Instituto de Física Teórica
Universidade Estadual Paulista

MASTER DISSERTATION

IFT–D.011/2014

Classical Solutions for D-branes in AdS

Segundo Parra Milián

Supervisor

Andrei Mikhailov

July 2014

To my family

Acknowledgements

I would like to acknowledge and extend my heartfelt gratitude to the following persons who have made the completion of this work possible:

to God who made all things possible,

to my family for their support and to being patients during these two years,

to my advisor Prof. Andrei Mikhailov for his vital support, to give me motivation and encourage me how to research,

to Wei He for his advices and to share part of his time answering my questions,

to Luan Vinicius for checking the english redaction and for helping me to write down the abstract in portuguese,

to my friends that I knew here in So Paulo; The Colombian guys Fernando Serna,Jose David, David, Cristian, Emerson, Luis Y. and Rodolfo, the Brazilian guys Fernando O. Paulo, Prieslei, Luan Koerich , etc. And the people from my country: Iván, Jhosep, Daniel, Luis A.,and to everybody that I did not mentioned because of the limited space, thanks a lot for these two great years,

to IFT-UNESP staff for their hospitality, and

to CAPES for financial support.

Abstract

The main result of this dissertation is the study of the classical solutions for D-brane in Anti-de Sitter(AdS) space. They arised from the study of the orbit solution of the stablizer subgroup of a point on the boundary of the AdS space. To reach this goal we started making a brief summary of superstring theories, then we presented some geometric facts of the AdS space. A discussion of the effective low-energy action of a bosonic D-brane was done, then we wrote the action of a single D3-brane in the AdS_5 space.

Unfortunately, the study of this particular D-brane solution is not complete and goes beyond the scope of this dissertation. We intend to complete this work in the future, with this in mind a summary of the marginal deformation of $\mathcal{N} = 4$ SYM field theory and its gravity dual were given in the last chapter.

Keywords: Stabilizer subgroup, orbit solution, D-branes and β-deformation.

Knowledge Area Superstring Theory and AdS/CFT correspondence

Resumo

No presente trabalho comprometemo-nos ao estudo de soluções clássicas para D-branas no espaço de Anti-de Sitter, as quais surgem da solução de órbita do subgrupo estabilizador de um ponto na fronteira no espaço AdS. Começamos por fazer uma breve revisão da teoria de supercordas e apresentamos algumas características da geometria do espaço AdS. Então fazemos uma discussão sobre a ação efetiva a baixas energias e, por último, escrevemos a ação de uma D3-brana no espaço AdS_5.

O estudo desta solução particular da D-brana não é completo e acaba por fugir ao escopo deste trabalho. Esperamos complementá-lo num futuro breve e, com isto em mente, concluímos a dissertação com um resumo de deformações marginais da teoria de campo $\mathcal{N} = 4$ SYM e sua gravidade dual.

Palavras chave: Subgrupo estabilizador, solução órbita, D-branas e deformação β.

Áreas do conhecimento: Teoria de supercordas e correspondência AdS/CFT.

Contents

Chapter 1

Introduction

Since the Anti-deSitter/Conformal Field Theory(AdS/CFT) correspondence [1], has come to light, much progress has been done towards its comprehension. For a review see [12]. What this correspondence states is the equivalence between gauge theory and gravity. In its original form, it states that $\mathcal{N} = 4$ $d = 4$ $SU(N)$ Super Yang-Mills (SYM) field theory is equivalent to type IIB string theory in $AdS_5 \times S^5$ in the $N \to \infty$ limit.

One of the features of the correspondence is to relate expetation values in string theory to coupling constants in the field theory, e.g. $g^2 \propto e^{\langle \Phi \rangle}$, where Φ is the dilaton field and g is the gauge coupling. If we change the gauge coupling constant, it will change the expectation value of the dilaton field. This leads us to imagine that the simplest extension of the original proposal arises when we consider supersymmetry preserving deformations of the $\mathcal{N} = 4$ SYM field theory. Another characteristic of the correspondence is that the mass m of a scalar supergravity field φ, which lives in the AdS subspace and couples to operators $\mathcal{O}$ who live on the boundary of AdS, with dimension Δ, is related by $m^2 = \Delta(\Delta - d)$, where d is the dimensionality of the space-time which the field theory lives in, where we see that marginal operators correspond to massles fields on the supergravity side.

These two characteristics lead us to believe, that we can deform the $\mathcal{N} = 4$ SYM field theory. In fact the systematic study for the case of exactly marginal deformations of this field theory was done by Leigh and Strassler [6] and its gravity dual had remained unknown until the Lunin and Maldacena's work [7], come to light.

Moreover, on the gravity side, we get $\mathcal{N} = 4$ SYM with $SU(N)$ gaug group by putting N D3-branes in type IIB string together. In this picture the vibrational modes of the brane become scalars. It might be thoguht that a fundamental open string ends on one of the N branes. This leads to $SU(N)$ gauge group and puts the scalars in the adjoint representation. As important as the fundamental string is to the gravitational side, so are the D-brane configurations. It is rather well known that on the gravity side many particular D-brane configurations exist. We will not list them.

Nevertheless, in this dissertation we study in detail a particular D3-brane solution obtained as an orbit of the superconformal symmetry group.

1.1 Plan of this work

With the facts presented above, this work was structured as follows

- In Chapter 2 we make a brief review of superstrings theories. We start with the Polyakov action, then we present its generalization in any curved space-time. Then, we list different formalisms in superstring theory, and briefly discuss the fermionic boundary conditions. We latter focus on the type IIB supergravity because of its relation with the AdS/CFT correspondence, whose whose characteristics used in the text we review.

- In chapter 3 we review some geometric facts about anti-de Sitter space.

- In chapter 4 we review some aspects of the bosonic D-brane. Firstly we see how D-branes appear in string theory, then we write down the low-energy effective action of the bosonic D-brane and. Finally as exercise we write down the explicit form of a bosonic D-brane in the $AdS_5 \times S^5$ background.

- In chapter 5 we present classical solutions for D-branes by studying the orbit of the stabilizer subgroup of a point on the boundary of the AdS_n space. Firstly, we start studying very well the stabilizer subgroup in the AdS_2 space. What we found is that its orbit solution is the trajectory of a charged particle in AdS_2 within a constant field strength. Finally, we argue that in the AdS_5 space we get classical solutions for D3-branes.

- In chapter 6 we explain what marginal deformation of $\mathcal{N} = 4$ SYM theory is, which was done by the first time by Leigh-Strassler in [6]. Actually [6], the latter discussed many models with fixed lines from the point of view of $\mathcal{N} = 1$

SYM. Here we have focused in the $\mathcal{N}=4$ SYM with $SU(N)$ gauge group, because of its importance with AdS/CFT correspondence. In the last section we present the gravity dual for the so-called β-deformation of the $\mathcal{N}=4$ SYM theory. This was done by Lunin-Maldacena in [7], also we present the Frolov's argument, [27].

We have left for the appendices A and B details that are important to the text but due to the limited space, were not cover in the chapter 6. We must emphasize that in this work we did not attempt to review supersymmetry, which we believe it is very well explained in [34].

Chapter 2

Brief summary of superstrings

In this chapter we make a brief review of superstring theory, we start with the Polyakov action, then we present its generalization in any curved spacetime. Then, we list different formalisms in superstring theory, and briefly discuss the fermionic boundary conditions, then we focus on the type IIB supergravity because of its relation with the AdS/CFT correspondence, which we summarize the facts that we will use in the following chapters

2.1 The type IIB Supergravity theory

The action of the bosonic string in d-dimensional Minkowski spacetime, which is suitable for quantization is the so called Polyakov action, see [8]

$$S_P = -\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma(-\gamma)^{1/2}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu, \tag{2.1}$$

where γ_{ab} is two-dimensional metric, with Lorentzian signature $(-,+)$ and $X^\mu = X^\mu(\sigma,\tau)$ are the spacetime coordinates which give the embedding of the two-dimensional surface(worldsheet) into the Minkowski spacetime. The constant α' is called the Regge slope, which has units of length squared, is related to the string tension T by $T = \frac{1}{2\pi\alpha'}$.

Roughly speaking, the quantization of the bosonic string is achieved by expanding in normal modes the solution of the classical equations of motion, then the coefficients are interpreted as creation and annihilation operators, so we have

- ⋆ For open bosonic string this gives rise to a spectrum made of a tachyon with mass $m^2 = -\frac{1}{\alpha'}$ and a massless vector plus an infinite tower of massive states,

- ⋆ For closed bosonic string one obtains, beyond the tachyon and a tower of massive states; we have a rank two tensor, which decomposes into a symmetric traceless tensor, an antisymmetric tensor and a scalar.

Requiring the absence of conformal anomaly in the quantization of both open and closed bosonic string theories, one finds that it requires a 26-dimensional spacetime to live and the presence of tachyons in their spectrum, which are troubles because the theory is unstable, see [8] for more details.

Nevertheless, it is important to consider the string action in any curved spacetime, the Polyakov action can be thought of as describing a coherent state of gravitons. This suggests a natural generalization: include backgrounds of the other massless string states as well, which are obtained in the quantization of closed bosonic string, the massless states include the metric $G_{\mu\nu}$, the dilaton Φ and the antisymmetric tensor $B_{\mu\nu}$. The action of the bosonic string in this background fields is

$$S = -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\sqrt{-g}\big[\big(g^{ab}G_{\mu\nu}(X) - i\epsilon^{ab}B_{\mu\nu}(X)\big)\partial_a X^\mu \partial_b X^\nu - \alpha' R\Phi(X)\big], \quad (2.2)$$

Moreover there are details on bosonic string theory that we skipped, like showing the conformal symmetry of the string worldsheet theory, etc. This details can be found in [8], which we suggest to reader.

An important extension is to include fermions in the theory, this leads to superstring theories, in which worldsheet supersymmetry is obtained by supplementing the bosonic fields $X^\mu(\tau,\sigma)$ with their superpartners, $\Psi^\mu(\tau,\sigma)$, they are two dimensional Majorana spinors. Superstring theories are consistently formulated in ten spacetime dimensions, in order to incorporate supersymmetry into string theory three basic approaches have been developed supersymmetry into string theory three basic approaches have been developed,

- ⋆ The Green-Schwarz(GS) formalism is supersymmetric in ten-dimensional Minkowski spacetime. It can be generalized to other background spacetime geometries,

- ⋆ The Ramond-Neveu-Schwarz(RNS) formalism is supersymmetric on the string worldsheet,

- ⋆ Pure spinors formalism is a new formalism, in which the superstring can be quantized in a manifestly super-Poincaré covariant manner, it has been introduced by Nathan Berkovits in [9].

When we consider the supersymmetric generalization of (2.1) and exploiting world-sheet superconformal symmetry one can write down the action in the form

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \big[\partial_+ X^\mu \partial_- X_\mu - i\psi_L \partial_+ \psi_L - i\psi_R \partial_- \psi_R\big], \tag{2.3}$$

where $+$ and $-$ indices refer to light-cone coordinates, $\sigma_\pm = \tau \pm \sigma$.

The equations of motion coming from (2.3) lead to an expansion of X^μ and ψ^μ in terms of left- and right- moving oscillation modes, the solutions of the equations of motion can be written schematically in the form

$$\begin{aligned} \partial_+ X^\mu &= \sum_n \alpha_n^\mu e^{-in\sigma_+}, \\ \psi_+^\mu &= \sum_n \psi_n^\mu e^{-in\sigma_+}, \end{aligned} \tag{2.4}$$

and analogously for the right-moving part. In the sum (2.4), for the X^μ n runs over integer values, while for the fermions n takes integer values in the Ramond(R) sector and half-integer values in the Neveu-Schwarz(NS) sector. Let us discuss briefly what are R and NS sectors, they rise when we impose consistent boundary conditions in the fermions $\psi_\pm$, taking the coordinate σ in the range $[0, \pi]$ for open strings and to be periodic for closed strings. For open strings one finds

- ⋆ Ramond boundary condition corresponds to choose the boundary condition, $\psi_+^\mu(l) = \psi_-^\mu(l)$, with $l = 0, \pi$ and it gives rise to spacetime fermions,

- ⋆ Neveu-Schwarz boundary condition corresponds to choose a relative sign in the boundary condition in one end, i.e. $\psi_+^\mu(\pi) = -\psi_-^\mu(\pi)$ and the another end to be $\psi_+^\mu(0) = \psi_-^\mu(0)$,and it gives rise to spacetime bosons.

While for closed strings, its boundary conditions give two sets of fermionic modes, corresponding to the left- and right- moving sectors. There are two possible periodicity conditions i.e. $\psi_\pm(\sigma) = \pm\psi_\pm(\sigma + 2\pi)$, the positive sign in the above relation describes periodic boundary conditions while the negative sign describes antiperiodic boundary conditions. It is possible to impose the periodicity(R) or antiperiodicity(NS) of the right- and left- movers separately, after that one finds four distint closed-string sectors. States in $NS-NS$ and $R-R$ sectors are spacetime bosons, while states in the $NS-R$ and $R-NS$ sectors are spacetime fermions.

It was realized that NS ground state has tachyonic state, in order to get rid of the tachyons we impose the Gliozzi-Scherk-Olive(GSO) projection this leads to space-time supersymmetric string theories, containing no tachyonic state, [10, 11].

After applying GSO projection in type II superstring, one finds two different theories with spacetime supersymmetric and tachyonic-free spectrum. These theories are type IIA and IIB superstring theory and their masssless spectra are, respectively, that of type IIA and type IIB supergravity in ten dimensions. Their spectrum share the $NS-NS$ sector, but differ by the content of their fermionic sector and $R-R$ sector. In what follows, we will focus on the type IIB superstring because it is directly related to the AdS/CFT duality.

The type IIB superstring massless spectrum includes the degrees of freedom of

- ⋆ Spacetime bosons:
  - NS-NS sector: the metric $G_{\mu\nu}$ associated to the graviton, the dilaton Φ and the 2-form $B_{\mu\nu}$.
  - R-R sector: a real scalar χ (or sometimes denoted $C_{(0)}$) known as the axion, a 2-form $C_{(2)\mu\nu}$ and a 4-form $C_{(4)\mu\nu\rho\lambda}$.
- ⋆ Spacetime fermions: they are the mixed sectors ($R\otimes NS$ and $NS\otimes R$), which are the fermionic superpartners of the previous fields; two Majorana-Weyl spinors known as gravitinos ($\phi_{1,\mu}, \phi_{2,\mu}$) and two spinors known as dilatinos (λ_1, λ_2), for more details see [10].

The low energy effective description of type IIB superstring theory, constructed in terms of the above fields, is type IIB supergravity(SUGRA) in ten dimensions, [10]. The self-dual field strength $F_5 = *F_5$ condition cannot be expressed in terms of covariant action, this means that it cannot be imposed by a lagrangian without introducing extra fields,[10, 29]. But the following comes close, that the action of the bosonic sector of the ten-dimensional type IIB SUGRA in the string metric splits into two parts[10],

$$S = S_{NS} + S_R. \tag{2.5}$$

The first term in (2.5), is the action for the (NS-NS) sector:

$$S_{NS} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\{R + 4(\nabla\Phi)^2 - \frac{1}{2}|H_3|^2\}, \tag{2.6}$$

where x^μ $(\mu = 0, 1, \ldots, 9)$ are 10-dimensional coordinates, we use the notation $G \equiv \det G_{\mu\nu}$. The $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ fields denote the 10-dimensional metric, NS-NS 2-form and dilaton, respectively. The NS-NS field strength is written as $H_3 = dB_2$ where $B_2 = \frac{1}{2!}B_{\mu\nu}dx^\mu \wedge dx^\nu$. With the subscript of a form we mean the rank of the tensor.

The second term in (2.5), is the action for the Ramond-Ramond(R-R) sector, S_R, can be written as

$$S_R = -\frac{1}{4\kappa_{10}^2}\int \left[d^{10}x\sqrt{-G}(|F_1|^2 + |F_3|^2 + \frac{1}{2}|F_5|^2) + C_4 \wedge H_3 \wedge F_3\right], \tag{2.7}$$

to write down (2.7), we used the following notation

$$\int \sqrt{-G}|F_p|^2 \equiv \frac{1}{p!}\int \sqrt{-G}G^{\mu_1\nu_1}\cdots G^{\mu_p\nu_p}F_{\mu_1\cdots\mu_p}F_{\nu_1\cdots\nu_p},$$

where the R-R field strengths F_{p+2} are defined from the (p+1)-form R-R potentials

$$C_{p+1} = \frac{1}{(p+1)!}C_{\mu_1\cdots\mu_{p+1}}dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}, \tag{2.8}$$

as

$$\begin{aligned} F_1 &= dC_0, \qquad F_2 = dC_1, \\ F_3 &= dC_2 + H_3 \wedge C_0, \qquad F_4 = dC_3 + H_3 \wedge C_1, \\ F_5 &= dC_4 + \frac{1}{2}(-C_2 \wedge H_3 + B_2 \wedge dC_2). \end{aligned} \tag{2.9}$$

The self-duality of F_5 is imposed as a constraint condition, this formalism is good enough for the study of classical theory, although not satisfactory for the study of quantum theory.The Type IIB SUGRA is invariant under the symmetry group $Sl(2, \boldsymbol{R})$, but this symmetry is not manifest in (2.6) and (2.7). To make the symmetry manifest, we must refine fields from the string metric $G_{\mu\nu}$ used in (2.6) to the Einstein metric[8]. For more details in type IIB SUGRA we suggest[10, 11].

2.2 The AdS/CFT correspondence

The Anti-de Sitter/Conformal Field Theory(AdS/CFT) correspondence states the equivalence (also referred to as duality) between the following theories [1, 12]

⋆ Type IIB superstring theory on $AdS_5 \times S^5$ space where both AdS_5 and S^5 subspaces have the same radius R, where the 5-form F_5 has integer flux

$$N = \int_{S^5} F_5, \tag{2.10}$$

where Eq. (2.10) is due to the existence of D3-branes, see Chapter 4, the flux of F_5 is quantized and where the string coupling is g_s;

- ⋆ $\mathcal{N} = 4$ SYM in 4-dimensions is a theory with four supersymmetries, namely 16 real supercharges. The field content is as follows: one vector field A_μ , six scalars ϕ^I with $I = 1, \dots, 6$ and four fermions, in section 6.2.1 we will discuss carefully this field theory. The lagrangian density is schematically of the form

$$\mathcal{L} = \frac{1}{g^2} \operatorname{Tr}\left[F^2 + (D\phi)^2 + \overline{\chi}\not{D}\chi + \sum_{IJ}[\phi^I\phi^J]^2 + \overline{\chi}\Gamma^I\phi^I\chi\right] + \theta \operatorname{Tr}[F \wedge F], \tag{2.11}$$

it contains two parameters, the Yang-Mills coupling g_{YM} and the theta angle,

this means that string theory in an asymptotically AdS space-time can be described exactly as a CFT ($\mathcal{N} = 4$ SYM) on the boundary of this space-time, with the following identifications between the parameters of both theories,

$$g^2_{YM} \propto g_s = e^{\langle\Phi\rangle},\ R^4 = 4\pi g_s N \alpha'^2. \tag{2.12}$$

In brief, equivalence includes a precise map between the states (and fields) on the superstring side and the local gauge invariant operators on $\mathcal{N} = 4$ SYM side, as well as a correspondence between the correlators in both theories.

In the strongest form of the correspondence, it is conjectured that it holds for all values of N and all regimes of coupling. Certain limits of the correspondence are, however, also highly non-trivial . The 't Hooft limit on the SYM-side, in which $\lambda \equiv g^2_{YM}N$ is fixed and $\lambda \gg \alpha'^2$ as $N \to \infty$ corresponds to classical string theory on $AdS_5 \times S^5$(no strings loops) on the AdS-side. A further limit $\lambda \to \infty$ reduces classical string theory to classical Type IIB SUGRA on $AdS_5 \times S^5$. Thus, strong coupling dynamics in SYM theory(at least in the large N limit) is mapped to classical low energy dynamics in SUGRA and string theory.

