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Topics on cosmological perturbation theory

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Abstract

In this thesis the Theory of Cosmological Perturbations is reviewed and three original topics, that are part of this huge branch of theoretical cosmology, are presented. We start by reviewing and deducing the needed formulas from first principles in chapter 2. After it, in chapter 3, we study in detail Quantum Field Theories in de Sitter spacetime that contain Higher Spin currents. We show that the existence of Higher Spin currents - even in the interacting case - can put further constraints on the n-point function, making it asymptotically gaussian in the far future. This result can be interpreted as the analog of Coleman-Mandula theorem for de Sitter spacetime. Chapter 4 is devoted to conformal inflationary models with the Higgs field playing the role of the Inflaton field. We construct models with a Weyl symmetry and a $SO(1,1)$ symmetry at high energies. It is verified what are the conditions to get an arbitrary value for the tensor to scalar ratio, which measures the amplitude of primordial gravitational waves in a given model. Also, we introduce a coupling, different from the conformal one, for the scalar field and the curvature tensor. This breaks the Weyl symmetry but we verify that there is a strong attractor towards the Starobinsky line. In the last chapter, we apply the back-reaction effect of long wavelength modes (modes with wavelength bigger than the Hubble radius) in some inflationary models and in the Ekpyrotic scenario. We check if this effect could prevent eternal inflation in the region where stochastic effects are important for these models. Some appendices, with detailed calculations, are also included in the end.

Resumo

Nesta tese a Teoria das Perturbações Cosmológicas é revisada e três tópicos originais, incluídos neste grande ramo da cosmologia teórica, são apresentados. Começamos introduzindo e deduzindo as fórmulas necessárias partindo de primeiros princípios no capítulo 2. Em seguida, no capítulo 3, estudamos em detalhe Teorias Quânticas de Campos em de Sitter que contêm correntes de spin alto. Mostramos que a existência de correntes de spin alto - mesmo em teorias com interação - pode colocar mais vínculos na função de n-pontos, tornando a teoria assintoticamente gaussiana no futuro longínquo. Este resultado pode ser interpretado como o análogo do teorema de Coleman-Mandula para o espaço-tempo de de Sitter. O capítulo 4 é dedicado a modelos inflacionários conformes com o campo de Higgs fazendo o papel de Inflaton. Modelos com simetria de Weyl e com simetria $SO(1,1)$ para valores altos da energia são construídos. Verificamos quais as condições necessárias para que se obtenha um valor arbitrário para a razão escalar tensorial, parâmetro que mede a intensidade de ondas gravitacionais primordiais em um dado modelo. Introduzimos também um acoplamento diferente do valor conforme para a interação do campo escalar com o tensor de curvatura. Isto quebra a simetria de Weyl, mas verificamos que existe um forte atrator na direção da linha de Starobinsky. No último capítulo, aplicamos o efeito do 'back reaction' dos modos com comprimentos de onda longo (maiores que o raio de Hubble) em alguns modelos inflacionários e no cenário Ekpyrótico. Checamos se este efeito pode prevenir a inflação eterna nas regiões onde efeitos estocásticos são importantes nestes modelos. Alguns apêndices, com cálculos detalhados, são incluídos no final.

1. Introduction

Cosmology is in its golden era. Many experiments, as the Planck satellite and the BICEP2 for example, have worked hard to collect new data and many other new projects, such as the Euclid satellite, are expected to spring up in the near future. The increase of data available provides new tests for theoretical models. With the new data, also new problems are now open to the community. Dark energy and dark matter are the biggest issues to cosmologists of this century. Also, pure theoretical problems as the initial singularity, the initial condition problem for Inflation and the definition of an observable in cosmological perturbation theory are still unsolved. In this rich landscape there are plenty of topics to tackle and the Theory of Cosmological Perturbations (TCP) provides a powerful tool to explore the primordial and late time universe. Two examples where the TCP is important are the formation and evolution of large scale structures and the anisotropies of the Cosmic Microwave Background (CMB).

In the TCP one usually starts by defining a manifold and adding quantum fields in it. The manifolds in this context are curved, also called curved spacetimes. After defining a curved spacetime one can introduce perturbations on it. A very well known example is the case of Inflation [1, 2, 3, 4]. In this scenario one starts with a curved spacetime covered by a Friedmann-Lemaître-Robertson-Walker (FLRW) metric. The field content is given by a scalar field which enters as a perfect fluid in the energy momentum tensor. The equation of state needed to know the dynamics and to explain the problems of the old Standard Cosmology, such as the horizon and flatness problems, is given - as we will see later - by $p \simeq -\rho$. In this case the Universe is expanding almost exponentially with time. By perturbing the field or the metric in Einstein equations and going up to first order in the perturbations it is also possible to explain with good accuracy the anisotropies observed in the CMB.

1.1. de Sitter Quantum Field Theories with higher spin currents

The case where $p = -\rho$ reduces to an exactly exponentially expanding Universe. This solution is known as de Sitter spacetime. The first topic we will address in this thesis is the Quantum Field Theory (QFT) in de Sitter spacetime. More precisely, we will see how higher spin (HS) currents - which lead to higher spin charges - can put further constraints on the Quantum Field [5]. Since Quantum Field Theories are hard to solve exactly even in flat spacetimes, and knowing that symmetries usually put some constraints on the spectrum of the theory[1], it is natural to think of new symmetries in order to simplify or even solve exactly some QFT. In fact, this idea was already very explored in the literature as we see below.

In $D > 2$ dimensional Minkowski space, the Coleman-Mandula theorem [6] asserts that HS symmetries constrain the scattering matrix of a theory to be trivial. In $D = 2$ dimensions, the presence of such symmetries implies that the scattering matrix has no particle production (particle number is conserved and the scattering is elastic) [7]. When combined with conformal symmetry, HS symmetries yield interesting classes of solvable rational CFTs in 2D [8], and in 3D CFTs they constrain the theory to have a free-field current algebra, *i.e.*, the theory is essentially free [9]. In

[1]See Appendix C for an example of the 4-point function in a Conformal Field Theory.

each of these cases, the presence of HS symmetries forces the underlying theory to behave as a free theory for at least a large set of physical observables.

In Chapter 3 we investigate consequences of HS symmetries in QFTs on a fixed de Sitter (dS) spacetime. de Sitter space provides the maximally symmetric cosmological model of an inflating spacetime, and is the lowest-order solution in the standard slow-roll expansion of inflation as we already mentioned. The natural set of QFT observables we investigate are the vacuum expectation values of operators inserted near the future and past conformal boundaries. In the context of inflation, observables located near the future asymptotic boundary of de Sitter correspond to late-time expectation values which provide the input for cosmological power spectra. We also consider the correlation functions of observables located near both the past and future asymptotic boundaries of global de Sitter. These correlators describe the global dS analogue of a scattering experiment.[2] Our basic tactic is to analyze how the Ward identities associated with HS symmetries constrain the correlation functions of these observables. In many ways, our investigation is similar in spirit to recent cosmological "consistency conditions" and "soft theorems" (see, e.g., [11, 12, 13, 14, 15] and references therein), though we emphasize that our analysis does not include gravitational back-reaction.

More concretely, our analysis proceeds as follows. For simplicity we consider the effect of HS symmetries on correlations of a scalar operator $\phi(x)$. We assume the theory admits a charge $Q_p^{(s)}$ which is the spin $s > 1$ analogue of a translation in a dS Poincaré chart. Although this chart provides a convenient interpretation for $Q_p^{(s)}$, the charge is in fact well-defined everywhere on dS. The action of $Q_p^{(s)}$ on $\phi(x)$ is local and may be written as a sum of local operators $\mathcal{O}_A(x)$ of the schematic form

$$\left[Q_p^{(s)}, \phi(x)\right] = \sum_A C_A \mathcal{O}_A(x). \tag{1.1}$$

In general, any operator with the correct quantum numbers may appear on the right-hand side of this expression, making a general analysis of (1.1) intractable. However, if we are interested in correlators of $\phi(x)$ near the conformal boundaries of dS (i.e., near past/future asymptotic infinity), we may expand (1.1) in a Fefferman-Graham expansion in powers of conformal time η. Only those operators which scale with η in the same way as $\phi(x)$ will contribute to the commutator asymptotically. Thus, for instance, for a scalar with characteristic scaling $\phi(x) = O(\eta^\Delta)$, $\Delta > 0$, as $\eta \to 0$, we may truncate the right-hand side to operators which likewise scale like $O(\eta^\Delta)$ when evaluating the commutator at asymptotically late times ($\eta \to 0$).

Here we consider the simple case when $\phi(x)$ and its descendants are the only operators in the theory which scale like $O(\eta^\Delta)$ as $\eta \to 0$. In this case the action of $Q_p^{(s)}$ on $\phi(x)$ becomes *asymptotically linear* in $\phi(x)$ near the conformal boundary, i.e. the action takes the form

$$\left[Q_p^{(s)}, \phi(x)\right]\Big|_{O(\eta^\Delta)} = \mathscr{D}(x)\phi(x)\Big|_{O(\eta^\Delta)}, \tag{1.2}$$

where $\mathscr{D}(x)$ is a differential operator. When this occurs, we show that this implies that the leading $O(\eta^{n\Delta})$ behavior of an n-pt correlation function of $\phi(x)$ is Gaussian, i.e., it is composed of 2-pt correlations. The same conclusion holds in global dS for correlation functions in which each operator is placed near one of the asymptotic boundaries. Thus $\phi(x)$ has trivial cosmological spectra (no bispectrum, Gaussian trispectrum, etc.), and also has no "scattering" in global dS (when measured with respect to equivalent initial/final vacua).

[2] Here we mean only that these correlations measure the transition amplitude for states constructed at asymptotically early/late times. We will not attempt to establish a rigorous notion of asymptotic particle states for global dS. See [10] for a construction of such particle states and their corresponding S-matrix for *perturbatively* interacting QFTs on global de Sitter space.

The assumption that the action of a HS charge is asymptotically linear is clearly very restrictive. However, we regard this assumption as the appropriate dS analogue of one of the assumptions of the Coleman-Mandula theorem: a symmetry of the S-matrix is one which maps n-particle states to n-particle states [6]. Said differently, a symmetry-generating charge acts linearly on the field redefinition-invariant parts of the asymptotic correlation functions. In dS QFT, the leading asymptotic behavior of vacuum correlation functions near the conformal boundaries is, at least in perturbation theory, field redefinition-invariant and the key input in the perturbative de Sitter S-matrix [10]. Thus, we regard our result as providing an analogue of the Coleman-Mandula theorem for dS QFT.

1.2. Inflation with the Higgs field

As we said, the equation of state $p \simeq -\rho$ is the one used in Inflation. The difference between the equality and the approximation in the equation of state can be quantified by the slow roll parameters and it is not trivial to show that the slow roll parameters are connected to the metric or field perturbations which, up to first order in the perturbations, lead to the anisotropies observed in the CMB spectrum[3].

Another important characteristic of Inflation is the presence of primordial gravitational waves that are originated from perturbations of the metric tensor at early times. These primordial gravitational waves are able to propagate until the current days and, in principle, could be detectable [16, 17]. In 2014 the BICEP2 team [18] had claimed the detection of primordial gravitational waves and made the community of theoretical cosmologists very excited. In 3 months the paper got more than 400 citations. In this period, the second original topic of this thesis was developed. The idea was to modify the T-models [19, 20] (following earlier work by [21, 22, 23, 24, 25, 26] where the initial motivation was superconformal symmetry) imposing, for small field values, the Higgs field potential. The idea of starting with the Higgs field and extending it to higher field values was done in [22, 23, 24] where the authors assume that the instability of the Higgs field is true for large field values. With this approach the authors were ”naturally" lead to cyclic cosmological models. In both ideas a Weyl symmetry (symmetry under rescaling the metric and the scalar fields by a local conformal factor) was considered. It turns out that it is possible to start with the Higgs field and to have a plateau-like region for large field values if a SO(1,1) symmetry is imposed in this region. However, the natural constructions lead to a very small value for the amplitude of the primordial gravitational waves which would be incompatible with the BICEP2 result. We show in appendix D why the standard flat potentials usually lead to a small value for the tensor to scalar ratio[4] and we construct non trivial potentials in Chapter 4 which can be made compatible to the BICEP2 result.

Unfortunately, the result of BICEP2 was corrected by the new data released by the Planck satellite in 2015 [27]. This new data was due to dust in our galaxy and according to the theory, dust could also polarize the primordial light in the very same way as the primordial gravitational waves. If one subtract this contamination, the value for the power spectrum of primordial gravitational waves is way down the limits of the current detectors.

However, the construction prescribed here shows that any value of r that may be found experimentally could be made consistent with the "conformal" inflation models.

[3]We show it in Chapter 2.

[4]The tensor to scalar ratio is the parameter used to quantify the amplitude of primordial gravitational waves.

1.3. Back-reaction of long wavelength perturbations

Another interesting problem one could attack with the TCP is the problem of Eternal Inflation. The basic idea of Eternal Inflation (EI) is simple to understand. In a region of large field values, *i.e.*, in a specific region for the initial condition, quantum effects may become as large as or bigger than the classical force, which makes the field to have a 50% chance to roll up its potential. So there is a possibility for the field to be stuck in this region. Also, the equations of motion show that the Universe is expanding in this region, therefore, we could have an eternal inflating Universe. The idea of EI is an old one [28, 29]. More recently, Alan Guth [30] showed that models with EI would generate a multiverse scenario. As pointed out by Alan Guth, in this scenario everything that could happen will happen and will happen an infinite number of times.

In the last Chapter of this thesis we will use the perturbations of long wavelengths to deal with this problem. This effect starts if we consider perturbations up to second order in Einstein equations. The second order perturbations enter as a effective energy momentum tensor for the background Einstein equations and it was discovered by R. Abramo, R. Brandenberger and V. Mukhanov in [31]. Accordingly to the modified equations of motion, this effect could be able to stop the almost exponential expansion of spacetime in a dynamical way since the equation of state for the perturbations is now $p \simeq \rho$. We applied this effect in the well know models of a power law potentials, the Starobinky model and in the Ekyprotic scenario [32].

1.4. Overview

The summary of the above paragraphs is: we will explore three projects related to perturbations in cosmological spacetimes up to second order. In the next chapter we will introduce the basics necessary to follow the upcoming chapters. Since Chapter 3 of this thesis is devoted to Quantum Field Theories in de Sitter spacetime, we start reviewing what is a de Sitter spacetime. In Chapter 4, we find a model for Inflation, so we also introduce how Inflation works. We will need to start with perturbations up to first order in order to take into account the anisotropies of the CMB. Going to second order in perturbations, we review the topic of back reaction of long wavelength modes, which is the basis for chapter 5 of this thesis.

The thesis is divided in 3 parts. Chapter 2 contains the basic facts to understand chapters 3, 4 and 5 and it is highlighted in the text that up to that point the reader is able to skip the rest of chapter 2 and read the chapter he/she is interested in. Chapters 3, 4 and 5 are supported by appendix C, D and E respectively. The chapters treat topics on TCP and are devoted to experts and advanced students and we direct state the results there. The appendices which supplements this thesis are devoted mainly to students which are interested in the details of the computations.

2. Basics of cosmological perturbations theory

In this chapter we review cosmological perturbations theory. The goal is to deduce the formulas used in the next chapters from first principles.

2.1. de Sitter spacetime and Inflation

The de Sitter spacetime is a solution of the Einstein's equation with a cosmological constant,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2.1}$$

Another way of defining it, which will be useful in order to introduce inflation later, is to consider Einstein's equation as

$$G_{\mu\nu} = T_{\mu\nu}, \tag{2.2}$$

where, $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$[1], is the energy momentum tensor of a perfect fluid. If $p = -\rho$, and ρ is constant[2], we see that we recover the (2.1).

So far we have only said that de Sitter spacetime is a solution of the above mentioned Einstein's equation. In order to find what solution it is, we start with a spacetime $\mathcal{M}$, covered with a metric g, *i.e.*, the set, $(\mathcal{M}, g)$. Impose homogeneity and isotropy, that is, the spacetime is symmetric under spacetime translations and rotations. The metric satisfying these symmetries is given by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric[3] :

$$ds^2 = -dt^2 + a(t)d\vec{x}^2. \tag{2.3}$$

Einstein's equations give the dynamics of the scale factor $a(t)$ of the above metric. After plugging (2.3) into (2.2), with the energy momentum tensor of a perfect fluid, we get the Friedmann's equation,

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3}. \tag{2.4}$$

Here k is the measure of the spatial curvature of the universe and, according to the recent analysis made by Planck collaboration [33], $k = 0.000 \pm 0.005$ (Planck data with Baryonic Acoustic Oscillations - BAO - data). H is the Hubble parameter defined by $\frac{\dot{a}}{a}$ and ρ is the energy density of a perfect fluid.

We also obtain, using the equation of state $p = -\rho$, where p is the pressure of the perfect fluid, the following equation for the scale factor

$$\ddot{a} - H^2 a = 0. \tag{2.5}$$

The general solution for equation (2.5) is

$$a(t) = Ae^{Ht} + Be^{-Ht}, \tag{2.6}$$

[1] In this thesis we use the mostly plus signature $\eta_{\mu\nu} = (-, +, +, +)$.

[2] In this chapter we use, $m_{pl} = \frac{1}{\sqrt{8\pi G}} = 1$, for the reduced Planck mass unless stated otherwise.

[3] See appendix A.

with A, B being constants of integration. Inserting this result in the Friedmann's equation we obtain the following constraint

$$4H^2AB = k. \tag{2.7}$$

So, in the relevant measured case, $k = 0$, one of the constants must be equal to zero. For $A = 0$ and $B \neq 0$ the universe contracts exponentially and it is not consistent with the current observations. If $A \neq 0$ and $B = 0$ we have an exponential expanding universe, which is consistent with current observations and is the paradigm for the early universe inflationary phase. The exponentially expanding solution is called *de Sitter solution*. For $k = 0$ the Friedmann's equation becomes

$$H = \left(\frac{\rho}{3}\right)^{1/2}, \tag{2.8}$$

which is a constant if ρ is constant. Thus, the conclusion we draw is that we can have a Universe expanding exponentially with time if this Universe has a cosmological constant or, equivalently, if it is filled with a perfect fluid satisfying the equation of state, $p = -\rho$, where ρ is a positive real constant. The reader interested only in chapter 3 of this thesis is invited to skip the rest of this introduction.

The importance of the de Sitter solution became more evident after the proposal of some authors, in the early 80's [1, 2, 3, 4], that a phase of accelerated exponential expansion immediately after the big bang could explain the isotropy observed in the Cosmic Microwave Background (CMB). The CMB was measured for the first time by Arno Penzias and Robert W. Wilson in 1964 [34]. The scenario of an accelerated exponential expansion was named by Inflation, and is also useful to explain the flatness of the spatial section of the Universe and the absence of monopoles (a reader interested in this issues can find very good reviews in [35, 36]).

The idea of considering a Universe filled with a perfect fluid is used in Inflation. In fact, if we consider a Universe filled with a classical scalar field we can write the following action,

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right) \tag{2.9}$$

and find the dynamics of this field by varying the action. By deriving the action with respect to the metric tensor, we find the energy momentum tensor for the scalar field

$$T^{\varphi}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial^\alpha\varphi\partial_\alpha\varphi + V(\varphi)\right). \tag{2.10}$$

If we assume the FLRW metric and a homogeneous scalar field, i.e, $\varphi(x,t) = \varphi(t)$, then we can write the energy density and the pressure as

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{2.11}$$

$$p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi). \tag{2.12}$$

We see that the equation of state

$$p_\varphi = w_\varphi\rho_\varphi, \tag{2.13}$$

becomes, $p_\varphi \approx -\rho_\varphi$, if, $V(\varphi) \gg \dot{\varphi}^2$. This is called the first *slow roll condition*. This condition can be rewriten in terms a slow roll parameter. In order to do that, we need the following Einstein equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_\varphi + 3p_\varphi) = H^2(1-\epsilon), \tag{2.14}$$

where,

$$\begin{aligned}\epsilon &\equiv \frac{3}{2}(w_\varphi + 1) \\ &= \frac{3}{2}(\frac{\rho_\varphi + p_\varphi}{\rho_\varphi}) \\ &= \frac{1}{2}\frac{\dot{\varphi}^2}{H^2},\end{aligned} \tag{2.15}$$

is called the first *slow roll parameter*. It is easy to see that if $\epsilon \to 0$ we recover the de Sitter exponential expansion. This happens when $\dot{\varphi}^2 \ll V(\varphi)$. The first slow roll parameter can also be written in terms of the potential. In fact, substituting

$$\dot{\varphi} \approx \frac{V'}{3H}, \qquad H^2 \approx \frac{1}{3M_{pl}^2}V, \tag{2.16}$$

into equation (2.15) we get

$$\epsilon = \frac{M_{pl}^2}{2}\left(\frac{V'(\varphi)}{V(\varphi)}\right)^2. \tag{2.17}$$

However, if we look to the equation of motion of a scalar field

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \tag{2.18}$$

we see that the accelerated expansion will only last for a small period of time if we do not impose the smallness of the second derivative of the scalar field. So, we are led to impose the *second slow roll condition*[4],

$$|\ddot{\varphi}| \ll |3H\dot{\varphi}|. \tag{2.19}$$

This condition can be expressed in terms of the *second slow roll parameter*. It is defined by

$$\tilde{\eta} \equiv -\frac{\ddot{\varphi}}{H\dot{\varphi}}. \tag{2.20}$$

We can rewrite equation (2.20) as

$$\tilde{\eta} = \epsilon - \frac{1}{2\epsilon}\frac{d\epsilon}{dN}. \tag{2.21}$$

In fact, up to a constant and using $dN \equiv Hdt$

$$\begin{aligned}\frac{d\ln\epsilon}{dN} &= \frac{2d\ln\dot{\varphi}}{dN} - \frac{2d\ln H}{dN} \\ &= 2\frac{\ddot{\varphi}}{H\dot{\varphi}} + 2\epsilon\end{aligned} \tag{2.22}$$

so

$$2\tilde{\eta} = 2\epsilon - \frac{d\ln\epsilon}{dN} \tag{2.23}$$

and the result (2.21) follows.

[4]Note that $V' \propto \sqrt{\epsilon}$ so it is already small.

Since

$$\begin{aligned}\frac{d\epsilon}{dN} &= \frac{2V'}{V}\left(\frac{V''}{V}\frac{\dot\varphi}{H}-\frac{V'^2}{V^2}\frac{\dot\varphi}{H}\right)\\ &= 2\sqrt{2\epsilon}\left(\frac{V''}{V}\sqrt{2\epsilon}-\frac{V'^2}{V^2}\sqrt{2\epsilon}\right)\\ &= 4\epsilon\left(\frac{V''}{V}-2\epsilon\right), \end{aligned}\tag{2.24}$$

if we plug this result back into (2.21) we see that $|\tilde\eta|\ll 1$ if and only if $V''/V\ll 1$. So the condition to have a small second slow roll parameter can be written as

$$\eta \doteq M_{pl}^2\frac{V''(\varphi)}{V(\varphi)}.$$

In summary,

$$\boxed{\epsilon = \frac{M_{pl}^2}{2}\left(\frac{V'(\varphi)}{V(\varphi)}\right)^2,}\tag{2.25}$$

$$\boxed{\eta = M_{pl}^2\frac{V''(\varphi)}{V(\varphi)}.}\tag{2.26}$$

Here we reintroduce M_{pl} only to make the slow-roll parameters manifestly dimensionless. Equations (2.25) and (2.26) will be used frequently in chapter 3.

2.2. 1^{st} order perturbations - Anisotropies in the CMB

By considering quantum perturbations of the scalar field (the same field used in Inflation to mimic a cosmological constant) it is also possible to explain the anisotropies observed in the CMB. Here we introduce from first principles the quantitative relevant parameters we look at in order to see if there are such anisotropies; they are called *power spectrum*, *scalar tilt* and *tensor to scalar ratio.*

2.2.1. The power spectrum for scalar perturbations and the scalar tilt

We start with the action (2.9),

$$S=\frac{1}{2}\int d^4x\sqrt{-g}\Big(R+g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-2V(\varphi)\Big).$$

This action is invariant under diffeomorphisms and we can use the following gauge fixing for the matter field and for the metric[5]:

$$\delta\varphi=0,\quad g_{ij}=a^2[(1-2\mathcal{R})\delta ij+h_{ij}],\quad \partial^i h_{ij}=h^i_i=0,\tag{2.27}$$

where $\mathcal{R}=\Psi+\frac{H}{\dot\varphi}\delta\varphi$ is a gauge invariant variable. With some effort we can expand the above action to second order in $\mathcal{R}$

$$S_{(2)}=\frac{1}{2}\int d^4x a^3\frac{\dot\varphi^2}{H^2}[\dot{\mathcal{R}}^2-a^{-2}(\partial_i\mathcal{R})^2].\tag{2.28}$$

[5] To fix a gauge in General Relativity is always a subtle task, see Appendix B for the details.

