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ZEROS OF CLASSICAL ORTHOGONAL POLYNOMIALS OF A DISCRETE VARIABLE

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ABSTRACT. In this paper we obtain sharp bounds for the zeros of classical orthogonal polynomials of a discrete variable, considered as functions of a parameter, by using a theorem of A. Markov and the so-called Hellmann-Feynman theorem. Comparisons with previous results for zeros of Hahn, Meixner, Kravchuk and Charlier polynomials are also presented.

1. INTRODUCTION

The behavior of zeros of the classical continuous orthogonal polynomials has been studied extensively, mainly because of their beautiful electrostatic interpretation and their important role as nodes of Gaussian quadrature formulae [2, 3, 10, 12, 13, 14, 18, 19, 25].

On the other hand, classical orthogonal polynomials of a discrete variable and their zeros are used in many applications ranging from the least-squares method of approximation [15], queueing theory [21, 33], lengths of weakly increasing subsequences of random words [20], totally asymmetric simple exclusion process [5] to cross-directional control on paper machines [27]. Despite the fact that Chebyshev [6, 28] emphasized the importance of the zeros of orthogonal polynomials of a discrete variable in 1855, results on monotonicity and limits of their zeros have been obtained only recently in [8, 18, 24, 26, 30, 31]. In the present paper we establish very sharp estimates for these zeros.

Since most of the classical orthogonal polynomials depend on parameters, one of the natural problems which arises is to study the zeros of these polynomials, considered as functions of the parameter. A natural tool to do so, is to employ a beautiful result of Andrei Markov [29] which says that the monotonicity of the zeros of these polynomials with respect to the parameter depends on the monotonicity of the logarithmic derivative of the weight function, with respect to the parameter, considered as a function of the variable in the interval of orthogonality. Another powerful tool to investigate the monotonic dependence of zeros of orthogonal polynomials is the so-called Hellmann-Feynman theorem [17, Section 7.3] which provides information about the behavior of the eigenvalues of the Jacobi matrix associated with the orthogonal polynomials, and these, as it is well known,

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coincide with the zeros of the polynomials. In this paper we make extensive use of these powerful methods to investigate the behavior of the zeros of four families of classical orthogonal polynomials of a discrete variable, those of Charlier, Meixner, Kravchuk and Hahn. This allows us to provide new and sharp bounds for their zeros.

We adopt the definitions and notations of these families of classical orthogonal polynomials of a discrete variable given in [23]:

- Charlier polynomials
$$C_n(x;a) = {}_2F_0\left(\begin{array}{c} -n,-x \\ - \end{array} \middle| -\frac{1}{a} \right),$$
whose orthogonality property is
$$\sum_{k=0}^{\infty} \frac{a^k}{k!} C_n(k;a) C_m(k;a) = a^{-n} e^a n!\, \delta_{mn}, \qquad a>0.$$
- Meixner polynomials
$$M_n(x;\beta,c) = {}_2F_1\left(\begin{array}{c} -n,-x \\ \beta \end{array} \middle| 1-\frac{1}{c} \right),$$
whose orthogonality property is
$$\sum_{k=0}^{\infty} \frac{(\beta)_k c^k}{k!} M_n(k;\beta,c) M_m(k;\beta,c) = \frac{c^{-r} r!}{(\beta)_r (1-c)^{\beta}} \delta_{mn}, \quad \beta>0, \quad 0<c<1.$$
- Kravchuk polynomials, defined by
$$K_n(x;p,N) = {}_2F_1\left(\begin{array}{c} -n,-x \\ -N \end{array} \middle| \frac{1}{p} \right), \quad n=0,1,\dots,N,$$
whose orthogonality property is
$$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} K_n(k;p,N) K_m(k;p,N)$$
$$= \frac{(-1)^n n!}{(-N)_n} \left(\frac{1-p}{p} \right)^n \delta_{mn}, \quad 0<p<1.$$
- Hahn polynomials
$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{array}{c} -n,-x,n+\alpha+\beta+1 \\ -N,\alpha+1 \end{array} \middle| 1 \right), \quad n=0,1,\dots,N,$$
whose orthogonality property is
$$\sum_{k=0}^{N} \binom{\alpha+k}{k} \binom{\beta+N-k}{N-k} Q_n(k;\alpha,\beta,N) Q_m(k;\alpha,\beta,N)$$
$$= \frac{(-1)^n (n+\alpha+\beta+1)_{N+1} (\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n (-N)_n N!} \delta_{nm}, \quad \alpha>-1, \quad \beta>-1.$$

The paper is organized as follows. In Section 2 we furnish the necessary background and theoretical results which will be employed in the investigation. The remaining sections are devoted to more specific and precise results concerning monotonicity and sharp limits for the zeros of the four aforementioned families of orthogonal polynomials. The analysis starts with Charlier and Meixner orthogonal polynomials, for which the support of orthogonality is unbounded. In these two cases we provide bounds for all the zeros, and we compare our results with the best known limits in Sections 3 and 4, respectively. Moreover, results on the extreme zeros of Kravchuk and Hahn polynomials are also provided in Sections 5 and 6. We also compare the results obtained in this paper with other limits, exemplifying in several tables the sharpness of our results.

2. Basic tools

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be the set of nonnegative integers and let $\mathcal{N}$ be $\mathbb{N}_0$ or a finite subset of the form $\{0, 1, \dots, N\}$. Let $\{p_n(x,\tau)\}_{n=0}^{\mathcal{N}}$ be a parametric sequence of orthonormal polynomials of a discrete variable, with respect to the positive weight function $\omega(x;\tau)$, for $\tau \in (\tau_1, \tau_2)$. That is, for every $n \in \mathcal{N}$ and for every $\tau \in (\tau_1, \tau_2)$, $p_n(x,\tau)$ is a polynomial of degree exactly n and

$$\sum_{k\in\mathcal{N}} \omega(k;\tau) p_n(k,\tau) p_m(k,\tau) = \delta_{nm} \quad \text{whenever} \quad m, n \in \mathcal{N}. \tag{2.1}$$

It suffices that the function $\omega(x;\tau)$ is defined only in $\mathcal{N}$ but in most cases ω is well defined for all real values of x in the convex hull $\mathcal{K}$ of $\mathcal{N}$. In what follows we shall suppose that ω obeys this property. It is well known that every $p_n(x,\tau)$ has n real distinct zeros which belong to $\mathcal{K}$ [7].

We begin with the first tool we shall apply in our studies. It is probably the first result ever proved which deals with monotonicity of zeros of orthogonal polynomials and was established by Andrey Markov in 1886. For a more general version of Markov's Theorem we refer the reader to Ismail's book [17, Theorem 7.1.1, p. 204]. An idea for a completely different proof, without use of Gaussian quadrature, which also allows a slight generalization, can be found in [10]. Here we state the result in a form appropriate for our needs in this paper.

Theorem A. *For every $\tau \in (\tau_1, \tau_2)$, let the polynomials $\{p_n(x,\tau)\}_{n=0}^{\infty}$ obey the orthogonality relation* (2.1) *with respect to the positive weight function $\omega(x;\tau)$ and assume that it has continuous first derivative $\omega_\tau(x;\tau)$ with respect to τ, for every $\tau \in (\tau_1, \tau_2)$ and $x \in \mathcal{K}$. Assume further that the series*

$$\sum_{k\in\mathcal{N}} k^j\, \omega_\tau(k;\tau), \qquad j = 0, \dots, 2n-1,$$

converge uniformly for τ in every compact subset of (τ_1, τ_2). Then the zeros of $p_n(x,\tau)$ are increasing (*decreasing*) *functions of $\tau \in (\tau_1, \tau_2)$ if $\partial\{\log \omega(x;\tau)\}/\partial\tau$ is an increasing* (*decreasing*) *function of x in $\mathcal{K}$.*

Consider the parametric sequence $\{s_k(x;\tau)\}_{k=0}^{\infty}$ of orthogonal polynomials. It is well known that it satisfies the three term recurrence relation

$$\begin{aligned} s_0(x;\tau) &= 1 \\ s_1(x,\tau) &= (x - \beta_0(\tau))/\gamma_0(\tau) \\ x s_k(x;\tau) &= \gamma_k(\tau) s_{k+1}(x;\tau) + \beta_k(\tau) s_k(x;\tau) + \delta_k(\tau) s_{k-1}(x;\tau), \quad k = 1, 2, \dots \end{aligned}$$

with $\gamma_k(\tau)\delta_{k+1}(\tau) > 0$.

We associate with this recurrence relation the $n\times n$ matrix

$$H_n(\tau)=\begin{pmatrix} \beta_0(\tau) & \gamma_0(\tau) & 0 & 0 & \dots & 0\\ \delta_1(\tau) & \beta_1(\tau) & \gamma_1(\tau) & 0 & \dots & 0\\ 0 & \delta_2(\tau) & \beta_2(\tau) & \gamma_2(\tau) & \dots & 0\\ 0 & & \ddots & \ddots & \ddots & 0\\ 0 & & & \delta_{n-2}(\tau) & \beta_{n-2}(\tau) & \gamma_{n-2}(\tau)\\ 0 & & & 0 & \delta_{n-1}(\tau) & \beta_{n-1}(\tau) \end{pmatrix}.$$

However, we can obtain a symmetric matrix when we consider the corresponding sequence $\{p_k(x,\tau)\}_{k=0}^{\infty}$ of parametric orthonormal polynomials which are given by

$$p_0(x;\tau)=1,\qquad p_k(x;\tau)=\left(\frac{\gamma_0\gamma_1\dots\gamma_{k-1}}{\delta_1\delta_2\dots\delta_k}\right)^{1/2}s_k(x;\tau)$$

and they satisfy the three term recurrence relation

$$xp_k(x;\tau)=a_k(\tau)p_{k+1}(x;\tau)+b_k(\tau)p_k(x;\tau)+a_{k-1}(\tau)p_{k-1}(x;\tau), \tag{2.2}$$

where

$$a_k(\tau)=\sqrt{\gamma_k(\tau)\delta_{k+1}(\tau)}\quad\text{and}\quad b_k(\tau)=\beta_k(\tau)\qquad k\geq 0.$$

Observe that the zeros of the polynomial $p_n(x;\tau)$ coincide with those of $s_n(x;\tau)$ and they are also the eigenvalues of the Jacobi matrix

$$J_n=J_n(\tau)=\begin{pmatrix} b_0(\tau) & a_0(\tau) & 0 & 0 & \dots & 0\\ a_0(\tau) & b_1(\tau) & a_1(\tau) & 0 & \dots & 0\\ 0 & a_1(\tau) & b_2(\tau) & a_2(\tau) & \dots & 0\\ 0 & & \ddots & \ddots & \ddots & 0\\ 0 & & & a_{n-3}(\tau) & b_{n-2}(\tau) & a_{n-2}(\tau)\\ 0 & & & 0 & a_{n-2}(\tau) & b_{n-1}(\tau) \end{pmatrix}. \tag{2.3}$$

Moreover, if $\lambda_j=\lambda_j(\tau)$ is a zero of $p_n(x;\tau)$ and

$$\mathcal{P}_j=(p_0(\lambda_j;\tau),p_1(\lambda_j;\tau),\dots,p_{n-1}(\lambda_j;\tau))^T,$$

then

$$J_n\mathcal{P}_j=\lambda_j\mathcal{P}_j.$$

Let us denote by $J_n'=J_n'(\tau)$ the Jacobi matrix whose entries are the derivatives of the corresponding entries of $J_n(\tau)$. We shall formulate the so-called Hellmann-Feynman theorem (see [17]) in a form which will be convenient for our needs.

