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Exploring the properties of the pure spinor b ghost

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Resumo

Esta tese é baseada em parte do meu trabalho de doutoramento e tem como objetivo apresentar uma análise detalhada de algumas propriedades recém abordadas do fantasma b composto no formalismo de espinores puros. Primeiramente será feita uma revisão dos formalismos mínimo e não-mínimo. Em seguida, será apresentada a construção do fantasma b passo a passo, incluindo correções quânticas. Por fim, serão estudadas em detalhes suas propriedades fundamentais, que vão desde a nilpotência até a definição de um possível conjugado, o fantasma c.

Palavras-chave: Teoria de supercordas; formalismo de espinores puros; fantasma b composto.

Área do conhecimento: Partículas elementares e campos.

Abstract

This thesis is based in part of my work during the Ph.D. and aims to present a detailed analysis of some newly studied properties of the composite non-minimal pure spinor b ghost. First, a review of the minimal and non-minimal pure spinor formalisms will be presented. Then, the construction of the non-minimal b ghost will be done step-by-step, including quantum corrections. Finally, some of its fundamental properties will be studied in detail, ranging from nilpotency until the definition of a possible canonical conjugate, the c ghost.

Keywords: Superstring theory; pure spinor formalism; composite b ghost.

Contents

Chapter 1

Introduction

The super Poincaré covariant quantization of the superstring was achieved in the year 2000, with the development of the pure spinor formalism [1]. It can be described as an *ad hoc* approach to the quantization of the string, in the sense that the gauge fixing procedure that provides the BRST-like approach has not yet been discovered. Allowing explicit Lorentz covariant computations in the elegant language of $D = 10$ superfields, the formalism gathers together the advantages of the other traditional superstring descriptions (RNS and Green-Schwarz) without most of their restrictions. The Green-Schwarz formulation [2] cannot be quantized in a Lorentz covariant manner, only in the (semi) light cone gauge, and the introduction of the interaction-point operators makes hard even the construction of vertex operators. As for the Ramond-Neveu-Schwarz (RNS) string [3, 4], amplitude computations require the sum over spin structures (implied by world-sheet supersymmetry and related to GSO projection), integration over super *moduli* space and the introduction of picture-changing and spin operators [5], lacking explicit space-time supersymmetry (the algebra closes up to a picture changing operation) and making the Ramond sector hard to deal with. Another advantage of the pure spinor formalism is the possibility of dealing with curved backgrounds that have Ramond-Ramond flux, feature that is most welcome concerning the latest developments of the AdS/CFT correspondence.

Despite all the good features of the pure spinor approach, its world-sheet origin is still unknown, as reparametrization symmetry is hidden. It is a well known fact that in gauge fixing such symmetry, a (b, c) system rises as the ghost-antighost pair. The c ghost is a conformal weight -1 field, as it comes from the general coordinate transformation parameter, and the b ghost, the conjugate of c, is a conformal weight $+2$ field. Concerning amplitudes, the fundamental objects of study in quantum strings, the c ghost appears at tree and 1-loop level. In these world-sheet topologies (respectively, the sphere and the torus), the conformal Killing symmetries can be removed by fixing some vertex positions. In the pure spinor formalism, given the amplitude prescriptions described in [1, 6, 7], a possible c ghost plays no role at all.

For the b ghost the story is different. In a BRST-like description, b ghost insertions lie in the heart of the BRST invariance of string loop amplitudes. The fundamental property is $\{Q, b\} = T$, where T is the energy-momentum tensor (since the BRST charge has ghost number $+1$, the b ghost must have ghost number -1). Combined with the Beltrami differentials, this property induces only a surface contribution in the *moduli* space integration. Therefore, understanding the properties of the b ghost is a fundamental task in providing a better understanding of the formalism and potential developments.

In the minimal pure spinor formalism, where the available ghost variables are the pure spinor λ^α and its conjugate ω_α, the b ghost is based upon a complicated chain of operators and can be implemented only in a picture raised manner [6], as there are no suitable ghost number -1 fields.

With the addition of the ghost fields $\left(\overline{\lambda}_\alpha, r_\alpha\right)$ and their conjugates $(\overline{\omega}^\alpha, s^\alpha)$, the so-called non-minimal pure spinor formalism enables a much simpler construction of the b ghost [7]. More than that, the theory can be interpreted as a twisted $\mathcal{N} = 2$ $\hat{c} = 3$ topological string, where the BRST charge and the b ghost are the fermionic generators, while the ghost number current and the energy-momentum tensor are the bosonic ones. This fact allowed the covariant computation of multiloop superstring amplitudes without picture changing operators, making the super Poincaré symmetry explicit in all the steps.

However, the general properties of b are non-trivial, as it is also non-trivially composed. Its rich structure has been explored over the years [8, 9, 10, 11, 12, 13], but it is not yet completely understood.

This thesis will present several of these properties in detail and it will be organized as follows. Chapter 2 reviews the basic concepts of the pure spinor formalism, introducing the pure spinor ghosts and some of their properties, and preparing the ground for the subsequent study on the b ghost. Chapter 3 will start with the classical definition of the b ghost, intuitively presenting its structure. Then, the quantum definition will be dealt with, including the ordering prescription and possible quantum corrections to the classical definition. Finally, some of the fundamental properties of b will be derived in details, including: primary field condition (which is clearly related to the ordering prescription); cohomology (an exclusion criterion will be established for the non-trivial cohomology of b); the topological string perspective; non-uniqueness (some deformations on the definition of the currents will be analyzed and constrained in order to not spoil the topological string algebra); and a candidate for the c ghost (the formalism does not have a natural conformal weight -1 field to act as the conjugate of b and its existence is intriguing, requiring an unusual construction). Chapter 4 summarizes the content of the thesis, presenting some research perspectives on the subject. Appendix A includes the conventions and some properties of the gamma matrices, while appendix B detailedly shows the free-field parametrization of the pure spinor constraints, including the computation of the complete set of OPE's related to the pure spinor fields.

Chapter 2

Review of the Pure Spinor Formalism

The goal of this chapter is to present the field content of the pure spinor formalism and some of its fundamental structures, introducing the basic tools that will be needed for the construction of the b ghost.

To motivate the pure spinor construction, it might be useful to explain that Siegel's proposal [14] for the covariant quantization of the superstring was to replace the fermionic constraints of the Green-Schwarz formalism, that includes both first and second class ones, by another set of constraints where the fundamental piece is

$$\begin{aligned} d_\alpha &\equiv p_\alpha - \frac{1}{2}\partial X^m \left(\theta\gamma_m\right)_\alpha - \frac{1}{8}\left(\theta\gamma^m\partial\theta\right)\left(\theta\gamma_m\right)_\alpha \\ &= 0. \end{aligned}$$

Here, X^m is the usual bosonic worldsheet scalar, with $m = 0, \ldots, 9$ the spacetime vector index, and $(\theta^\alpha, p_\alpha)$ is the fermionic conjugate pair, with $\alpha = 1, \ldots, 16$ the spacetime (chiral) spinor index. Note that d_α is the generator of the supersymmetric derivative. Although this approach succeeded for the superparticle [15], a suitable set of first class constraints was never found for the superstring. However incomplete, it also led Siegel to conjecture the integrated massless vertex to be

$$V_{\text{Siegel}} = \frac{1}{2\pi i}\oint \left\{\Pi^m A_m + \partial\theta^\alpha A_\alpha + d_\alpha W^\alpha\right\},$$

where A_m, A_α and W^α are the usual super-Yang-Mills fields (more details below).

As will be presented in the next sections, the pure spinor BRST-charge has a simple form, where the supersymmetric operator d_α appears multiplied by a constrained bosonic spinor ghost λ^α, enabling a natural superfield description of the cohomology. Supporting this structure, the massless vertex of the pure spinor approach is shown to be very similar to Siegel's proposal, but with a correction that comes from the ghost sector.

2.1 Matter fields

The matter content of the pure spinor superstring is described by the Green-Schwarz-Siegel action

$$S_m = \frac{1}{2\pi} \int d^2z \left(\frac{1}{2} \partial X^m \overline{\partial} X_m + p_\beta \overline{\partial} \theta^\beta \right), \tag{2.1}$$

with free field propagators given by:

$$X^m(z) X^n(y) \sim -\eta^{mn} \ln |z-y|^2, \tag{2.2a}$$
$$p_\alpha(z) \theta^\beta(y) \sim \sim \frac{\delta_\alpha^\beta}{z-y}. \tag{2.2b}$$

Observe that the fundamental length of the string is being fixed through $\alpha' = 2$ but it can be easily recovered by dimensional analysis.

The supersymmetric charge will be defined to be

$$q_\alpha \equiv \oint \left[p_\alpha + \frac{1}{2} \partial X^m (\theta\gamma_m)_\alpha + \frac{1}{24} (\theta\gamma_m\partial\theta)(\theta\gamma_m)_\alpha \right] \tag{2.3}$$

and the supersymmetry algebra is directly reproduced

$$\{q_\alpha, q_\beta\} = -i\gamma^m_{\alpha\beta} P_m. \tag{2.4}$$

Here, P_m is the usual momentum operator, with

$$P_m \equiv i \oint \partial X_m, \quad \text{and} \quad [P_m, X^n] = -i\delta^n_m.$$

The construction of the supersymmetric invariants follows:

$$\Pi^m = \partial X^m + \frac{1}{2} (\theta\gamma^m\partial\theta), \tag{2.5a}$$

$$d_\alpha = p_\alpha - \frac{1}{2}\partial X^m (\theta\gamma_m)_\alpha - \frac{1}{8}(\theta\gamma^m\partial\theta)(\theta\gamma_m)_\alpha . \quad (2.5b)$$

It is straightforward to obtain the OPE's among them through the fundamental ones given in (2.2):

$$\Pi^m(z)\,\Pi^n(y) \sim -\frac{\eta^{mn}}{(z-y)^2}, \quad (2.6a)$$

$$d_\alpha(z)\,\Pi^m(y) \sim \frac{\gamma^m_{\alpha\beta}\partial\theta^\beta}{(z-y)}, \quad (2.6b)$$

$$d_\alpha(z)\,d_\beta(y) \sim -\frac{\gamma^m_{\alpha\beta}\Pi_m}{(z-y)}. \quad (2.6c)$$

It will be useful also to present also the action of the the operators of (2.5) on a superfield $F(X,\theta)$:

$$\Pi_m(z)\,F(X,\theta;y) \sim -\frac{\partial_m F}{(z-y)}, \quad (2.7)$$

$$d_\alpha(z)\,F(X,\theta;y) \sim \frac{D_\alpha F}{(z-y)}. \quad (2.8)$$

Here,

$$D_\alpha \equiv \partial_\alpha + \frac{1}{2}\left(\gamma^m_{\alpha\beta}\theta^\beta\right)\partial_m, \quad (2.9)$$

with $\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}$, $\partial_m = \frac{\partial}{\partial X^m}$. Note that

$$\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m, \quad (2.10)$$

as expected from the OPE (2.6c).

Up to this point, the basic blocks of the matter sector have been introduced without mentioning the original superstring description. By that it is meant the two fundamental objects that arise naturally in the Green-Schwarz formulation: the Virasoro constraint and the fermionic constraints (related to κ-symmetry).

The Virasoro constraint ($\frac{1}{2}\Pi^m\Pi_m + d_\alpha\partial\theta^\alpha = 0$) is identified with the energy-momentum tensor T_{matter} of the theory, given by

$$T_{\text{matter}} = -\frac{1}{2}\partial X^m\partial X_m - p_\alpha\partial\theta^\alpha,$$

and the most direct piece of information one may extract from it is the central charge of the matter sector. Computing the OPE of T_{matter} with itself one obtains:

$$T_{\text{matter}}(z)\,T_{\text{matter}}(y) \sim -\frac{11}{(z-y)^4} + \frac{2T_{\text{matter}}}{(z-y)^2} + \frac{\partial T_{\text{matter}}}{(z-y)}. \tag{2.11}$$

Hence, the free matter action yields a negative central charge (-22).

The fermionic constraints ($d_\alpha = 0$) are a bit more subtle, as they satisfy (2.6c). This implies that only half of the components of d_α are first class constraints. Although one can use a gauge fixing procedure to solve them (light cone gauge), it breaks explicitly Lorentz symmetry because there is no simple way to covariantly split the first and second class constraints mixed in.

The vanishing of the central charge in string theory is related to the vanishing of world-sheet gravitational anomalies. The way the pure spinor formalism solves the conformal anomaly and deals with the fermionic constraints that are being ignored will be explained below.

2.2 Ghost fields

The fundamental ingredient in the construction of the formalism is a bosonic ghost λ^α that, in an indirect manner, implements the κ-symmetry generators of the Green-Schwarz superstring in a BRST fashion. The first step is to define the current

$$J_{\text{BRST}} \equiv \lambda^\alpha d_\alpha, \tag{2.12}$$

and the associated charge

$$Q = \oint J_{\text{BRST}}, \tag{2.13}$$

satisfying

$$\{Q, Q\} = -\oint (\lambda\gamma^m\lambda)\,\Pi_m. \tag{2.14}$$

Now, if one imposes

$$\lambda\gamma^m\lambda = 0, \tag{2.15}$$

(the $D = 10$ pure spinor constraint), the charge defined in (2.13) is nilpotent, being a BRST-like operator built out of the d_α.

To understand a bit more these pure spinors, it is interesting to start with an unconstrained bosonic spinor pair $(\Lambda^\alpha, \Omega_\alpha)$ satisfying

$$\Lambda^\alpha(z)\,\Omega_\beta(y) \sim \frac{\delta^\alpha_\beta}{z-y}. \tag{2.16}$$

Naively, one may try to project Λ^α into a pure spinor,

$$\lambda^\alpha \stackrel{?}{=} P^\alpha_\beta \Lambda^\beta. \tag{2.17}$$

Here, $P^\alpha_\beta = \delta^\alpha_\beta + K^\alpha_\beta$ plays the role of the projector, although its form is not known yet. Now, it is possible to constrain K^α_β by some general properties of projections. The simplest one is that

$$P^\alpha_\gamma P^\gamma_\beta \ = P^\alpha_\beta \quad \Rightarrow \quad K^\alpha_\beta = -K^\alpha_\gamma K^\gamma_\beta. \tag{2.18}$$

In other words, projected subspaces are invariant under the action of the projection. When this concept is applied to the pure spinor λ^α, that is $P^\alpha_\beta \lambda^\beta = \lambda^\alpha$, one readily notes that

$$K^\alpha_\beta \lambda^\beta = 0 \quad \Rightarrow \quad K^\alpha_\beta = K^\alpha_m \left(\gamma^m \lambda\right)_\beta, \tag{2.19}$$

since the pure spinor constraint is the only information available so far. Besides, equation (2.18) implies that

$$K^\alpha_m \left(\gamma_n \lambda\right)_\alpha = K'^\alpha_n \left(\gamma_m \lambda\right)_\alpha - \eta_{mn}.$$

A possible solution for K^α_β that satisfies all these constraints is given by

$$K^\alpha_\beta = -\frac{1}{2}\frac{\left(C\gamma_m\right)^\alpha \left(\gamma^m \lambda\right)_\beta}{C \cdot \lambda}, \tag{2.20}$$

where C_α is also a pure spinor and K^α_β depends recursively on the pure spinor which is being projected, which strictly speaking make it not a projector, although enough for the present purpose.

The next step is to associate Ω_a to the conjugate ω_α of λ^α. Clearly this is not a trivial relation, as λ^α is constrained. To acknowledge this, note that an explicitly Lorentz invariant action for the ghost sector,

$$S_\lambda = \frac{1}{2\pi}\int d^2 z \left(\omega_\alpha \bar{\partial} \lambda^\alpha\right), \tag{2.21}$$

has the gauge symmetry

$$\delta_\epsilon \omega_\alpha = \epsilon_m \left(\gamma^m \lambda\right)_\alpha . \tag{2.22}$$

It is now straightforward to write down an OPE between $\lambda^\alpha = P^\alpha_\beta \Lambda^\beta$ and $\Omega_\beta \sim \omega_\beta$,

$$\lambda^\alpha (z)\, \omega_\beta (y) \sim \frac{1}{z-y}\left[\delta^\alpha_\beta - \frac{1}{2}\frac{\left(C\gamma_m\right)^\alpha \left(\gamma^m\lambda\right)_\beta}{C\cdot\lambda}\right]. \tag{2.23}$$

As a consistency check, note that ω_α has no poles with $\lambda\gamma^m\lambda$. The meaning of C_α is not clear in this derivation and in principle troublesome, since it breaks Lorentz symmetry. In fact, it is directly related to the definition and the gauge fixing of ω_α. For a simple example on that see section B.3 of the appendix.

The simplest gauge invariant quantities that can be built out of ω_α are

$$T_\lambda = -\omega\partial\lambda, \quad N^{mn} = -\tfrac{1}{2}\omega\gamma^{mn}\lambda, \quad J_\lambda = -\omega\lambda,$$

respectively, the energy-momentum tensor, the Lorentz current and the ghost number current.

Due to the non linear form of the pure spinor constraint, there might be ordering contributions in their quantum version, such as $\partial^2 \ln\left(C\lambda\right)$ for T_λ or $\partial \ln\left(C\lambda\right)$ for the ghost number current[1]. Without knowing these ordering contributions one can determine, for example, the central charge associated to the pure spinor variables, that is directly read from the quartic pole in the OPE of the energy-momentum tensor with itself,

$$T_\lambda (z)\, T_\lambda (y) \sim \frac{c_\lambda/2}{\left(z-y\right)^4} + \ldots$$

$$\Rightarrow c_\lambda = 2\left[\delta^\alpha_\beta - \frac{1}{2}\frac{\left(C\gamma_m\right)^\alpha \left(\gamma^m\lambda\right)_\beta}{C\cdot\lambda}\right]\left[\delta^\beta_\alpha - \frac{1}{2}\frac{\left(C\gamma_n\right)^\beta \left(\gamma^n\lambda\right)_\alpha}{C\cdot\lambda}\right] = 22, \tag{2.24}$$

which precisely cancels the contribution coming from the matter part of the theory, equation (2.11). More than that, this result gives a hint on the number of independent components of λ^α: the pure spinor constraint (2.15) implies that only 11 components of λ^α are independent.

[1]This approach known as Y-Formalism [16, 9]: a pure spinor variable $Y_\alpha = \frac{C_\alpha}{C\lambda}$ is defined and all the relevant quantum operators are constructed and made work fixing those ordering contributions, but it will not be discussed here.

Instead of working with the constrained variable λ^α and its messy OPE, it is possible to use only free fields. One way of doing this is through the $U(5)$ decomposition of the $SO(10)$ spinors, the original formulation of the pure spinor formalism in [1]. This construction is presented in the appendix with a free-field parametrization of the pure spinor λ^α and the derivation of most of the results below.

For the minimal pure spinor formalism, the full set of OPE's of the ghost sector is given by:

$$\begin{aligned} T_\lambda(z)\,T_\lambda(y) &\sim \tfrac{11}{(z-y)^4} + 2\tfrac{T_\lambda}{(z-y)^2} + \tfrac{\partial T_\lambda}{(z-y)}, \quad T_\lambda(z)\,\lambda^\alpha(y) \sim \tfrac{\partial\lambda^\alpha}{(z-y)}, \\ T_\lambda(z)\,J_\lambda(y) &\sim \tfrac{8}{(z-y)^3} + \tfrac{J_\lambda}{(z-y)^2} + \tfrac{\partial J_\lambda}{(z-y)}, \quad J_\lambda(z)\,\lambda^\alpha(y) \sim \tfrac{\lambda^\alpha}{(z-y)}, \\ T_\lambda(z)\,N^{mn}(y) &\sim \tfrac{N^{mn}}{(z-y)^2} + \tfrac{\partial N^{mn}}{(z-y)}, \quad N^{mn}(z)\,\lambda^\alpha(y) \sim \tfrac{1}{2}\tfrac{(\gamma^{mn}\lambda)^\alpha}{(z-y)}, \\ N^{mn}(z)\,J_\lambda(y) &\sim \text{regular}, \quad J_\lambda(z)\,J_\lambda(y) \sim -\tfrac{4}{(z-y)^2}, \end{aligned}$$

$$N^{mn}(z)\,N^{pq}(y) \sim 6\frac{\eta^{m[p}\eta^{q]n}}{(z-y)^2} + 2\frac{\eta^{m[q}N^{p]n} + \eta^{n[p}N^{q]m}}{(z-y)}.$$

Next section presents a basic introduction to the cohomology of the pure spinor superstring.

2.3 Pure spinor cohomology

Having introduced the ghost fields, it is now time to discuss the pure spinor BRST charge of equation (2.13) and its cohomology.

The first unconventional thing to be noted is that it contains more than the (would be) 8 first class constraints of the Green-Schwarz superstring, since the pure spinor λ^α has eleven independent components.