Chapter 3

Geometry of Anti-de Sitter space

In this chapter we review some geometric facts about anti-de Sitter space.

3.1 Anti-de Sitter space as a solution of Einstein equations

When one tries to look for simplest vacuum solutions of Einstein's equations with cosmological constant,

$$G_{AB} + \Lambda g_{AB} = 0 \tag{3.1}$$

which are spacetimes of constant curvature. They are locally characterized by the condition

$$R_{ABCD} = \frac{R}{(d-1)d}(g_{AC}g_{BD} - g_{AD}g_{BC}) \tag{3.2}$$

where d is the spacetime dimension. Using (3.1) we see that,

$$G_{AB} = -Rg_{AB}\frac{d-2}{d} = -\Lambda g_{AB} \tag{3.3}$$

i.e. $R = \frac{2d}{d-2}\Lambda$ are the constant curvature solutions of the Einstein equations with cosmological constant. In particular for $\Lambda = 0$ we have flat Minkowski spacetime, for $\Lambda > 0$ positively curved, de-Sitter spacetime and for $\Lambda < 0$ anti-de-Sitter spacetime. The above three spacetimes are of maximal symmetric.

3.2 Geometry of AdS_n space

The n-dimensional anti-de Sitter space(AdS_n) can be represented as hyperboloid

$$X_{-1}^2 + X_0^2 - X_1^2 - \cdots - X_{n-1}^2 = 1, \tag{3.4}$$

in $\mathbb{R}^{2,n-1}$ with metric given by

$$ds^2 = dX_{-1}^2 + dX_0^2 - \sum_{i=1}^{n-1} dX_i^2, \tag{3.5}$$

its isometric group is $SO(2, n-1)$, we can parametrize the hyperboloid by ρ, T and $\boldsymbol{n}$ coordinates by setting

$$X_{-1} + iX_0 = e^{iT}\cosh\rho,\ \ X_a = \boldsymbol{n}_a \sinh\rho, (a = 1, \cdots, n-1). \tag{3.6}$$

Substituting this into (3.4), we obtain the metric on AdS_n

$$ds^2 = \cosh^2\rho dT^2 - d\rho^2 - \sinh^2\rho d\boldsymbol{n}_a^2. \tag{3.7}$$

By taking $\rho \geq 0$ and $0 \leq T \leq 2\pi$ the parametrization (3.4) covers the entire hyperboloid once. Therefore $(T, \rho, \boldsymbol{n}_a)$ are called global coordinates of AdS_n.

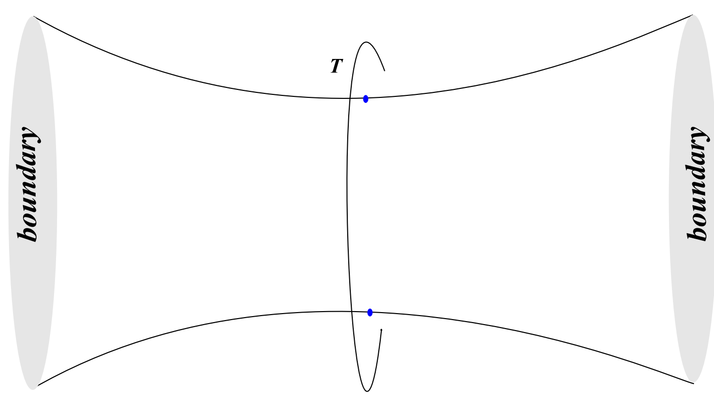

Figure 3.1: AdS_n space is realized as a hyperboloid in $\mathbb{R}^{2,n-1}$. The hyperboloid has closed timelike curves along the T direction. The blue point means $\rho = 0$ and the boundary is at $\rho = \infty$.

Since the metric (3.7) behaves near $\rho = 0$ as $ds^2 \simeq dT^2 - d\rho^2 - \sinh^2\rho\boldsymbol{n}_a^2$, the hyperboloid has the topology of $S^1 \times \mathbb{R}^{n-1}$, with S^1 representing closed timelike curves in the T direction. Fortunately we can go to the covering space and consider with T non-compact. When we talk about AdS we are always going to think about this covering space.

The boundary of the AdS_n space is the set of light-rays in $\mathbb{R}^{2,n-1}$, in this coordinate system it is at $\rho = \infty$, which is the product of S^1 parametrized by T and S^{n-2} parametrized by $\boldsymbol{n}$. i.e. $\partial AdS_n = S^1 \times S^{n-2}$.

In addition to the global parametrization (3.6) of AdS, there is another set of coordinates $(h, t, \overrightarrow{x})$ with $(h \geq 0, \overrightarrow{x} \in \mathbb{R}^{n-2})$. Let us made the following transformation

$$\begin{aligned} X_{-1} \pm X_{n-1} &= X_{\pm}, \quad h = \frac{1}{X_+} \\ x_0 &= \frac{X_0}{X_+}, \quad x_i = \frac{X_i}{X_+} \quad (i = 1, \cdots, n-2). \end{aligned} \tag{3.8}$$

the equation of the hyperboloid become $X_+X_- - X_\mu X^\mu = 1$, where μ runs over 0 to $n-2$, after some computations, we get

$$v = (X_{-1}, X_0, X_1, \cdots, X_{n-2}, X_{n-1}) = [\frac{1 - x^\mu x_\mu + h^2}{2h}, \frac{x_\mu}{h}, \frac{1 + x^\mu x_\mu - h^2}{2h}] \tag{3.9}$$

where $x^\mu x_\mu = (x^0)^2 - (x^i)^2$. These coordinates cover half of the hyperboloid (3.4). Substituting this into (3.5), we obtain another form of the AdS_n space metric

$$ds^2 = \frac{1}{h^2}\{dx^\mu dx_\mu - dh^2\}. \tag{3.10}$$

The coordinates $(h, t, \overrightarrow{x})$ are called the Poincaré coordinates. The boundary of AdS_n is at $h = 0$ and can be represented as

$$l(\tau) = \{\frac{1}{2}(1 - x^\mu x_\mu), x^\mu(\tau), \frac{1}{2}(1 + x^\mu x_\mu)\}. \tag{3.11}$$

where $l(\tau)$ is a one-parameter family of the light-like vectors of the hyperboloid. In this work we are interested in the AdS_5 space.

Chapter 4

D-branes

In this chapter we review some aspects of the bosonic D-brane, firstly we see how D-branes appear in string theory, then we write down the low-energy effective action of the bosonic D-brane and finally as exercise we write down the explicit form of a bosonic D-brane in the $AdS_5 \times S^5$ background.

4.1 The necessity of existence of D-branes

As we mentioned in Chapter 2 the bosonic string theory is described by the Polyakov string action, and it requires the string moves on 26 dimensions, see [8]. Closed strings are free to move arbitrarily through space, sometimes called *bulk*. Open strings have boundary conditions defined on the endpoints. By varying the Polyakov string action, we get an extra boundary term for an open string,

$$\delta S_{P,boundary} \propto \int d\tau \sqrt{-\gamma}\delta X^{\mu}\partial^{\sigma}X_{\mu}\Big|_{\sigma=0}^{\sigma=l}, \tag{4.1}$$

the boundary term vanishes if we have the following two conditions

- ⋆ Neumann boundary conditions:

$$\partial^{\sigma}X^{\mu}(\tau,0) = \partial^{\sigma}X^{\mu}(\tau,l) = 0, \tag{4.2}$$

It implies that the endpoints must move at the speed of light. Stated more covariantly,

$$n^a\partial_a X_{\mu} = 0, \tag{4.3}$$

where n^a is normal to the boundary of string worldsheet ∂M;

⋆ Dirichlet boundary condition:

$$\delta X^{\mu}(\tau, 0) = \delta X^{\mu}(\tau, l) = 0, \tag{4.4}$$

thus X^{μ} = constant at $\sigma = 0$ and l. In this case the endpoints of the string are fixed in space.

Nevertheless, we can choose $p+1$ Neumann boundary conditions for p spatial dimensions and time, and $d-p-1$ Dirichlet boundary conditions, where the letter d means the dimensionality of the spacetime; $d = 26$ in Bosonic string theory and $d = 10$ in superstring theory. This means that the endpoints of the string are constrained to live on a $p+1$-dimensional hyperplane in space-time. But different string endpoints could be on a different hyperplane [8], see Fig 4.1.

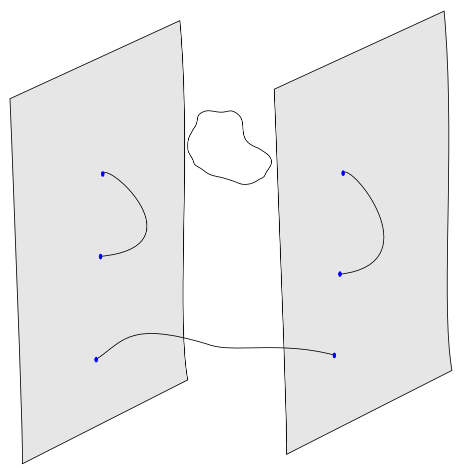

Figure 4.1: Two hyperplanes with open strings ending on them and one closed string moving in the bulk. Fig. extracted from [13].

In fact this hyperplane is not a rigid one, instead is a dynamical object, i.e. it can fluctuate and respond to external interactions, and that it has degrees of freedom living on it. The hyperplane was then called a Dp-brane, where the letters D stands for Dirichlet and p stands the number of spatial dimensions of the D-brane. For example, a D0-brane is a point particle.

4.1.1 About T-duality

Nevertheless, one way of motivating the necessity of D-branes is based on T-duality, which relates spacetime geometries possesing a compact isometry group. We end this section summarizing some facts about T-duality, and we point out to the reader [8, 13] for more details.

Under T-duality transformations, closed bosonic strings transform into closed strings of the same type in the T-dual geometry. In order to realise this fact consider the simplest example, namely the bosonic string with one of the 25 spatial direction forming a circle of radius R, i.e. the spacetime geometry is chosen to be $(\mathbb{R}^{24,1} \times S^1)$, sometimes one describes this as compactification on a circle of radius R. The momentum p^{25} takes the discrete values n/R, for $n \in \mathbb{Z}$. Under $\sigma \sim \sigma + 2\pi$, a closed string may now wind around the compact dimension [8]

$$X^{25}(\sigma + 2\pi) = X^{25} + 2\pi R w, \qquad\qquad w \in \mathbb{Z},$$

the integer w is the winding number. The mass spectrum in the remaining uncompactified $24 + 1$ dimensions is

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \widetilde{N} - 2), \tag{4.5}$$

where N,$\widetilde{N}$ denote the total levels on left- and right-moving sides, as before. Notice that the spectrum (4.5) is invariant under the exchange

$$n \longleftrightarrow w \qquad \text{and} \qquad R \longleftrightarrow R' \equiv \alpha'/R, \tag{4.6}$$

the string theory compactified on a circle of radius R' (with momenta and windings exchanged) is the 'T-dual' theory, and the process of going from one theory to the other will be referred to as 'T-dualising', i.e. the theory on R and the theory on $\frac{\alpha'}{R}$ are completely equivalent. The 26-dimensional coupling is related to the 25-dimensional one by $\kappa = (2\pi R)^{1/2}\kappa_{25}$, since the resulting 25-dimensional theory is supossed to have the same physics by T-duality, so the string coupling of the dual 25-dimensional theory is $\widetilde{\kappa_{25}} = \kappa_{25}\sqrt{\alpha'}/R$. Then after T-duality we get $X^{25} = X_L^{25} + X_R^{25} \mapsto X_L^{25} - X_R^{25}$.

The situation is different for open strings, however. The key is to focus on the type of boundary conditions imposed at the ends of the open strings. In order to find the T-dual of a open string with Neumann boundary conditions, let us consider that such string is moving in a 26-dimensional Minkowski spacetime, after solving its equation of motion, one finds that the solution can be written as, $X = X_L + X_R$. Let us imagine that we place X^{25} on a circle of radius R, the T-dual coordinate $\widetilde{X}^{25}$

has no dependence on τ in the zero mode sector, so we have no momentum and we see that the endpoints do not move in the $\widetilde{X}^{25}$ direction

$$\widetilde{X}^{25}(\pi) - \widetilde{X}^{25}(0) = \frac{2\pi\alpha' n}{R} = 2\pi n R'. \tag{4.7}$$

This is consistent with the fact that under T-duality normal and tangential derivatives get exchanged

$$\partial_n X^{25} = \partial_t \widetilde{X}^{25}. \tag{4.8}$$

We emphasize that T-duality have been applied to X^{25} coordinate, so the endpoints are to move in the another 24 spatial dimensions, which constitutes a hyperplane called a D-brane.

4.2 Action of the D-brane

D-branes exist in bosonic string and superstring theories, in the former one they are unstable. Indeed, as in the bosonic theory, adding D-branes to the type IIB vacuum configuration gives a theory that has closed strings in the bulk plus open strings that end on the D-branes

The low-energy effective action for the massless bosonic sector of D-brane of type II string theories is the sum of Dirac-Born-Infeld and Wess-Zumino terms

$$I = I_{DBI} + I_{WZ}. \tag{4.9}$$

The coupling of a D-brane to NS-NS closed string fields is the same DBI action as in the bosonic string. So let us introduce coordinates ξ^a, $a = 0, \ldots, p$ on the brane. The fields on the brane are the embedding $X^\mu(\xi)$ and the gauge field $A_a(\xi)$, then the DBI action is

$$I_{DBI} = -T_p \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \tag{4.10}$$

where G_{ab} and B_{ab} are the pullbacks of the spacetime fields to the brane, i.e.

$$\begin{aligned} G_{ab}(\xi) &= \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}(X(\xi)), \\ B_{ab}(\xi) &= \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}(X(\xi)), \end{aligned} \tag{4.11}$$

where T_p is the p-volume tension with mass dimension $p+1$. The Dp-brane action can be verified by a perturbative string calculation, see [14], which also gives a precise expression for the brane tension

$$T_p = g_s \tau_p = \frac{1}{\sqrt{\alpha'}} \frac{1}{(2\pi\sqrt{\alpha'})^p}. \tag{4.12}$$

The G_{ab} and B_{ab} are the components of the spacetime NS-NS fields parallel to the brane, i.e. G_{ab} and B_{ab} are the induced metric and antisymmetric tensor on the brane, and F_{ab} is the field strength of the world-volume $U(1)$ gauge field living on the brane.

All features of the action (4.10) can be understood from general reasoning. Considering first just the spacetime metric and the embedding, the simplest and lowest-derivative coordinate-invariant action is the integral

$$-T_p \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det(G_{ab})}, \tag{4.13}$$

it is the world-volume. Noting that this term has an implicit dependence on $X^\mu(\xi)$. Expanded around a flat D-brane it gives the action for the fluctuations

$$-T_p \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det(\eta_{ab} + \partial_a X^i \partial_b X^i)}, \tag{4.14}$$

where i runs over $p+1, \cdots, d-1$. The dilaton dependence $e^{-\Phi} = g_s^{-1}$ arises because this is an open string tree level action, and so this is the apropiate form of the dilaton, see [8] for more details.

The dependence on F_{ab} can be understood as follows, let us consider a D-brane extended in the X^1- and X^2- directions, with the other direction unspecified, and let there be a constant gauge field F_{12} on it. We can choose a gauge in which $A_2 = X^1 F_{12}$. Now consider T-duality along the X^2- direction. The Neumann condition in this direction is replaced by a Dirichlet condition . so the D-brane loses a dimension, However, the T-duality relation between the potential and coordinate implies that the D-brane is tilted in the $(1-2)-$ plane,

$$X'^2 = -2\pi\alpha' X^1 F_{12}, \tag{4.15}$$

the tilted angle is $\theta = \tan^{-1}(2\pi\alpha' F_{12})$. This gives a geometric factor in the D-brane world-volume action

$$S \sim \int_D ds = \int dX^1 \sqrt{1 - (\partial_1 X'^2)^2} = \int dX^1 \sqrt{1 - (2\pi\alpha' F_{12})^2}. \tag{4.16}$$

We can always boost the D-brane to be aligned with the coordinate axes and then rotate to bring $F_{\mu\nu}$ to block-diagonal form, and in this way we can reduce the problem to a product of factors like (4.16) giving a determinant

$$S \sim \int d^D X \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab})}. \tag{4.17}$$

This determinant form for the gauge field action is known as the Born-Infeld action. It was originally proposed , unsuccesfully, as a solution to the short-distance divergences of quantum electrodynamics.

The dependence on B_{ab} in (4.10) can be understood by the following argument. The closed string field $B_{\mu\nu}$ and the open string field A^μ appear in the string-world-sheet as

$$\frac{i}{4\pi\alpha'} \int_M d^2\sigma g^{1/2} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} + i \int_{\partial M} dX^\mu A_\mu, \tag{4.18}$$

associated with each of these fields is a spacetime gauge invariance, which must be preserved for the consistency of the spacetime theory. Since the arguments of $B_{\mu\nu}$ are the string coordinates, the gauge transformation take the form

$$\delta B_{\mu\nu} = \frac{\partial \Lambda_\nu}{\partial X^\mu} - \frac{\partial \Lambda_\mu}{\partial X^\nu}. \tag{4.19}$$

Let us work out just in the first term of (4.18), the variation of it lead us

$$\frac{i}{2\pi\alpha'} \int_M d^2\sigma g^{1/2} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu \frac{\partial \Lambda_\nu}{\partial X^\mu} = \frac{i}{2\pi\alpha'} \int_M d^2\sigma g^{1/2} \epsilon^{ab} \partial_a \Lambda_\nu \partial_b X^\nu, \tag{4.20}$$

assuming that our gauge parameter Λ vanishes in the infinite past and in the future past, and if the string under consideration is closed, then there is no boundary in σ^2 and the ∂_2 term gives no contribution, either. This demonstrates the gauge invariance for closed strings.

When we consider open string, the term ∂_2 term in (4.20) gives rise to boundary contributions that do not vanish. Since open strings end in Dp-branes, let us call the string coordinates along the brane X^m for $m = 0, 1, \cdots, p$ and string coordinates normal to the brane X^i for $i = p+1, ..., 10$

$$X^\mu = (X^m, X^i), \ \mu = (m, i). \tag{4.21}$$

We drop the ∂_1 term in (4.20) we hold the same assumption for Λ as we did for closed string. so we focus on

$$\frac{i}{2\pi\alpha'} \int d\sigma^1 \left[\Lambda_m \partial_1 X^m + \Lambda_i \partial_1 X^i \right]_{\sigma^2=0}^{\sigma^2=l}, \tag{4.22}$$

taking account that X^i are Dirichlet coordinates, it means $\partial_1 X^i = 0$ at both end-points, therefore (4.22) becomes

$$\frac{i}{2\pi\alpha'}\int d\sigma^1 \Lambda_m \partial_1 X^m \Big|_{\sigma^2=0}^{\sigma^2=l}, \tag{4.23}$$

So in the case of open strings, the variation of the first term plus the variation of the second term of (4.18) is

$$\frac{i}{2\pi\alpha'}\int d\sigma^1 \Lambda_m \partial_1 X^m \Big|_{\sigma^2=0}^{\sigma^2=l} + i\int_{\partial M} dX^\mu \delta A_\mu, \tag{4.24}$$

(4.24) tells us that in order to have consistency of the spacetime theory, A_μ must transform as

$$\delta A_m = -\frac{1}{2\pi\alpha'}\Lambda_m. \tag{4.25}$$

Therefore, we can say the following, whenever we vary $B_{\mu\nu}$ with a gauge parameter $\Lambda_\mu = (\Lambda_m, \Lambda_i)$, we must also vary the Maxwell field A_m living on the D-brane

$$\begin{aligned} \delta B_{\mu\nu} &= \frac{\partial \Lambda_\nu}{\partial X^\mu} - \frac{\partial \Lambda_\mu}{\partial X^\nu}, \\ \delta A_m &= -\frac{1}{2\pi\alpha'}\Lambda_m. \end{aligned} \tag{4.26}$$

Actually, the field strength F_{ab} living on the D-brane is not gauge invariant

$$\delta F_{ab} = \partial_a \delta A_b - \partial_b \delta A_a = -\frac{1}{2\pi\alpha'}(\partial_a \Lambda_b - \partial_b \Lambda_a) = -\frac{1}{2\pi\alpha'}B_{ab}, \tag{4.27}$$

this is significant, because it follows that the fully gauge invariant combination should be

$$2\pi\alpha' \mathcal{F}_{ab} = 2\pi\alpha' F_{ab} + B_{ab}. \tag{4.28}$$

On the D-brane, $\mathcal{F}_{ab}$ is the gauge invariant field strength. Thus the form (4.10) of the action is fully determined.