Theorem B. *For every zero $\lambda_j(\tau)$ of $p_n(x;\tau)$ we have*

$$\lambda_j'(\tau)=\frac{\mathcal{P}_j^TJ_n'\mathcal{P}_j}{\mathcal{P}_j^T\mathcal{P}_j}=\frac{\displaystyle\sum_{k=0}^{n-1}b_k'p_k^2(\lambda_j;\tau)+2\sum_{k=0}^{n-2}a_k'p_k(\lambda_j;\tau)p_{k+1}(\lambda_j;\tau)}{\displaystyle\sum_{k=0}^{n-1}p_k^2(\lambda_j;\tau)}.$$

Furthermore, if the sum in the numerator in the latter expression is positive, then the zeros $\lambda_j(\tau)$ of $p_n(x;\tau)$ are increasing functions of τ. This happens if J_n' is a positive definite matrix.

We need also the following corollary of the Perron-Frobenius theorem [16, Theorems 8.4.4 and 8.4.5]:

Theorem C. *Let $H_n(\tau)$ be an $n \times n$ tridiagonal matrix with positive off-diagonal elements. If the entries of $H_n(\tau)$ are increasing functions of τ, then the largest eigenvalue of $H_n(\tau)$ is an increasing function of τ.*

It is a well-known property of zeros of orthogonal polynomials with respect to a positive Borel measure that they are all real, distinct, belong to the convex hull of the support of the measure with respect to which the polynomials are orthogonal. Let the zeros of $C_n(x;a)$, $M_n(x;\beta,c)$, $K_n(x;p,N)$ and $Q_n(x;\alpha,\beta,N)$ be denoted by $c_{n,j}(a)$, $m_{n,j}(\beta,c)$, $\kappa_{n,j}(p,N)$ and $q_{n,j}(\alpha,\beta,N)$, respectively, all arranged in decreasing order, so that

$$0 < c_{n,n}(a) < c_{n,n-1}(a) < \cdots < c_{n,1}(a) < \infty, \quad a > 0,$$
$$0 < m_{n,n}(\beta,c) < m_{n,n-1}(\beta,c) < \cdots < m_{n,1}(\beta,c) < \infty, \quad \beta > 0, \quad c \in (0,1),$$
$$0 < \kappa_{n,n}(p,N) < \kappa_{n,n-1}(p,N) < \cdots < \kappa_{n,1}(p,N) < N, \quad p \in (0,1), \quad n \leq N,$$

and for $\alpha, \beta > -1$, $n \leq N$,

$$0 < q_{n,n}(\alpha,\beta,N) < q_{n,n-1}(\alpha,\beta,N) < \cdots < q_{n,1}(\alpha,\beta,N) < N.$$

Moreover, the zeros of any pair of orthogonal polynomials of consecutive degrees interlace. The following theorem summarizes known properties of the zeros of classical orthogonal polynomials of a discrete variable.

Theorem 2.1. *Let $\{G_n(x)\}$ be any of the four sequences of classical orthogonal polynomials of a discrete variable (Charlier, Meixner, Kravchuk and Hahn). Then, for any admissible values of the parameters, the distance between every two consecutive zeros of $G_n(x)$ is greater than one, so that the zeros of $G_n(x)$ interlace with those of $G_n(x+1)$ and $G_n(x-1)$ Moreover, for every $n, j \in \mathbb{N}$ with $1 \leq j \leq n$,*

- *$c_{n,j}(a)$ are increasing functions of a, for $a \in (0,\infty)$;*
- *$m_{n,j}(\beta,c)$ are increasing functions of both $\beta \in (0,\infty)$ and $c \in (0,1)$;*
- *$\kappa_{n,j}(p,N)$ are increasing functions of the parameter $p \in (0,1)$ and increasing functions of N in the sense that $\kappa_{n,j}(p,N) < \kappa_{n,j}(p,N+1)$ whenever $N \geq n$.*
- *$q_{n,j}(\alpha,\beta,N)$ are increasing functions of $\alpha \in (-1,\infty)$, decreasing functions of $\beta \in (-1,\infty)$ and increasing functions of N, that is, $q_{n,j}(\alpha,\beta,N) < q_{n,j}(\alpha,\beta,N+1)$ provided $N \geq n$.*

Proof. The first statements of the theorem follow immediately from a nice result of Krasikov and Zarkh [26, Theorem 1] . It states that if the real polynomial $p(x)$, with only distinct real zeros $a < \zeta_1 < \zeta_2 < \cdots < \zeta_n < b$ satisfies the difference equation

$$p(x+1) = 2A(x)p(x) - B(x)p(x-1)$$

and $B(x) > 0$ for $x \in (a,b)$, then $\zeta_{j+1} - \zeta_j > 1$ for every $j = 1, \ldots, n-1$. The classical orthogonal polynomials of a discrete variable are solutions of difference equations of the above form (see formulae (1.12.5), (1.9.5), (1.10.5) and (1.5.5) in [23]). It is straightforward to verify that the corresponding coefficients $B(x)$ are all positive in the convex hulls of the set of orthogonality of the corresponding $G_n(x)$.

The monotonicity of the zeros of Charlier, Meixner, Kravchuk and Hahn polynomials with respect to the parameters a, p, β, c, α and β, can be proved via Theorem A by computing the logarithmic derivative of the weight function and establishing its monotonicity with respect to the variable. We omit the straightforward technical details.

The monotonicity of the zeros $\kappa_{n,j}(p,N)$ of Kravchuk's polynomials with respect to N, that is, the inequality $\kappa_{n,j}(p,N) < \kappa_{n,j}(p,N+1)$, was established by L. Chihara and D. Stanton [8]. The monotonicity of the zeros of Hahn polynomials $q_{n,j}(\alpha,\beta,N)$ with respect to N have been proved in [30, Theorem 6]. $\square$

Below we illustrate the dependence of the zeros of the classical orthogonal polynomials of a discrete variable on the parameters. The figures show the zeros of fifth degree polynomials as functions of the corresponding parameters.

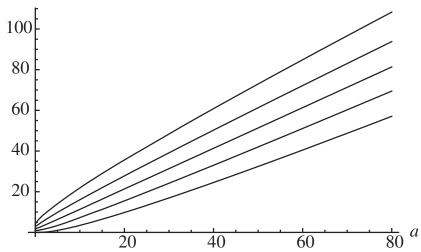

FIGURE 2.1. Zeros of $C_5(x;a)$.

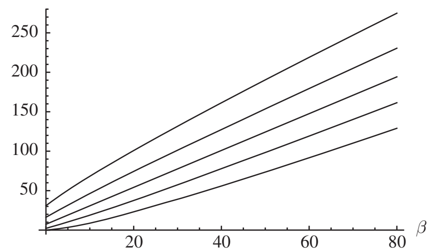

FIGURE 2.2. Zeros of $M_5(x;\beta,0.7)$.

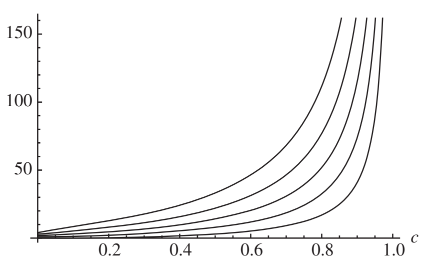

FIGURE 2.3. Zeros of $M_5(x;10,c)$.

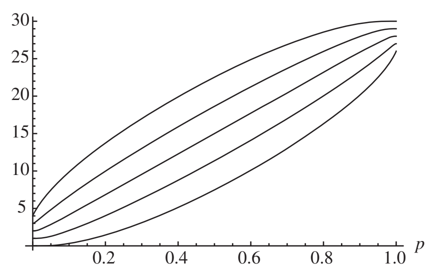

FIGURE 2.4. Zeros of $K_5(x;p,30)$.

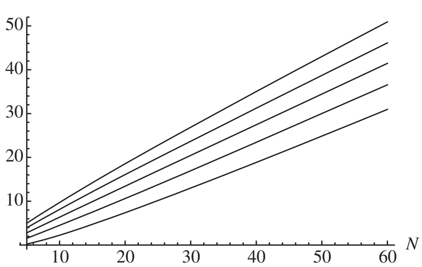

FIGURE 2.5. Zeros of $K_5(x;0.7,N)$.

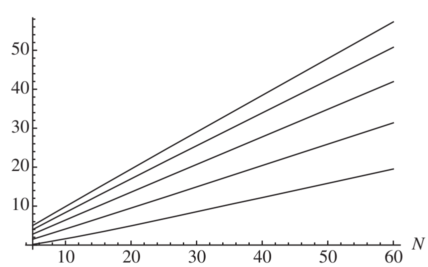

FIGURE 2.6. Zeros of $Q_5(x;10,2,N)$.

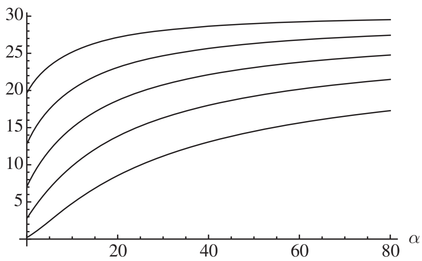

FIGURE 2.7. Zeros of $Q_5(x;\alpha,10,30)$.

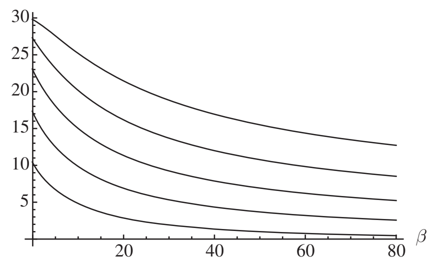

FIGURE 2.8. Zeros of $Q_5(x;10,\beta,30)$.