Another fact that is worth mentioning is the existence of the operator $\xi = \frac{C\theta}{C\lambda}$. Note that $\{Q, \xi\} = 1$ for any constant spinor C_α. Such an operator is potentially dangerous as it trivializes the cohomology of the BRST like charge: any BRST-closed operator $\mathcal{O}$, *i.e.*, $[Q, \mathcal{O}] = 0$, is BRST exact: $\mathcal{O} = \{Q, \xi\mathcal{O}\}$. From another point of view, the state ξ gives a hint on the space of allowed states and one way of avoiding it is prohibiting inverse powers of λ^α. In the RNS formalism, for example, the existence of ξ led to the introduction of the concept of small Hilbert space and picture changing operators [5].

Physical states will be defined to be in the ghost number 1 cohomology of (2.13) and to understand a bit more the origin of the pure spinor superstring spectrum, the massless case will be discussed in details.

The unintegrated massless vertex operator is given by

$$U_0 = \lambda^\alpha A_\alpha (X, \theta) \, . \tag{2.25}$$

As there is no negative conformal weight field available[2], A_α contains only the zero modes of the matter fields (that is only X^m and θ^α, not their derivatives).

The condition for U_0 to be in the cohomology of the BRST charge is the vanishing of

$$\{Q, U_0\} = \lambda^\alpha \lambda^\beta D_\alpha A_\beta, \tag{2.26}$$

where D_α was defined in (2.9). The Fierz decomposition of the symmetric product $\lambda^\alpha\lambda^\beta$ is given by

$$\lambda^\alpha \lambda^\beta = \frac{1}{16} (\lambda \gamma^m \lambda) \gamma_m^{\alpha\beta} + \frac{1}{5! \cdot 32} (\lambda \gamma^{mnpqr} \lambda) \gamma_{mnpqr}^{\alpha\beta}, \tag{2.27}$$

where the 3-form vanishes because γ^{mnp} is antisymmetric in the spinor indices. The first term on the right-hand side vanishes due to the pure spinor constraint. Therefore, the vanishing of (2.26) implies

$$(D\gamma_{mnpqr} A) = 0, \tag{2.28}$$

which is the linearized version of the super Yang-Mills equation of motion for the superfield A_α [17], the expected massless superstring spectrum.

As mentioned in the beginning of the chapter, the integrated version of the massless vertex operator closely resembles the one proposed by Siegel in [14] and is given by

$$V_0 = \oint \{\Pi^m A_m + \partial\theta^\alpha A_\alpha + d_\alpha W^\alpha + N^{mn} F_{mn}\} \, , \tag{2.29}$$

[2] In fact, an artificial construction has been recently proposed for a -1 conformal weight composite field ([12], section 3.6), but it is not yet fully understood.

where

$$\begin{aligned}
A_m &\equiv \frac{1}{8}\left(D_\alpha \gamma_m^{\alpha\beta} A_\beta\right), && (2.30a)\\
(\gamma_m W)_\alpha &\equiv (D_\alpha A_m - \partial_m A_\alpha), && (2.30b)\\
F_{mn} &\equiv \frac{1}{2}(\partial_m A_n - \partial_n A_m) && (2.30c)\\
&= \frac{1}{16}(\gamma_{mn})^\alpha{}_\beta D_\alpha W^\beta,
\end{aligned}$$

are the usual superfields built out of A_α. BRST-closedness of V is straightforward to demonstrate and $[Q, V_0]$ vanishes up to a surface term after using the pure spinor constraint and the equation of motion (2.28).

The massive spectrum is much harder to describe in this covariant fashion. For example, the unintegrated vertex of the first massive level is given by

$$\begin{aligned}
U_1 &= \partial\lambda^\alpha A_\alpha(X,\theta) + \lambda^\alpha \partial\theta^\beta B_{\alpha\beta}(X,\theta) + \lambda^\alpha d_\beta C_\alpha^\beta(X,\theta)\\
&\quad + \lambda^\alpha \Pi^m C_{\alpha m}(X,\theta) + \lambda^\alpha N^{mn} D_{\alpha mn}(X,\theta) + \lambda^\alpha J_\lambda E_\alpha(X,\theta). \qquad (2.31)
\end{aligned}$$

BRST-closedness will impose the equations of motion and the constraints among all the superfields present in U_1 [18]. Obviously, higher massive levels will involve more superfields and constraints making the search for a full superspace description of the spectrum almost impossible. If this is so, it could be asked how one knows that the pure spinor formalism is equivalent to the traditional superstring formalism, even at the cohomological level. This is a question that has been addressed long ago and proof that the pure spinor cohomology is equivalent to the light-cone Green-Schwarz spectrum was obtained in [19] through a complicated procedure, where the pure spinor variable was written in terms of $SO(8)$ variables, involving an infinite chain of ghost-for-ghosts. Later, the equivalence of the pure spinor spectrum with the traditional superstring formalisms was demonstrated in different ways [20, 21], involving field redefinitions and similarity transformations, but an explicit superfield description of the massive states was still lacking. Recently, a complete spectrum generating algebra was introduced, enabling a systematic description of the whole pure spinor spectrum in terms of $SO(8)$-covariant superfields [22]. In a separate work [23], a light-cone analysis of the pure spinor superstring was made, arguing that there is no other state in the pure spinor cohomology besides the ones described by the

DDF-operators[3], up to Lorentz transformations and BRST-exact contributions.

Next section will present the non-minimal variables of the pure spinor formalism. As will be discussed, these extra fields do not change the cohomology, but are fundamental ingredients in the construction of the non-minimal b ghost.

2.4 Non-minimal variables

The non minimal version of the pure spinor formalism includes a new set of ghosts, $\left(\overline{\lambda}_\alpha, r_\alpha\right)$. The former is also a pure spinor, that is

$$\overline{\lambda}\gamma^m\overline{\lambda} = 0, \tag{2.32}$$

whereas the latter is a fermionic spinor constrained through

$$\overline{\lambda}\gamma^m r = 0. \tag{2.33}$$

Both constraints imply that there are only 11 independent components in each spinor. Their conjugates are represented by $(\overline{\omega}^\alpha, s^\alpha)$ and are gauge transformed by

$$\begin{aligned} \delta_{\epsilon,\phi}\bar{\omega}^\alpha &= \epsilon^m\left(\gamma_m\bar{\lambda}\right)^\alpha + \phi^m\left(\gamma_m r\right)^\alpha, \\ \delta_\phi s^\alpha &= \phi^m\left(\gamma_m\bar{\lambda}\right)^\alpha. \end{aligned} \tag{2.34}$$

In a straight analogy with the minimal formalism, one can derive the OPE's

$$\overline{\lambda}_\beta(z)\,\overline{\omega}^\alpha(y) \sim \frac{1}{z-y}\left[\delta^\alpha_\beta - \frac{1}{2}\frac{\left(\overline{\lambda}\gamma_m\right)^\alpha\left(\gamma^m\overline{C}\right)_\beta}{\overline{\lambda}\cdot\overline{C}}\right], \tag{2.35}$$

and

$$r_\beta(z)\,s^\alpha(y) \sim \frac{1}{z-y}\left[\delta^\alpha_\beta - \frac{1}{2}\frac{\left(\overline{\lambda}\gamma_m\right)^\alpha\left(\gamma^m\overline{C}\right)_\beta}{\overline{\lambda}\cdot\overline{C}}\right]. \tag{2.36}$$

[3] The DDF operators were introduced in [24] for the bosonic string. It is important to mention that a DDF construction within the pure spinor formalism was already discussed in [25]. However, the approach of Mukhopadhyay has a important difference. The lack of an explicit expression for the DDF operators in [25], although sufficient for his purposes, makes the superfield description of the massive states incomplete and introduces BRST-exact terms in the creation/annihilation algebra that demand an extended argument for proving the validity of the construction.

Here, $\overline{C}^{\alpha}$ is a constant pure spinor. Due to (2.33), one can expect a non trivial OPE between r and $\overline{\omega}$. Indeed, it is straightforward to show that:

$$r_{\beta}(z)\,\overline{\omega}^{\alpha}(y) \sim \frac{1}{z-y}\left[\frac{1}{2}\left(r\cdot\overline{C}\right)\frac{\left(\overline{\lambda}\gamma_{m}\right)^{\alpha}\left(\gamma^{m}\overline{C}\right)_{\beta}}{\left(\overline{\lambda}\cdot\overline{C}\right)^{2}}-\frac{1}{2}\frac{\left(r\gamma_{m}\right)^{\alpha}\left(\gamma^{m}\overline{C}\right)_{\beta}}{\overline{\lambda}\cdot\overline{C}}\right]. \tag{2.37}$$

All of these relations were postulated in [9], where they were obtained requiring $\overline{\lambda}\gamma^{m}r$ and $\overline{\lambda}\gamma^{m}\overline{\lambda}$ to be regular with respect to $\overline{\omega}^{\alpha}$ and s^{α}.

There are several gauge invariant quantities that can be built out of $\overline{\omega}^{\alpha}$ and s^{α}.

$$\begin{array}{ccc} & \overline{N}^{mn}=\frac{1}{2}\left(\bar{\lambda}\gamma^{mn}\bar{\omega}-r\gamma^{mn}s\right), & \\ J_{\bar{\lambda}}=-\bar{\lambda}\bar{\omega}, & T_{\overline{\lambda}}=-\bar{\omega}\partial\bar{\lambda}-s\partial r, & \Phi=r\bar{\omega}, \\ S=\overline{\lambda}s, & S^{mn}=\frac{1}{2}\bar{\lambda}\gamma^{mn}s, & J_{r}=rs. \end{array} \tag{2.38}$$

Here, $\overline{N}^{mn}$ is the Lorentz generator, $T_{\overline{\lambda}}$ is the energy-momentum tensor, and $J_{\bar{\lambda}}$ and J_{r} are the ghost number currents. The quantum versions of the above objects are subject to ordering effects. Again, instead of working with (2.35), (2.36) and (2.37), it is much more convenient to use the free fields coming from the usual $U(5)$ decomposition. All of the ordering effects together with the relevant OPE's are given in appendix B.3. The results can be summarized as follows:

$$\begin{array}{rclrcl}
T_{\overline{\lambda}}(z)\,T_{\overline{\lambda}}(y) & \sim & 2\frac{T_{\overline{\lambda}}}{(z-y)^{2}}+\frac{\partial T_{\overline{\lambda}}}{(z-y)}, & \overline{N}^{mn}(z)\,T_{\overline{\lambda}}(y) & \sim & \frac{\overline{N}^{mn}}{(z-y)^{2}}, \\
J_{\bar{\lambda}}(z)\,T_{\overline{\lambda}}(y) & \sim & -\frac{11}{(z-y)^{3}}+\frac{\overline{J}_{\overline{\lambda}}}{(z-y)^{2}}, & S^{mn}(z)\,T_{\overline{\lambda}}(y) & \sim & \frac{S^{mn}}{(z-y)^{2}}, \\
J_{r}(z)\,T_{\overline{\lambda}}(y) & \sim & \frac{11}{(z-y)^{3}}+\frac{J_{r}}{(z-y)^{2}}, & \Phi(z)\,T_{\overline{\lambda}}(y) & \sim & \frac{\Phi}{(z-y)^{2}}, \\
\Phi(z)\,S(y) & \sim & -\frac{8}{(z-y)^{2}}-\frac{J_{\overline{\lambda}}+J_{r}}{(z-y)}, & S(z)\,T_{\overline{\lambda}}(y) & \sim & \frac{S}{(z-y)^{2}}, \\
T_{\overline{\lambda}}(z)\,r_{\alpha}(y) & \sim & \frac{\partial r_{\alpha}}{(z-y)}, & T_{\overline{\lambda}}(z)\,\overline{\lambda}_{\alpha}(y) & \sim & \frac{\partial\overline{\lambda}_{\alpha}}{(z-y)}, \\
\Phi(z)\,\overline{\lambda}_{\alpha}(y) & \sim & -\frac{r_{\alpha}}{(z-y)}, & \Phi(z)\,S^{mn}(y) & \sim & \frac{\overline{N}^{mn}}{(z-y)}, \\
\Phi(z)\,\Phi(y) & \sim & \text{regular}, & \overline{N}^{mn}(z)\,J_{\bar{\lambda}}(y) & \sim & \text{regular}, \\
\overline{N}^{mn}(z)\,\overline{N}^{pq}(y) & \sim & 2\frac{\eta^{m[q}\overline{N}^{p]n}+\eta^{n[p}\overline{N}^{q]m}}{(z-y)}, & \overline{N}^{mn}(z)\,\Phi(y) & \sim & \text{regular},
\end{array}$$

$$\begin{aligned}
J_{\bar{\lambda}}(z)\,J_{\bar{\lambda}}(y) &\sim -\tfrac{5}{(z-y)^2}, \quad J_{\bar{\lambda}}(z)\,J_r(y) \sim -\tfrac{3}{(z-y)^2},\\
\overline{N}^{mn}(z)\,J_r(y) &\sim \text{regular}, \quad \overline{N}^{mn}(z)\,S(y) \sim \text{regular},\\
\overline{N}^{mn}(z)\,\overline{\lambda}_\alpha(y) &\sim -\tfrac{1}{2}\tfrac{\left(\overline{\lambda}\gamma^{mn}\right)_\alpha}{(z-y)}, \quad J_r(z)\,J_r(y) \sim \tfrac{11}{(z-y)^2},\\
\overline{N}^{mn}(z)\,r_\alpha(y) &\sim -\tfrac{1}{2}\tfrac{(r\gamma^{mn})_\alpha}{(z-y)}, \quad J_{\bar{\lambda}}(z)\,\overline{\lambda}_\alpha(y) \sim \tfrac{\overline{\lambda}_\alpha}{(z-y)},\\
J_r(z)\,r_\alpha(y) &\sim \tfrac{r_\alpha(y)}{(z-y)}, \quad J_{\bar{\lambda}}(z)\,r_\alpha(y) \sim \text{regular},\\
J_r(z)\,\overline{\lambda}_\alpha(y) &\sim \text{regular}.
\end{aligned}$$

Note that there are no contributions to the central charge (any contribution coming from the non minimal sector would imply a conformal anomaly) and no contributions to the level of the Lorentz algebra[4].

The non-minimal ghosts enter the formalism in a very simple way, as the BRST charge is defined to be

$$Q \equiv \oint \underbrace{\left(\lambda^\alpha d_\alpha + \Phi\right)}_{J_{BRST}(z)}. \tag{2.39}$$

The same notation was used for the BRST charge in the minimal formalism, but from now on, only (2.39) will be referred to as Q. The cohomology of (2.39) is independent of $\left(\overline{\lambda}, \overline{\omega}, r, s\right)$, as can be seen from the quartet argument, and there is a state ξ that trivializes it,

$$\xi = \tfrac{\overline{\lambda}\cdot\theta}{\overline{\lambda}\cdot\lambda - r\cdot\theta}, \qquad \{Q, \xi\} = 1.$$

Since r_α and θ^α are grassmannian variables, ξ can be expanded in a finite power series in terms of $r\cdot\theta$. Besides, r_α has only 11 independent components, in such a way that

$$\xi = \frac{\overline{\lambda}\cdot\theta}{\overline{\lambda}\cdot\lambda}\sum_{n=0}^{11}\left(\frac{r\cdot\theta}{\overline{\lambda}\cdot\lambda}\right)^n.$$

Therefore, one way of avoiding the appearance of ξ is limiting the amount of inverse powers of $\overline{\lambda}\lambda$.

It is argued in [7] that the occurrence of ξ is directly related to the pure spinor integration measure in amplitude calculations, where a divergence occurs for inverse powers

[4]As pointed out in [9], there is a typo in the J_r OPE with itself in [7] and their result is confirmed here within the $U(5)$ decomposition. The quadratic pole in $\Phi(z)\,S(y)$ is also absent there.

greater than $\lambda^8\overline{\lambda}^{11}$. Concerning loop amplitudes, the obstruction is due to the fact that a genus $g > 1$ loop needs $3\,(g-1)$ b ghost insertions, increasing the divergence in $\overline{\lambda}\lambda$. In [8] a regularization scheme that overcomes this problem was developed, but its practical implementation still very difficult.

As will be shown in the next section, inverse powers of $\overline{\lambda}\lambda$ are a fundamental ingredient in the construction of the b ghost

Chapter 3

The b ghost

The role of the b ghost in string theory is very clear whenever one starts with an action that is $2D$ reparametrization invariant, as it is related to the modular description of the worldsheet topology.

In the path integral formulation, the b ghost insertions are of the form

$$B_\tau = \int d^2z \left\{ b\left(z\right) \partial_\tau g\left(z; \tau\right) \right\}, \tag{3.1}$$

where τ represents the *moduli* parameters and g is the worldsheet metric (more details can be found in [26] and references therein).

The key property of the b ghost is

$$\{Q, b\} = T, \tag{3.2}$$

where T is the full energy-momentum tensor of the theory under consideration. Together with equation (3.1), it ensures the BRST invariance of loop amplitudes up to surface terms in the *moduli* space integration. Thus, the b ghost coming from gauge fixing the reparametrization symmetry provides a natural way to build loops.

In the pure spinor formalism, however, one starts with an action in the conformal gauge, and there is no known gauge fixing procedure that accounts for the origin of the pure spinor BRST-like charge, and worldsheet reparametrization symmetry is hidden in this approach.

Therefore, in order to build an amplitude prescription in the pure spinor superstring that has the nice geometric interpretation of the traditional formalisms, it is necessary to have a field acting as a b ghost, as there is no fundamental one.

This chapter will present the construction of the b ghost in the non-minimal pure spinor formalism, together with some of its main properties, where special attention will be devoted to the most recent results.

3.1 Definition and Construction

As introduced initially in [6] for the minimal and extended in [7] to the non-minimal pure spinor formalism, the construction of the b ghost is based on a chain of operators satisfying some special relations, that will be reviewed below.

In the minimal formalism, there is a natural starting point. Since the BRST charge contains $\lambda^\alpha p_\alpha$ and part of the energy momentum tensor is given by $-p_\alpha\partial\theta^\alpha - \omega_\alpha\partial\lambda^\alpha$, one can expect that

$$b_{\rm cl} = -\omega_\alpha\partial\theta^\alpha + \cdots$$

The subscript *cl* means that only classical commutation relations are being used (no quantum ordering effects, that will be dealt with in the next subsection) and the $\ldots$ stand for possible extra contributions, as will be explained.

Although simple, this term has not the ω_α gauge invariance (in gauge fixing it, some of the $\partial\theta^\alpha$ components decouple). It can be directly seen from (2.23) that one can build a non covariant gauge invariant form of ω_α [16], given by

$$\tilde{\omega}_\alpha\,(C) = \omega_\alpha - \frac{(\omega\gamma_m C)\,(\gamma^m\lambda)_\alpha}{2C\lambda}.$$

Inserting it in the expression of $b_{\rm cl}$ above,

$$\begin{aligned} b_{\rm cl} &= -\tilde{\omega}_\alpha\partial\theta^\alpha + \ldots \\ &= -\frac{1}{4}N_{mn}\frac{(C\gamma^{mn}\partial\theta)}{C\lambda} - \frac{1}{4}J\frac{(C\partial\theta)}{C\lambda} + \ldots \end{aligned}$$

where, in the second line, the identity (A.3) was used in order to make the construction explicitly gauge invariant. The $\ldots$ stand for a term related to X^m, that closes the relation $\{Q, b_{\rm cl}\} = T$. It is easy to convince oneself that this extra part must be proportional to

$\Pi^m (\gamma_m d)_\alpha \left(\lambda^\beta\right)^{-1}$, since it provides a $(\partial X)^2$ contribution and has ghost number -1.

Now, the first link of the up-mentioned chain of operators is defined to be

$$G^\alpha = \frac{1}{2}\Pi^m (\gamma_m d)^\alpha - \frac{1}{4}N_{mn} (\gamma^{mn}\partial\theta)^\alpha - \frac{1}{4}J\partial\theta^\alpha, \tag{3.3}$$

satisfying

$$\{Q, G^\alpha\} = \lambda^\alpha (T_\lambda + T_{\text{matter}}),$$

and leading to

$$b_{\text{cl}} = \frac{C_\alpha G^\alpha}{C_\beta \lambda^\beta}.$$

Despite its origin, C_α can be any constant spinor. In the amplitude context, this form of b_{cl} is certainly dangerous due to the presence of picture changing operators proportional to $\delta(\lambda)$. This problem could be avoided in [6], where a picture raised version of the b ghost was developed, based on a complicated chain of operators $\left(H^{\alpha\beta}, K^{\alpha\beta\gamma}, L^{\alpha\beta\gamma\lambda}\right)$, that could implement the desired relation without poles in λ^α (there is also another proposal for the picture changing operators in the minimal formalism [27], where there is no such problem, but loop amplitudes, *i.e.*, amplitudes involving b ghost insertions, have not yet been calculated using them).