Up to this point we have dealt with the NS-NS background fields on D-brane, and we have left unspecified the R-R sector, since they do not appear in the DBI action. In the R-R sector, Dp-branes are considered as generalization of electromagnetic source for objects with charge density

$$\mu_p = \frac{1}{(2\pi)^p \sqrt{\alpha'}^{p+1}}, \tag{4.29}$$

where one finds that it is the same expression for the tension of the Dp-brane, which it means that Dp-branes carry R-R charges, [14]. The second term of the action in (4.9) is the Wess-Zumino term, which describes the coupling of the D-brane to the background R-R fields, this term simply states that a Dp-brane is the source of $C^{(p+1)}$, the $p+1$-form R-R potential

$$\mu_p \int C_{p+1}, \tag{4.30}$$

where the integral running over its world-volume. The couplings to the R-R background also includes corrections involving the gauge field on the brane. Like the Born-Infeld action, these can be explained via T-duality. Consider, a D-brane in the (1,2) plane, so

$$\int C_2 = \int dx^0(dx^1 C_{01} + dx^2 C_{02}) = \int dx^0 dx^1 (C_{01} + \partial_1 X^2 C_{02}). \tag{4.31}$$

Under a T-duality in the 2-direction this becomes

$$\int dx^0 dx^1 dx^2 (C_{012} + 2\pi\alpha' F_{12} C_0). \tag{4.32}$$

The generalization of (4.32) to Dp-brane with background field gives

$$\mu_p \int e^{\mathcal{F}} \wedge C, \tag{4.33}$$

where $\mathcal{F}_2 = 2\pi\alpha' F_2 + B_2$, $C = \sum_q C_q$ with $q = 0, 2, 4, 6, 8$. The expansion of the integrand involves forms of various rank; the integral picks out precisely the terms that are proportional to the volume for of the p-brane world volume.

4.2.1 The action of a D3-brane in AdS_5 space

Let us write the low-energy bosonic action of a single D3-brane with type IIB background fields, we mean the $AdS_5 \times S^5$ space. In the $AdS_5 \times S^5$ space the dilaton Φ is constant, and it is given by $e^{-\Phi} = \frac{1}{g_s}$ and $B_2 = C_2 = 0$ because of the the 10-dimensional space is maximally symmetric. The low-energy bosonic action of the D3-brane is

$$S = -\frac{T_3}{g_s} \int d^4\sigma \sqrt{-\det(G_{ab} + 2\pi\alpha' F_{ab})} + \mu_3 \int \{C_4 + \frac{1}{2} C_0 F_2 \wedge F_2\}, \tag{4.34}$$

where T_3, means the D3-brane tension, is given by

$$T_3 = \mu_3 = \frac{1}{(2\pi)^3 l_s^4}. \tag{4.35}$$

with $l_s^2 = \alpha'$. We write the metric of the AdS_5 subspace of $AdS_5 \times S^5$ space in Poincaré coordinates, the metric of $AdS_5 \times S^5$ space is

$$ds^2 = \frac{1}{h^2} dx^\mu dx_\mu - \frac{1}{h^2} dh^2 + d\Omega_5^2, \tag{4.36}$$

where $d\Omega_5^2$ is the metric of the S^5 subspace, let us make $r = \frac{1}{h}$, so we write down the ten-dimensional metric as

$$ds^2 = r^2 dx^\mu dx_\mu + r^{-2} \eta_{IJ} dr^I dr^J, \tag{4.37}$$

where the second term of (4.37) means; the sphere S^5 is located in the radial part of the AdS_5.

We have set the AdS_5 and S^5 subspaces radius to be $R = 1$, which can be recovered by dimensional argument in the final step. So the DBI action, first term, of the D3-brane action is

$$-\frac{T_3}{g_s} \int d^4\sigma \sqrt{-\det(r^2 \partial_a x \cdot \partial_b x + r^{-2} \eta_{IJ} \partial_a r^I \partial_b r^J + \sqrt{\frac{\pi}{g_s N}} F_{ab})}. \tag{4.38}$$

To write down the WZ term, which describes Ramond-Ramond fields couples to the brane. Because of $AdS_5 \times S^5$ we get that $F_5 = dC_4$, where the field strength F_5 is self dual, i.e. $F_5 = *F_5$, so

$$F_5 = \frac{4}{g_s}(1 + *) vol(AdS_5) \tag{4.39}$$

where $vol(AdS_5)$, means the volume of the AdS_5 subspace, which written in Poincaré coordinates is

$$\frac{1}{h^5} dh \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{4.40}$$

or could be written in terms of r as

$$vol(AdS_5) = r^3 dr \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = d(\frac{r^4}{4} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3). \tag{4.41}$$

So the WZ term is given by

$$\mu_3 \int_{\partial\Omega} C_4 = \mu_3 \int_\Omega dC_4 = \frac{\mu_3}{g_s} \int r^4 d^4\sigma. \tag{4.42}$$

Denote $C_0 = \chi$, we have that the D3-brane action is

$$\begin{aligned} S &= -\frac{\mu_3}{g_s} \int d^4\sigma \sqrt{-\det\left(r^2 \partial_a x \cdot \partial_b x + r^{-2} \eta_{IJ} \partial_a r^I \partial_b r^J + \sqrt{\frac{\pi}{g_s N}} F_{ab}\right)} \\ &+ \frac{\mu_3}{g_s} \int r^4 d^4\sigma + \frac{1}{2} \mu_3 \chi \int F_2 \wedge F_2, \end{aligned} \tag{4.43}$$

we can make a simplification about our action (4.43)

- ⋆ Ignoring the axion,$C_0 = \chi$, because of it contributes as a boundary term, that may not change the equations of motion;

- ⋆ supposing that we are interested in pure D3-branes, this means there is no lower dimensional branes in the D3-brane.

With these two assumptions will lead us to the following action

$$S = -\frac{\mu_3}{g_s}\int d^4\sigma r^4 \left\{ \sqrt{-\det(\partial_a x\cdot\partial_b x + r^{-4}\eta_{IJ}\partial_a r^I\partial_b r^J + \frac{1}{r^2}\sqrt{\frac{\pi}{g_s N}}F_{ab})} - 1 \right\}, \tag{4.44}$$

choosing the static gauge, which identify the world-volume coordinates of the D3-brane with the spacetime coordinates, i.e. $x^\mu(\sigma) = \delta^\mu_\alpha\sigma^\alpha$

$$S = -\frac{\mu_3}{g_s}\int r^4\{\sqrt{-\det(\eta_{\mu\nu} + r^{-4}\eta_{IJ}\partial_\mu r^I\partial_\nu r^J + \frac{1}{r^2}\sqrt{\frac{\pi}{g_s N}}F_{\mu\nu})} - 1\}d^4x, \tag{4.45}$$

where $\eta_{\mu\nu}$ is the flat metric. Let us assume that the D3-brane is at a fixed position on the five-sphere, then the five coordinates that correspond to the S^5 sphere do not contribute to the geometry, so the D3-brane action is given by

$$S = -\frac{\mu_3}{g_s}\int r^4\{\sqrt{-\det V_{\mu\nu}} - 1\}d^4x, \tag{4.46}$$

where we did $V_{\mu\nu} = \eta_{\mu\nu} - r^{-4}\partial_\mu r\partial_\nu r + \frac{1}{r^2}\sqrt{\frac{\pi}{g_s N}}F_{\mu\nu}$, we noticed that if r does not depend on the worldvolume coordinates, i.e. $r = \text{const}$, and in the limit $r\to\infty$, i.e. near the AdS_5 boundary, we get the Born Infeld(BI) action,

$$L_{BI} = b^2 - b^2\sqrt{-\det(\eta_{\mu\nu} + \frac{1}{b}f_{\mu\nu})}, \tag{4.47}$$

with $f_{\mu\nu} = \sqrt{\frac{\pi}{g_s N}}F_{\mu\nu}$ and $b = r^2$.

Chapter 5

Stabilizer subgroup of a point on the boundary of the AdS_n space

In this chapter we present classical solutions for D-branes by studying the orbit of the stabilizer subgroup of a point on the boundary of the AdS_n space. Firstly, we start studying very well the stabilizer subgroup in the AdS_2 space, what we found is that its orbit solution is the trajectory of a charged particle in AdS_2 within a constant field strength. Finally, we argue that in the AdS_5 space we get classical solutions for D3-branes.

5.1 Stabilizer subgroup of a point on the boundary of the AdS_2 space

Let us start this section by an introduction of the stabilizer subgroup of a group G. For a manifold X with a group action G, x is a apoint on X, the stabilizer is defined as the set of all elements in G that fix x:

$$G_x = \{g \in G | g.x = x\} \subset G. \tag{5.1}$$

For $AdS_n \times S^m$ spacetime, the group G is the isometry $SO(n-1,2) \times SO(m+1)$.

5.1.1 Warming up

In this stage we will warm up ourselves on how to compute such stabilizer subgroup by studying the simple case of stabilizer of AdS_2 spacetime. Recall that AdS_n space was introduced previously in Chapter 2. For $n = 2$, i.e. AdS_2 space, our hyperboloid reduces to

$$X_{-1}^2 + X_0^2 - X_1^2 = \eta_{MN} X^M X^N = 1, \tag{5.2}$$

which is in $\mathbb{R}^{2,1}$, in (5.2) M and N runs over $(-1, 0, 1)$.

Let be v a null vector in $\mathbb{R}^{2,1}$ and M the matrix which preserves the null vector v, i.e. $Mv = v$. As an example let us compute M, choosing a particular null vector v as

$$v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \tag{5.3}$$

the matrix M which preserves (5.3) must satisfy the properties

$$M^t \eta M = \eta, \qquad \det M = 1, \tag{5.4}$$

because of $v \in \mathbb{R}^{2,1}$ and the isometric group of $\mathbb{R}^{2,1}$ is $SO(2,1)$. After computations the matrix M is

$$M_{ij} = \begin{pmatrix} 1 - \frac{\kappa^2}{2} & -\kappa & \frac{\kappa^2}{2} \\ \kappa & 1 & -\kappa \\ \frac{-\kappa^2}{2} & -\kappa & 1 + \frac{\kappa^2}{2} \end{pmatrix}, \tag{5.5}$$

with $\kappa = \sqrt{2 - 2a}$, at $\kappa = 0$ we recover the identity matrix, so M can be written as $M = e^m$, where m is given by

$$m_{ij} = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\kappa \\ 0 & -\kappa & 0 \end{pmatrix}, \tag{5.6}$$

with $m^3 = 0$.

To see how points in the hyperboloid (5.2) transform with the action of M, let us write it in Poincaré coordinates

$$X = \left(\frac{1 - t^2 + h^2}{2h}, \frac{t}{h}, \frac{1 + t^2 - h^2}{2h}\right). \tag{5.7}$$

The action of M on X, i.e. $X \mapsto X' = MX$, where X' can be written in the form of (5.7) because of (5.4), but with the transformation $t \mapsto t'$ and $h \mapsto h'$, the (t', h') coordinates are given by

$$t' = \frac{t + \kappa(h^2 - t^2)}{(\kappa t - 1)^2 - \kappa^2 h^2}, \qquad h' = \frac{h}{(\kappa t - 1)^2 - \kappa^2 h^2}. \tag{5.8}$$

We can parametrize the null vectors in $\mathbb{R}^{2,1}$ as

$$l = (\cos\theta, \sin\theta, 1), \tag{5.9}$$

at $\theta = 0$ we recover (5.3). To get the null vectors that were not covererd by (5.9), we can use Lorentz transformation on it. The action of M on l, i.e. $l \mapsto l' = Ml$, is given by

$$l' = \alpha \begin{pmatrix} \cos\theta' \\ \sin\theta' \\ 1 \end{pmatrix}, \tag{5.10}$$

where α and θ' are

$$\begin{aligned} \alpha &= \frac{\kappa^2}{2}(1-\cos\theta) - \kappa\sin\theta + 1, \\ \tan\theta' &= \frac{\kappa(\cos\theta - 1) + \sin\theta}{\frac{\kappa^2}{2}(1-\cos\theta) - \kappa\sin\theta + \cos\theta}, \end{aligned} \tag{5.11}$$

where we can see that l' still satisfies the null condition $l'^M l'_M = 0$, this result follows from that M satisfies the properties (5.4).

Anyway, we found the matrix M which preserves (5.3), but from (5.10), it seems that we could look for a matrix M which preserves (5.3) up to a multiplicative factor, i.e. $Mv = \lambda v$. In the computation of the new M, we use (5.4), so M is

$$\widetilde{M}_{ij} = \begin{pmatrix} a & -\lambda\sqrt{\frac{1}{\lambda^2}+1-\frac{2a}{\lambda}} & \lambda - a \\ \sqrt{\frac{1}{\lambda^2}+1-\frac{2a}{\lambda}} & 1 & -\sqrt{\frac{1}{\lambda^2}+1-\frac{2a}{\lambda}} \\ a-\frac{1}{\lambda} & -\lambda\sqrt{\frac{1}{\lambda^2}+1-\frac{2a}{\lambda}} & \lambda+\frac{1}{\lambda}-a \end{pmatrix}, \tag{5.12}$$

where we can see that for $a = \lambda = 1$, one obtains the unitary matrix, this fact can be written as

$$\delta\widetilde{M} = \widetilde{M}_a \delta a + \widetilde{M}_\lambda \delta\lambda \tag{5.13}$$

with

$$\widetilde{M}_a = \left.\frac{\partial\widetilde{M}}{\partial a}\right|_{a=\lambda=1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \tag{5.14}$$

$$\widetilde{M}_\lambda = \left.\frac{\partial\widetilde{M}}{\partial \lambda}\right|_{a=\lambda=1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.15}$$

The action of the matrix $\widetilde{M}$ on (5.7) give us as result the transformations: $t \mapsto t'$ and $h \mapsto h'$, this fact is possible because of (5.4), $X' = \widetilde{M}X$ can be written as

(5.7), so h' and t' are given by

$$\begin{aligned} h' &= \frac{h}{\lambda\{(t\sqrt{\frac{1}{\lambda^2}-\frac{2a}{\lambda}+1}-1)^2-(\frac{1}{\lambda^2}-\frac{2a}{\lambda}+1)h^2\}}, \\ t' &= \frac{t+(h^2-t^2)\sqrt{\frac{1}{\lambda^2}-\frac{2a}{\lambda}+1}}{\lambda\{(t\sqrt{\frac{1}{\lambda^2}-\frac{2a}{\lambda}+1}-1)^2-(\frac{1}{\lambda^2}-\frac{2a}{\lambda}+1)h^2\}}. \end{aligned} \tag{5.16}$$

The action of $\widetilde{M}$, (5.12), on the lightray (5.9) give us an expresion like (5.10), but in this case, α and θ' are given by

$$\begin{aligned} \alpha &= (1-\cos\theta)(\frac{1}{\lambda}-a)+\lambda(1-\sin\theta\sqrt{\frac{1}{\lambda^2}+1-\frac{2a}{\lambda}}), \\ \tan\theta' &= \frac{(\cos\theta-1)\sqrt{\frac{1}{\lambda^2}+1-\frac{2a}{\lambda}}+\sin\theta}{a(\cos\theta-1)+\lambda(1-\sin\theta\sqrt{\frac{1}{\lambda^2}+1-\frac{2a}{\lambda}})}. \end{aligned} \tag{5.17}$$

We do not make any discussion about the relations that we got here, because it was just a warm up of how we would compute the stabilizer subgroup.

5.1.2 About stabilizer subgroup of the lightray in $\mathbb{R}^{2,1}$

Up to this point, working in the coordinates (5.2) lead us in a trouble. Since we do not have restriction in choosing our system of coordinates, let us choose the light-cone coordinates, $x_\pm$. For the AdS_2 space, the $x_\pm$ coordinates are given by

$$x_\pm = \frac{1}{\sqrt{2}}(X_{-1} \pm X_1), \tag{5.18}$$

with these coordinates the hyperboloid (5.2) is

$$2x_+x_- + X_0^2 = 1,$$

and the η_{MN} transform into the metric G_{MN} which is

$$G_{MN} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.19}$$

Let us recall that the lightrays in $\mathbb{R}^{2,n-1}$ can be written as

$$l(\tau) = \{\frac{1}{2}(1-x^\mu x_\mu), x^\mu(\tau), \frac{1}{2}(1+x^\mu x_\mu)\}, \tag{5.20}$$

in the case of $\mathbb{R}^{2,1}$ and using (5.18), we find that (5.20) reduces to

$$l = \{\frac{1}{\sqrt{2}}, \tau, -\frac{\tau^2}{\sqrt{2}}\}, \tag{5.21}$$

where we have chosen the parameter τ to be $\tau = x^0 = t - h$, the hyperboloid (5.2) in Poincaré coordinates was written in (5.7), with the definition (5.18), we find that

$$X = \{\frac{1}{h\sqrt{2}}, \frac{t}{h}, -\frac{t^2 - h^2}{h\sqrt{2}}\}. \tag{5.22}$$

The null vector given in (5.3) is recovered when we make $\tau = 0$ in (5.21), which in Poincaré coordinates corresponds to $h = t$ and we would take them to be zero

$$l_0 = \{\frac{1}{\sqrt{2}}, 0, 0\}, \tag{5.23}$$

The matrix M which preserves the lightray l_0 up to a multiplicative factor, i.e. $Mv = \lambda v$, is given by

$$M_{ij} = \begin{pmatrix} \lambda & b & -\frac{b^2}{2\lambda} \\ 0 & 1 & -\frac{b}{\lambda} \\ 0 & 0 & \frac{1}{\lambda} \end{pmatrix}, \tag{5.24}$$

where the matrix M satisfies (5.4).

The action of (5.24) on (5.21), i.e. $l \mapsto l' = Ml$ with l' given by

$$l' = \alpha \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \tau' \\ -\frac{\tau'^2}{\sqrt{2}} \end{pmatrix}, \tag{5.25}$$

where α and τ' are

$$\begin{aligned} \tau' &= \frac{\tau}{\lambda + \frac{b\tau}{\sqrt{2}}}, \\ \alpha &= \frac{1}{\lambda}(\lambda + \frac{b\tau}{\sqrt{2}})^2. \end{aligned} \tag{5.26}$$

The action of (5.24) on (5.22), i.e. $X \mapsto X' = MX$, give us the transformation $h \mapsto h'$ which is given by

$$h' = \frac{h}{\alpha + b\sqrt{\frac{2\alpha}{\lambda}}h} = \frac{(1 - \frac{b\tau'}{\sqrt{2}})h}{\alpha(1 - \frac{b\tau'}{\sqrt{2}}) + bh\sqrt{2}}. \tag{5.27}$$

Let us give some properties of the matrix M, (5.24). From M we see that for $\lambda = 1$ and $b = 0$ one obtains the unitary matrix, this fact can be written as

$$\delta M = M_b \delta b + M_\lambda \delta\lambda \tag{5.28}$$

with

$$M_b = \frac{\partial M}{\partial b}\bigg|_{b=0,\lambda=1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.29}$$

$$M_\lambda = \frac{\partial M}{\partial \lambda}\bigg|_{b=0,\lambda=1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5.30}$$

Here M_b and M_λ tell us how the parameters describing M is to start out from the identity, they obeys the commutation relation: $[M_b, M_\lambda] = -M_b$.