Finally, let us mention that in the present paper we obtain bounds for the zeros of classical orthogonal polynomials of a discrete variable by using appropriate limit relations with classical orthogonal polynomials. Thus, in what follows we use the following standard definitions and notations of the classical orthogonal polynomials of Jacobi, Laguerre and Hermite [23]:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array}\,\middle|\, \frac{1-x}{2}\right),$$
$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array}\,\middle|\, x\right),$$
$$H_n(x) = (2x)^n\,{}_2F_0\left(\begin{array}{c} -n/2, -(n-1)/2 \\ - \end{array}\,\middle|\, -\frac{1}{x^2}\right).$$

Their zeros are denoted by $x_{n,j}(\alpha,\beta)$, $x_{n,j}(\alpha)$ and $h_{n,j}$, $j=1,\ldots,n$, respectively, and are supposed to be arranged in decreasing order. In other words, we have

$$-1 < x_{n,n}(\alpha,\beta) < \cdots < x_{n,1}(\alpha,\beta) < 1,$$
$$0 < x_{n,n}(\alpha) < \cdots < x_{n,1}(\alpha),$$

and

$$h_{n,n} < \ldots < 0 < h_{n,[n/2]} < \cdots < h_{n,1}.$$

3. Charlier polynomials

Motivated by the limit relation [23, (2.12.1)]

$$\lim_{a\to\infty} (2a)^{\frac{n}{2}} C_n(a + x\sqrt{2a}; a) = (-1)^n H_n(x),$$

which implies

$$(c_{n,j}(a) - a)/\sqrt{2a} \to h_{n,j} \quad \text{as} \quad a \to \infty \tag{3.1}$$

and

$$(c_{n,j}(a) - a - n + 1)/\sqrt{2a} \to h_{n,j} \quad \text{as} \quad a \to \infty, \tag{3.2}$$

we obtain a result concerning the monotonicity, with respect to a, of the quantities that appear on the left-hand sides of the latter two limit relations.

Theorem 3.1. *Let $n \in \mathbb{N}$ and $a > 0$. Then, for every j, $1 \le j \le n$, the functions $(c_{n,j}(a) - a)/\sqrt{2a}$ decrease and $(c_{n,j}(a) - a - n + 1)/\sqrt{2a}$ increase for $a \in (0,\infty)$. Moreover, the inequalities*

$$a + \sqrt{2a}\, h_{n,j} \le c_{n,j}(a) \le a + n - 1 + \sqrt{2a}\, h_{n,j},$$

hold for $j = 1,\ldots,n$.

Proof. Since the polynomials satisfy the three term recurrence relation

$$-xC_n(x;a) = aC_{n+1}(x;a) - (n+a)C_n(x;a) + nC_{n-1}(x;a),$$

then the polynomials defined by $\tilde{C}_n(x;a) = (-1)^n(2a)^{\frac{n}{2}}C_n(a + x\sqrt{2a};a)$, whose zeros are $(c_{n,j}(a) - a)/\sqrt{2a}$, satisfy

$$x\tilde{C}_n(x;a) = \frac{1}{2}\tilde{C}_{n+1}(x;a) + \frac{n}{\sqrt{2a}}\tilde{C}_n(x;a) + n\tilde{C}_{n-1}(x;a),$$

so that the corresponding orthonormal polynomials satisfy a recurrence relation of the form (2.2) with $\tilde{a}_k(a) = \sqrt{(k+1)/2}$ and $\tilde{b}_k(a) = k/\sqrt{2a}$. Since $\tilde{a}'_k(a) = 0$ for all k and $\tilde{b}'_k(a) = -k(2a)^{-3/2}$ for $k = 0, \ldots, n-1$, then $J'_n(a)$ is a diagonal matrix whose entries are all negative. The Hellmann-Feynman theorem shows that the quantities $(c_{n,j}(a) - a)/\sqrt{2a}$ are decreasing functions of a. Since they converge to $h_{n,j}$, then $(c_{n,j}(a) - a)/\sqrt{2a} \geq h_{n,j}$. Therefore,

$$a + \sqrt{2a}h_{n,j} \leq c_{n,j}(a). \tag{3.3}$$

The polynomials $\hat{C}_k(x;a) = (-1)^k(2a)^{\frac{k}{2}}C_k(a+n-1+x\sqrt{2a};a)$, whose zeros are exactly $(c_{n,j}(a) - a - n + 1)/\sqrt{2a}$, satisfy the three term recurrence relation

$$x\hat{C}_k(x;a) = \frac{1}{2}\hat{C}_{k+1}(x;a) - \frac{n-k-1}{\sqrt{2a}}\hat{C}_k(x;a) + k\hat{C}_{k-1}(x;a)$$

and their corresponding orthonormal polynomials obey a three term recurrence relation of the form (2.2) with $\hat{a}_k(a) = \sqrt{(k+1)/2}$ and $\hat{b}_k(a) = (-n+k+1)/\sqrt{2a}$. Since, $\hat{a}'_k(a) = 0$ and $\hat{b}'_k(a) = (n-k-1)/(2a)^{3/2} > 0$ for $k = 0, \ldots, n-1$, then the Hellmann-Feynman theorem implies that $(c_{n,j}(a) - a - n + 1)/\sqrt{2a}$ are increasing functions of a. This, together with (3.2), yields the inequality

$$c_{n,j}(a) \leq a + n - 1 + \sqrt{2a}\,h_{n,j}. \tag{3.4}$$

□

Now we are in position to use various results concerning limits of zeros of Hermite polynomials to obtain explicit bound for the zeros of the Charlier polynomials. We begin with a result which concerns upper limits for half of the zeros of Charlier polynomials and lower limits for the other half. In order to this we use the fact that the zeros of Hermite polynomials are symmetric with respect to the origin and the following observation, given in [2, Eq. (1.5)]:

$$h_{n,j} \leq \sqrt{2n-2}\,\cos\frac{(j-1)\pi}{n-1}, \quad j = 1, \ldots, [n/2].$$

Then we immediately obtain

Corollary 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and $a > 0$. Then the inequalities*

$$c_{n,j}(a) \leq a + n - 1 + 2\sqrt{(n-1)a}\,\cos\frac{(j-1)\pi}{n-1}$$

and

$$c_{n,n+1-j}(a) \geq a - 2\sqrt{(n-1)a}\,\cos\frac{(j-1)\pi}{n-1},$$

hold for $j = 1, \ldots, [n/2]$. In particular, the zeros of the Charlier polynomials satisfy the inequalities

$$a - 2\sqrt{(n-1)a} \leq c_{n,j}(a) \leq a + n - 1 + 2\sqrt{(n-1)a}, \quad j = 1, \ldots, n.$$

Observe that the last inequality is obtained by an upper limit for the largest zero $h_{n,1}$ of the Hermite polynomial. Such sharp bounds are known and two of the best ones are given in [11, Theorem 2] and by Szegő [32, (6.32.6)]. Theorem 2 in [11] states that

$$h_{n,1}^2 \leq \frac{n^2 - \frac{3}{2}n + 2 + (n-2)\sqrt{n^2+n+4}}{n+4} \tag{3.5}$$

for every $n \in \mathbb{N}$ and the sharper estimate

$$h_{n,1}^2 \leq \frac{n^2 - \frac{5}{2}n + \frac{15}{2} + (n-3)\sqrt{n^2+n+10}}{n+3}, \tag{3.6}$$

holds if n is an odd number. Then (3.5) implies:

Corollary 3.2. *Let $n \in \mathbb{N}$, $n \geq 2$, and $a > 0$. Then*

$$c_{n,j}(a) \leq a + n - 1 + \sqrt{a\,\frac{2n^2 - 3n + 4 + 2(n-2)\sqrt{n^2+n+4}}{n+4}} \tag{3.7}$$

and

$$c_{n,j}(a) \geq a - \sqrt{a\,\frac{2n^2 - 3n + 4 + 2(n-2)\sqrt{n^2+n+4}}{n+4}}, \tag{3.8}$$

hold for every j with $1 \leq j \leq n$.

We do not state explicitly the obvious refinements if we use (3.6) for odd values of n.

Szegő's limits are given in [32, (6.32.6)] by

$$h_{n,1} \leq (2n+1)^{1/2} - 6^{-1/3}(2n+1)^{-1/6} i_1,$$

where i_1 is the lowest zero of the Airy function [32, 1.81] and, since $6^{-1/3} i_1 \geq 1.85575$, then

$$h_{n,1} \leq (2n+1)^{1/2} - 1.85575\,(2n+1)^{-1/6}. \tag{3.9}$$

Then, using (3.4) we state

Corollary 3.3. *Let $n \in \mathbb{N}$, $n \geq 2$, and $a > 0$. Then*

$$c_{n,j}(a) \leq a + n - 1 + \sqrt{(4n+2)a} - 1.85575\,\sqrt{2a}\,(2n+1)^{-1/6}$$

and

$$c_{n,j}(a) \geq a - \sqrt{(4n+2)a} + 1.85575\,\sqrt{2a}\,(2n+1)^{-1/6},$$

hold for every j with $1 \leq j \leq n$.

We remark that, when $n \leq 8$, the limits (3.5) and (3.6) are better than (3.9). Therefore, the same happens when we compare the results in Corollary 3.2 and Corollary 3.3.

A very interesting estimate which limits half of the zeros of Charlier polynomials is obtained by the following result for the zeros of Hermite polynomials, obtained in the same Theorem 2 in [11]. The latter states that

$$h_{n,j}^2 \geq \frac{n^2 - \frac{3}{2}n + 2 - (n-2)\sqrt{n^2+n+4}}{n+4} \quad \text{for} \quad 1 \leq j \leq [n/2]. \tag{3.10}$$

Thus, we obtain:

Corollary 3.4. *Let $n \in \mathbb{N}$, $n \geq 2$, and $a > 0$. Then*

$$c_{n,j}(a) \geq a + \sqrt{a\,\frac{2n^2-3n+4-2(n-2)\sqrt{n^2+n+4}}{n+4}}$$

and

$$c_{n,n+1-j}(a) \leq a + n - 1 - \sqrt{a\,\frac{2n^2-3n+4-2(n-2)\sqrt{n^2+n+4}}{n+4}},$$

hold for $j = 1, \ldots, [n/2]$.

We illustrate our results with the figure below. It shows the zeros of $C_5(x;a)$, in continuous line, and in dashed lines the extreme limits from Corollary 3.2 and the limits for the zeros "in the middle" from Corollary 3.4.

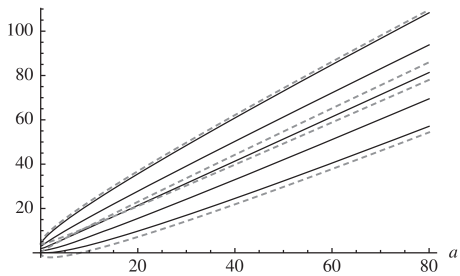

FIGURE 3.1. Zeros of $C_5(x;a)$, limits for the extreme zeros from Corollary 3.2 and center limits from Corollary 3.4.