Using the Mukhanov variables,

$$v = z\mathcal{R}, \quad z^2 \equiv a^2 \frac{\dot{\varphi}^2}{H^2} \tag{2.29}$$

we can rewrite[6]

$$S_{(2)} = \frac{1}{2}\int d\tau d^3x \Big[(v')^2 + (\partial_i v)^2 + \frac{z''}{z}v^2\Big]. \tag{2.30}$$

Note that the above action is similar to that of a harmonic oscillator time dependent mass. So, it is instructive to make a Fourier expansion of the Mukhanov variable,

$$v(\mathbf{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} v_{\mathbf{k}}(\tau) e^{i\mathbf{k}.\mathbf{x}}. \tag{2.31}$$

Inserting it back in the action (2.30), we have an equation of motion of a time dependent harmonic oscillator for each mode,

$$v_k^{''} + \Big(k^2 - \frac{z^{''}}{z}\Big)v_k = 0. \tag{2.32}$$

To solve this equation we need two conditions for v_k, since it is a second order differential equation. The quantum treatment, which is very similar to a quantum harmonic oscillator with time depend frequency, will give us the conditions we need. To see it, we promote $v(\mathbf{x}, \tau)$ to a quantum operator,

$$v(\mathbf{x}, \tau) \to \hat{v}(\mathbf{x}, \tau) = \int \frac{dk^3}{(2\pi)^3} \Big[v_k(\tau)\hat{a}_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}} + v_k^*(\tau)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}.\mathbf{x}}\Big], \tag{2.33}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the creation and annihilation operators, that is, they act for example in a state $|n_1, n_2, ...\rangle$ as

$$\hat{a}_1|n_1, n_2, ...\rangle = \sqrt{n_1}|(n_1 - 1), n_2, ...\rangle, \tag{2.34}$$

$$\hat{a}_1|n_1, n_2, ...\rangle = \sqrt{n_1 + 1}|(n_1 + 1), n_2, ...\rangle. \tag{2.35}$$

Alternatively

$$v_{\mathbf{k}} \to \hat{v}_{\mathbf{k}} = v_k(\tau)\hat{a}_{\mathbf{k}} + v_{-k}^*(\tau)\hat{a}_{-\mathbf{k}}^\dagger. \tag{2.36}$$

Now, the canonical quantization condition $[\hat{v}_{\mathbf{k}}, \hat{v}_{\mathbf{k}}^{'}] = i\hbar$, implies

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k'}}^\dagger] = (2\pi)^3\delta(\mathbf{k} - \mathbf{k'}) \tag{2.37}$$

if and only if

$$\langle v_k, v_k\rangle \equiv \frac{i}{\hbar}(v_k^* v_k^{'} - v_k^{*'} v_k) = 1. \tag{2.38}$$

This is the first condition for v_k. The second condition comes from a subtlety that happens in Quantum Field Theories on Curved Spacetimes, *i.e.*, the non uniqueness of the vacuum![7] A way to deal with this subtlety is noting that the FLRW spacetime reduces to Minkowski spacetime in the far past ($k \gg aH$). This condition imposes a constraint on the equation for $v_k(\tau)$,

$$v_k^{''} + k^2 v_k = 0. \tag{2.39}$$

[6] Here, τ is the conformal time defined by $d\tau = \frac{dt}{a(t)}$, and the prime means a derivative with respect to the conformal time.

[7] See [35] for details.

Thus in the far past we have a well known harmonic oscillator equation! To solve this equation we need two initial conditions. The first one is already stated in equation (2.38). The second one is given by,

$$\lim_{\tau\to-\infty} v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{2.40}$$

which is exactly what we expect from a harmonic oscillator in Minkowski spacetime.[8] The boundary conditions (2.38) and (2.40) completely fix the mode functions on all scales!

But, we wish to solve the general equation

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0. \tag{2.41}$$

For our purposes we can consider only the de Sitter limit $\epsilon \to 0$, *i.e.*, $\dot{\varphi}$ doesn't vary much. Then[9],

$$\frac{z''}{z} = \frac{a''}{a} = \frac{2}{\tau^2}. \tag{2.42}$$

One can verify by substitution that the solution for the equation

$$v_k'' + \left(k^2 - \frac{2}{\tau}\right)v_k = 0, \tag{2.43}$$

compatible with the conditions (2.38) and (2.40) is

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{2.44}$$

With these results, we can finally compute the important quantity we wish to, *i.e.*, the *power spectrum*. The definition of the power spectrum is given by

$$\langle \mathcal{R}_k \mathcal{R}_k' \rangle \doteq (2\pi)^3 \delta(k+k') P_{\mathcal{R}}(k), \qquad \Delta^2_{\mathcal{R}} = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k). \tag{2.45}$$

Here $P_{\mathcal{R}}$ is the scalar power spectrum and $\Delta_{\mathcal{R}}$ is the dimensionless power spectrum. To compute these power spectra we see that we only need to compute the two point function of the modes of the gauge invariant variable $\mathcal{R}_k$. Since $\mathcal{R}_k = \frac{v_k}{a}\frac{H}{\dot{\varphi}}$ (see appendix B), we find

$$\begin{aligned} \langle \mathcal{R}_k \mathcal{R}_{k'} \rangle &= (2\pi)^3 \delta(k+k') \frac{|v_k|^2}{a^2} \frac{H^2}{\dot{\varphi}^2} \\ &= (2\pi)^3 \delta(k+k') \frac{H^2}{2k^3}(1+k^2\tau^2)\frac{H^2}{\dot{\varphi}^2}. \end{aligned} \tag{2.46}$$

In the second line we used $k = a(t_\star)H(t_\star)$ since we are computing it when the modes are crossing the Hubble radius[10]. On super-Hubble scales $|k\tau| \ll 1$, the result is

$$\langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = (2\pi)^3 \delta(k+k') \frac{H_\star^2}{2k^3} \frac{H_\star^2}{\dot{\varphi}_\star^2}. \tag{2.47}$$

[8] Actually we would expect that a solution $\propto e^{ik\tau}$ is also allowed, but positivity of $\langle v_k, v_k \rangle$ implies that only equation (2.40) obey this restriction.

[9] Since $d\tau = \frac{dt}{e^{Ht}}$, we get $\tau = \frac{-1}{Ha}$ and then the result (2.42) easily follows.

[10] We compute the two point function in the moment of the crossing because $\dot{\mathcal{R}} \simeq 0$ on super Hubble scales. So the fluctuations which leave the Hubble radius during inflation are frozen on super Hubble scales and re-enter at late times causing the anisotropies observed in the CMB.

The dimensionless scalar power spectrum is given by

$$\begin{aligned}\Delta^2_{\mathcal{R}}(k) &= \frac{H^2}{(2\pi)^2}\frac{H^2}{\dot{\varphi}^2} \\ &= \frac{1}{8\pi^2}\frac{H^2}{M^2_{pl}}\frac{1}{\epsilon}\bigg|_{k=aH}. \end{aligned} \tag{2.48}$$

Now, a very important experimental parameter is the one that measures the scale dependence of the spectra. This scale follow from the time dependence of the Hubble parameter and is defined by the so called, *tilt*

$$n_s - 1 \equiv \frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k}. \tag{2.49}$$

So, if the tilt n_s is equal to 1, the power spectrum is scale invariant. Our goal now is to write the scalar tilt as a function of the slow-roll parameters. This relation will be used, without demonstration, in chapter 4.

First, consider

$$\frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k} = \frac{d\ln\Delta^2_{\mathcal{R}}}{dN}\frac{dN}{d\ln k}. \tag{2.50}$$

Now, the first term gives

$$\frac{d\ln\Delta^2_{\mathcal{R}}}{dN} = \left(2\frac{d\ln H}{dN} - \frac{d\ln\epsilon}{dN}\right) \tag{2.51}$$

Since

$$\begin{aligned}\frac{d\ln H}{dN} &= \frac{1}{H}\frac{dH}{dN} \\ &= \frac{1}{H^2}\frac{dH}{dt} \\ &= -\frac{1}{2}\frac{\dot{\varphi}^2}{H^2} \\ &= -\epsilon, \end{aligned} \tag{2.52}$$

where we used $dN = Hdt$ and $\dot{H} = -\frac{\dot{\varphi}^2}{2}$. Also,

$$\begin{aligned}\frac{d\ln\epsilon}{dN} &= \frac{2d\ln\dot{\varphi}}{dN} - \frac{2d\ln H}{dN} \\ &= 2\frac{\ddot{\varphi}}{H\dot{\varphi}} + 2\epsilon \\ &= -2\eta + 2\epsilon. \end{aligned} \tag{2.53}$$

where we have used equation (2.20). Then, the first term of equation (2.50) is

$$\frac{d\ln\Delta^2_s}{dN} = -4\epsilon + 2\eta. \tag{2.54}$$

The second term from equation (2.50) can be computed by noting that at the moment where modes cross the Hubble radius,

$$\ln k = \ln a + \ln H = N + \ln H. \tag{2.55}$$

Hence,

$$\frac{dN}{d\ln k} = \left(\frac{d\ln k}{dN}\right)^{-1} = \left(1 + \frac{d\ln H}{dN}\right)^{-1} \approx 1 + \epsilon. \tag{2.56}$$

Finally, to first order in the slow-roll parameters we obtain the desired formula,

$$\boxed{n_s - 1 = 2\eta - 4\epsilon.} \tag{2.57}$$

2.2.2. The tensor to scalar ratio

Another prediction from Inflation is the existence of primordial gravitational waves. They come from the tensor modes, h_{ij}, of the perturbed metric (2.27). By putting the perturbed metric into the action (2.9) and expanding up to second order in the tensor modes, one can find

$$S_h = \frac{M_{pl}^2}{8} \int d\tau d^3 x a^2 [(\partial_\tau h_{ij})^2 - (\partial_l h_{ij})^2]. \tag{2.58}$$

We reintroduce the Planck mass here in order to make the metric perturbations h_{ij} explicitly dimensionless. Up to the normalization factor, equation (2.58) is the same action as a massless scalar field in a FLRW spacetime.

Expanding the tensor perturbations in Fourier modes

$$h_{ij} = \int \frac{d^3 k}{(2\pi)^3} \sum_{s=\times,+} \mathbf{e}_{ij}^s h_{\vec{k}}^s(\tau) e^{i\vec{k}\cdot\vec{x}}, \tag{2.59}$$

we have the polarization vectors satisfying, $\mathbf{e}_{ii} = k^i \mathbf{e}_{ij} = 0$, as a consequence of $h_i^i = \partial^i h_{ij} = 0$. Also, the polarization vector are orthogonal, $e_{ij}^s(k) e_{ij}^{s'}(k) = 2\delta_{ss'}$. In terms of the tensor modes, the action (2.58) becomes

$$S_h = \sum_s \int d\tau d^3 k \frac{a^2 M_{pl}^2}{4} [(\partial_\tau h_{\vec{k}}^s)^2 - k^2 (h_{\vec{k}}^s)^2]. \tag{2.60}$$

In terms of a canonically normalized field

$$v_{\vec{k}}^s \equiv \frac{a}{2} M_{pl} h_{\vec{k}}^s, \tag{2.61}$$

the action becomes

$$S_v = \sum_s \frac{1}{2} \int d\tau d^3 k \left[(\partial_\tau v_{\vec{k}}^s)^2 - \left(k^2 - \frac{a''}{a} \right) (v_{\vec{k}}^s)^2 \right]. \tag{2.62}$$

This action is just two copies of action (2.30). Now, using

$$v_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau} \right), \tag{2.63}$$

and the definition of the tensor power spectrum

$$\langle h_k^s h_{k'}^s \rangle \doteq (2\pi)^3 \delta(k + k') P_h(k), \qquad \Delta_h^2 = \frac{k^3}{2\pi^2} P_h(k), \tag{2.64}$$

we get

$$\Delta_h^2(k) = \frac{4}{M_{pl}^2} \left(\frac{H_\star}{2\pi} \right)^2. \tag{2.65}$$

The tensor power spectrum is twice the tensor mode power spectrum,

$$\Delta_t^2 = 2\Delta_h^2(k) = \frac{2}{M_{pl}^2} \left(\frac{H_\star}{2\pi} \right)^2. \tag{2.66}$$

Finally, we are able to define the *tensor to scalar ratio* parameter,

$$r \equiv \frac{\Delta_t^2(k)}{\Delta_s^2(k)}. \tag{2.67}$$

Since the scalar power spectrum is measured ($\Delta_s^2 \equiv \Delta_{\mathcal{R}}^2 \approx 10^{-9}$ according to Planck data [33]), once r is inferred, we can find the tensor power spectrum. Also, since $\Delta_t^2 \propto H^2 \approx V$, the tensor to scalar ratio gives the scale of single field inflation

$$V^{1/4} \approx \left(\frac{r}{0.01}\right)^{1/4} 10^{16} GeV. \tag{2.68}$$

A last thing we can do is to rewrite the tensor to scalar ratio in terms of the first slow roll parameter. It is obtained by dividing (2.48) by (2.66). Then, we obtain

$$\boxed{r \equiv \frac{\Delta_t^2}{\Delta_s^2} = 16\epsilon_\star,} \tag{2.69}$$

where $\epsilon_\star$ is the slow roll parameter computed at Hubble radius crossing. The reader interested only in chapter 4 is invited to skip the next subsection.

2.3. 2^{nd} order perturbations and the back reaction of long wavelength effect

As we saw, in early universe cosmology we usually consider a homogenous and isotropic background spacetime and impose small amplitude cosmological fluctuations which are treated by linearizing the field equations about the background. However, Einstein field equations are highly nonlinear, and hence even at the classical level the fluctuations at second order influence the background. This is what we mean by *back reaction*.

The expansion parameter for cosmological perturbation theory is the amplitude of the fluctuations which is set by the size of the observed CMB anisotropies and is of the order 10^{-5}. Hence, the back reaction effect of any given fluctuation mode is tiny (of the order 10^{-10}). However, each fluctuation mode effects the background, and hence, for a long period of inflation the integrated effect of all of the modes can be important.

It is worth to mention that the back-reaction effect of sub-Hubble fluctuations is very small and approximately constant in time[11]. On the other hand, the back-reaction of long wavelength modes shows secular growth due to the increasing phase space of super-Hubble modes.

The gravitational back-reaction formalism is as follows [37, 31]. We begin with the full Einstein field equations

$$G_{\mu\nu} = T_{\mu\nu}\,, \tag{2.70}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the energy-momentum tensor of matter. The background metric $\bar{g}_{\mu\nu}(t)$ and matter $\bar{\varphi}(t)$ obey these (nonlinear) equations. Next, we introduce the linearized metric and matter fluctuations $\delta g_{\mu\nu}$ and $\delta\varphi$ which are both functions of space and time and which obey the linearized Einstein equations. However, the system of fields

$$\begin{aligned} g_{\mu\nu}(x,t) &= \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x,t) \\ \varphi(x,t) &= \bar{\varphi}(t) + \delta\varphi(x,t) \end{aligned} \tag{2.71}$$

does not satisfy the Einstein equations at second order.

If we are only interested in modifying the background at quadratic order we need to introduce second order correction terms $g^{(2)}_{\mu\nu}(t,x)$ and $\varphi^{(2)}(t,x)$ to metric and matter. Adding these terms to

[11] Quantum fluctuations are constantly generated - see appendix E - and the Hubble radius is almost constant during inflation.

the ansatz (2.71) for metric and matter, inserting into the Einstein equations (2.70), cancelling the linear terms using the linear fluctuation equations, moving all terms quadratic in the fluctuations to the right hand side, and taking the spatial average of the resulting equation yields an equation of motion for the corrected background metric

$$\bar{g}^{(br)}_{\mu\nu}(t) \;=\; \bar{g}_{\mu\nu}(t) + g^{(2)}_{\mu\nu}(t) \tag{2.72}$$

of the following form:

$$G_{\mu\nu}(\bar{g}^{(br)})(t) \;=\; 8\pi G\Big[T^{(0)}_{\mu\nu}(\bar{\varphi}(t)) + \tau_{\mu\nu}(t)\Big]\,, \tag{2.73}$$

where $\tau_{\mu\nu}(t)$ is quadratic in the cosmological fluctuations, and is called the "effective energy-momentum tensor of cosmological perturbations". The effective energy momentum tensor is obtained by integrating over the contributions of all fluctuation (Fourier) modes. For the reasons explained above we are only interested in the contribution of the super-Hubble modes (the sub-Hubble modes will contribute as a stochastic term).

As the background metric we take a spatially flat Friedmann-Robertson-Walker-Lemâitre metric given by the line element

$$ds^2 \;=\; a(\eta)^2(d\eta^2 - d\vec{x}^2)\,, \tag{2.74}$$

where η is conformal time. To evaluate $\tau_{\mu\nu}$ we work in longitudinal (conformal-Newtonian) gauge in which (for single component matter without anisotropic stress) the line element is

$$ds^2 \;=\; a(\eta)^2\Big[(1+2\Phi)d\eta^2 - (1-2\Phi)d\vec{x}^2\Big]\,, \tag{2.75}$$

where $\Phi(x,t)$ is the relativistic gravitational potential [38].

The metric and matter fluctuations are not independent. They are coupled via the Einstein constraint equations. In the background of a slowly rolling scalar field their connection is given by

$$\delta\varphi(k) \;=\; -\frac{2V}{V'}\Phi(k)\,, \tag{2.76}$$

where the argument k indicates that we are considering the Fourier modes of the fluctuations. The general expression for $\tau_{\mu\nu}$ is

$$\begin{aligned}\tau_{00} \;&=\; 12H\langle\Phi\dot{\Phi}\rangle - 3\langle\dot{\Phi}^2\rangle + 9a^{-2}\langle(\nabla\Phi)^2\rangle \\ &+\; \langle(\delta\dot{\varphi})^2\rangle + a^{-2}\langle(\nabla\delta\varphi)^2\rangle \\ &+\; \frac{1}{2}V''(\bar{\varphi})\langle\delta\varphi^2\rangle + 2V'(\bar{\varphi})\langle\Phi\delta\varphi\rangle,\end{aligned} \tag{2.77}$$

and

$$\begin{aligned}\tau_{ij} \;&=\; a^2\delta_{ij}\Big((24H^2+16\dot{H})\langle\Phi^2\rangle + 24H\langle\dot{\Phi}\Phi\rangle + \langle\dot{\Phi}^2\rangle + 4\langle\Phi\ddot{\Phi}\rangle - \frac{4}{3}a^{-2}\langle(\nabla\Phi)^2\rangle \\ &+\; 4\dot{\bar{\varphi}}^2\langle\phi^2\rangle + \langle(\delta\dot{\varphi})^2\rangle - a^{-2}\langle(\nabla\delta\Phi)^2\rangle - 4\dot{\bar{\varphi}}\langle\delta\dot{\varphi}\Phi\rangle \\ &-\; \frac{1}{2}V''(\bar{\varphi})\langle\delta\varphi^2\rangle + 2V'(\bar{\varphi})\langle\Phi\delta\varphi\rangle\Big).\end{aligned} \tag{2.78}$$

Here $\langle\rangle$ means spatial average, Φ is the metric perturbation and $\bar{\varphi}$ is the background inflaton field. Because of the slow roll conditions, terms containing $\dot{\Phi}$ can be neglected because they are proportional to $\dot{\zeta}$ (see equation (B.35)) which is conserved on super Hubble scales. Also, $\dot{\bar{\varphi}}^2$ and

$\dot{H}$ can be neglected due to the slow roll condition[12]. Gradient terms can also be neglected due to the super-Hubble condition ($k \ll Ha$). Also, using one of the Einstein equations

$$\dot{\Phi} + H\Phi = \frac{1}{2}\dot{\bar{\varphi}}\delta\varphi, \tag{2.79}$$

neglecting $\dot{\Phi}$[13] and using $\dot{\bar{\varphi}} \simeq \frac{V'}{3H}$ we have

$$\delta\varphi = -\frac{2V}{V'}\Phi. \tag{2.80}$$

Then, the components of the effective energy momentum tensor for long wavelength modes reduces to

$$\tau_{00} \;=\; \frac{1}{2}V''(\bar{\varphi})\langle\delta\varphi^2\rangle + 2V'(\bar{\varphi})\langle\Phi\delta\varphi\rangle, \tag{2.81}$$

and

$$\tau_{ij} \;=\; a^2\delta_{ij}\Big((24H^2)\langle\Phi^2\rangle - \frac{1}{2}V''(\bar{\varphi})\langle\delta\varphi^2\rangle + 4V'(\bar{\varphi})\langle\Phi\delta\varphi\rangle\Big). \tag{2.82}$$

So

$$\tau_0^0 \equiv \rho_{br}(t) \;\simeq\; \left[2\frac{V''V^2}{(V')^2} - 4V\right]\langle\Phi^2\rangle\,, \tag{2.83}$$

where $\langle\Phi^2\rangle$ is obtained by integrating over all of the super-Hubble modes. In the simple chaotic inflation model considered in [37, 31], the second term in (2.83) is larger in magnitude than the first, and hence the effective energy density of long wavelength cosmological perturbations is negative. We shall see that the same is true in the other models considered here. The effective pressure of cosmological perturbations is

$$p_{br} \;\simeq\; -\rho_{br}\,, \tag{2.84}$$

and hence long wavelength cosmological fluctuations affect the background geometry like a negative cosmological constant (with possible implications for a possible solution of the cosmological constant problem discussed in [39]). The physical reason why long wavelength fluctuations act as a negative cosmological constant is easy to understand: a matter fluctuation (with positive matter energy density) leads to a potential well (negative gravitational energy density), and on super-Hubble scales the magnitude of the gravitational energy is larger than that of the matter energy, hence leading to a negative effective energy density. Since no terms with spatial and temporal gradients contribute, the equation of state of $\tau_{\mu\nu}$ has to be that of a cosmological constant.

An important question to ask [40] is whether the effects of the contribution of super-Hubble modes to $\tau_{\mu\nu}$ are locally measurable. First, note that the idea of large wave length modes affecting the local dynamics is not a absurd one. Consider a black hole for example. When a massive particle falls in it, the mass cross the horizon and may forever disappear but the gravitational effects is still visible for a observer at infinity. Note that there needs to be an observer (the observer at infinity) to measure the change in the mass of the black hole when some matter is thrown into it. In a similar way, a physical clock field is required in order to locally measure the effects of the long wavelength contribution to $\tau_{\mu\nu}$. For purely adiabatic fluctuations, the effect is equivalent [41, 42] to a second order time-translation. However, in terms of a clock field χ, the effects of the back-reaction of super-Hubble modes is physically measurable [43], and it corresponds to a decrease in the local Hubble expansion rate [44]. In chapter 5 we will implicitly assume that we have a clock field present which plays the same role as the CMB plays in late time cosmology in setting the clock without producing curvature of space.

[12]See equation (B.26) where we show that $\dot{H}$ is proportional to $\dot{\bar{\varphi}}^2$.
[13]See appendix B for a justification.

3. Constraints from higher spin charges in de Sitter QFT's

Quantum Field Theories may have, despite the field and spacetime symmetries, higher spin currents which lead to higher spin charges. These charges enhance the symmetry group and the algebra of the theory putting further constraints on it. Here we investigate how such symmetries can constrain Quantum Field Theories in de Sitter spacetime. We start reviewing the de Sitter spacetime. Afterwards, we add quantum fields in it and we state some properties the fields need to satisfy in order to have a well defined quantum field theory in a curved spacetime. An example of a free quantum field theory with higher spin currents is given and it is shown how it constrains the n-point function. We go further and investigate how the presence of higher spin currents can constrain an interacting quantum field theory in de Sitter spacetime.

3.1. de Sitter QFTs

We use this preliminary section to establish our notation as well as review background material relevant to our study.

We consider $D = d + 1$ dimensional de Sitter spacetime dS_D with curvature radius ℓ. The spacetime is conveniently described as a hyperboloid in a $D+1$-dimensional Minkowski embedding space:

$$dS_D := \left\{ X \in \mathbb{R}^{D,1} \mid X \cdot X = \ell^2 \right\}. \tag{3.1}$$

This surface is preserved under the action of the embedding space Lorentz group $SO(D, 1)$, i.e. boosts and rotations in the embedding space which preserve the origin. This group is thus the isometry group of dS_D (the "dS group"). The entire manifold may be covered by the global coordinate chart

$$ds^2 = -d\tau^2 + \ell^2 \cosh^2(\tau/\ell) d\Omega_d^2, \tag{3.2}$$

where $d\Omega_d^2$ is the line element of the unit sphere S^d. This chart nicely displays the hyperboloid geometry of dS, and in particular the fact that the manifold may be foliated by compact Cauchy surfaces. The conformal boundary of the spacetime is composed of two disconnected components, future (past) conformal infinity $\mathscr{I}^{+(-)}$ located at $\tau \to +(-)\infty$, each conformal to S^d.

For our purposes, a more convenient set of coordinates is given by the (expanding) Poincaré chart:

$$ds^2 = \frac{\ell^2}{\eta^2} \left(-d\eta^2 + \delta_{ab} dx^a dx^b \right), \quad \eta \in (-\infty, 0),\ x^a \in \mathbb{R}^d. \tag{3.3}$$

Here η is conformal time, roman indices run over spatial dimensions $1, \ldots, D$, and δ_{ab} is the flat metric on $\mathbb{R}^d$. The expanding Poincaré chart covers only half of the dS manifold; the other half manifold may covered by a contracting Poincaré chart with conformal time $\eta \in (0, +\infty)$. In these coordinates the dS isometries may be described as: (i) translations and (ii) rotations on constant-η surfaces, (iii) the dilations $x^\mu \to \lambda x^\mu$, and (iv) special conformal transformations $x^\mu \to x^\mu + 2b^\nu x_\nu x^\mu - x_\nu x^\nu b^\mu$ generated by vectors b^μ tangent to constant-η surfaces.

With regard to any compact subset of spacetime, in the limit $\ell \to \infty$ the de Sitter geometry reduces to Minkowski spacetime. We refer to this limit, with all other dimensionful quantities held fixed, as the flat-space limit. The Poincaré coordinates (3.3) are not well-suited for this limit; instead we use the "proper time" coordinate t:

$$\eta = -\ell e^{-t/\ell}, \quad t = -\ell \ln\left(-\frac{\eta}{\ell}\right), \tag{3.4}$$

so that the line element becomes

$$ds^2 = -dt^2 + e^{2t/\ell}\delta_{ab}dx^a dx^b, \quad t \in \mathbb{R}. \tag{3.5}$$

Taking $\ell \to \infty$ with these coordinates fixed, the line element indeed reduces to that of Minkowski space.