In the non minimal formalism, the new links appear in a much simpler way. To include the non minimal energy momentum tensor, it is worth noting the symmetric form that (λ, p) and $(r, \overline{\omega})$ appear in the BRST charge (2.39), leading to

$$\left\{Q, -s^\alpha \partial\overline{\lambda}_\alpha\right\} = T_{\overline{\lambda}}.$$

The main difference from the minimal case is the replacement of C_α by $\overline{\lambda}_\alpha$ in b_{cl}, which explains, operationally, the generation of the chain of operators, since $\left[Q, \overline{\lambda}_\alpha\right] = -r_\alpha$. Starting with

$$b_{\text{cl}} = -s^\alpha \partial\overline{\lambda}_\alpha + \frac{\overline{\lambda}_\alpha G^\alpha}{\overline{\lambda}_\beta \lambda^\beta} + \ldots$$

the non minimal BRST charge acts as

$$\{Q, b_{\text{cl}}\} = T_\lambda + T_{\overline{\lambda}} + T_{\text{matter}} - \frac{\overline{\lambda}_\alpha r_\beta}{\left(\overline{\lambda}\lambda\right)^2} \left(\lambda^\alpha G^\beta - \lambda^\beta G^\alpha\right) + \ldots$$

which leads to the introduction of an operator $H^{\alpha\beta}$ satisfying

$$\left[Q, H^{\alpha\beta}\right] = \frac{1}{2}\left(\lambda^{\alpha} G^{\beta} - \lambda^{\beta} G^{\alpha}\right).$$

That is,

$$b_{\mathrm{cl}} = -s^{\alpha}\partial\overline{\lambda}_{\alpha} + \frac{\overline{\lambda}_{\alpha} G^{\alpha}}{\overline{\lambda}_{\beta}\lambda^{\beta}} - 2!\frac{\overline{\lambda}_{\alpha} r_{\beta}}{\left(\overline{\lambda}\lambda\right)^{2}} H^{\alpha\beta} + \ldots$$

Of course, $\overline{\lambda}_{\alpha} r_{\beta}\left(\overline{\lambda}\lambda\right)^{-2}$ generates an extra piece of the chain. And that is how it works. The extra terms generated are all of the form $\overline{\lambda}\,(r)^{n}\left(\overline{\lambda}\lambda\right)^{-n-1}\mathcal{O}_{(n)}$. Naively, being able to construct all the links, the chain must stop at $n = 11$ but, in fact, it happens at $n = 3$. It can be shown that

$$b_{\mathrm{cl}} = -s^{\alpha}\partial\overline{\lambda}_{\alpha} + \frac{1}{\left(\overline{\lambda}\lambda\right)}\overline{\lambda}_{\alpha} G^{\alpha} - \frac{2!}{\left(\overline{\lambda}\lambda\right)^{2}}\overline{\lambda}_{\alpha} r_{\beta} H^{\alpha\beta} - \frac{3!}{\left(\overline{\lambda}\lambda\right)^{3}}\overline{\lambda}_{\alpha} r_{\beta} r_{\gamma} K^{\alpha\beta\gamma} + \frac{4!}{\left(\overline{\lambda}\lambda\right)^{4}}\overline{\lambda}_{\alpha} r_{\beta} r_{\gamma} r_{\lambda} L^{\alpha\beta\gamma\lambda}$$

satisfies $\{Q, b_{\mathrm{cl}}\} = T_{\lambda} + T_{\overline{\lambda}} + T_{\mathrm{matter}}$, where

$$\begin{aligned}
H^{\alpha\beta} &= \frac{1}{4\cdot 96}\gamma^{\alpha\beta}_{mnp}\left(d\gamma^{mnp}d + 24N^{mn}\Pi^{p}\right), \\
K^{\alpha\beta\gamma} &= -\frac{1}{3\cdot 96}N^{mn}\left[\gamma^{\alpha\beta}_{mnp}\left(\gamma^{p}d\right)^{\gamma} + \gamma^{\beta\gamma}_{mnp}\left(\gamma^{p}d\right)^{\alpha} + \gamma^{\gamma\alpha}_{mnp}\left(\gamma^{p}d\right)^{\beta}\right], \\
L^{\alpha\beta\gamma\lambda} &= -\frac{1}{(96)^{2}}N^{mn}N^{rs}\eta^{pq}\left[\gamma^{\alpha\beta}_{mnp}\gamma^{\gamma\lambda}_{qrs} + \gamma^{\beta\gamma}_{mnp}\gamma^{\alpha\lambda}_{qrs} + \gamma^{\gamma\alpha}_{mnp}\gamma^{\beta\lambda}_{qrs}\right],
\end{aligned}$$

$$\left[Q, H^{\alpha\beta}\right] = \lambda^{[\alpha}G^{\beta]}, \quad \left\{Q, K^{\alpha\beta\gamma}\right\} = \lambda^{[\alpha}H^{\beta\gamma]}, \quad \left[Q, L^{\alpha\beta\gamma\lambda}\right] = \lambda^{[\alpha}K^{\beta\gamma\lambda]},$$

and $\lambda^{[\alpha}L^{\beta\gamma\lambda\sigma]} = 0$ ensures the end of the chain. The demonstration of the above relations makes heavy use of the gamma matrices identities given in the appendix, from (A.3) to (A.9), together with the constraints (2.32) and (2.33).

The quantum b ghost

All the construction so far was purely classical. However, in the quantum theory, one is dealing with products of operators that diverge when approach each other in the coordinate space and an ordering prescription is needed. In this work, the ordering is implemented through:

$$\left(A,B\right)\left(y\right)\equiv\frac{1}{2\pi i}\oint\frac{dz}{\left(z-y\right)}A\left(z\right)B\left(y\right). \tag{3.4}$$

More details on this ordering prescription can be found in [28].

The *full quantum version* of the b ghost in the non minimal pure spinor formalism can be cast as

$$b=b_{-1}+b_0+b_1+b_2+b_3, \tag{3.5}$$

where

$$\begin{aligned}
b_{-1}&\equiv-s^{\alpha}\partial\overline{\lambda}_{\alpha}, & O&\equiv-\partial\left(\frac{\overline{\lambda}_{\alpha}\overline{\lambda}_{\beta}}{\left(\overline{\lambda}\lambda\right)^{2}}\right)\lambda^{\alpha}\partial\theta^{\beta},\\
b_0&\equiv\left(\frac{\overline{\lambda}_{\alpha}}{\left(\overline{\lambda}\lambda\right)},G^{\alpha}\right)+O, & G^{\alpha}&=\tfrac{1}{2}\gamma_{m}^{\alpha\beta}\left(\Pi^{m},d_{\beta}\right)-\tfrac{1}{4}N_{mn}\left(\gamma^{mn}\partial\theta\right)^{\alpha}-\tfrac{1}{4}J\partial\theta^{\alpha}+4\partial^{2}\theta^{\alpha},\\
b_1&\equiv-2!\left(\frac{\overline{\lambda}_{\alpha}r_{\beta}}{\left(\overline{\lambda}\lambda\right)^{2}},H^{\alpha\beta}\right), & H^{\alpha\beta}&=\tfrac{1}{4\cdot 96}\gamma_{mnp}^{\alpha\beta}\left(d\gamma^{mnp}d+24N^{mn}\Pi^{p}\right),\\
b_2&\equiv-3!\left(\frac{\overline{\lambda}_{\alpha}r_{\beta}r_{\gamma}}{\left(\overline{\lambda}\lambda\right)^{3}},K^{\alpha\beta\gamma}\right), & K^{\alpha\beta\gamma}&=-\tfrac{1}{96}N_{mn}\gamma_{mnp}^{[\alpha\beta}\left(\gamma^{p}d\right)^{\gamma]},\\
b_3&\equiv4!\left(\frac{\overline{\lambda}_{\alpha}r_{\beta}r_{\gamma}r_{\lambda}}{\left(\overline{\lambda}\lambda\right)^{4}},L^{\alpha\beta\gamma\lambda}\right), & L^{\alpha\beta\gamma\lambda}&=-\tfrac{3}{(96)^{2}}\left(N^{mn},N^{rs}\right)\eta^{pq}\gamma_{mnp}^{[\alpha\beta}\gamma_{qrs}^{\gamma]\lambda}.
\end{aligned} \tag{3.6}$$

Note that the subscript n in b_n is the r charge q_r of the operators, defined as

$$\int dz\left\{J_{r}\left(z\right)\mathcal{O}\left(y\right)\right\}=q_{r}\left(\mathcal{O}\right)\mathcal{O}\left(y\right). \tag{3.7}$$

The building blocks of b_n satisfy the ordered version of the classical commutation relations:

$$\begin{aligned}
\left\{Q,-s^{\alpha}\partial\overline{\lambda}_{\alpha}\right\}&=T_{\overline{\lambda}}, & \left\{Q,G^{\alpha}\right\}&=\left(\lambda^{\alpha},T_{\lambda}+T_{\mathrm{matter}}\right),\\
\left[Q,H^{\alpha\beta}\right]&=\left(\lambda^{[\alpha},G^{\beta]}\right), & \left\{Q,K^{\alpha\beta\gamma}\right\}&=\left(\lambda^{[\alpha},H^{\beta\gamma]}\right),\\
\left[Q,L^{\alpha\beta\gamma\lambda}\right]&=\left(\lambda^{[\alpha},K^{\beta\gamma\lambda]}\right), & \left(\lambda^{[\alpha},L^{\beta\gamma\lambda\sigma]}\right)&=0.
\end{aligned}$$

There are some observations that must be made concerning the above operators:

- the ordering here plays a major role, allowing a correct manipulation of the quantum corrections to the b ghost. Obviously, a different ordering prescription must not conflict with $\{Q,b\}=T$. It turns out that both computations and results are clearer using (3.4);

- there is an explicit correction related to the ordering prescription chosen. The operator O defined above is required because

$$\left(\frac{\left(\overline{\lambda}_\alpha r_\beta - \overline{\lambda}_\beta r_\alpha\right)\lambda^\alpha}{\left(\overline{\lambda}\lambda\right)^2}, G^\beta\right) - \left(\frac{\overline{\lambda}_\alpha r_\beta}{\left(\overline{\lambda}\lambda\right)^2}, \left(\lambda^\alpha, G^\beta\right) - \left(\lambda^\beta, G^\alpha\right)\right) \neq 0,$$

and

$$\left(\frac{\overline{\lambda}_\alpha}{\left(\overline{\lambda}\lambda\right)}, \left(\lambda^\alpha, T_\lambda\right)\right) - T_\lambda \neq 0.$$

One can see that $\{Q, O\}$ precisely matches these inequalities. In [9], besides (3.4), an alternative prescription was used, that conveniently absorbs the operator O.

- the quantum contribution to G^α is proportional to $\partial^2\theta^\alpha$. The coefficient can be fixed through the $U(5)$ decomposition or, more directly, comparing the cubic pole between the energy momentum tensor and both sides of the equation $\{Q, G^\alpha\} = \left(\lambda^\alpha, T\right)$.

Therefore, the quantum version of the $b$ghost of (3.5) and (3.6) satisfy

$$\{Q, b\} = T, \tag{3.8}$$

where T is the full energy-momentum tensor of the theory, as desired.

Next section presents some of the basics properties of the $b$ghost of (3.5).

3.2 Properties

3.2.1 Primary Conformal Field

OPE computations are more systematic[1] within the prescription (3.4). As a sample, it will be shown here that the b ghost for the non minimal formalism is a primary field.

Concerning b_{-1}, the ordering does not matter and it is straightforward to see that

$$T(z)\, b_{-1}(y) \sim 2\frac{b_{-1}}{(z-y)^2} + \frac{\partial b_{-1}}{(z-y)}.$$

[1]See chapter 6 of [28], where the normal ordering is presented in details.

For b_0, however, there are some subtleties. Analysing G^α first,

$$T(z)\,G^\alpha(y) \sim 2\frac{G^\alpha}{(z-y)^2} + \frac{\partial G^\alpha}{(z-y)} + \frac{\partial\theta^\alpha}{(z-y)^3}. \tag{3.9}$$

Note that the cubic pole receives contributions from J_λ (the ghost current anomaly), $\partial^2\theta^\alpha$ and $\left(\Pi^m, \gamma_m^{\alpha\beta} d_\beta\right)$:

$$\begin{aligned}
T(z)\,J_\lambda(y) &\sim \frac{8}{(z-y)^3} + \frac{J_\lambda}{(z-y)^2} + \frac{\partial J_\lambda}{(z-y)},\\
T(z)\,\partial^2\theta^\alpha(y) &\sim 2\frac{\partial\theta^\alpha}{(z-y)^3} + 2\frac{\partial^2\theta^\alpha}{(z-y)^2} + \frac{\partial^3\theta^\alpha}{(z-y)},\\
T(z)\left(\Pi^m, \gamma_m^{\alpha\beta} d_\beta\right)(y) &\sim \left(\frac{\Pi^m}{(z-y)^2} + \frac{\partial\Pi^m}{(z-y)}, \gamma_m^{\alpha\beta} d_\beta\right)\\
&+ \left(\Pi^m, \frac{\gamma_m^{\alpha\beta} d_\beta}{(z-y)^2} + \frac{\gamma_m^{\alpha\beta}\partial d_\beta}{(z-y)}\right).
\end{aligned}$$

According to the ordering prescription, the first term in the last OPE can be rewritten as

$$\begin{aligned}
\frac{1}{2\pi i}\oint \frac{dw}{(w-y)}\left\{\frac{1}{(z-w)^2}\Pi^m(w) + \frac{1}{(z-w)}\partial\Pi^m(w)\right\}\gamma_m^{\alpha\beta} d_\beta(y) &=\\
-10\frac{\partial\theta^\alpha}{(z-y)^3} + \frac{\left(\Pi^m, \gamma_m^{\alpha\beta} d_\beta\right)}{(z-y)^2} + \frac{\left(\partial\Pi^m, \gamma_m^{\alpha\beta} d_\beta\right)}{(z-y)}&,
\end{aligned}$$

where (2.6c) is responsible for the cubic pole. Therefore,

$$T(z)\left(\Pi^m, \gamma_m^{\alpha\beta} d_\beta\right)(y) \sim -10\frac{\partial\theta^\alpha}{(z-y)^3} + 2\frac{\left(\Pi^m, \gamma_m^{\alpha\beta} d_\beta\right)}{(z-y)^2} + \frac{\partial\left(\Pi^m, \gamma_m^{\alpha\beta} d_\beta\right)}{(z-y)}.$$

Adding up all the contributions, equation (3.9) is reproduced. For the whole b_0,

$$\begin{aligned}
T(z)\,b_0(y) \sim& \left(\frac{1}{(z-y)}\partial\left(\frac{\overline{\lambda}_\alpha}{\overline{\lambda}\lambda}\right), G^\alpha\right)\\
&+ \left(\frac{\overline{\lambda}_\alpha}{\overline{\lambda}\lambda}, 2\frac{G^\alpha}{(z-y)^2} + \frac{\partial G^\alpha}{(z-y)} + \frac{\partial\theta^\alpha}{(z-y)^3}\right) + 2\frac{O}{(z-y)^2} + \frac{\partial O}{(z-y)}.
\end{aligned} \tag{3.10}$$

Again, the first term on the right-hand side can be rewritten as

$$\frac{1}{2\pi i}\oint\frac{dw}{(w-y)}\partial_w\left(\frac{\overline{\lambda}_\alpha}{\overline{\lambda}\lambda}\right)(w)\frac{G^\alpha(y)}{(z-w)}=\frac{1}{(z-y)}\left(\partial\left(\frac{\overline{\lambda}_\alpha}{\overline{\lambda}\lambda}\right),G^\alpha\right)-\frac{1}{(z-y)^3}\left(\frac{\overline{\lambda}_\alpha\partial\theta^\alpha}{\overline{\lambda}\lambda}\right).$$

Replacing this equation in (3.10), the cubic pole disappears, yielding a primary field.

For b_1, b_2 and b_3, there are no contributions like the one in b_0 (they are all proportional to the pure spinor constraints), therefore the b ghost given in (3.5) and (3.6) is a primary field:

$$T(z)\,b(y)\ \sim\ 2\frac{b}{(z-y)^2}+\frac{\partial b}{(z-y)}. \qquad (3.11)$$

3.2.2 Nilpotency

The results of this subsection are based on [11]. A previous demonstration on the nilpotency of the pure spinor b ghost was given in [10], which will be shown here to be incomplete.

The OPE of the b ghost with itself can be cast as

$$b(z)\,b(y)\sim\frac{O_0}{(z-y)^4}+\frac{O_1}{(z-y)^3}+\frac{O_2}{(z-y)^2}+\frac{O_3}{(z-y)}, \qquad (3.12)$$

for there are no (covariant, supersymmetric) negative conformal weight fields in the theory. Due to its anticommuting character, $b(z)\,b(y)=-b(y)\,b(z)$, implying that

$$b(z)\,b(y)\sim\frac{O_1}{(z-y)^3}+\frac{1}{2}\frac{\partial O_1}{(z-y)^2}+\frac{O_3}{(z-y)}. \qquad (3.13)$$

Furthermore, since $\{Q,b\}=T$ and b is a primary field of conformal weight 2,

$$\{Q,b(z)\}\,b(y)-b(z)\,\{Q,b(y)\}\ =\ T(z)\,b(y)-b(z)\,T(y) \qquad (3.14)$$

$$\sim\ \text{regular}, \qquad (3.15)$$

or, equivalently,

$$\{Q,b(z)\,b(y)\}\ \sim\ \frac{\{Q,O_1\}}{(z-y)^3}+\frac{1}{2}\frac{\partial\{Q,O_1\}}{(z-y)^2}+\frac{\{Q,O_3\}}{(z-y)}. \qquad (3.16)$$

Comparing equations (3.14) and (3.16), one concludes that O_1 and O_3 are BRST closed.

