IFT Instituto de Física Teórica
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DISSERTAÇÃO DE MESTRADO IFT–D.001/01

# Sobre Métodos de Solução de Modelos de Toda não Abelianos

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Dezembro de 2000

# Agradecimentos

Ao Prof. A. H. Zimerman pela orientação e apoio, aos professores J. F. Gomes e L. A. Ferreira pela gentil cooperação, valiosas discussões e esclarecimentos. Aos meus amigos do IFT os quais, sem saber, me ajudaram muito a sentir-me melhor em São Paulo ao longo destes 2 últimos anos.

Finalmente à FAPESP pelo apoio financeiro.

# Resumo

Nesta dissertação reconstruímos um modelo de Toda não abeliano baseado na álgebra afim $B_2(SO(5))$ a partir de sua representação de curvatura nula (ou das equações de Leznov-Saveliev) de forma a obtermos soluções. Desenvolvemos dois métodos sistemáticos: transformações de "gauge" fornecendo equações de primeira ordem (transformações de Backlund) e "dressing".

**Palavras Chaves**: Solitons, modelos integráveis, modelos de Toda não abelianos, álgebras afins.

**Áreas do conhecimento**: Física das partículas elementares e Campos; métodos matemáticos em Física; fenômenos não lineares.

# Abstract

A non abelian Toda model based on the affine $B_2(SO(5))$ álgebra is reconstructed in terms of its zero curvature representation (or its Leznov-Saveliev equations). Its solutions are obtained from two sistematic methods, namely, gauge transformations leading to first order differential equations (Backlund transformations) and "dressing".

# 1 Capítulo 1

## 1.1 Introdução

Modelos integraveis a duas dimenções fornecem um laboratório para testar novos métodos de construção de soluções exatas.

Recentemente alguns modelos, fornecendo uma generalização do modelo de Sine-Gordon complexo foram construidos [1] [2] com soluções solitónicas obtidas pelo método de "dressing" [3]

Estes modelos baseiam-se em grupos "simply laced" do tipo SL(r+1).

Em analogia a êsses foram construidos modelos asociados a álgebras "não simply laced" baseados por exemplo em grupos do tipo $B_n$ Ref. [4]

A finalidade da Tese presente é a de discutir mais explicitamente o caso de $B_2$ cujo modelo já foi construido em J.F. Gomes et.al. [4]

Reconstruimos êste modelo utilizando $B_2$ à partir da álgebra $A_3$ equipada com um automorfismo que descreve uma simetría do diagrama de Dynkin. Esta simetria permite a construção dos geradores de $B_2$ em termos dos geradores de $A_3$.

As soluções do modelo são estudadas primeiramente em termos de transformações de "gauge" (Sotkov) relacionando duas soluções distintas (transformações de Backlund). Este método caracteriza-se por reduzir equações diferenciais não lineares de segunda ordem a equações diferenciais de primeira ordem.

Utilizando êste formalismo obtivemos uma solução particular do problema.

Um segundo formalismo tambem baseado em transformações de "gauge" foi estudado. Este método denominado "dressing" envolve a teoria da representação das álgebras de Kac-Moody. e fornece a solução explícita em termos de elementos de matriz (funçoes tau) de certos operadores (funções de vértice) que caracterizam o modelo. A dependência espaço-temporal das funções tau é escrita em termos de uma série finita de exponenciais cuja ordem depende da álgebra em questão.

Os modelos baseados em SL(r+1) e descritos por J.F. Gomes et.al. tem por característica um truncamento da expansão das funções de vértice em primeira ordem. No caso de $B_2$ o truncamento ocorre em ordem superior (o que é verificado no caso mais simples em que se considera soluções de um vértice em cujo caso o truncamento ocorre em segunda ordem). Por exemplo no caso de soluções de duas funções de vértice, o truncamento pode ocorrer em ordem superior como discutiremos nas conclusões.

Esta dissertação está organizada da seguinte maneira.

Primeiramente introduzimos as transformações de Backlund para o modelo de Sine Gordon. Como ilustração elas são discutidas no formalismo das transformações canônicas bem como no formalismo algébrico. Em seguida a álgebra de Lie $A_3$ (=SL(4)) e suas simetrias são discutidas. Em particular, define-se a álgebra $B_2$ em termos dos geradores de $A_3$. A construção do modelo integrável em questão é discutida e as equações de movimento são derivadas, atraves da condição de curvatura nula e das equações de Leznov-Saveliev.

A construção de uma transformação de Backlund é então abordada e uma solução particular é obtida. Em seguida apresenta-se o método de "dressing", após a construção dos operadores de vértice e soluções particulares são discutidas.

Nos apêndices A,B,C,D,E e F são dados detalhes e verificação de certas passagens matemáticas discutidas no texto principal.

Nas conclusões apresentamos uma maneira alternativa de se resolver as equações para as funções tau.Desse modo conseguimos certas soluções interessantes para as equações (3.69) e mencionamos os problemas em aberto.

# 2 Capítulo 2

## 2.1 Equação de SINE-GORDON. Transformacões de Backlund

A equação de Sine -Gordon é definida pela densidade Langrangeana :

$$\pounds \ = \frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 + (\cos\phi - 1) \tag{2.1}$$

A equação de Euler-Langrange dá como equação de movimento

$$\phi_{tt} - \phi_{xx} - \sin\phi = 0 \tag{2.2}$$

que é a equação de Sine-Gordon.
O momento canonicamente conjugado é

$$\pi = \frac{\delta\pounds}{\delta\phi_t} = \phi_t \tag{2.3}$$

O tensor de energia momento é dado por

$$t^{uv} = \partial^u\phi\left(\frac{\partial\pounds}{\partial(\partial_v\phi)}\right) - g^{uv}\pounds \tag{2.4}$$

fornecendo como densidade Hamiltoniana

$$h = \quad t^{00} = \partial^t\phi\left(\frac{\partial\pounds}{\partial\left(\partial_t\phi\right)}\right) - g^{tt}\pounds = \partial^t\phi\partial_t\phi - g^{tt}\pounds$$

óu:

$$h = \ \frac{1}{2}\pi^2 + \frac{1}{2}\phi_x^2 + (1 - \cos\phi) \tag{2.5}$$

Já para a densidade de momento temos

$$p = t^{01} = \pi\phi_x \tag{2.6}$$

Queremos relacionar duas solucões distintas $\phi$ e $\phi'$ da equação de Sine Gordon (2.2). As hamiltonianas totais e momentos totais devem ter a mesma forma para ambas as solucões $\phi$

e $\phi'$e portanto as suas densidades de hamiltoniana e de momento devem diferir por derivadas totais em relacão á coordenada x.

Kodama e Wadati [5] propuseram que as duas solucões $\phi$ e $\phi'$, de (2.2) fossem obtidas uma da outra por uma transformação canônica.

Introduziram a seguinte função geratriz

$$W[\phi, \phi'] = \int_{-\infty}^{+\infty} \{\phi\phi'_x - 2a\cos\frac{1}{2}(\phi+\phi') + \frac{2}{a}\cos\frac{1}{2}(\phi-\phi')\}dx \tag{2.7}$$

onde a é uma constante arbitrária. Neste caso temos para o momento canônicamente conjugado a $\phi$ [6]

$$\pi(x,t) = \frac{\delta W}{\delta\phi} = \phi'_x + a\sin\frac{1}{2}(\phi+\phi') - \frac{1}{a}sin\frac{1}{2}(\phi-\phi') \tag{2.8a}$$

Analogamente:

$$\pi'(x,t) = -\frac{\delta W}{\delta\phi'} = \phi_x - a\sin\frac{1}{2}(\phi+\phi') - \frac{1}{a}sin\frac{1}{2}(\phi-\phi') \tag{2.8b}$$

Os momentos totais do nosso sistema são dados respectivamente por

$$P = \int_{-\infty}^{+\infty} \pi\phi_x dx$$

$$P = \int_{-\infty}^{+\infty} \phi'_x\phi_x + a\phi_x\sin\frac{1}{2}(\phi+\phi') - \frac{1}{a}\phi_x sin\frac{1}{2}(\phi-\phi')dx \tag{2.9a}$$

$$P\prime = \int_{-\infty}^{+\infty} \pi'\phi'_x dx$$

$$P\prime = \int_{-\infty}^{+\infty} \phi_x\phi'_x - \phi'_x a\sin\frac{1}{2}(\phi+\phi') - \frac{1}{a}\phi'_x sin\frac{1}{2}(\phi-\phi')\ dx \tag{2.9b}$$

A sua diferença dá

$P\prime - P = \int_{-\infty}^{+\infty} (\pi'\phi'_x - \pi\phi_x)\, dx$

$P\prime - P = \int_{-\infty}^{+\infty} -(\phi_x+\phi'_x)\ a\ \sin\frac{1}{2}(\phi+\phi') + \frac{1}{a}\ (\phi_x-\phi'_x)\ sin\frac{1}{2}(\phi-\phi')\quad dx$

ou

$$P\prime - P = \int_{-\infty}^{+\infty} \frac{d}{dx}\left(a\ \cos\frac{1}{2}(\phi+\phi') - \frac{1}{a}\cos\frac{1}{2}(\phi-\phi')\right) \tag{2.10}$$

Um cálculo análogo para as Hamiltonianas dá

H'-H= $\frac{1}{2}\int_{-\infty}^{+\infty}\left[\pi'^2+\phi_x'^2+2(1-\cos\phi')\right]dx-\frac{1}{2}\int_{-\infty}^{+\infty}\left[\pi^2+\phi_x^2+2(1-\cos\phi)\right]dx$

o resultado sendo

$$2H'-2H=\int_{-\infty}^{+\infty}\left[-2\,a\,(\phi_x+\phi_x')\sin\tfrac{1}{2}(\phi+\phi')-\tfrac{2}{a}(\phi_x-\phi_x')\,sin\,\tfrac{1}{2}(\phi-\phi')\right.$$

$$\left.+4\sin\tfrac{1}{2}(\phi+\phi')\,sin\,\tfrac{1}{2}(\phi-\phi')-2(\cos\phi'-\cos\phi)\right]dx$$

Usando o fato $\cos\phi'-\cos\phi=-2\sin\frac{1}{2}(\phi+\phi')\,sin\,\frac{1}{2}(\phi'-\phi)$
segue-se:

$$H\prime-H=\int_{-\infty}^{+\infty}\frac{d}{dx}\left(2a\,\cos\frac{(\phi+\phi')}{2}+\frac{2}{a}\,\cos\frac{(\phi-\phi')}{2}\right) \tag{2.11}$$

Utilizando (2.3) as expressões (2.8a) e (2.8b) são reescritas como

$$\phi_t=\phi_x'+a\sin\frac{1}{2}(\phi+\phi')-\frac{1}{a}sin\frac{1}{2}(\phi-\phi') \tag{2.12a}$$

$$\phi_t'=\phi_x-a\sin\frac{1}{2}(\phi+\phi')-\frac{1}{a}sin\frac{1}{2}(\phi-\phi') \tag{2.12b}$$

Substraindo (2.12a) e (2.12b), obtemos

$$\phi_t-\phi_t'=\phi_x'-\phi_x+2a\sin\frac{1}{2}(\phi+\phi')$$

ou:

$$\frac{1}{2}(\partial_t+\partial_x)(\phi-\phi')\;=\;a\sin\frac{1}{2}(\phi+\phi') \tag{2.13a}$$

Da mesma maneira, somando (2.12a) e (2.12b) segue-se

$$-\frac{1}{2}(\partial_t-\partial_x)(\phi+\phi')\;=\;\frac{1}{a}\sin\frac{1}{2}(\phi-\phi') \tag{2.13b}$$

E' facil de se verificar que aplicando o operador $(\partial_t-\partial_x)$ em (2.13a) e utilizando (2.13b) de que é satisfeita a equação de Sine-Gordon (2.2). As equações (2.13a) e (2.13b) constituem as transformacões de Backlund.

Reescrevamos as equações (2.13a) e (2.13b) nas coordenadas do cone de luz:

$$z=t+x \qquad \bar{z}=t-x$$

ou

$$t=\frac{z+\bar{z}}{2} \qquad x=\frac{z-\bar{z}}{2} \tag{2.14a}$$

por tanto

$$\partial = \frac{1}{2}(\partial_t + \partial_x)$$

$$\overline{\partial} = \frac{1}{2}(\partial_t - \partial_x) \tag{2.14b}$$

$$\partial(\phi - \phi') = a \sin\frac{1}{2}(\phi + \phi') \tag{2.13a'}$$

$$\overline{\partial}(\phi + \phi') = -\frac{1}{a}\sin\frac{1}{2}(\phi - \phi') \tag{2.13b'}$$

Evidentemente $\phi' = 0$ é uma solução da equação (2.2). Procuremos uma outra solução desta equação utilizando as transformacões de Backlund (2.13a') e (2.13b'). Neste caso temos

$$2\,\partial\left(\frac{1}{2}\phi\right) = a\sin\frac{1}{2}\phi \tag{2.15a}$$

$$2\overline{\partial}\left(\frac{1}{2}\phi\right) = -\frac{1}{a}\sin\frac{1}{2}\phi \tag{2.15b}$$

Podemos rescrever (2.15a) e (2.15b) na forma

$$\frac{\partial\left(\frac{1}{2}\phi\right)}{\sin\frac{1}{2}\phi} = \frac{a}{2} \tag{2.16a}$$

$$\frac{\overline{\partial}\left(\frac{1}{2}\phi\right)}{\sin\frac{1}{2}\phi} = -\frac{1}{2a} \tag{2.16b}$$

A integração das equações (2.16a) e (2.16b) dá

$$\ln\tan(\frac{\phi}{2}) = \frac{a}{2}z + f_1(\bar{z})$$

$$\ln\tan(\frac{\phi}{2}) = -\frac{1}{2a}\bar{z} + f_2(z)$$

Subtraindo segue-se

$$\frac{a}{2}z - f_2(z) = f_1(\bar{z}) + \frac{1}{2a}\bar{z} = cons\tan te = \delta$$

ou

$$\frac{a}{2}z - \frac{1}{2a}\bar{z} = f_1(\bar{z}) + f_2(z)$$

Somando segue

$$2\ln\tan(\frac{\phi}{2}) = \frac{a}{2}z + f_1(\bar{z}) - \frac{1}{2a}\bar{z} + f_2(z)$$

Por tanto:

$$\frac{\phi}{2} = \arctan e^{\frac{a}{2}z - \frac{1}{2a}\bar{z}} \tag{2.17}$$

ou utilizando (2.14a)

$$\phi = 2\arctan[e^{\frac{1}{2}(a+\frac{1}{a})\left(x+\frac{(a-\frac{1}{a})}{(a+\frac{1}{a})}t\right)}] \tag{2.17'}$$

Temos como solução de uma perturbação que viaja com velocidade

$$v = \frac{(a-\frac{1}{a})}{(a+\frac{1}{a})} \tag{2.17"}$$

Constitue a solução de <u>1 sóliton</u>.

# 3 Capítulo 3

## 3.1 Equações de Lax, Condição de Curvatura Nula, e Equações de Leznov-Saveliev

Considere as equações de Lax para as conexões $A$ e $\overline{A}$

$$(\partial + A)\psi = 0 \tag{3.1}$$

$$\left(\overline{\partial} + \overline{A}\right)\psi = 0 \tag{3.2}$$

Aplicando o operador $\overline{\partial}$ em (3.1) e $\partial$ em (3.2) e utilizando a condição de integrabilidade : $\overline{\partial}\,\partial = \partial\overline{\partial}$ , a compatibilidade das equações dá a condição de curvatura nula:

$$\left[\partial + A, \overline{\partial} + \overline{A}\right] = \partial\overline{A} - \overline{\partial}A + \left[A, \overline{A}\right] = 0 \tag{3.3}$$

Fazendo a " transformação de gauge ", ( "com a função de gauge" $\theta$ )

$$A^g = \theta A \theta^{-1} - \partial\theta\theta^{-1}$$

$$\overline{A}^g = \theta\overline{A}\theta^{-1} - \overline{\partial}\theta\theta^{-1} \tag{3.4}$$

é facil verificar que a condição de curvatura nula (3.3) continua válida para $A^g$ e $\overline{A}^g$, isto é

$$\partial\overline{A}^g - \overline{\partial}A^g + \left[A^g, \overline{A}^g\right] = 0 \tag{3.5}$$

Considere as equações de Lax

$$(\partial + A^g)\psi_g = 0 \qquad \left(\overline{\partial} + \overline{A}^g\right)\psi_g = 0 \tag{3.6}$$

$$\left(\partial + A^f\right)\psi_f = 0 \qquad \left(\overline{\partial} + \overline{A}^f\right)\psi_f = 0 \tag{3.7}$$

e considere

$$\psi_g = \theta\psi_f \tag{3.8}$$

Neste caso é facil verificar que $A^g$ e $A^f$, $\overline{A}^g$ e $\overline{A}^f$ estão relacionadas por uma 'transformação de gauge' do tipo (3.4) onde $\theta$ é a função de "gauge" correspondente.

As solucões de condição de curvatura nula são do tipo

$$A = -(\partial T)T^{-1} \qquad \overline{A} = -(\overline{\partial} T)T^{-1} \tag{3.9}$$

Considere agora os "potenciais de gauge"

$$A = -B\varepsilon_+ B^{-1} \qquad \overline{A} = -\overline{\partial} B B^{-1} + \varepsilon_- \tag{3.10}$$

com $\varepsilon_+$ e $\varepsilon_-$ operadores constantes em relação a z e $\overline{z}$

Temos $[A, \overline{A}] = (-B\varepsilon_+ B^{-1})\left(-\overline{\partial} B B^{-1} + \varepsilon_-\right) - \left(-\overline{\partial} B B^{-1} + \varepsilon_-\right)(-B\varepsilon_+ B^{-1})$

$= B\varepsilon_+ B^{-1}\overline{\partial} B B^{-1} - B\varepsilon_+ B^{-1}\varepsilon_- - \overline{\partial} B\varepsilon_+ B^{-1} + \varepsilon_- B\varepsilon_+ B^{-1}$

$$\partial\overline{A} - \overline{\partial} A = -\partial\left(\overline{\partial} B B^{-1}\right) + \left(\overline{\partial} B\right)\varepsilon_+ B^{-1} - B\varepsilon_+ B^{-1}\left(\overline{\partial} B\right) B^{-1}$$

onde usamos $\partial\varepsilon_- = \overline{\partial}\varepsilon_+ = 0$

Utilizando a condição de curvatura nula (3.3) segue :

$$\partial[\overline{\partial} B B^{-1}] = [\varepsilon_-, B\varepsilon_+ B^{-1}] \tag{3.11}$$

Podemos reescrever:

$$\partial\overline{\partial} B\ B^{-1} + \left(\overline{\partial} B\right)(\ \partial B^{-1}) = \partial\overline{\partial} B\ B^{-1} - \left(\overline{\partial} B\right) B^{-1}(\partial B) B^{-1}$$
$$= \varepsilon_- B\varepsilon_+ B^{-1} - B\varepsilon_+ B^{-1}$$

Multipliquemos á esquerda por $B^{-1}$ e á dereita por B

$$B^{-1}\ \partial\overline{\partial} B - B^{-1}\left(\overline{\partial} B\right) B^{-1}(\partial B) = B^{-1}\varepsilon_-\ B\ \varepsilon_+ - \varepsilon_+ B^{-1}\varepsilon_-\ B$$

ou:

$$B^{-1}\left(\partial\overline{\partial} B\right) + \left(\overline{\partial} B^{-1}\right)(\partial B) = [B^{-1}\varepsilon_-\ B, \varepsilon_+]$$

ou:

$$\overline{\partial}\left(\ B^{-1}\partial B\right) = -\ [\ \varepsilon_+, B^{-1}\varepsilon_- B] \tag{3.12}$$

Em resumo, no caso das conexões $A$ e $\overline{A}$ serem dadas por (3.10) com $\varepsilon_+$ e $\varepsilon_-$ operadores constantes, então a condição de curvatura nula pode-se reescrever na forma das equações (3.11) e (3.12). Estas equações foram propostas primeiramente por Leznov e Saveliev [7]

<u>Obtenção da Transformação de Backlund á partir de Transformações de Gauge</u>

Vamos dar agora uma formulação sugerida por G.Sotkov.

