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One-sided and two-sided Green's functions

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Abstract

We discuss the one-sided Green's function, associated with an initial value problem and the two-sided Green's function related to a boundary value problem. We present a specific calculation associated with a differential equation with constant coefficients. For both problems, we also present the Laplace integral transform as another methodology to calculate these Green's functions and conclude which is the most convenient one. An incursion in the so-called fractional Green's function is also presented. As an example, we discuss the isotropic harmonic oscillator.

1 Introduction

There are several methods to discuss a second-order linear partial differential equation. Among them we mention the simplest one, the method of separation of variables, and the method of integral transforms, particularly the Laplace transform, which is in many cases most convenient [1].

On the other hand, after the method of separation of variables, we get the following general second-order linear ordinary differential equation:

$$a_1(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + a_2(x)\frac{\mathrm{d}}{\mathrm{d}x}y(x) + a_3(x)y(x) = 0$$

on the interval $a < x < b$, whose corresponding nonhomogeneous one is given by

$$a_1(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + a_2(x)\frac{\mathrm{d}}{\mathrm{d}x}y(x) + a_3(x)y(x) = F(x),$$

where $F(x)$ is a forcing term. Assuming that $a_1(x)$ is a continuously differentiable positive function on this interval and that $a_2(x)$ and $a_3(x)$ are continuous functions, we can write the above ordinary differential equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right] + q(x)y(x) = f(x), \tag{1}$$

the so-called self-adjoint form known also as an ordinary differential equation in the Sturm-Liouville form. In this equation, $p(x)$ and $q(x)$ are continuous functions that are related to the coefficients $a_1(x)$, $a_2(x)$ and $a_3(x)$. The nonhomogeneous term, $f(x)$, is also related to $F(x)$ [1].

The methodology of the Laplace integral transform is adequate to discuss the one-sided Green's function[a] because the initial conditions, in general, are given in terms of the own function and the first derivative. A simple question arises when we discuss the two-sided Green's function associated with a problem involving boundary conditions, *i.e.*, is the

Laplace transform methodology convenient to discuss this problem? The answer depends on the sort of problem we are studying, as we will see in the following sections.

On the other hand, the fractional harmonic oscillator was discussed in a series of papers by Narahari *et al.* [2–5] where they presented the dynamic of the fractional harmonic oscillator, including also the damping, and by Tofighi [6] who discusses the intrinsic damping.

In this paper we discuss Eq. (1) associated with an initial value problem and a boundary value problem. In both cases, we present two methodologies, the Laplace integral transform and the Green's function methodology. After that we conclude which methodology is the most convenient one. We sum the paper up presenting the corresponding fractional case where we discuss the Green's function associated with the fractional harmonic oscillator. Finally, we present our concluding remarks.

2 One-sided Green's function

To solve the self-adjoint differential equation, we introduce the so-called one-sided Green's function, also called influence function, a two-parameter function, denoted by $\mathcal{G}_i(x|\xi)$, which describes the influence of a disturbance (also known as impulse) on the value of $y(x)$ at a point x, concentrated at the point ξ. If we know the Green's function, as we construct below, the solution of an initial value problem composed of the self-adjoint differential equation and the initial conditions can be written as follows:

$$y(x) = \int_a^x \mathcal{G}_i(x|\xi) f(\xi)\,\mathrm{d}\xi, \tag{2}$$

where $y(a) = 0 = y'(a)$.

For a fixed value of ξ, the one-sided Green's function is the solution of the corresponding homogeneous initial value problem, *i.e.*, for $x > \xi$ we have the homogeneous ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{G}_i(x|\xi)\right] + q(x)\mathcal{G}_i(x|\xi) = 0$$

and the initial conditions

$$\mathcal{G}_i(\xi|\xi) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{G}_i(x|\xi)\Big|_{x=\xi} = \frac{1}{p(\xi)}.$$

2.1 Constant coefficients

As a particular case, we consider the problem involving an ordinary differential equation with constant coefficients, which can represent a problem related to the classical harmonic oscillator and transmission lines, for example,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + 2b\frac{\mathrm{d}}{\mathrm{d}x}y(x) + \left(a^2 + b^2\right)y(x) = f(x), \tag{3}$$

where a and b are positive constants, and the homogeneous initial conditions are given by $y(0) = 0 = y'(0)$.