Given two points $X_i, X_j \in dS_D$ it is convenient to define the $SO(D,1)$-invariant *chordal distance*[1]

$$X_{ij} := \frac{1 - X_i \cdot X_j/\ell^2}{2}. \tag{3.6}$$

Different causal relationships between the points are encoded in X_{ij} as follows:[2]

$$\begin{aligned} \text{spacelike separation:} \qquad & X_{ij} > 0, \\ \text{null separation:} \qquad & X_{ij} = 0, \\ \text{timelike separation:} \qquad & X_{ij} < 0. \end{aligned} \tag{3.7}$$

The chordal distance may be expressed in Poincaré coordinates as

$$X_{ij} = \frac{|\vec{x}_i - \vec{x}_j|^2 - (\eta_i - \eta_j)^2}{4\eta_i\eta_j}. \tag{3.8}$$

We are interested in studying local dS QFTs which have standard properties of QFTs on curved spacetime (see e.g., [45, 46], as well as discussion in [47]). In particular, we restrict our attention to theories with the following properties:

1. **dS covariance:** The theory does not select a preferred direction or otherwise spoil the unitary representation of the dS isometry group. For each dS Killing vector field (KVF) ξ^μ there exists an isometry generator in the QFT of the form

$$G_\xi := \int d\Sigma(x) n^\mu \xi^\nu T_{\mu\nu}(x)\Big|_\Sigma, \tag{3.9}$$

where Σ is a Cauchy surface, n^μ is the future-pointing unit normal vector, and $T_{\mu\nu}(x)$ is the QFT stress-tensor. The generators satisfy the $SO(D,1)$ algebra inherited from the KVFs

$$[G_{\xi_1}, G_{\xi_2}] = -iG_{[\xi_1,\xi_2]}, \tag{3.10}$$

where for vector fields $[\xi_1, \xi_2]$ denotes the Lie bracket. The generators act on any quantum operator $\mathcal{O}(x)$ via a Lie derivative:

$$[G_\xi, \mathcal{O}(x)] = i\mathcal{L}_\xi \mathcal{O}(x). \tag{3.11}$$

[1] This is actually one quarter the chordal distance.

[2] Spacelike-separated points in dS which may be connected by a geodesic satisfy $X_{ij} \in (0, 1]$.

2. **dS-invariant states:** The theory admits at least one state invariant under the action of the dS isometry group $SO(D,1)$.[3]

3. **microlocal spectrum condition ("μSC"):** Correlation functions of the theory contain short-distance (ultraviolet) singularities consistent with the micro-local spectrum condition of [50]. Essentially, the μSC states that the only singularities of correlation functions are at coincident configurations, and are of "positive frequency" as measured in any locally flat coordinate chart. This assures that the singularities present in correlation functions look like those of the usual Minkowski vacuum. The μSC may be regarded as the generalization of the Hadamard condition to interacting theories [51, 50, 52].

4. **IR-regularity:** We assume that correlation functions of local operators do not grow too quickly as the chordal distance Z between two clusters of operators grows. For massive theories in dS, such correlation functions *decay* as the chordal separation Z increases [53, 54, 55]. For interacting massless theories, such correlation functions are known to grow in perturbation theory like a power of $(\log Z)$ (see [56, 57] for discussion of dS-invariant states, as well as [58, 59, 60, 61] for less-symmetric states). To be concrete, we assume that the correlation functions of local operators clustered into two groups separated by chordal distance $|Z| \gg 1$ may be bounded by cZ for some finite constant c.[4]

Given these assumptions, we expect that correlation functions of scalar operators with respect to a dS-invariant state Ω admit representations as generalized Mellin-Barnes integrals which take the form [53, 54, 56, 55]:

$$\langle \phi(x_1)\dots\phi(x_n)\rangle_\Omega = \left[\prod_{i<j}^{n} \int_{C_{ij}} \frac{d\mu_{ij}}{2\pi i} X_{ij}^{\mu_{ij}}\right] \mathcal{M}(\vec{\mu}). \tag{3.12}$$

There is one integration variable μ_{ij} for each pairing x_i, x_j. Each μ_{ij} is integrated along a Mellin-Barnes contour C_{ij}, i.e. a contour traversed from $-i\infty$ to $+i\infty$ in the left half-plane, which may be diverted to left or right to avoid pole singularities in $\mathcal{M}(\vec{\mu})$. We refer to $\mathcal{M}(\vec{\mu}) := \mathcal{M}(\mu_{12},\dots,\mu_{n-1,n})$ as the *Mellin amplitude* of the correlation function $\langle\phi(x_1)\dots\phi(x_n)\rangle_\Omega$. Mellin amplitudes contain pole singularities in the complex μ_{ij} planes. The contour integrals converge due to the fact that $\mathcal{M}(\vec{\mu})$ decays sufficiently fast – i.e. exponentially – as any $|\mathrm{Im}\,\mu_{ij}| \to \infty$. For the generalized Mellin transforms to be defined at timelike separations one must add the standard position-space $i\epsilon$ prescriptions to the X_{ij}. These $i\epsilon$ prescriptions will be unimportant to our analysis so we will suppress them.

The details of these Mellin-Barnes representations will not be important to our analysis. The key point to take away is that these representations define functions of the chordal distances regarded as independent variables. The set of chordal distances which describe n points in dS are not all independent as they are constrained so that all n points "fit" in dS_D. However, the Mellin-Barnes integrals (3.12) define functions of the X_{ij} over a larger domain. Further details of the Mellin-Barnes representation are presented in [53, 54, 56].[5]

[3] Not all theories satisfying (1) necessarily have such states. A well-known example is the massless, minimally-coupled Klein-Gordon field, when quantized via canonical quantization [48, 49].

[4] This assumption seems eminently reasonable for any theory which might called massive or massless. However, in de Sitter there also exist unitary, tachyonic theories which violate this assumption. The simplest example is a discrete set of free, tachyonic scalars [62]. The correlation functions of these theories can grow like a power of Z. However, we know of no interacting theories which violate assumption (4).

[5] Mellin transforms are a natural integral transform to consider whenever the underlying function has power-law asymptotics, As such, Mellin transforms are also useful tools in CFT and AdS/CFT (see, e.g., [63, 64] and references therein). Historically, Mellin transforms have played a key role in calculations of multi-loop Feynman diagrams in Minkowski space [65]).

3.2. Higher spin symmetries on dS

The higher-spin symmetries we study are generated by a local, covariantly conserved current $J_{\mu_1\ldots\mu_s}(x)$ of rank $s > 2$. Given a rank-$(s+1)$ current and a rank s Killing tensor $K^{\nu_1\cdots\nu_s}(x)$ we may define a spin s conserved charge

$$Q_K^{(s)} := \int d\Sigma(x)\, n^\mu K^{\nu_1\ldots\nu_s} J_{\mu\nu_1\ldots\nu_s}(x)\Big|_\Sigma. \tag{3.13}$$

Here Σ is a *global* dS Cauchy surface and n^μ is the future-pointing normal vector. We focus on Killing tensors composed of products of de Sitter KVFs, i.e.,

$$K^{\nu_1\ldots\nu_s}(x) = \xi_1^{\nu_1}\cdots\xi_s^{\nu_s}(x), \tag{3.14}$$

where each ξ_i^μ may be a distinct KVF. In this case it is easy to show that the charges $Q_K^{(s)}$ and the dS generators G_ξ have the commutators

$$\left[G_\xi, Q_K^{(s)}\right] = -iQ^{(s)}_{\mathcal{L}_\xi K}. \tag{3.15}$$

Since ξ is a dS KVF, the action of the lie derivative $\mathcal{L}_\xi$ on $K^{\nu_1\cdots\nu_s}(x)$ produces another Killing tensor of the form (3.14). Thus, the charges $Q_K^{(s)}$ and the dS generators G_ξ form a closed algebra which enlarges the $SO(D,1)$ algebra of the generators alone.

We will be primarily concerned with the higher-spin charges

$$Q_p^{(s)} := \int d\Sigma(x)\, n^\mu p^{\nu_1}\ldots p^{\nu_s} J_{\mu\nu_1\ldots\nu_s}(x)\Big|_\Sigma, \tag{3.16}$$

where p^μ is a KVF whose flow corresponds to translation along constant η surfaces in some Poincaré chart. Note that for a given p^μ the Poincaré time direction ∂_η is unique up to choice of sign (the sign corresponds to whether the Poincaré chart is expanding or contracting). It is convenient to adopt this expanding Poincaré chart to describe characteristics of $Q_p^{(s)}$. We emphasize, however, that $Q_p^{(s)}$ is defined over the global dS manifold. We normalize p^μ such that in this Poincaré chart $\delta_{ab}p^a p^b = 1$. Let us denote the dS generators in this chart by P_a (translations), D (dilations), R_{ab} (rotations), and K_a (SCTs). Then, for instance, $Q_p^{(1)}$ corresponds to a linear combination of the P_a's. From (3.15) it follows that the $Q_p^{(s)}$ enjoy simple commutation relations with some of the dS generators:

$$\left[P_a, Q_p^{(s)}\right] = 0, \quad \left[D, Q_p^{(s)}\right] = -sQ_p^{(s)}, \quad \left[R^\perp, Q_p^{(s)}\right] = 0. \tag{3.17}$$

Here $R^\perp$ represents the generators of rotations which preserves p^μ (these exist for $D \geq 4$). In general $Q_p^{(s)}$ are covariant under $SO(d)$ rotations in the sense that

$$R(\alpha)Q_p^{(s)}R^{-1}(\alpha) = Q^{(s)}_{R^{-1}p}. \tag{3.18}$$

On the other hand, the special conformal transformation generators K_a alter $Q_p^{(s)}$ in a complicated way, and $\left[K_a, Q_p^{(s)}\right]$ does not correspond to any $Q_p^{(s)}$.

The action of a HS charge $Q_p^{(s)}$ on a local operator $\mathcal{O}(x)$ is given by the commutator $\left[Q_p^{(s)}, \mathcal{O}(x)\right]$. For simplicity we will focus on the action of $Q_p^{(s)}$ on scalar operators. The most general action is given by the following:

Lemma 3.2.1 *Let $\phi(x)$ be a local scalar operator. The most general form of the commutator $\left[Q_p^{(s)}, \phi(x)\right]$ is*

$$\left[Q_p^{(s)}, \phi(x)\right] = \sum_A \frac{1}{\eta^{s-k}} C_A^{\mu_1\cdots\mu_k} \mathcal{O}^A_{\mu_1\cdots\mu_k}(x). \tag{3.19}$$

Here A is a collective index labeling operators $\mathcal{O}^A_{\mu_1\cdots\mu_k}(x)$ which transform covariantly under $SO(D,1)$, k is an integer (which depends on A) satisfying $0 \leq k \leq s$, and the $C_A^{\mu_1\cdots\mu_k}$ are constant coefficients. These coefficients are composed of products of p^μ and t^μ, where $t^\mu\partial_\mu = \partial_\eta$, such that there is an even (odd) number of p^μ's when s is even (odd).

Proof: To begin we write the general form of the commutator of $\phi(x)$ with a HS charge,

$$\left[Q_K^{(s)}, \phi(x)\right] = \sum_A \tilde{C}_A^{\mu_1\cdots\mu_k}(x) \mathcal{O}^A_{\mu_1\cdots\mu_k}(x), \tag{3.20}$$

where the $\tilde{C}_A^{\mu_1\cdots\mu_k}(x)$ are coefficient functions which are taken to have length dimension $k-s$; in order to make the dimensionality of this equation consistent, the operators $\mathcal{O}^A_{\mu_1\cdots\mu_s}(x)$ must have length dimension $1 - D/2 - k$. Consider the action of a dS generator on this commutator. From the Jacobi identity we obtain

$$\left[G_\xi, \left[Q_K^{(s)}, \phi(x)\right]\right] = \left[\left[G_\xi, Q_K^{(s)}\right], \phi(x)\right] + \left[Q_K^{(s)}, \left[G_\xi, \phi(x)\right]\right]. \tag{3.21}$$

We say that G_ξ preserves $Q_K^{(s)}$ when

$$\left[G_\xi, Q_K^{(s)}\right] = \epsilon Q_K^{(s)}, \tag{3.22}$$

for some constant ϵ. In this case we obtain from (3.21)

$$\left[G_\xi, \left[Q_K^{(s)}, \phi(x)\right]\right] = \left(i\mathcal{L}_\xi + \epsilon\right)\left[Q_K^{(s)}, \phi(x)\right]. \tag{3.23}$$

On the other hand, from the ansatz (3.20) it follows that

$$\left[G_\xi, \left[Q_K^{(s)}, \phi(x)\right]\right] = \sum_A \tilde{C}_A^{\mu_1\cdots\mu_k}(x) i\mathcal{L}_\xi \mathcal{O}^A_{\mu_1\cdots\mu_k}(x). \tag{3.24}$$

Taking the difference of these equations we obtain a constraint on the coefficient functions:

$$\left(i\mathcal{L}_\xi + \epsilon\right)\tilde{C}_A^{\mu_1\cdots\mu_k}(x) = 0. \tag{3.25}$$

We now apply this result to $Q_p^{(s)}$. There are several dS generators which preserve this charge. First consider the translation generators P_a for which $\epsilon = 0$. It follows from (3.25) that the $\tilde{C}_A^{\mu_1\cdots\mu_k}(x)$ cannot depend on the spatial variables x^a. The generator of dilations D also preserves $Q_P^{(s)}$, with $\epsilon = is$. In this case (3.25) requires that non-zero components of $\tilde{C}_A^{\mu_1\cdots\mu_k}(\eta)$ be $O(\eta^{k-s})$. For $D \geq 4$ there exist rotations which leave p^μ unchanged; the associated generators $R^\perp$ preserve $Q_p^{(s)}$ with $\epsilon = 0$. The existence of these generators implies that $\tilde{C}_A^{\mu_1\cdots\mu_k}$ are composed of tensors invariant under the $SO(d-1)$ rotations which preserve p^μ.

We may also consider the discrete parity transformations of $\mathbb{R}^d$ on the constant η surfaces. Those that preserve p^μ imply that, in all dimensions, the $\tilde{C}_A^{\mu_1\cdots\mu_k}(\eta)$ are equal to η^{k-s} times a constant tensor composed of δ^{ab}, p^μ, and $t^\mu = \delta^\mu_\eta$. The discrete transformation which acts as $p^\mu \to -p^\mu$ further requires the $\tilde{C}_A$ to have an even (odd) number of p^μ's when s is even (odd). Furthermore, any factor of δ^{ab} in $\tilde{C}_A^{\mu_1\cdots\mu_k}(\eta)$ may be re-cast as factor of the inverse metric $g^{\mu\nu}$, and this may be absorbed into the definition of the relevant operator.

Bringing everything together, we conclude that we may write the coefficients as

$$\tilde{C}_A^{\mu_1\cdots\mu_k}(\eta) = \frac{1}{\eta^{s-k}} C_A^{\mu_1\cdots\mu_k}, \tag{3.26}$$

where the $C_A^{\mu_1\cdots\mu_k}$ are constant coefficients composed of factors of p^μ and t^μ, with the additional requirement of s modulo 2 factors of p^μ. Up to this point we have proven the form (3.19), except that we have yet to limit the range of k.

Next we consider those dS generators which do not preserve $Q_p^{(s)}$. In this case the action of a generator on a higher-spin charge produces a new charge,

$$\left[G_\xi, Q_{K_1}^{(s)}\right] = \kappa Q_{K_2}^{(s)}, \tag{3.27}$$

for some constant κ. By (3.15) $K_2^{\mu_1\cdots\mu_s} \propto \mathcal{L}_\xi K_1^{\mu_1\cdots\mu_s}$. We may once again use the Jacobi identity to obtain a constraint satisfied by the coefficients functions, though now this constraint involves the coefficient functions corresponding to two HS charges. For the case (3.27) we obtain

$$-i\mathcal{L}_\xi C_{A,K_1}^{\mu_1\ldots\mu_k}(x) = \kappa C_{A,K_2}^{\mu_1\ldots\mu_k}(x). \tag{3.28}$$

In order to use (3.28), let us consider without loss of generality the case $p^\mu\partial_\mu = \partial_1$, and let s_1^μ be the KVF associated with the special conformal transformation with parameter $b^\mu \propto p^\mu$. It follows from the $SO(D,1)$ algebra satisfied by the KVFs that

$$(\mathcal{L}_{s_1})^{2s+1}(p^{\mu_1}\ldots p^{\mu_s}) = 0. \tag{3.29}$$

Denoting the SCT generator associated with s_1^μ by K_1 it then follows that

$$\Big[\underbrace{K_1, \Big[K_1, \ldots [K_1}_{2s+1}, Q_p^{(s)}]\ldots\Big]\Big] = 0. \tag{3.30}$$

Combining this result with (3.27) we obtain the following constraint on the coefficient functions:

$$(\mathcal{L}_{s_1})^{2s+1}\left(\eta^{k-s} C_A^{\mu_1\cdots\mu_k}\right) = 0. \tag{3.31}$$

Given the form of $C_A^{\mu_1\cdots\mu_k}$ this equation is satisfied only for $k \le s$. To see this we first note

$$\mathcal{L}_{s_1}^3 \eta^{-1} = 0, \quad \mathcal{L}_{s_1}^3 p^\mu = 0, \quad \mathcal{L}_{s_1}^3 t^\mu = 0, \tag{3.32}$$

from which it follows that (3.31) is satisfied for $0 \le k \le s$. However, $\mathcal{L}_{s_1}$ does not annihilate positive powers of η, and thus (3.31) is not satisfied for $k > s$. This proves the lemma. ■

Note that in order for the dimensions to be consistent in (3.19), the operators $\mathcal{O}^A_{\mu_1\ldots\mu_k}(x)$ must have length dimension $1 - D/2 - k$. The explicit factors of η and t^μ are allowed in (3.19) because p^μ selects a unique time coordinate η.

Let us compare lemma 3.2.1 to the analogous result in Minkowski QFT. If a Minkowski theory has a HS current then then one may construct, e.g., the HS charge

$$Q_1^{(s)} := \int d^d x J_{01\ldots1}(x)\Big|_{x^0=\text{const}}, \tag{3.33}$$

where we use standard Cartesian coordinates $\{x^0, x^1, \ldots, x^d\}$. For $s > 1$ this is the higher spin analogue of a translation along x^1. It is easy to show that in this case the action of $Q_1^{(s)}$ on a scalar field is of the form

$$\left[Q_1^{(s)}, \phi(x)\right] = \sum_A c_A \mathcal{O}^A_{1\ldots1}(x), \tag{3.34}$$

where c_A are constant coefficients. Comparing (3.34) to (3.19), we see that if a dS theory is to admit a smooth flatspace limit, it must be that terms involving t^μ in (3.19) must vanish as $\ell \to \infty$, either due to explicit factors of $1/\eta$ (which tend to zero in the limit), or because the operator vanishes in the limit.

Returning to the dS context, we assume that there exist dS-invariant states which are annihilated by $Q_p^{(s)}$. It follows that expectation values taken with respect to these states satisfy "charge conservation identities," or Ward identities, obtained by commuting $Q_p^{(s)}$ through the string of operators. E.g., for such a state Ω we may commute $Q_p^{(s)}$ through $\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle_\Omega$ to obtain

$$0 = \left\langle \left[Q_p^{(s)}, \phi(x_1)\right]\phi(x_2)\ldots\phi(x_n) + \cdots + \phi(x_1)\ldots\phi(x_{n-1})\left[Q_p^{(s)}, \phi(x_n)\right]\right\rangle_\Omega . \tag{3.35}$$

These Ward identities will be the central object of our study.

3.3. HS charges in free fields

In order to provide a concrete example of HS symmetries, in this section we review the HS symmetries present in complex Klein-Gordon theory on dS.

A massive, complex Klein-Gordon field on dS may be described by the classical action

$$S = \int d^D x \sqrt{-g}\left(-\nabla_\mu \phi^\dagger \nabla^\mu \phi(x) - M^2 \phi^\dagger \phi(x)\right), \quad M^2 > 0. \tag{3.36}$$

In general we let $M^2 = m^2 + \xi R(x)$, with $m^2 > 0$, ξ a coupling constant, and $R(x)$ the Ricci scalar which is constant on dS. The case $m^2 = 0$, $\xi = (d-1)/(4d)$ corresponds to a conformally-invariant theory; in terms of M^2 this "conformally coupled" mass is

$$M^2_{\text{c.c}}\ell^2 = \frac{d^2-1}{4}. \tag{3.37}$$

Upon quantization the theory possesses a unique dS-invariant state Ω satisfying the μSC [48, 49]. The composite operators below are defined by normal ordering with respect to Ω.

The reader is undoubtedly familiar with the lowest-spin currents in the theory, namely the spin-1 "Klein-Gordon current" and the stress tensor:

$$\begin{aligned} J_\mu(x) &= \phi^\dagger \overleftrightarrow{\nabla}_\mu \phi(x), &(3.38)\\ J_{\mu\nu}(x) &= 2\nabla_{(\mu}\phi^\dagger \nabla_{\nu)}\phi(x) + 2\xi \nabla_\mu \nabla_\nu \left(\phi^\dagger\phi(x)\right) - 2\xi R_{\mu\nu}\phi^\dagger\phi(x) \\ &\quad - g_{\mu\nu}\left(\nabla_\lambda \phi^\dagger \nabla^\lambda \phi(x) + 2\xi \Box\left(\phi^\dagger\phi(x)\right) + M^2\phi^\dagger\phi(x)\right), &(3.39)\end{aligned}$$

where $R_{\mu\nu} = (d/\ell^2) g_{\mu\nu}$ is the dS Ricci tensor. Perhaps less familiar is the fact that the theory admits symmetric, covariantly conserved currents of every rank which are of the form

$$J_{\mu_1\ldots\mu_n}(x) = \sum_{j=0}^n c_j \nabla_{(\mu_1}\ldots\nabla_{\mu_j}\phi^\dagger \nabla_{\mu_{j+1}}\ldots\nabla_{\mu_n)}\phi(x) + \text{traces}, \tag{3.40}$$

where the c_j are constants and "traces" denote terms composed of partial traces of the terms written, multiplied by appropriate factors of the metric.[6] The most tidy example is the spin-3 current, which with convenient normalization may be written

$$\begin{aligned} J_{\mu\nu\lambda}(x) \;=\; \frac{1}{4(d+2)}\Big[&(d-1)\left(\phi^\dagger\nabla_{(\mu}\nabla_\nu\nabla_{\lambda)}\phi(x) - \nabla_{(\mu}\nabla_\nu\nabla_{\lambda)}\phi^\dagger\phi(x)\right) \\ &- 3(3+d)\nabla_{(\mu}\phi^\dagger\overleftrightarrow{\nabla}_\nu\nabla_{\lambda)}\phi(x) + 6g_{(\mu\nu}\nabla^\alpha\phi^\dagger\overleftrightarrow{\nabla}_\lambda\nabla_{\alpha)}\phi(x) \\ &+ \left[6M^2-(d-1)(3d+2)\ell^{-2}\right]g_{(\mu\nu}J_{\lambda)}(x)\Big], \end{aligned} \tag{3.41}$$

and which has trace

$$g^{\mu\nu}J_{\mu\nu\lambda}(x) = \left(M^2 - M^2_{\text{c.c.}}\right)J_\lambda(x). \tag{3.42}$$

Unfortunately, we are unaware of explicit expressions for these HS currents for general M^2; for the conformally coupled case expressions for the currents may be obtained from known CFT results (see, e.g., [66, 67]).

Let us examine the action of the resulting HS charges on $\phi(x)$. A straightforward if tedious way to compute the commutator $\left[Q_p^{(s)}, \phi(x)\right]$ is by direct application of the canonical commutation relations. Expressed at equal times in Poincaré coordinates, these familiar relations are

$$[\phi(\eta,\vec{x}), \pi(\eta,\vec{y})] = i\left(\frac{\eta}{\ell}\right)^d \delta^d(\vec{x},\vec{y}), \tag{3.43}$$

where $\pi(x)$ is the momentum conjugate to $\phi(x)$ and $\delta^d(\vec{x},\vec{y})$ is the d-dimensional Dirac delta function. For example, for the spin-2 charge associated with the spin-3 current (3.41), diligent calculation yields

$$\left[Q_p^{(2)}, \phi(x)\right] = -i\frac{|\vec{p}|^2}{2(d+2)}\left(-\partial_\eta^2 + \frac{(d-1)}{\eta}\partial_\eta - \frac{M^2\ell^2}{\eta^2} + \Delta_s\right)\phi(x) + i\partial_p^2\phi(x). \tag{3.44}$$

Here $|\vec{p}|^2 = \delta_{ab}p^ap^b$, $\partial_p = p^\mu\partial_\mu$ is set to unity in the main text, and Δ_s is the Laplacian compatible with the flat metric on constant-η hypersurfaces. The terms in parenthesis are proportional to the KG wave operator and thus annihilate the field $\phi(x)$ on-shell. The final expression is then

$$\left[Q_p^{(2)}, \phi(x)\right] = i\partial_p^2\phi(x). \tag{3.45}$$

For general s it is more efficient to use lemma 3.2.1 in order to prove that the commutator is

$$\left[Q_p^{(s)}, \phi(x)\right] = i\partial_p^s\phi(x). \tag{3.46}$$

The argument is as follows. Since the currents are bi-linear in $\phi(x)$, and since the canonical commutation relations map $\phi\times\phi\to\mathbb{C}$, it follows that the right-hand side of (3.46) must be linear in $\phi(x)$. The commutator is a solution to the Klein-Gordon equation, the thus right-hand side of (3.46) must also be a solution. The only term which is linear in $\phi(x)$, a solution to the Klein-Gordon equation, and consistent with lemma 3.2.1 is $\partial_p^s\phi(x)$.

[6] In the classical field theory, the currents may also be made traceless when $M^2 = M^2_{\text{c.c.}}$. However, as in the familiar case of the stress tensor, we expect that this tracelessness may be spoiled in the quantum theory by anomalies due to the curved background. We thank E. Mottola for raising this point.