Considere as eqs. de Lax ( 3.6 e 3.7) com os potenciais de gauge da forma (3.10)

$$(\ \partial + A^g)\,\psi_g = 0 \qquad \left(\ \partial + A^f\right)\psi_f = 0$$

$$\left(\ \overline{\partial} + \overline{A}^g\right)\psi_g = 0 \qquad \left(\ \overline{\partial} + \overline{A}^f\right)\psi_f = 0 \tag{3.13}$$

com

$$A^g = -g\varepsilon_+ g^{-1} \qquad \overline{A}^g = -\left(\overline{\partial} g\right) g^{-1} + \varepsilon_- \tag{3.14}$$

$$A^f = -f\varepsilon_+ f^{-1} \qquad \overline{A}^f = -\left(\overline{\partial} f\right) f^{-1} + \varepsilon_- \tag{3.15}$$

e considere a transformação de "gauge"

$$\psi_g = \theta \psi_f \tag{3.16}$$

Introduzindo na primeira equação (3.14) segue-se:

$$\left(\overline{\partial} + \overline{A}^g\right)\theta\psi_f = \left(\overline{\partial} - \left(\overline{\partial} g\right) \ g^{-1} + \varepsilon_-\right)\theta\psi_f = 0$$

e aplicando á esquerda o operador $\theta^{-1}$ obtemos

$$\left[\theta^{-1}\overline{\partial}\theta - \theta^{-1}\left(\overline{\partial} g\right) \ g^{-1}\theta + \theta^{-1}\varepsilon_-\theta\right]\psi_f = 0$$

Comparando com a segunda equação de (3.14) segue

$$-\left(\overline{\partial} f\right) f^{-1} + \varepsilon_- = \theta^{-1}\overline{\partial}\theta - \theta^{-1}\left(\overline{\partial} g\right) g^{-1}\theta + \theta^{-1}\varepsilon_-\theta \tag{3.17}$$

Da mesma maneira substituindo na ecuação (3.13) e comparando com a equação para $\psi_f$ segue

$$-\theta^{-1} f\varepsilon_+ f^{-1}\theta = \ \theta^{-1}\partial\theta - \theta^{-1} g\varepsilon_+ g^{-1}\theta \tag{3.18}$$

Aplicando em (3.17) o operador $\theta$, segue-se

$$-\theta\left(\overline{\partial} f\right) f^{-1} + [\theta, \varepsilon_-] = \overline{\partial}\theta - \left(\overline{\partial} g\right) g^{-1}\theta \tag{3.19}$$

ou a sua equivalente

$$\theta f\left(\overline{\partial} f^{-1}\right) + [\theta, \varepsilon_-] = \overline{\partial}\theta - \left(\overline{\partial} g\right) g^{-1}\theta \tag{3.20}$$

Consideremos o "anzats"

$$\theta = x + gyf^{-1} \tag{3.21}$$

com x e y, são <u>matrizes constantes</u>. Introduzindo (3.21) em (3.20) segue-se

$$xf\overline{\partial} f^{-1} + [\theta, \varepsilon_-] = g\overline{\partial} g^{-1} x$$

ou

$$xf\overline{\partial} f^{-1} + \left[x + gyf^{-1}, \varepsilon_-\right] = g\overline{\partial} g^{-1} x \tag{3.22}$$

Da mesma maneira,considerando as equações (3.13) e a transformação de "gauge" (3.16) segue-se:

$$\partial\theta - g\varepsilon_+ g^{-1}\theta = -\theta f\varepsilon_+ f^{-1} \tag{3.23}$$

Introduziendo (3.18) em (3.23) vale

$$g^{-1}\partial g y + y\partial f^{-1} f - [\varepsilon_+, y] - [\varepsilon_+, g^{-1}xf] = 0 \tag{3.24}$$

Aplicando em (3.22) o operador $\partial$, e lembrando que x e y são constantes, segue utilizando para g e f as equações (3.11), (3.12):

$$x\partial\left(f\bar{\partial}f^{-1}\right) + \left[\partial\left(gyf^{-1}\right), \varepsilon_-\right] = \partial\left(g\bar{\partial}g^{-1}\right)x$$

ou

$$-x\partial\left(\bar{\partial}f\ f^{-1}\right) + \left[\partial\left(gyf^{-1}\right), \varepsilon_-\right] = -\partial\left(\bar{\partial}g\ g^{-1}\right)x$$

ou

$$-x[\varepsilon_-, f\varepsilon_+ f^{-1}] + \left[\partial\left(gyf^{-1}\right), \varepsilon_-\right] = -[\varepsilon_-, f\varepsilon_+ f^{-1}]x$$

ou

$$-x[\varepsilon_-, f\varepsilon_+ f^{-1}] + \left[\partial g\ yf^{-1} + gy\ \partial f^{-1}, \varepsilon_-\right] = -[\varepsilon_-, f\varepsilon_+ f^{-1}]x \tag{3.25}$$

Já a ecuação (3.24) reescreve-se como:

$$\partial g y + g y\partial f^{-1} f - g[\varepsilon_+, y] - g[\varepsilon_+, g^{-1}xf] = 0 \tag{3.26}$$

Substituindo em (3.25) segue:

$$-x[\varepsilon_-, f\varepsilon_+ f^{-1}] + \left[-gy\partial f^{-1} + g\left[\varepsilon_+, y\right] f^{-1} + g[\varepsilon_+, g^{-1}xf]f^{-1} + gy\partial f^{-1}, \varepsilon_-\right] = -[\varepsilon_-, g\varepsilon_+ g^{-1}] \tag{3.27}$$

Agora:

$$g[\varepsilon_+, g^{-1}xf]f^{-1} = g\varepsilon_+ g^{-1}xff^{-1} - gg^{-1}xf\varepsilon_+ f^{-1} = g\varepsilon_+ g^{-1}x - xf\varepsilon_+ f^{-1}$$

que introduzida em (3.27) dá:

$$x[\varepsilon_-, f\varepsilon_+ f^{-1}] + -gy\partial f^{-1} + \left[g\left[\varepsilon_+, y\right] f^{-1}, \varepsilon_-\right] + \left[g\varepsilon_+ g^{-1}x, \varepsilon_-\right] - \left[xf\varepsilon_+ f^{-1}, \varepsilon_-\right] = -[\varepsilon_-, g\varepsilon_+ g^{-1}]x$$

ou:

$$x[\varepsilon_-, f\varepsilon_+ f^{-1}] + \left[g\left[\varepsilon_+, y\right] f^{-1}, \varepsilon_-\right] + \left[g\varepsilon_+ g^{-1}, \varepsilon_-\right] x +$$

$$+ g\varepsilon_+ g^{-1}[x, \varepsilon_-] - x\left[f\varepsilon_+ f^{-1}, \varepsilon_-\right] - [x, \varepsilon_-]\, f\varepsilon_+ f^{-1}$$

$$= -[\varepsilon_-, g\varepsilon_+ g^{-1}]x \tag{3.28}$$

Esta é a condição para x e y de modo que g e f satisfaçam as equações (3.11) e (3.12).

As equações (3.22) e (3.24) com a condição (3.28) dão as transformações de Backlund. Em muitos casos, uma solução simples de (3.28) é dada por

$$[x, \varepsilon_-] = [y, \varepsilon_+] = 0 \tag{3.29}$$

### 3.1.1 Aplicação á Equação de Sine Gordon

Nas Variáveis de cone de luz (2.14a),(2.14b), a equação de Sine-Gordon (2.2) pode-se reescrever:

$$\partial\overline{\partial}\phi - \frac{1}{2}\sin\phi = 0 \tag{3.30}$$

Tomemos

$$\varepsilon_+ = E^0_\alpha + E^1_{-\alpha}, \qquad \varepsilon_- = E^0_{-\alpha} + E^{-1}_\alpha \tag{3.31}$$

onde $E^0_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $E^1_{-\alpha} = \lambda \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $E^0_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $E^{-1}_\alpha = \frac{1}{\lambda}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$h^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.32}$$

com $\lambda$ parâmetro espectral. Utilizando (3.10) obtemos

$$A = -e^{2\varphi}E^0_\alpha - e^{-2\varphi}E^1_{-\alpha}$$

$$\overline{A} = -h^0\overline{\partial}\varphi + E^0_{-\alpha} + E^{-1}_\alpha \tag{3.33}$$

Daí segue:

$$[A, \overline{A}] = -2e^{2\varphi}\overline{\partial}\varphi E^0_\alpha + 2e^{-2\varphi}\overline{\partial}\varphi E^1_{-\alpha} + \left(-e^{-2\varphi} + e^{2\varphi}\right)h^0 \tag{3.34}$$

$$\partial\overline{A} - \overline{\partial}A = -h^0\partial\overline{\partial}\varphi + 2\overline{\partial}\varphi e^{2\varphi}E^0_\alpha - 2\overline{\partial}\varphi e^{-2\varphi}E^1_{-\alpha} \tag{3.35}$$

Por tanto a condição de curvatura nula escreve-se:

$$\partial\overline{A} - \overline{\partial}A + \left[A, \overline{A}\right] = \left(\partial\overline{\partial}\varphi - 2\sinh(2\varphi)\right)h^0 \tag{3.36}$$

dando a equação de Sine Gordon (3.20) com a correspondencia $2\varphi \leftrightarrow \phi$

Estudemos agora as transformações de Backlund para este problema do ponto de vista de transformações de gauge.

Considere a solução simples de (3.29)

$$x = a\varepsilon_- + b \qquad\qquad y = c\varepsilon_+ + d$$

Escrevamos

$$f = e^{h\varphi} = \begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix} \qquad\qquad g = e^{h\varphi'} = \begin{pmatrix} e^{\varphi'} & 0 \\ 0 & e^{-\varphi'} \end{pmatrix}$$

Neste caso a eq. (3.22) dá

$$-\frac{1}{2}\overline{\partial}\left(\varphi' - \varphi\right) b = c \sinh\left(\varphi + \varphi'\right)$$

$$\frac{1}{2}\overline{\partial}\left(\varphi' + \varphi\right) a = d \sinh\left(\varphi - \varphi'\right) \tag{3.37}$$

enquanto que a equação (3.24) dá

$$\frac{d}{2}\partial\left(\varphi - \varphi'\right) = -a \sinh\left(\varphi + \varphi'\right)$$

$$-\frac{c}{2}\overline{\partial}\left(\varphi' - \varphi\right) = b \sinh\left(\varphi + \varphi'\right) \tag{3.38}$$

Temos a possibilidade de b=c=0 e então obtemos as transformações de Backlund:

$$\overline{\partial}\left(\varphi' + \varphi\right) = \frac{2d}{a} \sinh\left(\varphi - \varphi'\right)$$

$$\partial\left(\varphi - \varphi'\right) = -2\frac{a}{d} \sinh\left(\varphi + \varphi'\right) \tag{3.39}$$

ou a possibilidade a=d=0 e temos então as transformações de Backlund na forma:

$$\overline{\partial}\left(\varphi' - \varphi\right) = -\frac{2c}{b} \sinh\left(\varphi + \varphi'\right)$$

$$\partial\left(\varphi' + \varphi\right) = -\frac{2b}{c} \sinh\left(\varphi - \varphi'\right) \tag{3.40}$$

## 3.2 A álgebra $A_3$ dobrada

A algebra $A_3$ (=SL(4)), tem por geradores satisfazendo ás relações de comutação:

$$
\begin{aligned}
&[E_{\alpha1},E_{\alpha2}] = E_{\alpha1+\alpha2}\ ,\ [E_{\alpha2},E_{\alpha3}] = E_{\alpha2+\alpha3}\ ,\\
&[E_{\alpha1},E_{-\alpha1}] = h_1,\ [E_{\alpha2},E_{-\alpha2}] = h_2,\\
&[E_{\alpha3},E_{-\alpha3}] = h_3,\ [h_1,E_{\alpha1}] = 2E_{\alpha1}\ ,\\
&[h_1,E_{\alpha2}] = -E_{\alpha2},\ [h_2,E_{\alpha1}] = -E_{\alpha1}\ ,\\
&[h_2,E_{\alpha2}] = 2E_{\alpha2},[h_2,E_{\alpha3}] = -E_{\alpha3},\\
&[h_3,E_{\alpha2}] = 2E_{\alpha2},\ [h_3,E_{\alpha3}] = 2E_{\alpha3},\\
&[E_{\alpha1},E_{\alpha2+\alpha3}] = E_{\alpha1+\alpha2+\alpha3}\ ,\ [E_{-\alpha1-\alpha2},E_{\alpha1}] = E_{\alpha2}\ ,\\
&[E_{-\alpha1-\alpha2},E_{\alpha2}] = -E_{-\alpha1},\ [E_{-\alpha2-\alpha3},E_{\alpha2}] = E_{-\alpha3}\ ,\\
&[E_{\alpha_1+\alpha_2},E_{\alpha3}] = E_{\alpha1+\alpha2+\alpha3}\ ,\ [E_{-\alpha2-\alpha3},E_{\alpha3}] = -E_{-\alpha2}\\
&[E_{-\alpha1-\alpha2},E_{\alpha1+\alpha2+\alpha3}] = E_{\alpha3}\ ,\ [E_{-\alpha2-\alpha3},E_{\alpha1+\alpha2+\alpha3}] = -E_{\alpha1}\\
&[E_{-\alpha2-\alpha3},E_{-\alpha1}] = E_{-\alpha1-\alpha2-\alpha3}\ ,\ [E_{\alpha1+\alpha2},E_{-\alpha1}] = -E_{\alpha2}\\
&[E_{-\alpha1-\alpha2},E_{-\alpha3}] = -E_{-\alpha1-\alpha2-\alpha3}\ ,\ [E_{\alpha2+\alpha3},E_{-\alpha3}] = E_{\alpha2}\\
&[E_{\alpha2+\alpha3},E_{-\alpha1-\alpha2-\alpha3}] = E_{-\alpha1}\ ,\ [E_{\alpha1+\alpha2},E_{-\alpha1-\alpha2-\alpha3}] = -E_{\alpha3}\\
&[E_{\alpha1},E_{-\alpha1-\alpha2-\alpha3}] = -E_{-\alpha2-\alpha3}\ ,\ [E_{\alpha3},E_{-\alpha1-\alpha2-\alpha3}] = -E_{-\alpha1-\alpha2}\\
&[h_1,h_2] = 0\ ,\ [h_1,h_3] = 0,\ [h_2,h_3] = 0
\end{aligned}
\tag{3.41}
$$

O diagrama de Dynkin de $A_3$ é dado por

o—o—o
1 2 3

Ele possue a simetría $1 \longleftrightarrow 3$ e $2 \longleftrightarrow 2$ o qual induz o automorphismo $\sigma$ :

$$\sigma(E_{\alpha1}) = E_{\alpha3} \quad , \quad \sigma(E_{\alpha3}) = E_{\alpha1} \ , \quad \sigma^2 = 1\ , \quad \sigma(E_{\alpha2}) = E_{\alpha2} \tag{3.42}$$

A fim de vermos como o automorphismo $\sigma$ atua em $E_{\alpha1+\alpha2}$ notemos que

$$\sigma(E_{\alpha1+\alpha2}) = \sigma([E_{\alpha1},E_{\alpha2}]) = [E_{\alpha3},E_{\alpha2}] = -E_{\alpha2+\alpha3} \tag{3.43}$$

e portanto

$$\sigma(E_{\alpha1+\alpha2}) = -E_{\alpha2+\alpha3} \tag{3.44}$$

Como veremos adiante, a álgebra $A_3$ munida do automorfismo $\sigma$ é idéntica a $B_2$ = SO(5). Consideramos $A_3$ dobrada (folded) neste caso.

## 3.3 Relação entre a algebra $A_3$ dobrada e a algebra $B_2$

A fim de estudarmos a relação entre o $A_3$ dobrado e $B_2$ façamos a associação:

$$E_{\beta 1} = E_{\alpha 2}$$

$$E_{\beta 2} = E_{\alpha 1} + E_{\alpha 3} \tag{3.45}$$

Do ponto de vista do diagrama de Dynkin temos:

o=<=o
1,3 2

Agora:

$$[E_{\beta 1}, E_{\beta 2}] = [E_{\alpha 2}, E_{\alpha 1} + E_{\alpha 3}] = -E_{\alpha 1+\alpha 2} + E_{\alpha 2+\alpha 3} = E_{\beta 1+\beta 2} \tag{3.46}$$

Ainda:

$$[E_{\beta 1+\beta 2}, E_{\beta 2}] = [-E_{\alpha 1+\alpha 2} + E_{\alpha 2+\alpha 3}, E_{\alpha 1} + E_{\alpha 3}] = -2E_{\alpha 1+\alpha 2+\alpha 3} = -2E_{\beta 1+2\beta 2} \tag{3.47}$$

e por tanto

$$E_{\beta 1+2\beta 2} = E_{\alpha 1+\alpha 2+\alpha 3} \tag{3.48}$$

Quanto aos elementos de Cartan temos

$$[E_{\beta 1}, E_{-\beta 1}] = \widetilde{h}_1 \;=\; [E_{\alpha 2}, E_{-\alpha 2}] = \frac{2\alpha_2.h}{\alpha_2^2} = h_2 \tag{3.49}$$

$$[E_{\beta 2}, E_{-\beta 2}] = (\frac{2\alpha_1.h}{\alpha_1^2} + \frac{2\alpha_3.h}{\alpha_3^2}) = (h_1 + h_3) = \widetilde{h}_2 \tag{3.50}$$

onde $\widetilde{h}_1$ e $\widetilde{h}_2$ são os elementos de Cartan para $B_2 = SO(5)$

Pode-se verificar que valem as relações de comutação da álgebra $B_2$ : [8]

$$[E_{\beta 1}, E_{-\beta 1}] = \widetilde{h}_1, [E_{\beta 2}, E_{-\beta 2}] = \widetilde{h}_2, [E_{\beta 1+\beta 2}, E_{-\beta 1-\beta 2}] = 2\widetilde{h}_1 + \widetilde{h}_2$$

$$[E_{\beta 1+2\beta 2}, E_{-\beta 1-2\beta 2}] = \widetilde{h}_1 + \widetilde{h}_2$$

$$E_{\pm\beta 1} = \pm\frac{1}{2}[\widetilde{h}_1, E_{\pm\beta 1}] = \pm\frac{1}{2}\left[E_{\pm\beta 1}, \widetilde{h}_2\right] = \pm\frac{1}{2}[E_{\pm(\beta 1+\beta 2)}, E_{\mp\beta 2}]$$

$$E_{\pm\beta 2} = \pm[E_{\pm\beta 2}, \widetilde{h}_1] = \pm\frac{1}{2}\left[\widetilde{h}_2, E_{\pm\beta 2}\right] = \pm\frac{1}{2}[, E_{\mp\beta 1}, E_{\pm(\beta 1+\beta 2)}] \;=\; \pm[E_{\pm(\beta 1+2\beta 2)}, E_{\pm(\beta 1+\beta 2)}]$$

$$E_{\pm(\beta 1+\beta 2)} = \pm[\widetilde{h}_1, E_{\pm(\beta 1+\beta 2)}] = \pm\,[E_{\pm\beta 1}, E_{\pm\beta 2}] = \pm\left[E_{\mp\beta 2}\;, E_{\pm(\beta 1+2\beta 2)}\right]$$

$$E_{\pm(\beta 1+2\beta 2)} = \pm\frac{1}{2}[\widetilde{h}_2, E_{\pm(\beta 1+2\beta 2)}] = \pm\frac{1}{2}\left[E_{\pm\beta 2}\;, E_{\pm(\beta 1+\beta 2)}\right] \tag{3.51}$$

## 3.4 Algebra de Kac Moody

Seja uma álgebra de Lie definida pelos comutadores

$$[E_\alpha, E_\beta] = \varepsilon(\alpha,\beta)E_{\alpha+\beta} \qquad [h_\gamma, E_\beta] = 2\frac{\alpha_\gamma.\beta}{\alpha_\gamma^2}E_\beta$$

$$[E_\alpha, E_{-\alpha}] = h_\alpha \qquad [h_\gamma, h_b] = 0 \tag{3.52}$$

$\alpha,\beta$ são raixes, $\varepsilon(\alpha,\beta)$ é determinado por (3.41), $h_\gamma = \alpha_\gamma.H$ e tomamos a notação $\alpha_\gamma$ como uma raiz referida com nome $\gamma$.

Introduzimos a Loop Algebra

$$[E^m_\alpha, E^n_\beta] = \varepsilon(\alpha,\beta)\ E^{m+n}_{\alpha+\beta} \qquad [h^m_\alpha, E^n_\beta] = 2\frac{\alpha_\alpha.\beta}{\alpha_\alpha^2}E^{m+n}_\beta$$

$$[E^m_\alpha, E^n_{-\alpha}] = h^{m+n}_\alpha \qquad [h^m_\alpha, h^n_\beta] = 0 \tag{3.53}$$

com $E^m_\alpha = \lambda^m E_\alpha$, $h^m_\beta = \lambda^m\ h^m_a$ e $\lambda$ e um numero complexo arbitrário.

Ela pode-se generalizar a uma Algebra de Kac Moody mudando os comutadores entre os elementos de Cartan e os comutadores entre os geradores conjugados na seguinte forma

$$[E^m_\alpha, E^n_{-\alpha}] = h^{m+n}_\alpha + cm\delta_{m,-n} \qquad [h^m_\alpha, h^n_\beta] = cm\delta_{m,-n}tr\,(h_\alpha\ h_\beta) \tag{3.54}$$

onde c e um operador chamado de 'termo central' que comuta com todos os geradores.

Em resumo, temos que a Algebra de Kac Moody é definida pelos comutadores.

$$[E^m_\alpha, E^n_\beta] = \varepsilon(\alpha,\beta)E^{m+n}_{\alpha+\beta} \qquad [h^m_a, E^n_\alpha] = 2\frac{\alpha_a.\alpha}{\alpha_a^2}E^{m+n}_\alpha$$

$$[E^m_\alpha, E^n_{-\alpha}] = h^{m+n}_\alpha + cm\delta_{m,-n} \qquad [h^m_\alpha, h^n_\beta] = cm\delta_{m,-n}tr\,(h_\alpha\ h_\beta) \tag{3.55}$$

Utilizaremos a graduação

$$\mathbf{Q} = 2d + \frac{2\ \lambda_2\ .\ h}{\alpha_2^2} \tag{3.56}$$

com $[d,\Lambda^m] = m\Lambda^m$, onde $\Lambda^m$ é um gerador qualquer da álgebra de Kac-Moody e

$$\frac{2\lambda_2.\ \alpha_i}{\alpha_2^2} = \delta_{i,2} \tag{3.57}$$

O operador Q vai dar conta do grau de qualquer expressão que seja dada em funcão de geradores da Algebra de Kac Moody.

## 3.5 Construção de Modelo

Consideraremos os seguintes elementos constantes de grau $\pm 1$,

$$\varepsilon_+ = \left(E^0_{\alpha 1+\alpha 2} - E^0_{\alpha 2+\alpha 3}\right) + \left(E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3}\right)$$

$$\varepsilon_- = \left(E^0_{-\alpha 1-\alpha 2} - E^0_{-\alpha 2-\alpha 3}\right) + \left(E^{-1}_{\alpha 1+\alpha 2} - E^{-1}_{\alpha 2+\alpha 3}\right) \tag{3.58}$$

e o elemento de grupo construído a partir de geradores de grau zero

$$B = e^{\tilde{\chi}(E^0_{-\alpha 1}+E^0_{-\alpha 3})} e^{\varphi(h)} e^{\tilde{\psi}(E^0_{\alpha 1}+E^0_{\alpha 3})} \tag{3.59}$$

com

$$\varphi(h) = \varphi(h_2 + \frac{1}{2}h_1 + \frac{1}{2}h_3) + R(h_1 + h_3) + \nu c + \eta d \tag{3.60}$$

póde-se verificar que $\varepsilon_+, \varepsilon_-$ e B são invariantes pelo automorfismo $\sigma$.
Introduzimos as conexões A e $\bar{\text{A}}$ por meio de:

$$A = -B\varepsilon_+ B^{-1}$$

$$\overline{A} = -\partial_- B B^{-1} + \varepsilon_- \tag{3.61}$$

Fazendo o calculo para $B^{-1}\ \partial B$ e substituindo $\tilde{\psi} e^R \to \psi$, $\tilde{\chi}\ e^R \to \chi$, (ver apêndice A)
$B^{-1}\partial B\ |_{vinc}=$

$$= \partial\psi e^{-R}(E^0_{\alpha 1} + E^0_{\alpha 3}) + \left(h_2 + \frac{1}{2}h_1 + \frac{1}{2}h_3\right)\partial\varphi + \partial\nu c + \partial\eta d + \frac{\partial\chi}{\Delta}\ e^R(E^0_{-\alpha 1} + E^0_{-\alpha 3}) \tag{3.62}$$

onde introduzindos o vínculo

$$\partial R = \frac{\psi\partial\chi}{1+\chi\psi} \tag{3.63}$$

o que corresponde a tomar em $B^{-1}\ \partial B$ a corrente na direção de $(h_1 + h_3)$ de igual a zero. Esta é a chamada condição de "Black Hole", e pode-se provar que ela decorre do fato de $(h_1 + h_3)$ comutar com $\varepsilon_+$ e $\varepsilon_-$.