Taking now into account the specific form of the b ghost for the non-minimal pure spinor formalism, given in (3.5), it is a simple task to verify that the cubic poles are all proportional to the constraints (2.32) and (2.33). The possible terms will be listed below:

- b_{-1} may give rise to cubic poles only in the OPE with b_3, due to ordering effects. The different terms are proportional to

$$\begin{aligned}&\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\partial\overline{\lambda}\gamma^{pqr}r\right)\left(\overline{\lambda}\gamma^{mn}\lambda\right)\left(\overline{\lambda}\gamma_{qr}\lambda\right), \quad \left(\overline{\lambda}\gamma_{mnp}r\right)\left(\partial\overline{\lambda}\gamma^{pqr}r\right)\left(\overline{\lambda}\gamma^{mn}\gamma_{qr}\lambda\right),\\&\left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right)\left(r\gamma^{pqr}r\right)\left(\overline{\lambda}\gamma^{mn}\lambda\right)\left(\overline{\lambda}\gamma_{qr}\lambda\right), \quad \left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right)\left(r\gamma^{pqr}r\right)\left(\overline{\lambda}\gamma^{mn}\gamma_{qr}\lambda\right).\end{aligned} \tag{3.17}$$

- b_0 has cubic poles with itself, b_1, b_2 and b_3:

  - in $b_0\left(z\right)b_0\left(y\right)$, it comes from the multiple contractions of $\Pi^m\left(\gamma_m d\right)^\alpha$ with itself and from its single contraction with $\partial^2\theta^\beta$, both proportional to $\Pi^m\left(\overline{\lambda}\gamma_m\overline{\lambda}\right)$.
  - for $b_0\left(z\right)b_1\left(y\right)$, it will arise in the contractions of $\left(d\gamma^{mnp}d\right)\left(\overline{\lambda}\lambda\right)^{-2}$ with all the terms in b_0, being proportional to $\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mnp}d\right)$.
  - in the OPE $b_0\left(z\right)b_2\left(y\right)$, the multiple contractions of $N^{mn}\left(\gamma^p d\right)^\alpha$ will give cubic poles like:

$$\begin{gathered}\left(\overline{\lambda}\gamma_{mnp}r\right)N^{mn}\left(\overline{\lambda}\gamma^p r\right),\\\left(\overline{\lambda}\gamma_{mnp}r\right)J\left(\overline{\lambda}\gamma^{mn}\lambda\right)\left(\overline{\lambda}\gamma^p r\right),\\\partial\left[\left(\overline{\lambda}\gamma^{mn}\lambda\right)\overline{\lambda}_\alpha\right]\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\gamma^p r\right)^\alpha.\end{gathered} \tag{3.18}$$

  - finally, in $b_0\left(z\right)b_3\left(y\right)$, the cubic poles are of the form:

$$\begin{gathered}\left(\overline{\lambda}\partial\theta\right)\left(\overline{\lambda}\gamma^{mn}\lambda\right)\left(\overline{\lambda}\gamma_{qr}\lambda\right)\left(\overline{\lambda}\gamma_{mnp}r\right)\left(r\gamma^{pqr}r\right),\\\left(\overline{\lambda}\gamma^{mn}\partial\theta\right)\left(\overline{\lambda}\gamma_{qr}\lambda\right)\left(\overline{\lambda}\gamma_{mnp}r\right)\left(r\gamma^{pqr}r\right).\end{gathered} \tag{3.19}$$

- b_1 has cubic poles with itself, b_2 and b_3:

  - in $b_1\left(z\right)b_1\left(y\right)$, they are of the form

$$\begin{gathered}\partial\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mnp}r\right),\\\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mnq}r\right)N^p{}_q,\\\partial\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{pqr}r\right)\left(\overline{\lambda}\gamma^{mn}\lambda\right)\left(\overline{\lambda}\gamma_{qr}\lambda\right).\end{gathered} \tag{3.20}$$

– for $b_1(z)\,b_2(y)$, the only possible cubic poles are proportional to

$$\begin{gathered}\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma_{qrs}r\right)\left(\overline{\lambda}\gamma^{mn}\lambda\right)\left(r\gamma^{qrs}\gamma^{p}\partial\theta\right),\\ \left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mnq}r\right)\left(r\gamma_{q}\gamma^{p}\partial\theta\right).\end{gathered}\tag{3.21}$$

– the cubic poles arising in $b_1(z)\,b_3(y)$ come from the multiple contractions of $N^{mn}\Pi^{p}\left(\overline{\lambda}\lambda\right)^{-2}$ with b_3, and are given by

$$\begin{gathered}\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mnq}r\right)\left(r\gamma_{qrs}r\right)\left(\overline{\lambda}\gamma^{rs}\lambda\right)\Pi^{p},\\ \left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mn}\lambda\right)\Pi^{p}\left(\overline{\lambda}\gamma_{qrs}r\right)\left(\overline{\lambda}\gamma^{qr}\lambda\right)\left(r\gamma^{stu}r\right)\left(\overline{\lambda}\gamma_{tu}\lambda\right),\\ \left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mn}\lambda\right)\Pi^{p}\left(\overline{\lambda}\gamma_{qrs}r\right)\left(\overline{\lambda}\gamma^{qr}\gamma_{tu}\lambda\right)\left(r\gamma^{stu}r\right).\end{gathered}\tag{3.22}$$

- b_2 has cubic poles with itself and with b_3:

  – in $b_2(z)\,b_2(y)$, they are of the form

$$\begin{gathered}\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma_{qrs}r\right)\left(r\gamma^{pst}r\right)\Pi_{t}\eta^{mq}\eta^{nr},\\ \left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mn}\lambda\right)\left(\overline{\lambda}\gamma_{qrs}r\right)\left(\overline{\lambda}\gamma^{qr}\lambda\right)\left(r\gamma^{pst}r\right)\Pi_{t}.\end{gathered}\tag{3.23}$$

  – for $b_2(z)\,b_3(y)$, d_α appearing in b_2 is inert and there are only contractions involving the ghost Lorentz currents:

$$\begin{gathered}\left(\overline{\lambda}\gamma_{mnp}r\right)\left(r\gamma^{p}d\right)\left(\overline{\lambda}\gamma^{mnq}r\right)\left(r\gamma_{qrs}r\right)\left(\overline{\lambda}\gamma^{rs}\lambda\right),\\ \left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mn}\lambda\right)\left(r\gamma^{p}d\right)\left(\overline{\lambda}\gamma_{qrs}r\right)\left(\overline{\lambda}\gamma^{qr}\lambda\right)\left(r\gamma^{stu}r\right)\left(\overline{\lambda}\gamma_{tu}\lambda\right).\end{gathered}\tag{3.24}$$

- the cubic poles of $b_3(z)\,b_3(y)$ involve all possible contractions of the the Lorentz generators and will give similar results to the ones above, only with more r's.

Due to the pure spinor constraints,

$$\left(\overline{\lambda}\gamma^{mn}\right)_{\alpha}\left(\overline{\lambda}\gamma_{mnp}r\right)=\left(\overline{\lambda}\gamma^{mn}\right)_{\alpha}\left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right)=\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mnp}\right)^{\alpha}=0,\tag{3.25}$$

and every expression listed contains at least one of these types of contractions. Consequently, $O_1=0$ and

$$b(z)\,b(y)\sim\frac{O_3}{(z-y)}.\tag{3.26}$$

It is clear from (3.5), that O_3 can only be composed with supersymmetric invariants: matter fields (Π^m, d_α, $\partial\theta^\alpha$); ghost currents from the minimal sector (N^{mn}, J); ghost fields (λ^α,$\overline{\lambda}_\alpha$,$r_\alpha$); and, in principle, their partial derivatives.

In [10], the vanishing of O_3 has been argued as follows. The author assumed that all partial derivatives of r_α that may appear in the OPE (3.26) can be removed due to the pure spinor constraint, since

$$\overline{\lambda}\gamma^m\partial r = -\partial\overline{\lambda}\gamma^m r. \tag{3.27}$$

Based on that assumption, all the r_α dependence of O_3 could be made explicitly through

$$O_3 = \Omega + r_\alpha\Omega^\alpha + r_\alpha r_\beta\Omega^{\alpha\beta} + \ldots \tag{3.28}$$

where the Ω's are supersymmetric, ghost number -2, conformal weight 3, BRST closed operators. Since the BRST charge can be split into two pieces according to the r-charge

$$\begin{aligned} Q &= Q_0 + Q_1, && (3.29a)\\ Q_0 &= \oint(\lambda^\alpha d_\alpha), && (3.29b)\\ Q_1 &= \oint(\overline{\omega}^\alpha r_\alpha), && (3.29c) \end{aligned}$$

requiring $[Q, O_3] = 0$, implies $[Q_0, \Omega] = 0$. Then, it has been shown that there are no Ω with the above requisites satisfying $[Q_0, \Omega] = 0$, so it vanishes identically. Then, $\Omega = 0$ implies $[Q_0, \Omega^\alpha] = 0$. Again, this can be demonstrated to vanish. Pursuing this argument, the nilpotency of the b ghost was obtained in [10].

However, the absence of $\partial^n r_\alpha$ in O_3 is incorrect, as will be illustrated soon, which means that the cohomology argument of [10], summarized above, must be extended, as will now be done.

The computation of (3.26) is organized according to the r-charge of the operators, that is

$$O_3 = (bb)_0 + (bb)_1 + (bb)_2 + (bb)_3 + (bb)_4 + (bb)_5 + (bb)_6\,. \tag{3.30}$$

To make the expressions more clear, the ordering notation will be dropped.

The first term, $(bb)_0$, is given by

$$(bb)_0 \equiv \int dz \left\{ b_0(z)\, b_0(y) + b_{-1}(z)\, b_1(y) + b_1(z)\, b_{-1}(y) \right\} \tag{3.31}$$

$$\begin{aligned} &= \alpha_{01} \frac{N^{mn} \left(\overline{\lambda}\gamma_{mn}\partial\theta\right) \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^2} + \alpha_{02} \frac{\left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right) N^{mn}\Pi^p}{\left(\overline{\lambda}\lambda\right)^2} \\ &+ \alpha_{03} \frac{\Pi^m \left(\overline{\lambda}\partial\theta\right) \left(\overline{\lambda}\gamma_m d\right)}{\left(\overline{\lambda}\lambda\right)^2} + \alpha_{04} \frac{\left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right) \left(d\gamma^{mnp} d\right)}{\left(\overline{\lambda}\lambda\right)^2} \\ &+ \alpha_{05} \frac{\Pi^m \left(\overline{\lambda}\gamma_m\partial^2\overline{\lambda}\right)}{\left(\overline{\lambda}\lambda\right)^2} + \alpha_{06} \frac{\left(\overline{\lambda}\partial\theta\right) \left(\overline{\lambda}\partial^2\theta\right)}{\left(\overline{\lambda}\lambda\right)^2} + \alpha_{07} \frac{\left(\overline{\lambda}\partial\theta\right) \left(\partial\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^2}, \end{aligned} \tag{3.32}$$

where α_{0n} are just numerical coefficients. By a direct computation, it is relatively simple to show the vanishing of $(bb)_0$. It is enough to compute $[Q, (bb)_0]$ and use the BRST argument mentioned above. Note that $[Q, D] = 0$ implies the vanishing of

$$\begin{aligned} [Q_0, (bb)_0] &= \alpha_{01} \frac{N^{mn} \left(\overline{\lambda}\gamma_{mn}\partial\lambda\right) \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^2} - \alpha_{01} \frac{\frac{1}{2} \left(d\gamma^{mn}\lambda\right) \left(\overline{\lambda}\gamma_{mn}\partial\theta\right) \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^2} \\ &- \alpha_{01} \frac{N^{mn} \left(\overline{\lambda}\gamma_{mn}\partial\theta\right) \left(\overline{\lambda}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^2} - \alpha_{02} \frac{\left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right) \frac{1}{2} \left(d\gamma^{mn}\lambda\right) \Pi^p}{\left(\overline{\lambda}\lambda\right)^2} \\ &+ \alpha_{02} \frac{\left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right) N^{mn} \left(\lambda\gamma^p\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^2} - \alpha_{03} \frac{\Pi^m \left(\overline{\lambda}\partial\theta\right) \left(\overline{\lambda}\gamma_m\gamma_n\partial\theta\right) \Pi^n}{\left(\overline{\lambda}\lambda\right)^2} \\ &+ \alpha_{03} \frac{\Pi^m \left(\overline{\lambda}\partial\lambda\right) \left(\overline{\lambda}\gamma_m d\right)}{\left(\overline{\lambda}\lambda\right)^2} + \alpha_{03} \frac{\left(\lambda\gamma^m\partial\theta\right) \left(\overline{\lambda}\partial\theta\right) \left(\overline{\lambda}\gamma_m d\right)}{\left(\overline{\lambda}\lambda\right)^2} \\ &+ \alpha_{04} \frac{2 \left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right) \left(d\gamma^{mnp}\gamma^q\lambda\right) \Pi_q}{\left(\overline{\lambda}\lambda\right)^2} + \alpha_{06} \frac{\left(\overline{\lambda}\partial\lambda\right) \left(\overline{\lambda}\partial^2\theta\right)}{\left(\overline{\lambda}\lambda\right)^2} - \alpha_{06} \frac{\left(\overline{\lambda}\partial\theta\right) \left(\overline{\lambda}\partial^2\lambda\right)}{\left(\overline{\lambda}\lambda\right)^2} \\ &+ \alpha_{05} \frac{\left(\lambda\gamma^m\partial\theta\right) \left(\overline{\lambda}\gamma_m\partial^2\overline{\lambda}\right)}{\left(\overline{\lambda}\lambda\right)^2} + \alpha_{07} \frac{\left(\overline{\lambda}\partial\lambda\right) \left(\partial\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^2} - \alpha_{07} \frac{\left(\overline{\lambda}\partial\theta\right) \left(\partial\overline{\lambda}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^2}. \end{aligned}$$

The Lorentz generators N^{mn} appear in three terms. It is straightforward to check that they are not related by a Fierz decomposition of the spinors, implying that $\alpha_{01} = \alpha_{02} = 0$. Now, there is only one term that contributes with one d_α and two $\partial\theta^\alpha$, so $\alpha_{03} = 0$, which, on the other hand, imply that $\alpha_{04} = 0$, since the term with one d_α and one Π^m cannot be cancelled any more. The vanishing of α_{05}, α_{06} and α_{07} is evident, since they do not possibly cancel each other. *There is no linear combination of the above operators that can be annihilated by* Q_0*, therefore* $(bb)_0 = 0$.

The second term, $(bb)_1$, is

$$\begin{aligned} (bb)_1 &\equiv \int dz \left\{ b_0(z) b_1(y) + b_1(z) b_0(y) + b_{-1}(z) b_2(y) + b_2(z) b_{-1}(y) \right\} \quad (3.33) \\ &= \alpha_{11} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) N^{mn} \Pi^p \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^3} + \alpha_{12} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) N^{mn} \left(\partial\overline{\lambda}\gamma^p d\right)}{\left(\overline{\lambda}\lambda\right)^3} + \\ &+ \alpha_{13} \frac{\left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right) N^{mn} \left(r\gamma^p d\right)}{\left(\overline{\lambda}\lambda\right)^3} + \alpha_{14} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) \left(d\gamma^{mnp} d\right) \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^3} \\ &+ \alpha_{15} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) \Pi^m \left(\partial\overline{\lambda}\gamma^{np}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^3} + \alpha_{16} \frac{\left(\overline{\lambda}\gamma_m \partial^2\overline{\lambda}\right) \left(r\gamma^m d\right)}{\left(\overline{\lambda}\lambda\right)^3}. \quad (3.34) \end{aligned}$$

Since $[Q_1, (bb)_0] = 0$, $[Q_0, (bb)_1]$ must also vanish:

$$\begin{aligned} [Q_0, (bb)_1] &= \alpha_{11} \frac{\frac{1}{2}\left(\overline{\lambda}\gamma_{mnp} r\right) \left(d\gamma^{mn}\lambda\right) \Pi^p \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^3} - \alpha_{11} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) N^{mn} \left(\lambda\gamma^p\partial\theta\right) \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^3} \\ &- \alpha_{11} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) N^{mn} \Pi^p \left(\overline{\lambda}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^3} + \alpha_{12} \frac{\frac{1}{2}\left(\overline{\lambda}\gamma_{mnp} r\right) \left(d\gamma^{mn}\lambda\right) \left(\partial\overline{\lambda}\gamma^p d\right)}{\left(\overline{\lambda}\lambda\right)^3} \\ &+ \alpha_{12} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) N^{mn} \left(\partial\overline{\lambda}\gamma^p\gamma^q\lambda\right) \Pi_q}{\left(\overline{\lambda}\lambda\right)^3} - \alpha_{13} \frac{\frac{1}{2}\left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right) \left(d\gamma^{mn}\lambda\right) \left(r\gamma^p d\right)}{\left(\overline{\lambda}\lambda\right)^3} \\ &+ \alpha_{13} \frac{\left(\overline{\lambda}\gamma_{mnp}\partial\overline{\lambda}\right) N^{mn} \left(r\gamma^p\gamma^q\lambda\right) \Pi_q}{\left(\overline{\lambda}\lambda\right)^3} - \alpha_{14} \frac{2\left(\overline{\lambda}\gamma_{mnp} r\right) \left(d\gamma^{mnp}\gamma^q\lambda\right) \Pi_q \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^3} \\ &- \alpha_{14} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) \left(d\gamma^{mnp} d\right) \left(\overline{\lambda}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^3} - \alpha_{15} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) \left(\lambda\gamma^m\partial\theta\right) \left(\partial\overline{\lambda}\gamma^{np}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^3} \\ &- \alpha_{15} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right) \Pi^m \left(\partial\overline{\lambda}\gamma^{np}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^3} + \alpha_{16} \frac{\left(\overline{\lambda}\gamma_m \partial^2\overline{\lambda}\right) \left(r\gamma^m\gamma^n\lambda\right) \Pi_n}{\left(\overline{\lambda}\lambda\right)^3}. \end{aligned}$$

There is only one term that contains one Lorentz generator N^{mn} and two $\partial\theta^\alpha$, so $\alpha_{11} = 0$. Now, there are two other terms that contain N^{mn}, but they are unrelated to any Fierz decomposition, implying that $\alpha_{12} = \alpha_{13} = 0$. The remaining terms are obviously independent: $\alpha_{14} = 0$, since it is the only one with $(d\gamma^{mnp}d)$; $\alpha_{15} = 0$, as no other term contains two $\partial\theta^\alpha$; and $\alpha_{16} = 0$, for there is nothing else to cancel it. As $(bb)_0$, $(bb)_1$ is not

BRST closed for any set of coefficients α_{1n} and $(bb)_1 = 0$ is the single possibility left.

Going on,

$$(bb)_2 \equiv \int dz \left\{ b_0(z)\, b_2(y) + b_2(z)\, b_0(y) + b_1(z)\, b_1(y) + b_{-1}(z)\, b_3(y) + b_3(z)\, b_{-1}(y) \right\} \tag{3.35}$$

can be written as

$$\begin{aligned}
(bb)_2 \;=\;& \alpha_{21} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right)(r\gamma^p d)\, N^{mn} \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^4} + \alpha_{22} \frac{\left(\overline{\lambda}\gamma_m \partial r\right)(r\gamma^m d)\left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^4} \\
+\;& \alpha_{23} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right)\left(\partial\overline{\lambda}\gamma^{pqr} r\right) N^{mn} N_{qr}}{\left(\overline{\lambda}\lambda\right)^4} + \alpha_{24} \frac{\left(\overline{\lambda}\gamma_m \partial r\right)\left(\overline{\lambda}\gamma_n d\right)(r\gamma^{mn}\partial\theta)}{\left(\overline{\lambda}\lambda\right)^4} \\
+\;& \alpha_{25} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right)\left(\partial\overline{\lambda}\gamma_q r\right) N^{mn} N^{pq}}{\left(\overline{\lambda}\lambda\right)^4} + \alpha_{26} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right)\left(r\gamma^p \partial^2\overline{\lambda}\right) N^{mn}}{\left(\overline{\lambda}\lambda\right)^4} \\
+\;& \alpha_{27} \frac{\left(\overline{\lambda}\gamma_m \partial r\right)\left(r\gamma^m \partial^2\overline{\lambda}\right)}{\left(\overline{\lambda}\lambda\right)^4} + \alpha_{28} \frac{\left(r\gamma_m \partial^2\overline{\lambda}\right)\left(\overline{\lambda}\gamma^m \partial r\right)}{\left(\overline{\lambda}\lambda\right)^4}.
\end{aligned} \tag{3.36}$$

The last line of the expression is Q_0-closed. In computing $[Q_0, (bb)_2]$,

$$\begin{aligned}
[Q_0, (bb)_2] \;=\;& \alpha_{21} \frac{\frac{1}{2}\left(\overline{\lambda}\gamma_{mnp} r\right)(r\gamma^p d)(d\gamma^{mn}\lambda)\left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^4} - \alpha_{21} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right)(r\gamma^p d)\, N^{mn}\left(\overline{\lambda}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^4} \\
-\;& \alpha_{21} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right)(r\gamma^p \gamma^q \lambda)\, \Pi_q N^{mn}\left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^4} - \alpha_{22} \frac{\left(\overline{\lambda}\gamma_m \partial r\right)(r\gamma^m \gamma^n \lambda)\, \Pi_n \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^4} \\
-\;& \alpha_{22} \frac{\left(\overline{\lambda}\gamma_m \partial r\right)(r\gamma^m d)\left(\overline{\lambda}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^4} - \alpha_{23} \frac{\frac{1}{2}\left(\overline{\lambda}\gamma_{mnp} r\right)\left(\partial\overline{\lambda}\gamma^{pqr} r\right)(d\gamma^{mn}\lambda)\, N_{qr}}{\left(\overline{\lambda}\lambda\right)^4} \\
-\;& \alpha_{23} \frac{\frac{1}{2}\left(\overline{\lambda}\gamma_{mnp} r\right)\left(\partial\overline{\lambda}\gamma^{pqr} r\right) N^{mn}(d\gamma_{qr}\lambda)}{\left(\overline{\lambda}\lambda\right)^4} - \alpha_{24} \frac{\left(\overline{\lambda}\gamma_m \partial r\right)\left(\overline{\lambda}\gamma_n d\right)(r\gamma^{mn}\partial\lambda)}{\left(\overline{\lambda}\lambda\right)^4} \\
+\;& \alpha_{24} \frac{\left(\overline{\lambda}\gamma_m \partial r\right)\left(\overline{\lambda}\gamma_n \gamma_p \lambda\right)\Pi^p (r\gamma^{mn}\partial\theta)}{\left(\overline{\lambda}\lambda\right)^4} - \alpha_{25} \frac{\frac{1}{2}\left(\overline{\lambda}\gamma_{mnp} r\right)\left(\partial\overline{\lambda}\gamma_q r\right) N^{mn}(d\gamma^{pq}\lambda)}{\left(\overline{\lambda}\lambda\right)^4} \\
-\;& \alpha_{25} \frac{\frac{1}{2}\left(\overline{\lambda}\gamma_{mnp} r\right)\left(\partial\overline{\lambda}\gamma_q r\right)(d\gamma^{mn}\lambda)\, N^{pq}}{\left(\overline{\lambda}\lambda\right)^4} - \alpha_{26} \frac{\frac{1}{2}\left(\overline{\lambda}\gamma_{mnp} r\right)\left(r\gamma^p \partial^2\overline{\lambda}\right)(d\gamma^{mn}\lambda)}{\left(\overline{\lambda}\lambda\right)^4},
\end{aligned}$$

the terms that contain matter fields or the Lorentz current do not vanish for any set α_{2n} of coefficients: $\alpha_{21} = 0$, for it is the single term that contains N^{mn} and Π^m; $\alpha_{22} = \alpha_{24} = 0$, since they are the only ones that contribute with one Π^m and one $\partial\theta^\alpha$, but independently; $\alpha_{23} = \alpha_{25} = 0$, because they are the remaining (and also independent) terms containing the Lorentz generator; and $\alpha_{26} = 0$, for it is not BRST closed.

