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TESE DE DOUTORAMENTO

Quantum Aspects of Solitons and Deformations on $\mathbf{AdS}_4 \times CP^3(S^7)$

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Resumo

Nesta tese serão analisados dois problemas relacionados a espaços do tipo $\mathrm{AdS}_4 \times \mathcal{M}$. Na primeira parte, será discutida a ambiguidade no cálculo da correção da energia em 1-loop para a corda girando no espaço $\mathrm{AdS}_4 \times CP^3$, através de um método anteriormente utilizado na resolução de um problema similar no caso do kink em quatro dimensões na teoria de campos. Na segunda parte será estudada a deformação de um buraco negro no espaço $\mathrm{AdS}_4 \times S^7$. Através de um paradigma de membrana, será realizado o cálculo da condutividade no modelo de Ginzburg-Landau por meio da dualidade AdS-matéria condensada (AdS/CMT), desenvolvida recentemente.

Palavras Chaves: Supercordas; Conjectura (AdS/CFT); Soluções; deformações.

Áreas do conhecimento: Ciências Exatas e da Terra; Física de Partículas e Campos; Física Matemática.

Abstract

In this thesis we will analyze two problems for backgrounds of the type $\mathrm{AdS}_4 \times \mathcal{M}$. In the first part, we will discuss the ambiguities in the computation of the 1-loop correction for the energy of spinning strings on $\mathrm{AdS}_4 \times CP^3$, using a method that solved the problem on the four dimensional kink in field theory. In the second part we will study the deformation of a black hole in $\mathrm{AdS}_4 \times S^7$, and use a membrane paradigm-like calculation to obtain the conductivity on the Ginzburg-Landau model via the AdS-condensed matter duality (AdS/CMT), that has been recently developed.

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Chapter 1

Introduction

Probably one of the greatest success on string theory is Maldacena's AdS/CFT conjecture [1]. To argue about this success we do not need to enter into the discussion of its (still lacking) mathematical proof and the great number os passed tests, but that it is, arguely, the only development from string theory that has been well received in other areas of physics. It is used on a regular basis now, as a tool to understand strongly coupled systems (e.g. quark-gluon plasma, condensed matter systems, etc.). Since the AdS/CFT duality can be prescribed without the direct use of string theory [2], physisist from other areas can use it without a strong string theory background; and that may be the reason of its success.

The canonical example of the AdS/CFT duality is on the equivalence between strings on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N}$=4 Super Yang-Mills. Here it has passed all the tests so far, and even allowed to get into sectors of $\mathcal{N}$=4 SYM (BMN operators [3]) that were unknown, and lead to the development of a new area of research, integrability on $\mathcal{N}$=4. It was later geralized for other AdS backgrouns and because of that tha canonical case is now called $\mathrm{AdS}_5/\mathrm{CFT}_4$. Some examples that are less developed and less understood than the canonical one, are the ones that involve $\mathrm{AdS}_4 \times \mathcal{M}$ ($\mathcal{M}$ could be $\mathbb{CP}^3$ for string theory or S^7 for M-theory) backgrounds, and these were found to be dual to the ABJM (Aharony Bergman Jafferis Maldacena) model [4], which is a 3-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory.

In this thesis we will analyse one aspect for each of the AdS_4 backgrouns on strings and on M-Theory. The first one will be related to quantum corrections to the energy of long strings on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. The second one will be about deformations of a black hole on $\mathrm{AdS}_4 \times S^7$, and the aplication of the duality to see the effect of this deformation on the conductivity of the Ginzburg-Landau model (which can arise as a truncation of ABJM [5]). The last example is more related to a new application of the duality to condensed matter systems, known as AdS/CMT.

1.1 The energy for solitons on $\mathrm{AdS}_4 \times \mathbb{CP}^3$

There are several striking differences between the $\mathrm{AdS}_5/\mathrm{CFT}_4$ and the $\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities. We can start by looking at the background themselves. The $\mathrm{AdS}_5 \times S^5$ is a maximally supersymmetric space (invariant under the supergroup $PSU(2,2|4)$, i.e. 32 supercharges), while $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is not maximally supersymmetric (invariant under $OSp(6|4)$, 24 supercharges). The same symmetry groups appear on the field theory side of the dualities, respectively.

There is another difference between the dualities, in which chapter 4 part of this thesis will be focused, that is the intrincated interpolation between weak and strong couplings in $\mathrm{AdS}_4/\mathrm{CFT}_3$. Let us look at the magnon dispersion relation

$$E(p) = \sqrt{Q^2 + 4h(\lambda)\sin^2\tfrac{p}{2}}, \tag{1.1.1}$$

here Q is the magnon R-charge and $h(\lambda)$ is a function that is not fixed by symmetry. The fundamental magnon in $\mathrm{AdS}_5/\mathrm{CFT}_4$ has charge Q=1, while in $\mathrm{AdS}_4/\mathrm{CFT}_3$ it has $Q = \frac{1}{2}$. In the case of $\mathrm{AdS}_5/\mathrm{CFT}_4$ the function $h(\lambda)$ turned out to be simply $\sqrt{\lambda}/4\pi$. Now for the case of $\mathrm{AdS}_4/\mathrm{CFT}_3$ the function h is quite non-trivial. The weak and strong coupling behaviors are like this

$$h(\lambda) = \begin{cases} \lambda\,(1 + c_1\lambda^2 + c_2\lambda^4 + ...) & \text{for}\lambda \gg 1, \\ \sqrt{\frac{\lambda}{2}} + a_1 + \frac{a_2}{\sqrt{\lambda}} + ... & \text{for}\lambda \ll 1. \end{cases} \tag{1.1.2}$$

The λ-dependance of many other quantities like the S-matrix, the Bethe ansatz, the Zhukowsky map, the universal scaling function, etc., are also related between the $\mathrm{AdS}_5/\mathrm{CFT}_4$ and the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence by appropriately replacing λ by $h(\lambda)$. Despite this fact, the subleading terms seem to be scheme dependent.

The first indication of this scheme dependence was found in [6], where they computed the one-loop energy shift of the spinning folded string and the answer differed from the prediction of the conjectured Bethe equations. Two proposals came to solve this problem. One was using algebraic curve inspired regularization for the sum of the string frequencies [7]. In the other, they have shown that in the string regularization scheme, the function $h(\lambda)$ receives a one-loop correction (a_1 in ((1.1.2))), and when using this contribution in the Bethe equations then the string theory result was reproduced. A comparison of the different prescriptions for summing frequencies was carried out in [8].

It does not seem that consensus has been reached in the literature as to how this puzzle should be resolved. It is exactly the one of the focuses of the present work to atack the problem on the ambiguities on the summation of frequencies. In [9] we applied a physical principle, previously used to eliminate ambiguities in quantum corrections to the 2 dimensional kink [10]. On this procedure the calculation of the energy is done substracting the energy from the vacuum configuration (like the calculation of the energy on the Casimir's effect). Then, instead of calculating the sum, its derivative is calculated with respect to the physical mass scale in the theory. Since the degree of divergence is lowered by taking the derivative, the summation can be done without problems, then it is integrated back. The constant of integration is fixed by the observation: the vacuum energy should not depend on the topology (topologycal boundary conditions, which do not introduce boundary effects) when the mass is zero. That is the physical principle that allows to perform the calculation of the energy unambiguously.

Applying to the case of spinning strings moving in $AdS_4 \times \mathbb{CP}^3$, thought of as another kind of two dimensional soliton is not so straightforward, since we have a non-linear sigma model and also it is not easy to identify a scale (like the mass on the kink). After analysing the classical solution we find a way to go to the vacuum (trivial) sector, and so we find the energy to substract. We apply then the principle for the sumation. We find that this eliminates the ambiguities and selects the result compatible with AdS/CFT, providing a solid foundation for one of the previous calculations, which found agreement.

1.2 Deformations on $\mathrm{AdS}_4 \times S^7$ and conductivity via AdS/CMT

Still on $\mathrm{AdS}_4/\mathrm{CFT}_3$, there is another aspect that we can analyse. A little time after ABJM was found, Gomis, Rodriguez-Gomez, Van Raamsdonk and Verlinde computed a massive deformation of the model that preserves the full supersymmetry [11]. This massive-deformed model is denoted as mABJM, for short. Having a duality with M-theory on $\mathrm{AdS}_4 \times S^7$, this deformation should have an effect on the background. The massive deformation of $\mathrm{AdS}_4 \times S^7$ was found in [12], although the deformed background is not so user friendly to perform calculation using the duality.

Among many interesting aspects of the $\mathrm{AdS}_4 \times \mathcal{M}$ backgrounds, there is one that arose in the last years; that lies in the application of the duality to condensed matter systems, also known as AdS/CMT. The initial construction of this duality have been very phenomenological, but recently in [5], Mohammed, Murugan and Nastase did a more formal construction directly from ABJM. In their work they performed a consistent abelian truncation of mABJM, and obtained the Ginzburg-Landau model used to describe superconductivity. There is still a plenty of work to do in order to understand how this truncation affects the gravity dual, and so find the AdS/CMT duality from first principles.

Related to the deformation of $\mathrm{AdS}_4 \times S^7$ and its aplication on AdS/CMT is the second part of this thesis. In [13] we analyzed the effect of the massive deformation of the ABJM model on the calculation of conductivity of the dual theory. Since on these condensed matter systems temperature is present, it is implemented on the gravity dual by the insertion of a black hole on the background. Now, the massive deformed background alone is dificult enough to work with, so the insertion of the black hole does not not make things easier. In order to circumvent the difficulties presented by the dual geometry, at least partially, a membrane paradigm-like computation of the conductivity is adopted, which requires just to know the effect of the deformation on the *horizon* of a black hole in AdS_4. The deformation at the horizon itself is found by first deforming the flat space near the horizon, and then using the corresponding solution near the horizon as initial conditions for the Einstein's equations. The result shows an increase in the conductivity due to the deformation.

1.3 Presentation of the work

This work is organized as follows. In chapter 2 we will see a mild construction of the AdS_d space and the $\mathbb{CP}^n$, the former constructed from the sphere S^{n+1}. In chapter 3 a brief construction of AdS/CFT will be presented. There we will start with the canonical case of $\mathrm{AdS}_5/\mathrm{CFT}_4$ and show how the duality works, then we will show the ingredients of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ case. Chapter 4 will be about the computation of the correction to the energy of a soliton on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. First we will review the method for the computation of the energy correction and apply it to the case of the 2-dimensional kink, then we will use the same method to analyse the 1-loop energy correction of the spinning string on the $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background. In chapter 5 we will work in the deformation of $\mathrm{AdS}_4 \times S^7$, it will be performed on the horizon of a black hole on that background and then it will be truncated to four dimensions, that is to use have the gravity dual (phenomenologycally proposed) of the Ginzburg-Landau model, and then see the effect of this deformation on the conductivity. Finally in chapter 6 we will conclude and talk about possible perspectives and future work.

Chapter 2

Geometry preliminaries

In this chapter we will review some very basic notions of the spaces that we will be dealing with. Mainly, we will focus in finding and/or analysing the metric, now that they play a central role in the development of this work. In the first section we will focus on the Anti de Sitter space, on the second on $\mathbb{CP}^3$. The material that well be discussed here can be found on several books or notes on gravitation and geometry (e.g. [14] and [15])

2.1 The Anti de Sitter space

The space (AdS_d) is as a solution of the Einstein-Hilbert action with negative cosmological constant. This is a space of constant negative curvature and with a pseudo-Riemmanian metric (it can also be worked with an euclidean metric), with signature $(-++...+)$. One way to describe it, is like an embedding in a $d+1$ space with signature $(--+...+)$ and the following constraint

$$-x_0^2 + \sum_{i=1}^{d-1} x_i - x_{d+1}^2 = -R^2. \tag{2.1.1}$$

We can parametrize AdS_d using global co-ordinates

$$\begin{aligned} x_0 &= R\cosh\rho\cos\tau, \\ x_{d+1} &= R\cosh\rho\sin\tau, \\ x_i &= R\sinh\rho\hat{e}_i \qquad \sum_i \hat{e}_i^2 = 1 \end{aligned} \tag{2.1.2}$$

With these co-ordinates the metric will now looks like this

$$ds^2 = R^2\left(\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_{d-2}^2\right). \tag{2.1.3}$$

There is a co-ordinate transformation that can be taken, $r = R\sinh\rho$ and $t = R\tau$, that will lead to an often used form of the metric

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)dt^2 + \frac{dr^2}{\left(1-\frac{r^2}{R^2}\right)} + r^2 d\Omega^2_{d-2}. \tag{2.1.4}$$

Another system of co-ordinates are the Poincaré co-ordinates (Poincaré's patch). On this system the metric looks as follows

$$ds^2 = \frac{R^2}{x_0^2}\left(-dt^2 + \sum_{i=1}^{d-2} dx_i^2 + dx_0^2\right), \tag{2.1.5}$$

here (t, x_i) behave like the usual Minkowski co-ordinates, but $x_0 \in (0,\infty)$. These co-ordinates do not cover all AdS_d.

There is a lot more of what can be said about the structute of the AdS space, about its symmetries also, but that would go off the interest of this work. On the next section we will make a similar presentation for the $\mathbb{CP}^n$ space.

2.2 The Complex Projective space

Now we will find the metric of the Complex Projective space ($\mathbb{CP}^n$). There are several construct the Projective space, here we will construct its metric from the sphere (S^{n+1}). Let us start with the Complex space $\mathbb{C}^{n+1}$, which has the flat metric

$$ds_{2n+2} = dz^A d\bar{z}_A, \tag{2.2.6}$$

where $A = (0,\alpha)$ and $1 \le \alpha \le n$.

Introducing the inhomogeneous co-ordinates ($z^0 \neq 0$ patch)

$$\zeta^\alpha = \frac{z^\alpha}{z^0}. \tag{2.2.7}$$

And we will have the following set of definitions

$$\begin{aligned} z^0 &= e^{i\tau}|z^0| = e^{i\tau} r f^{-1/2} \\ r &= \sqrt{z^A \bar{z_A}} \\ f &= 1 + \zeta^\alpha \bar{\zeta}^{\bar{\alpha}}. \end{aligned} \tag{2.2.8}$$

We return to the flat metric, but now we will write ir on spherical co-ordinates

$$ds^2_{2n+1} = dr^2 + r^2 d\Omega^2_{2n+1}. \tag{2.2.9}$$

focusing on the unit radius sphere (S^{2n+1}), the metric can be written as follows

$$d\Omega_{2n+1}^2 = (d\tau + B)^2 + f^{-1} d\zeta^\alpha d\bar{\zeta}^{\bar{\alpha}} - f^{-2} \zeta^{\bar{\alpha}} \zeta^\beta d\zeta^\alpha d\bar{\zeta}^{\bar{\beta}}. \tag{2.2.10}$$

The way the metric was presented is looking S^{2n+1} as a $U(1)$ bundle on $\mathbb{CP}^n$. For the connection on $U(1)$ we have

$$B = \frac{1}{2} i f^{-1} (\zeta^\alpha d\bar{\zeta}^{\bar{\alpha}} - \bar{\zeta}^{\bar{\alpha}} d\zeta^\alpha). \tag{2.2.11}$$

For the other part of the metric of the sphere we have the Fubini-Study metric for $\mathbb{CP}^n$, which is

$$ds_{\mathbb{CP}^n} = f^{-1} d\zeta^\alpha d\bar{\zeta}^{\bar{\alpha}} - f^{-2} \zeta^{\bar{\alpha}} \zeta^\beta d\zeta^\alpha d\bar{\zeta}^{\bar{\beta}}. \tag{2.2.12}$$

B also works as a Kähler potential for the $\mathbb{CP}^n$ manifold:

$$J = dB = i g_{\alpha\bar{\beta}} d\zeta^\alpha \wedge d\bar{\zeta}^{\bar{\beta}}. \tag{2.2.13}$$

With the background spaces constructed we can now put theories on them, and more interesting is the fact that these theories on those backgrounds have duals that are quantum field theories, which do not seem related at first sight.

Chapter 3

The AdS/CFT correspondence

In string theory and in supersymmetric theories, dualities have frequently been present, although they usually relate theories of the same kind, in some way to say it. The AdS/CFT correspondence, or what would be called more generally gauge/gravity duality relates two theories that seem very disconnected: quantum field theory and general relativity. It involves several ingredients that makes it very appealing to theorists since its development by Maldacena [1], like supersymmetry, string theory, and extra dimensions, and it ties all these together.

The duality was originally derived for the type IIB superstring in the $\mathrm{AdS}_5 \times S^5$ background and then found that it is also present in other several cases. Here we will show briefly the derivation for this case, which is nowadays known as the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, generalize to describe the features of AdS/CFT dual pairs, and then discuss a little about another case known as $\mathrm{AdS}_4/\mathrm{CFT}_3$, which is closely related to our present work.

3.1 Heuristic derivation

3.1.1 General picture

In the type IIB superstring we have more than just open and closed strings, we also have the non-perturbative solitonic objects known as Dp-branes (where p are the spatial dimensions). At the beginning, these branes were thought as boundary conditions were the open strings end, but then it was shown that they are the same as some solutions of supergravity. In type IIB we can arrange a stack of N D3-branes and place all of them on the same place, there we will have open strings (which contain gauge fields on their spectrum) intercepting the branes and for the low energy limit of this configuration we will have:

$$\mathrm{D}{=}4,\ \mathcal{N}{=}4,\ SU(N)\ \text{SuperYang-Mills}.$$

Next, we proceed to separate the D3-branes. When we have the branes separated we can have pairs of open strings forming closed strings (which contain the graviton on their spectrum), these are no longer attached to the branes and can leave them. Now that we have closed strings, they can curve the background, and for the low energy limit of this configuration we will have:

$$\text{Type IIB superstring theory on } \mathrm{AdS}_4 \times S^5.$$

As we just saw, we never abandoned the type IIB theory, we just manipulated a little some of its ingredients. Another important point is that on both configurations we went to the low energy limit, and since these are the limits of the same theory, then Maldacena's conjecture is that:

$$\text{D=4, } \mathcal{N}\text{=4, } SU(N) \text{ SuperYang-Mills} = \text{IIB string theory on } \mathrm{AdS}_4 \times S^5.$$

This duality is still at the level of a conjecture since there is not and exact proof yet, although it has passed all the tests performed so far therefore is generally assumed as valid. On the next session we will review a more formal derivation and see more properties of it.

3.1.2 D-branes analysis and validity of the duality

Now we will derive $\mathrm{AdS}_5 \times S^5$ from the brane configuration, then see the match of the elements of both sided of the duality. Let us go back to our previous stack of N D3-branes, initially we have them laying on top of each other. At low energy, on the world-volume we have a $SU(N)$, $\mathcal{N} = 4$ supersymmetric gauge theory (we took into account the global $U(1)$ corresponding to the collective motion of the stack). The gauge coupling is $g_{YM}^2 = 2\pi g_s$. The gauge multiplet contains the vector, A_μ, six scalars ϕ_i, $i = 1, ..., 6$ (that represent the transverse motions), and four two-component Weyl fermions, λ_a, $a = 1, ..., 4$, the fermionic superpartners of the eight physical bosonic degrees of freedom. There is a $SO(6) \simeq SU(4)$ R-symmetry. The theory is conformally invariant, (i.e. it has vanishing β-function) with the conformal group being $SO(2, 4)$.

The low energy supergravity solution is:

$$\begin{aligned} ds^2 &= H^{-\frac{1}{2}} \eta_{\mu\mu} dx^\mu dx^\nu + H^{\frac{1}{2}} dx^i dx^i, \\ e^{2\Phi} &= g_s^2, \\ C_4 &= (H^{-1} - 1) g_s^{-1} dx^0 \wedge ... \wedge dx^3, \end{aligned} \tag{3.1.1}$$

where $\mu = 0, ..., 3$, $i = 4, ..., 9$, and for the harmonic function we have

$$H = 1 + \frac{4\pi g_s N \alpha'^2}{r^4}. \tag{3.1.2}$$

Next we take the near-horizon limit, $r \to 0$, by taking also $\alpha' \to 0$. For the limit we keep the quantity $u = r/\alpha'$ fixed. This limit takes the metric to

$$
\begin{aligned}
ds^2 &= \frac{u^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\frac{du^2}{u^2} + R^2 d\Omega_5^2; \\
{}^*dC_4 &= 16\pi\alpha'^2 N\epsilon_5
\end{aligned}
\tag{3.1.3}
$$

where $R^4 = 2g_{YM}N\alpha'^2$.

This previous solution is nothing more than $\mathrm{AdS}_5 \times S^5$, both with the same radius of curvature.