Having all these bounds for the zeros of Charlier polynomials, we compare them with the best limits known in the literature, obtained recently by Krasikov and Zarkh [26, Th.5]. There the authors prove that

$$c_{n,1} \quad < \quad (\sqrt{a}+\sqrt{n})^2 - \frac{3a^{1/6}(\sqrt{a}+\sqrt{n})^{2/3}}{2^{2/3}n^{1/6}} \quad \text{and} \tag{3.11}$$

$$c_{n,n} \quad > \quad (\sqrt{a}-\sqrt{n})^2 + \frac{3a^{1/6}(\sqrt{a}-\sqrt{n})^{2/3}}{2^{2/3}n^{1/6}} \tag{3.12}$$

under the restriction $n < a$. Therefore, our Corollaries 3.2 and 3.3 provide the first results which hold for all values of n and a. Let us compare our results with those in [26] when $a \to \infty$. It turns out that, if we fix $n \in \mathbb{N}$, $n \geq 3$, the bounds obtained in Corollary 3.3 are asymptotically sharper than (3.11) and (3.12). The vast number of numerical experiments we have performed show that when n is fixed, and a varies, the following happens: when $a < n$ only the limits provided in our results hold. The estimates (3.11) and (3.12) are sharper when a is in an interval (n, a_1). Finally, the estimates in Corollary 3.3 become sharper again for the remaining values of $a \in (a_1, \infty)$. The numerical Table 3.1 below illustrate this phenomena.

Moreover, obviously Corollary 3.1 furnishes limits not only for the extreme zeros, but for half of the zeros of Charlier polynomials.

In the Table 3.1 we also provide the numerical values of the extreme zeros of $C_5(x;a)$ and the limits, given in (3.12), (3.8), (3.7) and (3.11). Here and in what follows the better lower and upper bounds are highlighted in gray; also, when all the bounds are out of the interval of orthogonality, no selection is made.

TABLE 3.1. Table with the extreme zeros of $C_5(x;a)$ and the limits (3.12), (3.8), (3.7) and (3.11)

a	(3.12)	(3.8)	$c_{5,5}$	$c_{5,1}$	(3.7)	(3.11)
0.1		-0.8067	0.0000	4.4330	5.0067	
5		-1.4112	1.0000	14.2858	15.4112	
10	2.8735	0.9332	3.4568	21.8798	23.0668	22.6141
50	31.3089	29.7261	32.4089	72.9684	74.2739	74.3490
100	72.4871	71.3284	74.0665	131.3159	132.6716	133.1878
500	435.1245	435.8882	438.7849	566.6016	568.1118	570.5634

4. Meixner polynomials

The main result obtained in this section is motivated by the limit relation between Laguerre and Meixner polynomials [23, (2.9.2)]

$$\lim_{c\to 1} M_n\left(\frac{x}{1-c};\alpha+1,c\right)=\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}.$$

Let us define new polynomials with shifted argument, as follows:

$$\tilde{M}_n(x;\alpha,c):=(-1)^n M_n\left(\frac{\sqrt{c}\,x+c(\alpha+1)}{1-c};\alpha+1,c\right),$$

$$\hat{M}_n(x;\alpha,c):=(-1)^n M_n\left(\frac{\sqrt{c}\,x+c(n+\alpha)+n-1}{1-c};\alpha+1,c\right).$$

It is clear that the asymptotic formulae

$$\lim_{c\to 1}\tilde{M}_n(x;\alpha,c)=\frac{(-1)^n L_n^{(\alpha)}(x+\alpha+1)}{L_n^{(\alpha)}(0)}, \tag{4.1}$$

$$\lim_{c\to 1}\hat{M}_n(x;\alpha,c)=\frac{(-1)^n L_n^{(\alpha)}(x+\alpha+2n-1)}{L_n^{(\alpha)}(0)}, \tag{4.2}$$

hold.

Theorem 4.1. *Let $n\in\mathbb{N}$, $\beta>0$ and $0<c<1$. Then $((1-c)m_{n,j}(\beta,c)-c\beta)/\sqrt{c}$ are decreasing functions of c and $((1-c)m_{n,j}(\beta,c)-c\beta-n+1)/\sqrt{c}$ are increasing functions of c, when $c\in(0,\infty)$. Moreover, the inequalities*

$$\frac{\sqrt{c}}{1-c}\left(x_{n,j}(\beta-1)-\beta(1-\sqrt{c})\right)\le m_{n,j}(\beta,c),$$

$$m_{n,j}(\beta,c)\le\frac{\sqrt{c}}{1-c}\left(x_{n,j}(\beta-1)-\beta(1-\sqrt{c})+(n-1)\frac{(1-\sqrt{c})^2}{\sqrt{c}}\right),$$

hold for $j=1,\ldots,n$.

Proof. The three term recurrence relation of $M_n(x;\beta,c)$ is

$$(c-1)xM_n(x;\beta,c)=c(n+\beta)M_{n+1}(x;\beta,c) - (n+(n+\beta)c)M_n(x;\beta,c)+nM_{n-1}(x;\beta,c).$$

After substitutions, we obtain the coefficients of the recurrence relation (2.2) for the orthonormal polynomials, corresponding to $\tilde{M}_n(x;\alpha,c)$ and $\hat{M}_n(x;\alpha,c)$. These coefficients are

$$\tilde{a}_k(c) = \hat{a}_k(c) = \sqrt{(k+1)(k+\alpha+1)},$$
$$\tilde{b}_k(c) = k(1+c)/\sqrt{c} \quad \text{and} \quad \hat{b}_k(c) = -(n-1-k)(1+c)/\sqrt{c}.$$

The derivatives with respect to c are $\tilde{a}_k'(c) = \hat{a}_k'(c) = 0$, $\tilde{b}_k'(c) = -k(1-c)/(2c^{3/2}) \le 0$ and $\hat{b}_k'(c) = (n-1-k)(1-c)/(2c^{3/2}) \ge 0$. Therefore, by the Hellmann-Feynmann theorem, the zeros $\{(1-c)m_{n,j}(\alpha+1,c) - c(\alpha+1)\}/\sqrt{c}$ of $\tilde{M}_n(x;\alpha,c)$ are decreasing functions of c while the zeros $\{(1-c)m_{n,j}(\alpha+1,c) - c(\alpha+n) - n + 1\}/\sqrt{c}$ of $\hat{M}_n(x;\alpha,c)$ are increasing functions of c, for $c \in (0,1)$.

Formulae (4.1) and (4.2) imply that

$$\{(1-c)m_{n,j}(\alpha+1,c) - c(\alpha+1)\}/\sqrt{c} \to x_{n,j}(\alpha) - \alpha - 1$$

and

$$\{(1-c)m_{n,j}(\alpha+1,c) - c(\alpha+1) - n + 1\}/\sqrt{c} \to x_{n,j}(\alpha) - \alpha - 2n + 1, \quad c \to 1.$$

The monotonic behavior of the functions on the left-hand sides shows that

$$\{(1-c)m_{n,j}(\alpha+1,c) - c(\alpha+1)\}/\sqrt{c} \ge x_{n,j}(\alpha) - \alpha - 1$$

and

$$\{(1-c)m_{n,j}(\alpha+1,c) - c(\alpha+1) - n + 1\}/\sqrt{c} \le x_{n,j}(\alpha) - \alpha - 2n + 1.$$

These inequalities are equivalent to those in the statement of the theorem. $\square$

Observe that the inequalities in Theorem 4.1 coincide with those obtained by Ismail and Muldoon [18, Theorem 6.2] using slightly different arguments. In fact, Ismail and Muldoon used the Hellmann-Feynmann theorem to obtain bounds for the zeros of the polynomials $M_n\left(\sqrt{c}\,x/(1-c);\alpha+1,c\right)$ and it suffices to ensure that the corresponding orthonormal polynomials are such that the coefficients a_k in the recurrence relation do not depend on c. Then, they conclude, via an extension of the Hellmann-Feynmann theorem which ensures convexity of the zeros, that the derivatives obey certain inequalities and finally integrate these inequalities, having in mind the asymptotic values of these zeros when c converges to 1. Except for doing the transformation of Ismail and Muldoon, we performed additional translations of the zeros which guarantee the positive (negative) definiteness of the corresponding matrices J_n' and obtain their result in a slightly different way.

Now we use limits for the zeros of the Laguerre polynomials to obtain explicit bounds for the zeros of the Meixner polynomials.

The best limits for the extreme zeros of $L_n^{(\alpha)}(x)$, when α is large, are given in [25, Theorem 1] and read as follows:

$$x_{n,1}(\alpha) < (\sqrt{n+\alpha+1} + \sqrt{n}\,)^2 - 3\frac{(\sqrt{n+\alpha+1} + \sqrt{n}\,)^{4/3}}{(4\sqrt{n(n+\alpha+1)})^{1/3}} + 2$$

and

$$x_{n,n}(\alpha) > (\sqrt{n+\alpha+1} - \sqrt{n}\,)^2 + 3\frac{(\sqrt{n+\alpha+1} - \sqrt{n}\,)^{4/3}}{(4\sqrt{n(n+\alpha+1)})^{1/3}}.$$

They, together with Theorem 4.1 yield

Corollary 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$, $0 < c < 1$ and $\beta > 0$. Then*

$$\frac{\sqrt{c}}{1-c}\left((\sqrt{\nu}-\sqrt{n})^2 + 3\frac{(\sqrt{\nu}-\sqrt{n})^{4/3}}{4^{1/3}(n\nu)^{1/6}} - \beta(1-\sqrt{c})\right) < m_{n,j}(\beta, c)$$
$$< \frac{\sqrt{c}}{1-c}\left((\sqrt{\nu}+\sqrt{n})^2 - \frac{3(\sqrt{\nu}+\sqrt{n})^{4/3}}{4^{1/3}(n\nu)^{1/6}} + 2 - \beta(1-\sqrt{c}) + (n-1)\frac{(1-\sqrt{c})^2}{\sqrt{c}}\right),$$

hold for $1 \leq j \leq n$, where $\nu = n + \beta$.

The best limit for the largest zero if the Laguerre polynomials, when n goes to infinity, is given by Szegő [32, (6.32.6)]:

$$x_{n,1}(\alpha) \leq ((4n+2\alpha+2)^{1/2} - 6^{-1/3}(4n+2\alpha+2)^{-1/6} i_1)^2.$$

Again, as in the previous section, since $6^{-1/3} i_1 \geq 1.85575$, then

$$x_{n,1}(\alpha) \leq ((4n+2\alpha+2)^{1/2} - 1.85575(4n+2\alpha+2)^{-1/6})^2.$$

Therefore, we have:

Corollary 4.2. *Let $n \in \mathbb{N}$, $n \geq 2$, $0 < c < 1$ and $\beta > 0$. Then*

$$m_{n,j}(\beta, c) < \frac{\sqrt{c}}{1-c}\Bigg(((4n+2\beta)^{1/2} - 1.85575(4n+2\beta)^{-1/6})^2$$
$$-\beta(1-\sqrt{c}) + (n-1)\frac{(1-\sqrt{c})^2}{\sqrt{c}}\Bigg),$$

hold for $1 \leq j \leq n$.