So we realized that there are two kinds of transformations which preserve l_0, which we will describe them as follow. Let us start studying the transformations:

1. At $b = 0$, from (5.26) and (5.27) we have that τ and h transform as

$$\begin{aligned} \tau &\mapsto \frac{\tau}{\lambda}, \\ h &\mapsto \frac{h}{\lambda}, \end{aligned} \tag{5.31}$$

then, we can conclude that t transforms as $t \mapsto \frac{t}{\lambda}$.

So we find that at $b = 0$,the transformations which preserve the lightray l_0 are dilations, D_λ, where λ must not be zero, i.e. $\lambda \in \mathbb{R}_*$ and they form a group. The Eqs. (5.25) and (5.22) change to

$$\begin{aligned} l' &= \lambda\{\frac{1}{\sqrt{2}}, \frac{\tau}{\lambda}, -\frac{\tau^2}{\lambda^2\sqrt{2}}\} \\ X' &= \{\frac{\lambda}{h\sqrt{2}}, \frac{t}{h}, -\frac{t^2-h^2}{h\lambda\sqrt{2}}\}. \end{aligned} \tag{5.32}$$

2. The another kind of transformation preserving l_0 happens when we set $\lambda = 1$ in (5.24) and being $b \in \mathbb{R}$, which we named it S_b. The matrix (5.24) changes

to

$$\widetilde{\mathcal{M}}_{ij} = \begin{pmatrix} 1 & b & -\frac{b^2}{2} \\ \\ 0 & 1 & -b \\ \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.33}$$

$\widetilde{\mathcal{M}}_{ij}$ can be written as: $\widetilde{\mathcal{M}} = e^m$, where m is

$$m_{ij} = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & -b \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.34}$$

with $m^3 = 0$. So the equations (5.26) and (5.27) gave us the following result

$$\begin{aligned} \tau &\mapsto \frac{\tau}{1 + \frac{b\tau}{\sqrt{2}}}, \\ h &\mapsto \frac{h}{(1 + \frac{bt}{\sqrt{2}})^2 - (\frac{bh}{\sqrt{2}})^2}, \end{aligned} \tag{5.35}$$

then, we conclude that t transforms as

$$t \mapsto \frac{t + \frac{b(t^2 - h^2)}{\sqrt{2}}}{(1 + \frac{bt}{\sqrt{2}})^2 - (\frac{bh}{\sqrt{2}})^2}, \tag{5.36}$$

the transformation S_b on h and t, (5.35) and (5.36),can be written as

$$x'^{\mu} = \frac{x^{\mu} - a^{\mu} x^2}{1 - 2a^{\mu} x_{\mu} + a^2 x^2}, \tag{5.37}$$

where a^{μ}, x^{μ} and x'^{μ} are given by

$$\begin{aligned} a^{\mu} &= (-\frac{b}{\sqrt{2}}, 0), \\ x^{\mu} &= (t, h), \quad x'^{\mu} = (t', h'). \end{aligned} \tag{5.38}$$

Eq. (5.37) can be seen as a translation a^{μ}, preceded and followed by an inversion $\frac{x^{\mu}}{x^2}$, and were used the lorentzian metric.

Properties of D_λ and S_b transformations: Since we have two kinds of transformations, D_λ and S_b. It is important to state what properties they have:

⋆ The transformations S_b form a group, i.e.

$$\tau \xrightarrow{S_{b_1}} \frac{\tau}{1+\frac{b_1\tau}{\sqrt{2}}} \xrightarrow{S_{b_2}} \frac{\tau}{1+\frac{(b_1+b_2)\tau}{\sqrt{2}}},$$

$$x^\mu \xrightarrow{a_1} x'^\mu \xrightarrow{a_2} x''^\mu = \frac{x^\mu-(a_1^\mu+a_2^\mu)x^2}{1-2(a_1^\mu+a_2^\mu)x_\mu+(a_1+a_2)^2x^2},$$

we can write this statement as $S_{b_1+b_2}(\tau,h)=S_{b_1}\circ S_{b_2}(\tau,h)$;

⋆ The transformations D_λ and S_b do not commute, but satisfies $D_\lambda S_b = S_{\frac{b}{\lambda}} D_\lambda$, i.e.

$$t \xrightarrow{S_b} \frac{t+\frac{b(t^2-h^2)}{\sqrt{2}}}{(1+\frac{bt}{\sqrt{2}})^2-(\frac{bh}{\sqrt{2}})^2} \xrightarrow{D_\lambda} \frac{\frac{t}{\lambda}+\frac{b(t^2-h^2)}{\lambda^2\sqrt{2}}}{(1+\frac{bt}{\lambda\sqrt{2}})^2-(\frac{bh}{\lambda\sqrt{2}})^2},$$

$$t \xrightarrow{D_\lambda} \frac{t}{\lambda} \xrightarrow{S_{\frac{b}{\lambda}}} \frac{1}{\lambda}\frac{t+\frac{b(t^2-h^2)}{\lambda\sqrt{2}}}{(1+\frac{bt}{\lambda\sqrt{2}})^2-(\frac{bh}{\lambda\sqrt{2}})^2}.$$

Because of the above properties, we state that:

The stabilizer subgroup of the lightray l_0 is given by the semidirect product between $\mathbb{R}_*$ and $\mathbb{R}$, i.e.

$$\mathcal{G}=\mathbb{R}_*\ltimes\mathbb{R}\subset \mathrm{SO(2,1)} \tag{5.39}$$

5.1.3 Orbits from S_b transformation

The S_b transformation in (5.22), gives us that the x_- component of (5.22) is preserved, so we arrived at the following invariant:

$$t^2-h^2+\sqrt{2}h\kappa=0, \tag{5.40}$$

and we call it *orbit*. Let us study the orbit of our S_b transformation

- First we consider points outside of the boundary that are located in the axis $t=0$, the S_b transformation change the points $(0,h)$ to

$$\left(\frac{\frac{b}{\sqrt{2}}}{\frac{b^2}{2}-\frac{1}{h^2}},\frac{\frac{-1}{h}}{\frac{b^2}{2}-\frac{1}{h^2}}\right),$$

  where we can see that $\frac{b}{\sqrt{2}}\neq\pm\frac{1}{h}$. The orbit can be obtained from (5.40), let $\kappa=h_0$, we get: $t^2-(h-\frac{h_0}{\sqrt{2}})^2+(\frac{h_0}{\sqrt{2}})^2=0$, which is the graph of a hyperbola with center at $(0,\frac{h_0}{\sqrt{2}})$.

- From the previous discussion when, $\kappa = h_0$ picks up negative values, the hyperbola reaches the boundary of the AdS_2 space at $(t, h) = (0, 0)$.

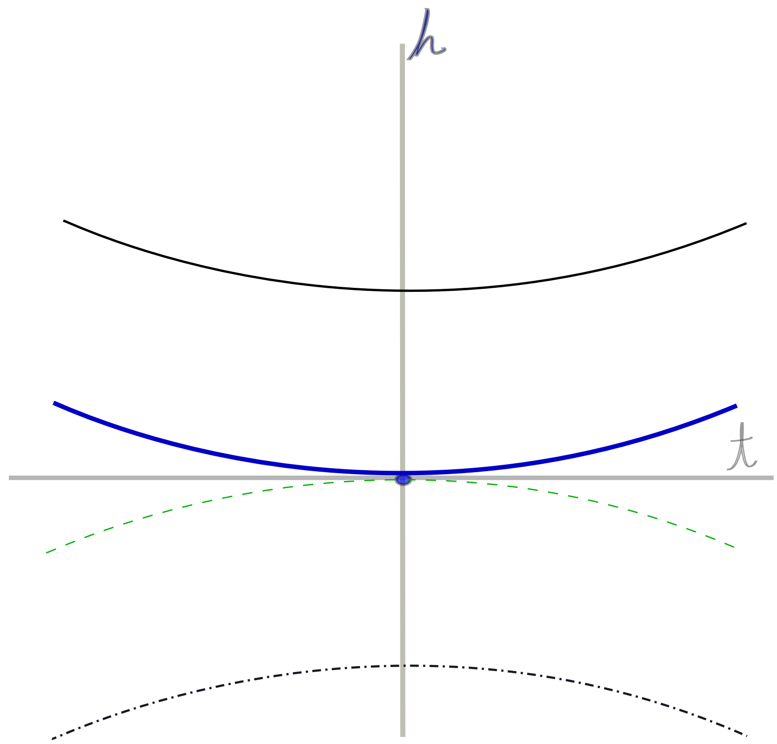

Figure 5.1: Solid black and blue curves represent the orbit solution, the blue curve can be obtained from the black one doing $\kappa \mapsto -\kappa$ and $h \mapsto -h$.

Let us make two proposition that would help us to understand, what exactly the hyperbola (5.40) means

Proposition 1. *The hyperbola* (5.40) *is not a geodesic in* AdS_2 *space*

Proof. The metric of AdS_n space in global coordinates was given in (3.6), in the case of AdS_2 space it is

$$ds^2 = (\zeta^2 + 1)dT^2 - \frac{1}{1+\zeta^2}d\zeta^2, \tag{5.41}$$

where we have substituted $\zeta = \sinh \rho$. Let be λ the proper time(affine parameter) for a massive particle(massless particle), so the geodesic equation is given by

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0, \tag{5.42}$$

so the differential equations for ζ and T are

$$\begin{aligned}
\frac{d^2T}{d\lambda^2} + \frac{2\zeta}{1+\zeta^2}\frac{dT}{d\lambda}\frac{d\zeta}{d\lambda} &= 0, \\
\frac{d^2\zeta}{d\lambda^2} + \zeta(1+\zeta^2)(\frac{dT}{d\lambda})^2 - \frac{\zeta}{1+\zeta^2}(\frac{d\zeta}{d\lambda})^2 &= 0.
\end{aligned} \tag{5.43}$$

The solution for these equations are:

$$\begin{aligned}
\zeta &= \frac{\sqrt{c^2+a^2}}{c}\sinh(c\lambda+k), \\
T &= \arctan\frac{a^2\tanh(c\lambda+k)+c^2}{ac(1-\tanh(c\lambda+k))} + s.
\end{aligned} \tag{5.44}$$

The relations (5.44) can be written as

$$\frac{a\zeta}{\sqrt{(c\zeta)^2+c^2+a^2}} = \tan(T+s-\arctan(\frac{c}{a})), \tag{5.45}$$

we can find one of the unknown integration constants,using the fact that: $\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda} = \epsilon$, where $\epsilon = 0$ for massless $\epsilon = 1$ and massive particle.

Let $\zeta = \tan\theta$, with this change the boundary of AdS_2 is at $\theta = \pi/2$, and (5.44) becomes:

$$\frac{\tan\theta}{\sqrt{1+\frac{c^2}{a^2}\sec^2\theta}} = \tan(T - \arctan\frac{c}{a}) \tag{5.46}$$

where $s = 0$.

The Poincaré coordinates (t,h) are related to the coordinates (ζ,T) by $t = \tan T, \quad h = \frac{1}{\cos T\sqrt{1+\zeta^2}}$, the Eq. (5.40) in the (ζ,T) coordinates is

$$-1 + \sin^2 T(1+\zeta^2) + \sqrt{2}\kappa\cos T\sqrt{1+\zeta^2} = 0 \tag{5.47}$$

where we can conclude whatever initial conditions we could have, equations (5.40) or (5.47) do not caracterize a geodesic in AdS_2. □

Proposition 2. *The hyperbola* (5.40) *is the equation of the trajectory of a charged particle under a constant field strength in AdS_2 space*

Proof. Let us consider the motion of a charged particle under a constant field strength in the Poincaré patch of AdS_2, its action is given by:

$$S = \int d\lambda\{\frac{1}{2\varrho h^2}[(\frac{dt}{d\lambda})^2 - (\frac{dh}{d\lambda})^2] + \varrho\frac{m^2}{2} + \frac{qE}{h}\frac{dt}{d\lambda}\}, \tag{5.48}$$

where $\varrho = \varrho(\lambda)$ is a Lagrange multiplier, the equations of motion for t and h are

$$\frac{d}{d\lambda}\left\{\frac{\dot{t}}{\varrho h^2} + \frac{qE}{h}\right\} = 0$$
$$-\dot{\varrho}\dot{h}h + \varrho h\ddot{h} - \varrho(\dot{t}^2 + \dot{h}^2) - \varrho^2 qEh\dot{t} \;=\; 0, \qquad (5.49)$$

and the equation for ϱ is given by

$$\dot{t}^2 - \dot{h}^2 - \varrho^2 h^2 m^2 = 0. \qquad (5.50)$$

Let us choose the static gauge, i.e. $t = \lambda$. So (5.50) becomes

$$\varrho^2 = \frac{1 - \dot{h}^2}{h^2 m^2} \qquad (5.51)$$

the solution of the equations of motion (5.49) is

$$(t - \frac{l}{a})^2 - (h - \frac{Eq}{a})^2 = \frac{-m^2}{a^2}. \qquad (5.52)$$

In the case when $l = 0$, we have a critical electric field, i.e. $(qE)^2 = m^2$, in such case (5.52) represents the trajectory of a charged particle in the Poincaré patch of the AdS_2 space. For a negative charged particle (5.52) becomes (5.40), with $\kappa < 0$.

The bellow figure represents the trajectory of a negative charged particle in the AdS_2 space in global coordinates at $(qE)^2 = m^2$.

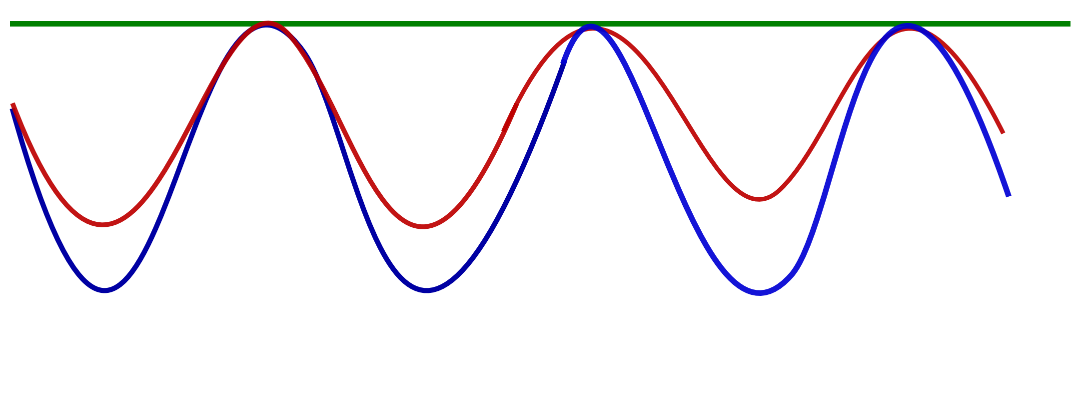

Figure 5.2: Trajectories of a charged particle within a constant field strength in AdS_2 space in global coordinates at two differents values of κ, green lines denote the boundary of AdS_2 space.

□

So we conclude that the orbit is the solution of the trajectory of a charged particle in AdS_2 space within constant field strength.

5.2 Stabilizer subgroup of a point on the boundary of the AdS_5 space

Here we use the same methods that were used in the previous section. But we turn back our attention to the AdS_5 space, let us choose as in the previous section an specific null direction l_0 in $\mathbb{R}^{2,4}$

$$l_0 = (\frac{1}{\sqrt{2}}, \mathbf{0}, 0), \tag{5.53}$$

where we have used the lightcone coordinates $x_{\pm} = \frac{X_{-1}+X_4}{\sqrt{2}}$, so the metric on $\mathbb{R}^{2,4}$ on lightcone coordinates is

$$G_{MN} = \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \eta_{\mu\nu} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}, \tag{5.54}$$

with $\eta_{\mu\nu} = diag(+,-,-,-)$, where μ runs from 0 to 3.

Without loss of generality we stated that the matrix that preserves (5.53) is given by

$$\Sigma_{MN} = \begin{pmatrix} \lambda & b_\mu & -\frac{b^2}{2\lambda} \\ & & \\ \mathbf{0} & \boldsymbol{I}_{4x4} & -\frac{b^\mu}{\lambda} \\ & & \\ 0 & \mathbf{0} & \frac{1}{\lambda} \end{pmatrix}, \tag{5.55}$$

with $b^2 = b_\mu b^\mu$,and b^μ being a 4-vector in $\mathbb{R}^{1,3}$ As we did in the previous section, let us write down the hyperboloid in $\mathbb{R}^{\,4,2}$, i.e.

$$G_{MN}X^M X^N = 1 \tag{5.56}$$

where the metric G_{MN} was given in (5.54), and the embedding X^M in Poincaré coordinates is given by

$$X = \{\frac{1}{\sqrt{2}h}, \frac{x_\mu}{h}, \frac{h^2 - x^\mu x_\mu}{\sqrt{2}h}\}. \tag{5.57}$$

The action of (5.55) on the embedding (5.57), i.e. $X \mapsto X' = \Sigma X$ give us the following results

$$h \mapsto h' = \frac{h}{\lambda + \sqrt{2}b\cdot x - \frac{b^\mu b_\mu}{2\lambda}(-x^\mu x_\mu + h^2)}, \tag{5.58}$$

$$x^\mu \mapsto x'^\mu = \frac{x^\mu - \frac{1}{\sqrt{2}\lambda} b^\mu(-x\cdot x + h^2)}{\lambda + \sqrt{2}b\cdot x - \frac{1}{2\lambda}(b\cdot b)(-x\cdot x + h^2)}, \tag{5.59}$$

$$\frac{-x^\mu x_\mu + h^2}{\sqrt{2}h\lambda} = \frac{-x'^\mu x'_\mu + h'^2}{\sqrt{2}h'} \tag{5.60}$$

As we discussed in the previous section for the AdS_2 space, when one finds the matrix Σ ,given in (5.55), which preserves l_0, (5.53). We get two kind of transformations:

- When we set the 4-vector $b^\mu = \mathbf{0}$, we have a dilation transformation, D_λ; with $\lambda \in \mathbb{R}_*$;
- For $\lambda = 1$, we have the $S_{\boldsymbol{b}}$ transformation with $b^\mu \in \mathbb{R}^{1,3}$, which forms a group. As we said for the AdS_2 space, S_b and D_λ do not commute, but satisfies $S_{\frac{b}{\lambda}} D_\lambda = D_\lambda S_b$.

Because of the above properties, we state that:

- The stabilizer subgroup of l_0 is given by the semidirect product between $\mathbb{R}_*$ and $\mathbb{R}^{1,3}$, i.e.

$$\mathcal{G} = \mathbb{R}_* \ltimes \mathbb{R}^{1,3} \tag{5.61}$$

Nevertheless, the stabilizer subgroup (5.61) for the lightray l_0 is not complete, since we have chosen an specific lightray. Here we just state that:

The stabilizer subgroup for any lightray in $\mathbb{R}^{2,4}$ is given by

$$\mathcal{G} = M_{\mu\nu} \bowtie \mathbb{R}_* \ltimes \mathbb{R}^{1,3} \subset \mathrm{SO(2,4)}, \tag{5.62}$$

where $M_{\mu\nu}$ stands for Lorentz transformationa, in comparison with the AdS_2 case, b has been promoted to a four-vector. What we get is that the orbit solution leads to classical solution for D3-branes,

$$-x^\mu x_\mu + h^2 - \sqrt{2}h\kappa = 0, \tag{5.63}$$

where κ being constant, which parametrizes the different classical solution for the D3-brane, and x^μ being the coordinates on the world-volume. The below figure show our orbit solution , D1-brane, in the AdS_3 space.

