2i-1}(s)+\frac{\gamma_{2i-1}(s)}{2}\right)\left(\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right)+\frac{\gamma_{2i-1}(s)}{2}\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right|\right\}ds \\ &= \sum_{i=1}^{n}\left\{\left(x_{2i-1}(0)+\frac{\delta_{2i-1}(0)}{2}\right)\left(\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right)+\frac{\delta_{2i-1}(0)}{2}\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right|\right\} \\ &+ \sum_{i=1}^{n}\int_0^t \left\{\left(f_{2i-1}(s)+\frac{\gamma_{2i-1}(s)}{2}\right)\left(\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right)+\frac{\gamma_{2i-1}(s)}{2}\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right|\right\}ds \\ &= \sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\lambda_{ji}(t,0)\left(x_{2i-1}(0)+\frac{\delta_{2i-1}(0)}{2}\right)\alpha_j+\int_0^t\sum_{j=1}^{n}\lambda_{ji}(t,s)\left(f_{2i-1}(s)+\frac{\gamma_{2i-1}(s)}{2}\right)\alpha_j ds\right\} \\ &+ \sum_{i=1}^{n}\left\{\frac{\delta_{2i-1}(0)}{2}\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right|+\int_0^t\frac{\gamma_{2i-1}(s)}{2}\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right|ds\right\}. \end{aligned}$$

That is,

$$C(R(t),\alpha)$$
$$= \sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\lambda_{ji}(t,0)\left(x_{2i-1}(0)+\frac{\delta_{2i-1}(0)}{2}\right)\alpha_j + \int_0^t \sum_{j=1}^{n}\lambda_{ji}(t,s)\left(f_{2i-1}(s)+\frac{\gamma_{2i-1}(s)}{2}\right)\alpha_j ds\right\}$$
$$+\frac{1}{2}\cdot\sum_{i=1}^{n}\left\{\delta_{2i-1}(0)\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right| + \int_0^t \gamma_{2i-1}(s)\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right| ds\right\}.$$

Since

$$\sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\lambda_{ij}(t,0)\left(x_{2j-1}(0)+\frac{\delta_{2j-1}(0)}{2}\right)\alpha_i + \int_0^t \sum_{j=1}^{n}\lambda_{ij}(t,s)\left(f_{2j-1}(s)+\frac{\gamma_{2j-1}(s)}{2}\right)\alpha_i ds\right\}$$
$$= \sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\lambda_{ji}(t,0)\left(x_{2i-1}(0)+\frac{\delta_{2i-1}(0)}{2}\right)\alpha_j + \int_0^t \sum_{j=1}^{n}\lambda_{ji}(t,s)\left(f_{2i-1}(s)+\frac{\gamma_{2i-1}(s)}{2}\right)\alpha_j ds\right\},$$

then,

$$C(R(t),\alpha)$$
$$= \sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\lambda_{ij}(t,0)\left(x_{2j-1}(0)+\frac{\delta_{2j-1}(0)}{2}\right)\alpha_i + \int_0^t \sum_{j=1}^{n}\lambda_{ij}(t,s)\left(f_{2j-1}(s)+\frac{\gamma_{2j-1}(s)}{2}\right)\alpha_i ds\right\}$$
$$+\frac{1}{2}\cdot\sum_{i=1}^{n}\left\{\delta_{2i-1}(0)\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right| + \int_0^t \gamma_{2i-1}(s)\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right| ds\right\}.$$

Therefore, $C(R(t),\alpha) = C(\widetilde{X}(t),\alpha)$ if and only if

$$\left(\sum_{i=1}^{n}|\alpha_i|\cdot\left|\sum_{j=1}^{n}\lambda_{ij}(t,0)\delta_{2j-1}(0) + \int_0^t\left(\sum_{j=1}^{n}\lambda_{ij}(t,s)\gamma_{2j-1}(s)\right)ds\right|\right)$$
$$= \sum_{i=1}^{n}\left\{\delta_{2i-1}(0)\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,0)\alpha_j\right| + \int_0^t \gamma_{2i-1}(s)\cdot\left|\sum_{j=1}^{n}\lambda_{ji}(t,s)\alpha_j\right| ds\right\}.$$

Thus, we obtained conditions to find the set of attainable states of differential inclusion by solution of generalized interval problem with conditions initial.

Final remarks

In this chapter we used the differentiability concept developed in Chapter 5 to present a concept of generalized interval differential equations and two concepts of solutions of a generalized interval differential equation, where one is called proper solution. We studied the solution of a particular case of generalized interval differential equation, which we called linear generalized interval differential equation, and with this particular case we presented a generalized interval problem with conditions initial, which under some conditions, helped us to find the set of attainable states of a classic differential inclusion.

Chapter 7

Conclusion

In this work we have presented new approaches to work with generalized interval under the algebraic, metric, and topology views. In each approach we have done we used a bijection between the set of generalized intervals and the set $\mathbb{R}^{2n}$ equipped with one of these structure. We have also provided a new approach to work with order relations in the set of generalized intervals.

By working with metric and order properties we have been able to introduce an optimization problem with proper-interval-valued cost function, which is related with a multi-objective optimization problem through their solutions. These properties, together with the topology ones, allowed us to state and prove a new version of Von Neumann's Theorem in interval context.

Via metric and algebraic properties we have introduced the concepts of limit, continuity, Lipschitz, and differentiability of generalized-interval-valued functions, which allowed us to elaborate the concepts of generalized interval differential equation, solution of a generalized interval differential equation, and proper solution of a generalized interval differential equation. We have also produced conditions that enabled us to find the set of attainable states of classic differential inclusion through a proper solution of a generalized interval differential equation.

Therefore, with the new tools that have been provided in this work, we generated a new point of view about interval analysis and, as a by product, we showed how to find the set of attainable states of a classic differential inclusion.

With these new concepts, we opened the way to study interval optimization and the behavior of dynamic systems with uncertainty.

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