Still, we write the limits from [11, Theorem 1] that are good when α or n is small. These limits are given by

$$x_{n,n}(\alpha) > \frac{2n^2 + n(\alpha-1) + 2(\alpha+1) - 2(n-1)\sqrt{n^2+(n+2)(\alpha+1)}}{n+2}$$

and

$$x_{n,1}(\alpha) < \frac{2n^2 + n(\alpha-1) + 2(\alpha+1) + 2(n-1)\sqrt{n^2+(n+2)(\alpha+1)}}{n+2}.$$

Therefore we obtain

Corollary 4.3. *Let $n \in \mathbb{N}$, $n \geq 2$, $0 < c < 1$ and $\beta > 0$. Then*

$$\frac{\sqrt{c}}{1-c}\left(\frac{2n^2 + n(\beta-2) + 2\beta - 2(n-1)\sqrt{n^2+(n+2)\beta}}{n+2} - \beta(1-\sqrt{c})\right) \tag{4.3}$$
$$\leq m_{n,j}(\beta, c) \leq \frac{\sqrt{c}}{1-c}\Bigg(\frac{2n^2 + n(\beta-2) + 2\beta + 2(n-1)\sqrt{n^2+(n+2)\beta}}{n+2}$$
$$-\beta(1-\sqrt{c}) + (n-1)\frac{(1-\sqrt{c})^2}{\sqrt{c}}\Bigg), \tag{4.4}$$

hold for $1 \leq j \leq n$.

We illustrate our results with figures for $n = 5$. The Figure 4.1 shows the zeros of $M_5(x; \beta, 0.7)$ in relation to β, in continuous line, and the corresponding limits from Corollary 4.3, in dashed line. In the same way, the Figure 4.2 shows the zeros of $M_5(x; 10, c)$ in relation to c and the corresponding limits from Corollary 4.3.

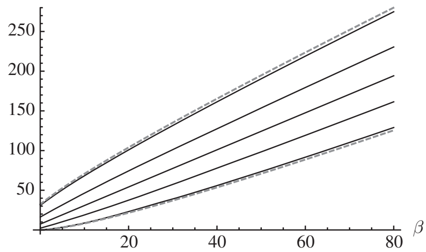

FIGURE 4.1. Zeros of $M_5(x;\beta,0.7)$ with limits from Corollary 4.3.

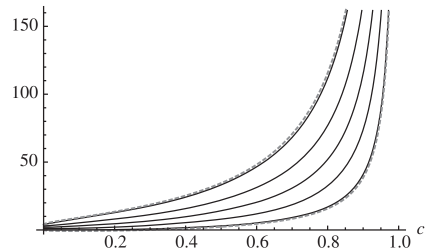

FIGURE 4.2. Zeros of $M_5(x;10,c)$ with limits from Corollary 4.3.

Krasikov and Zarkh [26, Theorem 7] proved that

$$m_{n,n}(\beta,c) > \mu_1 + \frac{3c^{1/6}\mu_1^{1/3}(\mu_1+\beta)^{1/3}}{2^{2/3}(1-c)^{1/3}n^{1/6}(n+\beta)^{1/6}}, \tag{4.5}$$

provided that $n < \beta c/(1-c)$, and
(4.6)

$$m_{n,1}(\beta,c) < \mu_2 - \begin{cases} \dfrac{3c^{1/6}\mu_2^{1/3}(\mu_2+\beta)^{1/3}}{2^{2/3}(1-c)^{1/3}n^{1/6}(n+\beta)^{1/6}}, & \mu_2 \le \mu_1 + \sqrt{\mu_1(\mu_1+\beta)}, \\ \dfrac{3c^{1/3}(\sqrt{\mu_1}+\sqrt{\mu_1+\beta})^{2/3}}{(1-c)^{2/3}}, & \mu_2 > \mu_1 + \sqrt{\mu_1(\mu_1+\beta)}, \end{cases}$$

where $\mu_1 = (\sqrt{n} - \sqrt{c(n+\beta)})^2/(1-c)$ and $\mu_2 = (\sqrt{n} + \sqrt{c(n+\beta)})^2/(1-c)$.

Let us compare asymptotically our results with the latter from [26]. For β and c fixed, the bounds (4.5) and (4.6) do not hold for large values of n. If $n \le 6$ and $\beta \to \infty$, the bounds from Corollary 4.3 are sharper than (4.5) and (4.6), while if $n \ge 6$ the bounds (4.5) and (4.6) are sharper than those obtained in the above corollaries. For $c \to 0$, our results are better. For $c \to 1$, the lower bound from Corollary 4.1 and (4.5) are asymptotically equal. Numerical experiments show that if $c \to 1$, the upper estimates in the above corollaries, especially in Corollary 4.1, are sharper than (4.6) for most values of n and β. The numerical tables below illustrate these phenomena.

TABLE 4.1. Table with the extreme zeros of $M_5(x;\beta,0.7)$ and the limits (4.5), (4.6), (4.3) and (4.4).

β	(4.5)	(4.3)	$m_{5,5}$	$m_{5,1}$	(4.4)	(4.6)
0.1		0.0118	0.0218	31.2943	32.6834	49.5072
5	2.5826	2.9145	3.4007	50.5901	52.6473	66.7876
20	21.4951	21.6617	23.1023	100.8055	103.9001	111.1347
100	163.3832	163.4497	167.4339	329.7718	335.4455	336.1549
500	993.2798	993.3690	1003.0964	1360.7678	1372.1928	1372.3100

TABLE 4.2. Table with the extreme zeros of $M_5(x;10,c)$ and the limits (4.5), (4.6), (4.3) and (4.4).

c	(4.5)	(4.3)	$m_{5,5}$	$m_{5,1}$	(4.4)	(4.6)
0.1		-0.7950	0.0258	8.4283	9.1108	9.2950
0.3		0.0409	0.7556	17.7329	18.6418	21.4749
0.5	2.3068	2.3280	3.0000	33.1590	34.5207	40.0471
0.7	7.9671	8.2040	9.0673	68.2082	70.6912	81.4756
0.9	37.4251	38.5348	40.9496	241.7715	249.9915	286.5523

5. Kravchuk polynomials

The main results in this section are motivated by the relation

$$\lim_{N\to\infty}\sqrt{\binom{N}{n}}K_n(pN+x\sqrt{2p(1-p)N};p,N)=\frac{(-1)^nH_n(x)}{\sqrt{2^nn!\left(\frac{p}{1-p}\right)^n}} \tag{5.1}$$

between the Kravchuk polynomials with a proper argument and Hermite polynomials, (see [23, (2.10.2)]).

We provide detailed results concerning the extreme zeros of $K_n(x;p,N)$.

Theorem 5.1. *Let $n,N\in\mathbb{N}$, with $n\le N$ and $0<p<1$.*

(i) *Let $0<p\le 1/2$. Then $\{\kappa_{n,1}(p,N)-pN-(1-2p)(n-1)\}/\sqrt{N}$ is an increasing function of N while $\{\kappa_{n,1}(p,N)-pN\}/\sqrt{N-n+1}$ is a decreasing function of N. Further, $\{\kappa_{n,n}(p,N)-pN\}/\sqrt{N}$ is a decreasing function of N while $\{\kappa_{n,n}(p,N)-pN-(1-2p)(n-1)\}/\sqrt{N-n+1}$ is an increasing function of N. Moreover, the inequalities*

$$\begin{aligned}
\kappa_{n,1}(p,N) &\le pN+(1-2p)(n-1)+\sqrt{2p(1-p)N}h_{n,1},\\
\kappa_{n,1}(p,N) &\ge pN+\sqrt{2p(1-p)(N-n+1)}h_{n,1},\\
\kappa_{n,n}(p,N) &\le pN+(1-2p)(n-1)-\sqrt{2p(1-p)(N-n+1)}h_{n,1},\\
\kappa_{n,n}(p,N) &\ge pN-\sqrt{2p(1-p)N}h_{n,1},
\end{aligned}$$

hold.

(ii) *Let $1/2\le p<1$. Then $\{\kappa_{n,1}(p,N)-pN\}/\sqrt{N}$ is an increasing function of N and $\{\kappa_{n,1}(p,N)-pN+(2p-1)(n-1)\}/\sqrt{N-n+1}$ is a decreasing function of N. Further, $\{\kappa_{n,n}(p,N)-pN+(2p-1)(n-1)\}/\sqrt{N}$ is a decreasing function of N while $\{\kappa_{n,n}(p,N)-pN\}/\sqrt{N-n+1}$ is an increasing function of N. Moreover,*

$$\begin{aligned}
\kappa_{n,1}(p,N) &\le pN+\sqrt{2p(1-p)N}h_{n,1},\\
\kappa_{n,1}(p,N) &\ge pN-(2p-1)(n-1)+\sqrt{2p(1-p)(N-n+1)}h_{n,1},\\
\kappa_{n,n}(p,N) &\le pN-\sqrt{2p(1-p)(N-n+1)}h_{n,1},\\
\kappa_{n,n}(p,N) &\ge pN-(2p-1)(n-1)-\sqrt{2p(1-p)N}h_{n,1}.
\end{aligned}$$

Proof. Recall that Kravchuk's polynomials $K_n(x;p,N)$ obey the three term recurrence relation

$$\begin{aligned} -xK_n(x;p,N) &= p(N-n)K_{n+1}(x;p,N) \\ &\quad - (p(N-n)+k(1-p))K_n(x;p,N) + n(1-p)K_{n-1}(x;p,N). \end{aligned} \tag{5.2}$$

The corresponding orthonormal polynomials are $\sqrt{\binom{N}{n}\left(\frac{p}{1-p}\right)^n}K_n(x;p,N)$. Set

$$\tilde{K}_n(x;p,N) = (-1)^n\sqrt{\binom{N}{n}\left(\frac{p}{1-p}\right)^n}K_n(pN + x\sqrt{2p(1-p)N};p,N).$$

Thus, by (5.1) we have

$$\lim_{N\to\infty}\tilde{K}_n(x;p,N) = \frac{H_n(x)}{\sqrt{2^n n!}}. \tag{5.3}$$

The Jacobi matrix of the form (2.3), associated with $\tilde{K}_n(x;p,N)$ has entries

$$\tilde{a}_j(N) = \sqrt{\frac{j+1}{2}\Big(1-\frac{j}{N}\Big)} \quad \text{and} \quad \tilde{b}_j(N) = \frac{j(1-2p)}{\sqrt{2p(1-p)N}}. \tag{5.4}$$

It is easy to see that the entries (5.4) are increasing functions of N when $1/2 \le p < 1$. Then, by the Perron-Frobenius theorem, the largest zero of $\tilde{K}_n(x;p,N)$, which is $\{\kappa_{n,1}(p,N) - pN\}/\sqrt{2p(1-p)N}$, is an increasing function of N, provided $1/2 \le p < 1$. Then, by (5.3), we have $\{\kappa_{n,1}(p,N) - pN\}/\sqrt{2p(1-p)N} \le h_{n,1}$.