Since we would like to use the gravity approximation of string theory, there is a limit for the validity of this duality:

- The background curvature must be larger than the string's length, meaning $R/\sqrt{\alpha'} >> 1$, on this way we will avoid string world-sheet quantum corrections. Then we have $g_{YM}N >> 1$.
- Quantum corrections on the string are governed by g_s, so this coupling must be small, thus $g_s \to 0$.

As we saw then, to work on gravity, we need $g_s \to 0$, $N \to \infty$, but $\lambda = 2\pi g_s N = g_{YM}^2 N$ fixed and large. We just set the conditions for gravity to be valid, now we can look at the implications of these conditions on the quantum field theory.

In the large N limit, 't Hooft showed that for gauge theories with adjoint fields the expansion parameters are the 't Hooft coupling $\lambda = g_{YM}^2 N$ and $1/N$. A diagram with V vertices, E propagators and F loops comes with a coefficient proportional to

$$N^{E-V+F}\lambda^{E-V} = N^{\chi}\lambda^{E-V}, \tag{3.1.4}$$

where $\chi = E - V + f$ is the Euler character of the surface ($\chi = 2 - 2g$, where g is the genus of the closed oriented surface) corresponding to the diagram. In the large N limit we see that any computation will be dominated by the surfaces of maximal χ or minimal genus, which are surfaces with the topology of a sphere (or a plane). All these planar diagrams will give a contribution of order N^2, while all other diagrams will be suppressed by powers of $1/N^2$. 't Hooft considered the perturbative expansion in λ, for this it has to be fixed and $\lambda << 1$, but in AdS/CFT we need $\lambda >> 1$ and fixed.

With this we see that the duality is valid for gravity and quantum gauge theory in opposite regimes, $\lambda >> 1$ and $\lambda << 1$ respectively.

3.1.3 The dictionary

Symmetries

Let us check now the symmetries on both systems. On the AdS side there are the $SO(4,2) \times SO(6)$ symmetries of the AdS_5 and S^5 spaces. In $\mathcal{N} = 4$ supersymmetric gauge theory, $SO(4,2)$ is the conformal group. The $SO(6)$ is the symmetry of the scalar field space A_m, with the four fermions transforming as the spinor representation. Equivalently, the fermions are in the fundamental representation of $SU(4) \simeq SO(6)$, and the six scalars in the antisymmetric product of two fundamentals. On both sides of the duality this bosonic symmetry group is extended by supersymmetry to the superconformal $PSU(2,2|4)$.

There is also an $SL(2,\mathbb{R})$ weak-strong duality on both sides, in the $D = 4$, $\mathcal{N} = 4$ gauge theory and in the IIB string theory.

Operator map

Now we will focus on the operators. In one side we have a CFT, the $\mathcal{N} = 4$ supersymmetric gauge theory, and an operator $\mathcal{O}$ will be characterized by a certain conformal dimension Δ and a representation index I_n for the $SO(6) \simeq SU(4)$ R-symmetry.

In the string theory in $\mathrm{AdS}_5 \times S^5$, $\mathcal{O}$ will correspond to a field, but we will go to the valid limit of the duality; not thinking about string theory, but gravity. We have a compact space S^5, then we can apply the Kaluza-Klein compactification. The fields will be expanded in the harmonic functions of the sphere, for the scalar field would be the following expansion

$$\phi(x,y) = \sum_n \sum_{I_n} \phi^{I_n}_{(n)}(x) Y^{I_n}_{(n)}(t), \tag{3.1.5}$$

where n is the level, x is a co-ordinate on AdS_5 and y a co-ordinate on S^5.

The field $\phi^{I_n}_{(n)}$ living in AdS_5, of mass m, correspond to an operator $\mathcal{O}^{I_n}_{(n)}$ in the 4 dimensional $\mathcal{N} = 4$ supersymmetric gauge theory, of dimension Δ. The field and the operator are on the same symmetry. For the relation between m and Δ we have

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2}. \tag{3.1.6}$$

We have that the tower of KK modes corresponds to the tower of operators, one with increasing mass and the other with increasing conformal dimension respectively, both of them labeled by the same n.

partition function

In order to compute observables (e.g. correlation functions) we need one last piece, and that is the most basic entry in the dictionary, which is the relation between the CFT partition function and the gravitational action. This result was found by Witten in [2]. The AdS/CFT prescription for the generating functional of correlation functions of operators corresponding to massless scalars in AdS_5 is

$$Z_{\mathcal{O}}[\phi_0]_{\mathrm{CFT}} = \int \mathcal{D}[\text{fields}] e^{-S + \int dx^4 \mathcal{O}(x)\phi_0(x)} = Z_{\text{class}}[\phi_0]_{\mathrm{AdS}} = e^{-S_{\text{sugra}}}, \tag{3.1.7}$$

where ϕ_0 is the solution of the massless Klein-Gordon equation near the boundary of AdS.

The solution ϕ_0 acts as a source for the operator $\mathcal{O}$, that lives on the boundary of AdS, and we can now compute correlation functions of $\mathcal{O}$ by taking derivatives with respect to ϕ_0.

Further tests

The dictionary and further testss that have been found so far between the theories are another evidence of its validity, here we summarize and enumerate some of them:

1. The symmetries on the two sides match.
2. The spectra of supersymmetric states match. This includes, for example, all modes of the graviton in $\text{AdS}_5 \times S^5$.
3. Amplitudes which are protected by supersymmetry and so can be compared between the two sides are equal.
4. When we perturb the duality in ways that break some of the supersymmetry and/or conformal symmetry, the geometry realizes the behaviors expected in the field theory, such as confinement.
5. There are higher symmetries on both sides, which allow some quantities to be calculated for all g, with apparent consistency.
6. Matching long string states can be identified on both sides.
7. The predictions of the duality for strongly coupled gauge theories can be compared with numerical calculations in those theories, using both light-cone diagonalization and lattice simulation (these are computationally challenging, but progressing).

Now that we made a brief review of the canonical case of AdS/CFT (also known as the $\text{AdS}_5/\text{CFT}_4$ duality), we can move forward and discuss a little about another case more related to the chapters to come, which is the $\text{AdS}_4/\text{CFT}_3$ duality.

3.2 The $\text{AdS}_4/\text{CFT}_3$ duality

We already saw how the duality works for the Type IIB superstring theory on $\text{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ SuperYang-Mills, it is the best understood example so far, it also holds in an appropriately adopted form for a second example of the AdS/CFT correspondence. This second example was developed in [4] by Aharony, Bergman, Jafferis and Maldacena (ABJM).

On the string theory side we have now:

Type IIA superstring theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$
with RR four-form flux $F_4 \sim N$ through AdS_4,
and RR two-form flux $F_2 \sim k$ through $\mathbb{CP}^3$.

On the supersymmetric gauge theory we have ABJM, which is:

D=3, $\mathcal{N} = 6$ superconformal Chern-Simons-matter theory,
with gauge group $U(N) \times U(N)$,
and Chern-Simons levels k and $-k$.

The parameters we have now on both theories are k and N, which take integer values. In ABJM theory, the Chern-Simons level k acts like a coupling constant. The fields can be rescaled in such a way that all interactions are suppressed by powers of $1/k$, so large k is the weak coupling regime. For the planar limit we now have

$$k, N \to \infty \ , \ \ \lambda = \frac{N}{k} \text{ fixed.} \tag{3.2.8}$$

On the gravity side, the string coupling constant and efective tension are given by

$$g_s \sim \left(\frac{N}{k^5}\right)^{1/4} = \frac{\lambda^{5/4}}{N} \ , \ \ \frac{R^2}{\alpha'} = 4\pi\sqrt{2\lambda}, \tag{3.2.9}$$

where R is the radius of $\mathbb{CP}^3$ and twice the radius of AdS_4.

The limits of validity work in a similar fashion that the $\mathrm{AdS}_5/\mathrm{CFT}_4$ case.

Looking more carefully at (3.2.9) we can find something else on this duality. If we are not in the 't Hooft limit but if we let $N >> k^5$, then the string coupling g_s becomes large. but strongly coupled IIA string theory is M-theory. So ABJM at arbitrary values of k and N is dual to

M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$,
with four-form flux $F_4 \sim N$ through AdS_4.

Translating to the brane analysis, ABJM is the world-volume theory of a stack of N M2-branes moving on $\mathbb{C}^4/\mathbb{Z}_k$. M/ABJM is the stronger duality, by taking the large k limit we can perform the analysis from the previous chapter and get $\mathbb{CP}^3$ with a $U(1)$ bundle as S^7.

3.2.1 $\mathcal{N} = 6$ Chern-Simons matter theory

ABJM, the dual gauge theory, is a three-dimensional superconformal Chern-Simons theory with product gauge group $U(N) \times \hat{U}(N)$ at Chern-Simons levels $\pm k$ and specific matter content. The entire field content is given by two gauge fields A_μ and $\hat{A}_\mu$, four complex scalar fields Y^A, and four Weyl-spinors ψ_A. The matter fields are $N \times N$ matrices transforming in the bi-fundamental representation of the gauge group.

The global symmetry group of ABJM theory, for Chern-Simons level $k > 2$ (since we are on the large k limit), is given by the orthosymplectic supergroup $OSp(6|4)$ and the "baryonic" $U(1)_b$. The bosonic components of $OSp(6|4)$ are the R-symmetry group $SO(6)_R \simeq SU(4)_R$ and the 3d conformal group $Sp(4) \simeq SO(2,3)$.

Arranging all the previous ingredients into an invariant action, we have the following

$$\begin{aligned} S_{ABJM} = \quad & \int d^3x \Big[\epsilon^{\mu\nu\lambda}\mathrm{tr}\big(A_\mu\partial_\nu A_\lambda + \tfrac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \tfrac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\big) \\ & \qquad -\mathrm{tr}(D_\mu Y)^\dagger D^\mu Y - i\mathrm{tr}\psi^\dagger \not{D}\psi - V_{\text{ferm}} - V_{\text{bos}}\Big], \end{aligned} \tag{3.2.10}$$

where the sextic bosonic and quartic mixed potentials are

$$\begin{aligned} V_{\text{bos}} = & -\frac{1}{12}\mathrm{tr}\Big[Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \\ & \qquad +4Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger\Big], \end{aligned} \tag{3.2.11}$$

$$\begin{aligned} V_{\text{ferm}} = & \frac{i}{2}\mathrm{tr}\Big[Y_A^\dagger Y^A \psi^{B\dagger}\psi_B - Y^A Y_A^\dagger \psi_B \psi^{B\dagger} + 2Y^A Y_B^\dagger \psi_A \psi^{B\dagger} - 2Y_A^\dagger Y^B \psi^{A\dagger}\psi_B \\ & \qquad -\epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D + \epsilon_{ABCD} Y^A \psi^{B\dagger} Y^C \psi^{D\dagger}\Big]. \end{aligned} \tag{3.2.12}$$

The covariant derivative acts on bi-fundamental fields as

$$D_\mu Y = \partial_\mu Y + iA_\mu Y - iY\hat{A}_\mu \tag{3.2.13}$$

while on anti-bi-fundamental fields it acts with A_μ and $\hat{A}_\mu$ interchanged. According to the M-theory interpretation, this theory describes the low-energy limit of N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity.

Now that we have the basic ingredients of the AdS/CFT duality and saw the components of the case of interest, $\mathrm{AdS}_4/\mathrm{CFT}_3$, we can move forward and analyse two problems on the $\mathrm{AdS}_4 \times \mathcal{M}$ type backgrounds.

Chapter 4

Solitons on $\mathrm{AdS}_4 \times \mathbb{CP}^3$

Quantum corrections to solitons have a long and complicated history, and it has proven difficult to find an algorithmic way to calculate them, due to regularization-dependent ambiguities. The most studied case so far, for being the simplest and easiest to analyze, is the kink in two dimensions. Studies of its quantum corrections started with [16] (see also [17, 18], and supersymmetric extensions started with [19–22]) and still go on (see [23] for basic techniques and references, and [24] for a review of recent results), due to the many subtleties present. In [10] a physical principle was proposed that eliminates the ambiguities and gives a quantum correction consistent (in the supersymmetric case) with supersymmetry.

A seemingly different area that has received a lot of attention lately is quantum corrections to classical (long) strings moving in gravitational backgrounds. The reasons for that interest are *usually* related to AdS/CFT, since one application has been to systems which have a field theory dual admitting a Bethe ansatz for the dual to the string. That is one motivation, although systems with Bethe ansatz are interesting in their own right, outside the existence of AdS/CFT. This is useful, since unlike other cases, when we need to invoke supersymmetry to match weak coupling field theory results to strong coupling gravity results, the Bethe ansatz allows one to have a prediction for the expected quantum correction at strong coupling. Then, provided we can trust AdS/CFT and the Bethe ansatz, we have a prediction for the expected quantum correction.

Of course, the classical string is just a type of solitonic solution in a two dimensional field theory (the sigma model of the string moving in the gravitational background), and as such a priori suffers from the same ambiguities as the largely studied kink. From this point of view, one should not be surprised that early calculations for the corrections to a spinning string in $AdS_4 \times \mathbb{CP}^3$ gave different results, and apparently incompatible with AdS/CFT [6, 7, 25, 26]. In [27], a calculation was proposed that matches with AdS/CFT expectations.[1] See [8, 28–34] for later related works.

[1] Later proposals were made on how it could be possible to match with AdS/CFT other calculations as well.

However, the calculation in [27] still suffers from the same *a priori* ambiguities, and it amounts to a particular choice of regularization for them, whose only justification is a posteriori, through matching with AdS/CFT. It is the purpose of this chapter to provide a justification for that calculation, as was done in [9], by taking the physical principle of [10] and applying it to classical strings. We will show that its use for the model in [27] eliminates the ambiguities implicitly hidden there, and thus offers the possibility of extending the same method to other classical string solutions.

This chapter is organized as follows. In section 4.1 we explain our method based on a physical principle, in section 4.2 we review how it applies to the case of the two dimensional kink, in section 4.3 we apply it to the spinning string in $AdS_4 \times \mathbb{CP}^3$.

4.1 The method

As is well known, one-loop corrections to the energy of the vacuum are equivalent (via the exponentiation of one-loop determinants) to sums of zero-point fluctuations, for a bosonic mode $\sum \frac{1}{2}\hbar\omega^B$. If we have fermionic modes, they will contribute with $-\sum \frac{1}{2}\hbar\omega^F$. These fluctuations give rise in particular to the Casimir energy, which is the difference in this zero-point energy between the infinite space and a space of finite size. Of course, this answer is a priori ambiguous ($\infty - \infty$), and moreover highly divergent. Generically, $\omega \sim n$ implies a quadratic (n^2) divergence.

The same idea applies when we calculate the quantum mass of some soliton, generically denoted $\phi_{sol}(x)$, with classical mass M. We have to calculate the fluctuations in the presence of the soliton, i.e. eigenmodes around $\phi_{sol}(x)$, and subtract the fluctuations in its absence (in the vacuum). An extra factor to take into account is renormalization. In terms of Feynman diagrams, we know we have counterterms, which correspond to renormalizing the parameters of the theory, for instance a bare mass parameter m_0 becomes the renormalized mass m. When going to the fluctuation representation, a useful way of encoding the counterterm for the energy, δM, is by the variation of the classical mass M when expressed in terms of unrenormalized parameters like m_0, vs. renormalized ones like m, with a result linear in $m_0 - m = \delta m$, where for δm we need to take the result of the one-loop Feynman diagramatic calculation.

Therefore generically the one-loop contribution to the quantum mass of a soliton is given by

$$E_1 = \frac{1}{2}\sum_n(\omega_n^B - \omega_n^F) - \frac{1}{2}\sum(\omega_n^{(0)B} - \tilde{\omega}_n^{(0)F}) + \delta M. \tag{4.1.1}$$

where the ω_n^B, ω_n^F are the frequencies coming from the bosonic and fermionic parts of the action, respectively, labelled by an integer n, and the (0) refers to the vacuum, i.e. without the soliton solution.

This expression contains ambiguities. The first type of ambiguities is due to the fact that we have generically the $\infty - \infty$ difference of quadratic divergences (if at large n,

$\omega_n \sim n$, then $\sum \omega_n \sim n^2$), which a priori will be linearly divergent ambiguities, not even constant ambiguities. Here we should note that we would be tempted to say that if we have something like, say, $M = \sum_n \sqrt{1+m^2/n^2} - \sum_n 1$, this is the same as $M = \sum_n m^2/(2n^2)$ which is finite. But this in fact amounts to a particular choice of regularization scheme. One needs more information to be precise about which regularization it is, but this would basically be part of the mode number regularization, if we would have instead of n, a k_n together with a relation between n and k. Mode number regularization means that we identify each mode in a sum with another mode in the other sum, effectively giving the summation operator as a common factor. In general however, we are not allowed to make the $\sum_n$ common if both sums are infinite. In terms of choosing a cut-off, there are always at least two ways to regularize, mode number cut-off (which corresponds to making the *sum* common) and energy/momentum cut-off. The second, choosing the same upper *energy* instead for the two sums (convert sum over k_n to integrals over k and identify the variable k in the two sums) gives different results if the sums are infinite [35]. Note that even in usual quantum field theory divergent integrals we can have this situation, just that usually one does not think about it. For instance, if $\int f$ and $\int g$ are UV divergent, then $\int^\Lambda (f-g)$ automatically means that we take the same cut-off Λ for f and g, but we could in principle choose $\int^\Lambda f - \int^{\Lambda+a} g$, giving a different result. There might be situations where this is necessary.

Yet another type of ambiguity is related to the existence of different types of possible boundary conditions, in turn determining different functions k_n, or $k(n)$ in the continuum limit.

But we want a physically unambiguous way to determine the correct regularization and boundary conditions. In [10] a physical principle was used to fix both. The principle can be simply formulated by saying that the non-trivial topology of the soliton boundary does not introduce any extra energy.

The first part of the method involves the notion of topological boundary condition, i.e. that the boundary condition should not introduce boundary-localized energy (surface effects), thus fixing one type of ambiguity. For scalar fields, the boundary conditions should be compatible with the classical solution (if the classical solution is antiperiodic, then so must the boundary condition for fluctuations), but for fermions and higher spins we need to be more careful. The method was described in detail in [10]: consider the symmetries of the action and the symmetries of the solution. For the kink, the action has a $\{\phi \to -\phi, \psi \to \gamma_3\psi\}$ symmetry and a $\psi \to -\psi$ symmetry, and the kink solution is antisymmetric in ϕ. Then e.g., the fluctuations around the kink solution have

$$\phi(-L/2) = -\phi(L/2); \quad \phi'(-L/2) = -\phi'(L/2); \quad \psi(-L/2) = (-1)^q \gamma_3 \psi(L/2) \qquad (4.1.2)$$

The second part is that when we take the classical soliton mass to zero, specifically by taking a relevant mass scale on which it depends, like the m above, to zero, the quantum mass of the soliton should also go to zero, such that there is no mass depending purely on topology, i.e. localized at the boundary. That in turn means that we can calculate instead of the soliton mass, its derivative with respect to the relevant mass scale m, thus reducing the UV divergence of the result, and obtaining a "derivative regularization". For instance, in the example above, $\partial M/\partial m = \sum_n m/(n^2\sqrt{1+m^2/n^2})$ is now indeed finite and unambigous.

We should emphasize that it is not guaranteed that this procedure eliminates ambiguities in general, since taking only one derivative may not reduce the divergence sufficiently. Nevertheless, we hope that in many cases of interest, the result is unambiguous. Note that taking more derivatives will in general reduce further the divergence, but it is not clear if there is a physical principle that will correspond to this modified prescription.

We can therefore define our procedure as follows: Find the soliton solution $\phi_{sol}(x)$, find the frequencies of fluctuations around it, and the renormalization of the relevant mass parameters. Then find the relevant mass parameter to define derivative regularization with respect to it, and topological boundary conditions. Ideally, the resulting quantum mass should be well-defined and unambigous, and we should be able to calculate it. We will see that in the simple $\lambda\phi^4$ kink case it is indeed true, however the string soliton case is more complicated. We can prove that the resulting answer is unambiguous, but we will still need to employ the same procedure used in [27] to calculate it.

4.2 One-loop mass for the kink in ϕ^4 theory

The first thing we want understand is the application of this method to a simple model, the kink solution of the ϕ^4 theory in two dimensions. That was done in [10], and here we will make a short review for it.

For the model we have the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4}(\phi^2 - \mu_0^2/\lambda)^2. \tag{4.2.3}$$

In this model, there are two degenerate vacuum states (trivial solutions) which are $\phi = \pm\mu_0/\sqrt{\lambda}$. According to the previous conditions, for the topologically non-trivial, localized solutions (kinks), we must have a limit $\pm\mu_0/\sqrt{\lambda}$ when $x \longrightarrow \pm\infty$.