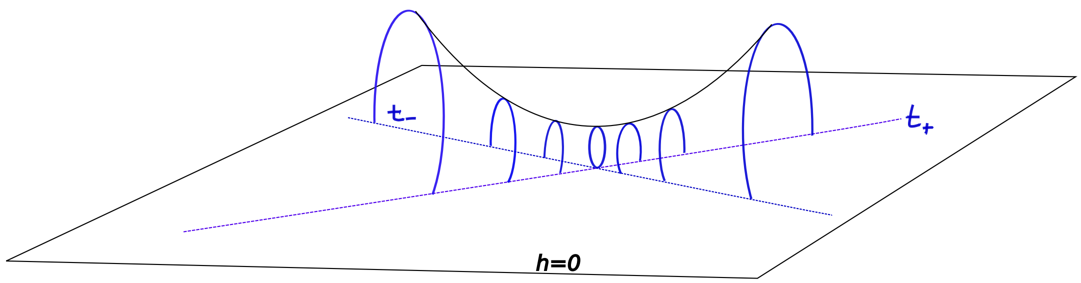

Figure 5.3: D1-brane solution in AdS_3 space, blue curves intersecting the boundary of AdS_3 were drawn at constant t, where $t_\pm = x \pm t$.

Chapter 6

Marginal deformation of $\mathcal{N} = 4$ SYM and its gravity dual

In this chapter we explain what marginal deformation of $\mathcal{N} = 4$ SYM theory is, which was done by the first time by Leigh-Strassler [6]. Actually the latter discussed many models with fixed lines from the point of view of $\mathcal{N} = 1$ SYM. Here we are focused in the $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group, due to its importance to the AdS/CFT correspondence. In the last section we present the gravity dual for the so-called β-deformation of the $\mathcal{N} = 4$ SYM theory. This was done by Lunin-Maldacena [7] also, we present the Frolov's argument to generate the Lunin-Maldacena background solution [27].

6.1 Conditions for scale invariance in Gauge Theories

In this section we present the conditions for scale invariance in gauge theories given by Leigh and Strassler [6]. Unfortunately their results are presented at one-loop level. It is argued to be exact at all orders in perturbation theory.

We notice that at classical level, the $U(1)_R$ current, the supercurrent and the energy-momentum tensor belong to the same multiplet (the supercurrent multiplet $\mathcal{J}_{\alpha\dot{\alpha}}$) [15]. Moreover the $U(1)_R$ current is the lowest component of the supercurrent multiplet. The classical equation for the supercurrent multiplet , $\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = 0$, contains the conservation of the three-currents, and also the relations

$$T^{\mu}{}_{\mu} = 0 \qquad\qquad J_{\alpha\dot{\alpha}} = 0, \tag{6.1}$$

which express the classical conformal and superconformal symmetries.

We are interested in a non-Abelian gauge theory with n chiral superfields Φ_i and a superpotential $\mathcal{W}(\Phi_i)$, so the derivative of the supercurrent multiplet is

$$\overline{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}}\Big|_{classical} = \frac{1}{3}D_\alpha\big(3\mathcal{W} - \sum \Phi_i \frac{\partial \mathcal{W}}{\partial \Phi_i}\big). \tag{6.2}$$

We can see that the right hand side of (6.2) vanishes if the superpotential is cubic in Φ_i or there is no superpotential $\mathcal{W} = 0$. This means that at the classical level this gauge theory is scale invariant.

But at the quantum level anomalies appears, which means that the symmetries are broken. The conservation of the $U(1)_R$ current is also lost at the quantum level and receives the name of chiral anomaly. For a pure gauge theory, we have that the one-loop result for the derivative of the $U(1)_R$ current, R_μ , is

$$\partial^\mu R_\mu = \frac{C_2(G)}{16\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu}. \tag{6.3}$$

The superfield genralization of (6.3) is

$$\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = -\frac{1}{3}\frac{3C_2(G)}{16\pi^2}D_\alpha(W^aW^a). \tag{6.4}$$

The generalization of (6.4) to the case of a Yang-Mills theory with matter fields and a superpotential $\mathcal{W}$ is

$$\overline{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = \overline{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}}\Big|_{classical} - \frac{1}{3}D_\alpha\Big(\frac{b_0}{16\pi^2}W^aW^a + \frac{\overline{D^2}}{8}\sum \gamma_i Z_i(\mu)\Phi_i^\dagger e^V \Phi_i\Big), \tag{6.5}$$

where $b_0 = 3C_2(G) - \sum_i T(R_i)$, $Z_i(\mu)$ is the wave function renormalization of the field Φ_i at Wilsonian cutoff μ and γ_i is its anomalous mass dimension. In the appendix A we summarized the Wilsonian approach in supersymmetric gauge field theories. We see that the anomalies cause additional dependence of the supercurrent operator on the Wilsonian cutoff μ.

The last term from the rhs of (6.5) is the equation of motion for each field Φ_i, multiplied by Φ_i, and corrected to account for the chiral (Konishi) gauge anomalies of the theory. A brief derivation of this anomaly is also presented in appendix A, after substituting (A.28) into the supercurrent anomaly, we get

$$\bar{D}^{\dot{\alpha}}\mathcal{J}_{\alpha\dot{\alpha}} = \frac{1}{3}D_\alpha\Big[-\frac{W^aW^a}{16\pi^2}\Big(b_0 - \sum_i T(R_i)\gamma_i\Big) + \Big(3W - \sum_i \Phi_i\frac{\partial W}{\partial \Phi_i}(1+\frac{1}{2}\gamma_i)\Big)\Big]. \tag{6.6}$$

Let us assume that the superpotential has the form of a polynomial

$$W(\Phi_i) = \sum_s h_s W^{(s)}(\Phi_i), \tag{6.7}$$

where $W^{(s)}$ as a product of d_s fields. Then we write the anomaly supercurrent as

$$\bar{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = \frac{1}{3} D_\alpha \Big[W^a W^a A_g + \sum_s W^{(s)} A_{h_s} \Big], \tag{6.8}$$

where the scaling coefficient A_g and A_{h_s} are

$$A_g = -\frac{1}{16\pi^2} \Big[b_0 - \sum_i T(R_i) \gamma_i \Big], \tag{6.9}$$

and

$$A_{h_s} = -h_s(d_s - 3) - \sum_i \frac{1}{2} h_s \gamma_i \frac{\partial \ln W^{(s)}}{\partial \ln \Phi_i}. \tag{6.10}$$

In order to have a theory with no scale dependence, the scaling coefficients must vanish. It is realized that (6.9) is proportional to the holomorphic β function,

$$\begin{aligned} \beta_h &= -\frac{g_h^3}{16\pi^2} \Big(b_0 - \sum_i T(R_i) \gamma_i \Big) \\ &= \frac{g_h^3}{16\pi^2} A_g, \end{aligned} \tag{6.11}$$

which can be obtained from (A.19) after computing $\mu_0 {}^{\partial}/_{\partial \mu_0}$ in both sides.

6.1.1 Some comments

From the results above, we can say that the condition for the theory to be marginal at one-loop requires the vanishing of the scaling coefficients A_g and A_{h_s}, see equations (6.9) and (6.10). It was also noted that these scaling coefficient would proportional to the β-functions of the couplings, if we have vanishing β-functions at all orders of perturbation theory, we will have a theory with no scale dependence, i.e. a fixed point, this means if we have n couplings $g, \{h_s\}$ we will have n constraints on the n couplings. If these constraints are linearly independent, then we expect their solutions to be isolated points in the space of coupling constants.

However, it may happen tha some of the β-functions be linearly dependent, e.g. see (6.9) and (6.10). Let p to be the number of linearly independent β-functions, with $p < n$, then the vanishing β_i impose p conditions on the couplings, and we expect the generic solution to be $n - p$ dimensional submanifold in the space of couplings, at least if any solution exists. One application of this statement is the Landau-Ginsburg model, the conditions for its fixed points is briefly described in [6].

Another example, in which we are interested, is supersymmetric theories. Where we have several simplifications, one of them is due to the non-renormalization of the superpotential. Let be the superpotential $\mathcal{W}$ be

$$\mathcal{W} = H_{ijk}\Phi^i\Phi^j\Phi^k, \tag{6.12}$$

the β-function for the Yukawa coupling is given by $\beta_H^{ijk} = H^{q(ij}\gamma_q^{k)}$ [16], which was roughly obtained in the previous section after analyzing the anomaly supercurrent, where γ_p^k is the anomalous dimension related to the $\langle\bar{\Phi}^k\Phi_p\rangle$ Z factor. At first-loop the anomalous dimension of the chiral superfield is $16\pi^2\gamma_n^{(1)m} = -P_n^m$, with $P_n^m = -\frac{1}{2}H^{mkl}H_{nkl} + 2g^2T(R_i)_n^m$ [17] The another simplification is due to NSVZ β-function. See appendix A equation (A.24).

To end this section we say that in supersymmetric gauge theories in order to find out a manifold of fixed points, we have to solve a set of linear equations, with the arguments given above, will be dependent of the couplings g, h_s and the anomalous dimensions. The possible solutions that we could get will be ruled out from loop calculations, i.e. in the weak coupling regime.

6.2 Marginal deformations of $\mathcal{N}=4$ SYM

In this section, we first discuss how $\mathcal{N}=4$ SYM theory arise after compactifiction in a six-dimensional torus. Then, we write down its lagrangian density in terms of $\mathcal{N}=1$ language, and, in the last section, we discussed the marginal deformation of $\mathcal{N}=4$ SYM, following the ideas from [6] and using the notation from [18].

6.2.1 The $\mathcal{N}=4$ SYM field theory

The lagrangian density for $\mathcal{N}=4$ SYM field theory in $d=4$ can be obtained from the YM theory in $d=10$ by dimensional reduction, this was done by the first time in [19], the lagrangian density in $d=10$ is

$$\mathcal{L}_{10} = \mathrm{Tr}(-\frac{1}{4}F_{MN}F^{MN} - \frac{i}{2}\bar{\Psi}\Gamma^M D_M\Psi), \tag{6.13}$$

where $F_{MN} = \partial_{[M}A_{N]} + igA_{[M}A_{N]}$, and Ψ satisfies the Majorana and Weyl conditions

$$\Psi = \Gamma^{11}\Psi, \qquad \Psi = C_{1,9}\Psi^*, \tag{6.14}$$

where $C_{1,9}$ is the charge conjugation operator.

Dimensional reduction on a six dimensional torus T^6 leads to a multiplet who has $SU(4)$ global symmetry, which is identified with R-symmetry.

In the bosonic sector we have a four-dimeansional real vector A_μ and six scalars, which can be read as

$$\begin{aligned} A^a_M &= A^a_\mu, \quad \text{with} \quad M=0,1,2,3 \\ \phi^{aAB} &= \frac{1}{\sqrt{2}}\Sigma^{MAB}A^a_M, \quad \text{with} \quad M=4,\dots,9, \\ \bar{\phi}^a_{AB} &= \frac{1}{\sqrt{2}}\bar{\Sigma}^M{}_{AB}A^a_M, \quad \text{with} \quad M=4,\dots,9, \end{aligned} \tag{6.15}$$

where Σ^{AB} is the six-dimensional generalization of the four-dimensional $\bar{\sigma}^{\mu\dot{\alpha}\beta}$, A,B index run over $\{1,2,3,4\}$ and a index denotes the gauge group element. The pure-YM part is

$$-\frac{1}{4}F_{MN}F^{MN} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}D_\mu\phi^{AB}D^\mu\bar{\phi}_{AB} + \frac{g^2}{16}[\phi^{AB},\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}], \tag{6.16}$$

where we have used $\Sigma^{iAB}\bar{\Sigma}^j{}_{AB} = 4\delta^{ij}$.

For the fermionic sector we used the following relations

$$\begin{aligned} \bar{\xi}\Gamma^\mu\psi &= \tilde{\xi}_{\dot{\alpha}A}\bar{\sigma}^{\mu\dot{\alpha}\beta}\psi^A{}_\beta + \xi^{\alpha A}\sigma^\mu{}_{\alpha\dot{\beta}}\tilde{\psi}^{\dot{\beta}}{}_A \\ \bar{\xi}\Gamma^{i+3}\psi &= -\Sigma^{iAB}\tilde{\xi}_{\dot{\alpha}A}\tilde{\psi}^{\dot{\alpha}}{}_B + \bar{\Sigma}^i{}_{AB}\xi^{\alpha A}\psi^B{}_\alpha, \quad \text{with} \quad i=1,\dots,6, \end{aligned} \tag{6.17}$$

where ξ and ψ are Weyl-Majorana spinors, $\alpha,\dot{\alpha}$ run over $\{1,2\}$. So the second term in (6.13) can be written as

$$\begin{aligned} i\bar{\Psi}\Gamma^M D_M\Psi &= 2i\tilde{\psi}_{\dot{\alpha}A}\bar{\sigma}^{\mu\dot{\alpha}\beta}D_\mu\psi^A{}_\beta + D_\mu(\dots) \\ &+ g\sqrt{2}\tilde{\psi}_{\dot{\alpha}A}[\phi^{AB},\tilde{\psi}^{\dot{\alpha}}{}_B] - g\sqrt{2}\psi^{\alpha A}[\bar{\phi}_{AB},\psi^B{}_\alpha], \end{aligned} \tag{6.18}$$

which results in four Majorana spinors where the second term in the rhs of (6.18) is a total derivative in four dimensional spacetime. Then the lagrangian density of $\mathcal{N}=4$ SYM in four dimensions up to a total derivative is

$$\begin{aligned} \mathcal{L}_4 &= -\frac{1}{4}F^{a\mu\nu}F^a{}_{\mu\nu} - \frac{1}{4}D_\mu\phi^{aAB}D^\mu\bar{\phi}^a{}_{AB} - i\tilde{\psi}^a_{\dot{\alpha}A}\bar{\sigma}^{\mu\dot{\alpha}\beta}D_\mu\psi^{aA}_\beta \\ &+ \operatorname{Tr}\left[\frac{g^2}{16}[\phi^{AB},\phi^{CD}][\bar{\phi}_{AB},\bar{\phi}_{CD}] - \frac{g}{\sqrt{2}}\tilde{\psi}_{\dot{\alpha}A}[\phi^{AB},\tilde{\psi}^{\dot{\alpha}}{}_B] + \frac{g}{\sqrt{2}}\psi^{\alpha A}[\bar{\phi}_{AB},\psi^B{}_\alpha]\right]. \end{aligned} \tag{6.19}$$

It is worth to notice that all the fields are in the adjoint representation of the gauge group. Since we have $SU(4)$ global symmetry, the six scalars ϕ^{aAB} transform in the **6** representation of $SU(4)$, the spinors ψ^{Aa} and $\tilde{\psi}^a_A$ transform in the **4** and $\bar{\mathbf{4}}$ representation of $SU(4)$, respectively.

In order to use the tools from the previous section, it is better to write down the lagrangian of the theory in a supersymmetry manifest way. So with this goal we write the $\mathcal{N}=4$ SYM lagrangian density in terms of the $\mathcal{N}=1$ SYM field theory, as follows: By taking one $\mathcal{N}=1$ vector superfield V and three $\mathcal{N}=1$ chiral superfields Φ_i all in the adjoint representation of the gauge group, where the six real scalars are joined in the three complex scalars of Φ_i. With this formulation we have $SU(3)\times U(1)$ subgroup of the $SU(4)$ global symmetry is manifest, with the $SU(3)$ rotating the chiral superfields Φ_i. So the lagrangian $\mathcal{L}_4$ is

$$\begin{aligned}\mathcal{L} &= \int d^4\theta\bar{\Phi}_i e^{2gV}\Phi_i + \frac{1}{16g^2}\Big(\int d^2\theta(W^aW^a) + h.c\Big)\\ &+ \Big(-\frac{ig\sqrt{2}}{3!}\int d^2\theta\epsilon_{ijk}\,\mathrm{Tr}(\Phi^i[\Phi^j,\Phi^k]) + h.c.\Big),\end{aligned} \tag{6.20}$$

where we have chosen canonical normalization for the kinetics terms of the Φ_i. Chosing the Wess-Zumino gauge, $V^2=0$, we get

$$\begin{aligned}\mathcal{L}_4 &= -\frac{1}{4}F^a_{\mu\nu}F^a{}_{\mu\nu} - i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda^a - i\bar{\psi}^a_i\bar{\sigma}^\mu D_\mu\psi^a_i - D_\mu A^a_i D^\mu A^{*a}_i\\ &+ \frac{g^2}{2}f_{abc}f_{ade}A^{*b}_i A^c_i A^{*d}_j A^e_j - \sqrt{2}gf_{abc}\big[A^{*a}_i\lambda^c\psi^b_i - \bar{\psi}^a_i\bar{\lambda}^b A^c_i\big]\\ &- \frac{g}{\sqrt{2}}\epsilon_{ijk}f_{abc}\big[\psi^c_i\psi^a_j A^b_k + \bar{\psi}^c_i\bar{\psi}^a_j A^{*b}_k\big]\\ &- \frac{g^2}{2}\epsilon_{ijk}\epsilon_{ilm}\big(f_{abc}A^a_j A^b_k\big)\big(f_{cde}A^{*d}_l A^{*e}_m\big).\end{aligned} \tag{6.21}$$

By the end, we have in (6.20) a $\mathcal{N}=1$ SYM theory with superpotential $\mathcal{W}$ given by

$$\mathcal{W} = -i\frac{g\sqrt{2}}{3!}\epsilon_{ijk}\,\mathrm{Tr}(\Phi^i[\Phi^j,\Phi^k]). \tag{6.22}$$

$\mathcal{N}=4$ SYM is marginal

It was shown that the gauge β-function for the $\mathcal{N}=4$ SYM field theory vanishes up to three loops [20], and since the gauge β-function is proportional to the trace anomaly $T^\mu{}_\mu \sim \beta(g)$, [21] it was argued that $\mathcal{N}=4$ SYM could be a conformal invariant theory, i.e. it has a vanishing gauge β-function to all orders.

Many attempts were done in order to check that $\mathcal{N}=4$ SYM is a conformal field theory. The first attempt was done by Sohnius and West [22]. They used anomaly arguments and what they believed a justifiable initial assumptions. The second one was done by Mandelstam [23], by using a light-cone superspace formulation. Nevertheless these attempts used methods which do not posses all of the symmetries of the theory. With this in mind P. L. White [24] used the method of Batalin and Vilkovisky and spectral sequences showing that all symmetries are preserved after quantization, and, therefore, there is no conformal anomaly.