For n fixed we consider, for $j = 0,\ldots,n$, the polynomials

$$\tilde{K}_j\Big(x + \frac{(n-1)(1-2p)}{\sqrt{2p(1-p)N}};p,N\Big), \quad (-1)^j\tilde{K}_j(-x;p,N), \quad \text{for} \quad 0 < p \le 1/2 \tag{5.5}$$

and

$$(-1)^j\tilde{K}_j\Big(-x - \frac{(n-1)(2p-1)}{\sqrt{2p(1-p)N}};p,N\Big) \quad \text{for} \quad 1/2 \le p < 1. \tag{5.6}$$

These polynomials satisfy recurrence relations of the form (2.2) with

$$a_j(N) = \sqrt{\frac{j+1}{2}\left(1-\frac{j}{N}\right)}$$

and $b_j(N)$ is, respectively,

$$\frac{(n-1-j)(2p-1)}{\sqrt{2p(1-p)N}}, \quad \frac{j(2p-1)}{\sqrt{2p(1-p)N}} \quad \text{and} \quad \frac{(n-1-j)(1-2p)}{\sqrt{2p(1-p)N}}.$$

Since these $a_j(N)$ and $b_j(N)$ are increasing functions of N, for $0 \le j \le n-1$, then by Theorem C, the largest zeros of the polynomials (5.5) and (5.6) are increasing functions of N. This fact, together with (5.3), yields the inequalities

$$\begin{aligned} \frac{\kappa_{n,1}(p,N) - pN - (1-2p)(n-1)}{\sqrt{2p(1-p)N}} &\le h_{n,1}, \\ \frac{-\kappa_{n,n}(p,N) + pN}{\sqrt{2p(1-p)N}} &\le -h_{n,n}, \\ \frac{-\kappa_{n,n}(p,N) + pN - (2p-1)(n-1)}{\sqrt{2p(1-p)N}} &\le -h_{n,n}. \end{aligned}$$

Using the fact that the zeros of Hermite polynomials satisfy $-h_{n,n} = h_{n,1}$, we can write these inequalities as in the statement of the theorem.

For $j = 0, \ldots, n$ and n fixed, consider the polynomials

(5.7)
$$\tilde{K}_j\Big(x\sqrt{1-\frac{n-1}{N}}; p, N\Big), \quad (-1)^j \tilde{K}_j\Big(-x\sqrt{1-\frac{n-1}{N}} + \frac{(n-1)(1-2p)}{\sqrt{2p(1-p)N}}; p, N\Big),$$

for $0 < p \leq 1/2$ and

$$\tilde{K}_j\Big(x\sqrt{1-\frac{n-1}{N}} - \frac{(n-1)(2p-1)}{\sqrt{2p(1-p)N}}; p, N\Big), \quad (-1)^j \tilde{K}_j\Big(-x\sqrt{1-\frac{n-1}{N}}; p, N\Big),$$

for $1/2 \leq p < 1$. These polynomials satisfy recurrence relations of the form (2.2) with

$$a_j(N) = \sqrt{\frac{(j+1)(N-j)}{2(N-n+1)}}$$

and $b_j(N)$ is, respectively,

$$\frac{j(1-2p)}{\sqrt{2p(1-p)(N-n+1)}}, \quad \frac{(n-1-j)(1-2p)}{\sqrt{2p(1-p)(N-n+1)}},$$

$$\frac{(n-1-j)(2p-1)}{\sqrt{2p(1-p)(N-n+1)}} \quad \text{and} \quad \frac{k(2p-1)}{\sqrt{2p(1-p)(N-n+1)}}.$$

Since these $a_j(N)$ and $b_j(N)$, for $0 \leq j \leq n-1$, are the entries of the Jacobi matrix $n \times n$ of the form (2.3) and are decreasing functions of N, then by Theorem C, the largest zeros of the polynomials (5.7) are decreasing functions of N.

The limit (5.3) together with the fact that the largest zeros of the polynomials (5.7) are decreasing functions of N imply the inequalities

$$\begin{aligned}
\frac{\kappa_{n,1}(p,N) - pN}{\sqrt{2p(1-p)(N-n+1)}} &\geq h_{n,1}, \\
\frac{-\kappa_{n,n}(p,N) + pN - (n-1)(1-2p)}{\sqrt{2p(1-p)(N-n+1)}} &\geq -h_{n,n}, \\
\frac{\kappa_{n,1}(p,N) - pN + (n-1)(2p-1)}{\sqrt{2p(1-p)(N-n+1)}} &\geq h_{n,1}, \\
\frac{-\kappa_{n,n}(p,N) + pN}{\sqrt{2p(1-p)(N-n+1)}} &\geq -h_{n,n}.
\end{aligned}$$

By the fact that $-h_{n,n} = h_{n,1}$ we can write these inequalities as the inequalities of the Theorem 5.1. □

We obtain sharp bounds for the zeros of Kravchuk polynomials using those for the largest zero of Hermite polynomial. The limit (3.5) from [11, Theorem 2] yields

Corollary 5.1. *Let $n, N \in \mathbb{N}$, with $n \leq N$, and $0 < p < 1$.*

(i) *If* $0 < p \leq 1/2$, *then*

$$\kappa_{n,1}(p,N) \leq pN + (1-2p)(n-1) + \sqrt{\frac{2p(1-p)N(n^2 - \frac{3}{2}n + 2 + (n-2)\sqrt{n^2+n+4})}{n+4}}, \tag{5.8}$$

$$\kappa_{n,n}(p,N) \geq pN - \sqrt{\frac{2p(1-p)N(n^2 - \frac{3}{2}n + 2 + (n-2)\sqrt{n^2+n+4})}{n+4}}. \tag{5.9}$$

(ii) *If* $1/2 \leq p < 1$, *then*

$$\kappa_{n,1}(p,N) \leq pN + \sqrt{\frac{2p(1-p)N(n^2 - \frac{3}{2}n + 2 + (n-2)\sqrt{n^2+n+4})}{n+4}}, \tag{5.10}$$

$$\kappa_{n,n}(p,N) \geq pN - (2p-1)(n-1) - \sqrt{\frac{2p(1-p)N(n^2 - \frac{3}{2}n + 2 + (n-2)\sqrt{n^2+n+4})}{n+4}}. \tag{5.11}$$

The limits (3.9), due to Szegő's [32, (6.32.6)], imply

Corollary 5.2. *Let* $n, N \in \mathbb{N}$, *with* $n \leq N$, *and* $0 < p < 1$.

(i) *If* $0 < p \leq 1/2$, *then*

$$\begin{aligned} \kappa_{n,1}(p,N) &\leq pN + (1-2p)(n-1) \\ &\quad + \sqrt{2p(1-p)N}((2n+1)^{1/2} - 1.85575\,(2n+1)^{-1/6}), \\ \kappa_{n,n}(p,N) &\geq pN - \sqrt{2p(1-p)N}((2n+1)^{1/2} - 1.85575\,(2n+1)^{-1/6}). \end{aligned}$$

(ii) *If* $1/2 \leq p < 1$, *then*

$$\begin{aligned} \kappa_{n,1}(p,N) &\leq pN + \sqrt{2p(1-p)N}((2n+1)^{1/2} - 1.85575\,(2n+1)^{-1/6}), \\ \kappa_{n,n}(p,N) &\geq pN - (2p-1)(n-1) \\ &\quad - \sqrt{2p(1-p)N}((2n+1)^{1/2} - 1.85575\,(2n+1)^{-1/6}). \end{aligned}$$

In the following, we show figures for $n = 5$ with the limits from Corollary 5.1. As throughout this paper the zeros are in continuous line and the limits in dashed line.

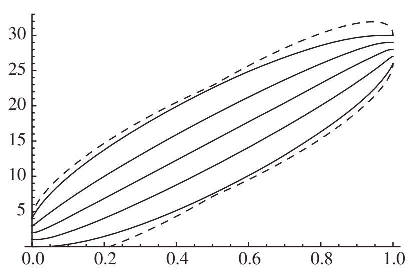

FIGURE 5.1. Zeros of $K_5(x; p, 30)$ with the limits from Corollary 5.1.

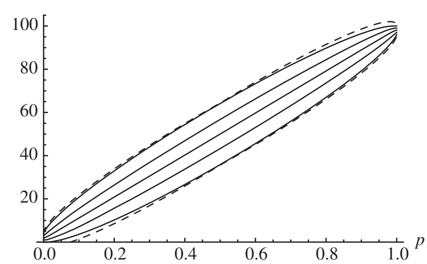

FIGURE 5.2. Zeros of $K_5(x; p, 100)$ with the limits from Corollary 5.1.

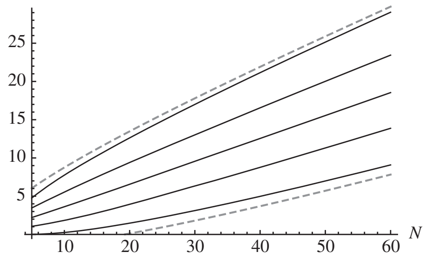

FIGURE 5.3. Zeros of $K_5(x; 0.3, N)$ with the limits from Corollary 5.1.

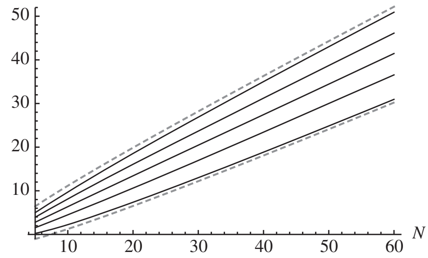

FIGURE 5.4. Zeros of $K_5(x; 0.7, N)$, with the limits from Corollary 5.1.