The model also has two non-trivial, stable, finite energy solutions, the so called kink and anti-kink

$$\phi_{K,\bar{K}} = \pm\mu_0/\sqrt{\lambda}\tanh[\mu_0(x-x_0)/\sqrt{2}], \tag{4.2.4}$$

they have a classical mass $M_0 = 2\sqrt{2}\mu_0^3/3\lambda$.

4.2.1 Fluctuations

Let us now analyze the quantum fluctuations for the model, by replacing on the Lagrangian the field like

$$\phi = \phi_{sol} + \eta, \tag{4.2.5}$$

where η is a small (quantum) fluctuation. For the mass we will take the renormalized one $\mu_0^2 = \mu^2 + \delta\mu$.

After replacing the previous redefinitions into the Lagrangian, it will take this form

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \eta)^2 - \mu^2\eta^2 - \mu\sqrt{\lambda}\eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{2}\delta\mu\left(\eta^2 + 2\frac{\mu}{\sqrt{\lambda}}\eta\right) + ... \tag{4.2.6}$$

We can see that the tree-level mass for η is $m^2 = 2\mu^2$, and for the correction $\delta\mu^2$ we have

$$\delta\mu^2 = -3i\lambda \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+m^2} = \frac{3\lambda}{2\pi}\int \frac{dk}{\sqrt{k^2+m^2}}. \tag{4.2.7}$$

Following, we can take a plane wave expansion for for the fluctuations around the solutions (let us call them ϕ_k now). Then, the equation of motion from the Lagrangian is

$$\left(-\frac{d^2}{dx^2} + V''(\phi_k)\right)\eta_n(x) = \omega_n \eta_n(x). \tag{4.2.8}$$

For the case of the trivial vacuum the operator takes the form $(-d^2/dx^2 + m^2)$, whose eigenvalues are

$$\tilde{\omega}_n = \sqrt{\tilde{k}_n^2 + m^2}, \tag{4.2.9}$$

and the allowed values for k_n come from the condition $\tilde{k}_n L = 2\pi n$, where L is the size of the one dimensional spatial box in which we put the system.

In the non-trivial case, the kink, the equations for motion have a chance as follows

$$\left(-\frac{d^2}{dx^2} - \mu_0^2 + 3\mu_0^2\tanh^2\left(\frac{\mu_0 x}{\sqrt{2}}\right)\right)\eta_n(x) = \omega_n\eta_n(x). \tag{4.2.10}$$

The frequencies have the same form as the trivial ones, $\sqrt{k_n^2+m^2}$, but now, the η's have the following asymtotic behavior

$$\eta_k(x) \sim \exp\left[i(kx \pm \delta(k)/2)\right] \tag{4.2.11}$$

where the explicit form of the phase shift is

$$\delta(k) = \left(2\pi - \arctan\left(\frac{3m \mid k \mid}{m^2 - 2k^2}\right)\right)\epsilon(k). \tag{4.2.12}$$

Then, the periodic boundary conditions for the k's

$$k_n L + \delta(k_n) = 2\pi n. \tag{4.2.13}$$

For the quantum correction to the soliton mass we have now

$$\frac{3m}{4\pi}\int \frac{dk}{(k^2+m^2)^{1/2}}\ , \qquad m^2 = 2\mu^2. \tag{4.2.14}$$

In the case of fermionic fluctuations (for a supersymmetric version of the kink), which are 2 component vectors, there is a further phase shift $\theta(k)$ giving a $e^{\pm i\theta(k)/2}$ relative factor at $\pm\infty$ between the two components.

4.2.2 1-loop correction for the energy

Now we have the necessary ingredients for the calculation of the one-loop correction to the energy. Substituting the frequencies in the expression for the 1-loop correction

$$E_1 = \frac{1}{2}\sum\omega - \frac{1}{2}\sum\tilde{\omega} + \delta M, \tag{4.2.15}$$

we obtain

$$E_1 = \frac{1}{2}\sum\sqrt{k_n^2+m^2} - \frac{1}{2}\sum\sqrt{\tilde{k}_n^2+m^2} + \delta M. \tag{4.2.16}$$

As we can see, even after the subtraction, the sum is linearly divergent. To apply the derivative regularization, we must find the mass parameter which takes the soliton mass to zero, now that when the mass is zero we are going from the non-trivial sector to the trivial one. Once this parameter is identified we take a derivative with respect to it. In this case, it is obvious, namely the mass parameter is m. We then differentiate the energy with respect to m, perform the summation and integrate back with respect to m.

That will get rid of both linearly and logarithmically divergent ambiguities. The physical principle then dictates that the constant of integration is zero.

Taking the derivative, we obtain

$$\frac{dE_1}{dm} = \frac{1}{2}\sum\frac{d\omega}{dm} - \frac{1}{2}\sum\frac{d\tilde{\omega}}{dm} + \frac{d\delta M}{dm}, \tag{4.2.17}$$

where

$$\begin{aligned}\frac{d\tilde{\omega}}{dm} &= \frac{m}{\sqrt{\tilde{k}_n^2+m^2}},\\ \frac{d\omega}{dm} &= \frac{1}{\sqrt{k_n^2+m^2}}\left(m + \frac{k_n^2}{Lm}\delta'(k_n)\right).\end{aligned} \tag{4.2.18}$$

The sums are less divergent now, and can be turned into integrals by taking into account the conditions for the allowed k_ns (4.2.13), obtaining a finite result

$$\begin{aligned}\frac{dE_1}{dm} &= -\frac{3m^2}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{(k^2+m^2)^{3/2}} + \frac{\sqrt{3}}{4}\\ &\quad -\frac{3m^2}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{1}{\sqrt{k^2+m^2}(m^2+4k^2)} = \frac{1}{4\sqrt{3}} - \frac{3}{2\pi}.\end{aligned} \tag{4.2.19}$$

Integrating back with respect to m we will get a constant of integration, but applying the physical principle, the constant is zero. Therefore finally the 1-loop energy correction is

$$E_1 = m\left(\frac{1}{4\sqrt{3}} - \frac{3}{2\pi}\right). \tag{4.2.20}$$

4.3 One-loop corrections to spinning strings on $\mathrm{AdS}_4 \times \mathbb{CP}^3$

4.3.1 Applying the method

We now try to apply the same method to the classical (long) string on the background $\mathrm{AdS}_4 \times \mathbb{CP}^3$. This can be thought of just as another 2d field theory, with action

$$S = \frac{R_{\mathrm{AdS}}^2}{4\pi} \int d\tau \int_0^{2\pi} d\sigma \, \sqrt{g} g^{ab} \left(G_{\mu\nu}^{\mathrm{AdS}} \partial_a X^\mu \partial_b X^\nu + 4 G_{\mu\nu}^{\mathbb{CP}^3} \partial_a X^\mu \partial_b X^\nu \right) . \tag{4.3.21}$$

As we have discussed, the first step is to understand the 2d vacuum and soliton solution. The (trivial) "vacuum" corresponds to the point-like string, equivalent to $\phi = \phi_0$=constant for the $\lambda\phi^4$ model. The non-trivial soliton whose mass we want to calculate is a spinning string solution, with non-trivial $X^\mu(\sigma, \tau)$.

The computation of the 1-loop energy correction for this soliton was done in different ways, obtaining different results. The calculation of [27] gave the correct result matching the expectation from AdS/CFT, but there was no *a priori* reason why it should be correct, given the implicit choice of regularization scheme needed to obtain it. We will therefore try to identify the ambiguities as above.

In order to apply our method, we note several complications with respect to the kink case. We note that (4.3.21) is a non-linear sigma model, and we have no potential, so formally it looks different from the kink. That however means we can avoid at least one ambiguity from the kink case. No potential means that the phase shifts $\delta(k)$ and $\theta(k)$ are not present, so at least the ambiguity of boundary conditions (related to non-zero $\delta(k)$ and $\theta(k)$) is not there. It is also lucky, since for the calculation of $\delta(k)$ and $\theta(k)$ we would need the full solutions, which as we will see are hard to find. The only ambiguity we still have is the UV divergence.

To deal with that, we need to define our physical regularization. But instead of the mass parameter m of the kink, we will have several parameters (which come in the solution), and we have to carefully analyze which can be varied in order to relate the energies and use the derivative regularization. Note however that there are no parameters *in the action* (4.3.21) (other than R_{AdS} which multiplies the whole action, so is not relevant), so the parameters of relevance will just characterize the vacuum solution. That also means that there are no counterterm contributions, since the only possible counterterm could be for R_{AdS}, which is not renormalized.

The parameter we want needs to be something that when equal to zero, takes the classical mass of the long string to zero, but also something that, like m for the kink, is normally non-zero in the vacuum.

An extra complication will be, as we will see, that it is not possible to find the full solutions for the eigenfrequencies, only as an expansion in a large parameter ω. But then it matters how n is related to ω; in particular, the expansion is not valid for $n > \omega$, which corresponds to the UV divergence we want to analyze. So the only goal we will have is to show that the physical derivative regularization obtained as above selects the regularization implicit in [27]. In order to actually compute the quantum correction, we will still need to use the same procedure as in [27].

In the following sections we will perform first the classical analysis of the model, then we will find the frequencies, and finally apply the derivative regularization.

4.3.2 Classical analysis for the string in $\mathrm{AdS}_4 \times \mathbb{CP}^3$

In this subsection we will take a look at the string on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, from a classical point of view. We will study about its conserved quantities and solutions (specifically the spinning string). We will see that there are several parameters present in this non-trivial solution, but there are relations between them due to the Virasoro constraints, so our search for the parameter that is nonzero in the vacuum, but takes the soliton mass to zero when it equals zero (the analog of the mass parameter m for the kink), will be highly constrained. The conserved quantities, like the energy, which here has the meaning of "soliton mass" modulo an additive constant, will be dependent on these parameters. An important technical detail is that the Virasoro constraints are complicated, so we can only solve them perturbatively in certain limits, hence the same will happen for the energy ("soliton mass").

Let us start by remembering that the metric on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ has two factors

$$ds^2 = R_{\mathrm{AdS}}^2 \left(ds^2_{\mathrm{AdS}_4} + 4 ds^2_{\mathbb{CP}^3} \right), \tag{4.3.22}$$

where the explicit expressions for the metrics are

$$\begin{aligned} ds^2_{\mathrm{AdS}_4} &= -\cosh^2\rho \, dt^2 + d\rho^2 + \sinh^2\rho \left(d\theta^2 + \sin^2\theta d\phi^2 \right), \qquad (4.3.23) \\ ds^2_{\mathbb{CP}^3} &= d\zeta_1^2 + \sin^2\zeta_1 \Big[d\zeta_2^2 + \cos^2\zeta_1 \left(d\tau_1 + \sin^2\zeta_2 \left(d\tau_2 + \sin^2\zeta_3 d\tau_3 \right) \right)^2 \\ &\quad + \sin^2\zeta_2 \left(d\zeta_3^2 + \cos^2\zeta_2 \left(d\tau_2 + \sin^2\zeta_3 d\tau_3 \right)^2 + \sin^2\zeta_3 \cos^2\zeta_3 d\tau_3^2 \right) \Big] \qquad (4.3.24) \end{aligned}$$

Here we have factored out the scale of the metric, R_{AdS}, which is the radius of $\mathbb{CP}^3$ which is twice the radius of AdS_4. This relative size is demanded by supersymmetry; and it is related to the 'tHooft coupling

$$R^2_{\mathrm{AdS}} = \sqrt{\bar{\lambda}} = \sqrt{2\pi^2\lambda} = \sqrt{2\pi^2 \frac{N}{k_{\mathrm{CS}}}} \tag{4.3.25}$$

which is very large (very large $\bar{\lambda}$, though finite).

The background admits five Killing vectors

$$i\partial_t \ , \ i\partial_\phi \ , \ i\partial_{\tau_1} \ , \ i\partial_{\tau_2} \ , \ i\partial_{\tau_3}. \tag{4.3.26}$$

Then, we have five conserved charges: the worldsheet energy E, the AdS-spin S and the $\mathbb{CP}^3$ momenta.

In Chapter 2, we saw how to see the S^7 metric as a $U(1)$ bundle over $\mathbb{CP}^3$, and that costruction will allows to see the relation between the $\mathbb{CP}^3$ co-ordinates and the scalar fields of the dual theory. Taking the complex co-ordinates for the sphere Z^A with $A = 1, ..., 4$, and the constrain $\sum_A |Z^A| = 1$. We go to the $Z^4 \neq 0$ patch, and will see the isometric directions of the metric ($\tau_{1,2,3}$) as phases of the new co-ordinates

$$Z^4 = e^{i\tau_4}|Z^4|\ ,\ Z^3 = e^{i(\tau_3+\tau_4)}|Z^3|\ ,\ Z^2 = e^{i(\tau_2+\tau_3+\tau_4)}|Z^2|\ ,\ Z^1 = e^{i(\tau_1+\tau_2+\tau_3+\tau_4)}|Z^1|. \tag{4.3.27}$$

We can now redefine the angles

$$\tau_1 = \phi_2 - \phi_1\ ,\ \tau_2 = \phi_3 - \phi_2\ ,\ \tau_3 = \phi_2 + \phi_1\ ,\ \tau_4 = \tau_0 - \frac{1}{2}(\phi_1 + \phi_2 + \phi_3) \tag{4.3.28}$$

and get

$$\begin{aligned} Z^1 &= e^{i(\tau_0+\frac{1}{2}(+\varphi_3+\varphi_2-\varphi_1))}|Z^1|,\ \ Z^2 = e^{i(\tau_0+\frac{1}{2}(+\varphi_3-\varphi_2+\varphi_1))}|Z^2|, \\ Z^3 &= e^{i(\tau_0+\frac{1}{2}(-\varphi_3+\varphi_2+\varphi_1))}|Z^3|,\ \ Z^4 = e^{i(\tau_0+\frac{1}{2}(-\varphi_3-\varphi_2-\varphi_1))}|Z^4|, \end{aligned} \tag{4.3.29}$$

which now can be identified one-to-one with the scalar fields Y^A of ABJM. The three Cartan generators fo $SO(6)$ are now represented by shifts of the φ's ($J_i = -i\frac{\partial}{\partial\varphi_i}$). Acting on the Z's we have

$$J_1(Z^A) = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}),\ \ J_2(Z^A) = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}),\ \ J_3(Z^A) = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}). \tag{4.3.30}$$

And we can see that the $SO(6)$ charges of the operator $\mathrm{Tr}[(Y^1 Y_4^\dagger)^J]$ (which represents the vacuum of the spin chain of ABJM) are matched by the charge of the product of J bilinears $[Z^1(Z^4)^\dagger]^J$.

The action (4.3.21) has a solitonic solution known as the spinning string, which was found in [36]. It is a rotating string lying in an $\mathrm{AdS}_3\times\mathrm{S}^1$ subspace of $\mathrm{AdS}_4 \times \mathbb{CP}^3$, which from the point of view of the 2d worldsheet looks like a soliton with

$$\begin{aligned} &\bar{t} = \kappa\tau,\ \ \bar{\rho} = \rho_*,\ \ \bar{\theta} = \frac{\pi}{2},\ \ \bar{\phi} = w\tau + k\sigma, \\ &\bar{\tau}_1 = \bar{\tau}_3 = \frac{1}{2}(\omega\tau + m\sigma),\ \ \bar{\tau}_2 = 0, \\ &\bar{\zeta}_1 = \frac{\pi}{4},\ \ \bar{\zeta}_2 = \frac{\pi}{2},\ \ \bar{\zeta}_3 = \frac{\pi}{2}. \end{aligned} \tag{4.3.31}$$

Unlike the kink case or usual quantum field theory, now we have also gravity on the worldsheet, which in the conformal gauge manifests itself in the presence of the Virasoro constraints $T_{ab} = 0$. For the solution (4.3.31), we have an equation of motion

$$w^2 - \kappa^2 - k^2 = 0, \tag{4.3.32}$$

and the Virasoro constraints reduce to

$$\begin{aligned} r_1^2 wk + \omega m &= 0, \\ -r_0^2\kappa^2 + r_1^2(w^2 + k^2) + \omega^2 m^2 &= 0. \end{aligned} \tag{4.3.33}$$

They can be solved perturbatively, as done in [27], in a certain limit that we will define shortly.

The charge densities are

$$\mathcal{E} = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, r_0^2\kappa = \mathrm{r}_0^2\kappa \ , \quad \mathcal{S} = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, r_1^2 \mathrm{w} = \mathrm{r}_1^2\mathrm{w} \ , \qquad \mathcal{J}_2 = \mathcal{J}_3 = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \omega = \omega, \tag{4.3.34}$$

so that the classical energy, spin and the charges under the second and third Cartan generators of $SO(6)$ are

$$E_0 = \sqrt{\bar\lambda}\, r_0^2\kappa \ , \qquad S = \sqrt{\bar\lambda}\, \mathrm{r}_1^2\mathrm{w} \ , \qquad J \equiv J_2 = J_3 = \sqrt{\bar\lambda}\,\omega \ , \tag{4.3.35}$$

where $r_0 = \cosh\rho_*$, $r_1 = \sinh\rho_*$.

The limit we use to solve the constraints (following [27]) and find some relations between the constants consists in taking large spin $\mathcal{S}$ and large angular momentum $\mathcal{J}$, with their ratio u (and also k) held fixed, i.e.

$$\mathcal{S},\ \mathcal{J} \to \infty \ , \qquad u = -\frac{m}{k} = \frac{\mathcal{S}}{\mathcal{J}} = \frac{S}{J} = \text{fixed} \ . \tag{4.3.36}$$

For this solution, the expansion of the classical energy at large $\mathcal{J} = \omega$ and thus large angular momentum $J = \sqrt{\bar\lambda}\mathcal{J} = \sqrt{\bar\lambda}\,\omega$ is given by

$$\begin{aligned} E_0 &= S + J + \frac{\bar\lambda}{2J}k^2 u(1+u) - \frac{\bar\lambda^2}{8J^3}k^4 u(1+u)(1+3u+u^2) \\ &\quad + \frac{\bar\lambda^3}{16J^5}k^6 u(1+u)(1+7u+13u^2+7u^3+u^4) + \mathcal{O}\left(\frac{1}{J^7}\right) \ . \end{aligned} \tag{4.3.37}$$

Later we will be able to see that this large ω limit is also needed to have a workable form for the eigenfrequencies around the classical solution.

On top of this limit, in the next subsection we will use another perturbative expansion which will have as a limit a trivial sector ("vacuum"). We will later see that we need to be only a bit away from this new limit (i.e., to be in the perturbative expansion) in order to be able to use our regularization procedure.

4.3.3 The vacuum solution

Since the two dimensional soliton we are interested in corresponds in spacetime to a long spinning string, it follows easily that the trivial solution ("vacuum") has to be a point-like string. Guided by the BMN limit [3], where we also have perturbations around a state with large J, we know that the eigenvalues of the Hamiltonian, corresponding to perturbations around a BPS state, are the equivalent of the soliton mass, and therefore we look for states of lowest $E - J$ as the vacuum. We then vary the parameters in the non-trivial solution to obtain such a vacuum. This Hamiltonian is, as we saw,

$$\begin{aligned} E_0 - J &= S + \frac{\bar{\lambda}}{2J}k^2 u(1+u) - \frac{\bar{\lambda}^2}{8J^3}k^4 u(1+u)(1+3u+u^2) \\ &+ \frac{\bar{\lambda}^3}{16J^5}k^6 u(1+u)(1+7u+13u^2+7u^3+u^4) + \mathcal{O}\left(\frac{1}{J^7}\right), \end{aligned} \tag{4.3.38}$$

where $S \sim r_1$ and $u \sim m$. The smallest value is then obtained for

$$\begin{aligned} r_1, m &\longrightarrow 0, \\ r_0 &\longrightarrow 1. \end{aligned} \tag{4.3.39}$$

which implies in particular very small S as well (relative, since we formally took $S \to \infty$ before, though note that $\bar{\lambda}$ is large in $S = \sqrt{\bar{\lambda}} r_1^2 w$), with everything else (J, ω, k, κ) kept fixed in this second limit.

Then we obtain the "soliton mass" in the vacuum $E - J = 0$, as we wanted.

Taking these limits directly on the spinning solution we indeed get then the point-like string, the trivial solution we were looking for. Now that we have both solutions we can proceed to analyze quantum fluctuations around them.