First of all, we write the ordinary differential equation in the corresponding self-adjoint ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{2bx}\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right]+\mathrm{e}^{2bx}\left(a^2+b^2\right)y(x)=\mathrm{e}^{2bx}f(x),$$

where we identify $p(x)\equiv\exp(2bx)$.

Thus, the one-sided Green's function, denoted by $\mathcal{G}_i(x|\xi)$, satisfies the following homogeneous ordinary differential equation for ξ fixed:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{2bx}\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{G}_i(x|\xi)\right]+\mathrm{e}^{2bx}\left(a^2+b^2\right)\mathcal{G}_i(x|\xi)=0$$

and the initial conditions

$$\mathcal{G}_i(\xi|\xi)=0\quad\text{and}\quad\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{G}_i(x|\xi)\Big|_{x=\xi}=\exp(-2b\xi).$$

Two linearly independent solutions of the homogeneous ordinary differential equation are given by

$$y_1(x)=\frac{\mathrm{e}^{-bx}}{a}\sin ax\quad\text{and}\quad y_2(x)=\frac{\mathrm{e}^{-bx}}{a}\cos ax$$

which furnishes for the general solution (the Green's function)

$$\mathcal{G}_i(x|\xi)=\frac{A(\xi)}{a}\mathrm{e}^{-bx}\sin ax+\frac{B(\xi)}{a}\mathrm{e}^{-bx}\cos ax,$$

where $A(\xi)$ and $B(\xi)$ must be calculated by means of the initial conditions, *i.e.*, by the following system:

$$\begin{aligned}&A(\xi)\sin a\xi+B(\xi)\cos a\xi=0,\\&\left(-\frac{b}{a}\sin a\xi+\cos a\xi\right)A(\xi)+\left(-\frac{b}{a}\cos a\xi-\sin a\xi\right)B(\xi)=\exp(-b\xi).\end{aligned}$$

Solving the system above, we obtain

$$\begin{aligned}&A(\xi)=\cos a\xi\exp(-b\xi),\\&-B(\xi)=\sin a\xi\exp(-b\xi)\end{aligned}$$

which furnish for the one-sided Green's function, for $x>\xi$,

$$\mathcal{G}_i(x|\xi)=\frac{\exp[-b(x+\xi)]}{a}\sin\bigl[a(x-\xi)\bigr]\tag{4}$$

and, as we know, satisfies the property $\mathcal{G}_i(x|\xi)=-\mathcal{G}_i(\xi|x)$.

Finally, the solution of our initial value problem is given by

$$y(x)=\frac{1}{a}\int_0^x\mathrm{e}^{-b(x-\xi)}\sin\bigl[a(x-\xi)\bigr]f(\xi)\,\mathrm{d}\xi.\tag{5}$$

2.2 Laplace transform

Another way to calculate this one-sided Green's function is by means of the methodology of Laplace transform. Multiplying Eq. (3) by the kernel of Laplace transform, $\exp(-sx)$ with $\mathrm{Re}(s) > 0$ and integrating from zero to infinity, we have

$$s^2F(s) - sy(0) - y'(0) + 2b\big[sF(s) - y(0)\big] + \big(a^2 + b^2\big)F(s) = G(s),$$

where

$$F(s) = \int_0^\infty \mathrm{e}^{-sx} y(x)\,\mathrm{d}x \quad \text{and} \quad G(s) = \int_0^\infty \mathrm{e}^{-sx} f(x)\,\mathrm{d}x.$$

Using the initial conditions, we obtain an algebraic equation whose solution is given by

$$F(s) = \frac{G(s)}{(s+b)^2 + a^2}.$$

By means of the convolution product and the relation involving the inverse Laplace transform

$$\mathfrak{L}^{-1}\left[\frac{1}{(s+b)^2 + a^2}\right] = \frac{\exp(-bx)}{a}\sin ax,$$

we get

$$y(x) = \frac{1}{a}\int_0^x \mathrm{e}^{-b(x-\xi)}\sin\big[a(x-\xi)\big]f(\xi)\,\mathrm{d}\xi$$

which is exactly Eq. (5).

3 Two-sided Green's function

In the case that we have a two-point boundary value problem, *i.e.*, when the boundary conditions are fixed on the extremes of the interval, we have a two-sided Green's function, also called Green's function, only.