3.4. HS charges with linear action

In this section we consider HS charges which act linearly on a scalar field $\phi(x)$. By this we mean that

$$\left[Q_p^{(s)}, \phi(x)\right] = \mathscr{D}(x)\phi(x), \tag{3.47}$$

where $\mathscr{D}(x)$ is a differential operator of the form

$$\mathscr{D}(x) = \sum_A \frac{1}{\eta^{s-k}} C_A^{\mu_1 \dots \mu_k} \mathscr{D}^{(A)}_{\mu_1 \dots \mu_k}, \tag{3.48}$$

where the coefficients $C_A^{\mu_1 \dots \mu_k}$ are as in lemma 3.2.1, and the $\mathscr{D}^{(A)}_{\mu_1 \dots \mu_k}$ are rank-k covariant differential operators composed of products of ∇_μ, $g_{\mu\nu}$, and $\Box$. Within this set-up we shall prove the following:

Lemma 3.4.1 *Consider a QFT satisfying the properties of §3.1 in spacetime dimension $D \geq 3$. Let Ω be a dS-invariant state which is annihilated by the HS charge $Q_p^{(s)}$. Suppose that the action of $Q_p^{(s)}$ on a scalar field $\phi(x)$ is linear, and furthermore that $\mathscr{D}(x)$ contains the term $(p^\mu \partial_\mu)^s$. Then the correlation functions $\langle \phi(x_1) \dots \phi(x_n) \rangle_\Omega$ are Gaussian.*

Proof. Consider the Ward identity associated with commuting $Q_p^{(s)}$ through the correlation function

$$F := \langle \phi(x_1) \dots \phi(x_n) \rangle_\Omega \, , \tag{3.49}$$

where no pair of points is null-separated. We may regard F as a function of the $n(n-1)/2$ chordal distances X_{ij}. Due to the linear action (3.47) of $Q_p^{(s)}$, this Ward identity may be written as

$$0 = \sum_{k=1}^{n} \mathscr{D}(x_k) F. \tag{3.50}$$

Unpacking this expression results in several terms. Let us focus on terms generated by $[p^\mu(\partial/\partial x_1^\mu)]^s$. From the derivatives

$$\begin{aligned} \left(p^\mu \frac{\partial}{\partial x_1^\mu}\right) X_{12} &= \frac{\vec{p} \cdot (\vec{x}_1 - \vec{x}_2)}{2\eta_1 \eta_2} = -\left(p^\mu \frac{\partial}{\partial x_2^\mu}\right) X_{12}, \\ \left(p^\mu \frac{\partial}{\partial x_1^\mu}\right)^2 X_{12} &= \frac{p^2}{2\eta_1 \eta_2} = \left(p^\mu \frac{\partial}{\partial x_2^\mu}\right)^2 X_{12}, \end{aligned} \tag{3.51}$$

it follows that that (3.50) contains the terms

$$\begin{aligned} T_k \;:=\; & \left(\frac{1}{2\eta_1\eta_2} \vec{p} \cdot (\vec{x}_1 - \vec{x}_2)\right)^{s-k} \left(\frac{1}{2\eta_1\eta_3} \vec{p} \cdot (\vec{x}_1 - \vec{x}_3)\right)^{k} \left(\frac{\partial}{\partial X_{12}}\right)^{s-k} \left(\frac{\partial}{\partial X_{13}}\right)^{k} F, \\ & \text{for} \quad k = 1, \dots, s-1. \end{aligned} \tag{3.52}$$

Each T_k depends on $\vec{p}\cdot(\vec{x}_1 - \vec{x}_2)$ and $\vec{p}\cdot(\vec{x}_1 - \vec{x}_3)$ in a distinct way which cannot arise from any other term in the Ward identity. In particular, these terms can only arise from the derivative operator $(p^\mu \partial/\partial x_1^\mu)^s$, and can only arise from one way of distributing the derivatives $\partial/\partial x_1^\mu$.[7] Thus, each T_k must vanish individually in order for the Ward identity to be satisfied. The first line of (3.52) does not vanish for general configurations of points, and thus the factor on the second line must vanish.

[7] The case $D = 2$ is different. In this case there is only one spatial dimension, so $p^\mu \partial_\mu$ does not have the same effect of selecting a preferred spatial direction. One may produce terms with the same coordinate dependence of T_k by acting with time derivatives ∂_{η_1}. As a result, it is no longer the case that the T_k must vanish.

When $n > 3$ we may repeat this argument for terms of the form T_k but with x_2, x_3 replaced with other combinations of points in $\{x_2, \dots x_n\}$. Ultimately we conclude that F must satisfy

$$\left(\frac{\partial}{\partial X_{1i}}\right)^{s-k}\left(\frac{\partial}{\partial X_{1j}}\right)^{k} F = 0, \quad i,j \in \{2,\dots,n\}, \quad k = 1,\dots,s-1. \tag{3.53}$$

We distinguish two ways the equalities (3.53) may be satisfied: i) F depends only on one chordal distance X_{1i}, or ii) F depends on more than one chordal distance X_{1i}, but must depend on each distance polynomially. In fact, the most general F is a sum of terms, each of which satisfies either (i) or (ii). However, possibility (ii) violates our assumption of IR regularity ((4) in §3.1). If F depends polynomially on all chordal distances involving x_1, then F grows polynomially in the chordal distance as x_1 is taken to large mutual chordal separation from the remaining points. Thus we conclude that F must be a sum of terms, each of which depends on only one chordal distance X_{1i}.

We can now repeat the argument for those terms in the Ward identity which are generated by $(p^\mu \partial/\partial x_j^\mu)^s$, $j = 2, \dots, n$, and which yield constraints similar to (3.53) but with x_1 swapped for another point. Eventually we are led to conclude that F is a sum of terms, each of which depends on only one chordal distance per spacetime point. Thus F is Gaussian. This concludes the proof. ■

3.5. HS charges with asymptotically linear action

Next we consider HS charges with a less restrictive form of action on scalar operators. Here we consider actions which become linear only in the neighborhood of the asymptotic boundaries. As operator expressions, the commutator $\left[Q_p^{(s)}, \phi(x)\right]$ and Ward identity may be expanded as a Laurent expansion with respect to conformal time η (or the coordinate q defined below) in the spirit of a Fefferman-Graham expansion. For simplicity we focus on the case where $\phi(x)$ is a scalar operator with characteristic leading behavior near the conformal boundary

$$\phi(x) = O(\eta^\Delta), \quad \Delta > 0, \quad \text{as } \eta \to 0. \tag{3.54}$$

Scalars with this asymptotic form, and with $0 < \Delta < d/2$, are often referred to as "light" fields, because in the canonical example of a Klein-Gordon theory such operators correspond to fields with mass of order ℓ^{-2}. Our results below are valid for any positive Δ.

Definition 3.5.1 *Let $\phi(x)$ be a scalar operator on dS_D with characteristic scaling $\phi(x) = O(\eta^\Delta)$, $\Delta > 0$, as $\eta \to 0$. Then the action of charge Q on $\phi(x)$ is* asymptotically linear *if the leading term in the commutator takes the form*

$$[Q, \phi(x)]\Big|_{O(\eta^\Delta)} = \mathscr{D}(x)\phi(x))\Big|_{O(\eta^\Delta)}, \tag{3.55}$$

where $\mathscr{D}(x)$ is a differential operator of the form described in lemma 3.4.1.

Theorem 3.5.2 *Let $\phi(x)$ be a scalar operator on dS_D, $D \geq 3$, with characteristic scaling $\phi(x) = O(\eta^\Delta)$, $\Delta > 0$, as $\eta \to 0$, and let Ω be a dS-invariant state annihilated by the HS charge $Q_p^{(s)}$. If the action of $Q_p^{(s)}$ on $\phi(x)$ is asymptotically linear and contains the term $(p^\mu\partial_\mu)^s$, then the leading $O(\eta^{n\Delta})$ behavior of the equal-time correlation functions $\langle\phi(\eta,\vec{x}_1)\dots\phi(\eta,\vec{x}_n)\rangle_\Omega$ is Gaussian.*

Proof. After a bit of simplification the proof is very similar to that of lemma 3.4.1. Let $F = \langle\phi(\eta,\vec{x}_1)\dots\phi(\eta,\vec{x}_n)\rangle_\Omega$ be an equal-time correlation function evaluated at n non-coincident points. The Ward identity now implies that as $\eta\to 0$

$$\sum_{k=1}^{n}\mathscr{D}(x_k)F = O(\eta^{n\Delta+\epsilon}),\quad \epsilon>0. \tag{3.56}$$

Once again we focus on the terms arising from the derivative $(p^\mu\partial/\partial x_1^\mu)^s$ within $\mathscr{D}(x_1)$. This yields terms of the form T_k as in (3.52). Due to their unique dependence on $\vec{p}\cdot(\vec{x}_1-\vec{x}_2)$ and $\vec{p}\cdot(\vec{x}_1-\vec{x}_3)$, each of these terms must satisfy the fall-off condition individually, i.e. each must be $O(\eta^{n\Delta+\epsilon})$. Following through the same logic as in the previous proof, we quickly conclude that

$$\left(\frac{\partial}{\partial X_{1i}}\right)^{s-k}\left(\frac{\partial}{\partial X_{1j}}\right)^{k}F = O(\eta^{n\Delta+\epsilon}),\quad i,j\in\{2,\dots,n\},\quad k=1,\dots,s-1. \tag{3.57}$$

To proceed further let $x_{ij} := |\vec{x}_i-\vec{x}_j|^2/4$ so that

$$X_{ij} = \frac{x_{ij}}{\eta^2}\ ; \tag{3.58}$$

then (3.57) may be written

$$\left(\frac{\partial}{\partial x_{1i}}\right)^{s-k}\left(\frac{\partial}{\partial x_{1j}}\right)^{k}F = O(\eta^{n\Delta-2s+\epsilon}),\quad i,j\in\{2,\dots,n\},\quad k=1,\dots,s-1. \tag{3.59}$$

We next expand F in a Laurent expansion with respect to η. The leading term is

$$F = \langle\phi(\eta,\vec{x}_1)\dots\phi(\eta,\vec{x}_n)\rangle_\Omega\Big|_{O(\eta^{n\Delta})} =: \eta^{n\Delta}f, \tag{3.60}$$

where f is a function of the x_{ij} but does not depend on η. We may write f explicitly in terms of Mellin amplitude $\mathcal{M}(\vec{\mu})$ of F as

$$f := \left[\prod_{i<j}^{n}\int_{C_{ij}}' \frac{d\mu_{ij}}{2\pi i}x_{ij}^{\mu_{ij}}\right]\mathcal{M}(\vec{\mu}), \tag{3.61}$$

where the prime on the integrals denotes that contours are traversed such that $(\sum_{i<j}^{n}\mu_{ij}+\frac{n\Delta}{2})=0$.[8] The key point is that f may be regarded as a function of independent variables x_{ij}. It follows that f satisfies

$$\left(\frac{\partial}{\partial x_{1i}}\right)^{s-k}\left(\frac{\partial}{\partial x_{1j}}\right)^{k}f = 0,\quad k=1,\dots,s-1. \tag{3.62}$$

The equation (3.62) is the same as (3.53). Thus the remainder of this proof mimics that of the previous section. In order for f to satisfy (3.62) it must be either Gaussian with respect to the spatial coordinates $\vec{x}_i$, or it must be polynomial in the distances x_{1i}. But here the polynomial form is ruled out by the simple fact that dS-invariance of the correlator demands that f behave under a dilation as

$$f(\lambda\vec{x}_1,\dots,\lambda\vec{x}_n) = \lambda^{-n\Delta}f(\vec{x}_1,\dots,\vec{x}_n). \tag{3.63}$$

[8] There will be singularities in $\mathcal{M}(\vec{\mu})$ at points along this set of contours, so (3.61) includes both residue and principal parts.

Thus f is Gaussian. ■

We can quickly extend this result to the case where operators are inserted in the neighborhood of both asymptotic boundaries of global dS. It is convenient to switch time coordinate to $q = -\ell^2/\eta$, yielding the line element

$$ds^2 = -\frac{\ell^2}{q^2}dq^2 + \frac{q^2}{\ell^2}d\vec{x}^2, \quad q \in \mathbb{R}. \tag{3.64}$$

This coordinate chart covers all of dS_D. In this chart the conformal boundary is composed of two copies of $\mathbb{R}^d$ located at $|q| \to \infty$, plus two points located at $q = 0$, $|\vec{x}| \to \infty$.

Theorem 3.5.3 *Let $\phi(x)$ be a scalar operator on dS_D, $D \geq 3$, with characteristic scaling $\phi(x) = O(q^{-\Delta})$, $\Delta > 0$, as $|q| \to \infty$, and let Ω be a dS-invariant state annihilated by the HS charge $Q_p^{(s)}$. Let the action of $Q_p^{(s)}$ on $\phi(x)$ be asymptotically linear as $|q| \to \infty$ and contain the term $(p^\mu \partial_\mu)^s$. Consider the correlation function $\langle \phi(q_1, \vec{x}_1) \ldots \phi(q_n, \vec{x}_n) \rangle_\Omega$, with each $q_1, \ldots, q_n$ equal to $\pm q$, evaluated a non-null separations as $|q| \to \infty$. The leading $O(q^{-n\Delta})$ behavior of this correlation function is Gaussian.*

The proof is essentially the same as for the previous theorem. As for the case of a single boundary, one considers points such that no $x_{ij} = 0$. This assures that the points x_i and x_j are not coincident (when $q_i = q_j$) or null-separated (when $q_i = -q_j$) as $|q| \to \infty$.

4. Conformal Inflation from the Higgs

An important parameter used to infer the existence of primordial gravitational waves is the tensor to scalar ratio, named by r. In chapter **??** we saw how it is defined from first principles. In this chapter we investigate how to obtain an arbitrary value for the tensor to scalar ratio in models with flat potentials, *i.e.*, with potentials of the form $V = A(1+Be^{-C\varphi})$ in the region of large field values ($\varphi \gg M_{pl}$). This is not trivial since the well known models with this behavior usually have a small value for r as, for example, $r \sim 0.01$ in the Starobinsky model. Another interesting thing we do is to construct models starting with the Higgs potential in the small field values region ($\varphi \ll M_{pl}$).

We begin generalizing the Starobinsky model in order to have a larger value for r. In the next sections we construct models starting from the Higgs potential and imposing a $SO(1,1)$ symmetry at large field values, it is a natural consequence to have a plateau-like potential in this region.

It is worth to note that this work was done when the BICEP2 data was released and it implied a $r \sim 0.2$. However, despite of the correction due to Planck data, which discarded a huge value for r, we think that it is worth to present the insights contained here since r could still be different than the simple Starobinsky value.

4.1. Starobinsky-like models and a general class of potentials.

The Starobinsky model is obtained by adding an R^2 correction term to the Einstein-Hilbert action, and it is equivalent to adding a scalar with a potential that is an exponentially corrected plateau[1]. We will explain this mechanism by generalizing to an R^p correction, where p is an arbitrary positive real number. We start with the action

$$S = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}(R + \beta_{p+1}R^{p+1}). \tag{4.1}$$

It can be rewritten by introducing an auxiliary scalar field α as

$$S = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left[R(1+(p+1)\beta_{p+1}\alpha) - p\beta_{p+1}\alpha^{\frac{p+1}{p}}\right]. \tag{4.2}$$

Defining the Einstein frame metric

$$g^E_{\mu\nu} = [1+(p+1)\beta_{p+1}\alpha]g_{\mu\nu} \equiv \Omega^{-2}g_{\mu\nu}\ , \tag{4.3}$$

and using the general formula for a Weyl rescaling in d dimensions

$$R[g_{\mu\nu}] = \Omega^{-2}\left[R[g^E_{\mu\nu}] - 2(d-1)g^{\mu\nu}_E\nabla^E_\mu\nabla^E_\nu\ln\Omega - (d-2)(d-1)g^{\mu\nu}_E(\nabla^E_\mu\ln\Omega)\nabla^E_\nu\ln\Omega\right]\ , \tag{4.4}$$

we obtain the Einstein plus scalar action

$$S = \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g}\left[R[g_E] - \frac{3(p+1)^2\beta_{p+1}^2}{2}g^{\mu\nu}_E\frac{\nabla^E_\mu\alpha\nabla^E_\nu\alpha}{[1+\beta_{p+1}(p+1)\alpha]^2} - V(\alpha)\right]\ , \tag{4.5}$$

[1]For a detailed calculation look at the appendix D.

where

$$V(\alpha) = p\beta_{p+1}\alpha^{\frac{p+1}{p}}. \tag{4.6}$$

Note that the auxiliary scalar has become dynamical, having a kinetic term. Defining the canonical scalar by

$$d\phi = \pm\sqrt{\frac{3}{2}}\frac{(p+1)\beta_{p+1}d\alpha}{1+\beta_{p+1}(p+1)\alpha} \Rightarrow \phi = -\sqrt{\frac{3}{2}}\ln[1+\beta_{p+1}(p+1)\alpha]\,, \tag{4.7}$$

the potential is

$$V(\phi) = p\beta_{p+1}(-1)^{1+\frac{1}{p}}\left[\frac{1-e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_{\rm Pl}}}}{\beta_{p+1}(p+1)}\right]^{\frac{p+1}{p}}. \tag{4.8}$$

In order for the potential to be real and positive definite for $\phi \geq 0$, we must have $p = 1/(2k+1)$, with k an integer. Then at large field value $\phi \to \infty$, we obtain

$$V \simeq \frac{p\beta_{p+1}}{[(p+1)\beta_{p+1}]^{\frac{p+1}{p}}}\left[1-\frac{p+1}{p}e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_{\rm Pl}}}+...\right]. \tag{4.9}$$

This method can be used for a general $f(R)$ action, but we will use it to reproduce a more general expansion at $\phi \to \infty$. We see that for the Starobinsky model $p = 1, k = 0$, as well as for the generalized Starobinsky model with general integer k, the power in the exponential is $\sqrt{\frac{2}{3}}$.

The above model suggests considering the more general expansion at $\phi \to \infty$,

$$V \simeq A\left[1-ce^{-a\frac{\phi}{M_{\rm Pl}}}\right]. \tag{4.10}$$

For this model, the inflationary slow-roll parameters are

$$\begin{aligned}\epsilon &\equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = \frac{a^2c^2}{2}e^{-2a\frac{\phi}{M_{\rm Pl}}} \ll |\eta| \\ \eta &= M_{\rm Pl}^2\frac{V''(\phi)}{V(\phi)} = -a^2ce^{-a\frac{\phi}{M_{\rm Pl}}}.\end{aligned} \tag{4.11}$$

Then the parameters (n_s, r) of the spectrum of CMB fluctuations are given by

$$\begin{aligned}n_s - 1 &= -6\epsilon + 2\eta \simeq 2\eta = -2a^2ce^{-a\frac{\phi}{M_{\rm Pl}}} \\ r &= 16\epsilon = 8a^2c^2e^{-2a\frac{\phi}{M_{\rm Pl}}} = \frac{2}{a^2}(n_s-1)^2.\end{aligned} \tag{4.12}$$

To fix the value of $e^{-a\frac{\phi}{M_{\rm Pl}}}$, we consider the number of e-foldings, N_e. Note that the value of ϕ is the value at horizon crossing (crossing outside the horizon, to return inside it now) for the scale of the CMB fluctuations. Inflation happens at large ϕ, decreasing as the Universe expands, so the start of inflation can correspond to a ϕ_0 which can be greater than ϕ, but for simplicity we will consider them equal. Then

$$\begin{aligned}N_e &= -\int_{\phi_0/M_{\rm Pl}}^{\phi_f/M_{\rm Pl}}\frac{d\phi/M_{\rm Pl}}{\sqrt{2\epsilon}} = -\frac{1}{ac}\int_{\phi_0/M_{\rm Pl}}^{\phi_f/M_{\rm Pl}}e^{a\frac{\phi}{M_{\rm Pl}}} \simeq \frac{1}{a^2c}e^{a\frac{\phi}{M_{\rm Pl}}} \\ &= \frac{1}{-\eta} = \frac{2}{1-n_s}\,,\end{aligned} \tag{4.13}$$

where we have used the fact that $e^{a\phi_f/M_{\rm Pl}} \sim 1$ in order for inflation to end, so we have neglected this term in N_e.

In conclusion, we obtain the constraints[2]

$$
\begin{aligned}
1-n_s &= \frac{2}{N_e} \\
r &= \frac{2}{a^2}(n_s-1)^2 \\
&= \frac{8}{a^2 N_e}.
\end{aligned}
\tag{4.14}
$$

Then, for instance with 50 e-folds, $N_e = 50$, n_s is fixed to be $n_s \simeq 0.9600$, which is in the middle of the allowed region by the Planck experiment [69], while 60 e-folds gives $n_s \simeq 0.9667$, again compatible with the data (for instance, Planck + WMAP gives 0.9603 ± 0.0073). For concreteness, we will take $N_e = 50$ from now on. With $a = \sqrt{2/3}$ as for the Starobinsky model (even in the case generalized to R^{p+1}), $r = 3(n_s-1)^2$, so $n_s - 1 \simeq -1/25$ gives $r \simeq 0.005$, too small for the measurement of BICEP2 (which excludes such a virtually zero r at at least 5σ).

But within the context of this more general model, we can fit even the central value of r of the (now discarded) BICEP2 experiment, of $r = 0.20$ at $n_s - 1 \simeq -1/25$, by $r \simeq 2/(25a)^2 \sim 0.20$ for $a \sim 1/8$.

To complete the analysis of (4.10), we consider the running of n_s,

$$
\frac{dn_s}{d\ln k} = -14\epsilon\eta + 24\epsilon^2 + 2\xi^2 \, , \tag{4.15}
$$

where

$$
\xi^2 \equiv M_{\rm Pl}^4 \frac{V'V'''}{V^2}. \tag{4.16}
$$

Then for (4.10) we have

$$
\xi^2 \simeq c^2 a^4 e^{-2a\frac{\phi}{M_{\rm Pl}}} \, , \tag{4.17}
$$

and $\xi^2 \sim \mathcal{O}(e^{-2a\frac{\phi}{M_{\rm Pl}}}) \gg \epsilon\eta, \epsilon^2$, so

$$
\frac{dn_s}{d\ln k} \simeq 2\xi^2 \simeq 2c^2 a^4 e^{-2a\frac{\phi}{M_{\rm Pl}}} \simeq \frac{(n_s-1)^2}{2}. \tag{4.18}
$$

Finally, the CMB normalization gives [33]

$$
\frac{H_{\rm inf}^2}{8\pi^2 \epsilon M_{\rm Pl}^2} \simeq 2.4 \times 10^{-9} \, , \tag{4.19}
$$

where $H_{\rm inf}^2 = V_{\rm inf}/3M_{\rm Pl}^2$. In our case, this is $A/3M_{\rm Pl}^2$, so that the constraint is

$$
\frac{A/M_{\rm Pl}^4}{24\pi^2\epsilon} \simeq 2.4 \times 10^{-9}. \tag{4.20}
$$

Replacing ϵ by $r/16$, we get

$$
\frac{A}{M_{\rm Pl}^4} \simeq \frac{3\pi^2}{2} r \times 2.4 \times 10^{-9}. \tag{4.21}
$$

Again, just to give a taste of what we can obtain when r is measured, using the central value of $r \simeq 0.2$ of BICEP2, we get

$$
\frac{A}{M_{\rm Pl}^4} \simeq 7.1 \times 10^{-9} \Rightarrow A^{1/4} \simeq 10^{-2} M_{\rm Pl}. \tag{4.22}
$$

[2]The analysis of this more general potential was obtained also in [68].

For the potential of the generalized Starobinsky model in (4.8), this gives the somewhat unnatural

$$\beta_{p+1} \approx (1.4 \times 10^8)^p M_{\rm Pl}^{-2p} \simeq (10^{-4} M_{\rm Pl})^{-2p}. \tag{4.23}$$

We now return to the question of how to obtain an $f(R)$ model, generalizing the Starobinsky model, that still allows for an a as small as $\sim 1/8$. It was essential that we had the canonical scalar related to the scalar α by (4.7) in order to obtain $a = \sqrt{\frac{2}{3}}$.

Consider $M_{\rm Pl} = 1$ for the moment, in order not to clutter formulas. We want to generalize (4.7) to

$$1 + \alpha = e^{-\sqrt{\frac{2}{3}}\frac{\phi}{b}} \Rightarrow d\phi = -\sqrt{\frac{3}{2}} b \frac{d\alpha}{1+\alpha} , \tag{4.24}$$

which gives $a = \sqrt{\frac{2}{3}}\frac{1}{b}$. We see that then the kinetic term must come in the Jordan frame from a factor $R[1+\alpha]^b$. Since moreover we want the potential to be (β^c is a constant)

$$V = \beta^c(-\alpha)^b = \beta^c \left[1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{b}}\right]^b \simeq \beta^c \left[1 - b e^{-\sqrt{\frac{2}{3}}\frac{\phi}{b}}\right] , \tag{4.25}$$

we obtain the action in Jordan frame (with a particular parametrization of the coefficient of the potential)

$$S = \frac{1}{2}\int d^4x \sqrt{-g} \left[R(1+\alpha)^b - \beta^{-(b-1)}\alpha^b\right] , \tag{4.26}$$

and we want $(-)^b$ to be real and positive, so we need $b = 2k+2$ to be an even integer. Eliminating the auxiliary scalar α gives the action

$$S = \frac{1}{2}\int d^4x \sqrt{-g} \frac{R}{\left(1 - \beta R^{\frac{1}{b-1}}\right)^{b-1}} , \tag{4.27}$$

i.e., an action of the type $f(R)$. Moreover,

$$f(R) = \frac{R}{\left(1 - \beta R^{\frac{1}{b-1}}\right)^{b-1}} \simeq R + (b-1)\beta R^{1+\frac{1}{b-1}} + \ldots , \tag{4.28}$$

so we see that with $b = 2k+2$, we obtain the same first term in the expansion as in the generalized Starobinsky model above.

In conclusion, just by allowing for a completion of the Starobinsky model via a particular infinite series that sums to the above $f(R)$, we can obtain any value for a, thus any value for r needed.