$(bb)_3$ can be cast as:

$$(bb)_3 \equiv \int dz \left\{ b_0(z)\, b_3(y) + b_3(z)\, b_0(y) + b_1(z)\, b_2(y) + b_2(z)\, b_1(y) \right\} \tag{3.37}$$

$$\begin{aligned} &= \alpha_{31} \frac{\left(\overline{\lambda}\gamma_{mnp} r\right)\left(r\gamma^{pqr} r\right) N^{mn} N_{qr} \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^5} + \alpha_{32} \frac{\left(r\gamma_{mnp} r\right)\left(\overline{\lambda}\gamma^{p}\partial r\right) N^{mn} \left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^5} \\ &+ \alpha_{33} \frac{\left(\overline{\lambda}\gamma_m\partial r\right)\left(r\gamma^m\partial r\right)\left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^5} + \alpha_{34} \frac{\left(\overline{\lambda}\gamma_m\partial r\right)\left(\overline{\lambda}\gamma_n\partial r\right)\left(r\gamma^{mn}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^5} \\ &+ \alpha_{35} \frac{\left(\overline{\lambda}\gamma_m\partial r\right)\left(\overline{\lambda}\gamma_n\partial r\right)\left(r\gamma^{mn}\lambda\right)\left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^6}. \end{aligned} \tag{3.38}$$

It is straightforward to see that the first two terms are not BRST closed. One of the contributions of the first one contains two Lorentz generators, that cannot be cancelled, so $\alpha_{31} = 0$. The same happens for the second one, which has a contribution in $[Q_0, (bb)_3]$ with one Lorentz generator, not balanced by any other, thus $\alpha_{32} = 0$. The result of the computation of $[Q_1, (bb)_2] + [Q_0, (bb)_3]$ with the remaining terms is

$$\begin{aligned} [Q_1, (bb)_2] + [Q_0, (bb)_3] &= \alpha_{27} \frac{4\left(\overline{\lambda}\gamma_m\partial r\right)\left(r\gamma^m\partial^2\overline{\lambda}\right)\left(r\lambda\right)}{\left(\overline{\lambda}\lambda\right)^5} - \alpha_{27} \frac{\left(r\gamma_m\partial r\right)\left(r\gamma^m\partial^2\overline{\lambda}\right)}{\left(\overline{\lambda}\lambda\right)^4} \\ &- \alpha_{27} \frac{\left(\overline{\lambda}\gamma_m\partial r\right)\left(r\gamma^m\partial^2 r\right)}{\left(\overline{\lambda}\lambda\right)^4} + \alpha_{28} \frac{4\left(\overline{\lambda}\gamma_m\partial^2\overline{\lambda}\right)\left(r\gamma^m\partial r\right)\left(r\lambda\right)}{\left(\overline{\lambda}\lambda\right)^5} \\ &- \alpha_{28} \frac{\left(r\gamma_m\partial^2\overline{\lambda}\right)\left(r\gamma^m\partial r\right)}{\left(\overline{\lambda}\lambda\right)^4} - \alpha_{34} \frac{\left(\overline{\lambda}\gamma_m\partial r\right)\left(\overline{\lambda}\gamma_n\partial r\right)\left(r\gamma^{mn}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^5} \\ &- \alpha_{28} \frac{\left(\overline{\lambda}\gamma_m\partial^2 r\right)\left(r\gamma^m\partial r\right)}{\left(\overline{\lambda}\lambda\right)^4} - \alpha_{33} \frac{\left(\overline{\lambda}\gamma_m\partial r\right)\left(r\gamma^m\partial r\right)\left(\overline{\lambda}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^5} \\ &- \alpha_{35} \frac{\left(\overline{\lambda}\gamma_m\partial r\right)\left(\overline{\lambda}\gamma_n\partial r\right)\left(r\gamma^{mn}\lambda\right)\left(\overline{\lambda}\partial\lambda\right)}{\left(\overline{\lambda}\lambda\right)^6}. \end{aligned}$$

Obviously, there is no nontrivial solution to $\{\alpha_{27}, \alpha_{28}, \alpha_{33}, \alpha_{34}, \alpha_{35}\}$ that may lead to the vanishing of this equation, thus $(bb)_2 = (bb)_3 = 0$. Note that

$$\frac{\left(\overline{\lambda}\gamma_m \partial r\right)\left(r\gamma^m \partial r\right)\left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^5} \tag{3.39}$$

does not allow the removal of partial derivatives acting on r, which contradicts the assumption of [10].

So far, the pure spinor constraints only have been used to reduce the number of independent terms in the OPE computation. It turns out that for $(bb)_4$, $(bb)_5$ and $(bb)_6$, all possible terms being generated vanish due to the constraints.

For

$$(bb)_4 \equiv \int dz \left\{ b_1(z)\, b_3(y) + b_3(z)\, b_1(y) + b_2(z)\, b_2(y) \right\}, \tag{3.40}$$

the simple poles are given by:

- terms with two N's and one Π, like

$$\frac{\left(\overline{\lambda}\gamma_{mnp} r\right)\left(\overline{\lambda}\gamma_{qrs} r\right)\left(r\gamma^{pqt} r\right) N^{mn} N^r{}_t \Pi^s}{\left(\overline{\lambda}\lambda\right)^6}. \tag{3.41}$$

  Since $(r\gamma^{mnp}r) = (r\gamma^m\gamma^n\gamma^p r)$ and $\left(\overline{\lambda}\gamma^{mnp}r\right)(r\gamma_p)^\alpha = (r\gamma^{mnp}r)\left(\overline{\lambda}\gamma_p\right)^\alpha$,

$$\left(\overline{\lambda}\gamma_{mnp} r\right)\left(\overline{\lambda}\gamma_{qrs} r\right)\left(r\gamma^{pqt} r\right) = \left(r\gamma_{mnp} r\right)\left(r\gamma_{qrs} r\right)\left(\overline{\lambda}\gamma^p\gamma^t\gamma^q\overline{\lambda}\right), \tag{3.42}$$

  which vanishes because $\left(\overline{\lambda}\gamma^{mnp}\overline{\lambda}\right) = 0$.

- terms with one N, one Π and one partial derivative (Taylor expansion of a quadratic pole), as

$$\frac{\left(\partial\overline{\lambda}\gamma_{mnp} r\right)\left(\overline{\lambda}\gamma_{qrs} r\right)\left(r\gamma^{pqr} r\right) N^{mn} \Pi^s}{\left(\overline{\lambda}\lambda\right)^6}, \tag{3.43}$$

  which vanishes, since

$$\begin{aligned}\left(\overline{\lambda}\gamma_{qrs} r\right)\left(r\gamma^{pqr} r\right) &= \left(\overline{\lambda}\gamma_{qr}\gamma_s r\right)\left(r\gamma^{qr}\gamma^p r\right)\\ &= 4\left(\overline{\lambda}\gamma^m r\right)\left(r\gamma_s\gamma_m\gamma^p r\right)\\ &- 2\left(\overline{\lambda}\gamma_s r\right)\left(r\gamma^p r\right) - 8\left(r\gamma_s r\right)\left(\overline{\lambda}\gamma^p r\right)\\ &= 0.\end{aligned} \tag{3.44}$$

- terms with one N and two d's, like

$$\frac{\left(\bar{\lambda}\gamma_{mnp}r\right)\left(\bar{\lambda}\gamma^{mqr}r\right)N^{n}{}_{r}\left(r\gamma^{p}d\right)\left(r\gamma_{q}d\right)}{\left(\bar{\lambda}\lambda\right)^{6}}. \tag{3.45}$$

Since $\bar{\lambda}\gamma^{mnp}r$ is equal to $\bar{\lambda}\gamma^{m}\gamma^{n}\gamma^{p}r$, this term is proportional to $\left(\bar{\lambda}\gamma^{m}\right)^{\alpha}\left(\bar{\lambda}\gamma_{m}\right)^{\beta}$, and, according to equation (A.9), it vanishes.

- terms with two d's and one partial derivative, such as

$$\frac{\left(\bar{\lambda}\gamma_{mnp}r\right)\left(\partial\bar{\lambda}\gamma^{mnq}r\right)\left(r\gamma^{p}d\right)\left(r\gamma_{q}d\right)}{\left(\bar{\lambda}\lambda\right)^{6}}. \tag{3.46}$$

Decomposing $\left(\partial\bar{\lambda}\gamma^{mnp}r\right)$ as $\left(\partial\bar{\lambda}\gamma^{mn}\gamma^{p}r\right)+\eta^{np}\left(\bar{\lambda}\gamma^{m}\partial r\right)-\eta^{mp}\left(\bar{\lambda}\gamma^{n}\partial r\right)$, it is possible to rewrite the expression as follows,

$$\begin{aligned}
\left(\bar{\lambda}\gamma_{mnp}r\right)\left(\partial\bar{\lambda}\gamma^{mnq}r\right) &= \left(\bar{\lambda}\gamma_{mn}\gamma_{p}r\right)\left(\partial\bar{\lambda}\gamma^{mn}\gamma^{q}r\right)+2\eta^{nq}\left(\bar{\lambda}\gamma_{mnp}r\right)\left(\bar{\lambda}\gamma^{m}\partial r\right) \\
&= \left(\bar{\lambda}\gamma_{m}\partial\bar{\lambda}\right)\left(r\gamma_{p}\gamma_{m}\gamma^{q}r\right)-2\left(\bar{\lambda}\gamma_{p}r\right)\left(\partial\bar{\lambda}\gamma^{q}r\right) \\
&- 8\left(\bar{\lambda}\gamma^{q}r\right)\left(\partial\bar{\lambda}\gamma_{p}r\right)+2\eta^{nq}\left(\bar{\lambda}\gamma_{m}\gamma_{np}r\right)\left(\bar{\lambda}\gamma^{m}\partial r\right) \\
&= 0,
\end{aligned} \tag{3.47}$$

showing that this term also vanishes.

- and terms with one Π and two partial derivatives (Taylor expansion of a cubic pole), like

$$\frac{\left(\partial\bar{\lambda}\gamma_{mnp}\partial r\right)\left(\bar{\lambda}\gamma^{mnq}r\right)\left(r\gamma^{p}{}_{qr}r\right)\Pi^{r}}{\left(\bar{\lambda}\lambda\right)^{6}}. \tag{3.48}$$

Decomposing $\left(\partial\bar{\lambda}\gamma^{mnp}\partial r\right)$ as $\left(\partial\bar{\lambda}\gamma^{mn}\gamma^{p}\partial r\right)-\eta^{np}\left(\partial\bar{\lambda}\gamma^{m}\partial r\right)+\eta^{mp}\left(\partial\bar{\lambda}\gamma^{n}\partial r\right)$, the expression

$$\left(\partial\bar{\lambda}\gamma_{mnp}\partial r\right)\left(\bar{\lambda}\gamma^{mnq}r\right)\left(r\gamma^{p}{}_{qr}r\right) \tag{3.49}$$

can be split into two pieces. One of them is similar to the ones presented before and also vanishes. The other one is proportional to

$$\left(\bar{\lambda}\gamma_{m}\partial r\right)\left(\bar{\lambda}\gamma_{n}\partial r\right)\left(r\gamma^{mnp}r\right) = \left(r\gamma_{m}\partial r\right)\left(\bar{\lambda}\gamma_{n}\partial r\right)\left(\bar{\lambda}\gamma^{mnp}r\right)$$

$$= \; -\left(r\gamma_m \partial r\right)\left(\overline{\lambda}\gamma_n \partial r\right)\left(\overline{\lambda}\gamma^n \gamma^{mp} r\right), \qquad (3.50)$$

and vanishes, since $\left(\overline{\lambda}\gamma^m\right)^\alpha \left(\overline{\lambda}\gamma_m\right)^\beta = 0$.

For

$$(bb)_5 \equiv \int dz \left\{b_3(z)\, b_2(y) + b_2(z)\, b_3(y)\right\}, \qquad (3.51)$$

all contributions to the simple pole will have d_α:

- there are terms with two N's, as

$$\frac{\left(\overline{\lambda}\gamma_{mnp} r\right)\left(r\gamma^p d\right)\left(\overline{\lambda}\gamma_{qrs} r\right)\left(r\gamma^{stu} r\right) N^{mq} \eta^{nr} N_{tu}}{\left(\overline{\lambda}\lambda\right)^7}. \qquad (3.52)$$

  Note that

$$\begin{aligned}\left(\overline{\lambda}\gamma_{mnp} r\right)\left(\overline{\lambda}\gamma_{qrs} r\right)\eta^{nr} &= \left(\overline{\lambda}\gamma_n \gamma_{mp} r\right)\left(\overline{\lambda}\gamma_r \gamma_{qs} r\right)\eta^{nr} \\ &= \left(\overline{\lambda}\gamma_m\right)^\alpha \left(\overline{\lambda}\gamma^m\right)^\beta (\ldots)_{\alpha\beta} \\ &= 0, \qquad (3.53)\end{aligned}$$

  gives a vanishing contribution.

- terms with one N and one partial derivative, as

$$\frac{\left(\overline{\lambda}\gamma_{mnp} \partial r\right)\left(r\gamma^p d\right)\left(\overline{\lambda}\gamma^{mnq} r\right)\left(r\gamma_{qrs} r\right) N^{rs}}{\left(\overline{\lambda}\lambda\right)^7}. \qquad (3.54)$$

  It is easy to extract the pure spinor constraint out of this expression:

$$\begin{aligned}\left(\overline{\lambda}\gamma_{mnp} \partial r\right)\left(\overline{\lambda}\gamma^{mnq} r\right) &= \left(\overline{\lambda}\gamma_{mn}\gamma_p \partial r\right)\left(\overline{\lambda}\gamma^{mn}\gamma^q r\right) \\ &- 2\left(\overline{\lambda}\gamma_m \partial r\right)\left(\overline{\lambda}\gamma^m \gamma^{nq} r\right)\eta_{np} \\ &= 4\left(\overline{\lambda}\gamma^m \overline{\lambda}\right)\left(\partial r \gamma_p \gamma_m \gamma^q r\right) - 10\left(\overline{\lambda}\gamma_p \partial r\right)\left(\overline{\lambda}\gamma^q r\right) \\ &- 2\left(\overline{\lambda}\gamma_m \partial r\right)\left(\overline{\lambda}\gamma^{m_0}\gamma^{nq} r\right)\eta_{np}. \\ &= 0. \qquad (3.55)\end{aligned}$$

- and terms with two partial derivatives, coming from the cubic poles, like

$$\frac{\left(\partial\overline{\lambda}\gamma_{mnp}\partial r\right)\left(r\gamma^{p}d\right)\left(\overline{\lambda}\gamma^{mnq}r\right)\left(r\gamma_{qrs}r\right)\left(\overline{\lambda}\gamma^{rs}\lambda\right)}{\left(\overline{\lambda}\lambda\right)^{8}}. \tag{3.56}$$

  Note that $\left(r\gamma_{qrs}r\right)\left(\overline{\lambda}\gamma^{rs}\lambda\right)$ has the same structure of (3.25) and also vanishes.

Finally, for the last term in the $b\left(z\right)b\left(y\right)$ OPE, where only the ghost fields appear,

$$\left(bb\right)_{6}\equiv\int dz\left\{b_{3}\left(z\right)b_{3}\left(y\right)\right\}, \tag{3.57}$$

- there are terms with three N's, like

$$\frac{\left(\overline{\lambda}\gamma_{mnp}r\right)\left(r\gamma^{pqr}r\right)\left(\overline{\lambda}\gamma^{mst}r\right)\left(r\gamma_{tuv}r\right)N_{qr}N^{n}{}_{s}N^{uv}}{\left(\overline{\lambda}\lambda\right)^{8}}. \tag{3.58}$$

  Since $\overline{\lambda}\gamma^{mnp}r=\overline{\lambda}\gamma^{m}\gamma^{n}\gamma^{p}r$, $\left(\overline{\lambda}\gamma_{mnp}r\right)\left(\overline{\lambda}\gamma^{mqr}r\right)$ vanishes, as shown above.

- terms with two N's and one partial derivative, like

$$\frac{\partial\left(\overline{\lambda}\gamma_{mnp}r\right)\left(r\gamma^{pqr}r\right)\left(\overline{\lambda}\gamma^{mns}r\right)\left(r\gamma_{stu}r\right)N_{qr}N^{tu}}{\left(\overline{\lambda}\lambda\right)^{8}}, \tag{3.59}$$

  which has the same structure presented before, being proportional to the pure spinor constraints.

- terms with one N and two partial derivatives, coming from triple poles, such as

$$\frac{\partial^{2}\left(\overline{\lambda}\gamma_{mnp}r\right)\left(r\gamma^{pqr}r\right)\left(\overline{\lambda}\gamma^{mns}r\right)\left(r\gamma_{qst}r\right)N_{r}{}^{t}}{\left(\overline{\lambda}\lambda\right)^{8}}, \tag{3.60}$$

  which are similar to the above ones and vanish.

- and terms with three partial derivatives, like

$$\frac{\left(\partial\overline{\lambda}\gamma_{mnp}\partial r\right)\left(r\gamma^{pqr}\partial r\right)\left(\overline{\lambda}\gamma^{mns}r\right)\left(r\gamma_{qrs}r\right)}{\left(\overline{\lambda}\lambda\right)^{8}}, \tag{3.61}$$

that can be rewritten as

$$\frac{\left(\partial\bar{\lambda}\gamma_m\partial r\right)\left(r\gamma^q\partial r\right)\left(\bar{\lambda}\gamma^{mnp}r\right)\left(r\gamma_{npq}r\right)}{\left(\bar{\lambda}\lambda\right)^8} \tag{3.62}$$

and vanish, since

$$\begin{aligned} \left(\bar{\lambda}\gamma^{mnp}r\right)\left(r\gamma_{npq}r\right) &= \left(\bar{\lambda}\gamma^{np}\gamma^m r\right)\left(r\gamma_{np}\gamma_q r\right) \\ &= \left(\bar{\lambda}\gamma^n r\right)\left(r\gamma^m\gamma_n\gamma_q r\right) \\ &- 2\left(\bar{\lambda}\gamma^m r\right)\left(r\gamma_q r\right) - 8\left(r\gamma^m r\right)\left(\bar{\lambda}\gamma_q r\right) \\ &= 0. \end{aligned} \tag{3.63}$$

Summarizing, in the OPE computation several terms vanish identically due to the pure spinor constraints (in particular, $(bb)_4$, $(bb)_5$ and $(bb)_6$ do not present nontrivial contributions). The remaining terms are excluded through the BRST argument, since they were shown to be not BRST closed. Therefore,

$$(bb)_1 = (bb)_2 = (bb)_3 = (bb)_4 = (bb)_5 = (bb)_6 = 0, \tag{3.64}$$

and the pure spinor b ghost is, indeed, nilpotent:

$$b(z)\, b(y) \sim \text{regular}. \tag{3.65}$$

3.3 Non-minimal pure spinor formalism as a $N = 2$ topological string

Another interesting property of the b ghost is the pole structure of its OPE with the BRST current:

$$J_{BRST}(z)\, b(y) \sim \frac{3}{(z-y)^3} + \frac{J}{(z-y)^2} + \frac{T}{(z-y)}, \tag{3.66}$$

where

$$J = J_\lambda + J_r - 2\frac{\bar{\lambda}\partial\lambda}{\bar{\lambda}\lambda} + 2\frac{r\partial\theta}{\bar{\lambda}\lambda} - 2\frac{(r\lambda)\left(\bar{\lambda}\partial\theta\right)}{\left(\bar{\lambda}\lambda\right)^2}.$$

$$= \quad J_{\lambda} - J_{\overline{\lambda}} - \left\{ Q, \left(S + 2\frac{\overline{\lambda}\partial\theta}{\overline{\lambda}\lambda} \right) \right\}. \tag{3.67}$$

With a BRST transformation, the U(1) current can be brought into a more natural form, without changing the ghost numbers of the BRST charge and the b ghost.