Analogamente consideramos a corrente $\overline{\partial} B\ B^{-1}$ (ver apêndice B) , a condição de "Black Hole"

$$\overline{\partial} R = \frac{\chi\overline{\partial}\psi}{1+\chi\psi} \tag{3.64}$$

dá

$$\overline{\partial}BB^{-1}\mid_{vinc} = \overline{\partial}\chi\, e^{-R}(E^{0}_{-\alpha1}+E^{0}_{-\alpha3})+\overline{\partial}\varphi\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)+\frac{\overline{\partial}\psi}{\Delta}e^{R}(E^{0}_{\alpha1}+E^{0}_{\alpha3})+\overline{\partial}\nu c+\overline{\partial}\eta d \tag{3.65}$$

As equações de movimento para os campos $\chi, \psi, \varphi, R$ são obtidas á partir da equação de curvatura nula

$$\partial\overline{A}-\overline{\partial}A+[A,\overline{A}]=0 \tag{3.66}$$

onde substituimos (3.59) em (3.61).

$$\begin{aligned}\overline{A} = & -e^{-\varphi}(1+2\chi\psi)(E^{0}_{\alpha1+\alpha2}-E^{0}_{\alpha2+\alpha3})-2\psi e^{-\varphi-R}(1+\chi\psi)E^{0}_{\alpha1+\alpha2+\alpha3}\\ & -e^{\varphi}(1+2\chi\psi)\left(E^{1}_{-\alpha1-\alpha2}-E^{1}_{-\alpha2-\alpha3}\right)-2\psi e^{-\varphi-R-\eta}(1+\chi\psi)E^{1}_{-\alpha2}\\ & +2\chi e^{-\varphi+R}E^{0}_{\alpha2}+2\chi e^{\varphi+R-\eta}E^{1}_{-\alpha1-\alpha2-\alpha3}\end{aligned} \tag{3.67}$$

$$\begin{aligned}A = & E^{0}_{-\alpha1-\alpha2}-E^{0}_{-\alpha2-\alpha3}+E^{-1}_{\alpha1+\alpha2}-E^{-1}_{\alpha2+\alpha3}+\partial\psi e^{-R}(E^{0}_{\alpha1}+E^{0}_{\alpha3})+\\ & +\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)\partial\varphi+\tfrac{\partial\psi}{\Delta}e^{R}(E^{0}_{-\alpha1}+E^{0}_{-\alpha3})+\partial\nu c+\partial\eta d\end{aligned} \tag{3.68}$$

no caso de Kac Moody. Introduzindo (3.67) e (3.68) em (3.66), obtemos:

$$\partial\overline{\partial}\varphi-2(1+2\psi\chi)e^{-\varphi}+2(1+2\psi\chi)e^{\varphi-\eta}=0$$

$$\partial\overline{\partial}\nu-2e^{\varphi-\eta}(1+2\psi\chi)=0$$

$$\partial\overline{\partial}\chi(1+\psi\chi)-\psi\partial\chi\overline{\partial}\chi+4\chi(1+\psi\chi)^2\left(\frac{e^{\varphi-\eta}+e^{-\varphi}}{2}\right)=0$$

$$\partial\overline{\partial}\psi(1+\psi\chi)-\chi\partial\psi\overline{\partial}\psi+4\psi(1+\psi\chi)^2\left(\frac{e^{-\varphi-\eta}+e^{-\varphi}}{2}\right)=0 \tag{3.69}$$

pode-se verificar facilmente que estas equações são derivaveis da Langrangiana

$$\mathcal{L}=2\frac{\partial\chi\overline{\partial}\psi}{(1+\psi\chi)}+\frac{1}{2}\partial\varphi\overline{\partial}\varphi-4(1+\chi\psi)\left(\frac{e^{-\varphi-\eta}+e^{\varphi}}{2}\right)+\overline{\partial}\eta\partial\nu \tag{3.70}$$

Pode-se verificar que as equações (3.69) seguem também das equações de Leznov-Saveliev ( Veja Apêndice E)

## 3.6 Solução as Equações (3.69) na aproximação de "Loop Algebra"

Tentemos resolver as equações (3.69) na aproximação de "Loop Algebra", (não consideramos $\nu$ e nem $\eta$). Neste caso temos

$$B^{-1}\partial B\,|_{vinc}= \partial\psi e^{-R}(E^{0}_{\alpha 1}+E^{0}_{\alpha 3})+\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\partial\varphi+\frac{\partial\chi}{\Delta}\,e^{R}(E^{0}_{-\alpha 1}+E^{0}_{-\alpha 3}) \tag{3.71}$$

$$\overline{\partial}BB^{-1}\,|_{vinc}= \overline{\partial}\chi\; e^{-R}(E^{0}_{-\alpha 1}+E^{0}_{-\alpha 3})+\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\overline{\partial}\varphi+\frac{\overline{\partial}\psi}{\Delta}e^{R}(E^{0}_{\alpha 1}+E^{0}_{\alpha 3}) \tag{3.72}$$

De agora em diante suprimimos a notação $|_{vinc}$, subentendendo $B^{-1}\ \partial B$ e $\overline{\partial}B\ B^{-1}$ vinculados.

Podemos reescrever $\varepsilon_+$ e $\varepsilon_-$ na forma matricial

$$\varepsilon_+=\left(E^{0}_{\alpha 1+\alpha 2}-E^{0}_{\alpha 2+\alpha 3}\right)+\left(E^{1}_{-\alpha 1-\alpha 2}-E^{1}_{-\alpha 2-\alpha 3}\right)=\begin{pmatrix}0 & \sigma_3\\ \lambda\sigma_3 & 0\end{pmatrix} \tag{3.73}$$

$$\varepsilon_-=\left(E^{0}_{-\alpha 1-\alpha 2}-E^{0}_{-\alpha 2-\alpha 3}\right)+\left(E^{-1}_{\alpha 1+\alpha 2}-E^{-1}_{\alpha 2+\alpha 3}\right)=\begin{pmatrix}0 & \lambda^{-1}\sigma_3\\ \sigma_3 & 0\end{pmatrix} \tag{3.74}$$

onde $\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ é a matriz de Pauli e $\lambda$ é o parâmetro de "Loop".

Utilizemos o método sugerido por G.Sotkov para formular a transformação de Backlund para o sistema.

Tomemos o "ansatz"

$$x=\alpha 1+\beta(h_1+h_3)+\gamma\varepsilon_-$$

$$y=\overline{\alpha}1+\overline{\beta}(h_1+h_3)+\overline{\gamma}\varepsilon_+$$

tais que [x,$\varepsilon_-$]=[y,$\varepsilon_+$]=0

Explícitamente

$$x=\begin{pmatrix}\alpha.1+\beta\sigma_3 & \lambda^{-1}\gamma\sigma_3\\ \gamma\sigma_3 & \alpha.1+\beta\sigma_3\end{pmatrix} \tag{3.75}$$

$$y=\begin{pmatrix}\overline{\alpha}.1+\overline{\beta}\sigma_3 & \overline{\gamma}\sigma_3\\ \overline{\gamma}\sigma_3 & \overline{\alpha}.1+\overline{\beta}\sigma_3\end{pmatrix} \tag{3.76}$$

$$\varphi(h)=\varphi(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3)+R(h_1+h_3)=\begin{pmatrix}e^{\frac{1}{2}\varphi+R} & 0 & 0 & 0\\ 0 & e^{\frac{1}{2}\varphi-R} & 0 & 0\\ 0 & 0 & e^{-\frac{1}{2}\varphi+R} & 0\\ 0 & 0 & 0 & e^{-\frac{1}{2}\varphi-R}\end{pmatrix} \tag{3.77}$$

e escrevemos as equações de primeira ordem

$$B^{-1}\partial B\ x \ = B^{-1}y\varepsilon_- - \varepsilon_- B^{-1}y \tag{3.78}$$

$$-\overline{\partial}BB^{-1}\ y \ = -Bx\varepsilon_+ + \varepsilon_+ Bx \tag{3.79}$$

$$B^{-1}\partial B = \begin{pmatrix} \frac{1}{2}\partial\varphi & \partial\psi e^{-R} & 0 & 0 \\ \frac{\partial\chi}{\Delta}\ e^{R} & \frac{1}{2}\partial\varphi & 0 & 0 \\ 0 & 0 & \frac{-1}{2}\partial\varphi & \partial\psi e^{-R} \\ 0 & 0 & \frac{\partial\chi}{\Delta}\ e^{R} & \frac{-1}{2}\partial\varphi \end{pmatrix} \tag{3.80}$$

$$\overline{\partial}BB^{-1} = \begin{pmatrix} \frac{1}{2}\overline{\partial}\varphi & \overline{\partial}\psi e^{R} & 0 & 0 \\ \frac{\partial\chi}{\Delta}\ e^{-R} & \frac{1}{2}\overline{\partial}\varphi & 0 & 0 \\ 0 & 0 & \frac{-1}{2}\overline{\partial}\varphi & \frac{\overline{\partial}\psi}{\Delta}e^{R} \\ 0 & 0 & \overline{\partial}\chi\ e^{-R} & \frac{-1}{2}\overline{\partial}\varphi \end{pmatrix} \tag{3.81}$$

onde

$$\Delta = (1+\psi\chi) \tag{3.82}$$

As equações (3.77) e (3.78) reescrevem-se

$$\frac{1}{2}(\alpha+\beta)\partial\varphi = \overline{\gamma}e^{-\widetilde{R}}(1+\chi\psi)^{1/2}(e^{-\varphi/2} - e^{\varphi/2})$$

$$\frac{1}{2}(\alpha-\beta)\partial\varphi = \overline{\gamma}e^{-\widetilde{R}}(1+\chi\psi)^{1/2}(e^{-\varphi/2} - e^{\varphi/2})$$

$$(\alpha-\beta)\frac{\partial\psi}{(1+\psi\chi)^{1/2}}e^{-\widetilde{R}} = -\overline{\gamma}\psi(e^{-\varphi/2} + e^{\varphi/2})$$

$$(\alpha+\beta)\frac{\overline{\partial}\chi}{(1+\psi\chi)^{1/2}}e^{\widetilde{R}} = -\overline{\gamma}\chi(e^{-\varphi/2} + e^{\varphi/2}) \tag{3.83}$$

$$\frac{1}{2}(\overline{\alpha}+\overline{\beta})\overline{\partial}\varphi = \gamma e^{\widetilde{R}}(1+\chi\psi)^{1/2}(e^{\varphi/2} - e^{-\varphi/2})$$

$$\frac{1}{2}(\overline{\alpha}-\overline{\beta})\overline{\partial}\varphi = \gamma e^{-\widetilde{R}}(1+\chi\psi)^{1/2}(e^{\varphi/2} - e^{-\varphi/2}) \tag{3.84}$$

$$(\overline{\alpha}-\overline{\beta})\frac{\overline{\partial}\psi}{(1+\psi\chi)^{1/2}}e^{\widetilde{R}} = \gamma\psi(e^{\varphi/2} + e^{-\varphi/2})$$

$$(\overline{\alpha}+\overline{\beta})\frac{\overline{\partial}\chi}{(1+\psi\chi)^{1/2}}e^{-\widetilde{R}}=\gamma\psi(e^{-\varphi/2}+e^{\varphi/2})$$

com

$$\widetilde{R}=R-\frac{1}{2}\ln(1+\chi\psi) \tag{3.85}$$

$$-\frac{1}{2}\gamma\partial\varphi=(\overline{\alpha}+\overline{\beta})e^{-\widetilde{R}}(1+\chi\psi)^{1/2}(e^{\varphi/2}-e^{-\varphi/2})$$

$$\frac{1}{2}\gamma\partial\varphi=-(\overline{\alpha}-\overline{\beta})e^{\widetilde{R}}(1+\chi\psi)^{1/2}(e^{\varphi/2}-e^{-\varphi/2})$$

$$-\gamma\frac{\partial\psi}{(1+\chi\psi)^{1/2}}e^{-\widetilde{R}}=(\overline{\alpha}-\overline{\beta})\psi(e^{\varphi/2}+e^{-\varphi/2})$$

$$\gamma\frac{\partial\chi}{(1+\chi\psi)^{1/2}}e^{\widetilde{R}}=(\overline{\alpha}+\overline{\beta})\chi(e^{\varphi/2}+e^{-\varphi/2}) \tag{3.86}$$

$$\frac{1}{2}\overline{\gamma}\overline{\partial}\varphi=(\alpha+\beta)e^{\widetilde{R}}(1+\chi\psi)^{1/2}(e^{\varphi/2}-e^{-\varphi/2})$$

$$-\frac{1}{2}\overline{\gamma}\overline{\partial}\varphi=-(\alpha-\beta)e^{-\widetilde{R}}(1+\chi\psi)^{1/2}(e^{\varphi/2}-e^{-\varphi/2})$$

$$\overline{\gamma}\frac{\overline{\partial}\psi}{(1+\chi\psi)^{1/2}}e^{\widetilde{R}}=(\alpha-\beta)\psi(e^{\varphi/2}+e^{-\varphi/2})$$

$$\overline{\gamma}\frac{\overline{\partial}\chi}{(1+\chi\psi)^{1/2}}e^{-\widetilde{R}}=(\alpha+\beta)\chi(e^{\varphi/2}+e^{-\varphi/2}) \tag{3.87}$$

As duas primeiras equações de (3.83), (3.86),(3.82b), (3.87) dão

$$\widetilde{R}=constante$$

Das duas primeiras equações de (3.83) e (3.86) obtemos

$$(\partial+\sigma\overline{\partial})\varphi=0 \quad onde\ \sigma=\frac{\overline{\gamma}(\overline{\alpha}-\overline{\beta})}{\gamma(\alpha+\beta)} \tag{3.88}$$

Análogamente as outras expressões dão:

$$(\partial+\sigma\overline{\partial})\psi=(\partial+\sigma\overline{\partial})\chi=0 \tag{3.89}$$

Então os campos são funções de $z - \frac{1}{\sigma}\overline{z}$

Apliquemos o operador $\overline{\partial}$ na 1ª equação (3.83). Utilizando o fato que $\widetilde{R} =$ constante, não é dificil mostrar que

$$\frac{1}{2}(\alpha + \beta)\overline{\partial}\partial\varphi = \frac{-\gamma\overline{\gamma}}{(\overline{\alpha} - \overline{\beta})}(e^{-\varphi} - e^{\varphi}) \tag{3.90}$$

Isto é, $\varphi$ satisfaz a equação de Sinh-Gordon. Comparando com a primeira equação (3.69) segue-se que $\chi\psi$ =constante

As duas primeiras equações (3.83) dão:

$$e^{2\widetilde{R}} = \frac{(\alpha - \beta)}{(\alpha + \beta)} \tag{3.91}$$

Multiplico a 3ª equação de (3.83) por $\chi$ e a 4ª equação por $\psi$ e divido uma pela outra obtendo:

$$\frac{(\alpha - \beta)\chi\partial\psi}{(\alpha + \beta)\psi\partial\chi}e^{-2\widetilde{R}} = 1 = \frac{\chi\partial\psi}{\psi\partial\chi} \tag{3.92}$$

onde utilizamos (3.91). De $\psi\chi$ =constante segue :

$$\psi\partial\chi + \chi\partial\psi = 0$$

enquanto (3.92) da

$$\psi\partial\chi - \chi\partial\psi = 0.$$

Estas 2 últimas equações dão $\psi\partial\chi = \chi\partial\psi = 0$. Da mesma forma temos $\psi\overline{\partial}\chi = \chi\overline{\partial}\psi = 0$. e por tanto $\psi$ e $\chi$ são constantes, As duas últimas equações (3.83) dizem que $\psi = \chi = 0$.

Em outras palavras, temos a solução para $\varphi$ satisfazendo a eq. de Sinh-Gordon e $\chi = \psi = 0$ , R=0.

## 3.7 Estudo dos Autoestados e Autovalores dos operadores $\varepsilon_+, \varepsilon_-$.

Vamos agora estudar os autoestados e autovalores correspondentes aos operadores $\varepsilon_+, \varepsilon_-$,eles são dados por (ver apêndice B):

$$F_1^+ = \sum_n E_{\alpha1}^n \lambda^{-2n} + \sum_n E_{\alpha3}^n \lambda^{-2n} + \sum_n E_{\alpha1+\alpha2+\alpha3}^n \lambda^{-2n-1} + \sum_n E_{-\alpha2}^n \lambda^{-2n+1}$$

$$F_1^- = \sum_n E_{\alpha1}^n \lambda^{-2n} + \sum_n E_{\alpha3}^n \lambda^{-2n} - \sum_n E_{\alpha1+\alpha2+\alpha3}^n \lambda^{-2n-1} - \sum_n E_{-\alpha2}^n \lambda^{-2n+1}$$

$$F_2^+ = \sum_n E_{\alpha2}^n \lambda^{-2n} + \sum_n E_{-\alpha3}^n \lambda^{-2n} + \sum_n E_{-\alpha1-\alpha2-\alpha3}^n \lambda^{-2n-1} + \sum_n E_{-\alpha1}^n \lambda^{-2n+1}$$

$$F_2^- = \sum_n E_{\alpha2}^n \lambda^{-2n-1} + \sum_n E_{-\alpha3}^n \lambda^{-2n} + \sum_n E_{-\alpha1-\alpha2-\alpha3}^n \lambda^{-2n+1} + \sum_n E_{-\alpha1}^n \lambda^{-2n}$$

$$F_4^+ = \sum_n h_1^n \lambda^{-2n} + 2\sum_n h_2^n \lambda^{-2n} + \sum_n h_3^n \lambda^{-2n} - \sum_n E_{\alpha1+\alpha2}^n \lambda^{-2n-1} + \\ + \sum_n E_{\alpha2+\alpha3}^n \lambda^{-2n-1} + \sum_n E_{-\alpha1-\alpha2}^n \lambda^{-2n+1} - \sum_n E_{-\alpha2-\alpha3}^n \lambda^{-2n+1} - c$$

$$F_4^- = \sum_n h_1^n \lambda^{-2n} + 2\sum_n h_2^n \lambda^{-2n} + \sum_n h_3^n \lambda^{-2n} + \sum_n E_{\alpha1+\alpha2}^n \lambda^{-2n-1} + \\ - \sum_n E_{\alpha2+\alpha3}^n \lambda^{-2n-1} - \sum_n E_{-\alpha1-\alpha2}^n \lambda^{-2n+1} + \sum_n E_{-\alpha2-\alpha3}^n \lambda^{-2n+1} - c$$

$$F_6^+ = \sum_n E_{\alpha2}^n \lambda^{-2n} - \sum_n E_{-\alpha1-\alpha2-\alpha3}^n \lambda^{-2n-1}$$

$$F_6^- = \sum_n E_{-\alpha2}^n \lambda^{2n} - \sum_n E_{\alpha1+\alpha2+\alpha3}^n \lambda^{2n-1} \tag{3.93}$$

e satisfazem á condição de serem invariantes pelo automorfismo $\sigma$.
Valem as relações

$$[\,\varepsilon^m, F_1^\pm] = \pm 2\lambda^{2m+1} F_1^\pm \qquad [\,\varepsilon, F_2^\pm] = \pm 2\lambda^{2m+1} F_2^\pm$$

$$[\,\varepsilon^m, F_4^\pm] = \pm 2\lambda^{2m+1} F_4^\pm \qquad [\,\varepsilon^m, F_6^\pm] = 0 \tag{3.94}$$

onde definimos

$$\varepsilon^m = \left(E_{\alpha1+\alpha2}^m - E_{\alpha2+\alpha3}^m\right) + \left(E_{-\alpha1-\alpha2}^{m+1} - E_{-\alpha2-\alpha3}^{m+1}\right) \tag{3.95}$$

Para m=0, $\varepsilon^m$ se reduz a $\varepsilon_+$ enquanto que para m=-1, $\varepsilon^{-1}$ se reduz a $\varepsilon_-$

Vamos reecrever as expressões (3.93),(3.94), e (3.95) en temos dos geradores (3.45),(3.46),(3.47),(3.4

$$\varepsilon^m = -E^m_{\beta1+\beta2} - E^m_{-(\beta1+\beta2)}$$

$$F_1^\pm = \sum_n E^n_{\beta2}\lambda^{-2n} \pm \sum_n E^n_{\beta1+2\beta2}\lambda^{-2n-1} \pm \sum_n E^n_{-\beta1}\lambda^{-2n+1}$$

$$F_2^\pm = \sum_n E^n_{\beta1}\lambda^{-2n-1} \mp \sum_n E^n_{-\beta2}\lambda^{-2n} + \sum_n E^n_{-\beta1-2\beta2}\lambda^{-2n+1}$$

$$F_4^\pm = \sum_n \widetilde{h}^n_2\lambda^{-2n} + 2\sum_n \widetilde{h}^n_1\lambda^{-2n} - \sum_n E^n_{\beta1+\beta2}\lambda^{-2n-1} \mp \sum_n E^n_{-\beta1-\beta2}\lambda^{-2n+1} - c$$

$$F_6^\pm = \sum_n E^n_{\pm\beta1}\lambda^{-2n} - \sum_n E^n_{\pm(\beta1+2\beta2)}\lambda^{-2n+1} \tag{3.96}$$

## 3.8 Metodo de "dressing"

Um método para determinar as soluções solitónicas consiste em vestir (to dress ) o vácuo para uma solução não trivial por meio de transformação de "gauge".

A condição de curvatura nula implica que as conexões A e Ā são puro "gauge".

As soluções de "vácuo" satisfazem

$$A_{vac} = \varepsilon_- + \partial\nu_0 c = \varepsilon_- + 2\bar{z}c$$

$$A_{vac} = -\varepsilon_+ \tag{3.97}$$

com $[\varepsilon_+, \varepsilon_-] = 2c$

Então

$T_0 = e^{\varepsilon_- z} e^{-\varepsilon_+ \bar{z}} \qquad T_0^{-1} = e^{\varepsilon_+ \bar{z}} e^{-\varepsilon_- z}$

$T_0^{-1}\partial T_0 = e^{\varepsilon_+ \bar{z}} e^{-\varepsilon_- z} \partial(e^{\varepsilon_- z}) e^{-\varepsilon_+ \bar{z}} = e^{\varepsilon_+ \bar{z}} \varepsilon_- e^{-\varepsilon_+ \bar{z}}$

$= \varepsilon_- + \bar{z}\,[\varepsilon_+, \varepsilon_-] = \varepsilon_- + 2\ c\ \bar{z} = A_{vac} = -\left(\partial T_0^{-1}\right) T_0$

Também

$-\ \bar{\partial} T_0^{-1} T_0 = T_0^{-1} \bar{\partial} T_0 = e^{\varepsilon_+ \bar{z}}\ e^{-\varepsilon_- \bar{z}}\ \bar{\partial}\ e^{\varepsilon_- z} e^{-\varepsilon_+ \bar{z}} = -\varepsilon_+ = \bar{A}_{vac}$

Póde-se escrever

$$(\partial + A_{vac}) T_0^{-1} = 0$$

$$(\bar{\partial} + \bar{A}_{vac}) T_0^{-1} = 0$$

Escrevamos A e Ā na forma

$$A = (T_o\theta)^{-1}\, \partial\,(T_o\theta) = \theta^{-1} T_o^{-1}\,(\partial T_o)\,\theta + \theta^{-1}\theta$$

ou:

$$A = \theta^{-1} A_{vac} \theta + \theta^{-1} \partial\theta$$

Da mesma maneira

$$\overline{A} = \theta^{-1} \overline{A}_{vac} \theta + \theta^{-1} \overline{\partial}\theta = (T_o\theta)^{-1}\, \overline{\partial}\,(T_o\theta)$$

O método de "dressing" baseia-se na hipótese da existência de duas transformações de gauge geradas por $\theta^{\pm}$ mapeando o vácuo numa configuracão não trivial:

$$A = (\theta^{\pm})^{-1} A_{vac} (\theta^{\pm}) + (\theta^{\pm})^{-1} \partial\theta^{\pm} = B^{-1} \partial B + \varepsilon_-$$

$$\overline{A} = (\theta^{\pm})^{-1}\overline{A}_{vac}(\theta^{\pm}) + (\theta^{\pm})^{-1}\overline{\partial}(\theta^{\pm}) = -B^{-1}\varepsilon_+ B \tag{3.98}$$

com $\theta^+$ contendo graus positivos e nulo:

$$\theta^+ = \theta_o\theta_> \; com\theta_> = e^{t^{(1)}+t^{(2)}+\dots}$$

onde $t^{(i)}$ é construído com o auxílio de geradores de grau i>0.
Então: $A = \theta_>^{-1}\theta_o^{-1}A_{vac}\;\theta_0\theta_> + \theta_>^{-1}\left(\theta_0^{-1}\partial\theta_0\right)\theta_> + \theta_>^{-1}\partial\theta_>$
ou

$$\begin{array}{ll}
\partial B\; B^{-1} + \varepsilon_- \;=\; \theta_>^{-1}\theta_o^{-1}\,(\varepsilon_- + 2\,\overline{z})\,\theta_0\theta_> + \theta_>^{-1}\theta_o^{-1}\partial\theta_0\theta_> + \theta_>^{-1}\partial\theta_> & \\
\text{Comparemos os 2 lados pelos diferentes graus.} & \\
\text{grau -1} \qquad \varepsilon_- = \theta_o^{-1}\varepsilon_-\theta \quad \rightarrow\text{permite a escolha } \theta_o = 1 & \\
\text{grau 0} \qquad B^{-1}\,\partial B = [\varepsilon_-, t^1] + 2\,\overline{z} & 
\end{array} \tag{3.99}$$