We consider the second-order ordinary differential equation in the Sturm-Liouville form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}y(x)\right] + q(x)y(x) = -f(x)$$

for $x_0 < x < x_1$ and the boundary conditions $y(x_0) = 0 = y(x_1)$, and comparing this problem with the corresponding one-sided Green's function, we have put $-f(x)$ in the place of $f(x)$ for convenience only.

The two-sided Green's function, denoted by $\mathcal{G}_b(x|\xi)$ for each ξ, satisfies the following problem consisting of homogeneous ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{G}_b(x|\xi)\right] + q(x)\mathcal{G}_b(x|\xi) = 0$$

for $x \neq \xi$ and the homogeneous boundary conditions

$$\mathcal{G}_b(x_0|\xi) = 0 = \mathcal{G}_b(x_1|\xi).$$

This Green's function must satisfy also the continuity

$$\mathcal{G}_b(\xi^+|\xi) = \mathcal{G}_b(\xi^-|\xi)$$

and a jump discontinuity on the first derivative [1]

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{G}_b(x|\xi)\Big|_{x=\xi^+} - \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{G}_b(x|\xi)\Big|_{x=\xi^-} = -\frac{1}{p(\xi)}.$$

The solution $y(x)$ can be interrelated as a displacement and be given by

$$y(x) = \int_{x_0}^{x_1} \mathcal{G}_b(x|\xi)f(\xi)\,\mathrm{d}\xi,$$

where $f(\xi)$ is a force per unit length. $\mathcal{G}_b(x|\xi)$ is the displacement at x due to a force of unit magnitude concentrated at ξ. In this case, we have $\mathcal{G}_b(x|\xi) = \mathcal{G}_b(\xi|x)$, the so-called reciprocity law.

3.1 Constant coefficients

As an example, we discuss the following ordinary differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y(x) + 2b\frac{\mathrm{d}}{\mathrm{d}x}y(x) + \left(a^2 + b^2\right)y(x) = -f(x)$$

with the homogeneous boundary conditions $y(x_0) = 0 = y(x_1)$.

First, we construct the corresponding Green's function, as we have seen before. Two linearly independent solutions of the homogeneous ordinary differential equation are

$$y_1(x) = \frac{\mathrm{e}^{-bx}}{a}\sin ax \quad \text{and} \quad y_2(x) = \frac{\mathrm{e}^{-bx}}{a}\cos ax.$$

Imposing that $y_1(x)$ satisfies $y_1(x_0) = 0$, we can write

$$y_1(x) = \frac{A(\xi)}{a}\mathrm{e}^{-bx}\sin\left[a(x - x_0)\right].$$

On the other hand, to satisfy $y_2(x_1) = 0$, we put

$$y_2(x) = \frac{B(\xi)}{a}\mathrm{e}^{-bx}\sin\left[a(x - x_1)\right],$$

where $A(\xi)$ and $B(\xi)$ must be determined.

Using the continuity at $x = \xi$ and the jump discontinuity of the first derivative, we obtain a system of two algebraic equations involving $A(\xi)$ and $B(\xi)$ whose solution is

$$A(\xi) = \frac{\sin[a(x_1 - \xi)]}{\sin[a(x_1 - x_0)]} \quad \text{and} \quad B(\xi) = -\frac{\sin[a(\xi - x_0)]}{\sin[a(x_1 - x_0)]}.$$

Thus, the Green's function is given by the following expression:

$$\mathcal{G}_b(x|\xi) = \frac{\exp[-b(x+\xi)]}{a\sin[a(x_1 - x_0)]}\begin{cases}\sin[a(x - x_0)]\sin[a(x_1 - \xi)], & x_0 < x < \xi,\\ \sin[a(\xi - x_0)]\sin[a(x_1 - x)], & \xi < x < x_1.\end{cases}$$

Using this Green's function, the solution of the boundary value problem can be written as follows:

$$y(x) = \int_{x_0}^{x_1} \mathcal{G}_b(x|\xi) f(\xi)\,\mathrm{d}\xi \cdot$$

3.2 Laplace transform

As we have already said, the Laplace transform converts the differential equation with constant coefficients into an algebraic equation whose solution is

$$F(s) = \frac{sy(0) + y'(0)}{(s+b)^2 + a^2} - \frac{G(s)}{(s+b)^2 + a^2},$$

where $F(s)$ and $G(s)$ are the corresponding Laplace transforms of $y(x)$ and $f(x)$, respectively.