4.3.4 The spectrum of quadratic fluctuations

Bosons

To find the characteristics frequencies we expand the action (4.3.21) around the solution (4.3.31). For the bosonic fluctuations we have six scalars corresponding to motion on $\mathbb{CP}^3$: one is massless, other four degrees of freedom give the same result,

$$p_0 = \sqrt{p_1^2 + \frac{1}{4}(\omega^2 - m^2)}\ , \tag{4.3.40}$$

and the last one gives

$$p_0 = \sqrt{p_1^2 + (\omega^2 - m^2)}\ . \tag{4.3.41}$$

From the scalars corresponding to motion in AdS space we find one massless degree of freedom, one massive one with

$$p_0 = \sqrt{p_1^2 + \kappa^2} \ , \tag{4.3.42}$$

and two fluctuations whose dispersion relation is given by the roots of the quartic equation

$$(p_0^2 - p_1^2)^2 + 4r_1^2\kappa^2 p_0^2 - 4\left(1 + r_1^2\right)\left(\sqrt{\kappa^2 + k^2}\ p_0 - kp_1\right)^2 = 0 \ . \tag{4.3.43}$$

We can find the explicit solutions to this equation (though they do not give much information), but only when we expand in large ω.

Fermions

For the fermionic part the spectrum contains four different frequencies, each being doubly-degenerate. Two such pairs have frequencies

$$(p_0)_{\pm 12} = \pm\frac{\mathrm{r}_0^2 k\kappa m}{2(m^2 + \mathrm{r}_1^2 k^2)} + \sqrt{(p_1 \pm b)^2 + (\omega^2 + k^2\mathrm{r}_1^2)} \ , \qquad b \equiv -\frac{\kappa m}{\mathrm{w}}\frac{\mathrm{w}^2 - \omega^2}{2(m^2 + \mathrm{r}_1^2 k^2)}, \tag{4.3.44}$$

while the frequencies of the other two pairs are solutions of the equation

$$(p_0^2 - p_1^2)^2 + \mathrm{r}_1^2\kappa^2 p_0^2 - \left(1 + \mathrm{r}_1^2\right)\left(\sqrt{\kappa^2 + k^2}\ p_0 - kp_1\right)^2 = 0 \ . \tag{4.3.45}$$

which can be solved in the same limit as in the bosonic case.

With the bosonic and fermionic frequencies we can start to calculate the quantum corrections, formally defined as in (4.1.1). But in order to do that, we must apply a regularization technique, specifically the derivative regularization previously defined. For that, we need to find the parameter that plays the role of m for us.

4.3.5 Physical limit and regularization

We want the parameter to lead to $E = J$ as it goes to zero, but be otherwise finite in the vacuum. Since

$$\begin{aligned} E_0 &= S + J + \frac{\bar{\lambda}}{2J}k^2 u(1+u) - \frac{\bar{\lambda}^2}{8J^3}k^4 u(1+u)(1+3u+u^2) \\ &\quad + \frac{\bar{\lambda}^3}{16J^5}k^6 u(1+u)(1+7u+13u^2+7u^3+u^4) + \mathcal{O}\left(\frac{1}{J^7}\right) \ , \end{aligned} \tag{4.3.46}$$

we could try u or k only, as we have $S = Ju$. Note one subtlety here: we have $E_0 = E_0(S, J, \bar{\lambda}, u, k)$, however $u = S/J$ so there is an ambiguity in the split of E_0 (how to we isolate the S dependence, when we could always write any u as S/J). We can consider that the S term is the one that is independent on k, which will be useful shortly.

However, u is not a good parameter, since it becomes always zero in the vacuum. On the other hand, k stays fixed in the vacuum, yet $k \to 0$ keeping everything else (u, J, ω, κ) fixed gives $E_0 \to S + J$. That is then not enough, and we need to supplement our original definition of the non-trivial vacuum with u small, and therefore also w, S, m small, i.e. in the perturbative expansion away from the vacuum.

Therefore k is the parameter that relates the two energies. One more subtlety to note is that, since we will use the large ω expansion, and since

$$u = \frac{S}{J} = \frac{r_1^2 w}{\omega} \lesssim \frac{1}{\omega}, \tag{4.3.47}$$

by doing the $1/\omega$ expansion first, we will not be able to match terms linear in u, as we will explain better later.

We should note that it was crucial that there were at least two parameters, J and k: J to guarantee a long string, with large J giving a perturbation theory, and k to differentiate with respect to it. It is our hope that this is more general for long strings, with something like J guaranteeing a long string, and something like k giving the "shape", allowing us to differentiate with respect to it.

We are finally ready for the calculation of the quantum correction to the energy.

4.3.6 Quantum correction to the energy

The one-loop energy correction was thought of as [37]

$$E_1 = \frac{1}{\kappa}\langle\Psi|H_2|\Psi\rangle, \tag{4.3.48}$$

where H_2 is the Hamiltonian for the quadratic fluctuations, but subtleties arose that were not well appreciated.

In order to understand what the issues are, we first review a few facts about previous calculations.

First, previous calculations have not taken into account the trivial sector or "vacuum" (cf. (4.1.1)), but considered only $E_1 = \frac{1}{2}\sum_n(\omega_n^B - \omega_n^F)$. Of course, at the classical level that does not matter, but it does matter at the quantum one-loop level. As we will see, removing the contribution of the trivial sector from the sum will help to the cancellation of some ambiguities.

Second, since one gets divergent sums in E_1, a regularization scheme is necessary, and various calculations gave regularization-dependent results [6–8, 25–27]. In the calculations of [6, 25, 26] the sum was turned into an integral, after which a cut-off was introduced and the integral sign given as a common factor, effectively choosing a form of energy/momentum cut-off regularization, as we explained in section 2. In [7] a different regularization was chosen, where one combines a mode number cut-off with a certain grouping of terms: instead of $\sum_n(\omega_n^{Bose} - \omega_n^{Fermi})$, one forms combinations called ω_n^{heavy} and ω_n^{light} and then a certain $n-$dependent combination of ω_n^{light} is added to ω_n^{heavy}, and the resulting sum over n is turned into an integral. This regularization gave a different result from the previous one. More recently, in [8], a modification of the regularization in [7] was given, with different combinations of ω_n^{heavy} and ω_n^{light}.

Yet another type of regularization was considered in [27], where a regularization method used successfully in the case $\mathrm{AdS}_5 \times S^5$ [38] was applied, together with a physically motivated redefinition of the coupling constant. The result of [27] is in agreement with AdS/CFT, so it was considered correct, but *a priori* we did not know which regularization scheme to choose to obtain an unambigous result, since as we saw different schemes can lead to different results, exactly as in the case of the 2d kink. We can use matching with AdS/CFT only as a kind of a posteriori check, exactly as one used the saturation of the BPS bound for the 2d supersymmetric kink (where both the mass and the central charge of the kink get renormalized in the same way).

In what follows we take a large ω expansion for both trivial and non-trivial sectors, and we will focus on the leading order in this expansion. As mentioned above, this will force us to take a small u expansion as well, and we can only say something about the leading term in the u expansion.

More importantly, in [27] it was explained that if we expand in $1/\omega$, since we can allow any value of $p_1 = n$, we have two regions for the expansion: a) $1/\omega \to 0$ with n fixed, i.e. $n \ll \omega$, for which we still have a discrete sum; and b) $n, \omega \to \infty$, with $x = n/\omega$=fixed, for which we can replace the sum with an integral. It was then noticed that while both regions contain divergences, the divergence of one can be identified with the divergence of the other, and can be dropped, obtaining a finite result. What we want to show here is that the ambiguity inherent in this procedure is removed by our method.

What we would have liked to do is take first the derivative with respect to k, and then do the sum over n, maybe with the same $1/\omega$ expansion, but this turns out to be prohibitively difficult, so we will be forced to follow the same analysis as [27] once we prove that our method eliminates the ambiguities.

We will start by analyzing region a), where we have discrete sums, and where

$$u \lesssim \frac{1}{\omega} \ll \frac{1}{n}. \tag{4.3.49}$$

The trivial sector ("vacuum") is simpler, and illustrates the point well, so we will start with it. The sum of bosonic frequencies (bosonic summand) in the trivial sector is

$$\sqrt{w^2 - n(2k-n)} + \sqrt{n(2k+n) + w^2} + 4\sqrt{n^2 + \frac{\kappa^2}{4}} + 2\sqrt{n^2 + \kappa^2}. \tag{4.3.50}$$

Replacing the perturbative solutions of the Virasoro constraints and expanding in ω, we get

$$6\omega + \frac{6n^2 + k^2(u(u+2)-1)}{\omega}. \tag{4.3.51}$$

A similar procedure for the fermionic summand (minus the sum of fermionic frequencies) gives

$$-6\omega - \frac{12n^2 + k^2(u+1)^2(u(u+2)-1)}{2\omega}. \tag{4.3.52}$$

Taking the sum of the two expressions to obtain the summand, we get terms like $n^2 - n^2$ and $\omega - \omega$ (since $\omega > n$, these are of the same type), which are ambiguous, but they will be cancelled after taking the derivative with respect to k. After the derivative with respect to k, the trivial sector summand $\tilde{e}(n)$ gives

$$\frac{\partial \tilde{e}(n)}{\partial k} \equiv \tilde{e}_k(n) = -\frac{k(1-u(2+u))^2}{\omega} + \mathcal{O}\left(\frac{1}{\omega^3}\right). \tag{4.3.53}$$

with no $n^2 - n^2$ and $\omega - \omega$ ambiguities.

Moving on to the non-trivial sector, the leading terms in the large ω expansion of the non-trivial sector summand are

$$\begin{aligned} e(n) = \frac{1}{2\omega}\Bigg[n\Bigg(3n - 4\sqrt{n^2 + k^2u(1+u)} + \sqrt{n^2 + 4k^2u(1+u)} \Bigg) \\ - k^2(1+u)(1+3u)\Bigg] + \mathcal{O}\left(\frac{1}{\omega^3}\right), \end{aligned} \tag{4.3.54}$$

and it can be seen that again terms like $n^2 - n^2$ appear, but they are again cancelled by taking the derivative with respect to k. After $\partial/\partial k$, the summand of the non-trivial sector gives

$$\frac{\partial e(n)}{\partial k} \equiv e_k(n) = -\frac{2k(1+u)(1+3u) - \left(\frac{4knu(1+u)}{\sqrt{n^2+k^2u(1+u)}} - \frac{4knu(1+u)}{\sqrt{n^2+4k^2u(1+u)}}\right)}{2\omega} + \mathcal{O}\left(\frac{1}{\omega^3}\right). \tag{4.3.55}$$

We note that even at $u = 0$, there is a constant piece that would give a divergence when summed over n, however it is the same one as in the trivial sector summand (4.3.53), so by subtracting the two we get rid of the last potential ambiguity.

We finally get

$$e_k(n) - \tilde{e}_k(n) = \frac{1}{\omega}\left(ku(u(u(4+u)-1)-8) + \frac{2knu(1+u)}{\sqrt{n^2+k^2u(1+u)}} - \frac{2knu(1+u)}{\sqrt{n^2+4k^2u(1+u)}} \right). \tag{4.3.56}$$

It would seem that we still have a divergence after we take the sum, but we need to remember that $u \ll 1/n$, so these terms linear in u do not give rise to divergences in this limit (or another way of saying it is that they belong to the omitted higher order terms in $1/\omega < 1/n$).

The final result for the one-loop correction to the energy coming from region a) is the sum over (4.3.56), integrated over k (with zero constant of integration).

There is a certain subtlety here, since in the end we want to calculate a correction to the energy that will turn out to have contributions linear in u, but as we mentioned, our only purpose (given our technical, i.e. calculational, limitations) is to show that the procedure of [27] becomes unambigous if we consider our physical principle.

Let us now analyze the result of [27] and compare to what we get. Expanding (4.3.54), now called $e^{sum}(n)$ to emphasize that we are in region a), at large n we get

$$e^{sum}(n) = \frac{1}{2\omega}\left(-k^2(1+u)(1+3u) - \frac{3}{2n^2}k^4u^2(1+u)^2 + \ldots\right) + \mathcal{O}\left(\frac{1}{\omega^3}\right), \qquad (4.3.57)$$

where the first term becomes divergent when summed over n (singular piece) and the second term becomes regular. The divergence and hidden ambiguities implicit in (4.3.57) were eliminated in our result (4.3.56).

On the other hand, in region b), with $\omega/n = x$ fixed, the expansion of the summand, now denoted $e^{int}(x)$ gives [27]:

$$\begin{aligned} e^{\rm int}(x) &= \frac{k^2(1+u)}{2\omega}\left(\frac{1+u(3+2x^2)}{(1+x^2)^{3/2}} - 2\frac{1+u(3+8x^2)}{(1+4x^2)^{3/2}}\right) \\ &- \frac{k^4(1+u)}{32\omega^3x^2}\Big[\frac{1}{(1+x^2)^{7/2}}\big(32u^2(1+u) + (7+u(77+u(221+135u)))x^2 \\ &\quad +4(-7+u(-7+u(29+21u)))x^4 + 16u(1+u(3+u))x^6 + 16u(1+u)x^8\big) \\ &\quad -\frac{8}{(1+4x^2)^{7/2}}\big(u^2(1+u) + (1+3u(5+u(11+5u)))x^2 \\ &\quad +8(-1+3u)(2+u(4+u))x^4 + 64u(2+3u)x^6 + 256u(1+u)x^8\big)\Big] + \mathcal{O}\left(\frac{1}{\omega^3}\right). \end{aligned} \qquad (4.3.58)$$

Note that in computing this expression we have also assumed the cancellation of $\infty - \infty$ terms that are a priori ambiguous, i.e. a priori the first term in the expansion would be ω, not $1/\omega$, but its coefficient is of the type $z - z$ and is k-independent, therefore disappears under our $\partial/\partial k$. [2]

[2] Note also that the result in (4.3.58) contains in the $1/\omega$ piece two subtracted terms linear in u that become log divergent at $x \to \infty$ after an integration in x. If one allows for cut-offs Λ_1, Λ_2 for the two subtracted terms such that $\Lambda_1/\Lambda_2 \to c \neq 1$, then we can still obtain an ambiguous result in the final answer (4.3.62). Such an ansatz, with $\Lambda_1/\Lambda_2 = 2$ instead of 1 for instance, leading to a difference of $2\ln 2$ in (4.3.62), was considered often starting with [7], but if we only allow $\Lambda_1 - \Lambda_2$ =finite, we don't have an ambiguity (more comments on that at the end of this section). Observe that in any case this term is

Then we can check that at $x \to 0$, the coefficient of the $1/\omega$ term becomes regular (constant), whereas from $1/\omega^3$ on, we have inverse powers of x at $x \to 0$, meaning a divergence in the integral $\int_0 dx$. Note that these singular terms all come multiplied by powers of u, so we cannot properly analyze them using our method, as $u < 1/\omega$ for us (for technical reasons).

However, we have

$$e_{sing}^{sum}(n) = e_{reg}^{int}\left(x = \frac{n}{\omega}\right), \tag{4.3.59}$$

as expected.

Similarly, in $e^{int}(x)$ have terms with inverse powers of x, which become singular (divergent) when integrated, but we can easily verify that

$$e_{sing}^{int}(x) = e_{reg}^{sum}(n = \omega x). \tag{4.3.60}$$

Due to this fact, in [27] it was proposed to just drop these singular terms, but this procedure hides a regularization ambiguity, since for instance we could expand in a slightly different parameter that ω and then by the same logic resolve to drop a different divergent piece from the total result. With our procedure, it becomes clear that result is unambigous and free of potential divergences, and we are in fact led to drop the singular terms of [27]. Indeed, the effect of summing over (4.3.56) and integrating over k with zero constant is (to leading order in u, which is what we can check) the same as just dropping the divergent terms in (4.3.57).

In conclusion, we see that there were a priori $\infty - \infty$ ambiguities that were hidden in the formal $1/\omega$ expansion procedure above, but we have checked that our physical principle just cancels them, and then we can continue with the same calculation as in [27]. Namely, the one-loop correction is now

$$E^{(1)} = E_{n=0} + \sum_{n\geq 1} e_{reg}^{sum} + \int dx e_{reg}^{int}(x), \tag{4.3.61}$$

where $E_{n=0}$ is the zero mode contribution. The terms giving odd powers of J are

$$E_{n=0} + \int dx e_{reg}^{int}(x) = S + J + \frac{\bar{h}^2(\bar{\lambda})k^2}{2J}u(1+u) + \mathcal{O}\left(\frac{1}{J^3}\right), \tag{4.3.62}$$

where

$$\bar{h}(\bar{\lambda}) = \sqrt{\bar{\lambda}} - \ln 2 + \mathcal{O}\left(\frac{1}{\sqrt{\bar{\lambda}}}\right) = 2\pi\left(\sqrt{\frac{\lambda}{2}} - \frac{\ln 2}{2\pi} + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right) = 2\pi h(\lambda), \tag{4.3.63}$$

linear in u, and the approximation we used was for $u \lesssim 1/\omega$, hence a term linear in u is really of at least one smaller order in $1/\omega$ in our case. Hence even in the case $\Lambda_1/\Lambda_2 \to c \neq 1$, we can at least claim that we have eliminated not only the *a priori* $\mathcal{O}(\omega)$ ambiguity that was implicit in the calculation, but also the ambiguity of the strict $1/\omega$ term (the piece not proportional to u), and to go beyond that we would need to avoid the constraint $u \lesssim 1/\omega$ which we needed solely in order to be able to calculate, but was not a theoretical restriction.

agrees with the value of $h(\lambda)$ argued in [27] to be predicted by AdS/CFT (though a direct calculation of quantum corrections to the dual to $h(\lambda)$ is still lacking).

Note however that changing both the $h(\lambda)$ above and the energy correction simultaneously could maintain agreement (see e.g. [8, 30]). Here we will assume, following [27], that the choice of $h(\lambda)$ above is unambigous (at least as long as the number of modes summed over in various terms differs only by a finite amount; in the heavy-light prescriptions used for instance in [7], some terms are summed over twice as many modes than other terms, due to some unitarity prescription).

On this chapter we analyzed the ambiguity in the summation of the frequencies for the spinning string on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ background, appliying a method that has not been used for these types of systems before. In the next chapter we will work on a diferent kind of problem but in the same type of background with an AdS4, then use AdS/CFT to see the effect on the dual field theory.

Chapter 5

Deformations of $\mathrm{AdS}_4 \times S^7$

A different application of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ duality stems from the utility of the ABJM model in testing various strongly coupled phenomena in planar condensed matter systems. The relation of ABJM to condensed matter system was effectively shown in [5] (see also [39]), via a consistent abelian truncation. One such application is the relatively poorly understood quantum critical phase that appears at nonzero temperatures around the $T = 0$ transition point between insulators and superconductors (superfluids). Since this system is effectively described by a conformal field theory (see for instance [40] and references therein), it is a prime candidate for application of the AdS/CFT toolbox.

Expanding on this point, the physics of particles (relevant on the insulator side of the transition) or vortices (relevant on the superconducting phase of the transition) suggest different behaviours for the conductivity of the system as a function of frequency, $\sigma(\omega)$. In [41] it was suggested that this situation could perhaps be captured by the gravity dual of the ABJM model by coupling the global $U(1)$ field to the Weyl tensor. Then, depending on the coupling, γ, one obtains either the particle-like or vortex-like behaviour for $\sigma(\omega)$. However, at nonzero temperatures, the conformal symmetry is broken by the scale T and understanding how this deformation affects quantities like the conductivity is essential. To this end finding a laboratory in which this can be done in a controlled setting that preserves as much of the other symmetries as possible is key.

In this chapter we study just such a laboratory, the massive deformation of the ABJM model of [11, 42], that is known to preserve the full $\mathcal{N} = 6$ supersymmetry. We want to generalize the calculation of [41] to include the effect of the mass deformation, however we will find that we can only calculate the *DC conductivity*, $\sigma(0)$. Even in this limited case though, in order to calculate this, we need to resort to a membrane paradigm-type calculation, similar to the one in [43], and extend its results to a more general set-up. An important technical point is that, in the absence of the exact solution for the black hole in the massive deformation of the gravity dual of ABJM, we need to rely on a perturbative approach to the massive deformation around the horizon of the black hole. We believe that this approach is new, novel and may indeed prove useful in other contexts as well.

The chapter itself is organized as follows. In section 5.1 we review the methods and results of [41]. In section 5.2 we show that by applying the Kubo formula, only now for a boundary term at the horizon of the AdS black hole, we obtain exactly the same DC conductivity as [41], and then review and generalize another membrane paradigm calculation as defined in [43]. In section 5.3 we perform the massive deformation of the black hole horizon, by first finding the zeroth and first order deformations, then using the Einstein's equations to compute the higher order one. We benchmark our procedure against the known AdS black hole with excellent agreement. Section 5.4 concerns itself with a calculation of the conductivity using the two versions of the membrane paradigm calculation.