Temos também:

$$\begin{array}{l}
\bar{\mathrm{A}} = \theta_>^{-1}\theta_0^{-1}\,(-\,\varepsilon_+)\,\theta_0\theta_> + \theta_>^{-1}\theta_0^{-1}\bar{\partial}\theta_0\theta_> + \theta_>^{-1}\bar{\partial}\theta_> = -B^{-1}\varepsilon_+ B \\
\text{analizando o grau 0 temos} \qquad 0 = 0 \\
\text{enquanto o grau 1 dá} \qquad -\,\varepsilon_+ + \bar{\partial}t_1 = -B^{-1}\varepsilon_+ B
\end{array} \tag{3.100}$$

Verifiquemos que (3.99) e (3.100) satisfazem à equação do movimento correta

$$\begin{aligned}
\bar{\partial}(B^{-1}\partial B) &\overset{(1)}{=} \left[\varepsilon_-, \bar{\partial}t_1\right] + 2 \overset{(2)}{=} [\varepsilon_-, -B^{-1}\varepsilon_+ B + \varepsilon_+] + 2 \\
&= -\,[\varepsilon_-, B^{-1}\varepsilon_+ B] + [\varepsilon_-, \varepsilon_+] + 2 \\
&= -\,[\varepsilon_-, B^{-1}\varepsilon_+ B] \text{ pois } [\varepsilon_-, \varepsilon_+] = -2
\end{aligned}$$

Temos também :

$$A = \theta_-^{-1}A_{vac}\theta_- = \theta_-^{-1}\,(\partial\theta_-) = \varepsilon_- + B^{-1}\partial B$$
$$\bar{A} = \theta_-^{-1}\bar{A}_{vac}\theta_- + \theta_-^{-1}\left(\bar{\partial}\theta_-\right) = -\;B^{-1}\varepsilon_+ B$$

com

$$\theta_- = \tilde{\theta}_o\;\theta_< \quad \text{e} \quad \theta_< = e^{t^{(-1)}+t^{(-2)}}$$

onde $t^{(-i)}$ e construído com geradores de graus negativos $(-i)$, $(i > 0)$
Reescrevemos

$$\begin{aligned}
\mathrm{A} &= \theta_<^{-1}\,\tilde{\theta}_o^{-1}(\varepsilon_- + 2\,\bar{z})\,\tilde{\theta}_o\,\theta_< + \theta_<^{-1}\,\tilde{\theta}_o^{-1}\;\partial\tilde{\theta}_o\,\theta_< + \theta_<^{-1}\partial\theta_< \\
&= \varepsilon_- + B^{-1}\partial B
\end{aligned}$$

Comparando os dois membros de grau zero obtemos:

$$2\,\bar{\mathrm{z}} + \tilde{\theta}_o^{-1}\;\partial\tilde{\theta}_o = B^{-1}\partial B$$

ou $\quad \tilde{\theta}_o = B\;e^{-2z\bar{z}}$

Comparando os dois membros de grau -1 temos

$$\varepsilon_- + [B^{-1}\partial B - 2\bar{z}\ c, t^{-1}] + \partial t^{-1} = \varepsilon_-$$

ou

$$[B^{-1}\partial B, t^{-1}] + \partial t^{-1} = 0$$

De maneira análoga temos

$$\bar{\mathrm{A}} = \theta_<^{-1}\tilde{\theta}_0^{-1}\,(-\,\varepsilon_+)\,\tilde{\theta}_0\theta_< + \theta_<^{-1}\left(\tilde{\theta}_0^{-1}\bar{\partial}\tilde{\theta}_0\right) + \theta_<^{-1}\bar{\partial}\theta_< = -B^{-1}\varepsilon_+ B$$

Comparando os dois membros de grau zero, temos:

$$[-B^{-1}\varepsilon_+ B, t^{-1}] + B^{-1}\bar{\partial}B - 2z = 0$$

Comparando os dois membros de grau 1

$$-B^{-1}\varepsilon_+ B = -B^{-1}\varepsilon_+ B$$

Escrevemos agora

$$(\partial + \mathrm{A}_{vac})\ \theta T = 0 \qquad (\bar{\partial} + \bar{\mathrm{A}}_{vac})\ \theta T = 0$$

com T a determinar. Multipliquemos à esquerda por $\theta^{-1}$

$$\theta^{-1}(\partial\theta T + \theta\partial T + A_{vac}\theta T) = 0$$

ou

$$(\theta^{-1}\partial\theta + \theta^{-1}A_{vac}\theta)T + \partial T = (\partial + A)T = 0$$

Considerando no lugar de $\theta$ as funções $\theta_+$ e $\theta_-$, temos:

$$\theta_+^{-1}(\partial + \mathrm{A}_{vac})\theta_+ T = 0 \ \rightarrow \theta_+ T = T_o^{-1}$$

$$\theta_-^{-1}(\partial + \mathrm{A}_{vac})\theta_- T\ g^{-1} = 0 \ \rightarrow \theta_- T g^{-1} = T_o^{-1}$$

onde g é um elemento de grupo constante. Daí segue

$\mathrm{T} = \theta_+^{-1}T_o^{-1}$

$\mathrm{T} = \theta_-^{-1}T_o^{-1}g$

e portanto

$$\theta_-\theta_+^{-1} = T_o^{-1}gT_o$$

$$\left(\tilde{\theta}_o\theta_-^{<}\right)\left(\theta_o\theta_>^{-1}\right) = T_o^{-1}gT_o$$

ou

$$\theta_-^{<}\ B\ \mathrm{e}^{-2z\bar{z}}\ \theta_>^{-1} = T_o^{-1}gT_o$$

com $\theta_o = 1$ e $T_o^{-1}$ solução do vácuo, isto é

$$(\partial + \mathrm{A}_{vac})T_o^{-1} = 0 \qquad (\bar{\partial} + \bar{\mathrm{A}}_{vac})T_o^{-1} = 0$$

Temos para o estado de peso mais alto $\mid \lambda_i >$

$$\theta_> \mid \lambda_i >= 0$$

e portanto

$$\theta_-^{<}\ B\ e^{-2z\bar{z}} \mid \lambda_i >= T_o^{-1}gT_o \mid \lambda_i > .$$

$1^0$) Projeção em $\mid 0 >$:

$$< 0 \mid e^{-2z\bar{z}+\nu\hat{c}} \mid 0 > = < 0 \mid T_o^{-1}gT_o \mid 0 >= e^{-2z\bar{z}+\nu}$$

$2^0$) Projeção em $\mid \lambda_i >$ com $i = 1,3$

$$= < \lambda_i \mid e^{\chi\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}\ e^{\lambda_2 H\varphi\ +(h_1+h_3)R+\nu c}\ e^{\psi\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)} \mid \lambda_i >= e^{\lambda_2\lambda_i\ \varphi\ +\ R-2z\bar{z}+\nu}$$

$$= < \lambda_i \mid T_o^{-1}gT_o \mid \lambda_i > \qquad \text{i=1,3}$$

3) $< \lambda_i \mid e^{\chi\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)} \; e^{\lambda_2 H\varphi+\nu c\,+(h_1+h_3)R-2z\bar{z}} \left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right) \mid \lambda_i >$
$= e^{\lambda_i \; \lambda_2\varphi \; +\nu+R-2z\bar{z}} < \lambda_i \mid (1+\psi \; (E^0_{\alpha 1}+E^0_{\alpha 3})) \left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right) \mid \lambda_i >$
$= \psi \; e^{\lambda_i \; \lambda_2 \; \varphi \; +\nu+R-2z\bar{z}} < \lambda_i \mid h_1+h_3 \mid \lambda_i >= \psi \; e^{\lambda_i \; \lambda_2 \; \varphi \; +\nu-2z\bar{z}+R}$
para i = 1 ou 3
Por sua vez esta expressão é igual a:
$< \lambda_i \mid T_o^{-1} g T_o \left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right) \mid \lambda_i >$ i=1,3
4) $< \lambda_i \mid \left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right) e^{\chi\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)} \; e^{\lambda_2 H\varphi\,+(h_1+h_3)R-2z\bar{z}+\nu c} \mid \lambda_i >$
$= e^{\lambda_i \; \lambda_2\varphi \; +\nu+R-2z\bar{z}} < \lambda_i \mid (E^0_{\alpha 1}+E^0_{\alpha 3}) \left(1+\chi\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right) \mid \lambda_i >$
$= \chi \; e^{\lambda_i \; \lambda_2\varphi \; +\nu+R-2z\bar{z}} < \lambda_i \mid h_1+h_3 \mid \lambda_i >$
$= \; < \lambda_i \mid (E^0_{\alpha 1}+E^0_{\alpha 3}) \, T_o^{-1} g T_o \mid \lambda_i >$ para i=1,3
Tomemos a autofunção F de $\varepsilon_\pm$ :

$[\varepsilon_\pm, F] = w_\pm F$
Então considerando g = $e^{aF}$ (a constante), $T_o^{-1} g T_o = e^{aF e^{w_+\bar{z}-w_- z}}$
Resumindo podemos escrever

$$\tau_0 = e^{\nu-2z\bar{z}} = \langle 0 \rfloor T^{-1} g T \lfloor 0 \rangle$$

$$\tau_1 = e^{\lambda_1\lambda_2\varphi+\nu-2z\bar{z}+R} = \langle \lambda_1 \rfloor T^{-1} g T \lfloor \lambda_1 \rangle$$

$$\tau_2 = e^{\lambda_2^2\varphi+\nu-2z\bar{z}} = \langle \lambda_2 \rfloor T^{-1} g T \lfloor \lambda_2 \rangle$$

$$\tau_\psi = \psi\tau_1 = \psi e^{\lambda_1\lambda_2\varphi+\nu-2z\bar{z}+R} = \langle \lambda_1 \rfloor T^{-1} g T \left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right) \lfloor \lambda_1 \rangle$$

$$\tau_\chi = \chi\tau_1 = \chi e^{\lambda_1\lambda_2\varphi+\nu-2z\bar{z}+R} = \langle \lambda_1 \rfloor \left(E^0_{\alpha 1}+E^0_{\alpha 3}\right) T^{-1} g T \lfloor \lambda_1 \rangle \qquad (3.101)$$

As equações de movimento em termos das funções tau escrevem-se
$\tau_1^2\tau_0^2\tau_2\partial\bar{\partial}\tau_2 - \tau_1^2\tau_0\tau_2^2\partial\bar{\partial}\tau_0 + \tau_1^2\partial\tau_0\tau_2^2\bar{\partial}\tau_0 - \tau_1^2\partial\tau_2\tau_0^2\bar{\partial}\tau_2 - 2\tau_0^3\tau_2\tau_1^2 - 4\tau_0^3\tau_2\tau_\chi\tau_\psi + 2\tau_2^3\tau_0\tau_1^2 + 4\tau_2^3\tau_0\tau_\chi\tau_\psi = 0$

$2\tau_1^2\tau_0^2 + \tau_1^2\tau_0\partial\bar{\partial}\tau_0 - \tau_1^2\partial\tau_0\bar{\partial}\tau_0 - 2\tau_1^2\tau_0\tau_2 - 4\tau_0\tau_2\tau_\chi\tau_\psi = 0$

$\tau_1^4\tau_2\tau_0\partial\bar{\partial}\tau_\chi - \tau_1^3\tau_2\tau_0\bar{\partial}\tau_\chi\partial\tau_1 - \tau_1^3\tau_2\tau_0\partial\tau_\chi\bar{\partial}\tau_1 - \tau_1^3\tau_2\tau_0\tau_\chi\partial\bar{\partial}\tau_1 + 2\tau_1^2\tau_2\tau_0\partial\tau_1\tau_\chi\bar{\partial}\tau_1 +$
$+\tau_1\tau_2\tau_0\tau_\chi^2\tau_\psi\bar{\partial}\partial\tau_1 + \tau_1^2\tau_2\tau_0\tau_\chi\tau_\psi\partial\bar{\partial}\tau_\chi + \tau_2\tau_0\tau_\chi^2\tau_\psi\partial\tau_1\bar{\partial}\tau_1 - \tau_1^2\tau_2\tau_0\tau_\psi\partial\tau_\chi\bar{\partial}\tau_\chi +$
$+2\tau_1^4\tau_2^2\tau_\chi + 2\tau_1^4\tau_0^2\tau_\chi + 4\tau_1^2\tau_2^2\tau_\chi^2\tau_\psi + 4\tau_1^2\tau_0^2\tau_\chi^2\tau_\psi + 2\tau_2^2\tau_\chi^3\tau_\psi^2 + 2\tau_0^2\tau_\chi^3\tau_\psi^2 = 0$

$\tau_1^4\tau_2\tau_0\partial\bar{\partial}\tau_\psi - \tau_1^3\tau_2\tau_0\bar{\partial}\tau_\psi\partial\tau_1 - \tau_1^3\tau_2\tau_0\partial\tau_\psi\bar{\partial}\tau_1 - \tau_1^3\tau_2\tau_0\tau_\psi\partial\bar{\partial}\tau_1 + 2\tau_1^2\tau_2\tau_0\partial\tau_1\tau_\psi\bar{\partial}\tau_1 +$
$+\tau_1\tau_2\tau_0\tau_\chi\tau_\psi^2\bar{\partial}\partial\tau_1 + \tau_1^2\tau_2\tau_0\tau_\chi\tau_\psi\partial\bar{\partial}\tau_\psi + \tau_2\tau_0\tau_\chi\tau_\psi^2\partial\tau_1\bar{\partial}\tau_1 - \tau_1^2\tau_2\tau_0\tau_\chi\partial\tau_\psi\bar{\partial}\tau_\psi +$

$$+2\tau_1^4\tau_2^2\tau_\psi + 2\tau_1^4\tau_0^2\tau_\psi + 4\tau_1^2\tau_2^2\tau_\psi^2\tau_\chi + 4\tau_1^2\tau_0^2\tau_\psi^2\tau_\chi + 2\tau_2^2\tau_\psi^3\tau_\chi^2 + 2\tau_0^2\tau_\psi^3\tau_\chi^2 = 0$$

$$\begin{aligned}&\partial\tau_1\tau_2\tau_0\tau_1^2 + 2\partial\tau_1\tau_2\tau_0\tau_\chi\tau_\psi - \tfrac{1}{2}\tau_1^3\partial\tau_2\tau_0 - \tfrac{1}{2}\tau_1\partial\tau_2\tau_0\tau_\chi\tau_\psi + \\ &-\tfrac{1}{2}\tau_1^3\tau_2\partial\tau_0 - \tfrac{1}{2}\tau_1\tau_2\partial\tau_0\tau_\chi\tau_\psi - \tau_\psi\tau_2\tau_0\partial\tau_\chi\tau_1 \ = 0\end{aligned} \tag{3.102}$$

## 3.9 Solução de um Vértice

Considere g=$e^{aF_4^+(\gamma)}$ com $F_4^+(\gamma)$ dado na expressão (3.93).

Então: $T_o^{-1} g T_o$ =$e^{a\rho F_4^+(\gamma)}$ com $\rho = e^{2\gamma\bar{z} - \frac{2}{\gamma}z}$

Façamos a expansão até a 2ª ordem:

$e^{a\rho F_4^+(\gamma)} = 1 + a\rho F_4^+(\gamma) + \frac{a^2\rho^2}{2!}\left(F_4^+(\gamma)\right)^2$

Temos

$$< \lambda_i \mid e^{a\rho F_4^+(\gamma)} \mid \lambda_i >= 1 + a\rho < \lambda_i \mid F_4^+(\gamma) \mid \lambda_i > + \frac{a^2\rho^2}{2!} < \lambda_i \mid \left(F_4^+(\gamma)\right)^2 \mid \lambda_i > \qquad (3.103)$$

com i=0,1,2. Utilizando $F_4^+(\gamma)$ da expressão (3.93) temos:

$$< \lambda_i \mid \left(F_4^+(\gamma)\right)^2 \mid \lambda_i >=< \lambda_i \mid \{ \sum_{n\geq 0, m\leq 0} [I_{m,n}]\gamma^{-2(m+n)} + 1\} \mid \lambda_i > \qquad (3.104)$$

onde $I_{m,n} = h_1^n h_1^m + h_3^n h_3^m + 4h_2^n h_2^m + +4h_1^n h_2^m + 4h_2^n h_3^m - E^n_{\alpha_1+\alpha_2} E^m_{-\alpha_1-\alpha_2} +$
$-E^n_{-\alpha_1-\alpha_2} E^m_{\alpha_1+\alpha_2} - E^n_{\alpha_2+\alpha_3} E^m_{-\alpha_2-\alpha_3} - E^n_{-\alpha_2-\alpha_3} E^m_{\alpha_2+\alpha_3} - 2h_1 - 4h_2 - 2h_3$

Podemos substituir os diferentes produtos de operadores pelos seus comutadores e usar

$[h_1^n, h_1^m] = [h_3^n, h_3^m] = [h_2^n, h_2^m] = 2n\ \delta_{n+m,0}$

$[h_1^n, h_2^m] = [h_2^n, h_3^m] = -n\ \delta_{n+m,0}$

$\left[E^n_{\alpha_1+\alpha_2}, E^m_{-\alpha_1-\alpha_2}\right] = h_1^{n+m} + h_2^{n+m} + n\ \delta_{n+m,0}$

$\left[E^n_{\alpha_2+\alpha_3}, E^m_{-\alpha_2-\alpha_3}\right] = h_2^{n+m} + h_3^{n+m} + n\ \delta_{n+m,0}$

Neste caso a equação (3.104) reduz-se a:

$< \lambda_0 \mid \left(F_4^+(\gamma)\right)^2 \mid \lambda_0> = < \lambda_0 \mid \{ \sum_{n>0} (2n + 2n + 8n - 4n - 4n - 4n) + 1 \} \mid \lambda_0 >$

$< \lambda_1 \mid \left(F_4^+(\gamma)\right)^2 \mid \lambda_1>=0$

$< \lambda_2 \mid \left(F_4^+(\gamma)\right)^2 \mid \lambda_2>=1$

$< \lambda_0 \mid F_4^+(\gamma) \mid \lambda_0>= -1$

$< \lambda_1 \mid F_4^+(\gamma) \mid \lambda_1>=0$

$< \lambda_2 \mid F_4^+(\gamma) \mid \lambda_2>=1$

Daí segue

$\tau_0 =< \lambda_0 \mid T_o^{-1} g T_o \mid \lambda_0> = e^{\nu - 2z\bar{z}} = 1 - a\rho(\gamma) + \frac{a^2}{2!}\rho^2(\gamma)$

$\tau_1 =< \lambda_1 \mid T_o^{-1} g T_o \mid \lambda_1> = 1 = e^{\frac{1}{2}\varphi\ +\ R - 2z\bar{z} + \nu}$

$\tau_2 =< \lambda_2 \mid T_o^{-1} g T_o \mid \lambda_2> = 1 + a\rho(\gamma) + \frac{a^2}{2!}\rho^2(\gamma) = e^{\varphi + \nu - 2z\bar{z}}$

$\chi = \psi = 0 \to R$ =constante

De $\tau_1 = 1$ segue : $\varphi = -2(\nu - 2z\bar{z}) - 2R$,que introduzida em $\tau_2$ dá

$e^{-2(\nu - 2z\bar{z}) - 2R} = 1 + a\rho(\gamma) + \frac{a^2}{2!}\rho^2(\gamma)$

ou:

$e^{-2R} = \left(1 + a\rho(\gamma) + \frac{a^2}{2!}\rho^2(\gamma)\right)\left(1 - a\rho(\gamma) + \frac{a^2}{2!}\rho^2(\gamma)\right)$

dando R não constante que é uma contradição.

A fim de evitarnos este problema escrevamos:

$< \lambda_i \mid \left(F_4^+(\gamma)\right)^2 \mid \lambda_i >= \lim_{\gamma_1 \to \gamma_2 = \gamma} < \lambda_i \mid F_4^+(\gamma_1)\ F_4^+(\gamma_2) \mid \lambda_i >=$
$= < \lambda_i \mid \sum_{n \geq 0, m \leq 0} [h_1^n h_1^m + h_3^n h_3^m + 4h_2^n h_2^m + +4h_1^n h_2^m + 4h_2^n h_3^m]\ \gamma_1^{-2n}\gamma_2^{-2m} +$
$- \left[E^n_{\alpha_1+\alpha_2} E^m_{-\alpha_1-\alpha_2} + E^n_{\alpha_2+\alpha_3} E^m_{-\alpha_2-\alpha_3}\right] \gamma_1^{-(2n+1)}\gamma_2^{-(2m-1)} +$
$- \left[E^n_{-\alpha_1-\alpha_2} E^m_{\alpha_1+\alpha_2} + E^n_{-\alpha_2-\alpha_3} E^m_{\alpha_2+\alpha_3}\right] \gamma_1^{-(2n-1)}\gamma_2^{-(2m+1)} + 1 \mid \lambda_i >$
Substituindo os diferentes produtos de operadores pelos seus comutadores segue
$\lim_{\gamma_1 \to \gamma_2 = \gamma} < \lambda_0 \mid F_4^+(\gamma_1)\ F_4^+(\gamma_2) \mid \lambda_0 >=$
$= \lim_{\gamma_1 \to \gamma_2} \sum_{n>0} 4\text{n} \left(\frac{\gamma_2}{\gamma_1}\right)^{2n} - \sum_{n>0} 2\left(\frac{\gamma_2}{\gamma_1} + \frac{\gamma_1}{\gamma_2}\right) \left(\frac{\gamma_2}{\gamma_1}\right)^{2n} n$
$= \lim_{\gamma_1 \to \gamma_2} \sum_{n>0} \left(4 - 2\frac{\gamma_2}{\gamma_1} - 2\frac{\gamma_1}{\gamma_2}\right) \left(\frac{\gamma_2}{\gamma_1}\right)^{2n} n = \lim_{\gamma_1 \to \gamma_2} 2\left(2 - \frac{\gamma_2}{\gamma_1} - \frac{\gamma_1}{\gamma_2}\right) \frac{\left(\frac{\gamma_2}{\gamma_1}\right)^2}{\left[1 - \left(\frac{\gamma_2}{\gamma_1}\right)^2\right]^2}$
$= \lim_{\gamma_1 \to \gamma_2} \frac{2}{\gamma_1\gamma_2} \frac{\left(2\gamma_1\gamma_2 - \gamma_2^2 - \gamma_1^2\right)\left(\frac{\gamma_2}{\gamma_1}\right)^2}{\left[1 - \left(\frac{\gamma_2}{\gamma_1}\right)^2\right]^2} = \lim_{\gamma_1 \to \gamma_2} - \frac{2}{\gamma_1\gamma_2} \frac{(\gamma_1 - \gamma_2)^2 \left(\frac{\gamma_2}{\gamma_1}\right)^2}{\left[1 - \frac{\gamma_2}{\gamma_1}\right]^2 \left[1 + \frac{\gamma_2}{\gamma_1}\right]^2} =$
$\lim_{\gamma_1 \to \gamma_2} - \frac{2}{\left(\frac{\gamma_2}{\gamma_1}\right)} \frac{\left(\frac{\gamma_2}{\gamma_1}\right)^2}{\left[1 + \frac{\gamma_2}{\gamma_1}\right]^2} = - \frac{1}{2}$
Daí segue
$< \lambda_0 \mid \left(F_4^+(\gamma)\right)^2 \mid \lambda_0 >= \frac{1}{2}$
$< \lambda_1 \mid \left(F_4^+(\gamma)\right)^2 \mid \lambda_1 >= -\frac{1}{2}$
$< \lambda_2 \mid \left(F_4^+(\gamma)\right)^2 \mid \lambda_2 >= \frac{1}{2}$
e portanto

$$e^{\nu - 2z\bar{z}} = 1 - a\rho(\gamma) + \frac{a^2}{4}\rho^2(\gamma)$$

$$e^{\frac{1}{2}\varphi\ +\ R - 2z\bar{z} + \nu} = 1 - \frac{a^2}{4}\rho^2(\gamma)$$

$$e^{\varphi\ -\ z\bar{z} + \nu} = 1 + a\rho(\gamma) + \frac{a^2}{4}\rho^2(\gamma) \tag{3.105}$$

Daí segue

$$e^{\varphi} = \frac{1 + a\rho(\gamma) + \frac{a^2}{4}\rho^2(\gamma)}{1 - a\rho(\gamma) + \frac{a^2}{4}\rho^2(\gamma)} \tag{3.106}$$

Também

$e^R = \frac{\left(1 - \frac{a^2}{4}\rho^2(\gamma)\right)\sqrt{1 - a\rho(\gamma) + \frac{a^2}{4}\rho^2(\gamma)}}{\left(1 - a\rho(\gamma) + \frac{a^2}{4}\rho^2(\gamma)\right)\sqrt{1 + a\rho(\gamma) + \frac{a^2}{4}\rho^2(\gamma)}} = \frac{\left(1 - \frac{a^2}{4}\rho^2(\gamma)\right)}{\sqrt{1 - a\rho(\gamma) + \frac{a^2}{4}\rho^2(\gamma)}\sqrt{1 + a\rho(\gamma) + \frac{a^2}{4}\rho^2(\gamma)}}$
$e^R = \frac{\left(1 - \frac{a^2}{4}\rho^2(\gamma)\right)}{\sqrt{(1 + a\rho(\gamma))^2 - a^2\rho^2(\gamma)}} = 1$
e portanto R=0 e consistente com $\chi = \psi = 0$

## 3.10 Solução com 2 vértices

Tomemos g $= e^{a_1 F_1^+(\gamma_1)} e^{a_2 F_2^-(\gamma_2)}$, Então:

$$T^{-1} g T = e^{a_1 \rho_1^+ F_1^+(\gamma_1)} e^{a_2 \rho_2^- F_2^-(\gamma_2)}$$

com $\rho_1^+ = e^{2\gamma_1 \bar{z} - \frac{2}{\gamma_1} z}$ $\quad \rho_2^- = e^{-2\gamma_2 \bar{z} + \frac{2}{\gamma_2} z}$

Calculemos $\tau$, expandindo $e^{a_1 \rho_1^+ F_1^+(\gamma_1)}$ e $e^{a_2 \rho_2^- F_2^-(\gamma_2)}$ até 1ª ordem cada um dêles.