Using the inverse Laplace transform and the convolution product, we get

$$\begin{aligned} y(x) &= y(0)\mathrm{e}^{-bx}\frac{\cos ax}{a} + y'(0)\mathrm{e}^{-bx}\frac{\sin ax}{a} \\ &\quad - \frac{1}{a}\int_0^x \mathrm{e}^{-b(x-\xi)} \sin\bigl[a(x-\xi)\bigr] f(\xi)\,\mathrm{d}\xi. \end{aligned}$$

Thus, substituting the boundary conditions, we obtain two algebraic equations, a system involving $y(0)$ and $y'(0)$. Solving this algebraic system, we obtain

$$\begin{aligned} y(x) &= \frac{-1/a}{\sin[a(x_1 - x_0)]}\int_{x_0}^{x} \mathrm{e}^{-b(x+t)} \sin\bigl[a(x_1 - x)\bigr]\sin\bigl[a(x_0 - t)\bigr] f(t)\,\mathrm{d}t \\ &\quad - \frac{1/a}{\sin[a(x_1 - x_0)]}\int_{x}^{x_1} \mathrm{e}^{-b(x+t)} \sin\bigl[a(x_1 - t)\bigr]\sin\bigl[a(x_0 - x)\bigr] f(t)\,\mathrm{d}t \end{aligned}$$

which can be rewritten as follows:

$$y(x) = \int_{x_0}^{x_1} \mathcal{G}_b(x|t) f(t)\,\mathrm{d}t,$$

where the Green's function is given by

$$\mathcal{G}_b(t|x) = \frac{\exp[-b(x+t)]}{a\sin[a(x_1 - x_0)]}\begin{cases} \sin[a(x_1 - x)]\sin[a(t - x_0)], & x_0 < t < x, \\ \sin[a(x_1 - t)]\sin[a(x - x_0)], & x < t < x_1, \end{cases}$$

which is the same expression as that obtained in Section 3.1.

At this point we conclude that for a problem involving initial conditions, the Laplace integral transform is more convenient since $y(0)$ and $y'(0)$ are known. On the other hand, *i.e.*, for a problem involving boundary conditions, the Sturm-Liouville, as opposed to the Laplace integral transform, is more convenient in the sense that the calculation is much more simple.

4 Fractional Green's function

Fractional calculus is one of the most accurate tools to refine the description of natural phenomena. The usual way to use this tool is to replace the integer-order derivatives of the partial differential equation that describes one specific phenomenon by a derivative of non-integer order. For many expected reasons, the solution of a fractional partial differential equation used to be much more complicated than the solution of the corresponding integer-order partial differential equation.

On the other hand, many important results and generalizations were obtained using this procedure in several areas such as fluid flow, diffuse transport, electrical networks, probability, biomathematics and others [7–12]. Here, as a generalization to the integer case, we present a calculation associated with the so-called fractional one-sided and two-sided Green's function relative to the fractional differential equation with constant coefficients, *i.e.*, we obtain the fractional Green's function for the fractional differential equation whose coefficients are constants. We discuss the problem by means of the Laplace integral transform, and as an application, we present explicitly the Green's function associated with the fractional harmonic oscillator.

4.1 Fractional one-sided Green's function

Let a, b and c be real constants. We present the solution of the fractional differential equation

$$a\mathrm{D}_x^{\alpha} y(x) + b\mathrm{D}_x^{\beta} y(x) + cy(x) = f(x), \tag{6}$$

where $1 < \alpha \leq 2$ and $0 < \beta \leq 1$ and the fractional derivatives are taken in the Caputo sense [13]. We also consider $y(0) = 0 = y'(0)$ as the initial conditions. In the case where $\alpha = 2$ and $\beta = 1$, we recover the results associated with the integer case, and taking $a = 0$ and $b \neq 0$, we recover the equation associated with the fractional relaxor-oscillator as discussed in [5].

Introducing the Laplace transform and using the initial conditions, we obtain an algebraic equation whose solution can be written as follows:

$$\mathfrak{L}\big[y(x)\big] = \frac{F(s)}{as^{\alpha} + bs^{\beta} + c},$$

where $F(s)$ is the Laplace transform of the $f(x)$. This expression can be manipulated, using the geometric series, to obtain

$$\mathfrak{L}\big[y(x)\big] = \frac{F(s)}{a} \sum_{k=0}^{\infty} \left(-\frac{b}{a}\right)^k \frac{s^{\beta k}}{(s^{\alpha} + c/a)^{k+1}}$$

which is valid for $|s^{-\lambda} c/a| < 1$.