To verify the interpretation of J as the ghost number current, it is worth noting that,

$$T(z)\,J(y) \sim -\frac{3}{(z-y)^3} + \frac{J}{(z-y)^2} + \frac{\partial J}{(z-y)}, \tag{3.68}$$

$$J(z)\,J_{BRST}(y) \sim \frac{J_{BRST}}{(z-y)}, \tag{3.69}$$

$$J(z)\,b(y) \sim -\frac{b}{(z-y)}, \tag{3.70}$$

$$J(z)\,J(y) \sim \frac{3}{(z-y)^2}. \tag{3.71}$$

Together, b, T, J_{BRST} and J may describe a twisted $\mathcal{N}=2$ $\hat{c}=3$ critical topological string [7]. The untwisted version would satisfy

$$T'(z)\,T'(y) \sim \frac{(9/2)}{(z-y)^4} + 2\frac{T'}{(z-y)^2} + \frac{\partial T'}{(z-y)}, \quad T'(z)\,J(y) \sim \frac{J}{(z-y)^2} + \frac{\partial J}{(z-y)},$$

$$T'(z)\,G^+(y) \sim \frac{3}{2}\frac{G^+}{(z-y)^2} + \frac{\partial G^+}{(z-y)}, \quad T'(z)\,G^-(y) \sim \frac{3}{2}\frac{G^-}{(z-y)^2} + \frac{\partial G^-}{(z-y)},$$

$$J(z)\,G^+(y) \sim \frac{G^+(y)}{(z-y)}, \quad J(z)\,G^-(y) \sim -\frac{G^-(y)}{(z-y)},$$

$$J(z)\,J(y) \sim \frac{3}{(z-y)^2}, \quad G^+(z)\,G^-(y) \sim \frac{3}{(z-y)^3} + \frac{J}{(z-y)^2} + \frac{T'+\frac{1}{2}\partial J}{(z-y)},$$

$$G^+(z)\,G^+(y) \sim \text{regular}, \quad G^-(z)\,G^-(y) \sim \text{regular},$$

where $G^+ = J_{BRST}$, $G^- = b$ and $T' = T - \frac{1}{2}\partial J$. The twist here means $T' \to T' - \frac{1}{2}\partial J$, which modifies the conformal weights of the ghosts λ and r from $\frac{1}{2}$ to 0 and turns the central charge off.

3.4 b ghost cohomology

It is interesting to point out that in the same manner that the BRST cohomology is non-trivial only for world-sheet scalars, the b ghost cohomology can be shown to be non-trivial only under a certain condition, that will now be derived.

Defining,

$$B \equiv \oint dzb\left(z\right), \tag{3.72}$$

it is direct to demonstrate through the OPE (3.66) that

$$\left\{B, J_{BRST}\left(z\right)\right\} = T\left(z\right) - \partial J\left(z\right). \tag{3.73}$$

Now, suppose that there is an operator V_{hg} satisfying

$$T\left(z\right)V_{hg}\left(y\right) \sim h\frac{V_{hg}}{\left(z-y\right)^2} + \frac{\partial V_{hg}}{\left(z-y\right)}, \tag{3.74a}$$

$$J\left(z\right)V_{hg}\left(y\right) \sim g\frac{V_{hg}}{\left(z-y\right)}, \tag{3.74b}$$

and that is annihilated by B, *i.e.*

$$\left[B, V_{hg}\right] = 0. \tag{3.75}$$

Then it follows that

$$\left\{B, J_{BRST}\left(z\right)V_{hg}\left(y\right)\right\} \sim \left(h+g\right)\frac{V_{hg}}{\left(z-y\right)^2} + \left(1-g\right)\frac{\partial V_{hg}}{\left(z-y\right)}, \tag{3.76}$$

showing that V_{hg} is B-exact for $\left(h+g\right) \neq 0$ and constituting an exclusion criterion for the non-trivial cohomology of B.

The cohomology of B will not be further discussed. Note that even the space where B acts is not yet understood. For example, one has to be concerned about poles em $\left(\overline{\lambda}\lambda\right)$ higher than 11, as there is not a simple regularization scheme that would allow a formal functional integration over its zero modes [8]. Note also that there is no natural candidate for an operator that trivializes the B cohomology.

3.5 Non-uniqueness

From equation (3.8), it is clear that the b ghost can be defined only up to BRST-exact terms. In this sense, it is not unique and it might be interesting to check whether the basic properties presented above are preserved with a BRST-exact deformed version of b.

For example, the simplest known version of the pure spinor b ghost,

$$b_{nc} = -s\partial\overline{\lambda} + \left(\frac{C_\alpha}{C\lambda}, G^\alpha\right) + 2\frac{(C\partial\lambda)\,(C\partial\theta)}{(C\lambda)^2}, \tag{3.77}$$

differs from (3.5) by a BRST-exact term [9]. It obviously satisfies $\{Q_0, b_{nc}\} = T$, but it is non-covariant due to the presence of the constant spinor C_α.

Performing the OPE computation of b_{nc} with itself, given by

$$b_{nc}(z)\, b_{nc}(y) \sim \frac{1}{(z-y)} \frac{(C\gamma_m C)}{2\,(C\lambda)^2} \left\{ \Pi^m \underbrace{\left(-\frac{1}{\alpha'}\Pi^2 - d\partial\theta\right)}_{A} - \frac{\alpha'}{8} \underbrace{(d\gamma^m \partial d)}_{D^m} + \ldots \right\}, \tag{3.78}$$

one readily observes that b_{nc} is nilpotent[2] only for $(C\gamma_m C) = 0$, that is, when C_α is a pure spinor.

Consider, now,

$$b' = b + [Q, \beta]\,. \tag{3.79}$$

Due to (3.65), it is clear that

$$b'(z)\, b'(y) \sim b(z)\,[Q, \beta(y)] + [Q, \beta(z)]\, b(y) + [Q, \beta(z)]\,[Q, \beta(y)]\,. \tag{3.80}$$

Note that the left-hand side of this relation can be written as

$$\{Q, \beta(z)\, b(y) - b(z)\, \beta(y) + \beta(z)\,[Q, \beta(y)]\} + T(z)\, \beta(y) - \beta(z)\, T(y)\,.$$

Requiring β to be a primary conformal weight 2 object,

$$T(z)\, \beta(y) - \beta(z)\, T(y) \sim \text{regular}, \tag{3.81}$$

and equation (3.80) is equivalent to

$$b'(z)\, b'(y) \sim \{Q, (\beta(z)\, b(y) - b(z)\, \beta(y) + \beta(z)\,[Q, \beta(y)])\}\,. \tag{3.82}$$

[2]Observe that A, the Virasoro constraint, and D^m, belong to Siegel's algebra [14]. As a possible pole in (3.78) should be BRST-closed, it is straightforward to determine the remaining terms appearing inside the curly brackets of (3.78), up to BRST-exact ones, like $\{Q_0, \partial\Pi^m \frac{C\partial\theta}{C\lambda}\}$.

There is no hope that b' will be nilpotent for a generic β, as in the non-covariant example above, and to understand the general case, it is useful to start with a simpler one.

3.5.1 $\beta = \left(S, \frac{\overline{\lambda}\partial\theta}{\overline{\lambda}\lambda}\right)$

Consider the particular covariant deformation

$$\beta = \left(S, \frac{\overline{\lambda}\partial\theta}{\overline{\lambda}\lambda}\right), \tag{3.83}$$

where S was defined in (2.38). It is straightforward to show that

$$\begin{aligned} [Q, \beta] &= -\left(\left(J_{\overline{\lambda}} + J_r\right), \frac{\overline{\lambda}\partial\theta}{\overline{\lambda}\lambda}\right) - \left(S, \frac{\overline{\lambda}\partial\lambda}{\overline{\lambda}\lambda}\right) \\ &\quad + \left(S, \frac{r\partial\theta}{\overline{\lambda}\lambda}\right) - \left(S, \frac{\left(\overline{\lambda}\partial\theta\right)\left(r\lambda\right)}{\left(\overline{\lambda}\lambda\right)^2}\right), \end{aligned} \tag{3.84}$$

and

$$\beta\left(z\right)\left[Q, \beta\left(y\right)\right] \sim \text{regular}. \tag{3.85}$$

There are no double poles, and possible simple poles are proportional to $\left(\overline{\lambda}\partial\theta\right)^2 = 0$.

The OPE between β and b is also simple to obtain. The action of S in b is to transform r_α in $\overline{\lambda}_\alpha$, making all the terms in the chain vanish due to the antisymmetric form of $H^{\alpha\beta}$, $K^{\alpha\beta\gamma}$ and $L^{\alpha\beta\gamma\lambda}$. There are simple poles proportional to $\left(\overline{\lambda}\partial\theta\right)^2$ and also quadratic poles related to the contraction between d_α and $\partial\theta^\beta$ (and, of course, simple poles coming from the Taylor expansion), but they always appear together with the constraints (2.32) and (2.33), implying that

$$\beta\left(z\right) b\left(y\right) \sim \text{regular}. \tag{3.86}$$

Looking back to expression (3.82) and using (3.85) and (3.86), nilpotency of the deformed b ghost

$$b_a \equiv b + a\left[Q_0, \left(S, \frac{\overline{\lambda}\partial\theta}{\overline{\lambda}\lambda}\right)\right], \tag{3.87}$$

follows directly, *i.e.*

$$b_a\left(z\right) b_a\left(y\right) \sim \text{regular}. \tag{3.88}$$

Here, a is just a numerical constant.

As a final check, the OPE computation of b_a with the BRST current results

$$J_{BRST}(z)\, b_a(y) \sim \frac{3}{(z-y)^3} + \frac{J_a}{(z-y)^2} + \frac{T}{(z-y)}, \tag{3.89}$$

where

$$J_a = J_\lambda - aJ_{\overline{\lambda}} + (1-a)\, J_r + (8a-2)\left(\frac{\overline{\lambda}\partial\lambda}{\overline{\lambda}\lambda} - \frac{r\partial\theta}{\overline{\lambda}\lambda} + \frac{(r\lambda)\left(\overline{\lambda}\partial\theta\right)}{\left(\overline{\lambda}\lambda\right)^2}\right). \tag{3.90}$$

Observe that

$$\begin{aligned} \oint dz \left\{ J_a(z) \frac{\overline{\lambda}_\alpha}{\left(\overline{\lambda}\lambda\right)}(y) \right\} &= -\frac{\overline{\lambda}_\alpha}{\left(\overline{\lambda}\lambda\right)}(y), \\ \oint dz \left\{ J_a(z) \frac{r_\alpha}{\left(\overline{\lambda}\lambda\right)}(y) \right\} &= 0, \end{aligned}$$

and the right-hand sides of the above equations do not depend on a.

Together, b_a, J_a, J_{BRST} and T satisfy a $\mathcal{N}=2$ $\hat{c}=3$ critical topological string algebra.

To illustrate how this particular example might be interesting, observe that (3.90) admits three simplifications, depending on the numerical choices of a:

- the first one is trivial, $a=0$, and corresponds to the usual construction, without deformations;
- the second choice is $a=1$, removing J_r from the $U(1)$ current. In this case, the combination $(J_\lambda - J_{\overline{\lambda}})$ is explicit, but does not appear alone.
- and the last one is $a=\frac{1}{4}$. With this particular choice, (3.90) is more conventional looking, since the unusual non-quadratic-terms vanish:

$$J_{\frac{1}{4}} = J_\lambda - \frac{1}{4}J_{\overline{\lambda}} + \frac{3}{4}J_r. \tag{3.91}$$

Note that the non-minimal variables become fractionally charged.

As a final comment before presenting the general discussion, note that β satisfies the criteria discussed in section 3.4 as a possible non-trivial object in the b ghost cohomology.

3.5.2 The invariance of the topological string algebra

Now, a more general class of deformations (defined as in (3.79)) will be analysed.

Requiring invariance of the $\mathcal{N} = 2\ \hat{c} = 3$ algebra, it will be shown that some constraints on the deformations must be imposed and β will be restricted to be:

- a commuting object, in order for b' to have definite statistics;
- an ghost number -2 object with respect to J (then b' will have a definite ghost charge), that is
$$J(z)\,\beta(y) \sim -2\frac{\beta}{(z-y)}. \tag{3.92}$$
- supersymmetric, which avoids the explicit introduction of objects that trivialize the cohomology. The two known examples are
$$\xi_1 = \frac{(C\theta)}{(C\lambda)}, \tag{3.93a}$$
$$\xi_2 = \frac{(\bar{\lambda}\theta)}{(\bar{\lambda}\lambda) - (r\theta)}, \tag{3.93b}$$
where $\{Q_0, \xi_1\} = \{Q_0, \xi_2\} = 1$ and C_α is any constant spinor[3];
- and, as already mentioned, a primary conformal weight 2 field,
$$T(z)\,\beta(y) \sim 2\frac{\beta}{(z-y)^2} + \frac{\partial\beta}{(z-y)}; \tag{3.94}$$

With this in mind, the impact of the deformation on the topological string algebra will be analyzed.

[3] The constructions with constant spinors in (3.77) and (3.93a) are a bit subtle, since they not globally defined in the pure spinor space. More details can be found in [27].

Assuming that the BRST current does not change, the first relation that will be presented is the OPE between J_{BRST} and β, that can be generically written as

$$J_{BRST}(z)\,\beta(y) \sim \frac{[Q_1,\beta] - y\,[Q_0,\beta]}{(z-y)^2} + \frac{[Q_0,\beta]}{(z-y)}, \tag{3.95}$$

where Q_n is defined to be

$$Q_n \equiv \oint dz z^n J_{BRST}(z)\,,$$

not to be confused with the r-charge used in the nilpotency analysis. Now, for example, Q_0 will represent the full BRST-charge.

Note that the cubic pole (3.95) vanishes, since there are no ghost number -1 anticommuting world-sheet scalars with the above requisites (for example, $(\overline{\lambda}\theta)$ is ruled out as it is not supersymmetric). Then, it follows that

$$J_{BRST}(z)\,[Q_0,\beta]\,(y) \sim \frac{J' - J}{(z-y)^2}, \tag{3.96}$$

for the BRST charge Q_0 is nilpotent. The quadratic pole does not have to vanish and the deformed $U(1)$ current is defined to be

$$J' = J - \{Q_0, [Q_1,\beta]\} \tag{3.97}$$

Therefore, (3.66) is reproduced with $J \to J'$ and $b \to b'$.

The next OPE, (3.68), is obviously preserved, since β is a primary conformal weight 2 field by assumption.

The J' OPE with itself is given generically by

$$J'(z)\,J'(y) \sim \left\{ \frac{3+\varphi}{(z-y)^2} + \frac{\frac{1}{2}\partial\varphi}{(z-y)} \right\}, \tag{3.98}$$

where the contribution of $J(z)\,J(y)$ was made explicit. Observe that,

$$\begin{aligned} J(z)\,\{Q_0,[Q_1,\beta(y)]\} &= \{Q_0, J(z)\,[Q_1,\beta(y)]\} + J_{BRST}(z)\,[Q_1,\beta(y)] \\ &= -\{Q_0,[Q_1,J(z)]\,\beta(y)\} - J_{BRST}(z)\,[Q_1,\beta]\,(y) \\ &\quad + \{Q_0,[Q_1,J(z)\,\beta(y)]\}\,. \end{aligned}$$

As $[Q_1, J(z)] = -zJ_{BRST}(z)$, the right-hand side can be rewritten as

$$\begin{aligned} J(z)\{Q_0,[Q_1,\beta(y)]\} &= \{Q_0,[Q_1,J(z)\beta(y)]\} \\ &\quad -z\{Q_0,J_{BRST}(z)\beta(y)\} + \{Q_1,J_{BRST}(z)\beta(y)\}\,. \end{aligned} \tag{3.99}$$

Noting that

$$\{Q_n, Q_m\} = 0, \tag{3.100}$$

for any $n, m \geq 0$, equation (3.95) implies that $J(z)\{Q_0,[Q_1,\beta(y)]\}$ is BRST-exact. Thus, replacing the definition (3.97) in left hand side of (3.98), φ is demonstrated to be a ghost number 0 BRST-exact world-sheet scalar, which cannot appear due to the hypothesis on β and shows that the OPE (3.71) is reproduced when $J \to J'$.

Going on, the OPE with J_{BRST} and $(J' - J)$ can be trivially shown to be regular, as its general form can be cast as

$$J_{BRST}(z)\{Q_1,[Q_0,\beta]\}(0) \sim \frac{[Q_1\{Q_1,[Q_0,\beta]\}]}{z^2} + \frac{[Q_0,\{Q_1,[Q_0,\beta]\}]}{z}. \tag{3.101}$$

Through (3.100), this result demonstrates that the deformations preserve (3.69).

The last OPE to be analysed is (3.70), with $J \to J'$ and $b \to b'$.

$$\begin{aligned} J'(z)\,b'(y) &= J(z)\,b(y) + \{Q_1,[Q_0,\beta(z)]\}\,b(y) \\ &\quad +J(z)[Q_0,\beta(y)] + \{Q_1,[Q_0,\beta(z)]\}[Q_0,\beta(y)]\,. \end{aligned} \tag{3.102}$$

Using (3.92), (3.95) and $\{Q_1, b(z)\} = J(z) + zT(z)$, the above equation can be rewritten as

$$\begin{aligned} J'(z)\,b'(y) &\sim -\frac{b'}{(z-y)} \\ &\quad +[Q_0,\{Q_1,[Q_0,\beta(z)]\}\beta(y)] \\ &\quad -[Q_0,\{Q_1,\beta(z)\,b(y)\}] \end{aligned} \tag{3.103}$$

In order for the topological algebra to be preserved, equations (3.82) and (3.103) impose some conditions on β and the following OPE's must hold up to BRST-closed poles:

$$\beta(z)\,b(y) \sim \text{regular}, \tag{3.104a}$$
$$\beta(z)\,[Q_0,\beta(y)] \sim \text{regular}, \tag{3.104b}$$
$$\beta(z)\,\{Q_0,[Q_1,\beta(y)]\} \sim \text{regular}. \tag{3.104c}$$

If this is the case,

$$\begin{aligned} b'(z)\,b'(y) &\sim \text{regular}, \\ J'(z)\,b'(y) &\sim -\frac{b'}{(z-y)}. \end{aligned}$$

Therefore, the $\mathcal{N}=2$ $\hat{c}=3$ critical topological string algebra is invariant under the self-consistent deformations of the b ghost and the $U(1)$ current, respectively, (3.79) and (3.97), as long as the requisites on β presented before equation (3.95) and in (3.104) are imposed. The example of subsection 3.5.1 satisfies all of these conditions.

3.6 The c ghost

The final feature to be investigated here about the b ghost is the existence of its conjugate, namely the c ghost.

In the topological string perspective, the existence of a c ghost in the non-minimal pure spinor formalism may seem to be meaningless. Indeed, the construction of the b ghost conjugate is very unusual and, more than that, unrequired. The reason is simple. First, one does not have a natural -1 conformal weight field to work with. Second, the amplitudes prescription (including the notion of unintegrated vertex, compared to the other superstring formalisms) is very well established without it.

Under these conditions, a c ghost like field is undoubtedly strange. It will be defined as

$$c \equiv -\frac{(r\lambda)}{\left(\partial\bar{\lambda}\lambda\right)+(r\partial\theta)}, \tag{3.105}$$

satisfying the relation

$$b(z)\,c(y) \sim \frac{1}{(z-y)}. \tag{3.106}$$

Note that c must have ghost number $+1$, since b is a ghost number -1 field.

To demonstrate that (3.105) is the conjugate of (3.5), observe that

$$(r\lambda) = -\left[Q_0, \left(\bar{\lambda}\lambda\right)\right]. \tag{3.107}$$

By a direct computation, one can derive

$$b(z)(r\lambda)(y) \sim -\frac{\left(\partial\bar{\lambda}\lambda\right) + (r\partial\theta)}{(z-y)}, \tag{3.108}$$

which is verified through

$$\begin{aligned} b(z)\left[Q_0, \left(\bar{\lambda}\lambda\right)\right](y) &= -\left\{Q_0, b(z)\left(\bar{\lambda}\lambda\right)(y)\right\} + \{Q_0, b(z)\}\left(\bar{\lambda}\lambda\right)(y) \\ &= -\left\{Q_0, b(z)\left(\bar{\lambda}\lambda\right)(y)\right\} + T(z)\left(\bar{\lambda}\lambda\right)(y) \\ &\sim \frac{\partial\left(\bar{\lambda}\lambda\right)}{(z-y)} - \frac{\left\{Q_0, \left(\bar{\lambda}\partial\theta\right)\right\}}{(z-y)} \\ &\sim \frac{\left(\partial\bar{\lambda}\lambda\right) + (r\partial\theta)}{(z-y)}, \end{aligned} \tag{3.109}$$

where (3.8) was used.

Since the b ghost is nilpotent, (3.65), the right hand side of the above equation does not have any poles with b, that is

$$b(z)\left(\partial\bar{\lambda}\lambda + r\partial\theta\right)(y) \sim \text{regular}. \tag{3.110}$$

Therefore, equation (3.106) is directly reproduced.