Segue então utilizando as expressões

$$< \lambda_i \mid F_1^+(\gamma_1)\ F_2^-(\gamma_2) \mid \lambda_i > =$$

$$< \lambda_i \mid \sum_{n\geq 0, m\leq 0} \left(h_1^{m+n} + n\delta_{m+n,0}\right) \gamma_1^{-2n} \gamma_2^{-2m}$$

$$+ \sum_{n\geq 0, m\leq 0} \left(h_1^{m+n} + h_2^{m+n} + h_3^{m+n} + n\delta_{m+n,0}\right) \gamma_1^{-(2n+1)} \gamma_2^{-(2m-1)}$$

$$+ \sum_{n>0, m<0} \left(-h_2^{m+n} + n\ \delta_{m+n,0}\right) \gamma_1^{-(2n-1)} \gamma_2^{-(2m+1)}$$

$$+ \sum_{n\geq 0, m\leq 0} \left(h_3^{m+n} + n\ \delta_{m+n,0}\right) \gamma_1^{-2n} \gamma_2^{-2m} \mid \lambda_i >$$

onde $\mid \lambda_i >$ pode ser $\mid \lambda_0 >, \mid \lambda_1 >, \mid \lambda_2 >, \mid \lambda_3 >$, etc.

Por tanto

$$< \lambda_0 \mid F_1^+(\gamma_1)\ F_2^-(\gamma_2) \mid \lambda_0 >$$

$$= \sum_{n=0}^{\infty} n \left(\frac{\gamma_2}{\gamma_1}\right)^{2n} + \sum_{n=0}^{\infty} n \frac{\gamma_2}{\gamma_1} \left(\frac{\gamma_2}{\gamma_1}\right)^{2n} + \sum_{n=1}^{\infty} n \frac{\gamma_1}{\gamma_2} \left(\frac{\gamma_2}{\gamma_1}\right)^{2n} + \sum_{n=0}^{\infty} n \left(\frac{\gamma_2}{\gamma_1}\right)^{2n}$$

$$= \left(2 + \frac{\gamma_2}{\gamma_1} + \frac{\gamma_1}{\gamma_2}\right) \sum_{n=0}^{\infty} n \left(\frac{\gamma_2}{\gamma_1}\right)^{2n}$$

Utilizando $x^n = \sum_{n=0}^{\infty} \frac{1}{1-x}$, $x\frac{\partial}{\partial x} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$

Segue-se $\qquad \sum_{n=0}^{\infty} n \left(\frac{\gamma_2}{\gamma_1}\right)^n = \frac{\gamma_1^2 \gamma_2^2}{(\gamma_1 - \gamma_2)^2}$

e portanto:

$$< \lambda_0 \mid F_1^+(\gamma_1)\ F_2^-(\gamma_2) \mid \lambda_0 > = \frac{\gamma_1 \gamma_2}{(\gamma_1 - \gamma_2)^2}$$

Da mesma maneira segue-se

$$< \lambda_1 \mid F_1^+(\gamma_1)\ F_2^-(\gamma_2) \mid \lambda_1 > = \frac{\gamma_1^2}{(\gamma_1 - \gamma_2)^2}$$

$$< \lambda_2 \mid F_1^+(\gamma_1)\ F_2^-(\gamma_2) \mid \lambda_2 > = \frac{\gamma_1 \gamma_2}{(\gamma_1 - \gamma_2)^2}$$

Portanto segue

$$e^{\nu - 2z\bar{z}} = < \lambda_0 \mid T_o^{-1} g T_o \mid \lambda_0 > = 1 + a_1 a_2 \rho_1^+\ \rho_2^- \frac{\gamma_1 \gamma_2}{(\gamma_1 - \gamma_2)^2} \tag{3.107}$$

$$e^{\varphi - 2z\bar{z} + \nu} = e^{\nu - 2z\bar{z}} \tag{3.108}$$

e portanto $\varphi = 0$

Também segue:

$$e^{\frac{1}{2}\varphi + R\ -\ z\bar{z} + \nu} = 1 + a_1 a_2 \rho_1^+\ \rho_2^- \frac{\gamma_1^2}{(\gamma_1 - \gamma_2)^2} \tag{3.109}$$

Temos tambem

$< \lambda_1 \mid T_o^{-1} g T_o \, (E^0_{-\alpha 1} + E^0_{-\alpha 3}) \mid \lambda_1 >$
$= a_1 < \lambda_1 \mid F_1^+ \, (\gamma_1) \, \rho_1^+ (E^0_{-\alpha 1} + E^0_{-\alpha 3}) \mid \lambda_1 >$
$= a_1 \rho_1^+ < \lambda_1 \mid \sum_{n=0}^{\infty} E^n_{\alpha 1} \gamma_1^{-2n} E^0_{-\alpha 1} + \sum_{n=0}^{\infty} E^n_{\alpha 3} \gamma_1^{-2n} E^0_{-\alpha 3} \mid \lambda_1 >$
$= a_1 \rho_1^+ < \lambda_1 \mid \sum_{n=0}^{\infty} \left( h_1^n \gamma_1^{-2n} + h_3^n \gamma_1^{-2n} \right) \mid \lambda_1 >= a_1 \rho_1^+$
Portanto

$$\psi = \frac{a_1 \rho_1^+}{1 + a_1 a_2 \rho_1^+ \, \rho_2^- \frac{\gamma_1^2}{(\gamma_1 - \gamma_2)^2}} \tag{3.110}$$

Da mesma maneira considerando:

$< \lambda_1 \mid (E^0_{\alpha 1} + E^0_{\alpha 3}) T_o^{-1} g T_o \mid \lambda_1 > =$

$a_2 \rho_2^- < \lambda_1 \mid E^0_{\alpha 1} \sum_{n=0}^{\infty} E^n_{-\alpha 1} \gamma_2^{-2n} \mid \lambda_1 > = a_2 \rho_2^-$
Portanto

$$\chi = \frac{a_2 \rho_2^-}{1 + a_1 a_2 \rho_1^+ \, \rho_2^- \frac{\gamma_1^2}{(\gamma_1 - \gamma_2)^2}} \tag{3.111}$$

Pode-se mostrar que (3.107)-(3.111) são soluções das equações (3.102) até a segunda ordem em $\rho$.

Entretanto não são soluções em ordem superior.

# 4 Capítulo 4

## 4.1 Conclusões

## 4.2

De acordo com sugestão do professor L.A.Ferreira, podemos utilizar o método de Hirota para expandir as funções tau em series de potencias de exponenciais que dão o comportamento espaço temporal dos elementos de matriz

$$< \lambda_i \mid T_o^{-1} g T_o \mid \lambda_i >$$

.

Introduzindo nas equações (3.102) podemos resolve-las utilizando o programa de " Mathemática ".

No caso das soluções de um vértice obtivemos como resultado as expressões (3.105), já obtidas analíticamente. Além disso obtivemos também soluções com $\chi = 0, \psi \neq 0, \varphi \neq 0$,ou $\chi \neq 0, \psi = 0, \varphi \neq 0$.

No caso de soluções com dois vértices, $\tau_0, \tau_1$ e $\tau_2$ são expandidas até segunda ordem nas exponenciais enquanto que $\tau_\chi$ e $\tau_\psi$ são expandidas até primeira ordem. Entretanto eles não são soluções de (3.102) em ordem superior.

Ao tomarmos expansões nas exponenciais em ordem superior para as funções tau, mais explicitamente, até quarta ordem, chegamos por cálculo computacional, à conclusão que $\tau_0, \tau_1$ e $\tau_2$ truncam até segumda ordem enquanto que $\tau_\chi$ e $\tau_\psi$ não. Por exemplo obtivemos :

$$\tau_0 = 1 + a_1 a_2 \frac{e^{2\left(-\frac{\bar{z}}{\gamma_1} + z\,\gamma_1\right) - 2\left(-\frac{\bar{z}}{\gamma_2} + z\,\gamma_2\right)} \gamma_1 \gamma_2}{(\gamma_1 - \gamma_2)^2} \tag{4.1}$$

$$\tau_2 = \tau_0 \tag{4.2}$$

$$\tau_1 = 1 + a_1 a_2 \frac{e^{2\left(-\frac{\bar{z}}{\gamma_1} + z\,\gamma_1\right) - 2\left(-\frac{\bar{z}}{\gamma_2} + z\,\gamma_2\right)} \gamma_2^2}{(\gamma_1 - \gamma_2)^2} \tag{4.3}$$

$$\tau_\chi = a_2 e^{-2\left(-\frac{\bar{z}}{\gamma_2} + z\,\gamma_2\right)} - a_1 a_2^2 \frac{e^{2\left(-\frac{\bar{z}}{\gamma_1} + z\,\gamma_1\right) - 4\left(-\frac{\bar{z}}{\gamma_2} + z\,\gamma_2\right)} \gamma_2}{(\gamma_1 - \gamma_2)} +$$

$$+ a_1^2 a_2^3 \frac{e^{4\left(-\frac{\bar{z}}{\gamma_1} + z\,\gamma_1\right) - 6\left(-\frac{\bar{z}}{\gamma_2} + z\,\gamma_2\right)} \gamma_1 \gamma_2^2}{(\gamma_1 - \gamma_2)^3} \tag{4.4}$$

$$\tau_\psi = a_1 e^{-2\left(-\frac{\bar{z}}{\gamma_1} + z\,\gamma_1\right)} - a_1^2 a_2 \frac{e^{4\left(-\frac{\bar{z}}{\gamma_1} + z\,\gamma_1\right) - 2\left(-\frac{\bar{z}}{\gamma_2} + z\,\gamma_2\right)} \gamma_2}{(\gamma_1 - \gamma_2)} +$$

$$+ a_1^3 a_2^2 \frac{e^{6\left(-\frac{\bar{z}}{\gamma_1} + z\,\gamma_1\right) - 4\left(-\frac{\bar{z}}{\gamma_2} + z\,\gamma_2\right)} \gamma_1 \gamma_2^2}{(\gamma_1 - \gamma_2)^3} \tag{4.5}$$

Entretanto estas não são soluções das equações (3.102).

Indo para ordens superiores verificamos que $\tau_0, \tau_1$ e $\tau_2$ permanecem inalteradas e que $\tau_\chi$ e $\tau_\psi$ recebem contribuções adicionais. Constatamos uma regularidade: cada termo da série que comparece em $\tau_\chi$ e $\tau_\psi$ é obtido do anterior pela multiplicação do fator:

$$-a_1 a_2 \frac{(\gamma_1\gamma_2)\; e^{-2\left(-\frac{\bar{z}}{\gamma_2}+z\,\gamma_2\right)}\; e^{2\left(-\frac{\bar{z}}{\gamma_1}+z\,\gamma_1\right)}}{(\gamma_1-\gamma_2)^3} \tag{4.6}$$

Dessa maneira $\tau_\chi$ e $\tau_\psi$ não são truncados e são expressos por meio de séries infinitas. Podemos soma-las obtendo:

$$\tau_\chi = \frac{a_2 e^{-2\left(-\frac{\bar{z}}{\gamma_2}+z\,\gamma_2\right)}\,(\gamma_1-\gamma_2+\gamma_2\,\tau_0)}{\gamma_1\,\tau_0} \tag{4.7}$$

$$\tau_\psi = \frac{a_1 e^{-2\left(-\frac{\bar{z}}{\gamma_1}+z\,\gamma_1\right)}\,(\gamma_1-\gamma_2+\gamma_2\,\tau_0)}{\gamma_1\,\tau_0} \tag{4.8}$$

Mostra-se que (4.1), (4.2), (4.3), (4.4), (4.5) verificam as equações (3.102) e portanto dão $\psi \neq 0, \chi \neq 0$ e $\varphi = 0$.

O passo seguinte é o de construir as soluções de tres e mais vértices.

# 5 Apêndice A.

## 5.1 Cálculo de $B^{-1}(\partial B)$.

Indiquemos $h = h_2 + \frac{1}{2}h_1 + \frac{1}{2}h_3,\; h' = h_1 + h_3$

$$B = \quad e^{\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}\; e^{\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\varphi+(h_1+h_3)R+\nu c+\eta d}e^{\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)} \tag{A.1}$$

Seja
$n = \; e^{\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)},\;\; a = e^{\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\varphi+(h_1+h_3)R+\nu c+\eta d},\; m = e^{\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}$
isto é $\quad B = nam$ ou $B^{-1} = m^{-1}a^{-1}n^{-1}$

*Seja* $I = B^{-1}(\partial B)$
$I = m^{-1}a^{-1}n^{-1}(\partial n)\, am + m^{-1}a^{-1}(\partial a)\, m + m^{-1}(\partial m)$
$m^{-1}(\partial m) = \left(\partial\widetilde{\psi}\right)(E^0_{\alpha 1} + E^0_{\alpha 3})$
$n^{-1}(\partial n) = (\partial\widetilde{\chi})\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)$
$a^{-1}(\partial a) = \left[\left(h_2 + \frac{1}{2}h_1 + \frac{1}{2}h_3\right)\partial\varphi + (h_1 + h_3)\,\partial R + (\partial\nu)\, c + (\partial\eta)\, d\right]$

Usando estas tres ultimas expresiones

$I = \left(\partial\widetilde{\psi}\right)(E^0_{\alpha 1} + E^0_{\alpha 3}) +$
$\left(h_2 + \frac{1}{2}h_1 + \frac{1}{2}h_3\right)\partial\varphi +$
$(\partial R)\, e^{-\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}(h_1 + h_3)\, e^{\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)} + (\partial\nu)\, c + (\partial\eta)\, d +$
$(\partial\widetilde{\chi})\, e^{-\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}e^{-(h_1+h_3)R}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)e^{(h_1+h_3)R}e^{\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}$

Seja

$\mathrm{M} = e^{-\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}(h_1 + h_3)\, e^{\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}$ e $\mathrm{N} = e^{-(h_1+h_3)R}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)e^{(h_1+h_3)R}$
$I = \left(\partial\widetilde{\psi}\right)(E^0_{\alpha 1} + E^0_{\alpha 3}) + \left(h_2 + \frac{1}{2}h_1 + \frac{1}{2}h_3\right)\partial\varphi +$
$(\partial R)\, M + (\partial\nu)\, c + (\partial\eta)\, d + (\partial\widetilde{\chi})\, e^{-\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}N e^{\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}$

(A.3)

pode-se verificar que

$$M = e^{-\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}(h_1 + h_3)\, e^{\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)} = h' + 2\widetilde{\psi}\left(E^0_{\alpha 1} + E^0_{\alpha 3}\right) \tag{A.4}$$

$$N = e^{-(h_1+h_3)R}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)e^{(h_1+h_3)R} = e^{2R}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right) \tag{A.5}$$

substituindo (A.4) e (A.5) em (A.3)

$$I = \left(\partial\widetilde{\psi}\right)(E^0_{\alpha 1} + E^0_{\alpha 3}) +$$
$$\left(h_2 + \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3\right)\partial\varphi + (\partial R)\left(h' + 2\widetilde{\psi}\,(E^0_{\alpha 1} + E^0_{\alpha 3})\right) + (\partial\nu)\,c + (\partial\eta)\,d +$$
$$(\partial\widetilde{\chi})\,e^{-\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}e^{2R}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)e^{\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)} \tag{A.6}$$

pode-se verificar que

$$e^{-\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)e^{\widetilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)} = \quad E^0_{-\alpha 1} + E^0_{-\alpha 3} - \widetilde{\psi}h' - \widetilde{\psi}^2\,(E^0_{\alpha 1} + E^0_{\alpha 3})$$
$$I = \left(\partial\widetilde{\psi}\right)(E^0_{\alpha 1} + E^0_{\alpha 3}) +$$
$$\left[\left(h_2 + \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3\right)\partial\varphi + (\partial R)\left(h' + 2\widetilde{\psi}\,(E^0_{\alpha 1} + E^0_{\alpha 3})\right) + (\partial\nu)\,c + (\partial\eta)\,d\right] +$$
$$(\partial\widetilde{\chi})\;e^{2R}\left[E^0_{-\alpha 1} + E^0_{-\alpha 3} - \widetilde{\psi}h' - \widetilde{\psi}^2\,(E^0_{\alpha 1} + E^0_{\alpha 3})\right] \tag{A.7}$$

$$I = \left[\left(\partial\widetilde{\psi}\right) + 2\widetilde{\psi}\,(\partial R) - \widetilde{\psi}^2\,(\partial\widetilde{\chi})\,e^{2R}\right](E^0_{\alpha 1} + E^0_{\alpha 3}) + (\partial\widetilde{\chi})\;e^{2R}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)$$
$$+\left(h_2 + \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3\right)\partial\varphi + (\partial R)\,(h') + (\partial\nu)\,c + (\partial\eta)\,d$$

fazendo as transformações $\widetilde{\psi} = \psi\;e^{-R}$, $\widetilde{\chi} = \chi\;e^{-R}$ temos

$$I = B^{-1}\,(\partial B) =$$
$$I = \left[(\partial\psi) + \psi\,(\partial R) + \psi^2\,(\chi\,(\partial R) - \partial\chi)\right](E^0_{\alpha 1} + E^0_{\alpha 3}) +$$
$$[(1 + \chi\psi)\,(\partial R) - \psi\partial\chi]\,(h_1 + h_3) +$$
$$(\partial\chi - \chi\partial R)\;e^R\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right) +$$
$$+\partial\varphi\left(h_2 + \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3\right) + (\partial\nu)\,c + (\partial\eta)\,d$$

Considerando o coeficiente em direção $(h_1 + h_3)$ igual a zero, isto é

$(1 + \chi\psi)\,(\partial R) - \psi\partial\chi = 0$ , isto e $\quad \partial R = \frac{\psi\partial\chi}{(1+\chi\psi)}$ $\quad$ finalmente temos

$$B^{-1}\,(\partial B)\,|_{vinc} =$$
$$= (\partial\psi)\,e^{-R}\,(E^0_{\alpha 1} + E^0_{\alpha 3}) + \tfrac{\partial\chi}{\Delta}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)$$
$$+\partial\varphi\left(h_2 + \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3\right) + (\partial\nu)\,c + (\partial\eta)\,d \tag{A.8}$$

# 6 Apêndice B.

## 6.1 Cálculo de $\overline{\partial}BB^{-1}$.

$$B = \quad e^{\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}\ e^{\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\varphi+(h_1+h_3)R+\nu c+\eta d}e^{\widetilde{\psi}\left(E^0_{\alpha1}+E^0_{\alpha3}\right)} \tag{B.1}$$

Seja $\mathrm{n} = e^{\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}$ $\mathrm{a}{=}e^{\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\varphi+(h_1+h_3)R+\nu c+\eta d}$ $\mathrm{m}{=}e^{\widetilde{\psi}\left(E^0_{\alpha1}+E^0_{\alpha3}\right)}$

(B.2)

$$\left(\overline{\partial}B\right) = \overline{\partial}\left(nam\right) = \left(\overline{\partial}n\right)am+ n\left(\overline{\partial}a\right)m+ na\left(\overline{\partial}m\right)$$
$$\left(\overline{\partial}m\right)m^{-1} = \left(\overline{\partial}\widetilde{\psi}\right)\left(E^0_{\alpha1}+E^0_{\alpha3}\right)$$

$$\left(\overline{\partial}n\right)n^{-1} = \left(\overline{\partial}\widetilde{\chi}\right)\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)$$
$$\left(\overline{\partial}a\right)a^{-1} = \left[\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)\overline{\partial}\varphi+(h_1+h_3)\,\overline{\partial}R+\left(\overline{\partial}\nu\right)c+\left(\overline{\partial}\eta\right)d\right]$$

Seja $J = \left(\overline{\partial}B\right)B^{-1}$

$$J = \left[\left(\overline{\partial}n\right)n^{-1}+n\left(\overline{\partial}a\right)a^{-1}n^{-1}+na\left(\overline{\partial}m\right)m^{-1}a^{-1}n^{-1}\right]$$