5.1 Conductivity for AdS black holes

To begin, and largely to establish our conventions and notation, let us review quickly the computation of the conductivity associated to a Maxwell field in an AdS black hole background following [41]. Materials possessing a quantum critical phase, like ultracold ^{87}Rb, exhibit a zero temperature phase transition in terms of the coupling g of the system, between a superconducting phase for $g < g_c$ and an insulator phase for $g > g_c$. Modelling the order parameter of the system as a bosonic (scalar) field, the material is typically described by the action

$$S = \int d^3x \left[|\partial_\tau \phi|^2 + v^2 |\vec{\nabla}\phi|^2 + (g - g_c)|\phi|^2 + \frac{n}{2}|\phi|^4 \right] . \tag{5.1.1}$$

From this action, it is clear that when $g < g_c$, the order parameter $\langle \phi \rangle \neq 0$, resulting in the usual superfluid properties of the material, while when $g > g_c$ we have that $\langle \phi \rangle = 0$, giving an insulator phase. Thus $(g = g_c, T = 0)$ is a critical point of the system, described by a $(2+1)-$dimensional conformal field theory. To apply the arsenal of the gauge/gravity correspondence to problems like this, it is assumed that this system is dual to some (quantum) gravitational theory in a one higher dimensional AdS_4 space. It is known that the action 5.1.1 comes from a truncation of the ABJM model [5] that we discussed about on chapter 3, and it has a gravity dual theory in $AdS_4 \times \mathbb{CP}^3$.

At nonzero temperatures a quantum critical phase opens up[1] around $g = g_c$ and it is expected that the physics of this quantum critical point should be governed by the zero temperature conformal field theory. However, a computation of the frequency-dependent conductivity $\sigma(\omega)$ using a particle/hole description which is generally valid on the insulator side of the phase transition, yields a functional form that *decreases to a minimum* before stabilizing. On the other hand, a similar computation using a vortex description, valid on the superfluid side of the phase transition, obtains a conductivity that *increases to a maximum* before stabilizing (see for instance figures 4 and 5 in [40]). There seems to be

[1]See e.g. figure 2 in [40].

no way to choose which behaviour should be realized using field theory arguments. A semi-phenomenological parameter will be introduced in the gravity dual that will govern this choice ($\gamma > 0$ for particle-like and $\gamma < 0$ for vortex-like).

Gravity dual

The problem appears to be better suited to a dual gravity description. There is a phenomenological proposal for a gravity dual to condensed matter systems like the one of interest. The proposal stars with the standard Einstein-Maxwell theory (with a negative cosmological constant) in four dimensions. The action is as follows

$$S_0 = \int d^4x\sqrt{-g}\left[\frac{1}{2\ell_{\mathrm{P}}^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{4\mathrm{g}_4^2}F_{ab}F^{ab}\right]. \tag{5.1.2}$$

The four-dimensional AdS vacuum solution of the above theory corresponds to the vacuum of the dual three-dimensional CFT. In a probe approximation for the background, we consider that the AdS black hole background is not modified by the gauge field perturbation. The theory then has AdS black hole solutions:

$$ds^2 = \frac{r^2}{L^2}(-f(r)\,dt^2 + dx^2 + dy^2) + \frac{L^2dr^2}{r^2f(r)}, \tag{5.1.3}$$

where $f(r) = 1 - r_0^3/r^3$. In these coordinates, the asymptotic boundary is at $r \to \infty$ and the event horizon, at $r = r_0$. This solution is dual to the boundary CFT at temperature T, where the temperature is given by the Hawking temperature of the black hole

$$T = \frac{3r_0}{4\pi L^2}\,. \tag{5.1.4}$$

It is also convenient to work with a new radial co-ordinate: $u = r_0/r$. In this co-ordinate system, the black hole metric becomes

$$ds^2 = \frac{r_0^2}{L^2u^2}(-f(u)\,dt^2 + dx^2 + dy^2) + \frac{L^2du^2}{u^2f(u)}, \tag{5.1.5}$$

where $f(u) = 1 - u^3$. Now the asymptotic boundary is at $u = 0$ and horizon at $u = 1$.

We can see that when $T \to 0$, we return to the original AdS_4 background, so this can show how the introduction of a black hole in the bulk theory induce finite temperature on the CFT. If we consider only terms that vanish on the AdS_4 background but that are nonzero in the AdS_4-black hole background corresponding to a finite temperature field theory, there is a unique contribution to the action: a coupling to the Weyl tensor (zero on AdS, non-zero on an AdS black hole). With this in mind, the action for the Maxwell field in the gravity dual can be taken to be

$$S_{vector} = \frac{1}{g_4^2}\int d^4x\sqrt{-g}\Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \gamma L^2C_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\Big]. \tag{5.1.6}$$

Further the factor of L^2 was introduced above so that the coupling γ is dimensionless. From this action, we find the generalized vector equations of motion

$$\nabla_\mu \left[F^{\mu\nu} - 4\gamma L^2 C^{\mu\nu\rho\sigma} F_{\rho\sigma}\right] = 0, . \tag{5.1.7}$$

Note that the AdS vacuum and (neutral) planar black hole solution (5.1.3) are still solutions of the modified metric equations produced by the new action.

DC conductivity

With this in place, the DC conductivity can now be calculated from a membrane-paradigm like calculation starting from a more general form of the action in (5.1.6),

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{8g_4^2} F_{\mu\nu} X^{\mu\nu\rho\sigma} F_{\rho\sigma} \right]. \tag{5.1.8}$$

The action (5.1.6) corresponds to a choice of

$$X_{\mu\nu}{}^{\rho\sigma} = 2\delta_{[\mu\nu]}{}^{[\rho\sigma]} - 8\gamma L^2 C_{\mu\nu}{}^{\rho\sigma}. \tag{5.1.9}$$

A natural current j^μ can be defined on the horizon by considering the horizon boundary term associated with (5.1.6) and varying it with respect to A_μ. Using the equations of motion, we find the conductivity as the proportionality constant in $j^x = \sigma F_{0x}$, giving

$$\sigma_0 \equiv \sigma(\omega = 0, k = 0) = \frac{1}{g_4^2} \sqrt{-g} \sqrt{-X^{xtxt} X^{xrxr}} \Big|_{r=r_H}, \tag{5.1.10}$$

which leads to[2]

$$\sigma_0 = \frac{1}{g_4^2}(1 + 4\gamma). \tag{5.1.11}$$

However, this membrane paradigm calculation is somewhat obscure.[3] We will return to another membrane-paradigm like computation in the next section, but first, let's note that one can also do the usual AdS/CFT calculation utilising the Kubo formula.

Let us first briefly derive the Kubo formula. If we have an electromagnetic field at the boundary and take the axial gauge $A_0 = 0$, in order to simplify the computations. We can compute the electric field at the boundary

$$E_j = F_{0j} = -\partial_t A_j = -i\omega A_j. \tag{5.1.12}$$

[2]The DC conductivity defined here is at $\omega = 0$; more generally, we will see shortly that the relevant parameter is $w = \omega/(4\pi T)$, hence the result is valid for $\omega/T \ll 1$. This is the quantity that will be of interest for us in the remaining sections.

[3]In particular, because it is not obvious that this definition of conductivity at the horizon is the same as the definition at the boundary.

For the expectation value of the current (at some direction x) we have

$$\langle J_x \rangle = \sigma E_x = -i\omega\sigma A_x \tag{5.1.13}$$

The gauge field A_μ enters as a external source. Linear response in quantum field theory give us an expression that will help us to derive a relation for the conductivity, that is

$$\delta\langle J_\mu \rangle = \langle J_\mu \rangle - \langle J_\mu \rangle_0 = -i \int_{-\infty}^{t} dt' d^3x \langle [J_\mu(\vec{x}, t) J_\nu(\vec{x}', t')] \rangle_0 A_\nu(\vec{x}', t'), \tag{5.1.14}$$

Going back to the axial gauge, $\langle J_i \rangle_0$ is the current on the quantum state with $A_i = 0$, that is

$$\langle J_\mu \rangle_0 = e^2 A_i \langle \psi^\dagger \psi \rangle_0 = e A_i \rho, \tag{5.1.15}$$

where ρ is the background charge density. Finaly we use the relation between E_i and A_i and replace in (5.1.14) to get

$$\sigma_{ij} = -\frac{e\rho}{i\omega}\delta_{ij} + \frac{G_{ij}(\vec{q}, \omega)}{i\omega}, \tag{5.1.16}$$

with the Fourier transform of the retarded Green's function given in terms of the current-current correlation function by

$$G_{ij}(\vec{q}, \omega) = -i \int_{-\infty}^{\infty} dt d^3x \theta(t) e^{i(\omega t - \vec{q}\cdot\vec{x})} \langle [J_\mu(\vec{x}, t) J_\nu(\vec{0}, 0)] \rangle. \tag{5.1.17}$$

When the quantum current $\rho = 0$ and there is no real part in G_{ij} we obtain the Kubo formula for conductivity. A more precise analysis shows that, in fact, there is no real part involved in the formula, so we have

$$\sigma(\omega) = -\mathrm{Im}\left(\frac{G_{yy}(\vec{q})}{\omega}\right), \tag{5.1.18}$$

to calculate the frequency-dependent conductivity.

Returning to the A_μ field, its fourier transformation gives

$$A_\mu(t, x, y, u) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{x}} A_\mu(\vec{q}, u), \tag{5.1.19}$$

where $\vec{q}\cdot\vec{x} = -t\omega + q^x x + q^y y$, with $q^\mu = (\omega, q, 0)$. The boundary condition at the horizon is

$$A_y(\vec{q}, u) = (1-u)^b F(\vec{q}, u), \tag{5.1.20}$$

at $u = 1$, with $F(\vec{q}, u)$ regular there. To compute the conductivity associated to A_y, we first calculate the action for A_y on-shell, obtaining the boundary term

$$S_{yy} = -\frac{1}{2g_4^2} \int d^3x \left[\sqrt{-g} g^{uu} g^{yy} (1 - 8\gamma L^2 C_{uy}{}^{uy}) A_y(u, \vec{x}) \partial_u A_y(u, \vec{x}) \right] \Big|_{\text{boundary}} . \quad (5.1.21)$$

Evaluating this expression on the AdS black hole solution, we get

$$S_{yy} = -\frac{2\pi T}{3g_4^2} \int d^3x \left[(1 - u^3)(1 + 4\gamma u^2) A_y(u, \vec{x}) \partial_u A_y(u, \vec{x}) \right] \Big|_{\text{boundary}} . \quad (5.1.22)$$

Considering that the boundary is only at $u = 0$ (infinity), we obtain

$$\begin{aligned} S_{yy} &= -\frac{2\pi T}{3g_4^2} \int d^3x \left[A_y(u, \vec{x}) \partial_u A_y(u, \vec{x}) \right] \Big|_{u=0} \\ &\equiv \int \frac{d^2\vec{q}}{(2\pi)^3} \frac{1}{2} A_y(-\vec{q}) G_{yy}(\vec{q}) A_y(\vec{q}), \end{aligned} \quad (5.1.23)$$

where

$$G_{yy}(\omega, q = 0) = -\frac{4\pi T}{3g_4^2} \frac{\partial_u A_y(u, \omega)}{A_y(u, \omega)} \Big|_{u \to 0} . \quad (5.1.24)$$

Substituting this into the Kubo formula (5.1.18) gives

$$\sigma = \frac{1}{3g_4^2} \mathrm{Im} \frac{\partial_u A_y}{w A_y} \Big|_{u \to 0} , \quad (5.1.25)$$

where $w \equiv \omega / 4\pi T$. Now, the equation of motion for A_y with sources on the boundary can be solved. Indeed, for small w (or, equivalently, small ω) the solution is found to be

$$A_y(u) \simeq (1 - u)^{-iw} \left[F_1(u) + w F_2(u) \right] , \quad (5.1.26)$$

where $F_1(u) = C$ (constant) and $F_2'(0) = iC(2 + 12\gamma)$. This, together with the fact that both $F_1(u)$ and $F_2(u)$ are well-behaved at the horizon $u = 1$, means that

$$\sigma(\omega \to 0) = \frac{1}{3g_4^2} \mathrm{Im} \left[i + \frac{F_2'(0)}{F_1(0)} \right] = \frac{1}{3g_4^2} [1 + (2 + 12\gamma)] = \frac{1 + 4\gamma}{g_4^2} , \quad (5.1.27)$$

exactly the same as the earlier membrane-paradigm calculation.

In ABJM theory (at $\gamma = 0$), we can find the meaning of g_4 as follows. The maximally gauged supergravity action in 4 dimensions has a coupling determined by L and the 4d Newton's constant $\kappa_N^{(4)}$,

$$g_4 \propto \frac{L^{-1}}{\kappa_N^{(4)-1}} = \frac{\kappa_N^{(4)}}{L} \quad (5.1.28)$$

where the proportionality factor is a numerical constant. Since by compactifying on $S^7/\mathbb{Z}_k$, with $L_{S^7} = 2L$,

$$\frac{1}{[\kappa_N^{(4)}]^{-1}} = \frac{\Omega_7 (L_{S^7})^7}{[\kappa_N^{(11)}]^2 k} = \frac{\Omega_7 g_s^{-3} l_s^{-9} 2^7 L^7}{k}, \tag{5.1.29}$$

where Ω_7 is the volume of the unit 7-sphere, we obtain

$$g_4 \propto g_s^{3/2} k^{1/2} \left(\frac{l_s}{L}\right)^{9/2} \propto \lambda^{1/4} N^{-1} \tag{5.1.30}$$

where in the last equality we wrote g_4 in terms of field theory parameters using $L/l_s = 5^{5/4}\sqrt{\pi}\lambda^{1/4}$ and $g_s = \lambda^{5/4}/N$.

5.2 Membrane paradigm and conductivity from horizon data

In order to extend the computation of the conductivity to the mass-deformed ABJM model, we will first need to modify the above membrane-paradigm computation to account for the fact that we are, as yet, unable to construct a black hole directly in the gravitational dual. The modification itself is easy enough to state; in (5.1.21), the boundary term was considered to be at $u = 0$ (at infinity), whereas now we want to consider the same boundary term at the horizon $u = 1$. This is a bit unusual, since it assumes that the sources for the bulk are *at the horizon* and that, strictly speaking, we have to ignore the normal boundary at $u = 0$, since otherwise we get a divergent contribution due essentially to the infinite ratio between the contributions from the two boundaries. Let's first benchmark this against the above case of the AdS$_4$ black hole to check that we are indeed on the right track before looking for a better justification for it.

For the boundary term at $u = 1$ only, from (5.1.22) we get

$$S_{yy} = -\frac{2\pi T}{3g_4^2}(1+4\gamma)\int d^3x \left[3(1-u)A_y(u,\vec{x})\partial_u A_y(u,\vec{x})\right]\Big|_{u\to 1}, \tag{5.2.31}$$

which leads to

$$G_{yy}(\omega, q=0) = -\frac{4\pi T}{3g_4^2}(1+4\gamma)\left[3(1-u)\frac{\partial_u A_y(u,\omega)}{A_y(u,\omega)}\right]\Big|_{u\to 1}. \tag{5.2.32}$$

This expression, together with the Kubo formula (5.1.18), evaluated at the horizon the approximation $\partial_u A_y/w \simeq i/(1-u)$ yields

$$\sigma(\omega \to 0) = \frac{1}{3g_4^2}(1+4\gamma)\mathrm{Im}\left[3(1-u)\frac{i}{1-u}\right] = \frac{1+4\gamma}{g_4^2}. \tag{5.2.33}$$

Not only is this the same result as the standard membrane paradigm computation but notice also that the factors of $1 + 4\gamma$ and -3 enter in a completely different way, so it is highly nontrivial that we get the same answer.[4]

To better understand this result, let's reconsider the general membrane paradigm analysis carried out in [43] where the same type of current at the horizon as in [41] was considered in the context of the action

$$S_{\rm em} = -\int_\Sigma d^{d+1}x\sqrt{-g}\frac{1}{4g_{d+1}^2(r)}F_{MN}F^{MN} \tag{5.2.34}$$

where $g_{d+1}(r)$ is an r-dependent coupling possibly arising from background fields. An for the background metric we have

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{ij}dx^i dx^j. \tag{5.2.35}$$

The bulk action results in a boundary term that can be cancelled by

$$S_{\rm bd} = \int_\Sigma d^dx\sqrt{-\gamma}\left(\frac{j^\mu}{\sqrt{-\gamma}}\right)A_\mu, \tag{5.2.36}$$

where $j^\mu = -\frac{1}{g_{d+1}^2}\sqrt{-g}F^{r\mu}$,
is obtained by varying the action with respect to $\partial_r A_\mu$, and $\gamma_{\mu\nu}$ is the induced metric on the stretched horizon Σ.

Note that the "Gauss's law" that we obtain by treating r as "time" becomes conservation of the currents $J^\mu_{\rm mb}$ and j^μ on any constant-r slice.

$$\partial_\mu J^\mu_{\rm mb} = \partial_\mu j^\mu = 0 \ . \tag{5.2.37}$$

While *a priori* the current $J^i_{\rm mb}$ (or j^i), which is determined by F^{ir}, and the electric field $E^i = F^{it}$ are independent variables, they are in fact proportional to each other at the horizon. This can be seen as follows. Since the horizon is a regular place for free in-falling observers, the electromagnetic field observed by them must be regular. This implies that near the horizon, A^M can only depend on r and t through their non-singular combination, the Eddington-Finklestein coordinate v defined by

$$dv = dt + \sqrt{\frac{g_{rr}}{g_{tt}}}dr \ . \tag{5.2.38}$$

This implies that

$$\partial_r A_i = \sqrt{\frac{g_{rr}}{g_{tt}}}\partial_t A_i \ , \quad r \to r_0 \tag{5.2.39}$$

[4]Also notice that the temperature has cancelled in the calculation of the DC conductivity, which as already explained, is still valid as long as $\omega/T \ll 1$.

and with gauge choice $A_r = 0$, we then have (with $r \to r_0$)

$$F_{ri} = \sqrt{\frac{g_{rr}}{g_{tt}}} F_{ti} \quad \to \quad J^i_{\text{mb}} = -\frac{1}{g^2_{d+1}} \sqrt{g^{tt}} F^i_t = \frac{1}{g^2_{d+1}} \hat{E}^i \tag{5.2.40}$$

Here $\hat{E}^i$ is an electric field measured in an orthonormal frame of a physical observer hovering just outside of the black hole. From (5.2.40), it is natural to interpret J^i_{mb} as the *response* of the horizon membrane to the electric field $\hat{E}^i$, leading to a membrane conductivity

$$\sigma_{\text{mb}} = \frac{1}{g^2_{d+1}(r_0)} \ . \tag{5.2.41}$$

Note that (unlike those arising from conventional quantum field theories) this conductivity is frequency-independent and depends only on the gauge coupling at the horizon. It is indeed compatible with (5.1.10) for $d = 3$ and $X_{\mu\nu}{}^{\rho\sigma} = 2\delta_{[\mu\nu]}{}^{[\rho\sigma]}$.

The equations of motion of the action (5.2.34) also imply

$$\partial_r j^i = 0 + \mathcal{O}(\omega F_{it}); \quad \partial_r F_{it} = 0 + \mathcal{O}(\omega j^i) \tag{5.2.42}$$

which in turn mean that the relation

$$j^i(r_H) = \frac{1}{g^2_{d+1}} \sqrt{\frac{-g}{g_{rr} g_{tt}}} g^{zz} \bigg|_{r_H} F_{it}(r_H) \tag{5.2.43}$$

is valid at all r, until infinity, with the same coefficient. Therefore the DC conductivity calculated at the horizon equals the one calculated at infinity, which is the usual AdS/CFT formula, with the result

$$\sigma(\omega = 0) = \frac{1}{g^2_{d+1}} \sqrt{\frac{-g}{g_{rr} g_{tt}}} g^{zz} \bigg|_{r_H} \tag{5.2.44}$$

which matches (5.2.41) for $d = 3$. In order to apply these methods to our action in (5.1.6), we shall consider the case when we can write, on the background solution,

$$S = -\int d^4x \sqrt{-g} \left(\frac{1}{4 g^2_{MN,4}(r)} F_{MN} F^{MN} \right) . \tag{5.2.45}$$

Moreover, for the conductivity associated to A_y, we need only the F_{ry} component to be nonzero in which case, following the membrane paradigm calculation, we obtain,

$$\sigma(\omega = 0) = \frac{1}{g^2_{ry,4}} \tag{5.2.46}$$

instead of (5.2.41) so that (5.2.42) still holds, implying that the membrane paradigm calculation still matches the AdS/CFT result as was indeed verified for pure ABJM theory.