Since we have used $\mathcal{N}=1$ language in order to describe $\mathcal{N}=4$ field theory, we can see this field theory as a subset of couplings, namely, the gauge coupling, in $\mathcal{N}=1$ SYM field theory, . Let us for a moment relax the coupling in the superpotential, making $h=g$ we recover $\mathcal{N}=4$ SYM, so the scaling coefficients are

$$A_g = 3C_2(G)\gamma \propto A_h = -\frac{3}{2}\gamma, \tag{6.23}$$

which imposes one condition on two couplings, namely $\gamma(g,h)=0$. For $g \gg h$ the superpotential is negligible and the theory is infrared(IR) free, so this corresponds to the statement that $\gamma(g \gg h)$ is positive. For $g \ll h$ we have a pure scalar theory which is IR free and has a Laundau pole, which correponds to have a negative $\gamma(g \ll h)$. If $\gamma(g,h)$ is continuous , these two regions must be separated by a curve where γ vanishes, which corresponds to $g=h$. Such behaviour is shown in Figure 1

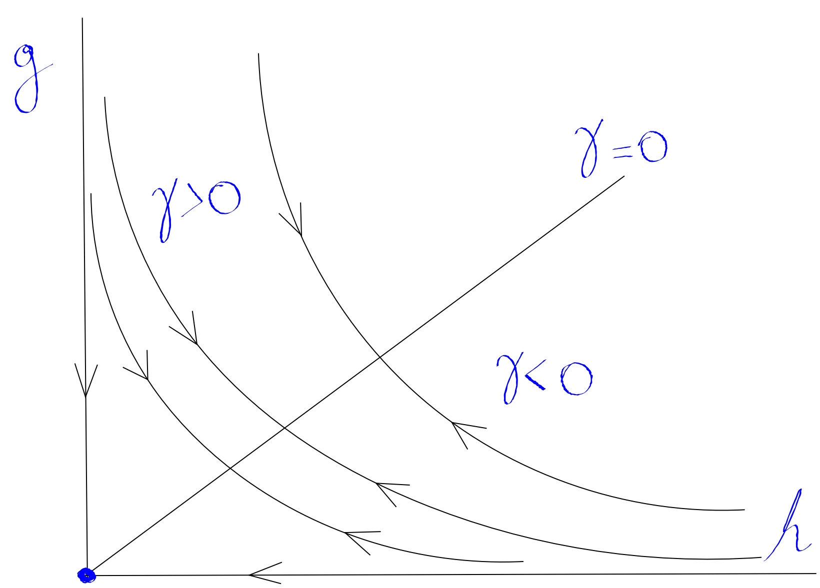

Figure 6.1: RG flow near the $\mathcal{N}=4$ SYM fixed line, arrows indicate flow toward the IR, Fig. taken from [6].

By the end we have used $\mathcal{N}=1$ arguments to prove the existence of a fixed line for $\mathcal{N}=4$ SYM field theory.

6.2.2 Marginal deformation of the $\mathcal{N}=4$ SYM

Previouly, we have written down the $\mathcal{N}=4$ SYM field theory in terms of $\mathcal{N}=1$ one, with superpotential

$$\mathcal{W}=-i\frac{g\sqrt{2}}{3!}\epsilon_{ijk}\,\mathrm{Tr}(\Phi^i[\Phi^j,\Phi^k])=\frac{g\sqrt{2}}{3!}f_{abc}\epsilon_{ijk}(\Phi^i_a\Phi^j_b\Phi^k_c). \tag{6.24}$$

Typically we can deform the superpotential (6.24) by adding to it,

$$\delta\mathcal{W}=\frac{h_1\sqrt{2}}{3!}f_{abc}\epsilon_{ijk}(\Phi^i_a\Phi^j_b\Phi^k_c)+\frac{h_{ijk}}{3!}d_{abc}(\Phi^i_a\Phi^j_b\Phi^k_c), \tag{6.25}$$

where $d_{abc}=\mathrm{Tr}[T_a\{T_b,T_c\}]$ and h_{ijk} is totally symmetric, at the end we have 12 couplings constants (g,h_1,h_{ijk}). The deformation (6.25) is marginal at classical level. The gauge groups in which d_{abc} is not vanishing are $SU(N\geq 3)$ and E_8. From here we discuss only the $SU(N)$ case, with $N\geq 3$. The superpotential (6.24) goes to

$$\mathcal{W} \;=\; \frac{g\sqrt{2}}{3!}f_{abc}\epsilon_{ijk}(\Phi^i_a\Phi^j_b\Phi^k_c)\mapsto(\sqrt{2}\lambda f_{abc}\epsilon_{ijk}+h_{ijk}d_{abc})\frac{1}{6}\Phi^i_a\Phi^j_b\Phi^k_c, \tag{6.26}$$

with $\lambda=g+h_1$, making $H^{ijk}_{abc}=\sqrt{2}\lambda f_{abc}\epsilon_{ijk}+h_{ijk}d_{abc}$, in such way that (6.26) can be written as

$$\tilde{\mathcal{W}}=\frac{1}{6}H^{ijk}_{abc}\Phi^i_a\Phi^j_b\Phi^k_c. \tag{6.27}$$

We are wondering in the conditions in which (6.27) is marginal at one-loop, the β-function for the Yukawa coupling H^{ijk} is proportional to the anomalous dimension of the chiral superfields, i.e. $\beta^{ijk}_H=H^{q(ij}\gamma^{k)}_q$, as we have from the analysis of the supercurrent anomaly. At one-loop level the anomalous dimension is

$$16\pi^2\gamma^{(1)m}_n=-P^m_n, \qquad \text{with} \qquad P^m_n=-\frac{1}{2}H^{mkl}H_{nkl}+2g^2T(R_i)^m_n, \tag{6.28}$$

in the case of one-loop gauge β-function, if we use holomorphic normalization of the field, is $16\pi^2\beta^{(1)}_g=-g^3b_0$ with $b_0=0$ in the case with three chiral superfields, but this result is obscure. Working with canonical normalization for the fields the exact gauge β-function is given by the NSVZ β-function, see appendix A for account, is

$$\beta_g=\frac{g_c^3}{16\pi^2}\frac{\sum_i T(R_i)\gamma_i}{1-\frac{g_c^2}{8\pi^2}C_2(G)}\propto\mathrm{Tr}\,\gamma. \tag{6.29}$$

So to have a marginal deformation we have to impose $\mathrm{Tr}\,\gamma = 0$ and $\beta_H^{ijk} = 0$, giving us 11 conditions. This way, we can expect at least a one dimensional manifold of the fixed point. As for the latter case with $h_1 = 0$ and $h_{ijk} = 0$, i.e for undeformed $\mathcal{N} = 4$ SYM field theory, this one dimensional manifold could be parametrized by the gauge coupling. This conclusion, though, is not compatible with the discussion given by Leigh and Strassler. So let us turn back to the pertubative analysis. After some computations, the one-loop anomalous dimension of the chiral superfields is

$$\gamma^{(1)ai}_{bj} = -\frac{1}{16\pi^2}\Big(2C_2(G)(\lambda^2 - g^2)\delta_{ij} + \frac{N^2-4}{N}\kappa^3 h_{ikl}\bar{h}_{jkl}\Big)\delta^a_b, \tag{6.30}$$

and the β-function for the Yukawa coupling is

$$\beta^{(1)ijk}_{abc} = -\frac{1}{16\pi^2}\Big\{6C_2(G)(\lambda^2 - g^2)H^{ijk}_{abc} + \frac{N^2-4}{N}\kappa^3\big(\sqrt{2}\lambda f_{abc}\tilde{h}_{ijk} + d_{abc}h^{(3)}_{ijk}\big)\Big\}, \tag{6.31}$$

where

$$\tilde{h}_{ijk} \;=\; \epsilon_{p(ij}h^{(2)}_{k)p} \qquad \text{with} \qquad h^{(2)}_{ip} = h_{ijk}\bar{h}_{pjk}, \tag{6.32}$$

and

$$h^{(3)}_{ijk} \;=\; \bar{h}_{pmn}(h_{pij}h_{kmn} + h_{pjk}h_{imn} + h_{pki}h_{jmn}). \tag{6.33}$$

We see that $\tilde{h}_{ijk} = \mathrm{Tr}(h^{(2)})\epsilon_{ijk}$ and $h^{(2)}_{ip}$ is hermitian, so that we can write (6.31) as

$$\beta^{(1)ijk}_{abc} = -\sqrt{2}\beta^{(1)}_{\lambda}\epsilon_{ijk}f_{abc} - \beta^{(1)}_{ijk}d_{abd}, \tag{6.34}$$

where

$$\begin{aligned}\beta^{(1)}_{\lambda} &= \frac{\lambda}{16\pi^2}\Big\{6C_2(G)(\lambda^2 - g^2) + \frac{N^2-4}{N}\kappa^3\,\mathrm{Tr}(h^{(2)})\Big\} \qquad \text{and,}\\ \beta^{(1)}_{ijk} &= \frac{1}{16\pi^2}\Big\{6C_2(G)(\lambda^2 - g^2)h_{ijk} + \frac{N^2-4}{N}\kappa^3 h^{(3)}_{ijk}\Big\}.\end{aligned} \tag{6.35}$$

If we make a reparametrization of the couplings g, λ, and h_{ijk} [18]. The NSVZ function and the β-function for λ, (6.35), simplify to an ordinary differential equation

$$\Big(\frac{1}{g^3} - \frac{2}{g}\Big)dg = -\frac{d\lambda}{\lambda},$$

which gives us the following flow

$$\frac{\lambda}{\lambda_0} = \frac{g^2}{g_0^2}\frac{e^{\frac{1}{4g^2}}}{e^{\frac{1}{4g_0^2}}}, \tag{6.36}$$

where $\lambda_0 = \lambda(\mu_0)$, so we can say that at UV limit we will not have an asymptotically free theory.

In order to have fixed points we demanded $\mathrm{Tr}\,\gamma = 0$. Then from (6.30) we have the condition

$$\mathrm{Tr}(h^{(2)}) = -6(\lambda^2 - g^2). \tag{6.37}$$

Substituting this into $\beta^{(1)}_{ijk}$, we get $3\,\mathrm{Tr}\,(h^{(2)})^2 = \big(\mathrm{Tr}\,h^{(2)}\big)^2$ [18]. Due to the hermitian propert of $h^{(2)}$ we may conclude that $h^{(2)}_{ij} = \upsilon^2\delta_{ij}$, which implies that we have the non-vanishing h_{ijk} couplings

$$h_{ijk} = \begin{cases} h & \text{if } i=j=k, \\ h' & \text{if } i\neq j\neq k\neq i, \\ 0 & \text{otherwise,} \end{cases}$$

and the anomalous dimension matrix γ , eq. (6.30), to be proportional to identity matrix. With this choice of the couplings h_{ijk} we get

$$\frac{\beta^{(1)}_{\lambda}}{\lambda} = \frac{\beta^{(1)}_{ijk}}{h_{ijk}}. \tag{6.38}$$

On our search for fixed point, we have only one condition on four couplings, and thus we have a three dimensional manifold of fixed points in the coupling constants space [6].

At $\mathrm{Tr}\,\gamma = 0$, we get that for a fixed point

$$2(\lambda^2 - g^2) + \upsilon^2 = 0 \Longrightarrow g^2 = \frac{1}{2}\big(h^2 + 2(\lambda^2 + h'^2)\big). \tag{6.39}$$

From this condition we see that in the IR limit, if we increase one of the couplings λ, h_{ijk}, we have that $2(\lambda^2 - g^2) + \upsilon^2 > 0$. Thus decreasing these couplings and increasing the gauge coupling in IR, we get $\mathrm{Tr}\,\gamma = 0$. Then we can say that in the weak coupling limit all fixed points that exist imply diagonal γ and are IR stable [6].

By the end, the superpotential for the deformed $\mathcal{N}=4$ SYM, eq. (6.26), is

$$\tilde{\mathcal{W}} = \Big[\sqrt{2}\lambda f_{abc} + h' d_{abc}\Big]\Phi^1_a\Phi^2_b\Phi^3_c + \frac{h}{6} d_{abc}\sum_{i=1}^{3}\Phi^i_a\Phi^i_b\Phi^i_c, \tag{6.40}$$

which is the superpotential written by Leigh and Strassler [6].

The β-deformation of the $\mathcal{N}=4$ SYM field theory

Since we got the superpotential written by Leigh and Strassler [6], let us make the reparametrization

$$-i\sqrt{2}\lambda = \frac{\tilde{h}}{2}(q+\bar{q}) \qquad \text{and} \qquad h' = \frac{\tilde{h}}{2}(q-\bar{q}), \tag{6.41}$$

with q being a complex number, which can be written as $q = e^{i\pi\beta}$. The lagrangian for the deformed $\mathcal{N}=4$ SYM is

$$\begin{aligned}\tilde{\mathcal{L}}_4 &= \int d^4\theta \bar{\Phi}_i e^{2gV}\Phi_i + \frac{1}{16g^2}\Big(\int d^2\theta (W^a W^a) + h.c\Big) \\ &+ \Big(\tilde{h}\int d^2\theta \, \mathrm{Tr}(q\Phi_1\Phi_2\Phi_3 - \bar{q}\Phi_1\Phi_3\Phi_2) + \frac{h}{3}\int d^2\theta \, \mathrm{Tr}\sum_{i=1}^{3}\Phi_i^3 + h.c.\Big).\end{aligned} \tag{6.42}$$

This is the lagrangian for the full Leigh-Strassler $\mathcal{N}=1$ deformation of the $\mathcal{N}=4$ SYM theory, which is invariant under the following transformation $\Phi^1 \to \Phi^2, \Phi^2 \to \Phi^3, \Phi^3 \to \Phi^1$ and $\Phi^1 \to \Phi^1, \Phi^2 \to \omega\Phi^2, \Phi^3 \to \omega^2\Phi^3$ where ω is a cubic root of unity.

From now on we will turn off h in (6.42). As a consequence, we have the so-called β-deformation, whose gravity dual has been found by Lunin and Maldacena,[7] . So we have that the resulting theory preserves $\mathcal{N}=1$ supersymmetry and a $U(1)\times U(1)$ non-R symmetry

$$\begin{aligned} U(1)_1 &: (\Phi_1,\Phi_2,\Phi_3) \to (\Phi_1, e^{i\varphi_1}\Phi_2, e^{-i\varphi_1}\Phi_3) \\ U(1)_2 &: (\Phi_1,\Phi_2,\Phi_3) \to (e^{-i\varphi_2}\Phi_1, e^{i\varphi_2}\Phi_2, \Phi_3). \end{aligned} \tag{6.43}$$

The deformation can be seen as a new definition of the product fields in the lagrangian

$$\Phi_i\Phi_j \mapsto \Phi_i * \Phi_j \equiv e^{i\pi\beta(Q_1^iQ_2^j - Q_2^iQ_1^j)}\Phi_i\Phi_j, \tag{6.44}$$

where (Q_1, Q_2) are the $U(1)_1 \times U(1)_2$ charges of the fields.

In the β-deformed theory, it was checked [25] that in the subspace where $\gamma^{(1)} = 0$, the anomalous dimensions $\gamma^{(2)}$ and $\gamma^{(3)}$ vanish. Nevertheless, one thing must be said about $\gamma^{(3)}$. This anomalous dimension does not vanish when non-planar diagrams are taken into account and it depends on the scheme that we have chosen to compute.

So, in general, the anomalous dimension γ is not known beyond three-loops in perturbation theory. However it was showed [26] that in the planar limit the condition on the couplings for the exact invariance of the β-deformed theory is $|\tilde{h}|^2 = g^2$. In the context of the AdS/CFT correspondence, this is the field theory described by the supergravity dual found by Lunin and Maldacena, which we will summarize in the next section. Strong coupling phase is described by the supergravity dual found by Lunin and Maldacena, which we summarized in the next section.

6.3 The gravity dual of marginal deformation of $\mathcal{N}=4$ SYM

We are going to finish this chapter discussing the gravity dual for the well-known β-deformation of the $\mathcal{N}=4$ SYM which was found by Lunin-Maldacena [7]. At the end we reviewed the Frolov's technique [27], which in fact he give a non-supersymmetric generalization of the Lunin-Maldacena solution.

6.3.1 An $SL(2,R)$ transfomation

The construction of the gravity dual of the β-deformed field theories is based on the following.

Let us consider that we know the gravity dual of the undeformed theory and that this geometry have $U(1)\times U(1)$ isometries. This means that the geometry contains a 2-torus parametrized by (φ_1, φ_2) coordinates. At the end, we have a two torus fibered on a eight-dimensional manifold. Let the real components of the metric and the B field on the 2-torus be denoted by g_{ij} and b_{12}, with $i,j=1,2$, such that the holographic description of the β-deformed theory is given by

$$\tau \equiv b_{12} + i\sqrt{g} \rightarrow \tau' = \frac{\tau}{1+\beta\tau}, \tag{6.45}$$

which is a transfomation of the Kälher modulus τ of the 2-torus, where $\sqrt{g}$ is the volume of the torus.

We see that equation (6.45) is an element of $SL(2,R)$. And it can be seen the equation (6.45) as the result of doing a T-duality on an one circle, a change of coordinates, followed by another T-duality, i.e. TsT tranformation, which is the Frolov's argument [27]. This $SL(2,R)$ tranformation leaves invariant the eight dimensional gravity theory, and it produces a non-singular metric if the original was not [7]. As an example, take the metric of $\mathbb{R}^4$

$$ds^2 = \sum_{i=1}^{2}(d\rho_i^2 + \rho_i^2 d\varphi_i^2). \tag{6.46}$$

From this, we have that the Kälher modulus is $\tau = i\rho_1\rho_2$ and after applying (6.45) to it, we have

$$\begin{aligned} ds^2 &= \sum_i d\rho_i^2 + \frac{1}{1+\beta^2\rho_1^2\rho_2^2}\sum_i \rho_i d\varphi_i^2, \\ b_{12} &= \frac{\beta\rho_1^2\rho_2^2}{1+\beta^2\rho_1^2\rho_2^2}, \\ e^{2\phi} &= e^{2\phi_0}\frac{1}{1+\beta^2\rho_1^2\rho_2^2} \end{aligned} \tag{6.47}$$

where the change of the dilation field ϕ_0 is due to that equation (6.45) leaves the eight-dimensional geometry invariant, not the ten-dimensional one.

6.3.2 The Lunin-Maldacena solution

In this section we review the exact solution found by Lunin and Maldacena [7] for deformations which preserve $U(1)\times U(1)$ global symmetry. They applied the method from the previous section to $AdS_5 \times S^5$ background, in particular to 2-torus inside the S^5 geometry.

Let us write the metric of S^5 like

$$\frac{ds^2}{R^2} = \sum_{i=1}^{3}(d\mu_i^2 + \mu_i^2 d\phi_i^2) \quad \text{with} \quad \sum_i \mu_i^2 = 1 \quad \text{and} \quad \mu_1 = c_\alpha, \mu_2 = s_\alpha c_\theta, \mu_3 = s_\alpha s_\theta, \tag{6.48}$$

where $s_\alpha = \sin\alpha$, $c_\alpha = \cos\alpha$ and so on. Let $\phi_1 = \psi - \varphi_2$, $\phi_2 = \psi + \varphi_1 + \varphi_2$ and $\phi_3 = \psi - \varphi_1$ in such way that the two $U(1)$ isometries act by shifting φ_1, φ_2. From this the τ parameter of the 2-torus is

$$\tau = i\sqrt{g} = iR^2[s_\alpha^2(c_\alpha^2 + s_\alpha^2 s_\theta^2 c_\theta^2)]^{1/2} = iR^2(\mu_1^2\mu_2^2 + \mu_1^2\mu_3^2 + \mu_2^2\mu_3^2)^{1/2}, \tag{6.49}$$

where $R^4 = 4\pi e^{\phi_0}N$. After doing the transformation (6.45), we have the gravity dual of the deformed theory,

$$\begin{aligned}
\frac{ds^2_{str}}{R^2} &= ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \widehat{\beta}^2 G\mu_1^2\mu_2^2\mu_3^2 \big(\sum_i d\phi_i\big)^2 && (6.50)\\
G^{-1} &= 1 + \widehat{\beta}^2(\mu_1^2\mu_2^2 + \mu_1^2\mu_3^2 + \mu_2^2\mu_3^2), \qquad \widehat{\beta} = R^2\beta, && (6.51)\\
e^{2\phi} &= e^{2\phi_0} G\\
B^{NS} &= \widehat{\beta} R^2 G(\mu_1^2\mu_2^2 d\phi_1 \wedge d\phi_2 + \mu_1^2\mu_3^2 d\phi_3 \wedge d\phi_1 + \mu_2^2\mu_3^2 d\phi_2 \wedge d\phi_3)\\
C_2 &= -3\beta(16\pi N) w_1 \wedge d\psi, \quad \text{with} \qquad dw_1 = c_\alpha s_\alpha^3 s_\theta c_\theta d\alpha \wedge d\theta\\
C_4 &= (16\pi N)(w_4 + G w_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3), \qquad \omega_{AdS_5} = dw_4\\
F_5 &= (16\pi N)(\omega_{AdS_5} + G\omega_{S^5}), \qquad \omega_{S^5} = dw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3,
\end{aligned}$$

where ω_{S^5} is the volume element of a unit radius S^5. Since β is real the axion field C_0 is constant and it is set to zero. The solution presented by Lunin and Maldacena has small curvature as long as

$$R\beta \gg 1, \qquad\qquad R \gg 1, \tag{6.52}$$

where the first inequality can be understood as the condition that the 2-torus does not become smaller than the string scale after the transformation. We see that at $\beta = 0$, one immediately recovers the $AdS_5 \times S^5$ background.