$= J1 + J2 + J3$ onde

$$\mathrm{J1} = \left(\overline{\partial}n\right)n^{-1} = \left(\overline{\partial}\widetilde{\chi}\right)\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)$$

$$\mathrm{J2} = n\left(\overline{\partial}a\right)a^{-1}n^{-1}$$
$$= e^{\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}\left[\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)\overline{\partial}\varphi+(h_1+h_3)\,\overline{\partial}R+\left(\overline{\partial}\nu\right)c+\left(\overline{\partial}\eta\right)d\right]\ e^{-\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}$$

$$\mathrm{J3} = na\left(\overline{\partial}m\right)m^{-1}a^{-1}n^{-1}$$

$$= e^{\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}\ e^{\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\varphi+(h_1+h_3)R+\nu c+\eta d}\ \left(\overline{\partial}\widetilde{\psi}\right)\left(E^0_{\alpha1}+E^0_{\alpha3}\right)$$
$$e^{-\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\varphi-(h_1+h_3)R-\nu c-\eta d}\ e^{-\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}$$
$$\mathrm{J2} = \left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)\overline{\partial}\varphi+\left(\overline{\partial}R\right)\left[e^{\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}\left(h_1+h_3\right)e^{-\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}\right]$$
$$+\left(\overline{\partial}\nu\right)c+\left(\overline{\partial}\eta\right)d$$

Seja $\mathrm{M} = e^{\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}\left(h_1+h_3\right)e^{-\widetilde{\chi}\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right)}$

usando $e^p\ \mathrm{b}\ e^{-p} = \mathrm{b} + [p,\mathrm{b}] + \frac{1}{2!}[p,[p,\mathrm{b}]] + \frac{1}{3!}[p,[p,[p,\mathrm{b}]]] + \ldots$ temos

$$\mathrm{M}{=}\ (h_1+h_3)+\widetilde{\chi}[\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right),(h_1+h_3)]+$$
$$+\tfrac{\widetilde{\chi}^2}{2!}\left[\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right),\left[\left(E^0_{-\alpha1}+E^0_{-\alpha3}\right),(\ h_1+h_3)\ \right]\right]+\ldots$$

usando $\left[\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),(h_1+h_3)\right]=2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)$ temos

$$M=(h_1+h_3)+\widetilde{\chi}[2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)] \\ +\frac{\widetilde{\chi}^2}{2!}\left[\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\ \right]+...$$

ou $M=(h_1+h_3)+2\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)$

$$J2=\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)\overline{\partial}\varphi \\ +\left(\overline{\partial}R\right)\left[(h_1+h_3)+2\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right] \\ +\left(\overline{\partial}\nu\right)c+\left(\overline{\partial}\eta\right)d$$

agora,

$$J3=e^{\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}e^{\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\varphi+(h_1+h_3)R+\nu c+\eta d} \\ \left(\overline{\partial}\widetilde{\psi}\right)\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)e^{-\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\varphi-(h_1+h_3)R-\nu c-\eta d}\ e^{-\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}$$

simplificando temos

$$J3=\left(\overline{\partial}\widetilde{\psi}\right)\ e^{\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}e^{(h_1+h_3)R}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)e^{-(h_1+h_3)R}\ e^{-\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}=$$

Seja $N=e^{(h_1+h_3)\mathrm{R}}\ \left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\ e^{-(h_1+h_3)\mathrm{R}}$

usando $e^p\ \mathrm{b}\ e^{-p}=\mathrm{b}+[p,\mathrm{b}]+\frac{1}{2!}[p,[p,\mathrm{b}]]+\frac{1}{3!}[p,[p,[p,\mathrm{b}]]]+...$

$$N=\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)+\left[(h_1+h_3)\mathrm{R},\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right] \\ +\frac{1}{2!}\left[(h_1+h_3)\,\mathrm{R},\left[(h_1+h_3)\,\mathrm{R},\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right]\right]+...$$

ordenando

$$N=\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)+\mathrm{R}\left[(h_1+h_3),\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right] \\ +\frac{(\mathrm{R})^2}{2!}\left[(h_1+h_3),\left[(h_1+h_3),\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right]\right]+...$$

agora, usando $\left[h_1+h_3,\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right]=2\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)$ temos

$$N=\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)+\mathrm{R}\left[2\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right]+\frac{(\mathrm{R})^2}{2!}\left[(h_1+h_3),\left[2\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right]\right]+...$$

$$=\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)+R\left[2\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right]+\frac{(R)^2}{2!}(2)\left[2\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right]+...$$

$$=\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\left[1+R\,(2)+\frac{(R)^2}{2!}(2)2+...\right]=\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)e^{2R}$$

$$N=\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)e^{2R}$$

$$J3=\left(\overline{\partial}\widetilde{\psi}\right)\ e^{\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\ e^{2R}\ e^{-\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}$$

$$J3=e^{2R}\left(\overline{\partial}\widetilde{\psi}\right)\ e^{\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)e^{-\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}$$

seja $P=e^{\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}\left[\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)e^{-2R}\right]\ e^{-\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}=$

usando $e^p\ \mathrm{b}\ e^{-p}=\mathrm{b}+[p,\mathrm{b}]+\frac{1}{2!}[p,[p,\mathrm{b}]]+\frac{1}{3!}[p,[p,[p,\mathrm{b}]]]+...$ temos

$$P=\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)+\left[\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right] \\ +\frac{1}{2!}\left[\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),\left[\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right]\right]+... \\ +\frac{1}{3!}\left[\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),\left[\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),\left[\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)\right]\right]\right]+...$$

usando $\left[\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right),\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right]=h_1+h_3$ temos

$$P=\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)+\widetilde{\chi}\left[-h_1-h_3\right]+\frac{\left(\widetilde{\chi}\right)^2}{2!}\left[\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),\left[-h_1-h_3\right]\right]$$

$$+\frac{(\widetilde{\chi})^3}{3!}\left[\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),\left[\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),-h_1-h_3\right]\right]+......$$

agora, usando $[E^0_{-\alpha 1}+E^0_{-\alpha 3},-h'] = 2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)$ temos

$$\text{P} = (E^0_{\alpha 1}+E^0_{\alpha 3})+\widetilde{\chi}\,[-h_1-h_3]+\frac{(\widetilde{\chi})^2}{2!}\left[-2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right]$$
$$-\frac{(\widetilde{\chi})^3}{3!}\left[\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right),-2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right]+......$$
$$\text{P} = (E^0_{\alpha 1}+E^0_{\alpha 3})+\widetilde{\chi}\,[-h_1-h_3]-\frac{(\widetilde{\chi})^2}{2!}\left[2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right]-0$$
$$\text{P} = (E^0_{\alpha 1}+E^0_{\alpha 3})-\widetilde{\chi}\,(h_1+h_3)-\widetilde{\chi}^2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)$$

$$J3 = e^{2R}\left(\overline{\partial}\widetilde{\psi}\right)\left[\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)-\widetilde{\chi}\,(h_1+h_3)-\widetilde{\chi}^2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right] \tag{B.3}$$

juntando (B.1), (B.2) e (B.3) temos

$$\begin{aligned}J =&\\ =&\left(\overline{\partial}B\right)B^{-1}=\left(\overline{\partial}\widetilde{\chi}\right)\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)+\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)\overline{\partial}\varphi\\ &+\left(\overline{\partial}R\right)\left[(h_1+h_3)+2\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right]+\left(\overline{\partial}\nu\right)c+\left(\overline{\partial}\eta\right)d\\ &+e^{2R}\left(\overline{\partial}\widetilde{\psi}\right)\left[(E^0_{\alpha 1}+E^0_{\alpha 3})-\widetilde{\chi}\,(h_1+h_3)-\widetilde{\chi}^2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right]\end{aligned} \tag{B.4}$$

ou em componentes,

$$\begin{aligned}J =&\left(\overline{\partial}B\right)B^{-1}\\ =&\left(\overline{\partial}\widetilde{\chi}\right)\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)+\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)\overline{\partial}\varphi+\\ &+\left(\overline{\partial}R\right)\left[(h_1+h_3)+2\widetilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right]\\ &+\left(\overline{\partial}\nu\right)c+\left(\overline{\partial}\eta\right)d+e^{2R}\left(\overline{\partial}\widetilde{\psi}\right)\left[(E^0_{\alpha 1}+E^0_{\alpha 3})-\widetilde{\chi}\,(h_1+h_3)-\widetilde{\chi}^2\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\right]\end{aligned} \tag{B.5}$$

temos

$$\begin{aligned}&\left(\overline{\partial}B\right)B^{-1}=\left[\left(\overline{\partial}\widetilde{\chi}\right)+\left(\overline{\partial}R\right)2\widetilde{\chi}-e^{2R}\left(\overline{\partial}\widetilde{\psi}\right)\widetilde{\chi}^2\right]\left(\mathrm{E}^0_{-\alpha 1}+\mathrm{E}^0_{-\alpha 3}\right)+\\ &+\left[\left(\overline{\partial}R\right)-e^{2R}\left(\overline{\partial}\widetilde{\psi}\right)\widetilde{\chi}\right](h_1+h_3)+\\ &+e^{2R}\left(\overline{\partial}\widetilde{\psi}\right)(\mathrm{E}^0_{\alpha 1}+\mathrm{E}^0_{\alpha 3})+\left(\mathrm{h}_2+\tfrac{1}{2}\mathrm{h}_1+\tfrac{1}{2}\mathrm{h}_3\right)\left(\overline{\partial}\varphi\right)+\left(\overline{\partial}\nu\right)\mathrm{c}+\left(\overline{\partial}\eta\right)\mathrm{d}\end{aligned}$$

fazendo a transformacão $\widetilde{\chi}->\chi e^{-R}$ , $\widetilde{\psi}->\psi e^{-R}$

tomando a componente em $\mathrm{E}^0_{-\alpha 1}+\mathrm{E}^0_{-\alpha 3}$ :

$$\left(\overline{\partial}\chi+\chi\overline{\partial}R-\chi^2\overline{\partial}\psi+\chi^2\psi\overline{\partial}R\right)e^{-R}$$

a componente em $h_1+h_3$ :

$$\overline{\partial}R-\chi\overline{\partial}\psi+\chi\psi\overline{\partial}R = -\chi\overline{\partial}\psi+(1+\chi\psi)\overline{\partial}R=0\ \rightarrow\overline{\partial}R=\frac{\chi\overline{\partial}\psi}{1+\chi\psi}$$

a componente em $E^0_{\alpha 1}+E^0_{\alpha 3}$

$\left(\overline{\partial}\psi - \psi\overline{\partial}R\right)e^R$
a componente em $E^0_{-\alpha 1}+E^0_{-\alpha 3}$ :

$$\left(\overline{\partial}\chi + \chi\overline{\partial}R - \chi^2\overline{\partial}\psi + \chi^2\psi\overline{\partial}R\right)e^{-R} =$$
$$=\left(\overline{\partial}\chi + \chi\left(\frac{\chi\overline{\partial}\psi}{1+\chi\psi}\right) - \chi^2\overline{\partial}\psi + \chi^2\psi\left(\frac{\chi\overline{\partial}\psi}{1+\chi\psi}\right)\right)e^{-R}$$
$$= e^{-R(x)}\overline{\partial}\chi(x)$$

a componente em $E^0_{\alpha 1}+E^0_{\alpha 3}$

$$\left(\overline{\partial}\psi - \psi\overline{\partial}R\right)e^R = \left(\overline{\partial}\psi(x) - \psi(x)\left(\frac{\chi(x)\overline{\partial}\psi(x)}{1+\chi(x)\psi(x)}\right)\right)e^{R(x)} = e^{R(x)}\frac{\overline{\partial}\psi(x)}{1+\chi(x)\psi(x)}$$

finalmente temos

$$\left(\overline{\partial}B\right)B^{-1} = \frac{\overline{\partial}\psi}{1+\chi\psi}e^R\left(E^0_{\alpha 1} + E^0_{\alpha 3}\right) + \overline{\partial}\chi\, e^{-R}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right) +$$
$$+\left(\mathrm{h}_2+\tfrac{1}{2}\mathrm{h}_1+\tfrac{1}{2}\mathrm{h}_3\right)\left(\overline{\partial}\varphi\right) + \left(\overline{\partial}\nu\right)\mathrm{c}+\left(\overline{\partial}\eta\right)\mathrm{d}$$

# 7 Apêndice C.

## 7.1 Cálculo de $\mathrm{B}^{-1}\ \varepsilon_+\ \mathrm{B}$.

Dados

$$B = \quad e^{\tilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}\ e^{\left(h_2+\frac{1}{2}h_1+\frac{1}{2}h_3\right)\varphi+(h_1+h_3)R+\nu c+\eta d}e^{\tilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)} \tag{C.1}$$

e

$$\varepsilon_+ = E^0_{\alpha 1+\alpha 2} - E^0_{\alpha 2+\alpha 3} + E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3} \tag{C.2}$$

temos

$$\mathrm{B}^{-1}\ \varepsilon_+\ \mathrm{B} = e^{\tilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}\ e^{\Lambda}\ e^{\tilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}\ \varepsilon_+\ e^{-\tilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}e^{-\Lambda}\ e^{-\tilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}$$

onde $\Lambda = \left(h_2 + \frac{1}{2}h_1 + \frac{1}{2}h_3\right)\varphi + (h_1+h_3)\,R + \nu c + \eta d$
E facil ver que $\mathrm{B}^{-1}\ \varepsilon_+\ \mathrm{B} = \widetilde{B}^{\ -1}\ \varepsilon_+\ \widetilde{B}$

$$\tag{C.3}$$

onde $\widetilde{B}=e^{\tilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}e^{\Omega}e^{\tilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}$ e
$\Omega = \left(h_2 + \frac{1}{2}h_1 + \frac{1}{2}h_3\right)\varphi + (h_1+h_3)\,R + \eta d$

pois c comuta com todos os geradores.

$\varepsilon_+ = E^0_{\alpha 1+\alpha 2} - E^0_{\alpha 2+\alpha 3} + E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3} = \begin{pmatrix} 0 & \sigma_3 \\ \lambda\sigma_3 & 0 \end{pmatrix}$
onde $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\widetilde{B}^{\ -1}\ \varepsilon_+\ \widetilde{B} =$

$$\widetilde{B} = e^{\tilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}e^{\Omega}e^{\tilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)} = e^{\tilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)}e^{h\varphi+h'R+\eta d}e^{\tilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}$$

$$\widetilde{B}^{-1} = e^{-\tilde{\psi}\left(E^0_{\alpha 1}+E^0_{\alpha 3}\right)}e^{-h\varphi-h'R-\eta d}e^{-\tilde{\chi}\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)} \tag{C.4}$$

Agora
$h = \frac{1}{2}A \qquad$ onde $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad$ e $\quad 1=\mathrm{I}_{2x2}$
então
$e^{\varphi h} = e^{\varphi\frac{A}{2}} = 1 + \frac{\varphi A}{2} + \frac{1}{2!}\left(\frac{\varphi A}{2}\right)^2 + \frac{1}{3!}\left(\frac{\varphi A}{2}\right)^3 + \ldots$
$\quad = \left[1 + \frac{1}{2!}\left(\frac{\varphi}{2}\right)^2 + \ldots\right] + \left[\frac{1}{2} + \frac{1}{3!}\left(\frac{\varphi}{2}\right)^3 + \ldots\right] A$

$= \cosh(\frac{\varphi}{2}) + A\sinh(\frac{\varphi}{2})$ isto é

$$e^{\varphi h} = \begin{pmatrix} M_1 I & 0 \\ 0 & M_2 I \end{pmatrix}$$

onde $M_1 = \left(\cosh\frac{1}{2}\varphi + \sinh\frac{1}{2}\varphi\right)$, $M_2 = \left(\cosh\frac{1}{2}\varphi - \sinh\frac{1}{2}\varphi\right)$, $I = I_{2x2}$

ou $e^{\varphi h} = \begin{pmatrix} e^{\frac{\varphi}{2}} I & 0 \\ 0 & e^{-\frac{\varphi}{2}} I \end{pmatrix}$ então $e^{-\varphi h} = \begin{pmatrix} e^{-\frac{\varphi}{2}} I & 0 \\ 0 & e^{\frac{\varphi}{2}} I \end{pmatrix}$

(C.5)

$$h' = h_1 + h_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$(h')^2 = 1 \qquad (h')^3 = h'$

$e^{h' R} = 1 + R\,h' + R^2\frac{1}{2!} + R^3\frac{1}{3!}h' + R^4\frac{1}{4!} + ...$

$= \left(1 + \frac{1}{2!}\left(\frac{R^2}{4}\right) + \frac{1}{4!}\left(\frac{R^4}{16}\right) + ...\right) + h'\left(R + \frac{R^3}{3!} + ...\right)$

$$e^{h'R} = \cosh R + g\sinh R = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$$

onde $N = \begin{pmatrix} \sinh R + \cosh R & 0 \\ 0 & \cosh R - \sinh R \end{pmatrix}$

$$e^{h'R} = \begin{pmatrix} e^R & 0 & 0 & 0 \\ 0 & e^{-R} & 0 & 0 \\ 0 & 0 & e^R & 0 \\ 0 & 0 & 0 & e^{-R} \end{pmatrix} \qquad e^{-h'R} = \begin{pmatrix} e^{-\frac{\varphi}{2}} & 0 & 0 & 0 \\ 0 & e^{-\frac{\varphi}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{\varphi}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{\varphi}{2}} \end{pmatrix}$$

(C.6)

reescrevendo (5)

$$\widetilde{B} = e^{\tilde{\chi}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)} e^{\Omega} e^{\tilde{\psi}\left(E^0_{\alpha 1} + E^0_{\alpha 3}\right)} = e^{\tilde{\chi}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)} e^{h\varphi + h'R} e^{\tilde{\psi}\left(E^0_{\alpha 1} + E^0_{\alpha 3}\right)} e^{\eta d} \qquad \text{(C.7)}$$

e facil mostrar que $e^{\tilde{\chi}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)} = 1 + \tilde{\chi}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)$ e

$$e^{\tilde{\psi}\left(E^0_{\alpha 1} + E^0_{\alpha 3}\right)} = 1 + \tilde{\psi}\left(E^0_{\alpha 1} + E^0_{\alpha 3}\right)$$

então

$$\widetilde{B} = \left[1 + \tilde{\chi}\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)\right] e^{h\,\varphi} e^{h'R} \left[1 + \tilde{\psi}\left(E^0_{\alpha 1} + E^0_{\alpha 3}\right)\right] e^{\eta d} \qquad \text{(C.8)}$$

onde

$$e^{h\,\varphi}e^{h'R} = \begin{pmatrix} e^{\frac{1}{2}\varphi}e^{R} & 0 & 0 & 0 \\ 0 & e^{\frac{1}{2}\varphi}e^{-R} & 0 & 0 \\ 0 & 0 & e^{-\frac{1}{2}\varphi}e^{R} & 0 \\ 0 & 0 & 0 & e^{-\frac{1}{2}\varphi}e^{-R} \end{pmatrix} \tag{C.9}$$

substituindo (C.9) em (C.8)

$$\widetilde{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tilde{\chi} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \tilde{\chi} & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}\varphi}e^{R} & 0 & 0 & 0 \\ 0 & e^{\frac{1}{2}\varphi}e^{-R} & 0 & 0 \\ 0 & 0 & e^{-\frac{1}{2}\varphi}e^{R} & 0 \\ 0 & 0 & 0 & e^{-\frac{1}{2}\varphi}e^{-R} \end{pmatrix} \begin{pmatrix} 1 & \tilde{\psi} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{\psi} \\ 0 & 0 & 0 & 1 \end{pmatrix} e^{\eta d} =$$

$$\widetilde{B} = \begin{pmatrix} e^{\frac{1}{2}\varphi+R} & \psi e^{\frac{1}{2}\varphi+R} & 0 & 0 \\ \tilde{\chi}\; e^{\frac{1}{2}\varphi+R} & \tilde{\chi}\;\psi e^{\frac{1}{2}\varphi+R} + e^{\frac{1}{2}\varphi-R} & 0 & 0 \\ 0 & 0 & e^{-\frac{1}{2}\varphi+R} & \psi e^{-\frac{1}{2}\varphi+R} \\ 0 & 0 & \tilde{\chi}\; e^{-\frac{1}{2}\varphi+R} & \tilde{\chi}\;\psi e^{-\frac{1}{2}\varphi+R} + e^{-\frac{1}{2}\varphi-R} \end{pmatrix} e^{\eta d}$$

$$\widetilde{B}^{-1} = e^{-\eta d}e^{-\tilde{\psi}\left(E^{0}_{\alpha 1}+E^{0}_{\alpha 3}\right)}e^{-h'R}e^{-h\,\varphi}e^{-\tilde{\chi}\left(E^{0}_{-\alpha 1}+E^{0}_{-\alpha 3}\right)}$$

$$e^{-h'R}e^{-h\,\varphi} = \begin{pmatrix} e^{-R} & 0 & 0 & 0 \\ 0 & e^{R} & 0 & 0 \\ 0 & 0 & e^{-R} & 0 \\ 0 & 0 & 0 & e^{+R} \end{pmatrix} \begin{pmatrix} e^{-\frac{\varphi}{2}} & 0 & 0 & 0 \\ 0 & e^{-\frac{\varphi}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{\varphi}{2}} & 0 \\ 0 & 0 & 0 & e^{\frac{\varphi}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{1}{2}\varphi-R} & 0 & 0 & 0 \\ 0 & e^{-\frac{1}{2}\varphi+R} & 0 & 0 \\ 0 & 0 & e^{\frac{1}{2}\varphi-R} & 0 \\ 0 & 0 & 0 & e^{\frac{1}{2}\varphi+R} \end{pmatrix}$$

$$\widetilde{B}^{-1} = e^{-\eta d}\begin{pmatrix} 1 & -\tilde{\psi} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\tilde{\psi} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}\varphi-R} & 0 & 0 & 0 \\ 0 & e^{-\frac{1}{2}\varphi+R} & 0 & 0 \\ 0 & 0 & e^{\frac{1}{2}\varphi-R} & 0 \\ 0 & 0 & 0 & e^{\frac{1}{2}\varphi+R} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\tilde{\chi} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\tilde{\chi} & 1 \end{pmatrix}$$

$$= e^{-\eta d}\begin{pmatrix} e^{-\frac{1}{2}\varphi-R} + \tilde{\psi}\; e^{-\frac{1}{2}\varphi+R}\tilde{\chi} & -\tilde{\psi}\; e^{-\frac{1}{2}\varphi+R} & 0 & 0 \\ -e^{-\frac{1}{2}\varphi+R}\tilde{\chi} & e^{-\frac{1}{2}\varphi+R} & 0 & 0 \\ 0 & 0 & e^{\frac{1}{2}\varphi-R} + \tilde{\psi}\; e^{\frac{1}{2}\varphi+R}\tilde{\chi} & -\tilde{\psi}\; e^{\frac{1}{2}\varphi+R} \\ 0 & 0 & -e^{\frac{1}{2}\varphi+R}\tilde{\chi} & e^{\frac{1}{2}\varphi+R} \end{pmatrix};$$