Using the Laplace transform of the generalized Mittag-Leffler function and its corresponding inverse [13],

$$\mathfrak{L}\big[t^{\mu-1} E_{\nu,\mu}^{\rho}\big(\lambda t^{\nu}\big)\big] = \frac{s^{\nu\rho-\mu}}{(s^{\nu} - \lambda)^{\rho}}$$

with $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\mu) > 0$, $\lambda \in \mathbb{C}$ and $|\lambda s^{-\nu}| < 1$, we have

$$y(x) = \frac{1}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k \int_0^x f(\xi)(x-\xi)^{(\alpha-\beta)k+\alpha-1}E^{k+1}_{\alpha,(\alpha-\beta)k+\alpha}\left[-\frac{c}{a}(x-\xi)^\alpha\right]\mathrm{d}\xi$$

which is the solution of the fractional differential equation.

Thus, the one-sided fractional Green's function can be written as follows:

$$\begin{aligned}\mathcal{G}(x|\xi) &= \frac{1}{a}(x-\xi)^{\alpha-1}E_{\alpha,\alpha}\left[-\frac{c}{a}(x-\xi)^\alpha\right] \\ &\quad + \frac{1}{a}\sum_{k=1}^{\infty}\left(-\frac{b}{a}\right)^k (x-\xi)^{(\alpha-\beta)k+\alpha-1}E^{k+1}_{\alpha,(\alpha-\beta)k+\alpha}\left[-\frac{c}{a}(x-\xi)^\alpha\right]. \qquad (7)\end{aligned}$$

Taking $b = 0$, $a = 1$ and $c = \omega^2$ in Eq. (7), we get

$$\mathcal{G}(x|\xi) = (x-\xi)^{\alpha-1}E_{\alpha,\alpha}\left[-\omega^2(x-\xi)^\alpha\right]$$

which is the fractional one-sided Green's function associated with the fractional harmonic oscillator. The one-sided Green's function associated with the classical harmonic oscillator is recovered by introducing $\alpha = 2$ in the last equation, *i.e.*,

$$\mathcal{G}(x|\xi) = (x-\xi)E_{2,2}\left[-\omega^2(x-\xi)^2\right], \quad x > \xi,$$

which can also be written as follows:

$$\mathcal{G}(x|\xi) = \frac{1}{\omega}\sin\left[\omega(x-\xi)\right],$$

which is the same expression as that obtained in Eq. (4) in the case $b = 0$.

To conclude this section, we plot, in Figure 1, a graphic $t = x - \xi \times t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha)$ for particular values of the parameter α to compare with the sine function. This graphic is plotted using the program (MatLab R2009a). For reader interested in the Mittag-Leffler

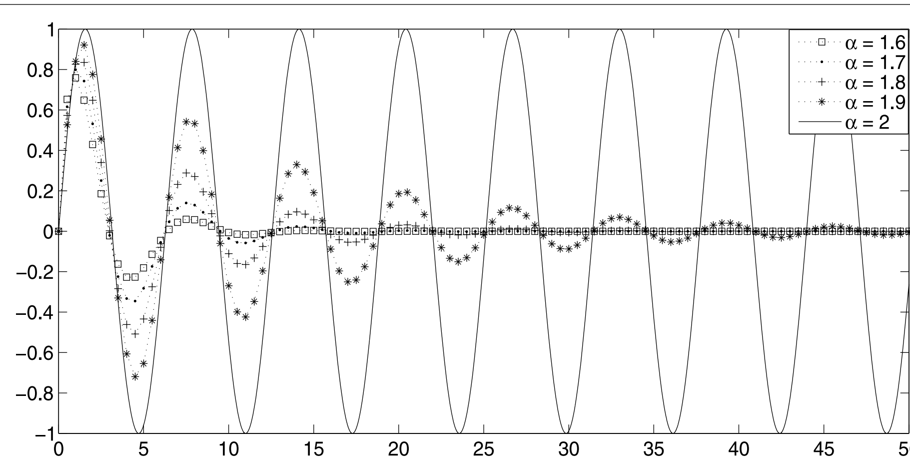

Figure 1 Graphics for $f(t) = t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha) \times t$ with $t \equiv x - \xi > 0$.

function, we suggest a recent nice book [14] and the paper [15] particularly regarding the asymptotic algebraic behavior of this function.