Moreover it was demostrated [28] that there is a relation between the NS-NS and RR fields of the Lunin-Maldacena solution. This relations establish that RR 3-form dC_2 is the Hodge dyal to the B^{NS}

$$B^{NS}_{ij} = c\epsilon_{ij}{}^{klm}\partial_k C_{lm}, \tag{6.53}$$

where c is some coefficient that in the full nonlinear solution becomes a function, but at the linearized level is a constant

$$c = -\frac{R^4}{16\pi N}. \tag{6.54}$$

The Frolov argument

Nevertheless it was noted by Frolov [27] that the solution (6.50) can be obtained from the original $AdS_5 \times S^5$ by performing a series of three TsT (T-duality, shift, T-duality) transformation to the three tori of S^5. In fact they studied a non-supersymmetric generalization of Lunin-Maldacena background by performing TsT transfomations on each of the three tori of S^5 but with different shift γ_i. After all, at equal γ_i, the deformation reduces to the Lunin-Maldacena one.

In order to recover the Lunin-Maldacena solution, Frolov parametrized the sphere S^5 in such way that its metric is

$$\frac{ds^2}{R^2} = \sum_{i=1}^{3} d\mu_i^2 + g_{ij} d\varphi_i^2 d\varphi_j^2, \tag{6.55}$$

where φ_i are the same coordinates that were used in (6.50), with $\varphi_3 = \psi$ and g_{ij} is given by

$$g_{ij} = \begin{pmatrix} \mu_2^2 + \mu_3^2 & \mu_2^2 & \mu_2^2 - \mu_3^2 \\ \mu_2^2 & \mu_1^2 + \mu_2^2 & \mu_2^2 - \mu_1^2 \\ \mu_2^2 - \mu_3^2 & \mu_2^2 - \mu_1^2 & 1 \end{pmatrix}. \tag{6.56}$$

The first TsT transfomation is applied on the pair (φ_1, φ_2) and it runs as follow. First we apply T-duality transformation, which is briefly described in the appendix B, on the circle parametrized by φ_1. We get T-tranformed metric $\tilde{g}_{ij}$

$$\tilde{g}_{ij} = \begin{pmatrix} \frac{1}{\mu_2^2+\mu_3^2} & 0 & 0 \\ 0 & \frac{\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2}{\mu_2^2+\mu_3^2} & \frac{2\mu_2^2\mu_3^2-\mu_1^2\mu_2^2-\mu_1^2\mu_3^2}{\mu_2^2+\mu_3^2} \\ 0 & \frac{2\mu_2^2\mu_3^2-\mu_1^2\mu_2^2-\mu_1^2\mu_3^2}{\mu_2^2+\mu_3^2} & 1 - \frac{(\mu_2^2-\mu_3^2)^2}{\mu_2^2+\mu_3^2} \end{pmatrix}, \tag{6.57}$$

and the skew-symmetric B-field b_{ij}

$$\tilde{b}_{ij} = \begin{pmatrix} 0 & \frac{\mu_2^2}{\mu_2^2+\mu_3^2} & \frac{\mu_2^2-\mu_3^2}{\mu_2^2+\mu_3^2} \\ -\frac{\mu_2^2}{\mu_2^2+\mu_3^2} & 0 & 0 \\ -\frac{\mu_2^2-\mu_3^2}{\mu_2^2+\mu_3^2} & 0 & 0 \end{pmatrix}. \tag{6.58}$$

Next, we make the shift on the angle φ_2

$$\varphi_2 \to \varphi_2 + \beta\varphi_1, \tag{6.59}$$

where β is any constant. The final step is to make again the T-duality transformation on the circle parametrized by φ_1. So if we apply TsT transformation on the pairs (φ_2, φ_3) and (φ_3, φ_1) at the end we will have Lunin-Maldacena solution for the NS-NS sector.

In order to recover the Lunin-Maldacena solution for the deformed fields strength and RR potentials , Frolov used the results from [29], which were briefly described in the appendix B, where the Γ matrix is given by

$$\Gamma_{mn} = \begin{pmatrix} 0 & \beta & -\beta \\ -\beta & 0 & \beta \\ \beta & -\beta & 0 \end{pmatrix}, \tag{6.60}$$

the deformed fields strenght $\tilde{F}$ are calculated from

$$\widetilde{F} = \exp(\frac{1}{2}\Gamma_{mn}\imath_m\imath_n)F_5 \qquad \text{with} \qquad F_5 = (16\pi N)(\omega_{AdS_5} + \omega_{S^5}), \tag{6.61}$$

where ω_{AdS_5} and $\omega_{S^5} = c_\alpha s_\alpha^3 s_\theta c_\theta d\alpha \wedge d\theta \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3$ are the volume forms for AdS_5 and S^5 subspaces of unit radius. After expanding the exponential , we see that the field strength $\widetilde{F}_1 = d\widetilde{\chi} = 0$, so we say χ is a pure gauge field, the deformed fields strength $\widetilde{F}_3$ and $\tilde{F}_5$ are

$$\begin{aligned} \widetilde{F}_3 &= \beta(\imath_1\imath_2 - \imath_1\imath_3 + \imath_2\imath_3)F_5 = -16\pi N\beta dw_1 \wedge \sum_i d\phi_i, \\ \tilde{F}_5 &= 16\pi N(\omega_{AdS_5} + \widetilde{\omega}_{S^5}), \end{aligned} \tag{6.62}$$

where

$$\begin{aligned} dw_1 &= s_\alpha^3 c_\alpha s_\theta c_\theta d\alpha \wedge d\theta, \\ \widetilde{\omega}_{AdS_5} &= \sqrt{\widetilde{G}_{AdS_5}} d\alpha \wedge d\theta \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 = G\omega_{AdS_5}. \end{aligned} \tag{6.63}$$

The deformed R-R fields, $\widetilde{C}_2$ and $\widetilde{C}_4$ follows directly from

$$\begin{aligned} \widetilde{F}_3 &= d\widetilde{C}_2 + \widetilde{H}_3 \wedge \widetilde{C}_0, \\ d\widetilde{C}_4 &= \widetilde{F}_5 + \widetilde{C}_2 \wedge d\widetilde{B}_2, \end{aligned}$$

setting $C_0 = \chi = 0$, the $\widetilde{C}_2$ and $\widetilde{C}_4$ fields are

$$\begin{aligned} \widetilde{C}_2 &= -16\pi N\beta w_1 \wedge \sum_i d\phi_i \\ \widetilde{C}_4 &= 16\pi N(w_4 + Gw_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3) \quad \text{with} \quad \omega_{AdS_5} = dw_4. \end{aligned} \tag{6.64}$$

If we allow the TsT transformation to have different shift parameters β_i we get a non-supersymmetric deformation of $AdS_5 \times S^5$. This was done by Frolov in [27].

Chapter 7

Conclusions and future perspectives

By studying the orbit solution of the stabilizer subgroup of a point on the boundary of the AdS_5 space, we found is that the orbit solution is a classical solution for a D-brane, and we argue that this is our novel result, since it was not given in any reference.

In order to get a field theory with scale independence the β-functions of the couplings cosntant of this theory must vanish. In particular for the case of supersymmetric gauge theories with matter fields, the gauge β-function is proportional to the anomalous mass dimension of the fields of the theory, in order to get a manifold of fixed points we demand the vanishing of the anomalous mass dimension in all orders in perturbation theory.

The Lunin Maldacena solution can also derived by applying a three series of TsT transformation with equal β on the three tori of S^5.

As future perspectives, we intend to make a systematic study of the intersection of our D-brane solution with boundary of the AdS space, since it was done for classical string in [2, 3, 4, 5]. Also we hope to find classical solutions for D-branes in β-deformed AdS space

Appendix A

The Wilsonian approach

In this appendix we make a brief review of the notion of Wilsonian Renormalization Group(RG), this approach appears as another tool, besides the well known 1PI correlation functions, for studying the quantum effective action. We basically follow the order presented in [30].

A Wilsonian effective action has the following generic aspect

$$S(\mu) = \int d^d x \sum_i \lambda_i^0(\mu)\Phi_i, \tag{A.1}$$

where Φ_i label an infinite set of local operators.

Normally a field theory is defined when we specify the bare parameters λ_i^0 at some cutoff scale μ, so all Green functions can be calculated as functions of λ_i^0 and μ. Let $|p_i| \sim \mu_0 \leq \mu$ be the typical momenta of the incoming particles in the process. We refer to these amplitudes as $\Gamma(p_i; \lambda_i^0, \mu)$.

Fixing these amplitudes to experimental values is a practical means of fixing the values of the couplings $\lambda_i^0(\mu)$. The computation of $\Gamma(p_i; \lambda_i^0, \mu)$ involves integrating over loop momenta q_μ in the range $\mu_0 \leq |q_\mu| \leq \mu$. For example for a $\lambda^0\phi^4/4!$ theory with cutoff μ, the 1PI 4-point function $\Gamma^4(\mu_0; \lambda^0, \mu)$ is given by, [31]

$$\Gamma^4(\mu_0; \lambda^0, \mu) = \lambda^0 + \frac{3}{16\pi^2}\lambda^{02}\left[\ln\frac{\mu_0}{\mu} - \frac{1}{2}\right] + \mathcal{O}(\lambda'^{\in}). \tag{A.2}$$

As we can see, at $\mu_0 \ll \mu$ the logarithm grows larger than tree level, which means that the one-loop term becomes comparable to the tree-level piece, making perturbation theory unreliable.

So it is convenient to define the theory at a different scale $\mu' < \mu$ such that tree-level results better approximate the physics. This is simple. It is possible to change the bare couplings λ_i^0 with the cutoff μ such that we keep the low-energy physics fixed.

$$\Gamma(\mu_0; \lambda^0(\mu), \mu) = \Gamma(\mu_0; \lambda^0(\mu'), \mu'). \tag{A.3}$$

The dependence of λ_i^0 on the defining scale μ, while keeping the low-energy physics fixed, is encoded in

$$\frac{d}{d\ln\mu}\lambda_i^0 = \beta_i^0(\lambda^0). \tag{A.4}$$

As an example, consider the perturbed free scalar field theory

$$S[\phi]_{free} = \int d^d x \frac{1}{2}\partial\phi\partial\phi. \tag{A.5}$$

This action is invariant under $\mu \mapsto \kappa\mu$ and $x \mapsto \kappa^{-1}x$, ϕ must escale as $\phi \mapsto \kappa\phi$, the mass dimension $[\phi] = 1$. Let us consider an operator made out of ϕ and its derivatives such that $\Phi \mapsto \kappa^{d_i}\Phi_i$, where d_i is the classical mass dimension of Φ. The perturbed action is as follows

$$S[\phi; \mu, \lambda_i] = \int d^d x \Big[\frac{Z(\mu)}{2}\partial\phi\partial\phi + \sum_i \lambda_i(\mu)\Phi_i\Big], \tag{A.6}$$

where $Z(\mu)$ is the wave function renomalization of the field ϕ_i. For the action to be invariant under the previous scaling, the coupling constants must have classical dimension $d - d_i$. Let us write $\lambda_i(\mu) = \mu^{d-d_i}\tilde{\lambda}_i(\mu)$ where $\tilde{\lambda}_i(\mu)$ is dimensionless, so that

$$\mu\frac{d\lambda_i}{d\mu} \;=\; \beta_i = (d - d_i)\lambda_i(\mu) + \beta_i^{quant}. \tag{A.7}$$

The limits that are of special importance are $\mu \to \infty$ ultraviolet (UV) and $\mu \to 0$ infrared (IR). If there exists a limit where the couplings do not run away we say that the theory has reached a fixed point. Theories at fixed point are important because they are scale invariant and conformal invariant, even

In the neighbourhood of a fixed point we have $\lambda_i = \lambda_i^* + \delta\lambda_i$, we can linearize the RG flows

$$\mu\frac{\lambda_i}{d\mu}\bigg|_{\lambda_j^*+\delta\lambda_j} = M_{ij}\delta\lambda_j + \mathcal{O}(\delta\lambda_|^{\in}), \tag{A.8}$$

and go to a diagonal basis $\delta\lambda_i \to \delta\tilde{\lambda}_i$, $M_{ij} \to \Delta_i\delta_{ij}$ such that, to linear order the *RG* flow is simply

$$\delta\tilde{\lambda}_i(\mu) = \Big(\frac{\mu}{\mu'}\Big)^{\Delta_i - d}\delta\tilde{\lambda}_i(\mu'), \tag{A.9}$$

the quantity Δ_i is called the scaling dimension of the operator associated to $\delta\tilde{\lambda}_i$. For an interacting Quantum Field Theory(QFT) it will be different. From the classical scaling dimension d_i,

$$\Delta_i = d_i + \gamma_i, \tag{A.10}$$

where $\gamma_i = -\partial \ln Z_i / \partial \ln \mu$ is called the anomalous mass dimension of the operator and its origin is purely quantum.

We can see from (A.9) that nearby to a fixed point, the couplings can be classified in the following way

- Relevant. If $\Delta_i < d$ the coupling decreases towards the UV. So in the vicinity of a UV fixed point all the these couplings vanish.
- Irrelevant. If $\Delta_i > d$ we have that as we approach a IR fixed point all irrelevant couplings disappear, and only relevant couplings remain.
- Marginal. When $\Delta_i = d$ we have that the couplings stay fixed under RG flow, so the β function vanish. In addition, we have to go beyond leading order if the couplings do not run to all orders, we called out as truly marginal coupling.

Let us apply this feautures into supersymmetric gauge theories.

A.1 The Wilsonian approach in supersymmetric gauge theories

Let us now consider the Wilsonian RGE for supersymmetric gauge theories with matter. At some cutoff μ_0 , the theory is specified by the bare lagrangian

$$\begin{aligned} \mathcal{L}(\mu_0) &= \Big(\frac{1}{16}\int d^2\theta \frac{1}{g_h^2(\mu_0)} W^a W^a + \text{h.c.}\Big) + Z(\mu_0)\int d^2\theta d^2\bar{\theta} \sum_i \Phi_i^\dagger e^{2V_i}\Phi_i, \\ &+ \Big(\int d^2\theta \mathcal{W}(\Phi_i;\lambda_i(\mu_0)) + h.c.\Big) \end{aligned} \tag{A.11}$$

where $V_i = V^a T_i^a$ is the vector superfield with T_i^a being the generators in the representation of the chiral superfield Φ_i, $D_{\dot{\alpha}}\Phi = 0$. With $W_\alpha = T^a W^a{}_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-2T^aV^a}D_\alpha e^{2T^aV^a}$ being the superfield strength, $\mathcal{W}(\Phi;\lambda_i(\mu_0))$ is the superpotential, and the holomorphic gauge coupling g_h is given by

$$\frac{1}{g_h^2(\mu_0)} = \frac{1}{g^2(\mu_0)} + i\frac{\Theta}{8\pi^2}. \tag{A.12}$$

It is conventional to set $Z(\mu_0) = 1$. Doing this we say that we have chosen to work with canonical normalization for the bare matter field kinetic terms. We mean by canonical normalization that the coupling constant is not present in front of the kinetic terms, and that it shows up in the covariant derivatives. It is useful taking canonical normalization, because is the conventional choice in field theory and it is useful when we compare lagrangians with different cutoffs with fixed normalization of the bare fields.

Now with this choise of the normalization when we change the cutoff from μ_0 to μ. The following question appears. How should the bare parameters be changed to keep the low energy physics fixed?. The answer is just the same action, but with renormalization gauge coupling $g_h(\mu_0) \to g_h(\mu)$

$$\begin{aligned} \mathcal{L}(\mu) &= \Big(\frac{1}{16}\int d^2\theta \frac{1}{g_h^2(\mu)} W^a W^a + \text{h.c.}\Big) + \int d^2\theta d^2\bar{\theta} \sum_i Z_i(\mu_0, \mu)\Phi_i^\dagger e^{2V_i}\Phi_i \\ &+ \Big(\int d^2\theta \mathcal{W}(\Phi_i; \lambda_i(\mu_0)) + h.c.\Big), \end{aligned} \tag{A.13}$$

where we have used the non-renormalization theorem for the superpotential automatically. Under RG flow $\mu_0 \to \mu < \mu_0$, the holomorphic gauge coupling gets renormalized as

$$\frac{8\pi^2}{g_h^2(\mu)} = \frac{8\pi^2}{g_h^2(\mu_0)} + f\Big(\frac{8\pi^2}{g_h^2(\mu_0)}, \ln\frac{\mu}{\mu_0}\Big), \tag{A.14}$$

where we have used that $f(\cdot,\cdot)$ must be holomorphic in $8\pi^2/g_h^2$, be periodic under shifts in Θ and must satisfy transitivity. With this consideration we get the perturbative running of the coupling constant

$$\frac{1}{g_h^2(\mu)} = \frac{1}{g_h^2(\mu_0)} + \frac{b_0}{8\pi^2}\ln\frac{\mu}{\mu_0}, \tag{A.15}$$

where $b_0 = 3C_2(G) - \sum_i T(R_i)$ is the coefficient of the one-loop beta function in the usual RG equation, with $C_2(G)$ being the quadratic Casimir of the adjoint representation and $T(R_i)$ is the quadratic Casimir of the representation in which Φ_i appears. Finally we can write down the lagrangian (A.13) as

$$\begin{aligned} \mathcal{L}(\mu) &= \Big\{\frac{1}{16}\int d^2\theta\Big(\frac{1}{g_h^2(\mu)} + \frac{b_0}{8\pi^2}\ln\frac{\mu}{\mu_0}\Big) W^a W^a + \text{h.c.}\Big\} \\ &+ \int d^2\theta d^2\bar{\theta}\sum_i Z_i(\mu_0,\mu)\Phi_i^\dagger e^{2V_i}\Phi_i + \Big(\int d^2\theta \mathcal{W}(\Phi_i;\lambda_i(\mu_0)) + h.c.\Big), \end{aligned} \tag{A.16}$$

so we got the one-loop law for the holomorphic gauge coupling constant.

However, as we see above, the change in $1/g_h^2$ is holomorphic only when the normalization for the matter field kinetic terms is allowed to change from 1 to $Z(\mu_0, \mu)$. In order to go back to canonical normalization for the matter fields, we just redefine $\Phi = Z(\mu_0, \mu)^{-1/2}\Phi'$. But the path integral measure $\mathcal{D}\Phi$ is not invariant under this change, i.e.

$$\mathcal{D}(Z(\mu_0, \mu)^{-1/2}\Phi') \neq \mathcal{D}\Phi',$$

and there is an anomalous Jacobian. When we make $Z = e^{i\alpha}$, i.e. a pure phase, the change of variable is just a phase rotation of all components of Φ, and the Jacobian is just the one assoociated with the chiral anomaly. This Jacobian is really known and has the property to be cutoff independent

$$\mathcal{D}(e^{-i\alpha/}\Phi')\mathcal{D}(e^{+i\alpha/}\bar{\Phi}') = \mathcal{D}\Phi'\mathcal{D}\bar{\Phi}' \exp\Big(\frac{1}{16}\int d^4y \int d^2\theta \frac{T(R_i)}{8\pi^2} \ln Z(\mu_0, \mu) W^a W^a\Big). \tag{A.17}$$

Therefore , if we want to keep canonical normalization for the matter fields when we change the cutoff from μ_0 to μ, the lagrangian at cutoff μ must be given by

$$\mathcal{L}'(\mu) = \Big(\frac{1}{16}\int d^2\theta \frac{1}{g_h'^2(\mu)} W^a W^a + \text{h.c.}\Big) + \int d^2\theta d^2\bar{\theta} \sum_i \Phi_i^\dagger e^{2V_i}\Phi_i, \tag{A.18}$$

where

$$\frac{1}{g_h'^2(\mu)} = \frac{1}{g_h^2(\mu_0)} + \frac{b_0}{8\pi^2}\ln\frac{\mu}{\mu_0} - \sum_i \frac{T(R_i)}{8\pi^2}\ln Z_i(\mu_0, \mu), \tag{A.19}$$

where in (A.18) we have turn off the superpotential term.