Now, let us compare our results with the sharper bounds found in the literature. Krasikov and Zarkh proved that [26, Theorem 6]

$$k_{n,n}(p,N) > (\sqrt{p(N-n)} + \sqrt{qn})^2 + \frac{3(pq)^{1/6}(\sqrt{p(N-n)} + \sqrt{qn})^{2/3}(N - (\sqrt{p(N-n)} + \sqrt{qn})^2)^{1/3}}{2^{2/3}n^{1/6}(N-n)^{1/6}}, \tag{5.12}$$

where $q = 1 - p$, provided that $n < pN$, and

$$k_{n,1}(p,N) < (\sqrt{p(N-n)} - \sqrt{qn})^2 - \frac{3(pq)^{1/6}(\sqrt{p(N-n)} - \sqrt{qn})^{2/3}(N - (\sqrt{p(N-n)} - \sqrt{qn})^2)^{1/3}}{2^{2/3}n^{1/6}(N-n)^{1/6}}, \tag{5.13}$$

where $q = 1 - p$, provided that $n < (1-p)N$.

Let us compare asymptotically, provided (5.12) and (5.13) hold. For a fixed value of p when N is large enough the bounds from Corollary 5.2 are sharper than (5.12) and (5.13); we remark that for $n \le 15$ the bounds from Corollary 5.1 are the sharpest when $N \to \infty$. When $p \to 0$ or $p \to 1$, the bounds from Corollaries 5.1 and 5.2 are sharper than (5.12) and (5.13) provided the latter hold. The numerical tables below confirm these observations.

TABLE 5.1. Table of the extreme zeros of $K_5(x; p, 30)$ and the limits (5.12), (5.13) and limits from Corollary 5.1.

p	(5.12)	Cor. 5.1	$\kappa_{5,5}$	$\kappa_{5,1}$	Cor. 5.1	(5.13)
0.0001		-0.1514	7.1×10^{-15}	4.0129	4.1592	4.3045
0.3	2.5796	1.8035	3.1147	17.0013	17.7965	17.5276
0.5	6.9333	7.1479	7.4661	22.5339	22.8521	23.0667
0.7	12.4724	12.2035	12.9987	26.8853	28.1965	27.4204
0.9999	25.6955	25.8408	25.9871	30.0000	30.1540	

TABLE 5.2. Table of the extreme zeros of $K_5(x; 0.3, N)$ and the limits (5.12), (5.13), (5.9) and (5.8).

N	(5.12)	(5.9)	$\kappa_{5,5}$	$\kappa_{5,1}$	(5.8)	(5.13)
5		-1.4380	0.0075	4.7614	6.0380	
10		-1.1549	0.2593	7.7528	8.7549	8.2300
50	6.4055	5.7093	6.9848	25.1477	25.8907	25.7871
100	17.3400	16.8610	18.1043	44.0398	44.7390	44.8992
500	120.1616	120.6203	121.8568	180.2960	180.9797	182.1080

TABLE 5.3. Table of the extreme zeros of $K_5(x; 0.7, N)$ and the limits (5.12), (5.13), (5.11) and (5.10).

N	(5.12)	(5.11)	$\kappa_{5,5}$	$\kappa_{5,1}$	(5.10)	(5.13)
5		-1.0380	0.2386	4.9925	6.4380	
10	1.7700	1.2451	2.2472	9.7407	11.1549	
50	24.2129	24.1093	24.8523	43.0152	44.2907	43.5945
100	55.1008	55.2610	55.9602	81.8957	83.1390	82.6600
500	317.8920	319.0203	319.7040	378.1432	379.3797	379.8384

TABLE 5.4. Table of the extreme zeros of $K_5(x; 0.5, N)$ and the limits (5.12), (5.13), (5.11) and (5.10).

N	(5.12)	(5.11)	$\kappa_{5,5}$	$\kappa_{5,1}$	(5.10)	(5.13)
5		-0.7056	0.0650	4.9350	5.7056	
10		0.4666	1.0000	9.0000	9.5334	
50	14.4814	14.8630	15.1226	34.8774	35.1370	35.5186
100	34.9994	35.6642	35.8725	64.1275	64.3358	65.0006
500	216.2194	217.9441	218.1282	281.8718	282.0559	283.7806

6. HAHN POLYNOMIALS

The results for the zeros of Hahn polynomials are motivated by the limit relation with the Jacobi polynomials [23, (2.5.1)]

$$\lim_{N\to\infty} Q_n(Nx; \alpha, \beta, N) = \frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)}.$$

Theorem 6.1. *Let n and N be positive integers with $n \le N$ and the real parameters α and β satisfy $\alpha, \beta > -1$. Then*

$$\frac{q_{n,1}(\alpha,\beta,N) + (1+\alpha)/2}{N + (\alpha+\beta+2)/2} \quad \text{and} \quad \frac{-q_{n,n}(\alpha,\beta,N) - (1+\alpha)/2}{N + (\alpha+\beta+2)/2} + \frac{n(n+\alpha)}{2n+\alpha+\beta}$$

are increasing functions of N. Moreover, the inequalities

$$q_{n,1}(\alpha,\beta,N) \le \Big(N + \frac{\alpha+\beta+2}{2}\Big)\Big(\frac{1 - x_{n,n}(\alpha,\beta)}{2}\Big) - \frac{1+\alpha}{2} \tag{6.1}$$

and

$$q_{n,n}(\alpha,\beta,N) \geq \Big(N + \frac{\alpha+\beta+2}{2}\Big)\Big(\frac{1-x_{n,1}(\alpha,\beta)}{2}\Big) - \frac{1+\alpha}{2} \tag{6.2}$$

hold.

Proof. The polynomials $Q_k(x;\alpha,\beta,N)$ satisfy the three term recurrence relation

$$\begin{aligned} &-x\, Q_k(x;\alpha,\beta,N) \\ &\quad = A_k Q_{k+1}(x;\alpha,\beta,N) - (A_k + C_k)Q_k(x;\alpha,\beta,N) + C_k Q_{k-1}(x;\alpha,\beta,N), \end{aligned} \tag{6.3}$$

where

$$A_k = \frac{(k+\alpha+\beta+1)(k+\alpha+1)(N-k)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)},$$
$$C_k = \frac{k(k+\alpha+\beta+N+1)(k+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)}.$$

Set

$$\tilde{Q}_k(x;\alpha,\beta,N) = (-1)^k Q_k\left(\Big(N + \frac{\alpha+\beta+2}{2}\Big)x - \frac{1+\alpha}{2};\alpha,\beta,N\right).$$

Then the above limit relation immediately yields

$$\lim_{N\to\infty} \tilde{Q}_k(x;\alpha,\beta,N) = \frac{(-1)^k P_k^{(\alpha,\beta)}(1-2x)}{P_k^{(\alpha,\beta)}(1)}. \tag{6.4}$$

Transforming (6.3) into a recurrence relation for the orthonormal polynomials $\tilde{Q}_k(x;\alpha,\beta,N)$, we see that the latter satisfy (2.2) with

$$\tilde{a}_k(N) = \frac{\sqrt{A_k C_{k+1}}}{N + (\alpha+\beta+2)/2}, \quad \tilde{b}_k(N) = \frac{A_k + C_k + (1+\alpha)/2}{N + (\alpha+\beta+2)/2}.$$

Observe that $\tilde{a}_k(N)$ and $\tilde{b}_k(N)$ are nonnegative for $k = 0,\dots,n-1$. Furthermore, the derivatives of $\tilde{a}_k(N)$ with respect to N are nonnegative, while $\tilde{b}_k(N)$ do not depend on N, so that their derivatives with respect to N vanish. Then, according to Theorem C, the largest zero of $\tilde{Q}_n(x;\alpha,\beta,N)$ is an increasing function of N.

The largest zero of $\tilde{Q}_n(x;\alpha,\beta,N)$ is $\{q_{n,1}(\alpha,\beta,N)+(1+\alpha)/2\}/\{N+(\alpha+\beta+2)/2\}$ and the largest zero of $P_n^{(\alpha,\beta)}(1-2x)$ is $\{1-x_{n,n}(\alpha,\beta)\}/2$. Then, by the preceding result and by the limit (6.4) we obtain

$$\frac{q_{n,1}(\alpha,\beta,N) + (1+\alpha)/2}{N + (\alpha+\beta+2)/2} \leq \frac{1-x_{n,n}(\alpha,\beta)}{2}.$$

This inequality implies the upper bound in the theorem.

Now, for n fixed, we work with the sequence of polynomials $\hat{Q}_k(x;\alpha,\beta,N)$, $k = 0,\dots,n-1$, where

$$\hat{Q}_k(x;\alpha,\beta,N) = Q_k\left(-\Big(N + \frac{\alpha+\beta+2}{2}\Big)\Big(x - \frac{n(n+\alpha)}{2n+\alpha+\beta}\Big) - \frac{1+\alpha}{2};\alpha,\beta,N\right).$$

Observe that the transformation in the argument depends on n. Then the following limit relation

$$\lim_{N\to\infty} \hat{Q}_k^{(\alpha,\beta)}(x;N) = \frac{P_k^{(\alpha,\beta)}(1 + 2x - \frac{2n(n+\alpha)}{2n+\alpha+\beta})}{P_k^{(\alpha,\beta)}(1)} \tag{6.5}$$

holds. Again, straightforward manipulations in the recurrence relation for Hahn polynomials imply that the entries of the Jacobi matrix of the orthonormal polynomials, corresponding $\hat{Q}_k(x;\alpha,\beta,N)$ are

$$\hat{a}_k = \sqrt{\frac{4(k+1)(1+k+\alpha)(1+k+\beta)(N-k)(2+k+N+\alpha+\beta)}{(1+2k+\alpha+\beta)(2+2k+\alpha+\beta)^2(3+2k+\alpha+\beta)(2N+2+\alpha+\beta)^2}}$$

and

$$\hat{b}_k = \frac{k(k+\alpha)}{(2k+\alpha+\beta)} + \frac{(n-k-1)(k(2n+\alpha+\beta)+(1+a)(n+\alpha+\beta)+n(1+\beta))}{(2n+\alpha+\beta)(2+2k+\alpha+\beta)},$$

for $k=0,\dots,n-1$.

We have that $\hat{a}_k = \hat{a}_k(N)$ and $\hat{b}_k$ are nonnegative for $n \geq 2$. Furthermore, $\hat{b}_k$ do not depend of N and the derivatives of $\hat{a}_k(N)$ with respect to N are nonnegative. Then, by the Perron-Frobenius theorem, the largest zero of $\hat{Q}_n^{(\alpha,\beta)}(x;N)$ is an increasing function of N.