4.2 Fractional two-sided Green's function

Let a, b and c be real constants. We present the solution of the fractional differential equation, Eq. (6), with the homogeneous boundary conditions, $y(x_0) = 0 = y(x_1)$.

By means of the Laplace transform, we can write

$$\mathfrak{L}\big[y(x)\big] = \frac{F(s) + s^{\alpha-1}y(0) + s^{\alpha-2}y'(0)}{as^\alpha + bs^\beta + c},$$

where $F(s)$ is the Laplace transform of the $f(x)$. This algebraic equation can be manipulated as follows:

$$\begin{aligned}\mathfrak{L}\big[y(x)\big] &= \frac{F(s)}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k \frac{s^{\beta k}}{(s^\alpha + c/a)^{k+1}} \\ &\quad + \frac{y(0)}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k \frac{s^{\beta k+\alpha-1}}{(s^\alpha + c/a)^{k+1}} + \frac{y'(0)}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k \frac{s^{\beta k+\alpha-2}}{(s^\alpha + c/a)^{k+1}}.\end{aligned}$$

From the inverse Laplace transform and the convolution theorem, we get

$$\begin{aligned}y(x) &= \frac{1}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k \int_0^x f(x-\xi)\xi^{(\alpha-\beta)k+\alpha-1}E^{k+1}_{\alpha,(\alpha-\beta)k+\alpha}\left(-\frac{c}{a}\xi^\alpha\right)\mathrm{d}\xi \\ &\quad + \frac{y(0)}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k \mathfrak{L}^{-1}\left[\frac{s^{\beta k+\alpha-1}}{(s^\alpha + c/a)^{k+1}}\right] + \frac{y'(0)}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k \mathfrak{L}^{-1}\left[\frac{s^{\beta k+\alpha-2}}{(s^\alpha + c/a)^{k+1}}\right]\end{aligned}$$

which can be rewritten in the following way:

$$\begin{aligned}y(x) &= \frac{1}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k \int_0^x f(\xi)(x-\xi)^{(\alpha-\beta)k+\alpha-1}E^{k+1}_{\alpha,(\alpha-\beta)k+\alpha}\left[-\frac{c}{a}(x-\xi)^\alpha\right]\mathrm{d}\xi \\ &\quad + \frac{y(0)}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k x^{(\alpha-\beta)k}E^{k+1}_{\alpha,(\alpha-\beta)k+1}\left(-\frac{c}{a}x^\alpha\right) \\ &\quad + \frac{y'(0)}{a}\sum_{k=0}^{\infty}\left(-\frac{b}{a}\right)^k x^{(\alpha-\beta)k+1}E^{k+1}_{\alpha,(\alpha-\beta)k+2}\left(-\frac{c}{a}x^\alpha\right).\end{aligned}$$

As we have already said in Section 3, to explicitly calculate the solution, we must substitute the homogeneous boundary conditions in the last equation and determine the $y(0)$ and $y'(0)$. Finally, we can get the respective fractional two-sided Green's function. We have shown this is a hard calculation. To obtain the results associated with the fractional harmonic oscillator, we introduce $b = 0$, $a = 1$ and $c = \omega^2$ in the last equation. Remember that to obtain the respective Green's function, we substitute $f(x)$ to the corresponding delta function.

5 Concluding remarks

We have presented and discussed the so-called one-sided and two-sided Green's function to study, respectively, an initial value problem and a boundary value problem. Besides that, we studied the same problems by means of the Laplace transform methodology in order to conclude which methodology was most accurate for each problem. We also obtained the fractional generalization of the one-sided and two-sided Green's function in terms of the generalized Mittag-Leffler function.

We conclude that for the initial value problem, the Laplace integral transform methodology is more convenient; on the other hand, for the boundary value problem, the two-sided Green's function provides a much more simple calculation. It is important to note that in the present manuscript we did not consider the problem involving physical dimensions. This problem has been discussed in a recent paper by Inizan [16].

A natural continuation of this work would be to study the problems involving partial differential equations with non-constant coefficients and their fractional versions, which could provide a better description of the phenomena related to those equations. Study in this direction is upcoming.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RFC had the idea of the work. ECO had the idea to consider the fractional version. AOC had the fractional harmonic oscillator, idea. All authors have made the calculations involved. All authors read and approved the final manuscript.

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Endnote
[a] A recent historical review about George Green can be found in ref. [17].

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