The canonical gauge coupling: NSVZ beta function

Actually, we have just worked with canonical normalization for the bare kinetic terms for the chiral superfield, leaving the kinetic gauge terms in the holomorphic gauge coupling. In order to write down in canonical normalization, we just make a change in the vector superfield $V_h = g_c V_c$. This means that the measure $\mathcal{D}\mathcal{V}$ will change. We just write the Jacobian and suggest to the reader the appendix of [32] for a quick derivation

$$\begin{aligned} \mathcal{D}V_h &= \mathcal{D}(e^{\ln g_c} V_c) \\ &= \mathcal{D}V_c \exp\Big(\frac{1}{16}\int d^4x d^2\theta \frac{2C_2(G)}{8\pi^2}\ln g_c W^a(g_c V_c) W^a(g_c V_c)\Big), \end{aligned} \tag{A.20}$$

so with the Jacobian (A.20), it is straightforward to get a relationship between the holomorphic and canonical gauge couplings. At a cutoff μ_0 we write down the partition function for pure Yang-Mills theory

$$\mathcal{Z} = \int \mathcal{D}V_c \exp\Big(- \frac{1}{16}\int d^4x d^2\theta \Big(\frac{1}{g_h^2} - \frac{2C_2(G)}{8\pi^2}\ln g_c\Big) W^a(g_cV_c)W^a(g_cV_c) + h.c.\Big). \tag{A.21}$$

In order to have a canonical normalization for the vector multiplet, we must consider that g_cV_c is a real vector superfield, and g_c must also be real. With this in mind, we get

$$\frac{1}{g_c^2} = \Re(\frac{1}{g_h^2}) - \frac{2C_2(G)}{8\pi^2}\ln g_c, \tag{A.22}$$

which is known in the literature as Shifman-Vainshtein formula.

As we said above, going to a lower cutoff μ the difference between ${}^1/_{g_h^2(\mu)}$ and ${}^1/_{g_h^2(\mu_0)}$ is exhausted at one-loop. Using (A.19) and (A.22) we get

$$\begin{aligned} \frac{1}{g_c^2(\mu)} + \frac{2C_2(G)}{8\pi^2}\ln g_c(\mu) &= \frac{1}{g_c^2(\mu_0)} + \frac{2C_2(G)}{8\pi^2}\ln g_c(\mu) + \frac{b_0}{8\pi^2}\ln\frac{\mu}{\mu_0} \\ &- \sum_i \frac{T(R_i)}{8\pi^2}\ln Z_i(\mu_0,\mu). \end{aligned} \tag{A.23}$$

After applying in both sides $\mu_0{}^{\partial}/_{\partial\mu_0}$, we get the Novikov-Shifman-Vainshtein and Zakharov (NSVZ) β function

$$\beta(g_c) = -\frac{g_c^3}{16\pi^2}\frac{3C_2(G) - \sum_i T(R_i)(1+\gamma_i)}{1 - \frac{g_c^2}{8\pi^2}C_2(G)}. \tag{A.24}$$

This β-function was calculated for the first time by Novikov et. al. [33], even though theirs results look somewhat osbscure. With this in mind Arkani et al [32], rederived this result. Actually this is what we have showed here.

About the Konishi Anomaly

In order to get the supersymmetric generalization of the familiar chiral anomaly, which is called Konishi anomaly, we promote $Z = e^{i\alpha}$ being a global tranformation to a local one. Beside this, we also promote α to be a chiral superfield.

When α is a constant, $\Phi_i = e^{-i\alpha/2}\Phi'_i$ defines by itself a global symmetry of

$$\mathcal{S} = \int d^4x d^2\theta(d^2\bar{\theta}\Phi_i^\dagger e^{2V_i}\Phi_i + \mathcal{W}(\Phi)) \tag{A.25}$$

for a vanishing superpotential $\mathcal{W} = 0$. In the case we promote α to be a chiral superfield, after absorving the factor of two, i.e. we have the following transformation

$$\Phi^i \mapsto e^{i\alpha(x,\theta,\overline{\theta})}\Phi^i. \tag{A.26}$$

This leads to a change in the action (A.25)

$$\begin{aligned} \delta_\alpha \mathcal{S} &= i\alpha(x,\theta,\overline{\theta})\frac{\delta\mathcal{S}(e^{i\alpha}\Phi)}{\delta i\alpha(x,\theta,\overline{\theta})} \\ &= i\alpha(x,\theta,\overline{\theta})\Big(-\frac{1}{4}\bar{D}^2\Phi_i^\dagger e^{2V_i}\Phi_i + \Phi_i\frac{\partial\mathcal{W}}{\partial\Phi_i}\Big), \end{aligned} \tag{A.27}$$

where we have used the chiral functional differentiation rule. As we said above, the transformation (A.26) produce an anomalous Jacobian, see (A.17). Also let us demand that the partition function $\mathcal{Z}$ to be independent of α, which leads us to a Ward identity that we will used below

$$0 = \Big\langle -\frac{1}{4}\bar{D}^2\Phi_i^\dagger e^{2V_i}\Phi_i + \Phi_i\frac{\partial W}{\partial\Phi_i} - \frac{1}{8\pi^2}T(R_i)W^aW^a\Big\rangle. \tag{A.28}$$

Appendix B

T-duality tranformation

In this appendix we summarize the T-duality transfomation used by Frolov in [27]. To begin with let us start recalling that the appropriate generalization of the world-sheet action that includes the NS-NS background fields, [13], is

$$
\begin{aligned}
S &= -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\big[\big(\sqrt{-\gamma}\gamma^{ab}G_{MN}(X) - \epsilon^{ab}B_{MN}(X)\big)\partial_a X^M\partial_b X^N \\
&- \sqrt{-\gamma}\alpha'\mathcal{R}\Phi(X)\big],
\end{aligned} \tag{B.1}
$$

where M, N run from 0 to 9 and $a, b = 0, 1$. If the background fields are independent of the circular coordinate, say X^9. The T-dual worldsheet can be derived by a duality transformation of the X^9 coordinate. The below results was presented in [13] and we present here its deduction. Let $h^{ab} = \sqrt{-\gamma}\gamma^{ab}$, the equation of motion for X^9 is

$$
p^a \equiv -\frac{1}{2\pi\alpha'}\frac{\partial\mathcal{L}}{\partial(\partial_a X^9)} = h^{ab}\partial_b X^M G_{9M} - \epsilon^{ab}\partial_b X^M B_{9M}, \tag{B.2}
$$

let us write the action (B.1) as

$$
\begin{aligned}
4\pi\alpha' S &= \int d^2\sigma\big[h^{ab}\big(-G_{99}\partial_a X^9\partial_b X^9 - 2G_{9\mu}\partial_a X^9\partial_b X^\mu - G_{\mu\nu}\partial_a X^\mu\partial_b X^\nu\big) \\
&+ \epsilon^{ab}\big(2B_{9M}\partial_a X^9\partial_b X^M + B_{\mu\nu}\partial_a X^\mu\partial_b X^\nu\big) + \alpha'\sqrt{-\gamma}R\Phi(X)\big],
\end{aligned} \tag{B.3}
$$

where $\mu, \nu = 0, \cdots, 8$, substituting (B.2) into (B.3) we get

$$
\begin{aligned}
4\pi\alpha' S &= \int d^2\sigma\big[\big(-2p^a\partial_a X^9 + h^{ab}G_{99}\partial_a X^9\partial_b X^9 - h^{ab}G_{\mu\nu}\partial_a X^\mu\partial_b X^\nu\big) \\
&+ \epsilon^{ab}B_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + \alpha'\sqrt{-\gamma}R\Phi(X)\big],
\end{aligned} \tag{B.4}
$$

we need to eliminate $\partial_b X^9 G_{99}$ from (B.4), from the definition of p^a

$$
h^{ab}\partial_b X^9 G_{99} = p^a - h^{ab}\partial_b X^\mu G_{9\mu} + \epsilon^{ab}\partial_b X^\mu B_{9\mu}, \tag{B.5}
$$

let us write down the second term in (B.4) as

$$\begin{aligned} h^{ab}G_{99}\partial_a X^9\partial_b X^9 &= \partial_a X^9 G_{99}\frac{h^{ab}}{G_{99}}\partial_b X^9 G_{99} \\ &= h^{ac}\partial_c X^9 G_{99}\frac{h_{ab}}{G_{99}}h^{bd}\partial_d X^9 G_{99} \end{aligned} \tag{B.6}$$

after some computations we get

$$\begin{aligned} h^{ab}G_{99}\partial_a X^9\partial_b X^9 &= \frac{p^a h_{ab} p^b}{G_{99}} - 2p^a\big(\partial_a X^\mu\frac{G_{9\mu}}{G_{99}} - h_{ab}\epsilon^{bd}\partial_d X^\mu\frac{B_{9\mu}}{G_{99}}\big) \\ &+ h^{ab}\partial_a X^\mu\partial_b X^\nu\frac{G_{9\mu}G_{9\nu}-B_{9\mu}B_{9\nu}}{G_{99}} - \epsilon^{ab}\partial_a X^\mu\partial_b X^\nu\frac{G_{9\mu}B_{9\nu}-G_{9\nu}B_{9\mu}}{G_{99}}. \end{aligned} \tag{B.7}$$

So with this expression (B.7), the action (B.4) becomes

$$\begin{aligned} 4\pi\alpha' S &= \int d^2\sigma\big[-2p^a\big(\partial_a X^N\frac{G_{9N}}{G_{99}} - h_{ab}\epsilon^{bd}\partial_d X^N\frac{B_{9N}}{G_{99}}\big) + \frac{h_{ab}p^a p^b}{G_{99}} \\ &- h^{ab}\partial_a X^\mu\partial_b X^\nu\big(G_{\mu\nu} - \frac{G_{9\mu}G_{9\nu}-B_{9\mu}B_{9\nu}}{G_{99}}\big) \\ &+ \epsilon^{ab}\partial_a X^M\partial_b X^N\big(B_{MN} - \frac{G_{9M}B_{9N}-G_{9N}B_{9M}}{G_{99}}\big) + \alpha'\sqrt{-\gamma}R\Phi(X)\big]. \end{aligned} \tag{B.8}$$

Since $\partial_a p^a = 0$, then the solution can be written as

$$p^a = \epsilon^{ab}\partial_b\widetilde{X}^9, \tag{B.9}$$

where $\widetilde{X}^9$ is the T-dual of X^9, so from this we have

$$\epsilon^{ab}\partial_b\widetilde{X}^9 = h^{ab}\partial_b X^M G_{9M} - \epsilon^{ab}\partial_b X^M B_{9M}. \tag{B.10}$$

Substituting (B.9) into (B.8) and using (B.10), after some computations we get the T-dual action, which has the same form as (B.1) and it is given by

$$S = -\frac{1}{4\pi\alpha'}\int_M d^2\sigma\big[\big(\sqrt{-\gamma}\gamma^{ab}\widetilde{G}_{MN}(X) - \epsilon^{ab}\widetilde{B}_{MN}(X)\big)\partial_a\widetilde{X}^M\partial_b\widetilde{X}^N - \sqrt{-\gamma}\alpha'\mathcal{R}\widetilde{\Phi}(X)\big], \tag{B.11}$$

where the transformed background fiels are

$$\begin{aligned} \widetilde{G}_{99} &= \frac{1}{G_{99}},\ \widetilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{9\mu}G_{9\nu}-B_{9\mu}B_{9\nu}}{G_{99}},\ \widetilde{G}_{9\mu} = \frac{B_{9\mu}}{G_{99}}, \\ \widetilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{G_{9\mu}B_{9\nu}-B_{9\mu}G_{9\nu}}{G_{99}},\ \widetilde{B}_{9\mu} = \frac{G_{9\mu}}{G_{99}},\ \text{and} \\ \widetilde{\Phi} &= \Phi - \frac{1}{2}\ln\frac{R^2}{\alpha'} = \Phi - \frac{1}{2}\ln G_{99}. \end{aligned} \tag{B.12}$$

B.1 T-duality in R-R fields

Since R-R fields couple to D-branes, one can use the T-duality properties of them to deduce the transformation rules [13]. Since the IIA and IIB theories have different R-R fields, T-duality must transform one set into the other. Again focus on T-duality in just the 9-direction, see [10] for more details. T-duality acts on the R-R field strengths and potentials, adding or subtracting the index for the dualized dimensions. Thus, if we start from the IIA string we get the IIB R-R fields as follows (up to signs),

$$\begin{aligned} C_9 &\rightarrow C_0, \\ C_\mu, C_{\mu\nu 9} &\rightarrow C_{\mu 9}, C_{\mu\nu}, \\ C_{\mu\nu\lambda} &\rightarrow C_{\mu\nu\lambda 9}, \end{aligned} \tag{B.13}$$

where μ stands for dimensions not affected by the T-duality.

However the formulas (B.13) are only valid for trivial NS-NS backgrounds ($B_{MN} = 0, G_{MN} = \eta_{MN}$ and constant Φ), [13]. With that in mind, we sume up the results from [29], without presenting the details. So the first step is to combine the R-R potentials C_p of the type II string theory with the NS-NS 2-form field B_2, in the following way

$$\begin{aligned} D_0 &\equiv C_0,\ D_1 \equiv C_1, \\ D_2 &\equiv C_2 + B_2 \wedge D_0,\ D_3 \equiv C_3 + B_2 \wedge C_1, \\ D_4 &\equiv C_4 + \frac{1}{2} B_2 \wedge C_2 + \frac{1}{2} B_2 \wedge B_2 \wedge C_0. \end{aligned} \tag{B.14}$$

After introducing potentials of higher degree, D_{p+1} $(p+1 = 5, \cdots, 8)$, as their electromagnetic duals. More precisely they introduce the sum of fields strengths

$$D \equiv \sum_{p=0}^{8} D_p, \quad F \equiv e^{-B_2} \wedge \sum_{p=0}^{8} dD_p = \sum_{p=0}^{8} F_{p+1}. \tag{B.15}$$

This ensures that F defined as above satisfies

$$F_{10-p} = (-1)^{\left[\frac{p-1}{2}\right]} * F_p, \tag{B.16}$$

where $\left[\frac{p-1}{2}\right]$ is the first integer greater than or equal to $\frac{p-1}{2}$. Next they introduce $2d$ fermionic operators ψ_i and $\psi^{i\dagger}$ satisfying

$$\{\psi_i, \psi^{j\dagger}\} = \delta_i^{\ j} \mathbf{1}, \quad \{\psi_i, \psi_j\} = \{\psi^{i\dagger}, \psi^{j\dagger}\} = 0, \quad (i, j = 1, \cdots, d) \tag{B.17}$$

with $(\psi_i)^\dagger = \psi^{i\dagger}$. They can be used to construct a 2^d dimensional fermion Fock space $\mathcal{F}$ spanned by

$$|\alpha\rangle = \psi^{i_1\dagger}\cdots\psi^{i_n\dagger}|0\rangle \qquad\qquad (n=0,\cdots,d), \tag{B.18}$$

in (B.18) the vacuum $|0\rangle$ is defined by $\psi_i|0\rangle = 0$ and $\langle 0|0\rangle = 1$, and α is a multi-index $\alpha = (i_1,\cdots,i_n)$ with $(i_1 < \cdots < i_n)$. The operator $\mathbf{1}$ in (B.17) denotes the identity map on $\mathcal{F}$.

Then it is useful to construct states in the Fock space $\mathcal{F}$ corresponding to the form fields D and F in (B.15). To this purpose [29] utilized the following one-to-one correspondence between the set of differential forms and the space of creation operators $\psi^{j\dagger}$ under which a differential form Ω

$$\Omega = \sum_n \frac{1}{n!}\Omega_{i_1\cdots i_n} dy^{i_1}\wedge dy^{i_n} = \sum_q\sum_n \frac{1}{n!}\Omega^{(q)}_{i_1,\cdots,i_n} dy^{i_1}\wedge dy^{i_n} \tag{B.19}$$

is mapped to the following operator

$$\mathbf{\Omega} \equiv \sum_n \frac{1}{n!}\Omega_{i_1\cdots i_n}\psi^{i_1\dagger}\cdots\psi^{i_n\dagger} = \sum_q\sum_n \frac{1}{n!}\Omega^{(q)}_{i_1,\cdots,i_n}\psi^{i_1\dagger}\cdots\psi^{i_n\dagger}. \tag{B.20}$$

The superscript (q) indicates that $\Omega^{(q)}_{i_1,\cdots,i_n}$ is a q-form for noncompact indices, whereas $i_1\cdots i_n$ are compact indices. Now, to each differential form Ω one can construct the state $|\Omega\rangle \in \mathcal{F}$ as

$$|\Omega\rangle \equiv \mathbf{\Omega}|0\rangle. \tag{B.21}$$

The main result of [29] is that the states corresponding to D and F in (B.15) transform under $SO(d,d,\mathbb{R})$, which is, as

$$|F\rangle \to \mathbf{\Lambda}|F\rangle, \qquad\qquad |D\rangle \to \mathbf{\Lambda}|D\rangle, \tag{B.22}$$

where

$$\mathbf{\Lambda}|\beta\rangle = \sum_\alpha |\alpha\rangle S_{\alpha\beta}(\Lambda), \tag{B.23}$$

with $|\alpha\rangle, |\beta\rangle \in \mathcal{F}$ and $S(\Lambda) = (S_{\alpha\beta}(\Lambda))$ is the spinor representation. Here Λ is a given $SO(d,d,\mathbb{R})$ matrix. [29] give the explicit construction of the operators $\mathbf{\Lambda}$ corresponding to the generators of $SO(d,d,\mathbb{R})$.

For instance, let us consider the case where $d=3$ and the matrix $T \in SO(3,3,\mathbb{R})$ has the following form

$$T = \begin{pmatrix} 1_3 & 0_3 \\ \Gamma & 1_3 \end{pmatrix}, \tag{B.24}$$

with 1_3 and 0_3 being 3×3 identity and zero matrices, respectively. And Γ being a 3×3 antisymmetric matrix. Also consider the case when there is no B_2 field in the background so the R-R fields D_m defined in (B.14) are simply equal to C_m. The operator $\mathbf{\Lambda}$ which acts on the Fock space, was constructed in [29] as

$$\mathbf{\Lambda} = \exp(\frac{1}{2}\Gamma_{mn}\psi_m\psi_n). \tag{B.25}$$

From the discussion given above one can see that the corresponding operator acting on the differential form is

$$\boldsymbol{T} = \exp(\frac{1}{2}\Gamma_{mn}\imath_m\imath_n), \tag{B.26}$$

where $\imath_m$ is the contraction with the isometric direction $\frac{\partial}{\partial\phi_m}$. Such a contraction takes a n-form to an $(n-1)$-form. For instance,

$$\imath_m(d\phi_1\wedge\cdots d\phi_{m-1}\wedge d\phi_m\wedge d\phi_{m+1}\wedge\cdots\wedge d\phi_n) = (-1)^{m-1}(d\phi_1\wedge\cdots d\phi_{m-1}\wedge d\phi_{m+1}\wedge\cdots\wedge d\phi_n), \tag{B.27}$$

where here m is fixed. And $\imath_m$ has the property: $\imath_m\imath_m = 0$.

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