4.2. Class of models with Weyl invariance and approximate $SO(1,1)$ invariance, reducing to Higgs.

We start with a model with both local Weyl symmetry and $SO(1,1)$ invariance, where the Planck scale appears when choosing a gauge, and otherwise there is no dimensionful parameter[3],

$$S = \frac{1}{2}\int d^4x \sqrt{-g} \left[\partial_\mu \chi \partial^\mu \chi - \partial_\mu \phi \partial^\mu \phi + \frac{\chi^2 - \phi^2}{6} R - \frac{\lambda}{18}(\phi^2 - \chi^2)^2\right] . \tag{4.29}$$

[3]See equation (D.73) where we rewrite the action (4.29) in a form where the $SO(1,1)$ symmetry is obvious.

The coupling of 1/6 of the scalar to the Einstein term was chosen such as to have the conformal value, and the potential was chosen to be quartic, all in all giving the local Weyl symmetry under

$$g_{\mu\nu} \to e^{-2\sigma(x)} g_{\mu\nu}; \quad \chi \to e^{\sigma(x)}\chi, \qquad \phi \to e^{\sigma(x)}\phi. \tag{4.30}$$

Since $\sqrt{-g} \to e^{-4\sigma(x)}\sqrt{-g}$, the potential needs to have scaling dimension 4 in the scalar fields. Moreover, imposing the $SO(1,1)$ symmetry of $\chi^2 - \phi^2$, which is a Lorentz type symmetry (in fact, in [22, 23, 24] the motivation for this model was 2-time physics, written covariantly in $(4,2)$ Minkowski dimensions, and the $SO(1,1)$ is a remnant of the $SO(4,2)$ Lorentz invariance), we are forced to take also the potential $(\phi^2 - \chi^2)$. The $SO(1,1)$ can also be obtained from a model with conformal invariance, again $SO(4,2)$, the initial motivation of [21, 25, 26].

It would seem like χ is a ghost, but because we have a *local* Weyl symmetry, we can put one scalar degree of freedom to zero, such as to get rid of the ghost. Choosing a gauge for the local scaling invariance will also necessarily introduce a scale, the Planck scale (it is obvious that in order to fix the scaling transformation we must choose a scale). Of course, since we have only one scale, physics is independent of this scale as it should be, since the scale is a gauge choice (calling $M_{\rm Pl}$ to be $1m$ or $10^{19}GeV$ or something else is meaningless unless there is an independent definition of what is $1m$ or $1GeV$ or some other scale).

One simple gauge choice is the *Einstein gauge* (E-gauge) $\chi^2 - \phi^2 = 6M_{\rm Pl}^2$, which is solved in terms of the canonically normalized field φ by

$$\chi = \sqrt{6}M_{\rm Pl}\cosh\frac{\varphi}{\sqrt{6}M_{\rm Pl}}; \quad \phi = \sqrt{6}M_{\rm Pl}\sinh\frac{\varphi}{\sqrt{6}M_{\rm Pl}}. \tag{4.31}$$

In terms of it, the action is the Einstein action, with a canonical scalar and a cosmological constant,

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \lambda M_{\rm Pl}^4\right]. \tag{4.32}$$

Another simple gauge choice is the *physical Jordan gauge or c-gauge*, $\chi(x) = \sqrt{6}M_{\rm Pl}$. In terms of it, the action is

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R\left(1 - \frac{\phi^2}{6M_{\rm Pl}^2}\right) - \frac{1}{2}\partial_\mu\phi\partial_\mu\phi - \frac{\lambda}{36}(\phi^2 - 6M_{\rm Pl}^2)^2\right]. \tag{4.33}$$

We can go to the Einstein metric by $g^E_{\mu\nu} = (1 - \phi^2/6M_{\rm Pl}^2)g^J_{\mu\nu}$ and then define the canonically normalized field φ by $d\varphi/d\phi = (1 - \phi^2/6M_{\rm Pl}^2)^{-1}$, and reobtain (4.32). But one thing one can note now is that the potential for the field ϕ with the conformal coupling to R looks like the Higgs potential, just with scalar VEV $v^2 = 6M_{\rm Pl}^2$, instead of the experimentally known $v \simeq 246GeV$.

Otherwise one could identify ϕ with a physical Higgs field, but the Higgs VEV of $\sqrt{6}M_{\rm Pl}$ was the result of the $SO(1,1)$ symmetry, and in the Einstein frame it led to an exactly flat potential (cosmological constant). If we completely discard the $SO(1,1)$ symmetry as in [70, 71] and insist instead on having the Higgs potential from low energies, $\lambda(H^\dagger H - v^2)^2$, appear in the physical Jordan gauge, we are led to $\lambda(\phi^2 - \omega^2\chi^2)^2$, generalized with the addition of a term $\lambda'\chi^4$ (in order to have the most general quartic potential for ϕ and χ) that reduces to a cosmological constant. Note however that for the kinetic terms and R-coupling terms we must insist on the same $SO(1,1)$ symmetric form, if we want to remain with only the physical Higgs and the scalar VEV after gauge fixing, which seems contrived if we completely abandon the symmetry in the potential.

The attraction of this construction is that now in the Lagrangean for the Standard Model coupled to gravity there are no explicit mass scales, and the fundamental theory is local Weyl-invariant.

Masses in the Standard Model come from coupling to the Higgs, and the Higgs VEV and Planck mass now come from choosing a gauge in a Weyl-invariant theory.

However, the absence of the $SO(1,1)$ symmetry means that we are naturally led towards a cyclic cosmology. Indeed, in the Einstein gauge for the local Weyl symmetry defined by (4.31), the potential is now

$$V \sim \lambda M_{\rm Pl}^4 \left(\sinh^2 \frac{\varphi}{\sqrt{6}M_{\rm Pl}} - \omega^2 \cosh^2 \frac{\varphi}{\sqrt{6}M_{\rm Pl}}\right)^2 - \lambda' M_{\rm Pl}^4 \cosh^4 \frac{\varphi}{\sqrt{6}M_{\rm Pl}} \approx C M_{\rm Pl}^4 \exp \frac{4\varphi}{\sqrt{6}M_{\rm Pl}} , \tag{4.34}$$

i.e. an exponential of the canonical scalar at large field value. This generically leads to cyclic cosmology, and we see that the origin of this is the non-cancellation of the leading exponentials. In the $SO(1,1)$ symmetric form, there was no $\lambda'\chi^4$ term and $\omega = 1$ at high field values, leading to a cancellation of the leading exponentials in $\cosh^2 - \sinh^2$, giving in fact a constant.

Besides the fact that requiring $SO(1,1)$ symmetric kinetic terms but no remnant of the symmetry in the potential is unnatural, we also have the issue that general cyclic cosmologies are incompatible with a large value of the tensor to scalar ratio of CMB fluctuations, r [72, 73]. So, if in the future we measure a big value for r it would be ideal to find a model that gives inflation at high energies, and a natural way for that is to have an approximate $SO(1,1)$ at high energies, which we take to translate into large field values.

So we ask what is the most general potential depending on ϕ and χ with the desired properties: local Weyl symmetry, to reduce at large field values (large energies) to the $SO(1,1)$ symmetric form, and at small field values (small energies) to the Higgs form. From just local Weyl symmetry, the most general form is $f(\phi/\chi)\phi^4$, since $\sqrt{-g}\phi^4$ and ϕ/χ are both local Weyl invariant. But we want also to reduce to the $SO(1,1)$ invariant form $(\phi^2 - \chi^2)^2$ at large field values, so we must have

$$V = \lambda \left[\tilde{f}(\phi/\chi)\phi^2 - \tilde{g}(\phi/\chi)\chi^2\right]^2 , \tag{4.35}$$

or in another parametrization

$$V = \lambda f(\phi/\chi) \left[\phi^2 - g(\phi/\chi)\chi^2\right]^2 . \tag{4.36}$$

Since in the Einstein gauge $\phi/\chi = \tanh \varphi/\sqrt{6}M_{\rm Pl}$ which goes to 1 at large φ, we must require $g(1) = 1$ for $SO(1,1)$ symmetry at large field values. In [19, 20] it was considered only $g(x) \equiv 1$, hence the possible connection with the Higgs was not considered.

Now the condition that we obtain the Higgs potential at low field values (low energies) is that $g(0) \simeq \omega^2$, where $\omega = 246GeV/\sqrt{6}M_{\rm Pl}$. Moreover, the function $g(x)$ must give at small energies subleading corrections to the Higgs potential. For the simplest function, a polynomial plus a constant, we can take

$$g(x) = \omega^2 + (1 - \omega^2)x^n , \tag{4.37}$$

and we must impose $n > 2$. Indeed, if $n = 1$ (and $f(x) = 1$) we get at small field values a potential $\propto [\phi^2 - (\omega^2 + (1-\omega^2)\phi/\chi)\chi^2]^2$ and in the Einstein gauge at small field the leading terms in the potential become

$$V \simeq \lambda \left[(1-\omega^2)(\varphi^2 - \sqrt{6}\varphi M_{\rm Pl})^2 - 6\omega^2 M_{\rm Pl}^2\right]^2 , \tag{4.38}$$

so the linear term will dominate over the quadratic term in the square bracket. In the $n = 2$ case we can check that actually the good ϕ^2 term cancels in the square bracket and the potential is simply $V \simeq 36\lambda\omega^4 M_{\rm Pl}^4$, so again is not good. For $n > 2$ instead, at small field the higher order corrections are suppressed by $M_{\rm Pl}$, i.e. we get approximately

$$V \simeq \lambda \left[\varphi^2 - 6\omega^2 M_{\rm Pl}^2 - 6M_{\rm Pl}^2 \left(\frac{\varphi}{\sqrt{6}M_{\rm Pl}}\right)^n\right]^2 \simeq \left[\varphi^2 - 6\omega^2 M_{\rm Pl}^2\right]^2 , \tag{4.39}$$

where we have used that $\omega^2 \ll 1$, and we have dropped all terms coming from higher orders in the expansion of $\cosh(\varphi/\sqrt{6}M_{\rm Pl})$ and $\sinh(\varphi/\sqrt{6}M_{\rm Pl})$.

Finally now, we can have an overall function $f(\phi/\chi)$, but if we want to have the Higgs potential at low energies, we need again to restrict its form. We write it as $f(x) = 1 + F(x)$, and a simple possible form for $F(x)$ would be a polynomial, $F(x) = c_n x^n$, but by the same logic as above, we need to have $n > 2$, if not we mess up the Higgs potential at small field values. In the next section we will study the resulting inflation from these models, and we will see other possible relevant examples for $F(x)$.

We have not addressed the issue of how can these models arise from a fundamental theory? The potentials described need to be only effective potentials, i.e. including quantum corrections. As we saw, the scale for the corrections to the Higgs potential is the Planck scale, so naturally these are quantum gravity corrections, which could in principle come, for example, from string theory. The quantum theory is supposed to be valid at large field values, which is generically the case for scalars arising from string models. Moreover, the energy density at large field values is very large, which again implies that we are in the quantum gravity region. At these high energies, we have as usual the local Weyl symmetry, but also the $SO(1,1)$ symmetry which gets deformed, so it would be natural to assume that $\varphi \to \infty$ is a special point that has the full symmetry, but the symmetry is not protected away from it. However, the understanding of this is left for future work.

Finally, instead of the scalar ϕ we have the Higgs, which is a doublet, whereas χ would be a singlet under $SU(2)$, and then ϕ from the above discussion, the field relevant for inflation, would be its norm $\phi = \sqrt{H^\dagger H}$. The action is then

$$\begin{aligned} S &= \frac{1}{2}\int d^4x\sqrt{-g}\left[\partial_\mu\chi\partial^\mu\chi - \partial_\mu H^\dagger\partial^\mu H + \frac{\chi^2 - H^\dagger H}{6}R \right.\\ &\left. -\frac{\lambda}{18}\left(1 + F(\sqrt{H^\dagger H}/\chi)\right)\left(H^\dagger H - g(\sqrt{H^\dagger H}/\chi)\chi^2\right)^2 + \mathcal{L}'_{SM}\right] , \end{aligned} \tag{4.40}$$

where $\mathcal{L}'_{SM}$ is the Standard Model Lagrangean other than the Higgs kinetic and potential term.

4.3. Inflation in these models.

We next turn to inflation in these models. We start with models with $F(x) = 0$ ($f(x) = 1$). For the simple $g(x)$ in (4.37), we obtain for the potential at $\varphi \to \infty$, using that $\omega^2 \ll 1$,

$$V \simeq 9(n-2)^2\lambda M_{\rm Pl}^4\left[1 - 2ne^{-\frac{2\varphi}{\sqrt{6}M_{\rm Pl}}}\right] , \tag{4.41}$$

which means, in the parametrization of section 2, that $a = \sqrt{2/3}$ as in the Starobinsky model, and $c = 2n$. That gives

$$\begin{aligned} \epsilon &\simeq \frac{4n^2}{3}e^{-\frac{4\varphi}{\sqrt{6}M_{\rm Pl}}} \\ \eta &\simeq -\frac{4n}{3}e^{-\frac{2\varphi}{\sqrt{6}M_{\rm Pl}}} , \end{aligned} \tag{4.42}$$

and then $n_s - 1 \simeq 2\eta$, $r \simeq 16\epsilon$, but more relevantly we get

$$\begin{aligned} 1 - n_s &\simeq \frac{2}{N_e} \\ r &\simeq 3(n_s - 1)^2 \simeq \frac{12}{N_e} , \end{aligned} \tag{4.43}$$

exactly as in the Starobinsky model. As we saw for the parametrization in section 2, we also have $dn_s/d\ln k \simeq (n_s-1)^2/2$ and $A^{1/4} \simeq 10^{-2} M_{\rm Pl}$, which translates now into $\lambda^{1/4}\sqrt{3(n-2)} \sim 10^{-2}$.

A two-parameter generalization with

$$g(x) = \omega^2(1-x^m) + x^n \ , \tag{4.44}$$

reducing to the previous case for $m = n$, gives however still (4.41) at $\varphi \to \infty$, so obtains nothing new.

Another simple case with $F(x) = 0$ and

$$g(x) = \omega^2 + (1-\omega^2)\sin\left(\frac{\pi}{2}x^n\right) \ , \tag{4.45}$$

that still satisfies the condition to interpolate between the Higgs potential and the inflationary potential, gives for the potential at $\varphi \to \infty$

$$V \simeq 36\lambda\left(1 - \frac{n^2\pi^2}{4}e^{-\frac{2\phi}{\sqrt{6}M_{\rm Pl}}}\right) . \tag{4.46}$$

Then we have again $a = \sqrt{2/3}$ as in the Starobinsky model, but now $c = n^2\pi^2/4$. We get therefore again the same Starobinsky model predictions $r \simeq 3(1-n_s)^2$, $1-n_s \simeq 2/N_e$ and $dn_s/d\ln k \simeq (1-n_s)^2/2$. Again $A^{1/4} \simeq 10^{-2}M_{\rm Pl}$ translates into $\sqrt{6}\lambda^{1/4} \sim 10^{-2}$.

Finally, consider an $F(x) = c_p x^p$ with $p > 2$ and $g(x)$ in (4.37). Then the potential at $\varphi \to \infty$ is

$$V \simeq \lambda M_{\rm Pl}^4 9(n-2)^2(1+c_p)\left[1 - 2\left(n + \frac{p}{1+c_p}\right)e^{-\frac{2\phi}{\sqrt{6}M_{\rm Pl}}}\right] \ , \tag{4.47}$$

so once again we obtain the predictions of the Starobinsky model.

We see that the predictions of these models are robust and generically give the same as the Starobinsky model. The reason is that we have functions of $\phi/\chi = \tanh(\varphi/\sqrt{6}M_{\rm Pl})$ which is $\simeq 1 - 2e^{-\frac{2\varphi}{\sqrt{6}M_{\rm Pl}}}$ at $\varphi \to \infty$, so any well-defined Taylor expansion in ϕ/χ would give the same $a = \sqrt{2/3}$.

In order to obtain a different a we need functions which have a somewhat singular behaviour at $\phi/\chi \to 1$. One obvious, yet somewhat unnatural example for the function $F(x)$ that would give a general a and preserves the Higgs potential at small φ would be

$$F(x) = c_p\left\{\tanh\left[a\sqrt{\frac{3}{2}}\tanh^{-1}(x)\right]\right\}^p \ , \tag{4.48}$$

that gives at $\varphi \to \infty$

$$F \simeq c_p\left[1 - 2pe^{-a\frac{\varphi}{M_{\rm Pl}}}\right] . \tag{4.49}$$

A more plausible example made up of only logs and powers is

$$F(x) = c_{\gamma,p}\left[\ln\left(\frac{2}{1+x^p}\right)\right]^\gamma . \tag{4.50}$$

At $\varphi \to \infty$ it gives

$$F \simeq c_{\gamma,p}p^\gamma e^{-\sqrt{\frac{2}{3}}\frac{\gamma\phi}{M_{\rm Pl}}} \ , \tag{4.51}$$

which implies a general $a = \gamma\sqrt{2/3}$. Moreover, at $\varphi \to 0$, we get

$$F \simeq c_{\gamma,p}(\ln 2)^{\gamma}\left[1 - \frac{\gamma}{\ln 2}\left(\frac{\varphi}{\sqrt{6}M_{\rm Pl}}\right)^{p}\right], \tag{4.52}$$

so with $p > 2$ we don't perturb the Higgs potential, and we just rescale the coupling $\lambda \to \lambda[1 + c_{\gamma,p}(\ln 2)^{\alpha}]$.

The general conditions on $F(x)$ can be described as $F(1-x) \propto x^{\alpha_1}$ and $F(x) \simeq c_1 + c_2 x^p$, $p > 2$, for $x \to 0$.

If someday we measure a large value for r comparable, for example, with the BICEP2 data which implies $a \sim 1/8$, we note that with these $F(x)$'s, any normal $g(x)$ like the ones given above would do the job. It is the case since the corrections coming from $F(x)$ would be leading with respect to the corrections coming from $g(x)$, which have $a = \sqrt{2/3}$.

In conclusion, with not too singular functions $g(x)$ and $F(x)$ we obtain the predictions of the Starobinsky model, but with ones with a more singular behaviour at $x = 1$ like for instance (4.50) we can obtain the generalized Starobinsky model of section 2, with arbitrary a, being able to fit any value for the tensor to scalar ratio.

4.4. General coupling ξ and attractors.

The exact $SO(1,1)$ symmetric model in the physical Jordan gauge is (4.33), and now consider the deformation of the potential with the parametrization in terms of $\tilde{f}, \tilde{g}$ in (4.35). The action in physical Jordan gauge then becomes

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R\left(1 - \frac{\phi^2}{6M_{\rm Pl}^2}\right) - \frac{1}{2}\partial_\mu\phi\partial_\mu\phi - \right. \\ &\left. - \left(\tilde{f}\left(\frac{\phi}{\sqrt{6}M_{\rm Pl}}\right)\phi^2 - \tilde{g}\left(\frac{\phi}{\sqrt{6}M_{\rm Pl}}\right)6M_{\rm Pl}^2\right)^2\right]. \end{aligned} \tag{4.53}$$

But we want to study the case of a general coupling $\xi < 0$ of the scalars to gravity $\xi\phi^2 R$, away from the conformal coupling $\xi = -1/6$. For consistency we replace $-\phi^2/(6M_{\rm Pl}^2)$ with $\xi\phi^2/M_{\rm Pl}^2$ everywhere, to obtain

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R\left(1 + \xi\frac{\phi^2}{M_{\rm Pl}^2}\right) - \frac{1}{2}\partial_\mu\phi\partial_\mu\phi - \right. \\ &\left. -36\left(\tilde{f}\left(\frac{\sqrt{|\xi|}\phi}{M_{\rm Pl}}\right)\xi\phi^2 + \tilde{g}\left(\frac{\sqrt{|\xi|}\phi}{M_{\rm Pl}}\right)M_{\rm Pl}^2\right)^2\right]. \end{aligned} \tag{4.54}$$

We can go to the Einstein frame $g_{\mu\nu}^E = (1+\xi\phi^2)g_{\mu\nu}$ and obtain

$$\begin{aligned} S &= \int d^4x\sqrt{-g_E}\left[\frac{M_{\rm Pl}^2}{2}R[g_E] - \frac{1}{2}\left[\frac{1+\xi\phi^2/M_{\rm Pl}^2 + 6\xi^2\phi^2/M_{\rm Pl}^2}{(1+\xi\phi^2/M_{\rm Pl}^2)^2}\right]g_E^{\mu\nu}\nabla_\mu^E\phi\nabla_\nu^E\phi - \right. \\ &\left. - 36(1+\xi\phi^2/M_{\rm Pl}^2)^2\left(\tilde{f}\left(\frac{\sqrt{|\xi|}\phi}{M_{\rm Pl}}\right)\xi\phi^2 + \tilde{g}\left(\frac{\sqrt{|\xi|}\phi}{M_{\rm Pl}}\right)M_{\rm Pl}^2\right)^2\right]; \end{aligned} \tag{4.55}$$

and define a canonical scalar field φ by

$$\frac{d\varphi}{d\phi} = \frac{\sqrt{1+\xi\phi^2/M_{\rm Pl}^2 + 6\xi^2\phi^2/M_{\rm Pl}^2}}{1+\xi\phi^2/M_{\rm Pl}^2}, \tag{4.56}$$

to finally obtain the action

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g_E}\left[\frac{M_{\rm Pl}^2}{2}R[g_E]-\frac{1}{2}g_E^{\mu\nu}\nabla_\mu^E\varphi\nabla_\nu^E\varphi-\right.\\
&\left.-36\left(\tilde f\left(\frac{\sqrt{|\xi|}\phi(\varphi)}{M_{\rm Pl}}\right)\xi\phi^2+\tilde g\left(\frac{\sqrt{|\xi|}\phi(\varphi)}{M_{\rm Pl}}\right)M_{\rm Pl}^2\right)^2\right]. \qquad (4.57)
\end{aligned}
$$

The potential is a function of $\frac{\sqrt{|\xi|}\phi(\varphi)}{M_{\rm Pl}}$ as in [19, 20], but of a form restricted by our conditions. The analysis then follows in a similar way. If ξ is not extremely small such as to be able to ignore the non-minimal coupling of the scalar to gravity, then inflation will occur at $\varphi\to\infty$, like in the $\xi=-1/6$ case, because of a plateau behaviour. From (4.56), this is seen to be where $1+\xi\phi^2/M_{\rm Pl}^2\ll 1$, and in that region we obtain

$$
\frac{d\varphi}{d\phi}\simeq\frac{\sqrt{6}|\xi|\phi/M_{\rm Pl}}{1+\xi\phi^2/M_{\rm Pl}^2}\,, \qquad (4.58)
$$

solved by

$$
\phi^2=\frac{M_{\rm Pl}^2}{|\xi|}\left(1-e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}}\right). \qquad (4.59)
$$

We see that this is again the same asymptotic functional form (except for the overall constant) as in the conformal case $\xi=-1/6$, which as we explained in the last section was the reason why we generically obtained the Starobinsky potential with $a=\sqrt{2/3}$, for functions $f(x)$ and $g(x)$ that are not too singular at $x=1$.

Therefore by the same arguments as in the previous sections we will obtain the Starobinsky point also for this non-conformal $\xi<0$. The only remaining issue is to define the small value of ξ at which we can consider chaotic inflation. The argument in [19, 20] can be carried over with little modifications. For a smooth potential function at $\phi=0$ we can Taylor expand and thus consider only a quadratic potential $V\simeq m^2\phi^2/2$ and chaotic inflation for that, for which the number of e-folds is $N_e=\phi^2/4M_{\rm Pl}^2$, so for 60 e-folds we obtain $\phi_{60}\simeq 15M_{\rm Pl}$. Then the condition to be able to ignore the non-minimal coupling to gravity is if $|\xi|\phi_{60}^2/M_{\rm Pl}^2\ll 1$, so as to never need to reach the region (4.58). That gives $|\xi|\ll 4\times 10^{-3}$. If however $|\xi|\gg 4\times 10^{-3}$, we are basically at the Starobinsky point. Therefore the Starobinsky point is a strong attractor in terms of a nonzero ξ, with even a small ξ driving us away from the chaotic inflation point towards the Starobinsky point.

In the case of functions $f(x)$ and $g(x)$ that are sufficiently singular at $x=1$ to give a potential with a general a, the same analysis follows. For $|\xi|\ll 4\times 10^{-3}$ we can Taylor expand the potential at $\phi=0$ and obtain chaotic inflation, but for $|\xi|\gg 4\times 10^{-3}$ we obtain (4.59). Combined with the condition $F(1-x)\propto x^{\alpha_1}$, we get in the inflation region

$$
F\left(\frac{|\xi|\phi^2}{M_{\rm Pl}^2}\right)\propto e^{-\sqrt{\frac{2}{3}}\frac{\alpha_1\varphi}{M_{\rm Pl}}}\,, \qquad (4.60)
$$

as before. So in the generalized Starobinsky case we also get a strong attractor behaviour towards the Starobinsky line, exactly as in the case of the Starobinsky point.

5. Eternal Inflation and the back reaction of long wavelength modes

The idea of this chapter is to find the value for the inflaton field in which *stochastic effects* (self-reproduction scale) and the regime where the back-reaction effect of long wavelengths becomes important. Then, to compare both effects and see if this effect can avoid an eternal inflationary universe. At the self-reproduction scale, quantum oscillations of the scalar field could make the field slowly-roll to higher or lower values, originating many expanding universes (*multiverse*). These universes could inflate forever generating *eternal inflation*. With as many universes as we want inflating forever, any kind of universe is possible. This would make inflation in this scale an *unpredictable scenario*. However, if one include *backreactions* effects of longwavelength - this effect implies a negative "energy density" as we saw in chapter **??** - it could, in principle, prevent the universes to inflate forever. All the universes generated would be similar to each other. The small differences between them would be given by the size of the quantum fluctuations.