Usando as propiedades de d,

$$\begin{aligned} e^{-\eta d}\varepsilon_{+}e^{\eta d} &= e^{-\eta d}\left(E^{0}_{\alpha 1+\alpha 2} - E^{0}_{\alpha 2+\alpha 3} + E^{1}_{-\alpha 1-\alpha 2} - E^{1}_{-\alpha 2-\alpha 3}\right)e^{\eta d} \\ &= e^{-\eta d}\left(E^{1}_{-\alpha 1-\alpha 2} - E^{1}_{-\alpha 2-\alpha 3}\right)e^{\eta d} \end{aligned}$$

usando a identidade $e^p \, b \, e^{-p} = b + [p,b] + \frac{1}{2!}[p,[p,b]] + \frac{1}{3!}[p,[p,[p,b]]] + ...$

temos

$$e^{-\eta d}\left(E^1_{-\alpha 1-\alpha 2}\right)e^{\eta d} =$$
$$= E^1_{-\alpha 1-\alpha 2} - \eta[d, E^1_{-\alpha 1-\alpha 2}] + \frac{(-\eta)^2}{2!}\left[d,[d,E^1_{-\alpha 1-\alpha 2}]\right] + ...$$

sustiruindo

$$[d, E^1_{-\alpha 1-\alpha 2}] = E^1_{-\alpha 1-\alpha 2}$$
$$\left[d,[d,E^1_{-\alpha 1-\alpha 2}]\right] = \left[d, E^1_{-\alpha 1-\alpha 2}\right] = E^1_{-\alpha 1-\alpha 2}$$

então

$$e^{-\eta d}\left(E^1_{-\alpha 1-\alpha 2}\right)e^{\eta d}$$
$$= E^1_{-\alpha 1-\alpha 2} - \eta E^1_{-\alpha 1-\alpha 2} + \frac{(-\eta)^2}{2!}E^1_{-\alpha 1-\alpha 2} + \frac{(-\eta)^3}{3!}E^1_{-\alpha 1-\alpha 2} + ... = e^{-\eta}\, E^1_{-\alpha 1-\alpha 2} \quad ....(*)$$

$$e^{-\eta d}\left(E^1_{-\alpha 1-\alpha 2}\right)e^{\eta d} = \quad e^{-\eta}\, E^1_{-\alpha 1-\alpha 2}$$

e analogamente $\quad e^{-\eta d}\left(E^1_{-\alpha 2-\alpha 3}\right)e^{\eta d} = \quad e^{-\eta}\, E^1_{-\alpha 2-\alpha 3}$

então

$$e^{-\eta d}\varepsilon_+ e^{\eta d} = e^{-\eta d}\left(E^0_{\alpha 1+\alpha 2} - E^0_{\alpha 2+\alpha 3} + E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3}\right)e^{\eta d}$$
$$= E^0_{\alpha 1+\alpha 2} - E^0_{\alpha 2+\alpha 3} + \left(E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3}\right)e^{-\eta}$$

$$e^{-\eta d}\varepsilon_+ e^{\eta d} = E^0_{\alpha 1+\alpha 2} - E^0_{\alpha 2+\alpha 3} + \left(E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3}\right)e^{-\eta} =$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \lambda e^{-\eta} & 0 & 0 & 0 \\ 0 & -\lambda e^{-\eta} & 0 & 0 \end{pmatrix}$$

$$[d, E^1_{-\alpha 1-\alpha 2}] = E^1_{-\alpha 1-\alpha 2}$$

$\widetilde{B}^{-1}\varepsilon_+\widetilde{B} = \begin{bmatrix} 0 & \widetilde{P} \\ \widetilde{Q} & 0 \end{bmatrix}$ onde

$$\widetilde{P} = \begin{bmatrix} e^{-\varphi} + 2\,\tilde{\psi}\, e^{-\varphi+2R}\tilde{\chi} & 2\left(e^{-\frac{1}{2}\varphi-R} + \tilde{\psi}\, e^{-\frac{1}{2}\varphi+R}\tilde{\chi}\right)\tilde{\psi}\, e^{-\frac{1}{2}\varphi+R} \\ -2e^{-\varphi+2R}\tilde{\chi} & -2\,\tilde{\psi}\, e^{-\varphi+2R}\tilde{\chi} - e^{-\varphi} \end{bmatrix} \quad \text{e}$$

$$\widetilde{Q} = \begin{bmatrix} \lambda e^{-\eta+\varphi} + 2\,\tilde{\psi}\,\lambda\tilde{\chi}e^{\varphi+2R-\eta} & 2\left(e^{\frac{1}{2}\varphi-R} + \tilde{\psi}\, e^{\frac{1}{2}\varphi+R}\tilde{\chi}\right)\lambda\,\tilde{\psi}\, e^{-\eta+\frac{1}{2}\varphi+R} \\ -2\,\tilde{\chi}\,\lambda e^{\varphi+2R-\eta} & -2\,\tilde{\psi}\,\lambda\tilde{\chi}\, e^{\varphi+2R-\eta} - \lambda e^{-\eta+\varphi} \end{bmatrix}$$

com as mudanças $\tilde{\psi} = \psi\, e^{-R}$, $\tilde{\chi} = \chi\, e^{-R}$ temos

$B^{-1}\,\varepsilon_+\,B =$

$$= \begin{pmatrix} 0 & 0 & e^{-\varphi}(1+2\psi\chi) & 2e^{-\varphi-R}(1+\ \psi\,\chi)\psi \\ 0 & 0 & -2\,\chi\, e^{-\varphi+R} & -e^{-\varphi}(1+2\psi\chi) \\ \lambda e^{-\eta+\varphi}(1+2\psi\chi) & 2e^{\varphi-R-\eta}(1+\ \psi\,\chi)\,\lambda\,\psi & 0 & 0 \\ -2\,\chi\,\lambda\, e^{R+\varphi-\eta} & -\lambda e^{-\eta+\varphi}(1+2\psi\chi) & 0 & 0 \end{pmatrix}$$

ou

$$B^{-1}\varepsilon_+ B = \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}$$

onde $P = \begin{bmatrix} e^{-\varphi}(1+2\psi\chi) & 2e^{-\varphi-R}(1+\psi\chi)\psi \\ -2\chi e^{-\varphi+R} & -e^{-\varphi}(1+2\psi\chi) \end{bmatrix}$

e $\quad Q = \begin{bmatrix} \lambda e^{-\eta+\varphi}(1+2\psi\chi) & 2e^{\varphi-R-\eta}(1+\psi\chi)\lambda\psi \\ -2\chi\lambda e^{R+\varphi-\eta} & -\lambda e^{-\eta+\varphi}(1+2\psi\chi) \end{bmatrix}$

Reescrevendo em forma de operadores,

$\mathrm{B}^{-1}\ \varepsilon_+\ \mathrm{B} =$

$$\begin{aligned} &e^{-\varphi}(1+2\psi\chi)\,E_{\alpha1+\alpha2} - e^{-\varphi}(1+2\psi\chi)\,E_{\alpha2+\alpha3} + \\ &+2e^{-\varphi-R}(1+\ \psi\ \chi)\,\psi E_{\alpha1+\alpha2+\alpha3} - 2\ \chi\ e^{-\varphi+R}E_{\alpha2} + \\ &e^{-\eta+\varphi}(1+2\psi\chi)\,E^1_{-\alpha1-\alpha2} - e^{-\eta+\varphi}(1+2\psi\chi)\,E^1_{-\alpha2-\alpha3} + \\ &-2\ \chi\ e^{R+\varphi-\eta}E^1_{-\alpha1-\alpha2-\alpha3} + 2e^{\varphi-R-\eta}(1+\ \psi\ \chi)\,\psi E^1_{-\alpha2} \end{aligned} \quad (C.10)$$

# 8 Apêndice D.

## 8.1 Contribuição do termo central em $\left[B^{-1}\varepsilon_{+}B, \varepsilon_{-}\right]$

No apêndice C, mostrou-se que $B^{-1}\varepsilon_{+}B = \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}$
onde

$$P = \begin{bmatrix} e^{-\varphi}(1+2\psi\chi) & 2e^{-\varphi-R}(1+\psi\chi)\psi \\ -2\chi e^{-\varphi+R} & -e^{-\varphi}(1+2\psi\chi) \end{bmatrix} \qquad \text{e}$$

$$Q = \begin{bmatrix} \lambda e^{-\eta+\varphi}(1+2\psi\chi) & 2e^{\varphi-R-\eta}(1+\psi\chi)\lambda\psi \\ -2\chi\lambda e^{R+\varphi-\eta} & -\lambda e^{-\eta+\varphi}(1+2\psi\chi) \end{bmatrix}$$

nos sabemos que $\varepsilon_{-} = \begin{bmatrix} 0 & \frac{1}{\lambda}\sigma_3 \\ \sigma_3 & 0 \end{bmatrix}$
reescrevendo $B^{-1}\varepsilon_{+}B$ em termos de operadores

$$\begin{aligned} B^{-1}\varepsilon_{+}B = {} & e^{-\varphi}(1+2\psi\chi)E^{0}_{\alpha1+\alpha2} - e^{-\varphi}(1+2\psi\chi)E^{0}_{\alpha2+\alpha3} - 2\chi e^{-\varphi+R}E^{0}_{\alpha2} \\ & +2e^{-\varphi-R}(1+\psi\chi)\psi E^{0}_{\alpha1+\alpha2+\alpha3} + \lambda e^{-\eta+\varphi}(1+2\psi\chi)E^{0}_{-\alpha1-\alpha2} \\ & -\lambda e^{-\eta+\varphi}(1+2\psi\chi)E^{0}_{-\alpha2-\alpha3} + 2e^{\varphi-R-\eta}(1+\psi\chi)\lambda\psi E^{0}_{-\alpha2} \\ & -2\chi\lambda e^{R+\varphi-\eta}E^{0}_{-\alpha1-\alpha2-\alpha3} \end{aligned}$$

e em termos de geradores de Kac Moody

$$\begin{aligned} B^{-1}\varepsilon_{+}B = {} & e^{-\varphi}(1+2\psi\chi)E^{0}_{\alpha1+\alpha2} - e^{-\varphi}(1+2\psi\chi)E^{0}_{\alpha2+\alpha3} \\ & -2\chi e^{-\varphi+R}E^{0}_{\alpha2} + 2e^{-\varphi-R}(1+\psi\chi)\psi E^{0}_{\alpha1+\alpha2+\alpha3} + \\ & e^{-\eta+\varphi}(1+2\psi\chi)E^{1}_{-\alpha1-\alpha2} - e^{-\eta+\varphi}(1+2\psi\chi)E^{1}_{-\alpha2-\alpha3} \\ & +2e^{\varphi-R-\eta}(1+\psi\chi)\psi E^{1}_{-\alpha2} - 2\chi e^{R+\varphi-\eta}E^{1}_{-\alpha1-\alpha2-\alpha3} \end{aligned}$$

$$\varepsilon_{-} = E^{-1}_{\alpha1+\alpha2} - E^{-1}_{\alpha2+\alpha3} + E^{0}_{-\alpha1-\alpha2} - E^{0}_{-\alpha2-\alpha3}$$

Definição: $[M,N]_c =$ parte de $[M,N]$ com termo central
$[M,N]_{nc} =$ parte de $[M,N]$ sem termo central

$[B^{-1}\varepsilon_{+}B, \varepsilon_{-}]_c$
$= [\, e^{-\eta+\varphi}(1+2\psi\chi)E^{1}_{-\alpha1-\alpha2} - e^{-\eta+\varphi}(1+2\psi\chi)E^{1}_{-\alpha2-\alpha3}, E^{-1}_{\alpha1+\alpha2} - E^{-1}_{\alpha2+\alpha3}]_c$
usando $[E^{m}_{\alpha}, E^{n}_{-\alpha}]_c = cm\delta_{m,-n}$ temos $[E^{1}_{\alpha}, E^{-1}_{-\alpha}] = c$

$$\begin{aligned} [B^{-1}\varepsilon_{+}B, \varepsilon_{-}]_c &= e^{-\eta+\varphi}(1+2\psi\chi)\left[E^{1}_{-\alpha1-\alpha2}, E^{-1}_{\alpha1+\alpha2}\right]_c + e^{-\eta+\varphi}(1+2\psi\chi)\left[E^{1}_{-\alpha2-\alpha3}, E^{-1}_{\alpha2+\alpha3}\right]_c \\ &= e^{-\eta+\varphi}(1+2\psi\chi)\,c + e^{-\eta+\varphi}(1+2\psi\chi)\,c \end{aligned}$$

isto é $$[B^{-1}\varepsilon_{+}B, \varepsilon_{-}]_c = 2\, e^{-\eta+\varphi}(1+2\psi\chi)\,c$$

## 8.2 Apêndice E. Verificação das equações (73) utilizando as equações de Leznov-Saveliev

Considere a equação de Leznov-Saveliev

$$\overline{\partial}(B^{-1}\partial B) = [B^{-1}\varepsilon_+ B, \varepsilon_-]$$

e calculemos o lado direito. Façamos primeiro os cálulos sem levar em conta o termo central.

No Apêndice C mostrou-se que

$B^{-1}\varepsilon_+ B = \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}$ onde

$$P = \begin{bmatrix} e^{-\varphi}(1+2\psi\chi) & 2e^{-\varphi-R}(1+\psi\chi)\psi \\ -2\chi e^{-\varphi+R} & -e^{-\varphi}(1+2\psi\chi) \end{bmatrix},$$

$$Q = \begin{bmatrix} \lambda e^{-\eta+\varphi}(1+2\psi\chi) & 2e^{\varphi-R-\eta}(1+\psi\chi)\lambda\psi \\ -2\chi\lambda e^{R+\varphi-\eta} & -\lambda e^{-\eta+\varphi}(1+2\psi\chi) \end{bmatrix}$$

enquanto que $\varepsilon_- = \begin{bmatrix} 0 & \frac{1}{\lambda}\sigma_3 \\ 1 & 0 \end{bmatrix}$

então $\qquad [B^{-1}\varepsilon_+ B, \varepsilon_-]_{nc} = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \qquad ......(\alpha)$

onde M =

$$= \begin{bmatrix} -e^{-\eta+\varphi} - 2e^{-\eta+\varphi}\psi\chi + 2\psi\chi e^{-\varphi} + e^{-\varphi} & -2\psi e^{-\varphi-R} - 2\psi^2 e^{-\varphi-R}\chi - 2\psi e^{\varphi-R-\eta} - 2\psi^2 e^{\varphi-R-\eta}\chi \\ -2\chi e^{-\varphi+R} - 2\chi e^{R+\varphi-\eta} & -e^{-\eta+\varphi} - 2e^{-\eta+\varphi}\psi\chi + 2\psi\chi e^{-\varphi} + e^{-\varphi} \end{bmatrix}$$

e N =

$$= \begin{bmatrix} e^{-\eta+\varphi} + 2e^{-\eta+\varphi}\psi\chi - 2\psi\chi e^{-\varphi} - e^{-\varphi} & -2\psi e^{-\varphi-R} - 2\psi^2 e^{-\varphi-R}\chi - 2\psi e^{\varphi-R-\eta} - 2\psi^2 e^{\varphi-R-\eta}\chi \\ -2\chi e^{-\varphi+R} - 2\chi e^{R+\varphi-\eta} & e^{-\eta+\varphi} + 2e^{-\eta+\varphi}\psi\chi - 2\psi\chi e^{-\varphi} - e^{-\varphi} \end{bmatrix}$$

No Apêndice D mostrou-se que a contribuição da parte central ao comutador é dada por

$$[B^{-1}\varepsilon_+ B, \varepsilon_-]_c = 2\, e^{-\eta+\varphi}(1+2\psi\chi)\, c$$

Lembrando que $\quad h = h_2 + \frac{1}{2}h_1 + \frac{1}{2}h_3 = \begin{pmatrix} \frac{1}{2}I & 0 \\ 0 & -\frac{1}{2}I \end{pmatrix} \quad$ com $I = I_{2x2}$

reescrevemos o resultado $(\alpha)$ em forma de operadores:

$$\begin{aligned} &[B^{-1}\varepsilon_+ B, \varepsilon_-] = \\ &\left(-2\psi e^{-\varphi-R} - 2\psi^2 e^{-\varphi-R}\chi - 2\psi e^{\varphi-R-\eta} - 2\psi^2 e^{\varphi-R-\eta}\chi\right)(E^0_{\alpha 1} + E^0_{\alpha 3}) \\ &+ (-e^{-\eta+\varphi} - 2e^{-\eta+\varphi}\psi\chi + 2\psi\chi e^{-\varphi} + e^{-\varphi})\left(2\left(h_2 + \tfrac{1}{2}h_1 + \tfrac{1}{2}h_3\right)\right) \\ &+ \left(-2\chi e^{-\varphi+R} - 2\chi e^{R+\varphi-\eta}\right)\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right) + 2e^{-\eta+\varphi}(1+2\psi\chi)\, c \end{aligned} \tag{E.1}$$

Agora, no Apêndice A mostrou-se que

$$B^{-1}\overline{\partial}B = \left[\left(\overline{\partial}\psi\right)e^{-R}\right](E^0_{\alpha 1} + E^0_{\alpha 3}) + \left[\tfrac{\overline{\partial}\chi}{\Delta}e^R\right]\left(E^0_{-\alpha 1} + E^0_{-\alpha 3}\right)$$

$$+\overline{\partial}\varphi\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)+\left(\overline{\partial}\nu\right)c+\left(\overline{\partial}n\right)d$$

(E.2)

então

$$\partial(B^{-1}\overline{\partial}B)=\left(\partial\left(\overline{\partial}\psi e^{-R}\right)\right)(E^0_{\alpha 1}+E^0_{\alpha 3})+\partial_+\left(\tfrac{\overline{\partial}\chi}{\Delta}\right)\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)+$$
$$+\left(\partial\,\overline{\partial}\,\varphi\right)\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)+\left(\partial\,\overline{\partial}\,\nu\right)c+\left(\partial\,\overline{\partial}\,n\right)d$$

nesta última expresão temos que a componente de $\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)$ é

$$\partial\left(\overline{\partial}\psi e^{-R}\right)=\left(\partial\,\overline{\partial}\,\psi-\left(\overline{\partial}\psi\right)(\partial R)\right)e^{-R}$$

e a componente de $\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)$ é

$$\partial\left(\tfrac{\overline{\partial}\chi}{\Delta}e^R\right)=\frac{-\psi\left(\overline{\partial}\chi\right)(\partial\chi)+\left(\partial\ \overline{\partial}\chi\right)+\chi\psi\left(\partial\,\overline{\partial}\chi\right)}{(1+\chi\psi)^2}e^R$$

Os resultados de cada lado na equação de Leznov-Saveliev são:

$$[B^{-1}\varepsilon_+B,\varepsilon_-]=$$
$$\left(-2\psi e^{-\varphi-R}-2\psi^2e^{-\varphi-R}\chi-2\psi e^{\varphi-R-\eta}-2\psi^2e^{\varphi-R-\eta}\chi\right)(E^0_{\alpha 1}+E^0_{\alpha 3})$$
$$+\left(-e^{-\eta+\varphi}-2e^{-\eta+\varphi}\psi\chi+2\psi\chi e^{-\varphi}+e^{-\varphi}\right)\left(2\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)\right)$$
$$+\left(-2\chi e^{-\varphi+R}-2\chi e^{R+\varphi-\eta}\right)\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)\ +2e^{-\eta+\varphi}(1+2\psi\chi)\,c$$

(E.3)

$$\partial(B^{-1}\overline{\partial}B)=$$
$$\left(\partial\,\overline{\partial}\psi-\left(\overline{\partial}\psi\right)(\partial R)\right)e^{-R}(E^0_{\alpha 1}+E^0_{\alpha 3})+$$
$$+\left[\frac{-\psi\left(\overline{\partial}\chi\right)(\partial\chi)+\left(\partial\,\overline{\partial}\,\chi\right)+\chi\psi\left(\partial\,\overline{\partial}\chi\right)}{(1+\chi\psi)^2}\right]e^R\left(E^0_{-\alpha 1}+E^0_{-\alpha 3}\right)+$$
$$+\left(\partial\,\overline{\partial}\varphi\right)\left(h_2+\tfrac{1}{2}h_1+\tfrac{1}{2}h_3\right)+\left(\partial\,\overline{\partial}\,\nu\right)c+\left(\partial\,\overline{\partial}\,n\right)d$$

(E.4)

por tanto, igualando as respectivas componentes, obtemos as equações

$$\left(-2\psi e^{-\varphi-R}-2\psi^2e^{-\varphi-R}\chi-2\psi e^{\varphi-R-\eta}-2\psi^2e^{\varphi-R-\eta}\chi\right)$$
$$=\left(\partial\,\overline{\partial}\,\psi-\left(\overline{\partial}\psi\right)(\partial R)\right)e^{-R}$$

(E.5.1)

$$\left[\frac{-\psi\left(\overline{\partial}\chi\right)(\partial\chi)+\left(\partial\,\overline{\partial}\,\chi\right)+\chi\psi\left(\partial\,\overline{\partial}\,\chi\right)}{(1+\chi\psi)^2}e^R\right]=\left(-2\chi e^{-\varphi+R}-2\chi e^{R+\varphi-\eta}\right)$$

(E.5.2)

$$\left(\partial\,\overline{\partial}\,\varphi\right)=2\left(-e^{-\eta+\varphi}-2e^{-\eta+\varphi}\psi\chi+2\psi\chi e^{-\varphi}+e^{-\varphi}\right)$$

(E.5.3)

$$2e^{-\eta+\varphi}(1+2\psi\chi)=\partial\,\overline{\partial}\,\nu$$

(E.5.4)

$$\partial\,\overline{\partial}\,\eta=0$$

(E.5.5)

na aproximacão de Loop Algebra consideramos $\eta=0$, e simplificando as equações temos

$$-2\psi\left(e^{-\varphi}+e^{\varphi}\right)(1+\psi\chi)=\left(\partial\,\overline{\partial}\,\psi-\left(\overline{\partial}\psi\right)(\partial R)\right)$$

(E.6.1)

$$\frac{-\psi\left(\overline{\partial}\chi\right)(\partial\chi)+\left(\partial\,\overline{\partial}\,\chi\right)+\chi\psi\left(\partial\,\overline{\partial}\,\chi\right)}{(1+\chi\psi)^2}=-2\chi\left(e^{-\varphi}+e^{+\varphi}\right)$$

(E.6.2)

$$\left(\partial\,\overline{\partial}\,\varphi\right)=2\left(-(1+2\psi\chi)\left(e^{\varphi}-\frac{1}{e^{\varphi}}\right)\right)$$

(E.6.3)

$$2e^{\varphi}(1+2\psi\chi)=\partial\,\overline{\partial}\,\nu$$

(E.6.4)

isto é

$$-2\psi\left(e^{-\varphi}+e^{\varphi}\right)(1+\psi\chi)=\left(\partial\,\overline{\partial}\,\psi-\left(\overline{\partial}\psi\right)(\partial R)\right)$$

(E.7.1)

$$-\psi\left(\overline{\partial}\chi\right)(\partial\chi)+(1+\chi\psi)\left(\partial\,\overline{\partial}\,\chi\right)+2\chi(1+\chi\psi)^2\left(e^{-\varphi}+e^{+\varphi}\right)=0$$

(E.7.2)

$$\left(\partial\,\overline{\partial}\,\varphi\right)+2\left((1+2\psi\chi)\left(e^{\varphi}-e^{-\varphi}\right)\right)=0 \quad \text{ou}$$

$$\left(\partial\,\overline{\partial}\,\varphi\right)+2(1+2\psi\chi)e^{\varphi}-2(1+2\psi\chi)e^{-\varphi}$$

(E.7.3)

$$\partial\,\overline{\partial}\,\nu-2e^{\varphi}(1+2\psi\chi)=0$$

(E.7.4)

substituindo na primeira equação $(E.7.1)$ $\partial R = \frac{\chi(\partial\psi)}{(1+\chi\psi)}$ :

$$-2\psi\left(e^{-\varphi}+e^{\varphi}\right)(1+\psi\chi) = \partial\,\overline{\partial}\,\psi - \frac{\chi\left(\overline{\partial}\psi\right)(\partial\psi)}{(1+\chi\psi)} \quad \text{ou}$$

$$-2\psi\left(e^{-\varphi}+e^{\varphi}\right)(1+\psi\chi)^2 = (1+\chi\psi)\ \ \partial\,\overline{\partial}\,\psi - \chi\left(\overline{\partial}\psi\right)(\partial\psi)$$

$$(1+\chi\psi)\ \ \partial\,\overline{\partial}\,\psi - \chi\left(\overline{\partial}\psi\right)(\partial\psi) + 2\,\psi\,(1+\psi\chi)^2\left(e^{-\varphi}+e^{\varphi}\right) \qquad \text{ou}$$

$$-\chi\left(\overline{\partial}\psi\right)(\partial\psi) + (1+\chi\psi)\ \ \partial\,\overline{\partial}\,\psi + 2\,\psi\,(1+\psi\chi)^2\left(e^{-\varphi}+e^{\varphi}\right)$$

(E.7.1')

# 9 Apêndice F.

## 9.1 Cálculo dos autovalores dos operadores $\varepsilon_+, \varepsilon_-$.

1. Cálculo do autovalor $w_+$ na equação $[\ \varepsilon_+, F_1^+\ ] = w_+ F_1^+$ onde

$F_1^+ = \sum_n E^n_{\alpha 1} z^{-2n} + \sum_n E^n_{\alpha 3} z^{-2n} + \sum_n E^n_{\alpha 1+\alpha 2+\alpha 3} z^{-2n-1} + \sum_n E^n_{-\alpha 2} z^{-2n+1}$.