Note also that

$$\{Q_0, c\} = c\partial c, \tag{3.111}$$

the usual BRST relation between the c ghost and the BRST charge, and that c is a supersymmetric Lorentz scalar.

The analogous construction of the c ghost as the conjugate of (3.87) is

$$c_a \equiv -\frac{(r\lambda)}{\partial\left(\bar{\lambda}\lambda\right) + (a-1)\left\{Q_0, \left(\bar{\lambda}\partial\theta\right)\right\}}, \tag{3.112}$$

satisfying

$$b_a(z)\,c_a(y) \sim \frac{1}{(z-y)}. \tag{3.113}$$

Although interesting, the strange form of (3.105) may be pathological, in the sense that one now is able to construct an entire new class of composite operators that trivialize the cohomology, *e.g.*

$$\xi \equiv \frac{\overline{\lambda}\partial\theta}{\overline{\lambda}\partial\lambda - r\partial\theta} \Rightarrow \{Q_0, \xi\} = 1. \tag{3.114}$$

It is clear, however, that this construction is highly artificial and cannot emerge naturally in any known process for the pure spinor formalism. From the conformal field theory point of view, this kind of construction is very unusual. Note that the denominator in (3.105) contains derivatives of world-sheet scalars, which implies that, wherever they vanish, the c ghost is singular.

Note also that the existence of a composite field satisfying

$$b(z)\,c(y) \sim \frac{1}{(z-y)},$$

trivializes the cohomology of b. In the twisting picture, the BRST current J_{BRST} and the b ghost exchange roles in different twists. Then, it might be useful to understand the cohomology of the pure spinor b ghost and study the Siegel's gauge implementation on the physical vertices (*e.g.* [29]).

Chapter 4

Conclusion

In this thesis, an extensive study on the properties of the non-minimal pure spinor b ghost was presented. Being a composite operator, even supposed to be simple properties are not easy to work out. In doing so, however, the b ghost has been shown to be structurally rich.

Some of these properties were addressed along the Ph.D. project and can be summarized as follows:

- nilpotency, which is crucial in the topological string interpretation of the non-minimal pure spinor formalism [11]. The previous demonstration was incomplete. By presenting an explicit counter example, the proof could be carried out using cohomology arguments.

- the non-uniqueness of b, as it is defined up to BRST-exact terms. This property has been mildly explored in the formalism. A set of consistency criteria were established in order for these ambiguities not to spoil the $\mathcal{N} = 2$ topological string algebra [12].

- and the introduction the c ghost. The existence of the canonical conjugate of the pure spinor b ghost is still a mystery. Based on an artificial construction (and, up to the knowledge of the author, unexplored in 2d CFT's), a composite c ghost operator was found, satisfying the expected properties [12].

All these results are potentially interesting in the pure spinor context as we are continuing to dissect the very basic structures of the non-fundamental b. It is clear though that there isn't so far a complete understanding of the role that the pure spinor b ghost plays when the relation with the other superstring formalisms is concerned. Recent works have made a huge progression towards this direction [13, 30], but the full picture is not yet transparent.

There are other features about b that may also help to clarify different aspects of the formalism: the b ghost cohomology (also related to Siegel's gauge, *e.g.* [29]); similarity transformations that could simplify the b structure, possibly related to what is done in [13]; the role of the newly introduced c ghost and any possible relations with the natural ghost system that would arise in the usual gauge fixing of the reparametrization symmetry.

The pure spinor formalism also has been consistently shown to provide an adequate framework for describing the superstrings beyond flat space. Clearly, part of the task is to compute amplitudes in such backgrounds and the b ghost is a fundamental piece. Recent works have made some progress on the construction of the b ghost in curved backgrounds [32, 33, 34], but there is plenty to be understood yet.

Appendix A

Conventions and useful properties

A.1 Conventions

Indices:

$$\begin{cases} m, n, \ldots = 0, \ldots, 9 & \text{space-time vector indices,} \\ \alpha, \beta, \ldots = 1, \ldots, 16 & \text{space-time spinor indices,} \\ a, b, \ldots = 1, \ldots, 5 & U(5)\text{vector indices.} \end{cases}$$

The indices antisymmetrization is represented by the square brackets, meaning

$$[I_1 \ldots I_n] = \frac{1}{n!} \left(I_1 \ldots I_n + \text{all antisymmetric permutations} \right).$$

For example,

$$\gamma^{[m} \gamma^{n]} = \frac{1}{2} \left(\gamma^m \gamma^n - \gamma^n \gamma^m \right) = \gamma^{mn},$$

or,

$$\lambda^{[\alpha} H^{\beta\gamma]} = \frac{1}{3!} \left(\lambda^\alpha H^{\beta\gamma} - \lambda^\alpha H^{\gamma\beta} + \lambda^\beta H^{\gamma\alpha} - \lambda^\beta H^{\alpha\gamma} + \lambda^\gamma H^{\alpha\beta} - \lambda^\gamma H^{\beta\alpha} \right).$$

Concerning OPE's, the right-hand sides of the equations are always evaluated at the coordinate of the second entry, that is,

$$A(z) B(y) \sim \frac{C}{(z-y)^2} + \frac{D}{(z-y)}$$

means $C = C(y)$ and $D = D(y)$.

A.2 Gamma matrices

The gamma matrices $\gamma^m_{\alpha\beta}$ and $\gamma_m^{\alpha\beta}$ satisfy

$$\{\gamma^m, \gamma^n\}^\alpha{}_\beta = (\gamma^m)^{\alpha\sigma} \gamma^n_{\sigma\beta} + (\gamma^n)^{\alpha\sigma} \gamma^m_{\sigma\beta} = 2\eta^{mn}\delta^\alpha_\beta. \tag{A.1}$$

The Fierz decompositions of bispinors are given by

$$\begin{aligned} \chi^\alpha \psi^\beta &= \tfrac{1}{16}\gamma_m^{\alpha\beta}(\chi\gamma^m\psi) + \tfrac{1}{3!16}\gamma_{mnp}^{\alpha\beta}(\chi\gamma^{mnp}\psi) + \tfrac{1}{5!16}\left(\tfrac{1}{2}\right)\gamma_{mnpqr}^{\alpha\beta}(\chi\gamma^{mnpqr}\psi), \\ \chi_\alpha \psi^\beta &= \tfrac{1}{16}\delta^\beta_\alpha(\chi\psi) - \tfrac{1}{2!16}(\gamma_{mn})^\beta{}_\alpha(\chi\gamma^{mn}\psi) + \tfrac{1}{4!16}(\gamma_{mnpq})^\beta{}_\alpha(\chi\gamma^{mnpq}\psi), \end{aligned} \tag{A.2}$$

where

$$\gamma_m^{\alpha\beta} = \gamma_m^{\beta\alpha}, \quad \gamma_{mnp}^{\alpha\beta} = -\gamma_{mnp}^{\beta\alpha}, \quad \gamma_{mnpqr}^{\alpha\beta} = \gamma_{mnpqr}^{\beta\alpha}.$$

The main gamma matrix identity that is being used in this work is

$$(\gamma^{mn})^\alpha{}_\beta (\gamma_{mn})^\gamma{}_\lambda = 4\gamma^m_{\beta\lambda}\gamma_m^{\alpha\gamma} - 2\delta^\alpha_\beta\delta^\gamma_\lambda - 8\delta^\alpha_\lambda\delta^\gamma_\beta, \tag{A.3}$$

which can be deduced using (A.2). The other relevant one is given by

$$\eta_{mn}\left(\gamma^m_{\alpha\beta}\gamma^n_{\gamma\lambda} + \gamma^m_{\alpha\gamma}\gamma^n_{\beta\lambda} + \gamma^m_{\alpha\lambda}\gamma^n_{\gamma\beta}\right) = 0. \tag{A.4}$$

There are several other identities that can be derived from (A.3):

$$(\gamma^{mn})^\alpha{}_\beta\, \gamma^{\gamma\lambda}_{mnp} = 2(\gamma^m)^{\alpha\gamma}(\gamma_{pm})^\lambda{}_\beta + 6\gamma_p^{\alpha\gamma}\delta^\lambda_\beta - (\gamma \leftrightarrow \lambda), \tag{A.5}$$

$$(\gamma_{mn})^\alpha{}_\beta\, \gamma^{mnp}_{\gamma\lambda} = -2(\gamma_m)_{\beta\lambda}(\gamma^{pm})^\alpha{}_\gamma + 6\gamma^p_{\beta\lambda}\delta^\alpha_\gamma - (\gamma \leftrightarrow \lambda), \tag{A.6}$$

$$\gamma^{\alpha\beta}_{mnp}(\gamma^{mnp})^{\gamma\lambda} = 12\left[\gamma_m^{\alpha\lambda}(\gamma^m)^{\beta\gamma} - \gamma_m^{\alpha\gamma}(\gamma^m)^{\beta\lambda}\right], \tag{A.7}$$

$$\gamma^{\alpha\beta}_{mnp}\gamma^{mnp}_{\gamma\lambda} = 48\left(\delta^\alpha_\gamma\delta^\beta_\lambda - \delta^\alpha_\lambda\delta^\beta_\gamma\right). \tag{A.8}$$

All of them are very helpful in extracting the pure spinor constraints out of product of bispinors containing space-time vector indices contracted. For example:

$$\begin{aligned}\left(\bar{\lambda}\gamma_{mnp}r\right)\left(\bar{\lambda}\gamma^{mn}\lambda\right) &= 2\left(\bar{\lambda}\gamma^{m}\bar{\lambda}\right)\left(r\gamma_{pm}\lambda\right)+6\left(r\lambda\right)\left(\bar{\lambda}\gamma_{p}\bar{\lambda}\right)\\ &\quad -2\left(\bar{\lambda}\gamma^{m}r\right)\left(\bar{\lambda}\gamma_{pm}\lambda\right)-6\left(\bar{\lambda}\lambda\right)\left(\bar{\lambda}\gamma_{p}r\right)\\ &= 0.\end{aligned}$$

The last identity that is often used in the calculations is

$$\gamma^{m}\gamma^{n_1\ldots n_k}\gamma_{m}=\left(-1\right)^{k}\left(10-2k\right)\gamma^{n_1\ldots n_k}, \tag{A.9}$$

which is particularly useful since it implies that $\left(\gamma^{m}\lambda\right)_{\alpha}\left(\gamma_{m}\lambda\right)_{\beta}=0$ for λ being a pure spinor.

A.3 Ordering considerations

This part of the text is intended to present some aspects of the ordering prescription that is being used in this work.

Classical relations between currents are now corrected with ordering contributions. For example,

$$N_{\mathrm{cl}}^{mn}\left(\gamma_{n}\lambda\right)_{\alpha}=\frac{1}{2}J_{\mathrm{cl}}\left(\gamma^{m}\lambda\right)_{\alpha}$$

is valid for any pure spinor λ. Its quantum version is given by

$$\left(N^{mn},\lambda^{\beta}\right)\gamma_{\alpha\beta}^{p}\eta_{np}-\frac{1}{2}\left(J,\lambda^{\beta}\right)\gamma_{\alpha\beta}^{m}=2\left(\gamma^{m}\partial\lambda\right)_{\alpha},$$

showing that the some of the 45 Lorentz generators can be written in terms of the others (in fact, only 10 are independent components).

Another important example is the equation

$$4\lambda^{\alpha}T_{\mathrm{cl}}+J_{\mathrm{cl}}\partial\lambda^{\alpha}+N_{\mathrm{cl}}^{mn}\left(\gamma_{mn}\partial\lambda\right)^{\alpha}=0,$$

which establishes a connection between the energy momentum tensor and the other currents. Implementing the ordering leads to

$$(\lambda^\alpha, T) + 4\partial^2\lambda^\alpha = -\frac{1}{4}\left(J, \partial\lambda^\alpha\right) - \frac{1}{4}\left(N_{mn}, (\gamma^{mn}\partial\lambda)^\alpha\right).$$

This relation appears in the construction of the quantum b ghost, as well as

$$\left(\frac{1}{4}\right)\gamma^{\beta\alpha}_{mnp}\left(N^{mn}, \lambda\gamma^p\partial\theta\right) = 8\partial\lambda^{[\alpha}\partial\theta^{\beta]} + \left(\lambda^{[\alpha}, N_{mn}\left(\gamma^{mn}\partial\theta\right)^{\beta]}\right) + \left(\lambda^{[\alpha}, J\partial\theta^{\beta]}\right),$$

which is the ordered version of

$$\gamma^{\alpha\beta}_{mnp}N^{mn}_{\mathrm{cl}}\left(\lambda\gamma^p\partial\theta\right) + 4\lambda^{[\alpha}N^{mn}_{\mathrm{cl}}\left(\gamma_{mn}\partial\theta\right)^{\beta]} + 4\lambda^{[\alpha}J_{\mathrm{cl}}\partial\theta^{\beta]} = 0.$$

A further application is the Sugawara construction of the energy momentum tensor for the minimal ghost sector,

$$T_\lambda = -\frac{1}{20}\left(N^{mn}, N_{mn}\right) - \frac{1}{8}\left(J, J\right) + \partial J, \tag{A.10}$$

which correctly reproduces the related OPE's.

Appendix B

$SO(10)$ to $U(5)$: solving explicitly the constraints

B.1 Spinorial projectors

Given an $SO(10)$ chiral spinor λ^α (antichiral $\overline{\lambda}_\alpha$), one can write down its $U(5)$ components through the use of some projectors P_I^α and $(P_I^\alpha)^{-1} \equiv P_\alpha^I$, where I generically indicates the $U(5)$ indices, defined in such a way that

$$\begin{aligned}
\lambda^\alpha &= P_+^\alpha \lambda^+ + \tfrac{1}{2} P_{ab}^\alpha \lambda^{ab} + P^{\alpha a} \lambda_a, & \lambda^+ &= P_\alpha^+ \lambda^\alpha, \\
\lambda^{ab} &= P_\alpha^{ab} \lambda^\alpha, & \lambda_a &= P_{\alpha a} \lambda^\alpha, \\
\overline{\lambda}_\alpha &= P_\alpha^+ \overline{\lambda}_+ + \tfrac{1}{2} P_\alpha^{ab} \overline{\lambda}_{ab} + P_{\alpha a} \overline{\lambda}^a, & \overline{\lambda}_+ &= P_+^\alpha \overline{\lambda}_\alpha, \\
\overline{\lambda}_{ab} &= P_{ab}^\alpha \overline{\lambda}_\alpha, & \overline{\lambda}^a &= P^{\alpha a} \overline{\lambda}_\alpha.
\end{aligned} \tag{B.1}$$

Being invertible,

$$\begin{aligned}
\delta_\beta^\alpha &= P_+^\alpha P_\beta^+ + \tfrac{1}{2} P_{ab}^\alpha P_\beta^{ab} + P^{\alpha a} P_{\beta a}, & P_+^\alpha P_\alpha^+ &= 1, \\
P_{ab}^\alpha P_\alpha^{cd} &= \delta_a^c \delta_b^d - \delta_b^c \delta_a^d, & P^{\alpha a} P_{\alpha b} &= \delta_b^a,
\end{aligned}$$

and $P_I^\alpha P_\alpha^J = 0$ for $I \neq J$.

Using the projectors, one can define the g-matrices,

$$
\begin{aligned}
(g^a)^{\alpha\beta} &\equiv \frac{1}{\sqrt{2}}\left(P_+^\alpha P^{\beta a} + P_+^\beta P^{\alpha a} + \frac{1}{4}\epsilon^{abcde} P_{bc}^\alpha P_{de}^\beta\right), \\
(g^a)_{\alpha\beta} &\equiv \frac{1}{\sqrt{2}}\left(P_{\beta b} P_\alpha^{ba} + P_{\alpha b} P_\beta^{ba}\right), \\
(g_a)_{\alpha\beta} &\equiv \frac{1}{\sqrt{2}}\left(P_\alpha^+ P_{\beta a} + P_\beta^+ P_{\alpha a} + \frac{1}{4}\epsilon_{abcde} P_\alpha^{bc} P_\beta^{de}\right), \\
(g_a)^{\alpha\beta} &\equiv \frac{1}{\sqrt{2}}\left(P^{\alpha b} P_{ba}^\beta + P^{\beta b} P_{ba}^\alpha\right),
\end{aligned}
$$

where ϵ^{abcde} and ϵ_{abcde} are the totally antisymmetric $U(5)$ tensors, with $\epsilon^{12345} = \epsilon_{12345} = 1$. It is straightforward to verify the following algebra:

$$
\begin{aligned}
\{g^a, g_b\}^\alpha{}_\beta &= (g^a)^{\alpha\gamma}(g_b)_{\gamma\beta} + (g_b)^{\alpha\gamma}(g^a)_{\gamma\beta} = 2\delta_\beta^\alpha \delta_b^a, \\
\left\{g^a, g^b\right\}^\alpha{}_\beta &= (g^a)^{\alpha\gamma}\left(g^b\right)_{\gamma\beta} + \left(g^b\right)^{\alpha\gamma}(g^a)_{\gamma\beta} = 0, \\
\{g_a, g_b\}^\alpha{}_\beta &= (g_a)^{\alpha\gamma}(g_b)_{\gamma\beta} + (g_b)^{\alpha\gamma}(g_a)_{\gamma\beta} = 0.
\end{aligned}
\tag{B.2}
$$

B.2 $SO(10)$ vectors

Given a $SO(10)$ vector N^m, the $U(5)$ decomposition used in this work is:

$$
n^a \equiv \tfrac{1}{\sqrt{2}}\left(N^{2a-1} + iN^{2a}\right), \quad n_a \equiv \tfrac{1}{\sqrt{2}}\left(N^{2a-1} + -iN^{2a}\right).
$$

Therefore, the scalar product between N^m e P^m in the $U(5)$ representation is given by $N^m P_m = n^a p_a + n_a p^a$. In a similar manner, the relation between the g-matrices and the γ^m matrices is:

$$
g^a \equiv \tfrac{1}{\sqrt{2}}\left(\gamma^{2a-1} + i\gamma^{2a}\right), \quad g_a \equiv \tfrac{1}{\sqrt{2}}\left(\gamma^{2a-1} - i\gamma^{2a}\right) . \tag{B.3}
$$

Note that the Dirac algebra follows from the above definition and equation (B.2).

For a rank-2 antisymmetric tensor N^{mn}, the $U(5)$ decomposition is obtained through the antisymmetric product of two vectors:

$$
\begin{aligned}
n^{ab} &\equiv \tfrac{1}{2}\left(N^{2a-1\,2b-1} - N^{2a\,2b} + iN^{2a\,2b-1} + iN^{2a-1\,2b}\right),\\
n_{ab} &\equiv \tfrac{1}{2}\left(N^{2a-1\,2b-1} - N^{2a\,2b} - iN^{2a\,2b-1} - iN^{2a-1\,2b}\right),\\
n^{a}_{b} &\equiv \tfrac{1}{2}\left(N^{2a-1\,2b-1} + N^{2a\,2b} + iN^{2a\,2b-1} - iN^{2a-1\,2b}\right) + \tfrac{i}{5}\delta^{a}_{b}\textstyle\sum_{c=1}^{5} N^{2c-1\,2c},\\
n &\equiv \tfrac{i}{5}\textstyle\sum_{a=1}^{5} N^{2a-1\,2a}.
\end{aligned}
\tag{B.4}
$$

Higher ranks will not be necessary here.

B.3 Minimal formalism

B.3.1 Pure spinors and gauge invariance

The translation of the pure spinor constraint $\lambda\gamma^m\lambda = 0$ to the $U(5)$ language is

$$
\lambda_a\lambda^{ab} = 0 \quad \text{and} \quad \lambda^{+}\lambda_a = -\frac{1}{8}\epsilon_{abcde}\lambda^{bc}\lambda^{de}, \tag{B.5}
$$

which easily follows from the g-matrices definition and their relation to the gamma matrices (B.3).

The pure spinor action is a curved $\beta\gamma$ system [31]. If one forgets for a moment the pure spinor constraint, the action would be

$$
S_{\lambda} = \frac{1}{2\pi}\int d^2z \left(\omega_\alpha\bar{\partial}\lambda^\alpha\right),
$$

where (λ, ω) have (holomorphic) conformal weight $(0, 1)$.

Due to the above constraints, an action that describes a pure spinor λ^α with manifest $SO(10)$ symmetry must be gauge invariant for $\delta_\epsilon\omega_\alpha = \epsilon_m\left(\gamma^m\lambda\right)_\alpha$, where ω_α is the conjugate of λ^α. In the $U(5)$ notation,

$$
\delta\omega_\alpha = \left\{\epsilon^a\left(g_a\lambda\right)_\alpha + \epsilon_a\left(g^a\lambda\right)_\alpha\right\} \Rightarrow \delta\omega^a = \sqrt{2}\left\{\epsilon^a\lambda^{+} + \epsilon_b\lambda^{ab}\right\}.
$$

Therefore, the most natural gauge choice is $\omega^a = 0$, which will be used from now on.