We therefore feel confident in using it for the massive deformation. Having said that, one potential issue could be the fact that we will need to deal with a more general metric background of the form

$$ds^2 = -g_{tt}(r,t)dt^2 + g_{rr}(r,t)dr^2 + 2g_{rt}(r,t)dr\,dt + g_{yy}(r)(dx^2 + dy^2) \tag{5.2.47}$$

However, it can be shown that the off-diagonal terms do not influence the result. Indeed, we will see shortly that the same is true for the calculation of the conductivity.

We should also point out that an explicit time dependence in the metric, as in (5.2.47), modifies (5.2.42) by replacing the "0" on the right hand side of the first equation by time derivatives of the metric. This would imply that we could not use (5.2.42) as an argument to show that the conductivity defined at the horizon matches the one defined at the boundary. However, this does not necessarily mean they are different, since the *integrated* $r-$dependence for $\sigma = j^i/F_{it}$ could still give the same result at the horizon and the boundary. This is as expected from the validity of AdS/CFT, which is, after all, a relation between the boundary field theory (the conductivity being a boundary observable) and bulk gravitational physics, and the membrane paradigm which implies that the this bulk gravitational physics should be describable in terms of just a fictitious membrane at the horizon. Moreover, the time dependence cannot last forever. The metric must "relax" to a stationary one after some time has passed, as expected from general principles of black hole physics. Thus although, in principle, there could be an explicit time dependence in the bulk metric, we can assume this dependence will die out way before we "shake" the system with the gauge fields A_μ^A we are using as probes.

5.3 Massive deformation for ABJM black hole

As alluded to in the introduction to this article, the ABJM model admits a mass deformation that, while obviously breaking its conformal symmetry, preserves its full $\mathcal{N} = 6$ supersymmetry [11, 42]. Unfortunately, the gravity dual to this massive deformation is, to say the least, very complicated (see [12, 44] for more details), and finding an exact black hole solution in this case seems hopeless. What we *can* do however, is to consider the deformation of the background in which the M2-branes of the ABJM model live. Recall that the pure ABJM model arises from the near-horizon limit of the back-reacted geometry produced by M2-branes moving in the background $\mathbb{R}^{2,1} \times \mathbb{C}^4/\mathbb{Z}_k$, i.e. 11-dimensional flat space with a $\mathbb{Z}_k$ identification. The background corresponding to the mass-deformed ABJM model is obtained from the maximally supersymmetric pp-wave of type IIB string theory by a sequence of T-dualizing and lifting to M-theory [44, 45]. To add a black hole to this construction is a notoriously difficult task.

5.3.1 First order deformation near the horizon

Fortunately, as argued earlier, it will suffice for us to consider just a mass-deformation of the horizon of the AdS_4 black hole. To zeroth order, we can consider the horizon of the black hole to be flat, so we can trivially add a pp-wave to it by replacing the flat space with the pp-wave space, in the coordinates around the spherical horizon. The mass deformation is then implemented T-dualizing to obtain a first order solution. To obtain a (back-reacted) solution around the horizon of the black hole with the mass deformation, we will input this first order solution as initial data into the Einstein's equations, and obtain the required metric and Weyl tensor at the horizon. We will show how to carry this out explicitly in the next subsection and, for now, restrict our attention to obtaining the first order solution to be used as an initial data.

To superpose the pp-wave on to the AdS black hole, we must first obtain flat space in usual Minkowski coordinates. To do that, we will take a near-horizon limit and then perform a double Wick rotation. We detail these steps as follows. The starting point is the $AdS_4 \times S^7$ black hole metric in Poincaré coordinates

$$ds^2 = \frac{r^2}{L^2}\left[-\left(1-\frac{r_0^3}{r^3}\right)dt^2 + d\vec{x}^2\right] + L^2\frac{dr^2}{r^2\left(1-\frac{r_0^3}{r^3}\right)} + L^2 d\Omega_7^2, \tag{5.3.48}$$

where $\vec{x} = (x_1, x_2)$. Since the near horizon geometry corresponds to $r - r_0 \ll r_0, L$, we should also consider $L\,\Omega_7$ large, and approximate it with the flat $d\vec{y_7}^2$. Changing the radial coordinate r to $\rho \equiv 2\sqrt{r/r_0 - 1}$ and expanding in powers of ρ yields the near horizon geometry,

$$ds_M^2 \simeq \frac{r_0^2}{L^2}\left[-\frac{3}{4}\rho^2 dt^2 + dx_1^2 + dx_2^2\right] + \frac{L^2}{3}d\rho^2 + d\vec{y}_7^2. \tag{5.3.49}$$

Since

$$g_{tt} = -\frac{3r_0^2}{4L^2}\rho^2, \qquad g_{x_1x_1} = \frac{r_0^2}{L^2,} \qquad g_{\rho\rho} = \frac{L^2}{3}, \tag{5.3.50}$$

reducing to string theory on x_2 and T-dualizing along x_1 yields

$$\begin{aligned}
\tilde{g}_{x_1x_1} &= \frac{1}{g_{x_1x_1}} = \frac{L^2}{r_0^2}, \\
\tilde{g}_{\rho\rho} &= g_{\rho\rho} = \frac{L^2}{3}, \\
\tilde{g}_{tt} &= g_{tt} = -\frac{3r_0^2\rho^2}{4L^2}, \\
\tilde{g}_{\mu x_1} &= 0, \\
\tilde{\phi} &= -\frac{1}{2}\ln\left(\frac{r_0^2}{L^2}\right).
\end{aligned} \tag{5.3.51}$$

The type IIB metric can then be read off as

$$ds^2_{IIB} = -\frac{3r_0^2}{4L^2}\rho^2 dt^2 + \frac{L^2}{r_0^2}dx_1^2 + \frac{L^2}{3}d\rho^2 + dy_7^2. \tag{5.3.52}$$

For notational convenience, in what follows, we define the new variables,

$$\tau \equiv \frac{3r_0}{2L^2}t, \qquad \bar{\rho} \equiv \frac{L}{\sqrt{3}}\rho, \qquad \bar{x}_1 \equiv \frac{L}{r_0}x_1, \tag{5.3.53}$$

in terms of which,

$$ds^2_{IIB} = -\bar{\rho}^2 d\tau^2 + d\bar{x}_1^2 + d\bar{\rho}^2 + dy_7^2. \tag{5.3.54}$$

Wick rotating to $\tau = i\theta$ yields

$$ds^2 = \bar{\rho}^2 d\theta^2 + d\bar{x}_1^2 + d\bar{\rho}^2 + dy_7^2. \tag{5.3.55}$$

Now we Wick rotate back along a Euclidean coordinate by first defining

$$z_1 \equiv \bar{\rho}\cos\theta = \bar{\rho}\cos(-i\tau) = \bar{\rho}\cosh\tau \tag{5.3.56}$$
$$z_2 \equiv \bar{\rho}\sin\theta = \bar{\rho}\sin(-i\tau) = -i\bar{\rho}\sinh\tau. \tag{5.3.57}$$

and rotating to $z_2 = -i\bar{\tau}$, to give the new set of variables

$$z_1 = \bar{\rho}\cosh\tau, \tag{5.3.58}$$
$$\bar{\tau} = \bar{\rho}\sinh\tau. \tag{5.3.59}$$

Near $\tau \sim 0$ we have $z_1 \simeq \bar{\rho}$ and $\bar{\tau} \simeq \bar{\rho}\tau$, so that $\bar{\rho} \simeq z_1$ and $\tau \simeq \bar{\tau}/\bar{\rho}$. Thus, expanding in powers of τ

$$d\tau^2 = \frac{d\bar{\tau}^2}{\bar{\rho}^2} + \mathcal{O}(\tau), \tag{5.3.60}$$
$$d\bar{\rho}^2 = dz_1^2 + \mathcal{O}(\tau), \tag{5.3.61}$$

yields

$$ds^2 = -d\bar{\tau}^2 + d\bar{x}_1^2 + dz_1^2 + dy_7^2. \tag{5.3.62}$$

Now that we have the near horizon metric explicitly written as a flat metric, we simply add the pp-wave to the horizon by replacing (5.3.62) with

$$ds^2 = -d\bar{\tau}^2 + d\bar{x}_1^2 - \mu^2\left(z_1^2 + \vec{y}_7^2\right)(d\bar{\tau} + d\bar{x}_1)^2 + dz_1^2 + dy_7^2, \tag{5.3.63}$$

and turning on an $F_{+z_1\cdots}$, where the lightcone coordinate $x^+ \equiv \bar{x}_1 + \bar{\tau} = \bar{x}_1 + \bar{\rho}\sinh\tau$ and $z_1 = \bar{\rho}\cosh\tau$, and the ellipses are three of the $\vec{y}$. Note that μ, the mass parameter of the pp-wave added, corresponds in the gauge theory to the mass deformation parameter of the same name, as was argued in [44, 45]. In particular, from the identification in eq. 2.28 of [45] with the calculation of the same quantity in gauge theory, we can see that μ is indeed the same parameter. Going back from the $(z_1, \bar{\tau})$ variables to the original $(\bar{\rho}, \tau)$ ones gives the IIB wave solution

$$ds^2 = -(d(\bar{\rho}\sinh\tau))^2 + d\bar{x}_1^2 - \mu^2\left(\bar{\rho}^2\cosh^2\tau + \vec{y}_7^2\right)(d(\bar{\rho}\sinh\tau) + d\bar{x}_1)^2 + (d(\bar{\rho}\cosh\tau))^2 + dy_7^2. \tag{5.3.64}$$

In order to arrive at the desired eleven dimensional metric, we now need to T-dualize it to IIA and then lift it to M-theory.

Let us perform the process of going back to IIA. We start expanding out the differentials in (5.3.64), the nonzero components of the metric are:

$$\begin{aligned}
g_{\tau\tau} &= -\bar{\rho}^2\left(1 + \mu^2\cosh^2\tau\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)\right),\\
g_{\bar{x}_1\bar{x}_1} &= 1 - \mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right),\\
g_{\bar{\rho}\bar{\rho}} &= 1 - \mu^2\sinh^2\tau\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right),\\
g_{\bar{\rho}\tau} &= -\mu^2\bar{\rho}\cosh\tau\sinh\tau\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right),\\
g_{\bar{x}_1\tau} &= -\mu^2\bar{\rho}\cosh\tau(\bar{\rho}^2\cosh^2\tau + y_7^2),\\
g_{\bar{\rho}\bar{x}_1} &= -\mu^2\sinh\tau\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right).
\end{aligned} \tag{5.3.65}$$

Now we will T-dualize on the $\bar{x}_1$ direction to IIA. After application of the Buscher rules the new metric components will be

$$\begin{aligned}
\tilde{g}_{\bar{x}_1\bar{x}_1} &= 1/g_{\bar{x}_1\bar{x}_1} = \frac{1}{1 - \mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)},\\
\tilde{g}_{\tau\tau} &= g_{\tau\tau} - (g_{\tau\bar{x}_1})^2/g_{\bar{x}_1\bar{x}_1} = -\frac{\bar{\rho}^2\left(1 + \mu^2\sinh^2\tau\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)\right)}{1 - \mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)},\\
\tilde{g}_{\bar{\rho}\bar{\rho}} &= g_{\bar{\rho}\bar{\rho}} - (g_{\bar{\rho}\bar{x}_1})^2/g_{\bar{x}_1\bar{x}_1} = \frac{1 - \mu^2\cosh^2\tau\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)}{1 - \mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)},\\
\tilde{g}_{\bar{\rho}\tau} &= g_{\bar{\rho}\tau} - (g_{\bar{\rho}\bar{x}_1}g_{\tau\bar{x}_1})/g_{\bar{x}_1\bar{x}_1} = -\frac{\mu^2\bar{\rho}\cosh\tau\sinh\tau\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)}{1 - \mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)},\\
\tilde{\phi} &= -\frac{1}{2}\log|g_{\bar{x}_1\bar{x}_1}| = -\frac{1}{2}\log\left(1 - \mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)\right),\\
\tilde{B}_{\tau\bar{x}_1} &= g_{\tau\bar{x}_1}/g_{\bar{x}_1\bar{x}_1} = \frac{-\mu^2\bar{\rho}\cosh\tau(\bar{\rho}^2\cosh^2\tau + y_7^2)}{1 - \mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)},\\
\tilde{B}_{\bar{\rho}\bar{x}_1} &= g_{\bar{\rho}\bar{x}_1}/g_{\bar{x}_1\bar{x}_1} = \frac{-\mu^2\sinh\tau\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)}{1 - \mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)}.
\end{aligned} \tag{5.3.66}$$

Then, the mass-deformed metric on type IIA has the following form:

$$\begin{aligned} ds^2_{IIA} &= \frac{1}{1-\mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)}\Big(-\bar{\rho}^2 d\tau^2 + d\bar{\rho}^2 + d\bar{x}_1^2 \\ &\quad -\mu^2\left(\bar{\rho}^2\cosh^2\tau + y_7^2\right)\left(d(\bar{\rho}\cosh\tau)\right)^2\Big) + d\vec{y}_7^2. \end{aligned} \tag{5.3.67}$$

Now, returning to the original (ρ, t, x_1) variables using (5.3.53), and defining

$$H \equiv 1 - \mu^2\left(\frac{L^2}{3}\rho^2\cosh^2\left[\frac{3tr_0}{2L^2}\right] + y_7^2\right) \tag{5.3.68}$$

puts the type IIA metric above into the form

$$ds^2_{IIA} = H^{-1}\left[-\frac{3r_0^2}{4L^2}\rho^2 dt^2 + \frac{L^2}{3}d\rho^2 + \frac{L^2}{r_0^2}dx_1^2 - (1-H)\frac{L^2}{3}\left(d\left(\rho\cosh\frac{3r_0 t}{2L^2}\right)\right)^2\right] + d\vec{y}_7^2. \tag{5.3.69}$$

Finally, lifting to M-theory gives

$$\begin{aligned} ds^2_M &= H^{-2/3}\Bigg[-\frac{3r_0^2}{4L^2}\rho^2 dt^2 + \frac{L^2}{3}d\rho^2 + \frac{L^2}{r_0^2}dx_1^2 + dx_{11}^2 \\ &\quad -(1-H)\frac{L^2}{3}\left(d\left(\rho\cosh\frac{3r_0 t}{2L^2}\right)\right)^2\Bigg] + H^{1/3}d\vec{y}_7^2. \end{aligned} \tag{5.3.70}$$

This metric corresponds to mass-deforming the black hole solution, though only near the horizon. We also note that (5.3.62) only corresponded to the metric near the horizon for small $\bar{\tau}$, or equivalently for small t, so we will still need to be careful to use it only in black hole calculations in this region.

Having, through considerable ingenuity, obtained the eleven dimensional metric, in order to apply the procedure for calculating conductivity, we first need to dimensionally reduce it to four dimensions with a cosmological constant. The question at hand then is whether we can perform a consistent truncation to four dimensions. The general ansatz for the truncation of the metric on an n-sphere S^n down to d dimensions is [46] (here $D = d + n$)

$$ds^2_D = \Delta^{-\frac{2}{d-2}}ds^2_d + \Delta^{\beta}T^{-1}_{AB}DY^A DY^B \tag{5.3.71}$$

where

$$\begin{aligned} \beta &= \frac{2}{n-1}\frac{d-1}{d-2}, \\ \Delta^{2-\beta n} &= T^{AB}Y_A Y_B, \\ DY^A &= dY^A + B^{AB}Y^B. \end{aligned} \tag{5.3.72}$$

Here $Y^A(y)$, with $A = 0, ..., n$, the scalar spherical harmonics on the sphere parametrized by $\vec{y}$ intrinsic coordinates, are also coordinates for the Euclidean embedding of the sphere, i.e. $Y^A Y^A = 1$. In our case, $d = 4, n = 7$ and $B^{AB} = 0$, giving

$$\begin{aligned} ds_{11}^2 &= \Delta^{-1} ds_4^2 + \Delta^{1/2} T_{AB}^{-1} dY^A dY^B, \\ \Delta &= (T^{AB} Y_A Y_B)^{-2/3}. \end{aligned} \tag{5.3.73}$$

This also matches with another expression found in [47], for $T_{AB} = X_A \delta_{AB}$, with

$$ds_D^2 = (X_A Y_A^2)^{\frac{2}{d-1}} ds_d^2 + (X_A Y_A^2)^{-\frac{d-3}{d-1}} (X_A^{-1} dY_A^2). \tag{5.3.74}$$

In our case, both this and the former expressions reduce to:

$$ds_{11}^2 = (X_A Y_A^2)^{2/3} ds_4^2 + (X_A Y_A^2)^{-1/3} X_A^{-1} dY_A^2 \tag{5.3.75}$$

where $A = 0, ..., 7$. The question is whether the metric (5.3.70) can be described as this kind of consistent truncation to four dimensions on a deformed 7-sphere since we need a metric solving a four dimensional action obtained by consistent truncation. It is easy to see that, in general, it can not. However, as we will now demonstrate, it is possible to do so in a special limit.

The spherical harmonics Y^A are coordinates for the embedding of the sphere in 8-dimensional Euclidean space, i.e. $Y_A Y^A = L^2$. We will consider a small patch of the sphere, namely near the "North Pole", $Y^0 = L$. Then we can consider an intrinsic parametrization of the sphere, i.e. some set of coordinates $\vec{y}$ which match the $\vec{y}_7$ coordinates in our metric. Taking the y_i's to be small means that the spherical harmonic components are $\overrightarrow{Y_7}(\vec{y}_7) \simeq \overrightarrow{y_7}$ so that, from the sphere constraint, we have

$$Y^0 = \sqrt{L^2 - (\overrightarrow{Y_7})^2} \simeq \sqrt{L^2 - (\overrightarrow{y_7})^2}. \tag{5.3.76}$$

The factor Δ in the metric is then

$$X_A Y_A^2 = X_0 \left(L^2 - (\overrightarrow{y_7})^2 \right) + X_7^i (y_7^i)^2, \tag{5.3.77}$$

and, reading off equation from (5.3.70), this should match with H^{-1}. Expanding both sides in powers of $(y_7^i)^2$ and matching coefficients, yields the eight scalars

$$\begin{aligned} X_0 L^2 &= \frac{1}{1 - \mu^2 \rho^2 \frac{L^2}{3} \cosh^2 \left[\frac{3tr_0}{2L^2} \right]}, \\ X_7^i &= |X_7| = X_0 + \mu^2 (X_0 L^2)^2, \end{aligned} \tag{5.3.78}$$

in four dimensions. Finally then, the four dimensional metric is

$$ds_4^2 = -dt^2 \left(\frac{3r_0^2}{4L^2} \rho^2 + \frac{\mu^2 \rho^4 r_0^2}{4} \cosh^2 \left[\frac{3tr_0}{2L^2} \right] \sinh^2 \left[\frac{3tr_0}{2L^2} \right] \right)$$

$$
\begin{aligned}
&+d\rho^2 \frac{L^2}{3}\left(1-\mu^2\frac{L^2}{3}\rho^2\cosh^4\left[\frac{3tr_0}{2L^2}\right]\right)\\
&-d\rho\, dt\, \mu^2 r_0 \frac{L^2}{3}\rho^3\cosh^3\left[\frac{3tr_0}{2L^2}\right]\sinh\left[\frac{3tr_0}{2L^2}\right]+\frac{L^2}{r_0^2}dx_1^2+dx_{11}^2. \qquad (5.3.79)
\end{aligned}
$$

Some points about this metric deserve further clarification. Obviously, as we claimed earlier, (5.3.70) is not of the reduced type (5.3.75). However, because the two match up to quadratic order in the internal y coordinates[5], there is a solution that looks like the reduction ansatz (5.3.75) and whose four-dimensional metric is a small perturbation of (5.3.81). We also note that we only need to define the (arbitrary) initial solution on a *Cauchy surface*, and so we needn't worry about satisfying the *exact* equations of motion for the initial solution that we define here.