5.1. Stochastic Dynamics of the Background Field

Let us begin with a brief review of stochastic inflation [74]. To put the discussion into context, consider the space-time sketch of inflationary cosmology shown in Fig. 1. Here, the horizontal axis gives the physical distance and the vertical axis is physical time t. The inflationary phase lasts from initial time t_i until the final time t_R, the time of *reheating*. The solid curve which is (almost) vertical in the inflationary phase denotes the Hubble radius. The dotted curves show the wavelengths of various fluctuation modes which exit the Hubble radius during inflation.

The formalism of stochastic inflationary dynamics describes the effect of modes exiting the Hubble radius during the evolution of the effective background field which is the full coarse grained field over the Hubble volume (see e.g. [75] for a modern view on the formalism of stochastic inflation). In slight abuse of notation we will also denote the effective background field (and not just the full field) by φ. Taking into account the effects of modes crossing the Hubble radius which are now entering the sea of long wavelength modes, the equation of motion for φ becomes[1]

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) \,=\, \frac{3}{2\pi} H^3 \xi(t)\,, \tag{5.1}$$

which assumes that H is approximately constant and where the prime denotes the derivative with respect to φ, and where $\xi(t)$ is a Gaussian random variable with unit variance which takes on different values in different Hubble patches. If $\xi(t)$ is positive, then the source term in (5.1) will drive φ up the potential, but if $\xi(t)$ is negative, then the stochastic source will reinforce the classical force driving φ down its potential.

The *stochastic region* of field space is defined to be the one for which the classical force term (the right hand side of (5.1)) exceeds the classical force in magnitude, i.e.

$$\frac{3}{2\pi} H^3 \,\geq\, |V'(\varphi)|\,. \tag{5.2}$$

[1]For a justification of the stochastic term see appendix E.

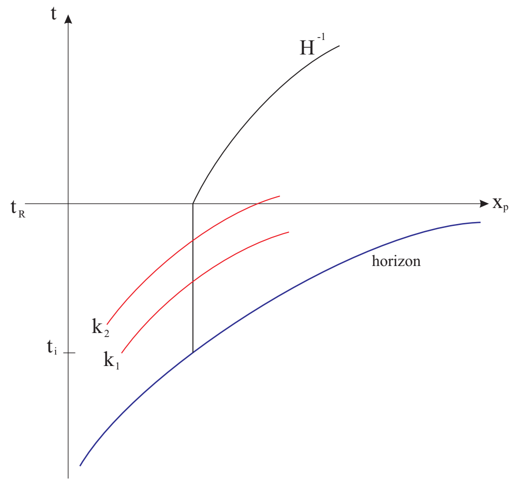

Figure 5.1.: Space-time sketch of an inflationary cosmology. The vertical axis is time, the horizontal axis is physical distance. The inflationary phase lasts from t_i until t_R, and it is during this period that the fluctuations we consider here exit the Hubble radius.

For example, for the simplest chaotic inflation model with potential

$$V(\varphi) \;=\; \frac{1}{2}m^2\varphi^2, \tag{5.3}$$

the condition (5.2) becomes

$$\left|\frac{\varphi}{m_{pl}}\right| \;\geq\; (96\pi^2)^{1/4}\left(\frac{m_{pl}}{m}\right)^{1/2}, \tag{5.4}$$

where m_{pl} is the reduced Planck mass defined in terms of Newton's constant G via $m_{pl} = (\sqrt{8\pi G})^{-1}$.

Note that for the normalization of the mass m which is consistent with the observed amplitude of CMB anisotropies, the stochastic region of field space is far beyond the field values that influence the period of inflation which is observationally accessible to us. They do, however, correspond to energy densities which are still much lower than Planck densities.

5.2. Case 1: Power-Law Inflation

We first consider large field power-law inflation models with potential

$$V(\varphi) \;=\; \lambda m^{4-n}\varphi^n\,, \tag{5.5}$$

where λ is a dimensionless constant and m is a mass scale. This class of potentials include the simple chaotic inflation models with $n = 2$ and $n = 4$, and axion monodromy inflation models which n can be a real number in the range $0 < n < 2$ [76]. In the case of $n \neq 4$ we can without loss of generality set $\lambda = 1$. However, sometimes it is convenient to keep λ and instead replace m by m_{pl}. With the first choice, the condition to obtain small density fluctuations is $m \ll m_{pl}$, in the second case $\lambda \ll 1$. Note that the region of slow roll inflation corresponds to trans-Planckian field value

$$|\varphi| \;>\; \alpha(n)m_{pl}\,, \tag{5.6}$$

where the $\alpha(n)$ is of the order 1.

Taking the derivative of (5.5) to obtain the classical force and comparing with the stochastic force given by the right hand side of (5.1) we get the following field range for which the stochastic force dominates over the classical one:

$$|\varphi| \;>\; \left(2\pi\sqrt{3}n\right)^f \lambda^{-f/2}\left(\frac{m_{pl}}{m}\right)^{(2-\frac{1}{2}n)f} m_{pl}\,, \tag{5.7}$$

where the exponent f is

$$f \;=\; \frac{1}{n/2+1}\,. \tag{5.8}$$

It is easy to see that in the case $n = 2$ and $\lambda = 1/2$ we recover the condition (5.2).

If we are in a region of space in which stochastic effects drive φ up the potential, the increase in the potential energy over one Hubble time H^{-1} is then given by

$$\Delta V \;\simeq\; \Delta\varphi V'\,, \tag{5.9}$$

where

$$\Delta\varphi \;=\; \frac{H}{2\pi} \tag{5.10}$$

is the change in φ over one Hubble time step (the coefficient in (5.10) is consistent with the coefficient of the stochastic term in (5.1)).

As discussed in the previous section, the back-reaction of super-Hubble modes leads to a negative contribution to the energy density which grows in time during a period of accelerated expansion since modes are exiting the Hubble radius and increasing the sea of infrared modes. In order to compare the change in the energy density due to back-reaction with the change due to stochastic evolution we need to evaluate the change in $\langle\Phi^2\rangle$ over a Hubble time, the same time interval considered above for the stochastic effect.

The starting point is the following expression for the contribution to $\langle\Phi^2\rangle$ from super-Hubble Fourier modes $\Phi(k)$:

$$\langle\Phi^2\rangle(t) \;=\; 4\pi\int_0^{k_H(t)} k^2|\Phi(k)|^2 dk\,, \tag{5.11}$$

where $k_H(t)$ is the comoving wavenumber corresponding to Hubble radius crossing at time t. The change in $\langle\Phi^2\rangle$ over one Hubble time is then given by

$$\Delta\langle\Phi^2\rangle \;=\; H^{-1}4\pi k_H^2|\Phi(k_H(t))|^2\frac{dk_H(t)}{dt}\,. \tag{5.12}$$

In the case of an exponentially expanding background we have

$$k_H(t) \;=\; a(t)H\,. \tag{5.13}$$

The corrections to this formula in the case of slow-roll inflation are negligible.

In the case of inflation, the value of $\Phi(k)$ at Hubble radius crossing is given by the vacuum initial conditions [77, 78]. We use[2]

$$\begin{aligned}\zeta(k) &\simeq \frac{2}{3}\frac{1}{1+w}\Phi(k)\\ \zeta(k) &= z^{-1}v(k)\,.\end{aligned} \tag{5.14}$$

[2]The first expression for ζ was deduced in appendix B.

The first expression shows how the gravitational potential Φ is related to the curvature fluctuation variable ζ (which is conserved in an expanding universe on super-Hubble scales). The second one shows that ζ is related to the canonical fluctuation variable v [79, 80] via the background variable z which is given by

$$z = \frac{a\dot{\bar{\varphi}}}{H}, \tag{5.15}$$

and

$$v = a\left[\delta\varphi + \frac{\dot{\bar{\varphi}}}{H}\Phi\right]. \tag{5.16}$$

In the above, the equation of state parameter w is the ratio of pressure to energy density. Using vacuum initial conditions for $v(k)$ (see [78] for details)

$$v_i(k) = \frac{1}{\sqrt{2k}}, \tag{5.17}$$

we then obtain

$$\begin{aligned} \Delta\langle\Phi^2\rangle &= H^{-1}\pi a H \dot{a} H \frac{9(1+w)^2}{2} \\ &= \frac{9\pi}{2} a\dot{a}H\left(1 + \frac{\dot{\bar{\varphi}}^2/2 - V}{\dot{\bar{\varphi}}^2/2 + V}\right)^2 \\ &= \frac{9\pi}{2} a^2 H^2\left(1 + \frac{\dot{\bar{\varphi}}^2/2 - V}{\dot{\bar{\varphi}}^2/2 + V}\right)^2 \\ &= \frac{9}{2}\pi H^4 \frac{(\dot{\bar{\varphi}})^2}{V^2}. \end{aligned} \tag{5.18}$$

Here we used the fact that in a slow rolling model of power law inflation $\dot{\bar{\varphi}} \propto H$. In fact, since

$$\dot{\bar{\varphi}} \simeq -\frac{V'}{3H} \tag{5.19}$$

and

$$H^2 = \frac{V}{3m_{pl}^2}, \tag{5.20}$$

using the power law potential we find that

$$\dot{\bar{\varphi}} = \frac{n}{\bar{\varphi}} H m_{pl}^2. \tag{5.21}$$

Since in the slow roll region $\bar{\varphi} \approx m_{pl}$, we have $\dot{\bar{\varphi}} \propto H$.

Inserting (5.18) back into the expression (2.83) for the effective energy density of back-reaction we then obtain (after using the slow-roll equation of motion to replace $\dot{\bar{\varphi}}$ in terms of V' and V):

$$\Delta\rho_{br} = \frac{\pi}{3}\left[V''V - 2(V')^2\right] m_{pl}^{-2}. \tag{5.22}$$

We are now able to compare the magnitude of the increase in energy density due to stochastic rolling up the potential with the decrease due to the increase in ρ_{br}. Combining the above equations (5.9), (5.10) and (5.22) yields

$$\frac{|\Delta\rho_{br}|}{\Delta V} = \frac{2\pi^2}{\sqrt{3}}\sqrt{\lambda} m^{2-\frac{1}{2}n}(n+1)\varphi^{\frac{1}{2}n-1} m_{pl}^{-1}. \tag{5.23}$$

Without loss of generality we can set $m = m_{pl}$ and represent the small slope of the potential (which is required in order for the cosmological fluctuations induced by inflation not to exceed the observational upper bound) by a small value $\lambda \ll 1$ of the coupling constant.

In the case $n = 2$ it is clear from (5.23) that $|\Delta\rho_{br}|$ is smaller than ΔV for all field values. Hence, back-reaction cannot prevent eternal inflation. For $n > 2$ there is a critical field value φ_c beyond which $|\delta\rho_{br}|$ exceeds ΔV:

$$\varphi_c \;=\; \left(\frac{\sqrt{3}}{2\pi^2}\frac{1}{n+1}\right)^{\frac{1}{n/2-1}} \lambda^{-\frac{1}{n-2}} m_{pl} \tag{5.24}$$

which corresponds to trans-Planckian energy densities. Once again, back-reaction cannot prevent eternal inflation. Finally, for $n < 2$ the exponents reverse sign and the condition for $\Delta\rho_{br}$ to dominate becomes an upper bound for $|\varphi|$:

$$|\varphi|^{\tilde{n}} \;<\; \frac{1}{3}\lambda^{1/2}(n+1)m_{pl}^{\tilde{n}}\,, \tag{5.25}$$

where $\tilde{n} \equiv 1 - n/2$. This is not the field range for inflation.

In conclusion, we find that in no version of simple power law inflation models back-reaction of long wavelength fluctuations can prevent eternal inflation.

5.3. Case 2: Starobinsky Inflation

Starobinsky's initial model of exponential expanding spacetime was based on a higher derivative gravitational Lagrangian [1]. After a conformal transformation, it corresponds to Einstein gravity in the presence of a scalar matter field φ with exponential potential[3]

$$V(\varphi) \;=\; A(1 - e^{-b\varphi})^2\,, \tag{5.26}$$

where the $A \ll m_{pl}^4$ and $b \sim m_{pl}^{-1}$.

The region of inflation once again corresponds to trans-Planckian field values where the potential energy is approximately given by A. As in the case of power law inflation discussed in the previous section, we first determine the field range where stochastic effects dominate. Demanding that the stochastic force amplitude exceeds the classical force yields the condition

$$\varphi \;>\; b^{-1}\ln\left[\frac{3}{4\pi}\left(\frac{1}{3}\right)^{3/2}\left(\frac{A}{m_{pl}^4}\right)^{1/2}(bm_{pl})^{-1}\right]. \tag{5.27}$$

Now we can turn to a comparison of the increase in potential energy due to stochastic rolling up the potential to the change in the energy density of back-reaction. Making use of (5.9) and (5.10), expressing H in terms of V, and making the approximation $V \simeq A$ we obtain

$$\Delta V \;\simeq\; \frac{1}{\sqrt{3}\pi} bA^{3/2}e^{-b\varphi}m_{pl}^{-1}\,. \tag{5.28}$$

On the other hand, from (2.83) and (5.18), the change in the energy density of back-reaction is given by

$$\Delta\rho_{br} \simeq -\frac{2\pi}{3}A^2b^2e^{-b\varphi}m_{pl}^{-2}\,. \tag{5.29}$$

The ratio is

$$\frac{|\Delta\rho_{br}|}{\Delta V} \;=\; \frac{2\sqrt{3}}{3}\pi^2 A^{1/2}bm_{pl}^{-1} \tag{5.30}$$

[3]See appendix D

which is much smaller than unity for the values of A and b which need to be chosen to get successful inflation. Hence, we conclude that also in Starobinksy inflation back-reaction terms are too weak to prevent eternal inflation.

5.4. Case 3: Cyclic Ekpyrotic Scenario

Finally, we turn to the "dark energy phase" of the cyclic Ekpyrotic scenario. The Ekpyrotic scenario is an alternative to inflation for producing the observed inhomogeneities and anisotropies [81]. It is based on the Horava-Witten scenario of heterotic M-theory [82], a higher dimensional model. At the effective field theory level it reduces to the theory of a scalar field (in the higher dimensional picture it corresponds to the separation of parallel branes) coupled to Einstein gravity. The potential of the scalar field is argued to be a negative exponential. This setup leads to a bouncing cosmology. According to the Ekpyrotic scenario, the universe begins in a phase of contraction in which the scalar field is rolling down the potential. Since the potential is negative, one obtains an equation of state with $w \gg 1$, where the equation of state parameter w is the ratio of energy density and pressure. Once the scalar field drops below $\varphi = 0$ (which corresponds to the brane separation approaching the string scale), a cosmological bounce is assumed to take place during which regular matter and radiation are produced, leading to a Standard Big Bang phase of expansion during which φ climbs back up the potential (while being a subdominant form of matter). Cosmological fluctuations are created during the phase of contraction. As long as an almost massless entropy field is present (and this completely natural from the higher dimensional point of view [83]), an almost scale-invariant spectrum of curvature perturbations is generated [84, 85, 86, 87].

By introducing a slight lift of the potential, i.e. by choosing

$$V(\varphi) \,=\, C - V_0 e^{-a\varphi}\,, \tag{5.31}$$

the Ekpyrotic scenario becomes "cyclic" [88]. Once φ during the phase of cosmic expansion reaches values with $V(\varphi) > 0$, a period of accelerated expansion will start. The scenario thus includes dark energy. Based on the classical equation of motion for φ we would conclude that φ will eventually turn around and start to decrease. This leads to a phase of contraction: the Ekpyrotic scenario has become cyclic. Figure 2 presents a sketch of space-time in the Ekpyrotic scenario. The vertical axis is time, the horizontal is comoving distance. The bounce time is taken to be $t = 0$. The Hubble radius decreases very fast before the bounce and then slowly increases in the Standard Big-Bang phase after the bounce. For the cyclic Ekpyrotic scenario to work the way it is meant to, the constant C corresponds to the currently observed cosmological constant, whereas the value of V_0 corresponds typically to a very high scale (e.g. the energy scale of Grand Unification). Also, the value of a will be given by the inverse string scale.

A question to ask is whether stochastic effects analogous to the ones which drive stochastic eternal inflation will lead to an eternal stochastic growth of φ in the Ekpyrotic scenario, thus leading to an "Ekpyrotic landscape". Intuitively we would not expect this to happen since the stochastic effects are highly suppressed in the dark energy phase (since $C <<< m_{pl}^4$) whereas the cosmological fluctuations (and hence their back-reaction effects) are not suppressed compared to the case of inflation. In fact, when the fluctuations exit the Hubble radius in the accelerating phase, they are not in the vacuum state, unlike in the case of Starobinsky inflation. Hence, the formula for the energy density of back-reaction is different. These effects lead to an enhancement of the back-reaction "force" compared to the stochastic one. In the following we will show that these expectations are indeed borne out.

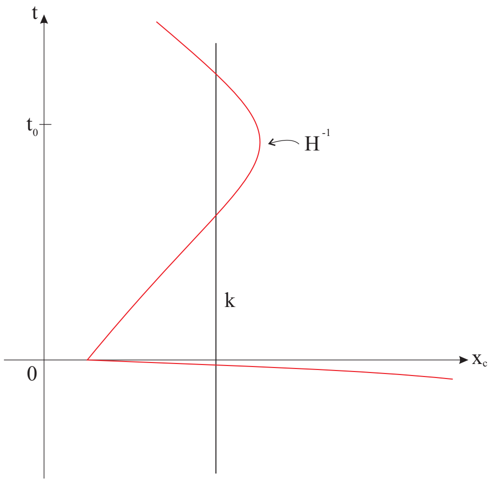

Figure 5.2.: Space-time sketch of the cyclic Ekpyrotic scenario. The vertical axis is time, the horizontal is comoving physical distance. Fluctuations are generated during the phase of contraction for $t < 0$. The bounce occurs at $t = 0$ and is followed by an expanding phase. Close to the present time, we enter the dark energy phase driven by the term C in the potential (5.31). The fluctuations which exit the Hubble radius during this phase are not vacuum perturbations, but the evolved ones and have a much larger amplitude than vacuum fluctuations would have.

The value of φ which corresponds to $V = 0$ will be denoted by φ_0 and is given by

$$e^{-a\varphi_0} = \frac{C}{V_0}\,. \tag{5.32}$$

For field values significantly larger than φ_0 we can approximate the value of the potential by $V = C$. In this case, the increase in potential energy while rolling up the potential for one Hubble time is given by

$$\Delta V \simeq \frac{1}{2\pi}\left(\frac{1}{3}\right)^{1/2} C^{3/2} a m_{pl}^{-1} e^{-a\delta\varphi}\,, \tag{5.33}$$

where

$$\varphi \equiv \varphi_0 + \delta\varphi\,. \tag{5.34}$$

Turning to the evaluation of the change in the energy density due to back-reaction, it is important to note that the fluctuations which are exiting the Hubble radius during the dark energy phase are not the vacuum ones, but the ones which have evolved and have produced the structure which we see on large scales. We can use the observed power spectrum

$$P(k) = k_H^3|\Phi(k)|^2 \sim 1 \tag{5.35}$$

and

$$\frac{dk_H}{dt} = \dot{a}H = aH^2 = k_H H, \tag{5.36}$$

into (5.12) to get

$$\Delta\langle\Phi^2\rangle \sim 1. \tag{5.37}$$

Hence from (2.83)

$$\Delta\rho_{br} \;=\; 2\Big[\frac{V''V^2}{(V')^2} - 2V\Big]\,. \tag{5.38}$$

Inserting the form of the Ekpyrotic potential and considering large field values we find

$$\Delta\rho_{br} \;\sim\; -2Ce^{a\delta\varphi}\,. \tag{5.39}$$

Comparing (5.33) and (5.39) we find

$$\frac{|\Delta\rho_{br}|}{\Delta V} \;=\; 4\pi\sqrt{3}\frac{a^{-1}m_{pl}}{C^{1/2}}e^{2a\delta\varphi}\,. \tag{5.40}$$

Since the scale C corresponds to the current dark energy scale, the coefficient in front of the exponential in the above equation is many orders of magnitude larger than 1. Hence we conclude that in the cyclic Ekpyrotic scenario back-reaction prevents the stochastic growth of φ and hence that there is no eternal expansion in the late time dark energy phase for this model.

6. Conclusions

We saw that the Theory of Cosmological Perturbations is a rich field to explore. Topics such as de Sitter QFT, Inflation and the back reaction of long wavelength perturbations were explored in detail and interesting results were obtained.

In Chapter D.3, we explored de Sitter QFT which contain Higher Spin currents. The important result obtained was that even interacting theories which act linearly on the scalar field ϕ at late times will be assimptotically gaussian. This result is the analog of the Coleman-Mandula theorem for QFT in flat spacetimes. We gave an example of a free theory which contains higher spin currents and we showed explicitly that it is free by computing the correlators of the n-point functions. More precisely, we showed that the additional symmetries constrain the n-point functions to be a sum of products of 2-point functions.

The result is also interesting because it opens at least two possibilities to explore. The first one is to verify if the slighted broken dilations and special conformal transformations - necessary for Inflation in order to get the small anisotropies in the CMB - modifies our result. If the answer is that the result is not modified we can state that many Inflationary models as the Chaotic model and the Higgs Inflation do not possess a bi-spectrum or a tri-spectrum, *i.e.*, non-gaussianities would not be present in such models. The second one would be to check if the ideas used here could be implemented in AdS QFT. Since plenty of results are known from the AdS/CFT correspondence we could potentially find explicitly examples of interacting theories which will be free at late times.

In Chapter D.4, we built inflationary models which reduce to the Higgs field at small field values ($\varphi \ll M_{pl}$). We were able to modify the Starobinsky which usually predicts a small value for the tensor to scalar ratio, in order to get an arbitrary one. Also, many models starting from the Higgs field were constructed. The main idea was to start with the Higgs field as the inflaton in a model with local Weyl symmetry, where the Planck mass appears by fixing a gauge, and with $SO(1,1)$ invariance at large field values (in the inflationary region). We have shown that in these models, defined by two functions $f(x)$ and $g(x)$, generically we obtain the predictions of the original Starobinsky model, but we can find functions that give a generalized version of the Starobinsky model, for which we can obtain any value of the tensor to scalar ratio of CMB fluctuations r. The potential is approximated in the inflationary region by a general exponentially-corrected plateau, which can be obtained from a generalized form of the Starobinsky model, with an infinite series of R^p corrections that sum to a function $f(R)$.

The functions $f(x)$ and $g(x)$ were analyzed from the point of view of the need to interpolate between the inflationary region and the Higgs potential, and that the inflationary region should arise in a consistent quantum gravity theory. Of course, specific functions would arise in the case of specific models, that would describe in particular how does the $SO(1,1)$ invariance gets broken at low energies, but we did not attempt here to construct such models. This is left for future work. If we modify the non-minimal coupling ξ of the scalar field with gravity away from the conformal point, the Starobinsky line is a strong attractor, as in the original Starobinsky point, so it seems that the local Weyl invariance is not really essential to the inflationary predictions.

In the last chapter we verified if the back reaction of long wavelength perturbations in the stochastic region, where Eternal Inflation could happen, can prevent EI in the models of power law potentials, Starobinsky model and the Ekpyrotic scenario.

Stochastic effects will lead to the effective scalar field climbing up the potential in some regions of space. This leads to an increase in the energy density. On the other hand, the back-reaction of fluctuations which have already exited the Hubble radius will lead to a decrease in the effective energy density. In chapter D.5, we compared the magnitude of the two effects in various cosmologies with an accelerating phase.

The conclusion is that for power law potentials and in the Starobinsky model of Inflation, the back-reaction would be too small to prevent eternal inflation. However, in the Ekpyrotic scenario, the late time accelerated expansion is prevented by the back-reaction of long wavelength perturbations.

A. FLRW metric

This appendix is mostly based on the General Relativity unpublished notes by George Matsas.

In order to construct the FLRW metric we start imposing homogeneity and isotropy in large scales, *i.e.*, there are observers that cannot select a preferred direction. In order to find these observers consider:

I) A foliation of the spacetime, Σ_τ.

II) Cover the surface Σ_τ by a coordinate system $\{x^i\}$ with $i = 1, 2, 3$.

III) For each $\tau = \tau_0 = const.$, assume that at each point $p \in \Sigma_{\tau_0}$ a time-like line emerge from it in a plane orthogonal to the plane Σ_{τ_0}.

IV) By construction, for each time-line defined above, $x^i = const.$

V) The time coordinate is defined by the proper time for each time-like line. The position vector is $x^\mu = (\tau, x^i)$ and the 4-velocity of the observers is

$$u = u^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial x^\mu}{\partial \tau}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \tau}. \tag{A.1}$$

VI) u^μ is orthogonal to Σ_τ for all τ. This is necessary to guarantee isotropy.

VII) By restricting the metric of the spacetime g to the hypersurface Σ_τ we obtain the metric h. So we find the set (Σ_τ, h), where $h(v, w) = g(v, w)$ for all vectors $v, w \in \Sigma_\tau$ and $h(u, w) = 0$ for all $u \perp \Sigma_\tau$ and $\forall w \in \Sigma_\tau$.

It is easy to check that, $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$, satisfy the criteria to be the metric induced in the hypersurface Σ_τ.

Homogeneity guarantees the existence of isometries that map each point of Σ_τ into another point Σ_τ, while isotropy must forbid a preferred direction in Σ_τ.

Given a metric we can construct the Riemann tensor $\tilde{R}_{\mu\nu\alpha\beta}$ on Σ_τ. The result is

$$\begin{aligned} \tilde{R}_{\mu\nu\alpha\beta} &= \tilde{R}_{\mu\alpha}h_{\nu\beta} - \tilde{R}_{\nu\alpha}h_{\mu\beta} + \tilde{R}_{\nu\beta}h_{\mu\alpha} \\ &- \frac{1}{2}\tilde{R}(h_{\mu\alpha}h_{\nu\beta} - h_{\nu\alpha}h_{\mu\beta}), \end{aligned} \tag{A.2}$$

where $\tilde{R}_\mu^{(\ \nu)} \equiv \tilde{R}_{\mu\sigma\alpha\beta}h^{\nu\sigma}h^{\alpha\nu}$ is the Ricci tensor in 3 dimensions[1], i.e, on Σ_τ. We see from equation (A.2) that if we know the metric $h_{\mu\nu}$ and the Ricci tensor in three dimensions we also have the Riemann tensor.