The largest zero of $\hat{Q}_n(x;\alpha,\beta,N)$ is

$$\frac{-q_{n,n}(\alpha,\beta,N)-(1+\alpha)/2}{N+(\alpha+\beta+2)/2} + \frac{n(n+\alpha)}{2n+\alpha+\beta}$$

and the largest zero of $P_n^{(\alpha,\beta)}\left(1+2x-\frac{2n(n+\alpha)}{2n+\alpha+\beta}\right)$ is

$$\frac{x_{n,1}(\alpha,\beta)-1}{2} + \frac{n(n+\alpha)}{2n+\alpha+\beta}.$$

Then the preceding result and by the limit (6.5) imply

$$\frac{-q_{n,n}(\alpha,\beta,N)-(1+\alpha)/2}{N+(\alpha+\beta+2)/2} + \frac{n(n+\alpha)}{2n+\alpha+\beta} \leq \frac{x_{n,1}(\alpha,\beta)-1}{2} + \frac{n(n+\alpha)}{2n+\alpha+\beta}.$$

This inequality yields the upper bound in the statement of the theorem. □

We use two recent results on limits of zeros of Jacobi polynomials. The first one was obtained in [11, Theorem 1] and states that

$$\begin{aligned} x_{n,n}(\alpha,\beta) &\geq \frac{B-4(n-1)\sqrt{\Delta}}{A}, \\ x_{n,1}(\alpha,\beta) &\leq \frac{B+4(n-1)\sqrt{\Delta}}{A}, \end{aligned}$$

where

$$\begin{aligned} B &= (\beta-\alpha)((\alpha+\beta+6)n+2(\alpha+\beta)), \\ A &= (2n+\alpha+\beta)(n(2n+\alpha+\beta)+2(2+\alpha+\beta)), \\ \Delta &= n^2(n+\alpha+\beta+1)^2+(\alpha+1)(\beta+1)(n^2+(\alpha+\beta+4)n+2(\alpha+\beta)). \end{aligned}$$

Thus, we obtain

Corollary 6.1. *Let $n, N \in \mathbb{N}$, with $n \leq N$, $\alpha,\beta > -1$ and let A, B and Δ be defined as above. Then*

$$q_{n,n}(\alpha,\beta,N) \geq \left(N+\frac{\alpha+\beta+2}{2}\right)\left(\frac{A-B-4(n-1)\sqrt{\Delta}}{2A}\right) - \frac{1+\alpha}{2} \tag{6.6}$$

and

$$q_{n,1}(\alpha,\beta,N) \le \left(N + \frac{\alpha+\beta+2}{2}\right)\left(\frac{A - B + 4(n-1)\sqrt{\Delta}}{2A}\right) - \frac{1+\alpha}{2}. \tag{6.7}$$

We employ the following limits for the zeros of the Jacobi polynomials, obtained by Krasikov [25, Theorem 2]:

$$\begin{aligned} x_{n,n}(\alpha,\beta) &\ge C + 3\frac{(1-C^2)^{2/3}}{(2R)^{1/3}}, \\ x_{n,1}(\alpha,\beta) &\le D - 3\frac{(1-D^2)^{2/3}}{(2R)^{1/3}} + \frac{4(\alpha-\beta)(\alpha+\beta+2)}{((2n+\alpha+\beta+1)^2+2\alpha+2\beta+3)^{3/2}}, \end{aligned}$$

where

$$\begin{aligned} R &= \sqrt{((2n+\alpha+\beta+1)^2 - (\alpha-\beta)^2 + 2\alpha+2\beta+3)(4n^2+4n(\alpha+\beta+1))}, \\ C &= -(R + (\alpha-\beta)(\alpha+\beta+2))((2n+\alpha+\beta+1)^2 + 2\alpha+2\beta+3)^{-1}, \\ D &= (R - (\alpha-\beta)(\alpha+\beta+2))((2n+\alpha+\beta+1)^2 + 2\alpha+2\beta+3)^{-1}. \end{aligned}$$

These yield:

Corollary 6.2. *Let $n, N \in \mathbb{N}$, with $n \le N$ and $\alpha,\beta > -1$. Let C, D and R be defined as above. Then,*

$$\begin{aligned} q_{n,1}(\alpha,\beta,N) &\le \left(N + \frac{\alpha+\beta+2}{2}\right)\left(\frac{2R^{1/3} - 2CR^{1/3} - 3(1-C^2)^{2/3}}{2(2R)^{1/3}}\right) - \frac{1+\alpha}{2}, \\ q_{n,n}(\alpha,\beta,N) &\ge \left(N + \frac{\alpha+\beta+2}{2}\right)\left(\frac{2R^{1/3} - 2DR^{1/3} + 3(1-D^2)^{2/3}}{2(2R)^{1/3}}\right. \\ &\qquad \left. - \frac{4(\alpha-\beta)(\alpha+\beta+2)}{2((2n+\alpha+\beta+1)^2+2\alpha+2\beta+3)^{3/2}}\right) - \frac{1+\alpha}{2}. \end{aligned}$$

The figures below illustrate our results.

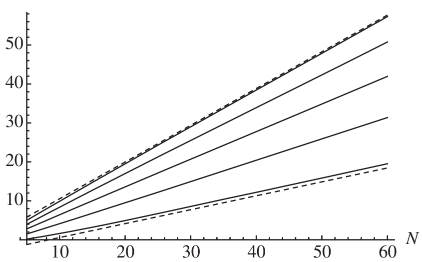

FIGURE 6.1. Zeros of $Q_5(x; 10, 2, N)$ with the limits of the Corollary 6.1.

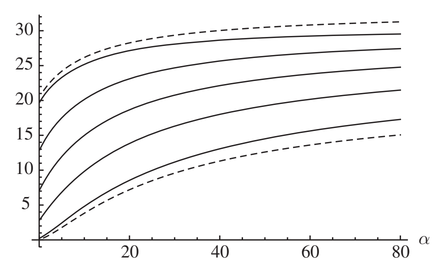

FIGURE 6.2. Zeros of $Q_5(x; \alpha, 10, 30)$ with the limits from Corollary 6.1.

Finally, we analyse these results, comparing them with the limits obtained by Krasikov and Zarkh [26, Theorem 10] who proved that, if

$$\begin{aligned}
V_1 &= \sqrt{n(n+\alpha+\beta+1)+(1+\alpha)(1+\beta)},\\
V_2 &= \sqrt{n(n+\alpha+\beta+1)((n+\alpha+\beta+2)N-n(n+\alpha+\beta+1))},\\
U_1 &= N((n+\alpha+\beta+2)-N)-2n(n+\alpha+\beta+1),\\
U_2 &= (n+\alpha+\beta+2)((n+\alpha+\beta+2)-N)+2n(n+\alpha+\beta+1),
\end{aligned}$$

then

$$\begin{aligned}
q_{n,n}(\alpha,\beta,N) &< \frac{S_1-2V_1V_2}{S_2}\\
&\quad -\frac{3}{2S_2}\left(\frac{((\alpha-\beta)V_2-U_1V_1)^2((\alpha-\beta)V_2-U_2V_1)^2}{V_1V_2S_2}\right)
\end{aligned} \tag{6.8}$$

provided that $n(n+\alpha+\beta+1)<(1+\beta)N$, and

$$\begin{aligned}
q_{n,1}(\alpha,\beta,N) &> \frac{S_1-2V_1V_2}{S_2}\\
&\quad +\frac{3}{2S_2}\left(\frac{((\alpha-\beta)V_2+U_1V_1)^2((\alpha-\beta)V_2+U_2V_1)^2}{5V_1V_2S_2}\right)
\end{aligned} \tag{6.9}$$

provided that $n\geq 5$ and $n(n+\alpha+\beta+1)<(1+\alpha)N$.

Obviously, these limits are valid only when, roughly speaking, $n<\sqrt{(1+\beta)N}$ or $n<\sqrt{(1+\alpha)N}$.

The results in the above corollaries are better than (6.8) and (6.9) when N is large enough. In particular, in the case $\alpha=\beta=0$ which corresponds with the discrete Chebyshev polynomials in the terminology of [32] or Gram polynomials in the terminology of [9]. When $\alpha\to\infty$ the lower limit (6.8) is not applicable because the restriction $n(n+\alpha+\beta+1)<(1+\beta)N$ is not satisfied and the upper limit (6.9) is better than the one given in (6.7) only for a very restricted part of the (n,β) plane. Similarly, when $\beta\to\infty$, the restriction $n(n+\alpha+\beta+1)<(1+\beta)N$ fails to hold, so that the lower limit (6.8) cannot be used. In this case the upper limit (6.6) is better than (6.8) for almost all values of n and α. The tables below exemplify some of these questions.

TABLE 6.1. Table of the extreme zeros of $Q_5(x;10,2,N)$ and the limits (6.8), (6.9), (6.6) and (6.7).

N	(6.8)	(6.6)	$q_{5,5}$	$q_{5,1}$	(6.7)	(6.9)
5		-1.2126	0.1659	4.9975	5.8174	
10	0.4624	0.5738	1.5604	9.9130	10.5330	
50	10.7916	14.8650	15.8455	47.8746	48.2575	48.9034
100	24.6288	32.7290	34.2895	94.9150	95.4133	96.6926
500	136.4521	175.6412	182.5365	470.6930	472.6592	478.9188

TABLE 6.2. Table of the extreme zeros of $Q_5(x;\alpha,10,30)$ and the limits (6.8), (6.9), (6.6) and (6.7).

α	(6.8)	(6.6)	$q_{5,5}$	$q_{5,1}$	(6.7)	(6.9)
-0.5		0.0142	0.0911	19.0850	19.8198	
5		1.7727	2.4417	23.2988	24.1896	
10	2.6382	3.8297	4.8068	25.1932	26.1703	26.4774
50	11.3974	12.6026	14.5032	29.0000	30.4896	30.0000
200	18.8195	18.4064	21.3024	29.9415	32.1985	

TABLE 6.3. Table of the extreme zeros of $Q_5(x;10,\beta,30)$ and the limits (6.8), (6.9), (6.6) and (6.7).

β	(6.8)	(6.6)	$q_{5,5}$	$q_{5,1}$	(6.7)	(6.9)
-0.5	7.1672	10.1802	10.9150	29.9089	29.9858	
5	4.0120	5.8104	6.7012	27.5583	28.2273	28.5616
10	2.6382	3.8297	4.8068	25.1932	26.1703	26.4774
50		-0.4896	1.0000	15.4968	17.3974	
200		-2.1985	0.0585	8.6976	11.5936	

TABLE 6.4. Table of the extreme zeros of Gram or discrete Chebyshev polynomials $Q_5(x;0,0,N)$ and the limits (6.8), (6.9), (6.6) and (6.7).

N	(6.8)	(6.6)	$q_{5,5}$	$q_{5,1}$	(6.7)	(6.9)
5		-0.2325	0.0087	4.9913	5.2325	
10		-0.0096	0.1440	9.8560	10.0096	
50	0.4889	1.7736	1.9203	48.0797	48.2264	49.2099
100	1.4649	4.0026	4.2520	95.7480	95.9974	97.7767
500	9.2312	21.8346	23.0048	476.9952	478.1654	486.7627

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