Returning to the original coordinate u with $\rho = 2(1/u-1)^{1/2}$, we can now see the full metric as the AdS black hole metric plus the massive deformation metric

$$
ds^2 \;=\; ds^2_{AdS_4\,BH}+\mu^2 ds^2_{\mathrm{MD}}, \qquad (5.3.80)
$$

where

$$
\begin{aligned}
ds^2_{\mathrm{MD}} \;=\; &-dt^2 r_0^2\frac{(1-u)^2}{u^2}\cosh^2\left(\frac{3tr_0}{2L^2}\right)\sinh^2\left(\frac{3tr_0}{2L^2}\right)-du^2\frac{4L^4}{9u^4}\cosh^4\left(\frac{3tr_0}{2L^2}\right)\\
&-du\,dt\frac{8L^2 r_0(1-u)}{3u^3}\cosh^3\left(\frac{3tr_0}{2L^2}\right)\sinh\left(\frac{3tr_0}{2L^2}\right).
\end{aligned}
\qquad (5.3.81)
$$

This metric (5.3.80) is our first order solution that we now employ as the *initial data* to the Einstein's equations to obtain the back-reacted metric and Weyl tensor at the horizon subject, of course, to the caveat that the solution corresponds to the mass deformation of the horizon only near $t = 0$.

Note that we now have a metric with scalars turned on near the horizon. Thus, our Einstein-Maxwell action with the Weyl coupling term needs to be further modified in order to account for the backreaction of the scalar fields. As argued in the introduction, we do not have the luxury of having an exact black hole solution for the gravity dual of the mass-deformed ABJM theory. What we do have, however, is the leading order mass deformation at the horizon, sourced by the scalars T_{AB}, which is sufficient to compute the membrane DC conductivity.

In order to find the correct coupling of the scalars to gravity and the Maxwell fields, we focus on the bosonic part the of the four-dimensional gauged supergravity action. We

[5] Note that the equations of motion only involve second derivatives, hence in the neighbourhood of the North Pole we only need to have the solution up to quadratic order in y in order to satisfy them.

are therefore led to consider the action

$$I = \int d^4x\sqrt{-g}\left[R - \frac{1}{2}m^2\left(\mathrm{Tr}(T)^2 - 2\mathrm{Tr}(T^2)\right) - \frac{1}{4}\mathrm{Tr}\left(\partial_\mu T^{-1}\partial^\mu T\right) + \frac{1}{g_4^2}\left(-\frac{1}{4}T_{AB}F^A_{\mu\nu}F^{B\mu\nu} + \gamma L^2 C_{\mu\nu\rho\sigma}F^{A\mu\nu}F^{A\rho\sigma}\right)\right] \tag{5.3.82}$$

with $T_{AB} \equiv L^2 X_A \delta_{AB}$, with $A = 0, \cdots, 7$. The L^2 factor is included in the definition of the T_{AB} fields for dimensional reasons. We close this subsection by writing down the Einstein's equations that follow from the action (5.3.82) which we will use to compute the membrane conductivity. Recalling that we are considering the situation where the gauge fields are taken as probes, that is, that the putative black hole background is not modified by the gauge perturbation, the Einstein's equations that follow from (5.3.82) are:

$$R_{\mu\nu} = \mathcal{T}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{T}, \tag{5.3.83}$$

where the energy momentum tensor $\mathcal{T}_{\mu\nu}$ given by

$$\mathcal{T}_{\mu\nu} = \frac{1}{8}\mathrm{Tr}(\partial_\mu T^{-1}\partial_\nu T + \partial_\nu T^{-1}\partial_\mu T) - g_{\mu\nu}U\,, \tag{5.3.84}$$

with $\mathcal{T} = \mathcal{T}^\mu{}_\mu$, and

$$U \equiv \frac{1}{4}m^2\left(\mathrm{Tr}(T)^2 - 2\mathrm{Tr}(T^2)\right) + \frac{1}{2}\mathrm{Tr}(\partial_\rho T^{-1}\partial^\rho T). \tag{5.3.85}$$

As a final remark we should also give the relation between the parameter m^2 and the AdS scale L. The most straightforward way to understand this relation is to compare the Ricci scalars of the pure AdS_4 black hole and the mass-deformed one. In the pure case, $R = 4\Lambda = -12/L^2$, whereas with the mass deformation turned on, we have $R = -\mathcal{T}$. In the $\mu \to 0$ limit, from (5.3.78) we have that $T_{AB} \to \delta_{AB}$, giving $R \to 48m^2$. Since this has to match with $-12/L^2$, we obtain

$$m^2 = -\frac{1}{4L^2}\,. \tag{5.3.86}$$

5.3.2 Integrating the Einstein's equations to get higher order deformations

Now that we have obtained a first order solution, we need to use the Einstein's equations to find the required second- and higher-order solution near the horizon. Indeed, in the action (5.1.6), we have not just the near-horizon metric, but the Weyl tensor as well, containing second order derivatives of the metric. To specify a solution of the Einstein's equations, which are second order differential equations, we need to specify the metric and its first derivative (in a non-tangential direction) on a codimension-one Cauchy surface. Therefore, if the coordinate away from the surface, i.e. the foliation direction, is called v, and the surface is at $v = 0$, we need to specify the two sets of functions $g_{\mu\nu}(\vec{x}, v = 0)$ and $\partial_v g_{\mu\nu}(\vec{x}, v = 0)$, with $\vec{x}$ coordinates tangent to the Cauchy surface. Then the Einstein's equations determine the full solution by integration, and in particular, we can obtain at least some of the second order derivatives algebraically (without the need to integrate them). We are accustomed with the situation when v is actually time, and we use a time foliation such that the Cauchy surface is at $t = 0$, but in our case we are actually interested in the situation when v is the radial direction away from the horizon, ρ, and the Cauchy surface is at the horizon, $\rho = 0$.

This situation is a bit subtle, since components of the metric are actually divergent or zero at the horizon, hence to check that we are applying it correctly, we first test it on an example when we actually know the full solution, the AdS black hole of (5.1.3). We will keep only the zeroth and first order solution, and deduce the second order, and in particular the Weyl tensor, from the Einstein's equations.

As mentioned before, we already know the full metric (5.1.3), but we will take it in the near horizon limit ($u \to 1$) and its first derivative as the only information given. Note however, that g_{tt} is zero at the horizon, whereas g_{rr} is infinite. So we have non-zero components of the metric that behave as

$$g_{\alpha\beta} = \frac{g^{(-1)}_{\alpha\beta}}{(u-1)} + g^{(0)}_{\alpha\beta} + g^{(1)}_{\alpha\beta}(u-1) + \mathcal{O}(u-1)^2, \tag{5.3.87}$$

The first derivatives with respect to the coordinate u will be denoted with a prime ($g'_{\alpha\beta}$). In this case therefore we see that giving the metric and its first derivative (as functions of the remaining coordinates) on a Cauchy surfaces needs to be replaced by giving the first two coefficients. For g_{rr} this means $g^{(-1)}$ and $g^{(0)}$, whereas for the rest it is $g^{(0)}$ and $g^{(1)}$. We should also comment on a more general case (even though we are not aware of such examples for the Einstein-Hilbert action): if it happens that some component starts at Laurent expansion order $g^{(-p)}$, we would give $g^{(-p)}$ and $g^{(-p+1)}$, whereas if it starts at $g^{(p)}$, we would need to specify at least $g^{(p)}$ ($g^{(p-1)} = 0$ could be a valid specification, depending on the case at hand). In the case of the AdS black hole test solution, we will take these first two coefficients in the components of $g_{\alpha\beta}$ as known, but we will use their explicit expressions just at the end of the calculations.

For the Einstein's equations, and later for the Weyl tensor, the primary ingredient is the Riemann tensor. It is practical to write the Riemann tensor with the second derivatives of the metric explicitly identified, that will make solving of the algebraic equations simpler. The expression for the Riemann tensor that will be used is

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(\partial_{\alpha\delta}g_{\beta\gamma} - \partial_{\alpha\gamma}g_{\beta\delta} + \partial_{\beta\gamma}g_{\alpha\delta} - \partial_{\beta\delta}g_{\alpha\gamma}) + \Gamma^{\epsilon}_{\alpha\delta}\Gamma_{\beta\epsilon\gamma} - \Gamma^{\epsilon}_{\alpha\gamma}\Gamma_{\beta\epsilon\delta}, \tag{5.3.88}$$

where, by definition, the Christoffel symbols contain first derivatives only.

Let us recall the Einstein equations with a cosmological constant and extract one piece of information before the full calculation. They are

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + g_{\alpha\beta}\Lambda = 0. \tag{5.3.89}$$

We can take the trace in the previous equation and obtain the curvature scalar without major effort. In four dimensions it is $R = 4\Lambda$. For the full metric the Ricci scalar is $-\frac{12}{L^2}$, and combining these last two expressions yields $\Lambda = -\frac{3}{L^2}$.

The Ricci tensor is now

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta}, \tag{5.3.90}$$

which contains inside the second derivatives of the metric (so from here we will have our algebraic equations) and can also be used to calculate the Schouten tensor. In four dimensions, the Schouten tensor has the following expression

$$S_{\alpha\beta} = \frac{1}{2}\left(R_{\alpha\beta} - \frac{R}{6}g_{\alpha\beta}\right). \tag{5.3.91}$$

Using the Einstein's equations here will lead to $S_{\alpha\beta} = \frac{\Lambda}{6}g_{\alpha\beta}$. The Schouten tensor is useful in this case to simplify the calculation of the Weyl tensor (it also contains all the matter information, as we will see later), which can be written as

$$C_{\alpha\beta}{}^{\gamma\delta} = R_{\alpha\beta}{}^{\gamma\delta} - 4S^{[\gamma}_{[\alpha}\delta^{\delta]}_{\beta]}. \tag{5.3.92}$$

Let us calculate the Weyl tensor component $C_{uy}{}^{uy}$ that was used to obtain the conductivity in the massless case. Here we have

$$C_{uy}{}^{uy} = R_{uy}{}^{uy} - \frac{\Lambda}{3}. \tag{5.3.93}$$

The Einstein's equations will play its role here as the relation to find $R_{uy}{}^{uy}$ algebraically. We will have to look at the equation for the component R_{yy} of the Ricci tensor,

$$R_{yy} = g^{tt}R_{tyty} + g^{uu}R_{uyuy} + g^{xx}R_{xyxy}. \tag{5.3.94}$$

We go then to equation (5.3.90) for R_{yy}, and rising the y index we obtain

$$R_{uy}{}^{uy} = \Lambda - g^{yy}(g^{tt}R_{tyty} + g^{xx}R_{xyxy}). \tag{5.3.95}$$

Using (5.3.88) and the assumption that we have a static and spherically symmetric black hole solution – which is in fact true for the full solution (5.1.3) –, we see that $R_{uy}{}^{uy}$, and therefore $C_{uy}{}^{uy}$, are determined solely in terms of the metric and its first derivatives.

After adding everything up and replacing the respective values of the metric and its first derivatives, we have the following expression for the sought Weyl tensor component

$$C_{uy}{}^{uy} = -\frac{1}{2L^2} + \mathcal{O}(1-u), \tag{5.3.96}$$

in the near horizon limit. This is in complete agreement with the result one obtains from the full solution (5.1.3) given by

$$C_{uy}{}^{uy} = -\frac{u^3}{2L^2}. \tag{5.3.97}$$

5.3.3 The higher order deformation at the horizon

We now move to apply the procedure defined and tested in the last subsection to our four dimensional mass-deformed metric near the horizon of the black hole.

This case will present more ingredients than the previous one, but the method to obtain the second derivatives is applied in the same way. The massive deformation for the AdS_4 black hole enters in the metric like

$$ds^2 = ds^2_{AdS_4-BH} + \mu^2 ds^2_{MD} \tag{5.3.98}$$

where $ds^2_{AdS_4-BH}$ is the near horizon expansion of the form (5.3.87) of (5.1.3) and the massive deformation is given in (5.3.81).

Now the desired component of the Weyl tensor is C^{uyuy} (see eq.(B.0.14)) which, written in terms of the Riemann and Schouten tensors components is

$$\begin{aligned} C^{uyuy} \;=\;& (g^{yy})^2 \left((g^{uu})^2 R_{uyuy} + 2g^{uu}g^{tu}R_{tyuy} + (g^{tu})^2 R_{tyty}\right) \\ & -g^{yy}\left(g^{uu}(S^u{}_u + S^y{}_y) + g^{tu}S^u{}_t\right). \end{aligned} \tag{5.3.99}$$

Like in the previous case, assuming also that we have spherical symmetry in the full solution (i.e. the metric does not depend on y, nor x) only the R_{uyuy} component involves the second derivatives $\partial_u^2 g_{yy}$; the other components R_{tyuy} and R_{tyty} only contain the metric and its first derivatives.

But there is a caveat: we need to assume that g_{yy} is t-independent in the full solution, not only in the Cauchy solution, if not, we could have unknown second time derivatives of the metric to calculate as well. We believe however that these are reasonable assumptions, though at this moment we cannot rigorously prove them.

The Einstein's equations (5.3.83) now involve an energy-momentum tensor. From those equations we will now obtain the Schouten tensor, that will capture just matter content

$$S_{\alpha\beta} = \frac{1}{2}\left(\mathcal{T}_{\alpha\beta} - \frac{1}{3}\mathcal{T} g_{\alpha\beta}\right). \tag{5.3.100}$$

Let us write again the Ricci tensor component that we need for this case,

$$R_{yy} = g^{tt}R_{tyty} + g^{uu}R_{uyuy} + 2g^{tu}R_{tyuy} + g^{xx}R_{xyxy}. \tag{5.3.101}$$

From here we see that the combination $g^{uu}R_{uyuy} + 2g^{tu}R_{tyuy}$, that also appears in the Weyl tensor (5.3.99), can be written in terms of R_{tyty} and R_{xyxy} which do not involve any second derivatives of the metric; so only these two components need to be computed. Thus, we have managed to write C^{uyuy} completely in terms of the metric and its first derivatives as desired.

Making all the replacements in (5.3.99) the component for the mass deformed Weyl tensor in the near horizon limit reads

$$C^{uyuy} = \left(\frac{3}{2r_0^2L^2} + \mu^2\frac{21}{2r_0^2}\right)(u-1). \tag{5.3.102}$$

As a consistency check we see that, after using (5.1.3) also, the $\mu \to 0$ limit above gives the correct answer for the massless case (5.3.96).

5.4 Conductivity of the mass-deformed system

We finally return to the calculation of the conductivity. There are two (equivalent) ways to calculate it using the membrane paradigm, as we saw in section 5.2: one, using a version of Kubo's formula at the horizon, and the other using formula (5.2.46), obtained by adding a surface term involving the current.

There is one more observation to be made: the conductivity is usually defined with respect to a $U(1)$ (abelian) field, whereas we have an $SO(8)$ (nonabelian) field, so the conductivity will depend on how we embed the abelian field inside $SO(8)$. In principle there are many ways to do this but we saw that, due to our approximation of being near the North Pole of the S^7 of the compactification that generates $SO(8)$, the diagonal gauge field associated with X^0 (A^0, for index $A = 0$) is special, and different than the diagonal gauge fields associated with X^i (A^i, for $A = i$). At the horizon, the X^i scalars "feel" the massive deformation while the X^0 field does not (see equation (5.4.104)).

From section 5.2, it follows that we need only compute the effective coupling $g^2_{ry,4}$ at the horizon (see eq. 5.2.45) with the mass deformation now turned on. In this case,

$$\begin{aligned}
g^{uu} &= \frac{3(1-u)}{L^2} + \mathcal{O}(1-u)^2\,, \quad g^{yy} = \frac{L^2}{r_0^2} + \mathcal{O}(1-u),\\
C^{uyuy} &= \left(\frac{3}{2r_0^2L^2} + \mu^2\frac{21}{2r_0^2}\right)(u-1) + \mathcal{O}(1-u)^2,\\
X_0 &= \frac{1}{L^2} + \mathcal{O}(1-u)\,, \quad X_i = \frac{1}{L^2} + \mu^2 + \mathcal{O}(1-u),
\end{aligned} \tag{5.4.103}$$

near the horizon. Concentrating on the gauge field part of the action (5.3.82), for $T_{AB} = L^2 X_A \delta_{AB}$, the terms involving F_{ry} are

$$-\frac{1}{4}\int d^4x\sqrt{-g}F^A_{ry}F^{A,ry}\Big[L^2X_A - 8\gamma C^{ry}{}_{ry}\Big]. \tag{5.4.104}$$

Therefore using (5.2.46) and the near horizon behavior listed in equations (5.4.103) through (5.4.104), we find that

$$\begin{aligned}\sigma^{(0)} &= \frac{1}{g_4^2}\left(1+4\gamma+28\gamma\mu^2L^2\right) &(5.4.105)\\ \sigma^{(i)} &= \frac{1}{g_4^2}\left(1+\mu^2L^2+4\gamma\left(1+7\mu^2L^2\right)\right), &(5.4.106)\end{aligned}$$

where the (0) and (i) superscripts on the left hand side denote the conductivities associated with the A^0_μ and A^i_μ gauge fields respectively.[6] One potential problem that we alluded to earlier was that now we also have a non-vanishing off-diagonal term in the metric g_{rt}. However, its effects cancel each other out at the horizon and do not contribute in the case of a scalar field. We could go one step further and show also that in the case of the gauge field, the above calculation for the conductivity goes through without modification due to a similar type of cancellations that occur at the horizon. Notice also that since X^0 does not “see” the mass deformation at the horizon, $\sigma^{(0)}$ is blind to the effect of μ^2 as we turn off the Weyl coupling γ.

Finally, we can use the Kubo formula (5.1.18) for the boundary term at the horizon, just as in the $\mu = 0$ case. Here, we have X^{uyty} nonzero in the background, so the action for the A_y field is now

$$S_{yy} = -\frac{1}{2g_4^2}\int d^3x\sqrt{-g}\left(X^{uyuy}A_y\partial_uA_y + X^{uyty}A_y\partial_tA_y\right)\Big|_{u\to1}. \tag{5.4.107}$$

As before, the associated Green's function

$$G^{(A)}_{yy} = -\frac{3r_0}{L^2g_4^2}\left(1+4\gamma+h_A(\mu^2)\right)(1-u)\frac{\partial_uA^A_y(\omega)}{A^A_y(\omega)}\Big|_{u\to1}, \tag{5.4.108}$$

where the index A is used to describe the gauge field whose conductivity we are calculating, leading to

$$h_0(\mu^2) = 28\gamma\mu^2L^2$$

[6]Note that μ corresponds exactly to the mass parameter in the gauge theory, as already noted. Then in the gauge theory, the corrections to the dimensionless conductivity (defined at zero frequency and momentum) must also come in a dimensionless combination of μ and a quantum scale in the nonconformal massive ABJM theory, corresponding to L, just like in the "hard-wall" model for QCD of Polchinski and Strassler, and in other theories with a mass gap that are conformal in the UV, L corresponds to $\Lambda^{-1}_{\mathrm{QCD}}$. Also, g_4 was related to gauge theory parameters in (5.1.30) and γ is a semi-phenomenological parameter from the point of view of gauge theory.

$$h_k(\mu^2) \quad = \quad \mu^2 L^2 + 28\gamma\mu^2 L^2. \tag{5.4.109}$$

The equation of motion for A_y near the horizon is then

$$\left.\frac{\partial_u A_y(\omega)}{A_y(\omega)}\right|_{u\to 1} = i\omega\frac{L^2}{3r_0}\frac{1}{1-u} + i\omega\mathcal{O}((1-u)^0). \tag{5.4.110}$$

Consequently then,

$$\sigma^{(A)} = \frac{1}{g_4^2}\left(1 + 4\gamma + h_A(\mu^2)\right). \tag{5.4.111}$$

We saw then how the mass deformation affects the AC conductivity, on the event horizon limit. Since we do not have the full gravity dual it was not possible to compute the whole conductivity but this leave the question open for further developments on that part.

Chapter 6

Conclusions

In this thesis we have worked in two different problems, both of them related by their respective backgrounds. although, like mentioned in chapter 3 the greater duality is M/ABJM for the M-theory on $\mathrm{AdS}_4 \times S^7$, and string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ is just a case of this bigger theory. Both of them dual to the same ABJM for different values of the coupling.

In chapter 4 we have proposed to apply the physical principle developed in [10] for elimination of ambiguities in the quantum corrections to the energy of two dimensional solitons, to the case of classical (long) strings moving in gravitational backgrounds, taking as a primer the case of the spinning string in $AdS_4 \times \mathbb{CP}^3$. In that case, it was found that there existed a certain regularization dependence, giving rise to different results (e.g [27] and [6]). A procedure was devised in [27] that gave a result consistent with AdS/CFT, but the regularization issue was hidden, without a clear physical principle to explain the choice. As the long history of the quantum corrections to the energy of two dimensional kinks has shown, just because a certain regularization choice seems natural is no guarantee that it is correct, and one needs some physical input to justify it. It is not however obvious that the physical principle from [10], which was natural from the point of view of 2d field theories, is related to the preservation of integrability by quantum corrections in worldsheet theories. We can only mention that the principle proved enough to preserve integrability in the sine-Gordon case (i.e., to obtain the 1-loop corrections compatible with the known exact formula, see [10]), which is a standard example of an integrable system.