The Ricci tensor can be seen as a linear operator acting on the vector space dual to the tangent vector space of a point $p \in \Sigma_\tau$. So

$$\tilde{R} : \tilde{V}_p^* \to \tilde{V}_p^* \tag{A.3}$$

[1]Actually, all the tilde in this appendix means that the object in consideration is in 3 dimensions.

Since it is symmetric, the eigenvalues are real. Also, isotropy states that the eigenvalues must be equal. Then

$$\tilde{R}_{\mu}^{\ \nu} = K h_{\mu}^{\ \nu} = K \delta_{\mu}^{\ \nu}. \tag{A.4}$$

Inserting (A.4) into (A.2) we obtain

$$\boxed{\tilde{R}_{\mu\nu\alpha\beta} = K h_{\alpha[\mu} h_{\nu]\beta} = 0.} \tag{A.5}$$

Also

$$\boxed{\tilde{R} = 3K}. \tag{A.6}$$

Now, homogeneity implies that the curvature scalar $\tilde{R}$ cannot depend of the position. This fact can be seen better by using the Bianchi identity

$$\tilde{\nabla}_{[\sigma} \tilde{R}_{\mu\nu]\alpha\beta} = (\tilde{\nabla}_{[\sigma} K) h_{|\sigma|\mu} h_{\nu]\beta}, \tag{A.7}$$

which implies K to be constant. There are 3 possible homogeneous and isotropic universes which we can consider:

1) 3-spheres ($K > 0$),
2) 3-planes ($K = 0$),
3) 3-hyperboles ($K < 0$).

Let's construct a metric for the case 3) as an example. The 3-hyperboloid, $\tilde{H}$, can be thought as a surface

$$x^2 + y^2 + z^2 - w^2 = -R^2, \tag{A.8}$$

embedded in a 4-plane $(\mathbb{R}, \delta)$ where

$$d\ell^2 = dx^2 + dy^2 + dz^2 + dw^2. \tag{A.9}$$

The metric over the 3-hyperboloid is then

$$d\ell^2 = dx^2 + dy^2 + dz^2 + \frac{xdx + ydy + zdz}{(R^2 + x^2 + y^2 + z^2)}. \tag{A.10}$$

If we make the following change of coordinates on $\tilde{H} = \{\chi, \theta, \phi\}$

$$\begin{aligned} x &= R \sinh\chi \sin\theta \cos\phi, \\ y &= R \sinh\chi \sin\theta \sin\phi, \\ z &= R \sinh\chi \cos\theta, \end{aligned} \tag{A.11}$$

where $0 \leq \chi < \infty$, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$. Noting that $w = \pm R \cosh\chi$ and computing dx, dy and dz from the above expressions we finally get

$$d\ell^2 = R^2 (d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi \sin^2\theta d\phi^2) \tag{A.12}$$

with $0 \leq \chi < \infty$, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$ for $\tilde{H}$. With a analogous treatment it is possible to obtain

$$d\ell^2 = R^2 (d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2), \tag{A.13}$$

where $0 \leq \chi < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$ for the 3-sphere $\tilde{S}$ and

$$d\ell^2 = R^2(d\chi^2 + \chi^2 d\theta^2 + \chi^2 \sin^2\theta + d\phi^2) \tag{A.14}$$

with $0 \leq \chi < \infty$, $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$ for the 3-plane.

Finally we use that $g_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu$, and the facts that, $u_\mu = (1,0,0,0)$ and $u^\mu u_\mu = -1$. Then

$$\begin{aligned} g_{00} &= g(\partial_\tau, \partial_\tau) = h(\partial_\tau, \partial_\tau) - u \otimes u(\partial_\tau, \partial_\tau) = -1, && \text{(A.15)} \\ g_{i0} &= g_{\partial_\tau, \partial_i} = h(\partial_\tau, \partial_\tau) - u \otimes u(\partial_\tau, \partial_j) = 0, && \text{(A.16)} \\ g_{ij} &= g_{\partial_\tau, \partial_i} = h(\partial_\tau, \partial_\tau) - u \otimes u(\partial_i, \partial_j) = h_{ij}. && \text{(A.17)} \end{aligned}$$

From the above facts we conclude that the most general metric that covers a homogeneous and isotropic spacetime is given by

$$\boxed{ds^2 = -d\tau^2 + a^2(\tau) d\ell^2}. \tag{A.18}$$

Here $R^2 \to a^2(\tau)$ because R could be different at each Σ_τ.

B. Introducing perturbations

The goal of this appendix is to introduce from first principles two gauge invariant variables which were important in Chapter 2 and Chapter 5 of this Thesis.

In order to take into account the small anisotropies observed in the CMB, we make perturbations around the FLRW metric

$$ds^2 = -(1+2\Phi)dt^2 + 2a(t)B_j dx^j dt + a^2(t)[(1-2\Psi)\delta_{ij} + 2H_{ij}]dx^i dx^j. \tag{B.1}$$

Here we see that for $\Phi = B_j = \Psi = H_{ij} = 0$ we recover the FRLW metric. Another important thing to know is that we can use the Scalar, Vector, Tensor (SVT) decomposition, *i.e.*,

$$B_j = \partial_j B + S_j; \qquad \partial^j S_j = 0, \tag{B.2}$$
$$H_{ij} = \partial_i \partial_j H + 2\partial_{(i}P_{j)} + h_{ij}; \qquad \partial^i P_i = h^i_i = \partial^i h_{ij} = 0. \tag{B.3}$$

Even knowing that $\partial_j B$ is a vector, we will call it the scalar part of B_j. The same thing applies for the tensor H_{ij}, where we call $\partial_i \partial_j H$ the scalar and $\partial_{(i}P_{j)}$ the vector part of H_{ij}. The SVT decomposition simplifies the computations, since up to first order in perturbations the scalar, vector and tensor perturbations do not mix, so we can deal with them separately. Also, the vector part is diluted exponentially in a exponential expanding universe [89].

The hard part is to deal with the scalar perturbations, so we will treat it here in order to find gauge invariant variables which are important in this thesis. We leave a more complete treatment for references such that [35, 90].

The scalar perturbed metric is

$$ds^2 = -(1+2\Phi)dt^2 + 2a(t)\partial_j B dx^j dt + a^2(t)(1-2\Psi)\delta_{ij} dx^i dx^j. \tag{B.4}$$

One can see how the scalar perturbations transform under a gauge transformation (diffeomorphism transformation)

$$t \rightarrow t + b, \tag{B.5}$$
$$x^i \rightarrow x^i + \delta^{ij} c_j, \tag{B.6}$$

where b and c_j are small parameters. By inserting this perturbations in the metric and using that the transformed interval $d\tilde{s}^2$ is invariant, i.e,

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{B.7}$$

For example, since

$$t' = t + b,$$
$$dt' = (1+\dot{b})dt \tag{B.8}$$
$$\Phi(t') = \Phi(t+b) = \Phi(t) + b\dot{\Phi} \tag{B.9}$$

and

$$(1+2\Phi)dt^2 = (1+2\Phi(t'))dt'^2 \tag{B.10}$$

we have

$$\begin{aligned}(1+2\Phi)dt^2 &= (1+2\Phi+2b\dot{\Phi})(1+2\dot{b}+\dot{b}^2)dt^2 \\ &\approx (1+2(\dot{b}+\Phi))dt^2. \end{aligned} \tag{B.11}$$

We have used the fact that b and Φ are small parameters. From (B.11) we see that if, $\Phi \to \Phi - \dot{b}$, the g_{00} component is invariant. A similar treatment gives

$$B \to B + a^{-1}b - a\dot{c} \tag{B.12}$$
$$E \to E - c \tag{B.13}$$
$$\Psi \to \Psi + Hb. \tag{B.14}$$

Since metric and matter are constrained by Einstein's equation, when we introduce a perturbed metric it will induce perturbations on the matter content contained in the energy momentum tensor. In fact, the energy momentum tensor becomes (see [35] for a complete treatment)

$$T^0_0 = -(\bar{\rho}+\delta\rho) \tag{B.15}$$
$$T^0_i = (\bar{\rho}+p)aw_i \tag{B.16}$$
$$T^i_0 = -(\bar{\rho}+p)(w^i - B^i)/a \tag{B.17}$$
$$T^i_j = \delta^i_j(p+\delta p) + \Sigma^i_j. \tag{B.18}$$

Here $\bar{\rho}$ is the background energy density which is a function only of the time coordinate, w_i is the spatial velocity of a perfect fluid in a frame where the 3-momentum density ($q_i = \partial_i \delta q + s_i \equiv (\bar{\rho}+\bar{p})aw_i = T^0_i$) vanishes and Σ^i_j is the anisotropic tensor.

Again, under

$$t \to t + b \tag{B.19}$$

we have

$$\bar{\rho} \to \bar{\rho} + b\dot{\bar{\rho}}. \tag{B.20}$$

Then, it is clear that if

$$\delta\rho \to \delta\rho - b\dot{\bar{\rho}} \tag{B.21}$$

the T^0_0 component of the energy momentum tensor is invariant. So if

$$\delta\rho \to \delta\rho - b\dot{\bar{\rho}}, \qquad \delta p \to \delta p - b\dot{\bar{p}}, \tag{B.22}$$

all components of the energy momentum tensor are invariant.

Finally, we can define a gauge invariant variable, called the *curvature perturbation*

$$\boxed{-\zeta \equiv \Psi + \frac{H}{\dot{\bar{\rho}}}\delta\rho.} \tag{B.23}$$

Another important gauge invariant variable, in the linearized theory, is the *comoving curvature perturbation*, defined by

$$\boxed{\mathcal{R} = \Psi - \frac{H}{\bar{\rho}+\bar{p}}\delta q,} \tag{B.24}$$

where δq comes from the scalar part of the vector q_i and $\partial^i s_i = 0$.

In a slow-roll phase, $\bar{\rho} = \frac{\dot{\varphi}^2}{2} + V(\varphi)$, so

$$\begin{aligned} \delta\rho &= \dot{\varphi}\delta\dot{\varphi} + V'\delta\varphi \\ &\approx V'\delta\varphi \\ &\approx -3H\dot{\bar{\varphi}}\delta\varphi. \end{aligned} \tag{B.25}$$

Also, from Friedmann's equation, we can find

$$\dot{H} = -\frac{1}{2}(\bar{\rho} + \bar{p}) = -\frac{\dot{\bar{\varphi}}^2}{2}. \tag{B.26}$$

Since $H^2 = \bar{\rho}/3$, then

$$\begin{aligned} \dot{\bar{\rho}} &= 6H\dot{H} \\ &= -3H\dot{\bar{\varphi}}^2. \end{aligned} \tag{B.27}$$

From (B.25) and (B.27) we conclude that

$$\frac{\delta\rho}{\dot{\bar{\rho}}} = \frac{\delta\varphi}{\dot{\bar{\varphi}}}. \tag{B.28}$$

So in a slow-roll phase, we can write

$$\boxed{-\zeta \approx \Psi + \frac{H}{\dot{\bar{\varphi}}}\delta\varphi.} \tag{B.29}$$

It is possible to find a similar expression for the comoving curvature during slow-roll period. It is only necessary to know that during inflation $T_i^0 = -\dot{\bar{\varphi}}\partial_i\delta\varphi$[1]. Using it with (B.26) we get,

$$\boxed{\mathcal{R} = \Psi + \frac{H}{\dot{\bar{\varphi}}}\delta\varphi.} \tag{B.30}$$

A last thing we should do is to explain how the ζ variable of chapter 5 was achieved. We will not do it in detailed, but mention how it can be done. First of all, Einstein's equations constrains the metric perturbations with the inflaton field perturbations. The important equation for us is[2],

$$\dot{\Phi} + H\Phi = \frac{1}{2}\dot{\bar{\varphi}}\delta\varphi. \tag{B.31}$$

Another point is that, if at first order in fluctuations of the fields there is no anisotropic stress present in the matter, Einstein's equations constrains $\Psi = \Phi$[3]. So

$$-\zeta \approx \Phi + \frac{H}{\dot{\bar{\varphi}}}\delta\varphi. \tag{B.32}$$

Substituting (B.31) into (B.32) and remembering that $\dot{\bar{\varphi}}^2 = (\bar{\rho} + \bar{p})$, we get

$$\begin{aligned} -\zeta &\approx \Phi + \frac{2H(\dot{\Phi} + H\Phi)}{(\bar{\rho} + \bar{p})}, \\ &= \Phi + \frac{2H^2(H^{-1}\dot{\Phi} + \Phi)}{(\bar{\rho} + \bar{p})} \\ &= \Phi + \frac{2}{3}\frac{(H^{-1}\dot{\Phi} + \Phi)}{(1 + w)}, \end{aligned} \tag{B.33}$$

[1] Since $\partial_i\dot{\bar{\varphi}} = 0$, then $\dot{\bar{\varphi}}\partial_i\delta\varphi = \partial_i(\dot{\bar{\varphi}}\delta\varphi)$. So, $\delta q = \dot{\bar{\varphi}}\delta\varphi$.

[2] The reader interested in the full set of Einstein's equations is referred to [78].

[3] See [78] for details.

where we used the Friedmann's equation $H^2 = \rho/3$. So

$$\zeta \approx \Phi + \frac{2}{3}\frac{(H^{-1}\dot{\Phi} + \Phi)}{(1+w)}. \tag{B.34}$$

During inflation $w \simeq -1$ so the second term dominates. Also, if we go to momentum space $\dot{\Phi}(x) \to k\Phi(k)$. So, for long wavelengths ($k << H$) during inflation we have

$$\boxed{\zeta(k) \approx \frac{2}{3}\frac{\Phi(k)}{(1+w)}.} \tag{B.35}$$

This is equation (5.14).

C. Conformal and higher spin charges constraints

We start reviewing how conformal symmetry constrains the 4-point function of a CFT[1]. These computations can be viewed as a warm up to the reader interested in Chapter D.3. After it, we introduce a higher spin current in the CFT under consideration and we see that it enhances the spacetime symmetry group putting further constraints on the theory. For it we follow closely the reference [9]. Then, we compute explicitly the spin-3 current in a free massive scalar theory.

C.1. Conformal constraints

Suppose that we have a quantum field theory which is invariant under conformal group of transformations. The conformal group is composed by spacetime rotations and translations, dilations and special conformal transformations. In a quantum field theory the important objects we are interested in are the n-point functions. Let's see how the 4-point function is constrained by the conformal symmetries:

1) Rotations and translations imply that the 4-point function must be a function which depends only of the distance between points, say, $f(x_{12}, x_{13}, x_{14,}x_{23,}x_{24,}x_{34})$. Here $x_{ij} = x_i - x_j$.

2) Scaling transformation implies that the 4-point function is invariant under $x \to x' = \lambda x$. Since $\phi(x) \to \phi(x') = \lambda^{\Delta}\phi(\lambda x)$, under dilations, we see that

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = f\left(\frac{x_{ij}}{x_{kl}}\right) \times \prod_{i<j}^{4} x_{ij}^{-\frac{2\Delta}{3}},$$

i.e, it has to be a fraction of ratios that doesn't vary under dilations times a factor which will give the right λ factor that will cancel the transformation of the field (here we are assuming that all the fields have the same conformal weight).

3) Last but not least, we need to see what special conformal transformations implies. We assume primary fields,

$$\phi(x) = \left|\frac{\partial \boldsymbol{x'}}{\partial \boldsymbol{x}}\right|^{\Delta/d} \phi(x').$$

Under special conformal transformations $x^\mu \to x^{\mu'} = \frac{x^\mu - b^\mu(\boldsymbol{x}.\boldsymbol{x})}{1-2(\boldsymbol{b}.\boldsymbol{x})+(\boldsymbol{b}.\boldsymbol{b})(\boldsymbol{x}.\boldsymbol{x})}$.

$$\phi(x) = \gamma^{-\Delta}\phi(x'),$$

where $\gamma = 1 - 2\boldsymbol{b}.\boldsymbol{x} + b^2\boldsymbol{x}^2$. Since $x'_{ij} = \frac{x_{ij}}{\gamma_i^{1/2}\gamma_j^{1/2}}$, then

$$\begin{aligned}
\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= (\gamma_1\gamma_2\gamma_3\gamma_4)^{-\Delta} f\left(\frac{x'_{ij}}{x'_{kl}}\right) \times \prod_{i<j}^{4} (x'_{ij})^{-\frac{2\Delta}{3}}, \\
&= (\gamma_1\gamma_2\gamma_3\gamma_4)^{-\Delta} \times \prod_{i<j}^{4} x_{ij}^{-\frac{2\Delta}{3}} (\gamma_1\gamma_2\gamma_3\gamma_4)^{\Delta} \times f\left(\frac{x'_{ij}}{x'_{kl}}\right). \qquad \text{(C.1)}
\end{aligned}$$

[1] The 2 and 3-point functions are uniquely determined by the conformal symmetries. The reader interested on it can look at [91].

We see that the 4-point function is invariant under all the conformal transformations if the function , $f\left(\frac{x'_{ij}}{x'_{kl}}\right)$, is also invariant under special conformal transformations. Since $x'_{ij} = \frac{x_{ij}}{\gamma_i^{\Delta_i/2}\gamma_j^{\Delta_j/2}}$,we see that in order to have f invariant, we need to have all x_i's in the numerator and in the denominator. Then we see that the most general form of the 4-point function which preserves all the conformal transformation are:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = f\left(\frac{x_{ij}}{x_{ik}}\frac{x_{kl}}{x_{jl}}, \frac{x_{ij}}{x_{il}}\frac{x_{kl}}{x_{jk}}\right) \times \prod_{i<j}^{4} x_{ij}^{-\frac{2\Delta}{3}}. \tag{C.2}$$

C.1.1. Higher spin current constraint

Suppose now that the quantum theory under consideration posses higher spin currents and that the commutator of the field with the spin-3 charge is

$$[Q^{(3)}, \phi(x)] = \partial^3\phi(x). \tag{C.3}$$

Admitting that the vacuum is invariant under the symmetry $Q^{(3)}$, *i.e.*, $\exp(i\theta Q^{(3)})|\Omega\rangle = |\Omega\rangle$[2] and noting that infinitesimally, $\exp(i\theta Q^{(3)}) \approx 1 + i\theta Q^{(3)}$, we have

$$\begin{aligned}\exp(i\theta Q^{(3)})|\Omega\rangle &\approx (1 + i\theta Q^{(3)})|\Omega\rangle \\ &= |\Omega\rangle.\end{aligned} \tag{C.4}$$

So we conclude that if the vacuum is invariant under $Q^{(3)}$, then $Q^{(3)}|\Omega\rangle = 0$. In other words, the charge annihilates the vacuum [3].

Now, assuming that the charge annihilates the vacuum and inserting the commutator (C.3) into the 4-point function we get, in momentum space:

$$\sum_{i=1}^{4} k_i^3 = -3(k_1 - k_2)(k_1 - k_3)(k_2 - k_3) = 0. \tag{C.5}$$

This implies that the four point function in momentum space is

$$\begin{aligned}\langle\phi_1\phi_2\phi_3\phi_4\rangle &= f(k_1, k_2, k_3, k_4) \times \big(\delta(k_1 + k_2)\delta(k_3 + k_4) \\ &+ \delta(k_1 + k_3)\delta(k_2 + k_4) + \delta(k_1 + k_4)\delta(k_2 + k_3)\big).\end{aligned} \tag{C.6}$$

Going back to position space we get:

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \int dk_1dk_2dk_3dk_4 e^{ik_1.x_1}e^{ik_2.x_2}e^{ik_3.x_3}e^{ik_4.x_4} f(k_1, k_2, k_3, k_4) \\ &\times \delta(k_1 + k_2)\delta(k_3 + k_4) + perm. \qquad (C.7) \\ &= \int dk_1dk_2 e^{ik_1.(x_1-x_3)}e^{ik_2.(x_2-x_4)} f(k_1, k_2) + perm. \\ &= f(x_{13}, x_{24}) + f(x_{12}, x_{34}) + f(x_{14}, x_{23}).\end{aligned} \tag{C.8}$$

[2] One can find that a symmetry operator Q is represented on the physical Hilbert space by $\exp(i\theta Q)$, where θ is a real parameter. See [92] for details.

[3] The arguments here are valid for any charge Q that leaves the vacuum invariant.

C.1.2. Conclusion

Now, if we compare ($C.8$) and ($C.2$), we see that we cannot construct any term of the form of cross-ratios, so in equation ($C.2$), the function f must be a constant. In order to have the 4-point function invariant under dilations and special conformal transformations we must have a product of the distance between points that are in the argument of the function f. So

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \frac{C}{x_{13}^{\Delta}x_{24}^{\Delta}} + perm. \\ &= \langle\phi(x_1)\phi(x_3)\rangle\langle\phi(x_2)\phi(x_4)\rangle + perm.\end{aligned}$$

We see explicitly that the presence of higher spin currents put further constraints in the CFT we considered such that in the end the theory is free, i.e, the 4-point function can be written as products of 2-point functions.

C.2. Example of higher spin currents

C.2.1. Spin-1 charge

As we saw above, the commutator of a higher spin charge with a quantum field can put further constraints in the n-point functions we are interested in. Let's start seeing how to compute this commutator. For it, let's start computing the spin-1 charge associated to the energy-momentum tensor of a massive, complex scalar field:

$$T^{\mu\nu}(y) = \nabla^{\mu}\phi^{\dagger}(y)\nabla^{\nu}\phi(y) - g^{\mu\nu}\nabla^{\alpha}\phi^{\dagger}(y)\nabla_{\alpha}\phi(y). \tag{C.9}$$

The charge associated with this current is given by

$$Q_{\zeta}^{(1)} = \int d^d y\sqrt{h}n_{\mu}(y)\zeta_{\nu}T^{\mu\nu}(y) \tag{C.10}$$

where n^{μ} is a future directed four vector perpendicular to the plane of foliation of the spacetime, ζ_{ν}, is a Killing vector field and $\sqrt{h}$ is the determinant of the spatial section. Here we will use the Poincaré patch (3.3), so

$$n_{\mu} = (\eta/\ell, 0, 0, ..., 0), \tag{C.11}$$

$$\sqrt{h} = \left(\frac{\ell}{\eta}\right)^{d}. \tag{C.12}$$

With it we have

$$\begin{aligned}[Q_{\zeta}^{(1)}, \phi(x)] &= [\int d^d y\sqrt{h}n_{\mu}(y)\zeta_{\nu}T^{\mu\nu}(y), \phi(x)] \\ &= \int d^d y t_{\mu}\zeta_{\nu}\left(\frac{\ell}{\eta}\right)^{d-1}\Big(\nabla^{\mu}\phi^{\dagger}(y)[\nabla^{\nu}\phi(y), \phi(x)] + [\nabla^{\mu}\phi^{\dagger}(y), \phi(x)]\nabla^{\nu}\phi(y) \\ &- g^{\mu\nu}\nabla^{\alpha}\phi^{\dagger}(y)[\nabla_{\alpha}\phi(y), \phi(x)]\Big).\end{aligned} \tag{C.13}$$

here $t_{\mu} = (1, 0, 0, ..., 0)$. The quantum fields satisfy the commutation relations:

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = [\pi(\vec{x}, t), \pi(\vec{y}, t)] = 0, \tag{C.14}$$

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{d}(\vec{x}, \vec{y}), \tag{C.15}$$

and similar relations for the complex conjugate. Here $\pi(x) = \frac{\eta}{\ell}\sqrt{\hbar}\partial_0\phi^\dagger(x)$. From now on $\ell = 1$. Since $\nabla^\mu\phi(y) = \partial^\mu\phi(y)$ we get:

$$
\begin{aligned}
[Q^{(1)}_\zeta, \phi(x)] &= \int d^d y t_\mu \zeta_0 \eta^{1-d} \nabla^\mu\phi(y)[\nabla^0\phi(y), \phi(x)] && \text{(C.16)}\\
&+ \int d^d y t_0 \zeta_\nu [\eta^{1-d}\nabla^0\phi(y), \phi(x)]\nabla^\nu\phi(y) && \text{(C.17)}\\
&- \int d^d y t_\mu \zeta^\mu \eta^{1-d} g^{\mu\nu}\nabla^0\phi(y)[\nabla_0\phi(y), \phi(x)]. && \text{(C.18)}
\end{aligned}
$$

Choosing and $\zeta^\nu = (0, 1, 0, ..., 0)$, we get

$$
\begin{aligned}
[Q^{(1)}_\zeta, \phi(x)] &= i\int d^d y \nabla^1\phi(y)\delta^d(\vec{x}, \vec{y}) && \text{(C.19)}\\
&= i\nabla^1\phi(x). && \text{(C.20)}
\end{aligned}
$$

C.3. Spin-3 current of a massive complex scalar field

The higher spin currents present in a free, massive and complex scalar field theory are global symmetries that are, by construction, conserved. In order to construct it, we add the most general terms we have at hand. The currents need to be symmetric, $J_{\mu\nu\alpha} = J_{(\mu\nu\alpha)}$, and conserved, $\nabla^\mu J_{\mu\nu\alpha} = 0$. Let's start first writing down the most general symmetric spin-3 current,

$$
\begin{aligned}
J_{\mu\nu\lambda} &= a_1\phi^\dagger\nabla_{(\mu}\nabla_\nu\nabla_{\lambda)}\phi + a_2\nabla_{(\mu}\nabla_\nu\phi^\dagger\nabla_{\lambda)}\phi + a_3\gamma_{(\mu\nu}\nabla_\alpha\phi^\dagger\nabla_{\lambda)}\nabla^\alpha\phi && \text{(C.21)}\\
&+ [a_4\ell^{-2} + a_5 M^2]\gamma_{(\mu\nu}\phi^\dagger\nabla_{\lambda)}\phi - (\phi^\dagger \leftrightarrow \phi). && \text{(C.22)}
\end{aligned}
$$

We want this current to be conserved, i.e, $\nabla^\lambda J_{\mu\nu\lambda} = 0$. Before starting, it is worth noting in a maximally symmetric spacetime in any dimensions,

$$
\begin{aligned}
[\nabla_\nu, \nabla_\lambda]\nabla_\mu\phi &= -R^\sigma_{\mu\nu\lambda}\nabla_\sigma\\
&= -\frac{R}{d(d-1)}(\gamma_{\sigma\nu}\gamma_{\mu\lambda} - \gamma_{\sigma\lambda}\gamma_{\mu\nu})\nabla^\sigma\phi\\
&= -\frac{R}{d(d-1)}(\gamma_{\mu\lambda}\nabla_\nu\phi - \gamma_{\mu\nu}\nabla_\lambda\phi)\\
&= -\ell^{-2}(\gamma_{\mu\lambda}\nabla_\nu\phi - \gamma_{\mu\nu}\nabla_\lambda\phi) && \text{(C.23)}
\end{aligned}
$$

where $R = \ell^{-2}d(d-1)$. Also

$$\Box\nabla_\mu\phi = \nabla_\mu(\Box + d\ell^{-2})\phi. \tag{C.24}$$

Again, let's do it term by term.