$\varepsilon_+ = E^0_{\alpha 1+\alpha 2} - E^0_{\alpha 2+\alpha 3} + E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3}$

porém $[\ \varepsilon_+, F_1^+\ ]$ não aparecem termos centrais, e podemos trabalhar nesta parte com a representação matricial:

$$\varepsilon_+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \end{pmatrix}$$

$$[\ \varepsilon_+, F_1^+\ ] = w_+ F_1^+$$

Os comutadores de $\varepsilon_+$ com os termos de $F_1^+$ são

$[\varepsilon_+, E^0_{\alpha 1}] = E^0_{\alpha 1+\alpha 2+\alpha 3} + E^1_{-\alpha 2}$ $\quad [\varepsilon_+, E^0_{\alpha 3}] = E^0_{\alpha 1+\alpha 2+\alpha 3} + E^1_{-\alpha 2}$

$[\varepsilon_+, E^0_{\alpha 1+\alpha 2+\alpha 3}] = E^1_{\alpha 1} + E^1_{\alpha 3}$ $\quad [\varepsilon_+, E^0_{-\alpha 2}] = E^0_{\alpha 1} + E^0_{\alpha 3}$ porém, substituindo

em

$$[\ \varepsilon_+, F_1^+\ ] = \sum_n [\varepsilon_+, E^n_{\alpha 1}] z^{-2n} + \sum_n [\varepsilon_+, E^n_{\alpha 3}] z^{-2n} + \sum_n [\varepsilon_+, E^n_{\alpha 1+\alpha 2+\alpha 3}]\, z^{-2n-1} + \sum_n [\varepsilon_+, E^n_{-\alpha 2}] z^{-2n+1}$$

temos

$$= \sum_n \left(E^n_{\alpha 1+\alpha 2+\alpha 3} + E^{n+1}_{-\alpha 2}\right) z^{-2n} + \sum_n \left(E^n_{\alpha 1+\alpha 2+\alpha 3} + E^{n+1}_{-\alpha 2}\right) z^{-2n} +$$

$$\sum_n \left(E^{n+1}_{\alpha 1} + E^{n+1}_{\alpha 3}\right) z^{-2n-1} + \sum_n \left(E^n_{\alpha 1} + E^n_{\alpha 3}\right) z^{-2n+1} =$$

$$= 2 \sum_n E^n_{\alpha 1+\alpha 2+\alpha 3}\ z^{-2n} + E^{n+1}_{-\alpha 2}\ z^{-2n} +$$

$$+ \sum_n \left(E^{n+1}_{\alpha 1} + E^{n+1}_{\alpha 3}\right) z^{-2n-1} + \sum_n \left(E^n_{\alpha 1} + E^n_{\alpha 3}\right) z^{-2n+1}$$

efetuando a mudanza $m = n + 1$ nos graus n+1 de "loop", temos

$$= 2 \sum_n E^n_{\alpha 1+\alpha 2+\alpha 3}\ z^{-2n} + E^m_{-\alpha 2}\ z^{-2(m-1)}$$

$$+ \sum_m \left(E^m_{\alpha 1} + E^m_{\alpha 3}\right) z^{-2(m-1)-1} + \sum_n \left(E^n_{\alpha 1} + E^n_{\alpha 3}\right) z^{-2n+1}$$

simplificando os exponentes,

$$= 2 \sum_n E^n_{\alpha 1+\alpha 2+\alpha 3}\ z^{-2n} + 2 \sum_n E^m_{-\alpha 2}\ z^{-2m+2}$$

$$+ \sum_m \left(E^m_{\alpha 1} + E^m_{\alpha 3}\right) z^{-2m+1} + \sum_n \left(E^n_{\alpha 1} + E^n_{\alpha 3}\right) z^{-2n+1}$$

e fatorando 2z, temos

$$= 2\ z\ \{\sum_n E^n_{\alpha 1} z^{-2n} + \sum_n E^n_{\alpha 3} z^{-2n} + \sum_n E^n_{\alpha 1+\alpha 2+\alpha 3} z^{-2n-1} + \sum_n E^n_{-\alpha 2} z^{-2n+1}\}$$

$[\ \varepsilon_+, F_1^+\ ] = 2zF_1^+$

2. Cálculo de $w_+$ para $[\ \varepsilon_+, F_2^+\ ] = w_+F_2^+$ onde

$$F_2^+ = \sum_n E^n_{\alpha 2} z^{-2n-1} - \sum_n E^n_{-\alpha 3} z^{-2n} + \sum_n E^n_{-\alpha 1-\alpha 2-\alpha 3}\ z^{-2n+1} - \sum_n E^n_{-\alpha 1} z^{-2n}$$

Também aqui não aparecem termos centrais,

$[\ \varepsilon_+, F_2^+\ ] = w_+F_2^+$

Temos

$$[\varepsilon_+, E^0_{\alpha 2}] = -E^1_{-\alpha 1} - E^1_{-\alpha 3} \qquad [\varepsilon_+, E^0_{-\alpha 3}] = -E^0_{\alpha 2} - E^1_{-\alpha 1-\alpha 2-\alpha 3}$$
$$[\varepsilon_+, E^0_{-\alpha 1-\alpha 2-\alpha 3}] = -E^0_{-\alpha 1} - E^0_{-\alpha 3} \qquad [\varepsilon_+, E^0_{-\alpha 1}] = -E^0_{\alpha 2} - E^1_{-\alpha 1-\alpha 2-\alpha 3}$$

Substituindo os comutadores no mesmo ordem,

$$[\ \varepsilon_+, F_2^+\ ] = \sum_n\ [\varepsilon_+, E^n_{\alpha 2}] z^{-2n-1} - \sum_n\ [\varepsilon_+, E^n_{-\alpha 3}] z^{-2n}$$
$$+ \sum_n\ [\varepsilon_+, E^n_{-\alpha 1-\alpha 2-\alpha 3}]\ z^{-2n+1} - \sum_n\ [\varepsilon_+, E^n_{-\alpha 1}] z^{-2n}$$

$$[\ \varepsilon_+, F_2^+\ ] = \sum_n\ \left(-E^{n+1}_{-\alpha 1} - E^{n+1}_{-\alpha 3}\right) z^{-2n-1} - \sum_n\ \left(-E^n_{\alpha 2} - E^{n+1}_{-\alpha 1-\alpha 2-\alpha 3}\right) z^{-2n}$$
$$+ \sum_n\ \left(-E^n_{-\alpha 1} - E^n_{-\alpha 3}\right) z^{-2n+1} - \sum_n\ \left(-E^n_{\alpha 2} - E^{n+1}_{-\alpha 1-\alpha 2-\alpha 3}\right) z^{-2n} =$$
$$= -\sum_n\ \left(E^{n+1}_{-\alpha 1} + E^{n+1}_{-\alpha 3}\right) z^{-2n-1} + \sum_n\ \left(E^n_{\alpha 2} + E^{n+1}_{-\alpha 1-\alpha 2-\alpha 3}\right) z^{-2n}$$
$$-\sum_n\ \left(E^n_{-\alpha 1} + E^n_{-\alpha 3}\right) z^{-2n+1} + \sum_n\ \left(E^n_{\alpha 2} + E^{n+1}_{-\alpha 1-\alpha 2-\alpha 3}\right) z^{-2n}$$
$$= -2\sum_n \left(E^{n+1}_{-\alpha 1} + E^{n+1}_{-\alpha 3}\right) z^{-2n-1} + 2\sum_n \left(E^n_{\alpha 2} + E^{n+1}_{-\alpha 1-\alpha 2-\alpha 3}\right) z^{-2n} =$$
$$-2\sum_n \left(E^{n+1}_{-\alpha 1} + E^{n+1}_{-\alpha 3}\right) z^{-2n-1} + 2\sum_n E^n_{\alpha 2} z^{-2n} + 2\sum_n E^{n+1}_{-\alpha 1-\alpha 2-\alpha 3} z^{-2n}$$

a mudança n=m-1 nos graus n+1 de "loop", temos

$$-2\sum_m \left(E^m_{-\alpha 1} + E^m_{-\alpha 3}\right) z^{-2m+1} + 2\sum_n E^n_{\alpha 2} z^{-2n} + 2\sum_m E^m_{-\alpha 1-\alpha 2-\alpha 3} z^{-2m+2}$$

fatorando 2z

$$2z\left[\sum_n - \left(E^n_{-\alpha 1} + E^n_{-\alpha 3}\right) z^{-2n} + \sum_n E^n_{\alpha 2} z^{-2n-1} + \sum_n E^n_{-\alpha 1-\alpha 2-\alpha 3} z^{-2n+1}\right]$$

$[\ \varepsilon_+, F_2^+\ ] = 2z\ F_2^+$

3. Cálculo de $w_+$ para $[\ \varepsilon_+, F_4^+\ ] = w_+F_4^+$ onde

$$\varepsilon_+ = E^0_{\alpha 1+\alpha 2} - E^0_{\alpha 2+\alpha 3} + E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3}$$

$$F_4^+ = \sum_n h_1^n z^{-2n} + 2\sum_n h_2^n z^{-2n} + \sum_n h_3^n z^{-2n} - \sum_n E^n_{\alpha 1+\alpha 2} z^{-2n-1}$$
$$+ \sum_n E^n_{\alpha 2+\alpha 3} z^{-2n-1} + \sum_n E^n_{-\alpha 1-\alpha 2}\ z^{-2n+1} - \sum_n E^n_{-\alpha 2-\alpha 3}\ z^{-2n+1} - c$$

Comutadores sem usar termos centrais

$[\ \varepsilon_+, h_1^0\ ] = E^1_{-\alpha 1-\alpha 2} + E^1_{-\alpha 2-\alpha 3} - E^0_{\alpha 1+\alpha 2} - E^0_{\alpha 2+\alpha 3}$

$[\ \varepsilon_+, h_2^0\ ] = E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3} - E^0_{\alpha 1+\alpha 2} + E^0_{\alpha 2+\alpha 3}$

$[\ \varepsilon_+, h_3^0\ ] = -\ E^1_{-\alpha 1-\alpha 2} - E^1_{-\alpha 2-\alpha 3} + E^0_{\alpha 1+\alpha 2} + E^0_{\alpha 2+\alpha 3}$

$[\ \varepsilon_+, E^0_{\alpha 1+\alpha 2}\ ] = -\ h_1^1 - h_2^1$

$[\ \varepsilon_+, E^0_{-\alpha 1-\alpha 2}\ ] = h_1^0 + h_2^0$

$[\ \varepsilon_+, E^n_{\alpha 2+\alpha 3}\ ] = h_2^1 + h_3^1$

$[\ \varepsilon_+, E^0_{-\alpha 2-\alpha 3}\ ] = -h_2^0 - h_3^0$

As contribuições da álgebra de Kac Moody ao comutador $[\ \varepsilon_+, F_4^+\ ]$

$$\varepsilon_+ = E^0_{\alpha1+\alpha2} - E^0_{\alpha2+\alpha3} + E^1_{-\alpha1-\alpha2} - E^1_{-\alpha2-\alpha3}$$

$[\ E^1_{-\alpha1-\alpha2}, E^n_{\alpha1+\alpha2}\ ] = -[\ E^n_{\alpha1+\alpha2}, E^1_{-\alpha1-\alpha2}]$ usando $[E^m_\alpha, E^n_{-\alpha}] = H^{m+n}_\alpha + c\, m\, \delta_{m,-n}$

temos $[\ E^n_{\alpha1+\alpha2}, E^1_{-\alpha1-\alpha2}] = \left(H^{n+1}_{\alpha1+\alpha2} + cn\delta_{n,-1}\right)$ isto é

$[\ E^1_{-\alpha1-\alpha2}, E^n_{\alpha1+\alpha2}\ ] = -\ H^{n+1}_{\alpha1+\alpha2} - cn\ \delta_{n,-1}$ aparece termo central se $n = -1$ ....$(\alpha)$

$[\ E^1_{-\alpha2-\alpha3}, E^n_{\alpha2+\alpha3}\ ] = -\ [\ E^n_{\alpha2+\alpha3}, E^1_{-\alpha2-\alpha3}] = -\left(H^{n+1}_{\alpha2+\alpha3} + c\ n\ \delta_{n,-1}\right)$

$[\ E^1_{-\alpha2-\alpha3}, E^n_{\alpha2+\alpha3}\ ] = -H^{n+1}_{\alpha2+\alpha3} - c\ n\ \delta_{n,-1}$ aparece termo central se n = -1 ....$(\beta)$

Tabela de comutadores incluindo Kac Moody
usando $\alpha$ e $\beta$ temos

$$\varepsilon_+ = E^0_{\alpha1+\alpha2} - E^0_{\alpha2+\alpha3} + E^1_{-\alpha1-\alpha2} - E^1_{-\alpha2-\alpha3}$$

$[\ \varepsilon_+, E^n_{\alpha1+\alpha2}\ ] = [\ E^0_{\alpha1+\alpha2} - E^0_{\alpha2+\alpha3} + E^1_{-\alpha1-\alpha2} - E^1_{-\alpha2-\alpha3}, E^n_{\alpha1+\alpha2}\ ] =$
$= -\ h_1^{n+1} - h_2^{n+1} - cn\ \delta_{n,-1}$
usando $(\alpha)$ :
$[\ \varepsilon_+, E^n_{\alpha2+\alpha3}\ ] = [\ E^0_{\alpha1+\alpha2} - E^0_{\alpha2+\alpha3} + E^1_{-\alpha1-\alpha2} - E^1_{-\alpha2-\alpha3}, E^n_{\alpha2+\alpha3}]$
$= h_2^{n+1} + h_3^{n+1} + c\ n\ \delta_{n,-1}$
usando $(\beta)$ :
$[\ \varepsilon_+, h_1^0\ ] = E^1_{-\alpha1-\alpha2} + E^1_{-\alpha2-\alpha3} - E^0_{\alpha1+\alpha2} - E^0_{\alpha2+\alpha3}$

$[\ \varepsilon_+, h_2^0\ ] = E^1_{-\alpha1-\alpha2} - E^1_{-\alpha2-\alpha3} - E^0_{\alpha1+\alpha2} + E^0_{\alpha2+\alpha3}$
$[\ \varepsilon_+, h_3^0\ ] = -\ E^1_{-\alpha1-\alpha2} - E^1_{-\alpha2-\alpha3} + E^0_{\alpha1+\alpha2} + E^0_{\alpha2+\alpha3}$
$[\ \varepsilon_+, E^n_{\alpha1+\alpha2}\ ] = -\ h_1^{n+1} - h_2^{n+1} - cn\ \delta_{n,-1}$
$[\ \varepsilon_+, E^n_{-\alpha1-\alpha2}\ ] = h_1^n + h_2^n$
$[\ \varepsilon_+, E^n_{\alpha2+\alpha3}\ ] = h_2^{n+1} + h_3^{n+1} + c\ n\ \delta_{n,-1}$
$[\ \varepsilon_+, E^n_{-\alpha2-\alpha3}\ ] = -h_2^n - h_3^n$
Agora, $[\ \varepsilon_+, F_4^+] =$

$$= \sum_n [\varepsilon_+, h_1^n]\ z^{-2n} + 2\sum_n\ [\varepsilon_+, h_2^n]\ z^{-2n} + \sum_n\ [\varepsilon_+, h_3^n]\ z^{-2n} +$$
$$- \sum_n\ [\varepsilon_+, E^n_{\alpha1+\alpha2}]\ z^{-2n-1} + \sum_n\ [\varepsilon_+, E^n_{\alpha2+\alpha3}]\ z^{-2n-1}$$
$$+ \sum_n\ [\varepsilon_+, E^n_{-\alpha1-\alpha2}]\ z^{-2n+1} - \sum_n\ [\varepsilon_+, E^n_{-\alpha2-\alpha3}]\ z^{-2n+1}$$

substituindo os comutadores em ordem

$$= \sum_n \left(E^{n+1}_{-\alpha1-\alpha2} + E^{n+1}_{-\alpha2-\alpha3} - E^n_{\alpha1+\alpha2} - E^n_{\alpha2+\alpha3}\right) z^{-2n}$$
$$+2\sum_n \left(E^{n+1}_{-\alpha1-\alpha2} - E^{n+1}_{-\alpha2-\alpha3} - E^n_{\alpha1+\alpha2} + E^n_{\alpha2+\alpha3}\right)\ z^{-2n}$$
$$+ \sum_n \left(-E^{n+1}_{-\alpha1-\alpha2} - E^{n+1}_{-\alpha2-\alpha3} + E^n_{\alpha1+\alpha2} + E^n_{\alpha2+\alpha3}\right) z^{-2n} +$$

$$- \sum_n \left(-h_1^{n+1} - h_2^{n+1} - c\ n\ \delta_{n,-1}\right)\ z^{-2n-1}$$
$$+ \sum_n \left(h_2^{n+1} + h_3^{n+1} + c\ n\ \delta_{n,-1}\right) z^{-2n-1}$$
$$+ \sum_n \left(h_1^n + h_2^n\right)\ z^{-2n+1}$$

$- \sum_n \left(-h_2^n - h_3^n\right) z^{-2n+1} =$

dão

$$= 2\sum_n \left(E^{n+1}_{-\alpha 1-\alpha 2} - E^{n+1}_{-\alpha 2-\alpha 3} - E^{n}_{\alpha 1+\alpha 2} + E^{n}_{\alpha 2+\alpha 3}\right) \ z^{-2n} +$$
$$+\sum_n \left(h_1^{n+1} + h_2^{n+1}\right) \ z^{-2n-1} - cz^{-2(-1)-1}$$
$$+\sum_n \left(h_2^{n+1} + h_3^{n+1}\right) z^{-2n-1} - cz^{-2(-1)-1} + \sum_n \left(h_1^n + h_2^n\right) \ z^{-2n+1}$$
$$+\sum_n \left( h_2^n + h_3^n\right) z^{-2n+1} =$$
$$= 2\sum_n \left(E^{n+1}_{-\alpha 1-\alpha 2} - E^{n+1}_{-\alpha 2-\alpha 3} - E^{n}_{\alpha 1+\alpha 2} + E^{n}_{\alpha 2+\alpha 3}\right) \ z^{-2n} +$$
$$+\sum_n \left(h_1^{n+1} + 2h_2^{n+1} + h_3^{n+1}\right) \ z^{-2n-1} + \sum_n \left(h_1^n + 2h_2^n + h_3^n\right) \ z^{-2n+1} - 2cz$$

com a mudança m=n+1 nos operadores $h_1^{n+1}$ temos

$$= 2\sum_n \left(E^{n+1}_{-\alpha 1-\alpha 2} - E^{n+1}_{-\alpha 2-\alpha 3} - E^{n}_{\alpha 1+\alpha 2} + E^{n}_{\alpha 2+\alpha 3}\right) \ z^{-2n} +$$
$$+2\sum_n \left(h_1^n + 2h_2^n + h_3^n\right) \ z^{-2n+1}$$
$$= 2\sum_n \left(E^{n+1}_{-\alpha 1-\alpha 2} - E^{n+1}_{-\alpha 2-\alpha 3}\right) \ z^{-2n} +$$
$$+2\sum_n \left(-E^{n}_{\alpha 1+\alpha 2} + E^{n}_{\alpha 2+\alpha 3}\right) \ z^{-2n} + 2\sum_n \left(h_1^n + 2h_2^n + h_3^n\right) \ z^{-2n+1} - 2cz$$

fazendo m=n+1 na primeira soma,

$$= 2\sum_n \left(E^{m}_{-\alpha 1-\alpha 2} - E^{m}_{-\alpha 2-\alpha 3}\right) \ z^{-2m+2} + \ 2\sum_n \left(-E^{n}_{\alpha 1+\alpha 2} + E^{n}_{\alpha 2+\alpha 3}\right) \ z^{-2n} +$$
$$+ 2\sum_n \left(h_1^n + 2h_2^n + h_3^n\right) \ z^{-2n+1} - 2c \ z$$

fatorando 2z

$$= 2z \quad [\sum_n \left(E^{m}_{-\alpha 1-\alpha 2} - E^{m}_{-\alpha 2-\alpha 3}\right) \ z^{-2m+1} + \sum_n \left(-E^{n}_{\alpha 1+\alpha 2} + E^{n}_{\alpha 2+\alpha 3}\right) \ z^{-2n-1}$$
$$+ \sum_n \left(h_1^n + 2h_2^n + h_3^n\right) \ z^{-2n} - c]$$

isto é $\quad [\ \varepsilon_+, F_4^+ \ ] = 2z \ F_4^+$

4. Cálculo de $w_+$ na equação $[\ \varepsilon_+, F_6^+ \ ] = w_+ F_6^+$ onde

$$F_6^+ = \sum_n E^{n}_{-\alpha 2} \ z^{2n} - \sum_n E^{n}_{\alpha 1+\alpha 2+\alpha 3} \ z^{2n-1}$$

usamos representação matricial, pois $\varepsilon_+$ e $F_6^+$ não tem geradores em comun.

$[\ \varepsilon_+, F_6^+ \ ] = \sum_n [\ \varepsilon_+, E^{n}_{-\alpha 2}] \ z^{2n} - \sum_n [\ \varepsilon_+, E^{n}_{\alpha 1+\alpha 2+\alpha 3}] \ z^{2n-1}$

$$\varepsilon_+ = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \end{pmatrix} \qquad E^{0}_{-\alpha 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E^{0}_{\alpha 1+\alpha 2+\alpha 3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left[\varepsilon_+, E^0_{-\alpha 2}\right] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E^0_{\alpha 1} + E^0_{\alpha 3},$$

$$\left[\varepsilon_+, E^0_{\alpha 1+\alpha 2+\alpha 3}\right] = \begin{pmatrix} 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \end{pmatrix} = E^1_{\alpha 1} + E^1_{\alpha 3}$$

$$[\ \varepsilon_+, F^+_6\ ] = \sum_n [\ \varepsilon_+, E^n_{-\alpha 2}]\ z^{2n} - \sum_n [\ \varepsilon_+, E^n_{\alpha 1+\alpha 2+\alpha 3}]\ z^{2n-1}$$

$$= \sum_n \left(E^n_{\alpha 1} + E^n_{\alpha 3}\right) z^n - \sum_n \left(E^{n+1}_{\alpha 1} + E^{n+1}_{\alpha 3}\right) z^{n+1}$$

mudando o índice de soma $n = m - 1$ na segunda soma,

$$= \sum_n \left(E^n_{\alpha 1} + E^n_{\alpha 3}\right) z^n - \sum_n \left(E^m_{\alpha 1} + E^m_{\alpha 3}\right) z^n = 0$$