It was our goal to justify the choice in [27] by a physical principle which can be applied to other cases of long strings as well. We have found that technical reasons limit how far we can calculate with our method in this case, but we can check that to leading order in u our procedure eliminates the ambiguities, and therefore justifies the choice in [27], leading to the result consistent with AdS/CFT. We hope to apply the same methods to other long strings in the future.

The next chapter went ahead to the $\mathrm{AdS}_4 \times S^7$ background. We analyzed the effect of a massive deformation of the ABJM model on the conductivity calculated from the proposed gravity dual. However, unlike the computation in [41], since we don't have the full A_y defined from the horizon to the boundary, nor even an explicit formula for the gravity dual, we needed to use membrane paradigm-type calculations. In particular we developed a novel calculation based on an extension of the Kubo formula, only using a boundary term at the horizon, together with a corresponding extension of the membrane paradigm method presented in [43].

Crucial to a computation of the conductivity is knowledge of the mass-deformed black hole solution. However, given the complications of the background dual to the mass-deformed ABJM model, such an exact solution remains unknown. Nevertheless, we were able to compute the conductivity knowing only the effect of the mass deformation on the near horizon region of the AdS_4 black hole. In order to determine the mass deformation of the solution near the horizon, we employed a two-step process: First, we input the zeroth and first order solutions as Cauchy data in the Einstein equations, and to find the correct ones that correspond to the mass deformation, superposed a pp-wave on the (approximately flat) horizon and T-dualized. Then, in order to find the higher order solution near the horizon, we developed a method of using Einstein's equations and the Cauchy data. This procedure was benchmarked against the known AdS black hole solution with excellent agreement. Finally, we obtained the same value for the mass-deformed DC conductivity using both membrane paradigm calculations, with both exhibiting an increase as a function of μ^2.

Of course, given the motivation for this work, we would have wanted to obtain $\sigma(\omega)$ at nonzero μ, with the hope that the mass deformation could replace the deformation by the Weyl tensor coupling γ at least in some regime (which should include $\mu \ll T$ so that we still have an approximate conformal field theory at finite temperature). But, due to the technical complications of the background geometry illustrated above, we could only calculate the DC conductivity, $\sigma(0)$. In that respect, all we are able to say is that the mass deformation gives a positive contribution, like a positive γ, but are unable to speculate any further on what happens at nonzero ω. It goes without saying that it would be of enormous interest to extend the calculation to finite ω in the future.

Appendix A

Membrane paradigm in the presence of g_{rt}

In the metric we obtain, we have an off-diagonal component g_{rt}, so in this Appendix we study its effect on the formulas for the membrane paradigm.

A.0.1 Infalling condition

We would like to find the infalling boundary condition for a metric of the form

$$ds^2 = -g_{tt}(r,t)dt^2 + g_{rr}(r,t)dr^2 + 2g_{rt}(r,t)dr\,dt + g_{yy}(r)(dx^2 + dy^2). \tag{A.0.1}$$

Define first the tortoise coordinate r^* such that ds^2 can be written as

$$ds^2 = dvdu + g_{yy}(r)(dx^2 + dy^2) \tag{A.0.2}$$

where $dv = dr^* + dt$, $du = dr^* - dt$. To find dr^* in terms of dr we write

$$dv = dt + dr^* \equiv dt + a\,dr \tag{A.0.3}$$

$$du = dt - dr^* \equiv b\,dt + d\,dr \tag{A.0.4}$$

which gives a quadratic equation for a, namely, $a^2 g_{tt} + 2g_{rt}a - g_{rr} = 0$. Solving for a yields

$$dr^* = a\,dr = \left(-\frac{g_{rt}}{g_{tt}} \pm \sqrt{\frac{g_{rt}^2}{g_{tt}^2} + \frac{g_{rr}}{g_{tt}}}\right) dr. \tag{A.0.5}$$

Consider now a scalar field ϕ near the horizon of a black hole described by the metric (A.0.1). The infalling condition at the horizon means that ϕ can only depend on r and t through the non-singular combination given by the Eddington-Finkelstein coordinate v, defined in (A.0.3). Thus, for fixed v, we have $dt = -adr$, and

$$d\phi(v) = \partial_r\phi(r,t)dr + \partial_t\phi(r,t)dt = (\partial_r\phi(r,t) - a\partial_t\phi(r,t))\,dr = 0 \tag{A.0.6}$$

yielding the relation

$$\partial_r \phi(r,t) = a\, \partial_t \phi(r,t) \tag{A.0.7}$$

but now with a as defined in (A.0.5). Note that in order to recover the usual infalling condition $\partial_r \phi = \sqrt{g_{rr}/g_{tt}}\, \partial_t \phi$ when $g_{rt} = 0$, we need to take the + solution in (A.0.5).

A.0.2 Scalar field calculation

To see the effect of the off-diagonal metric on a membrane paradigm-type calculation, we look at the example of a scalar field.

Consider a massless bulk scalar field with action

$$S = -\frac{1}{2} \int_{r>r_0} d^{d+1}x \sqrt{-g} \frac{1}{g_{d+1}^2(r,t)} \partial_\mu \phi \partial^\mu \phi \tag{A.0.8}$$

where the horizon is at $r = r_0$, and we now have allowed for an r and t dependent scalar coupling $g_{d+1}^2(r,t)$. This is to account for the fact that our background fields can now depend on t as well (see for example equations (5.3.78) and (5.3.81)). Following [48], we need to add a boundary term at the horizon to the one that arises from variations of this action. This term is given by

$$S_{\text{surf}} = \int_\Sigma d^d x \sqrt{-\gamma} \left(\frac{\Pi(r_0, x)}{\sqrt{-\gamma}} \right) \phi(r_0, x) \tag{A.0.9}$$

where $\gamma_{\mu\nu}$ is the metric induced at the stretched horizon Σ, and Π is the momentum conjugate to ϕ with respect to a foliation in the r-direction, i.e.:

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_r \phi)} = -\frac{\sqrt{-g}}{g_{d+1}^2(r,t)} g^{r\mu} \partial_\mu \phi. \tag{A.0.10}$$

The "membrane ϕ-charge" is therefore

$$\Pi_{\text{mb}} \equiv \frac{\Pi(r_0, x)}{\sqrt{-\gamma}} = -\frac{1}{q(r,t)} \left(g_{rr} + \frac{g_{rt}^2}{g_{tt}} \right)^{1/2} \left(g^{rr} \partial_r \phi + g^{rt} \partial_t \phi \right) \tag{A.0.11}$$

where in the last equality we have used the form of the metric (A.0.1). Now we use the infalling condition (A.0.7), obtaining

$$\begin{aligned} \Pi_{\text{mb}} &= -\frac{1}{g_{d+1}^2(r,t)} \sqrt{\frac{\Delta}{g_{tt}}} \left(-\frac{g^{rr} g_{rt}}{g_{tt}} + \frac{g^{rr}}{g_{tt}} \sqrt{\Delta} + g^{rt} \right) \partial_t \phi \\ &= -\frac{1}{g_{d+1}^2(r,t)} \frac{1}{\sqrt{g_{tt}}} \partial_t \phi \end{aligned} \tag{A.0.12}$$

where $\Delta \equiv g_{rr}g_{tt} + g_{rt}^2$. If go to the frame of an observer hovering just outside the horizon with proper time τ, then

$$\Pi_{\rm mb} = -\frac{1}{g_{d+1}^2(r,t)}\partial_\tau \phi. \tag{A.0.13}$$

Therefore, the effect of the off-diagonal metric component g_{rt} gets completely cancelled out and the membrane response $\Pi_{\rm mb}$ is the same as in the case of a diagonal metric (see e.g. [43]).

Appendix B

Conductivity using J^i vs. F_{ti} relation in the presence of g_{rt}

Consider now a generalization of the action in (5.1.8), namely

$$S \;=\; \int d^4x\sqrt{-g}\left[-\frac{1}{8g_4^2}X_{AB}^{\mu\nu\rho\sigma}F_{\mu\nu}^{A}F_{\rho\sigma}^{B}\right] \tag{B.0.1}$$

with

$$X_{AB\,\mu\nu}{}^{\rho\sigma} \;\equiv\; 2\delta_{[\mu\nu]}{}^{[\rho\sigma]}T_{AB} - 8\gamma L^2\delta_{AB}C_{\mu\nu}{}^{\rho\sigma}. \tag{B.0.2}$$

Generalizing the result of section 3, we need to add the surface term

$$S_{\text{surf}} = \int_\Sigma d^3x\sqrt{-\gamma}\left(\frac{j_A^\mu}{\sqrt{-\gamma}}\right)A_\mu^A \tag{B.0.3}$$

with

$$j_A^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_r A_\mu^A)} = -\frac{1}{2g_4^2}\sqrt{-g}X_{AB}^{r\mu\nu\rho}F_{\nu\rho}^{B}. \tag{B.0.4}$$

Thus, the membrane current in this case is

$$J_{\text{mb}}^\mu \equiv \left(\frac{j_A^\mu}{\sqrt{-\gamma}}\right) = -\frac{1}{2g_4^2}\frac{1}{\sqrt{g^{rr}}}X_{AB}^{r\mu\nu\rho}F_{\nu\rho}^{B}\Big|_{r=r_0}. \tag{B.0.5}$$

Using the ansatz (A.0.1), the only non-zero components of the X tensor that enter in the right hand side above for J_{mb}^i are X^{riri} and X^{riti}, yielding

$$J_{\text{mb}}^i \;=\; -\frac{1}{g_4^2}\frac{1}{\sqrt{g^{rr}}}\left(X_{AB}^{riri}F_{ri}^{B} + X_{AB}^{riti}F_{ti}^{B}\right)\Big|_{r=r_0} \tag{B.0.6}$$

(no sum over i). Now, for the A_i component of the gauge fields (we omit the scalar indices A, B for now), the infalling condition (A.0.7) at the horizon is

$$\partial_r A_i = a\,\partial_t A_i\ , \quad r \to r_0 \tag{B.0.7}$$

which, using the $A_r = 0$ gauge and the condition that $A_t = 0$ vanishes at the horizon[1], yields

$$F_{ri} = a\,F_{ti} \quad \text{as } r \to r_0. \tag{B.0.8}$$

This implies that J^i_{mb} is proportional to F_{ti} at the horizon as expected, i.e.

$$J^i_{\text{mb}} \;=\; -\frac{1}{g_4^2}\frac{1}{\sqrt{g^{rr}}}\left(aX^{riri} + X^{riti}\right)F_{ti}\Big|_{r=r_0}. \tag{B.0.9}$$

The factor in parenthesis after some algebra becomes

$$aX^{riri} + X^{riti} = \Delta^{-1/2}g^{ii} - 8\gamma L^2\left(\frac{\Delta^{1/2}}{g_{tt}}C^{riri} - \frac{g_{rt}}{g_{tt}}C^{riri} + C^{riti}\right). \tag{B.0.10}$$

For our ansatz (A.0.1), we have

$$\frac{C^{riti}}{C^{riri}} = \frac{g_{rt}}{g_{tt}}\,, \tag{B.0.11}$$

which implies that the last two terms inside the parentheses in (B.0.10) cancel each other out, giving

$$aX^{riri} + X^{riti} \;=\; \Delta^{-1/2}g^{ii} - 8\gamma L^2\frac{\Delta^{1/2}}{g_{tt}}C^{riri}. \tag{B.0.12}$$

After some more algebra, we arrive at

$$J^i_{\text{mb}} \;=\; -\frac{1}{g_4^2}\frac{\left(g^{rr}g^{ii} - 8\gamma L^2 C^{riri}\right)}{\sqrt{g_{tt}}g^{rr}}g_{ii}F_t{}^i \tag{B.0.13}$$

where we also used the fact that $g_{ij} = g_{ii}\delta_{ij}$ to write $F_{ti} = g_{ii}F_t{}^i$.

In an orthonormal frame of a physical observer just outside the horizon, with proper time τ, we have that $F_t{}^i = -\sqrt{g_{tt}}F^{\tau i} = -\sqrt{g_{tt}}\hat{E}^i$, where $\hat{E}^i$ is the electric field measured by such observer.

Reinserting the scalar indices A, B, and using $T_{AB} = L^2 X_A \delta_{AB}$, we arrive at a conductivity from the membrane paradigm given by

$$\sigma^A_{\text{mb}} = \frac{L^2}{g_4^2}\left(X_A - \frac{8\gamma g_{ii}C^{riri}}{g^{rr}}\right)\Big|_{r\to r_0}. \tag{B.0.14}$$

[1] This is required in order to have a nonsingular gauge connection (see, for example, [49])

Using the formulas at the horizon (5.4.103) to (5.4.104), we obtain

$$\sigma^{(0)}_{\rm mb} = \frac{1}{g_4^2}\left(1+4\gamma+28\gamma\mu^2L^2\right) \tag{B.0.15}$$

$$\sigma^{(k)}_{\rm mb} = \frac{1}{g_4^2}\left(1+4\gamma+\mu^2L^2+28\gamma\mu^2L^2\right) \tag{B.0.16}$$

which are the same we obtained using (5.2.46).

We can readily check that both of the formulas above give the correct result when the mass deformation is absent, namely [41]

$$\sigma^{A}_{\rm mb} \to \frac{1}{g_4^2}(1+4\gamma). \tag{B.0.17}$$

Recall that the relation we just obtained here corresponds to the conductivity defined on the horizon membrane. One way to relate to the conductivity at the boundary is to study the r dependence of the current j^i and electric field F_{ti}, by looking at the equation of motion and Bianchi identities involving $\partial_r j^i$ and $\partial_r F_{ti}$ along the lines of Appendix B in [43]. Namely, in our case we have

$$\begin{aligned}\partial_r j^i &= GX^{itri}\partial_t F_{ri} + GX^{itti}\partial_t F_{ti} + GX^{ijij}\partial_j F_{ij} \\ &\quad +\partial_t(GX^{itri})F_{ri} + \partial_t(GX^{itti})F_{ti} \end{aligned}\tag{B.0.18}$$

$$\partial_r F_{ti} = \partial_i F_{tr} + \partial_t F_{ri} \tag{B.0.19}$$

where $G \equiv \sqrt{-g}/g_4^2$ and $i \neq j$ (i.e., since the boundary is 2+1 dimensional, $i = y$ and $j = x$ or vice-versa). From here we immediately see that (B.0.19) implies that F_{ti} becomes r-independent in the $k_\mu \to 0$ limit, and that the first three terms in (B.0.18) will be suppressed in this limit. However, the last two terms in (B.0.18) will remain finite since they do not involve space-time derivatives of the gauge fields, but only time derivatives of the metric. Since the near horizon metric (5.3.80), (5.3.81), obtained by mass-deforming the flat horizon, does have an explicit time dependence – at least for small t – , we cannot rule out, in principle, the possibility that the full metric in the bulk also is time dependent, and we have a nontrivial $j^i(r)$, but as we explained in the text, this is not necessarily a problem.

Appendix C

Conductivity using the Kubo formula at the horizon

In this Appendix we give details about the calculation using the Kubo formula

$$\sigma(\omega) = -\mathrm{Im}\left(\frac{G_{yy}(\omega, \vec{q})}{\omega}\right) \tag{C.0.1}$$

at the horizon. The boundary term in the action is (we omit the scalar indices for now)

$$S_{yy} = -\frac{1}{2g_4^2}\int d^3x\sqrt{-g}X^{uy\mu y}A_y\partial_\mu A_y\bigg|_{u\to 1} \qquad \text{(sum over } \mu\text{)}. \tag{C.0.2}$$

Note that in the massless case, i.e. for the pure AdS_4 black hole, the integrand above reduces to $\sqrt{-g}X^{uyuy}A_y\partial_u A_y$. In our case however, we have an extra term coming from the fact that X^{uyty} does not vanish in the background (A.0.1). Therefore, S_{yy} is

$$S_{yy} = -\frac{1}{2g_4^2}\int d^3x\sqrt{-g}\left(X^{uyuy}A_y\partial_u A_y + X^{uyty}A_y\partial_t A_y\right)\bigg|_{u\to 1}. \tag{C.0.3}$$

Since we are interested in the conductivity at $\omega = 0$, one might naively think that we can simply ignore this second term above on the grounds that the time derivative will produce a term proportional to ω in the Green's function G_{yy}. However, the first term $\partial_u A_y$, when evaluated on-shell, will also be proportional to ω, and in fact gives a (leading) finite contribution to the conductivity (C.0.1). Therefore, we can not simply ignore $\mathcal{O}(\omega)$ extra terms right from the beginning.

Thus, it seems that we need to go back to the Einstein's equations to obtain the back-reacted Weyl component C^{uyty} at the horizon. However, we don't need to, since, in the general background (A.0.1), we have

$$X^{uyty} = -\frac{g_{ut}}{g_{tt}}X^{uyuy}. \tag{C.0.4}$$

To simplify even more the integrand in (C.0.3), we can also take advantage of the infalling condition at the horizon (valid for $u \to 1$) given in (B.0.7) to write $\partial_t A_y$ in terms of $\partial_u A_y$. Thus, all in all we have

$$S_{yy} = -\frac{1}{2g_4^2} \int d^3x \sqrt{-g} X^{uyuy} A_y \partial_u A_y \left(1 + \frac{u^2}{r_0 a(u)} \frac{g_{ut}}{g_{tt}}\right) \Bigg|_{u \to 1} \tag{C.0.5}$$

where

$$a(u) = \frac{u^2}{r_0} \left(\frac{g_{ut}}{g_{tt}} + \sqrt{\frac{g_{ut}^2}{g_{tt}^2} + \frac{g_{uu}}{g_{tt}}} \right). \tag{C.0.6}$$

The second term inside the parentheses in (C.0.5) was not present in the massless case, when g_{ut} is zero. However, expanding about $u = 1$ gives

$$1 + \frac{u^2}{r_0 a(u)} \frac{g_{ut}}{g_{tt}} = 1 + \mu \, \mathcal{O}(1-u)^{1/2} \tag{C.0.7}$$

which makes it irrelevant for the computation of S_{yy} at the horizon.

The Green's function we need is therefore

$$G_{yy}^{(A)} = -\frac{3r_0}{L^2 g_4^2} \left(1 + 4\gamma + h_A(\mu^2)\right)(1-u) \frac{\partial_u A_y^A(\omega)}{A_y^A(\omega)} \Bigg|_{u \to 1} \tag{C.0.8}$$

where h_A was defined in (5.4.109). The last step is to obtain A_y near the horizon. The equation of motion for A_y is

$$A_y'' + \alpha A_y' + \beta A_y = 0 \tag{C.0.9}$$

where

$$\alpha \equiv \frac{\partial_t \left(\sqrt{-g} X^{tyuy}\right) + \partial_u \left(\sqrt{-g} X^{uyuy}\right) - 2i\omega \sqrt{-g} X^{tyuy}}{\sqrt{-g} X^{uyuy}} \tag{C.0.10}$$

$$\beta \equiv -\frac{q^2 \sqrt{-g} X^{xyxy} + \omega^2 \sqrt{-g} X^{tyty} + i\omega \left(\partial_t + \partial_u\right) \left(\sqrt{-g} X^{tyty}\right)}{\sqrt{-g} X^{uyuy}}. \tag{C.0.11}$$

Inserting the ansatz $A_y = (1-u)^b F(u)$ into the equation of motion we again obtain $b = \pm \frac{iL^2 \omega}{3r_0}$ near $u = 1$ (independent of μ!). Using the + (infalling) value for b again into the equation for A_y near $u = 1$, we get

$$\frac{F'(1)}{F(1)} \simeq i\omega \left[\frac{L^2(3 - 28\gamma)}{6r_0(1+4\gamma)} + \mathcal{O}(\mu^2) \right] + \mathcal{O}(\omega^2) \tag{C.0.12}$$

for $q^2 = 0$ and small ω. Therefore

$$\frac{\partial_u A_y(\omega)}{A_y(\omega)} \Bigg|_{u \to 1} = i\omega \frac{L^2}{3r_0} \frac{1}{1-u} + i\omega \left(\frac{L^2(3-28\gamma)}{6r_0(1+4\gamma)} + \mathcal{O}(\mu^2) \right) + \mathcal{O}(\omega^2). \tag{C.0.13}$$

Putting all this into the Kubo formula (C.0.1) we obtain once again

$$\sigma^{(A)} = \frac{1}{g_4^2}\left(1 + 4\gamma + h_A(\mu^2)\right) \tag{C.0.14}$$

which matches the other results we obtained.

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