B.3.2 Classical currents

The simplest gauge invariant quantities are the Lorentz current, $N^{mn} = -\frac{1}{2}\omega\gamma^{mn}\lambda$, the ghost number current, $J_\lambda = -\omega\lambda$, and the energy-momentum tensor, $T_\lambda = -\omega\partial\lambda$. In a gauge fixed ($\omega^a = 0$) Wick-rotated description ($SO(9,1)$ to $SO(10)$), it is straightforward to see (using (B.4) and (B.3)) that:

$$
\begin{aligned}
n^{ab} &= -\omega_+\lambda^{ab} + \tfrac{1}{\lambda^+}\left(\lambda^{ac}\lambda^{bd}\omega_{cd} - \tfrac{1}{2}\lambda^{ab}\lambda^{cd}\omega_{cd}\right), \qquad & n_{ab} &= \lambda^+\omega_{ab}, \\
n &= \tfrac{1}{5}\left(\tfrac{5}{2}\lambda^+\omega_+ + \tfrac{1}{4}\lambda^{ab}\omega_{ab}\right), & n^a_b &= \lambda^{ac}\omega_{bc} - \tfrac{1}{5}\delta^a_b\lambda^{cd}\omega_{cd}, \\
T_\lambda &= -\omega_+\partial\lambda^+ - \tfrac{1}{2}\omega_{ab}\partial\lambda^{ab}, & J_\lambda &= -\lambda^+\omega_+ - \tfrac{1}{2}\lambda^{ab}\omega_{ab}.
\end{aligned}
$$

B.3.3 Free fields and quantum currents

From the above construction, the OPE's within the free field description of the pure spinors are easily determined:

$$\lambda^+(z)\,\omega_+(y) \sim \tfrac{1}{(z-y)}, \quad \lambda^{ab}(z)\,\omega_{cd}(y) \sim \tfrac{\left(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c\right)}{(z-y)}. \tag{B.6}$$

λ^+ will be parametrized here as e^u, where u is a chiral scalar field. Its conjugate, ω_+, will be $: e^{-u}\partial t :$ and the gauge fixed action,

$$S_\lambda = \frac{1}{2\pi}\int d^2z \left(\partial t\bar{\partial}u + \frac{1}{2}\omega_{ab}\bar{\partial}\lambda^{ab}\right),$$

will be accompanied by a chirality constraint on t and u, whose propagator is given by

$$t(z)\,u(y) \sim -\ln(z-y). \tag{B.7}$$

The chosen parametrization and possible ordering contributions introduce some freedom in the construction of the quantum currents. Being aware of that, the quantum versions of the gauge invariant objects above are given by:

$$n^{ab} = e^{-u}\left[-\lambda^{ab}\partial t + \lambda^{ac}\lambda^{bd}\omega_{cd} - \tfrac{1}{2}\lambda^{ab}\lambda^{cd}\omega_{cd} + B\lambda^{ab}\partial u + C\partial\lambda^{ab}\right], \quad n_{ab} = e^u\omega_{ab},$$

$$n = \tfrac{1}{5}\left(\tfrac{5}{2}\partial t + \tfrac{1}{4}\lambda^{ab}\omega_{ab}\right) + A\partial u, \quad n^a_b = \lambda^{ac}\omega_{bc} - \tfrac{1}{5}\delta^a_b\lambda^{cd}\omega_{cd},$$

$$T_\lambda = -\partial t \partial u - \tfrac{1}{2}\omega_{ab}\partial\lambda^{ab} + E\partial^2 u, \quad J_\lambda = -\partial t - \tfrac{1}{2}\lambda^{ab}\omega_{ab} + D\partial u.$$

A, B, C, D and E are constants to be fixed according to the criteria explained below.

In developing the pure spinor formalism, Berkovits [1] argued that the ghost contribution to the Lorentz current would be such that

$$N^{mn}(z)\,N^{pq}(y) \sim -3\frac{(\eta^{mq}\eta^{pn} - \eta^{mp}\eta^{nq})}{(z-y)^2} + \frac{(\eta^{mp}N^{nq} + \eta^{nq}N^{mp} - \eta^{mq}N^{np} - \eta^{np}N^{mq})}{(z-y)}.$$

In the $SO(9,1) \to SO(10) \to U(5)$ *decomposition*, this can be translated to:

$$n^a_b(z)\,n^c_d(y) \sim -3\frac{\left(\delta^a_d\delta^c_b - \frac{1}{5}\delta^a_b\delta^c_d\right)}{(z-y)^2} + \frac{\left(\delta^a_d n^c_b - \delta^c_b n^a_d\right)}{(z-y)}, \qquad n(z)\,n^a_b(y) \sim \text{regular},$$

$$n^{ab}(z)\,n_{cd}(y) \sim -6\frac{\delta^{[a}_d\delta^{b]}_c}{(z-y)^2} + 4\frac{\delta^{[a}_d\delta^{b]}_c n}{(z-y)} + 4\frac{\delta^{[b}_{[d}n^{a]}_{c]}}{(z-y)}, \qquad n^{ab}(z)\,n^{cd}(y) \sim \text{regular},$$

$$n^{ab}(z)\,n^c_d(y) \sim 2\frac{\left(\delta^{[b}_d n^{a]c} - \frac{1}{5}\delta^c_d n^{ab}\right)}{(z-y)}, \qquad n_{ab}(z)\,n_{cd}(y) \sim \text{regular},$$

$$n_{ab}(z)\,n^c_d(y) \sim 2\frac{\left(\delta^c_{[a}n_{b]d} + \frac{1}{5}\delta^c_d n_{ab}\right)}{(z-y)}, \qquad n(z)\,n(y) \sim -\frac{3}{5}\frac{1}{(z-y)^2},$$

$$n(z)\,n^{ab}(y) \sim \frac{2}{5}\frac{n^{ab}}{(z-y)}, \qquad n(z)\,n_{ab}(y) \sim -\frac{2}{5}\frac{n_{ab}}{(z-y)},$$

Furthermore, one expects the ghost number current to be a scalar and the Lorentz current to be a primary field (physical requirements), *i.e.*

$$J_\lambda(z)\,N^{mn}(y) \sim \textit{regular}, \qquad N^{mn}(z)\,T_\lambda(y) \sim \frac{N^{mn}}{(z-y)^2}.$$

The above set of OPE's is enough to fix the quantum contributions mentioned before. The actual calculation gives:

- Energy-momentum tensor:

$$T_\lambda(z)\,T_\lambda(y) \sim \frac{11}{(z-y)^4} + 2\frac{T}{(z-y)^2} + \frac{\partial T}{(z-y)}, \qquad n(z)\,T_\lambda(y) \sim \frac{(1-E)}{(z-y)^3} + \frac{n}{(z-y)^2},$$

$$n^{ab}(z)\,T_\lambda(y) \sim \frac{(2E-2)}{(z-y)^3} + \frac{n^{ab}}{(z-y)^2}, \quad n_{ab}(z)\,T_\lambda(y) \sim \frac{n_{ab}}{(z-y)^2},$$

$$J_\lambda(z)\,T_\lambda(y) \sim \frac{(2E-10)}{(z-y)^3} + \frac{J}{(z-y)^2}, \quad n^a_b(z)\,T_\lambda(y) \sim \frac{n^a_b}{(z-y)^2}.$$

- Ghost number current:

$$J_\lambda(z)\,n^{ab}(y)\sim(B+C+D+3)\,\tfrac{e^{-s}\lambda^{ab}}{(z-y)^2},\qquad J_\lambda(z)\,n(y)\sim\tfrac{\left(1+A-\frac{D}{2}\right)}{(z-y)^2},$$

$$J_\lambda(z)\,n_{ab}(y)\sim\text{regular.}\quad J_\lambda(z)\,J_\lambda(y)\sim\tfrac{(2D-10)}{(z-y)^2},\quad J_\lambda(z)\,n^a_b(y)\sim\text{regular},$$

- Lorentz currents:

$$n(z)\,n^{ab}(y)\sim\left(\tfrac{3}{10}+A-\tfrac{B}{2}-\tfrac{C}{10}\right)\tfrac{e^{-u}\lambda^{ab}}{(z-y)^2}+\tfrac{2}{5}\tfrac{n^{ab}}{(z-y)},\qquad n(z)\,n^a_b(y)\sim\text{regular},$$

$$n^a_b(z)\,n^c_d(y)\sim-3\tfrac{\left(\delta^a_d\delta^c_b-\frac{1}{5}\delta^a_b\delta^c_d\right)}{(z-y)^2}+\tfrac{\left(\delta^a_d n^c_b-\delta^c_b n^a_d\right)}{(z-y)},\qquad n^{ab}(z)\,n^{cd}(y)\sim\text{regular},$$

$$n_{ab}(z)\,n^c_d(y)\sim 2\tfrac{\left(\delta^c_{[a}n_{b]d}+\frac{1}{5}\delta^c_d n_{ab}\right)}{(z-y)},\qquad n_{ab}(z)\,n_{cd}(y)\sim\text{regular},$$

$$n(z)\,n_{ab}(y)\sim-\tfrac{2}{5}\tfrac{1}{(z-y)}n_{ab},\qquad n(z)\,n(y)\sim-\tfrac{\left(A+\frac{1}{10}\right)}{(z-y)^2},$$

$$n^{ab}(z)\,n^c_d(y)\sim 2(C+2)\,\frac{\left(\frac{1}{5}\delta^c_d\lambda^{ab}-\delta^{[b}_d\lambda^{a]c}\right)}{(z-y)^2}+2\frac{\left(\delta^{[b}_d n^{a]c}-\frac{1}{5}\delta^c_d n^{ab}\right)}{(z-y)},$$

$$n^{ab}(z)\,n_{cd}(y)\sim 2(C-1)\,\frac{\delta^{[a}_d\delta^{b]}_c}{(z-y)^2}+4\frac{\delta^{[a}_d\delta^{b]}_c n}{(z-y)}+4\frac{\delta^{[b}_{[d}n^{a]}_{c]}}{(z-y)}$$
$$+2(1-2A-B-C)\,\frac{\delta^{[a}_d\delta^{b]}_c\partial u}{(z-y)}.$$

By simple comparison, the ordering contributions are found to be $A=\frac{1}{2}$, $B=2$, $C=-2$, $D=3$, and $E=1$.

It remains to determine the constant C_α appearing in (2.23) in terms of the $U(5)$ variables. A direct computation gives,

$$\lambda^\alpha(z)\,\omega_\beta(y)\sim\frac{1}{z-y}\left[\delta^\alpha_\beta-\frac{\gamma^{\alpha+}_m(\gamma^m\lambda)_\beta}{2\lambda^+}\right],\tag{B.8}$$

as first presented in [1]. That is, in gauge fixing $\omega^a=0$, one is automatically choosing $C_\alpha=C_+$.

B.4 Non minimal formalism

The (minimal) pure spinor formalism was generalized to the so-called non minimal version [7], which was built with the addition of $\bar{\lambda}_\alpha$ and r_α and, respectively, their conjugates, $\bar{\omega}^\alpha$ and s^α, subject to the constraints (2.32) and (2.33). In the $U(5)$ decomposition, they can be rewritten as

$$\bar{\lambda}^a \bar{\lambda}_{ab} = 0, \qquad \bar{\lambda}_+ \bar{\lambda}^a = -\frac{1}{8} \epsilon^{abcde} \bar{\lambda}_{bc} \bar{\lambda}_{de},$$
$$r^a \bar{\lambda}_{ab} + r_{ab} \bar{\lambda}^a = 0, \qquad \bar{\lambda}_+ r^a = -\bar{\lambda}^a r_+ - \frac{1}{4} \epsilon^{abcde} \bar{\lambda}_{bc} r_{de}.$$

Ignoring the constraints for a moment, the non minimal action can be cast as

$$S = \frac{1}{2\pi} \int d^2 z \left(\bar{\omega}^\alpha \bar{\partial} \bar{\lambda}_\alpha + s^\alpha \bar{\partial} r_\alpha \right),$$

where $\left(\bar{\lambda}, \bar{\omega}\right)$ and (r, s) have conformal weight $(0, 1)$. When taken into account, the constraints imply a gauge invariance of the action, given by $\delta_{\epsilon,\phi} \bar{\omega}^\alpha = \epsilon^m \left(\gamma_m \bar{\lambda}\right)^\alpha + \phi^m \left(\gamma_m r\right)^\alpha$ and $\delta s^\alpha = \phi^m \left(\gamma_m \bar{\lambda}\right)^\alpha$. As in the minimal case, it is easy to see that $\bar{\omega}_a = s_a = 0$ is a natural gauge choice, which will be assumed from now on.

B.4.1 Classical currents

The gauge invariant currents that can be built are given in (2.38). Note that

$$S^{mn} \left(\frac{r \gamma_{mn} \lambda}{\bar{\lambda} \lambda} \right) + S \left(\frac{r \lambda}{\bar{\lambda} \lambda} \right) - 4 J_r = 0$$

$$\overline{N}^{mn} \left(\frac{r \gamma_{mn} \lambda}{\bar{\lambda} \lambda} \right) - J_{\bar{\lambda}} \left(\frac{r \lambda}{\bar{\lambda} \lambda} \right) + 3 J_r \left(\frac{r \lambda}{\bar{\lambda} \lambda} \right) + 4 \Phi = 0,$$

meaning that they are not all independent. The $U(5)$ decomposition follows directly from (B.1), (B.3) and (B.4):

$$\begin{aligned} \bar{n}_{ab} = -\bar{\omega}^+ \bar{\lambda}_{ab} + r_{ab} s^+ + \frac{1}{\bar{\lambda}_+} \left(\bar{\lambda}_{ac} \bar{\lambda}_{bd} - \frac{1}{2} \bar{\lambda}_{ab} \bar{\lambda}_{cd} \right) \left(\bar{\omega}^{cd} + \frac{r_+}{\bar{\lambda}_+} s^{cd} \right) \\ + \frac{1}{\bar{\lambda}_+} \left(2 \bar{\lambda}_{c[a} r_{b]d} + \frac{1}{2} \bar{\lambda}_{ab} r_{cd} + -\frac{1}{2} r_{ab} \bar{\lambda}_{cd} \right) s^{cd}, \end{aligned}$$

$$\bar{n}^{ab}=\bar{\lambda}_+\bar{\omega}^{ab}-r_+s^{ab},\qquad \bar{n}^a_b=\tfrac{1}{5}\delta^a_b\left(\overline{\lambda}_{cd}\overline{\omega}^{cd}-r_{cd}s^{cd}\right)-\left(\overline{\lambda}_{ac}\overline{\omega}^{bc}-r_{bc}s^{ac}\right),$$

$$J_r=r_+s^++\tfrac{1}{2}r_{ab}s^{ab},\qquad \bar{n}=\tfrac{1}{5}\left(\tfrac{5}{2}r_+s^++\tfrac{1}{4}r_{ab}s^{ab}-\tfrac{5}{2}\overline{\lambda}_+\overline{\omega}^+-\tfrac{1}{4}\overline{\lambda}_{ab}\bar{\omega}^{ab}\right),$$

$$J_{\bar{\lambda}}=-\bar{\lambda}_+\bar{\omega}^+-\tfrac{1}{2}\bar{\lambda}_{ab}\bar{\omega}^{ab},\qquad T_{\bar{\lambda}}=-\bar{\omega}^+\partial\bar{\lambda}_+-\tfrac{1}{2}\bar{\omega}^{ab}\partial\bar{\lambda}_{ab}-s^+\partial r_+-\tfrac{1}{2}s^{ab}\partial r_{ab},$$

$$s^{ab}=\bar{\lambda}_+s^{ab},\qquad s^a_b=\tfrac{1}{5}\delta^a_b\overline{\lambda}_{cd}s^{cd}-\overline{\lambda}_{ac}s^{bc},$$

$$s=-\tfrac{1}{5}\left(\tfrac{5}{2}\overline{\lambda}_+s^++\tfrac{1}{4}\overline{\lambda}_{ab}s^{ab}\right),\qquad s_{ab}=-s^+\overline{\lambda}_{ab}+\tfrac{1}{\overline{\lambda}_+}\left(\overline{\lambda}_{ac}\overline{\lambda}_{bd}-\tfrac{1}{2}\overline{\lambda}_{ab}\overline{\lambda}_{cd}\right)s^{cd},$$

$$S=\bar{\lambda}_+s^++\tfrac{1}{2}\bar{\lambda}_{ab}s^{ab},\qquad \Phi=r_+\overline{\omega}^++\tfrac{1}{2}r_{ab}\overline{\omega}^{ab}.$$

Here, s is one of the $U(5)$ components of S^{mn}, not to be confused with $S=\overline{\lambda}s$.

B.4.2 Free fields and quantum currents

In the gauge fixed version, the non minimal fields are just free (β,γ) and (b,c) systems, satisfying

$$\overline{\lambda}_+(z)\,\overline{\omega}^+(y)\sim\tfrac{1}{(z-y)},\quad \overline{\lambda}_{cd}(z)\,\bar{\omega}^{ab}(y)\sim\tfrac{\left(\delta^a_c\delta^b_d-\delta^a_d\delta^b_c\right)}{(z-y)},$$

$$r_+(z)\,s^+(y)\sim\tfrac{1}{(z-y)},\quad r_{cd}(z)\,s^{ab}(y)\sim\tfrac{\left(\delta^a_c\delta^b_d-\delta^a_d\delta^b_c\right)}{(z-y)}.$$

Proceeding as in the minimal case above, it is convenient to parametrize $\overline{\lambda}_+$ as $e^{\overline{u}}$, where $\overline{u}$ is a chiral scalar. Its conjugate, $\overline{\omega}^+$ can be described as $:e^{-\overline{u}}\partial\overline{t}:$, such that

$$\overline{t}(z)\,\overline{u}(y)\sim-\ln(z-y)\,. \tag{B.9}$$

Imposing a chirality constraint on $\overline{u}$ and $\overline{t}$, the gauge fixed action is just

$$S_{\overline{\lambda}}=\frac{1}{2\pi}\int d^2z\left(\partial\overline{t}\overline{\partial}\overline{u}+\frac{1}{2}\overline{\omega}^{ab}\overline{\partial\lambda}_{ab}+s^+\overline{\partial}r_++\frac{1}{2}s^{ab}\overline{\partial}r_{ab}\right). \tag{B.10}$$

The quantum contributions may be determined in the same manner as done for the minimal fields. The ones subject to ordering effects are:

$$\overline{n}=(\overline{n})_{cl}+\overline{A}\partial\overline{u},\qquad \overline{n}_{ab}=(\overline{n}_{ab})_{cl}+\overline{B}\overline{\lambda}_{ab}\partial\overline{u}+\overline{C}\partial\overline{\lambda}_{ab},\qquad J_{\overline{\lambda}}=(J_{\overline{\lambda}})_{cl}+\overline{D}\partial\overline{u},$$

$$J_r=(J_r)_{cl}+\overline{E}\partial\overline{u},\qquad \Phi=(\Phi)_{cl}+\overline{F}e^{-\overline{u}}\partial r_++\overline{G}e^{-\overline{u}}r_+\partial\overline{u},\qquad T_{\overline{\lambda}}=(T_{\overline{\lambda}})_{cl}+\overline{H}\partial^2\overline{u},$$

where the subscript cl means just their naive classical version built with the quantum free fields of (B.10) and $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$, $\overline{E}$, $\overline{F}$, $\overline{G}$ and $\overline{H}$ are constants to be fixed requiring

$$\overline{N}^{mn}(z)\, J_r(y) \sim \text{regular}, \qquad \overline{N}^{mn}(z)\, \Phi(y) \sim \text{regular}, \qquad \overline{N}^{mn}(z)\, J_{\bar{\lambda}}(y) \sim \text{regular},$$

$$\overline{N}^{mn}(z)\, T_{\overline{\lambda}}(y) \sim \tfrac{\overline{N}^{mn}}{(z-y)^2}, \qquad \overline{N}^{mn}(z)\, \overline{N}^{pq}(y) \sim \tfrac{\eta^{m[q}\overline{N}^{p]n}+\eta^{n[p}\overline{N}^{q]m}}{(z-y)}.$$

The result is:

$$\begin{array}{c} \overline{A} = -\frac{1}{4}, \quad \overline{B} = \frac{1}{2}, \quad \overline{C} = 0, \quad \overline{D} = \frac{5}{2}, \\ \overline{E} = -3, \quad \overline{F} = -3, \quad \overline{G} = \frac{5}{2}, \quad \overline{H} = -\frac{1}{2}. \end{array}$$

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