The last term which multiplies a_4 and a_5 is

$$\gamma_{(\mu\nu}\phi^\dagger\nabla_{\lambda)}\phi = \gamma_{(\mu\nu}J_{\lambda)} \tag{C.25}$$

Since $\nabla^\mu J_\mu = 0$, we have

$$
\begin{aligned}
\nabla^\mu\gamma_{(\mu\nu}J_{\lambda)} &= \nabla_{(\nu}J_{\lambda)}\\
&= \nabla_{(\mu}\phi^\dagger\nabla_{\nu)}\phi + \phi^\dagger\nabla_\mu\nabla_\nu\phi - (\phi^\dagger \leftrightarrow \phi)\\
&= \phi^\dagger\nabla_\mu\nabla_\nu\phi - (\phi^\dagger \leftrightarrow \phi). && \text{(C.26)}
\end{aligned}
$$

The term which multiplies a_3 can be easily work out if we define

$$V_\lambda \equiv \nabla_\alpha\phi^\dagger\nabla_\lambda\nabla^\alpha\phi - (\phi^\dagger \leftrightarrow \phi). \tag{C.27}$$

Now

$$\begin{aligned}\nabla^\lambda V_\lambda &= \nabla^\lambda\nabla_\alpha\phi^\dagger\nabla_\lambda\nabla^\alpha\phi + \nabla^\alpha\phi^\dagger\Box\nabla_\alpha\phi - (\phi^\dagger \leftrightarrow \phi)\\ &= (M^2 + d\ell^{-2})\nabla^\alpha\phi^\dagger\nabla_\alpha\phi - (\phi^\dagger \leftrightarrow \phi)\\ &= 0.\end{aligned} \tag{C.28}$$

Also

$$\begin{aligned}\nabla_{(\mu}V_{\lambda)} &= \nabla_{(\mu}\nabla^\alpha\phi^\dagger\nabla_{\nu)}\nabla_\alpha\phi + \nabla^\alpha\phi^\dagger\nabla_{(\mu}\nabla_{\nu)}\nabla_\alpha\phi - (\phi^\dagger \leftrightarrow \phi)\\ &= \nabla^\alpha\phi^\dagger\nabla_{(\mu}\nabla_{\nu)}\nabla_\alpha\phi - (\phi^\dagger \leftrightarrow \phi).\end{aligned} \tag{C.29}$$

Note that for any vector such that $\nabla^\mu W_\mu = 0$, we have

$$\nabla^\lambda\gamma_{(\mu\nu}W_{\lambda)} = \frac{2}{3}\nabla_{(\mu}W_{\nu)}. \tag{C.30}$$

Another important thing that we will use later is

$$\begin{aligned}\gamma^{\mu\nu}(\gamma_{(\mu\nu}W_{\lambda)}) &= \frac{1}{3}\gamma^{\mu\nu}(\gamma_{\mu\nu}W_\lambda + 2\gamma_{\lambda(\mu}W_{\nu)})\\ &= \frac{d+3}{3}W_\lambda.\end{aligned} \tag{C.31}$$

The term multiplied by a_2 gives

$$\begin{aligned}\nabla^\lambda(\nabla_{(\mu}\nabla_\nu\phi^\dagger\nabla_{\lambda)}\phi) &= \frac{1}{3}\nabla^\lambda(\nabla_\mu\nabla_\nu\phi^\dagger\nabla_\lambda\phi + \nabla_\mu\nabla_\lambda\phi^\dagger\nabla_\nu\phi + \nabla_\nu\nabla_\lambda\phi^\dagger\nabla_\mu\phi - (\phi^\dagger \leftrightarrow \phi))\\ &= \frac{1}{3}(\nabla^\lambda\nabla_\mu\nabla_\nu\phi^\dagger\nabla_\lambda + \nabla_\mu\nabla_\nu\phi^\dagger\Box\phi\\ &+ 2\Box\nabla_{(\mu}\phi^\dagger\nabla_{\nu)}\phi + 2\nabla_{(\mu}\nabla_\lambda\phi^\dagger\nabla^\lambda\nabla_{\nu)}\phi - (\phi^\dagger \leftrightarrow \phi))\\ &= \frac{1}{3}(\nabla^\lambda\nabla_{(\mu}\nabla_{\nu)}\phi^\dagger\nabla_\lambda\phi - M^2\phi^\dagger\nabla_{(\mu}\nabla_{\nu)}\phi - (\phi^\dagger \leftrightarrow \phi))\\ &= \frac{1}{3}\nabla_{(\mu}V_{\nu)} - \frac{M^2}{3}\nabla_{(\mu}J_{\nu)} - (\phi^\dagger \leftrightarrow \phi).\end{aligned} \tag{C.32}$$

The trace of this term is

$$\begin{aligned}\gamma^{\mu\nu}\nabla_{(\mu}\nabla_\nu\phi^\dagger\nabla_{\lambda)}\phi &= \frac{1}{3}\gamma^{\mu\nu}[\nabla_\lambda\phi^\dagger\nabla_\mu\nabla_\nu\phi + 2\nabla_{(\mu}\phi^\dagger\nabla_{\nu)}\nabla_\lambda\phi] - (\phi^\dagger \leftrightarrow \phi)\\ &= -\frac{M^2}{3}J_\lambda + \frac{2}{3}V_\lambda.\end{aligned} \tag{C.33}$$

The first term gives

$$\begin{aligned}\nabla^\lambda(\phi\dagger\nabla_{(\mu}\nabla_\nu\nabla_{\lambda)}\phi) &= \frac{1}{3}\nabla^\lambda(\phi^\dagger\nabla_\mu\nabla_\nu\nabla_\lambda\phi + \phi^\dagger\nabla_\nu\nabla_\mu\nabla_\lambda\phi + \phi^\dagger\nabla_\lambda\nabla_\mu\nabla_\nu\phi)\\ &= \frac{1}{3}(\nabla^\lambda\phi^\dagger\nabla_\mu\nabla_\nu\nabla_\lambda\phi + \phi^\dagger\nabla^\lambda\nabla_\mu\nabla_\nu\nabla_\lambda\phi\\ &+ \nabla^\lambda\phi^\dagger\nabla_\nu\nabla_\mu\nabla_\lambda\phi + \phi^\dagger\nabla^\lambda\nabla_\nu\nabla_\mu\nabla_\lambda\phi\\ &+ \nabla^\lambda\phi^\dagger\nabla_\lambda\nabla_\mu\nabla_\nu\phi + \phi^\dagger\Box\nabla_\mu\nabla_\nu\phi)\\ &= (\frac{1}{3}\phi^\dagger\Box\nabla_\mu\nabla_\nu\phi) + \frac{2}{3}\phi^\dagger\nabla^\lambda\nabla_{(\mu}\nabla_{\nu)}\nabla_\lambda\phi - (\phi^\dagger \leftrightarrow \phi))\\ &+ \nabla_{(\mu}V_{\nu)}.\end{aligned} \tag{C.34}$$

We can work out further the two first terms in the final expression above. In fact, noting that

$$\begin{aligned}
\Box\nabla_\mu\nabla_\nu\phi &= \gamma^{\alpha\beta}\nabla_\alpha[\nabla_\beta,\nabla_\mu]\nabla_\nu\phi+\gamma^{\alpha\beta}[\nabla_\alpha,\nabla_\mu]\nabla_\nu\nabla_\beta\phi \\
&+ \gamma^{\alpha\beta}\nabla_\mu[\nabla_\alpha,\nabla_\nu]\nabla_\beta\phi+M^2\nabla_\mu\nabla_\nu\phi \\
&= \gamma^{\alpha\beta}\big(-\nabla_\alpha R^\lambda_{\nu\beta\mu}\nabla_\lambda\phi-R^\lambda_{\nu\alpha\mu}\nabla_\lambda\nabla_\beta\phi \\
&- R^\lambda_{\beta\alpha\mu}\nabla_\nu\nabla_\lambda\phi-\nabla_\mu R^\lambda_{\beta\alpha\nu}\nabla_\lambda\phi\big)+M^2\nabla_\mu\nabla_\nu\phi \\
&= -2\ell^{-2}\gamma^{\alpha\beta}\big(\nabla_\alpha(\gamma^\lambda_\beta\gamma_{\mu\nu}-\gamma^\lambda_\mu\gamma_{\beta\nu})\nabla_\lambda\phi \\
&+ \nabla\mu(\gamma^\lambda_\alpha\gamma_{\beta\nu}-\gamma^\lambda_\nu\gamma_{\alpha\beta})\nabla_\lambda\phi\big)+M^2\nabla_\mu\nabla_\nu\phi \\
&= -2\ell^{-2}\big(\gamma_{\mu\nu}\Box\phi-\nabla_\mu\nabla_\nu\phi+\nabla_\mu\nabla_\nu\phi-(d+1)\nabla_\mu\nabla_\nu\phi\big)+M^2\nabla_\mu\nabla_\nu\phi \\
&= 2(d+1)\ell^{-2}\nabla_\mu\nabla_\nu\phi-2M^2\ell^{-2}\gamma_{\mu\nu}\phi+M^2\nabla_\mu\nabla_\nu\phi. \qquad \text{(C.35)}
\end{aligned}$$

It is easy to check that

$$\gamma^{\mu\nu}\Box\nabla_\mu\nabla_\nu\phi=M^4\phi. \qquad \text{(C.36)}$$

Also

$$\begin{aligned}
\nabla^\lambda\nabla_\mu\nabla_\lambda\nabla_\nu\phi &= \nabla^\lambda[\nabla_\mu,\nabla_\lambda]\nabla_\nu\phi+\Box\nabla_\mu\nabla_\nu\phi \\
&= -\nabla^\lambda R^\alpha_{\nu\mu\lambda}\nabla_\alpha\phi+\Box\nabla_\mu\nabla_\nu\phi \\
&= -\ell^{-2}\nabla^\lambda(\gamma^\alpha_\mu\gamma_{\nu\lambda}-\gamma^\alpha_\lambda\gamma_{\mu\nu})\nabla_\alpha\phi+\Box\nabla_\mu\nabla_\nu\phi \\
&= -\ell^{-2}\nabla_\mu\nabla_\nu\phi+\gamma_{\mu\nu}M^2\ell^{-2}\phi+\Box\nabla_\mu\nabla_\nu\phi \\
&= (2d+1)\ell^{-2}\nabla_\mu\nabla_\nu\phi-M^2\ell^{-2}\gamma_{\mu\nu}\phi+M^2\nabla_\mu\nabla_\mu\phi. \qquad \text{(C.37)}
\end{aligned}$$

Substituting (C.35) and (C.37) inside (C.34) we get

$$\nabla^\lambda(\phi^\dagger\nabla_{(\mu}\nabla_\nu\nabla_{\lambda)}\phi)=[(2d+\frac{4}{3})\ell^{-2}+M^2]\nabla_{(\mu}J_{\nu)}+\nabla_{(\mu}V_{\nu)}, \qquad \text{(C.38)}$$

and

$$\begin{aligned}
\gamma^{\mu\nu}(\phi^\dagger\nabla_{(\mu}\nabla_\nu\nabla_{\lambda)}\phi) &= \frac{1}{3}\gamma^{\mu\nu}\phi^\dagger(\nabla_\lambda\nabla_\mu\nabla_\nu+2\nabla_{(\mu}\nabla_{\nu)}\nabla_\lambda)\phi-(\phi^\dagger\leftrightarrow\phi) \\
&= \frac{1}{3}\phi^\dagger(M^2\nabla_\lambda+2\Box\nabla_\lambda)\phi-(\phi^\dagger\leftrightarrow\phi) \\
&= \frac{1}{3}\phi^\dagger(3M^2+2d\ell^{-2})\nabla_\lambda\phi-(\phi^\dagger\leftrightarrow\phi) \\
&= (M^2+\frac{2d}{3}\ell^{-2})J_\lambda. \qquad \text{(C.39)}
\end{aligned}$$

Returning everything to $J_{\mu\nu\lambda}$ we have

$$\begin{aligned}
\nabla^\lambda J_{\mu\nu\lambda} &= a_1\{[(2d+\frac{4}{3})\ell^{-2}+M^2]\nabla_{(\mu}J_{\nu)}+\nabla_{(\mu}V_{\nu)}\} \\
&+ a_2\Big(\frac{M^2}{3}\nabla_{(\mu}J_{\nu)}+\frac{1}{3}\nabla_{(\mu}V_{\nu)}\Big)+\frac{2a_3}{3}\nabla_{(\mu}V_{\nu)} \\
&+ \frac{2}{3}(M^2a_4+\ell^{-2}a_5)\nabla_{(\mu}J_{\nu)} \\
&= \nabla_{(\mu}J_{\nu)}\Big\{M^2\Big(a_1+\frac{1}{3}a_2+\frac{2}{3}a_4\Big)+\ell^{-2}\Big(a_1(2d+\frac{4}{3})+\frac{2}{3}a_5\Big)\Big\} \\
&+ \nabla_{(\mu}V_{\nu)}[a_1+\frac{1}{3}a_2+\frac{2}{3}a_3]. \qquad \text{(C.40)}
\end{aligned}$$

Also,

$$\begin{aligned}
\gamma^{\mu\nu}J_{\mu\nu\lambda} &= a_1\Big(M^2+\frac{2d}{3}\ell^{-2}\Big)J_\lambda + a_2\Big[\frac{2}{3}V_\lambda - \frac{M^2}{3}J_\lambda\Big] \\
&+ \frac{(d+3)}{3}a_3V_\lambda + \frac{(d+3)}{3}\Big(M^2a_4+\ell^{-2}a_5\Big)J_\lambda \\
&= J_\lambda\Big\{M^2\Big[a_1-\frac{1}{3}a_2+\frac{(d+3)}{3}a_4\Big]+\ell^{-2}\Big(\frac{2d}{3}a_1+\frac{(d+3)}{3}a_5\Big)\Big\} \\
&+ V_\lambda\Big(\frac{2}{3}a_2+\frac{(d+3)}{3}a_3\Big). \qquad \text{(C.41)}
\end{aligned}$$

Now, if we impose $\nabla^\mu J_{\mu\nu\lambda}=0$ and use Mathematica to solve the system for the coefficients we obtain

$$a_2 = -(3a_1+2a_3), \qquad a_5 = -(3d+2)a_1 + M^2\ell^{-2}(a_3-a_4). \qquad \text{(C.42)}$$

So

$$\begin{aligned}
J_{\mu\nu\lambda} &= a_1\phi^\dagger\nabla_{(\mu}\nabla_\nu\nabla_{\lambda)}\phi - (3a_1+2a_3)\nabla_{(\mu}\nabla_\nu\phi^\dagger\nabla_{\lambda)}\phi + a_3\gamma_{(\mu\nu}\nabla_\alpha\phi^\dagger\nabla_{\lambda)}\nabla^\alpha\phi \qquad \text{(C.43)}\\
&+ [-(3d+2)a_1\ell^{-2}+a_3M^2]\gamma_{(\mu\nu}\phi^\dagger\nabla_{\lambda)}\phi - (\phi^\dagger\leftrightarrow\phi). \qquad \text{(C.44)}
\end{aligned}$$

Another thing we can impose is $\gamma^{\mu\nu}J_{\mu\nu\lambda}=0$. This will give another equation to solve, which is enough since an overall normalization is not important. A traceless current requires

$$(i) \qquad 6a_1+(1-d)a_3 = 0, \qquad \text{(C.45)}$$

$$(ii) \qquad 3a_1(d+1)(d+2) - M^2\ell^2(6a_1+(d+5)a_3) = 0. \qquad \text{(C.46)}$$

For $d\neq 1$ we have

$$a_3 = \Big(\frac{6}{d-1}\Big)a_1, \qquad \text{(C.47)}$$

$$\gamma^{\mu\nu} = a_1\frac{(2+d)}{(d-1)}(d^2-1-4M^2\ell^2), \qquad \text{(C.48)}$$

so, excluding the trivial solution $a_1=0$, we have $\gamma^{\mu\nu}J_{\mu\nu\lambda}=0$ iff

$$M^2\ell^2 = \frac{(d^2-1)}{M^2}, \qquad \text{(C.49)}$$

i.e, for the case of conformally coupled mass!

For $d=1$, $a_1=0$,

$$\gamma^{\mu\nu}J_{\mu\nu\lambda} = 2a_3M^2 \qquad \text{(C.50)}$$

so it is zero if $M^2=0$, which is also the conformally coupled case or if $a_3=0$ and then there will be no current.

Finally, we conclude stating that, for all d, up to normalization,

$$\begin{aligned}
J_{\mu\nu\lambda} &= (d-1)\phi^\dagger\nabla_{(\mu}\nabla_\nu\nabla_{\lambda)}\phi - 3(3+d)\nabla_{(\mu}\nabla_\nu\phi^\dagger\nabla_{\lambda)}\phi + 6\gamma_{(\mu\nu}\nabla_\alpha\phi^\dagger\nabla_{\lambda)}\nabla^\alpha\phi \\
&+ [-(d-1)(3d+2)\ell^{-2}+6M^2]\gamma_{(\mu\nu}\phi^\dagger\nabla_{\lambda)}\phi - (\phi^\dagger\leftrightarrow\phi), \qquad \text{(C.51)}
\end{aligned}$$

and

$$\gamma^{\mu\nu}J_{\mu\nu\lambda} = (2+d)(d^2-1-4M^2\ell^2)\ell^{-2}. \qquad \text{(C.52)}$$

C.3.1. The commutator

As we saw in (C.21)

$$\begin{aligned} J_{\mu\nu\lambda} &= a_1\phi^\dagger\nabla_{(\mu}\nabla_\nu\nabla_{\lambda)}\phi + a_2\nabla_{(\mu}\nabla_\nu\phi^\dagger\nabla_{\lambda)}\phi + a_3\gamma_{(\mu\nu}\nabla_\alpha\phi^\dagger\nabla_{\lambda)}\nabla^\alpha\phi \\ &+ [a_4\ell^{-2} + a_5M^2]\gamma_{(\mu\nu}\phi^\dagger\nabla_{\lambda)}\phi - (\phi^\dagger \leftrightarrow \phi). \end{aligned} \tag{C.53}$$

The object we are interested in is the commutator of the spin-2 charge construct with the above current

$$\begin{aligned} [Q^{(2)}, \phi(x)] &= \int d^dy p^i p^j \eta^{1-d}[J_{\eta ij}(y), \phi(x)] \\ &= \int d^dy p^i p^j \eta^{1-d}\Big(a_1[\phi^\dagger\nabla_{(\eta}\nabla_i\nabla_{j)}\phi(y), \phi(x)] \\ &+ a_2[\nabla_{(\eta}\nabla_i\phi^\dagger\nabla_{j)}\phi(y), \phi(x)] \\ &+ \frac{a_3}{3}[\gamma_{ij}\nabla_\alpha\phi\nabla_\eta\nabla^\alpha\phi^\dagger(y), \phi(x)] \\ &+ \frac{a_4}{3}[\gamma_{ij}\phi^\dagger\nabla_\eta\phi(y), \phi(x)] \\ &- (\phi^\dagger \leftrightarrow \phi). \end{aligned} \tag{C.54}$$

Before start computing, it is good to note that the only nonzero Levi-Civita conections which will be important here are $\Gamma^\eta_{\eta\eta} = \Gamma^i_{i\eta} = \Gamma^\eta_{ii} = -1/\eta$. Then,

$$\nabla_i\nabla_j = \partial_i\partial_j + \delta_{ij}\frac{1}{\eta}\partial_\eta, \tag{C.55}$$

$$\nabla_i\nabla_\eta = \partial_i\partial_\eta + \frac{1}{\eta}\partial_i, \tag{C.56}$$

$$\nabla_\eta\nabla_i = \partial_\eta\partial_i + \frac{1}{\eta}\partial_i, \tag{C.57}$$

$$\nabla_\eta\nabla_\eta = \partial_\eta\partial_\eta + \frac{1}{\eta}\partial_\eta. \tag{C.58}$$

Now we do it term by term. First note that

$$\begin{aligned} \nabla_\mu\nabla_\nu\nabla_\lambda\phi &= (\partial_\mu\nabla_\nu\nabla_\lambda - \Gamma^\alpha_{\mu\nu}\nabla_\alpha\nabla_\lambda - \Gamma^\alpha_{\mu\lambda}\nabla_\alpha\nabla_\nu)\phi \\ &= \Big(\partial_\mu(\partial_\nu\partial_\lambda - \Gamma^\alpha_{\nu\lambda}\partial_\alpha) - \Gamma^\alpha_{\mu\nu}(\partial_\alpha\partial_\lambda - \Gamma^\beta_{\alpha\lambda}\partial_\beta) - \Gamma^\alpha_{\mu\lambda}(\partial_\alpha\partial_\nu - \Gamma^\beta_{\alpha\nu}\partial_\beta)\Big)\phi \\ &= \Big(\partial_\mu\partial_\nu\partial_\lambda - \partial_\mu(\Gamma^\alpha_{\nu\lambda}\partial_\alpha) - \Gamma^\alpha_{\mu\nu}(\partial_\alpha\partial_\lambda - \Gamma^\beta_{\alpha\lambda}\partial_\beta) \\ &- \Gamma^\alpha_{\mu\lambda}(\partial_\alpha\partial_\nu - \Gamma^\beta_{\alpha\nu}\partial_\beta)\Big)\phi. \end{aligned} \tag{C.59}$$

So

$$\begin{aligned} \nabla_i\nabla_j\nabla_\eta\phi &= \Big(\partial_i\partial_j\partial_\eta - \partial_i(\Gamma^j_{j\eta}\partial_j) - \Gamma^\eta_{ij}\partial_\eta\partial_\eta + \Gamma^\eta_{ij}\Gamma^\eta_{\eta\eta}\partial_\eta \\ &- \Gamma^i_{i\eta}\partial_i\partial_j + \Gamma^i_{i\eta}\Gamma^\eta_{ij}\partial_\eta\Big)\phi \\ &= \Big(\partial_i\partial_j\partial_\eta + \frac{\delta_{ij}}{\eta}\partial^2_\eta + \frac{2\delta_{ij}}{\eta^2}\partial_\eta + \frac{2}{\eta}\partial_i\partial_j\Big)\phi \\ &= \nabla_j\nabla_i\nabla_\eta\phi. \end{aligned} \tag{C.60}$$

Also

$$\begin{aligned}\nabla_\eta \nabla_i \nabla_j \phi &= \Big(\partial_i\partial_j\partial_\eta - \partial_\eta(\Gamma^\eta_{ij}\partial_\eta) - \Gamma^i_{\eta i}\partial_i\partial_j + \Gamma^i_{\eta i}\Gamma^\eta_{ij}\partial_\eta \\ &- \Gamma^j_{\eta j}\partial_i\partial_j + \Gamma^j_{\eta j}\Gamma^\eta_{ji}\partial_\eta\Big)\phi \\ &= \Big(\partial_i\partial_j\partial_\eta + \frac{\delta_{ij}}{\eta}\partial^2_\eta + \frac{\delta_{ij}}{\eta^2}\partial_\eta + \frac{2}{\eta}\partial_i\partial_j\Big)\phi \end{aligned} \tag{C.61}$$

We are now ready to compute the commutators (for the first term only the ($\phi^\dagger \leftrightarrow \phi$) term is important here, that's why we have a minus sign below)

$$\begin{aligned} -a_1\eta^{1-d}[\phi\nabla_{(\eta}\nabla_i\nabla_{j)}\phi^\dagger(y),\phi(x)] &= -\frac{a_1}{3}\eta^{1-d}\phi[\nabla_\eta\nabla_i\nabla_j\phi^\dagger(y) + \nabla_i\nabla_j\nabla_\eta\phi^\dagger(y) \\ &+ \nabla_j\nabla_i\nabla_\eta\phi^\dagger(y),\phi(x)] \\ &= -\frac{a_1}{3}\eta^{1-d}\phi[(\partial_i\partial_j\partial_\eta + \frac{\delta_{ij}}{\eta^2}\partial_\eta + \frac{\delta_{ij}}{\eta}\partial^2_\eta + \frac{2}{\eta}\partial_i\partial_j)\phi^\dagger(y),\phi(x)] \\ &- \frac{2}{3}a_1\phi(y)\eta^{1-d}[(\partial_i\partial_j\partial_\eta + \frac{2\delta_{ij}}{\eta^2}\partial_\eta + \frac{\delta_{ij}}{\eta}\partial^2_\eta + \frac{2}{\eta}\partial_i\partial_j)\phi^\dagger(y),\phi(x)] \\ &= ia_1\phi(y)\Big(\partial_i\partial_j\delta(\vec{x}-\vec{y}) + \frac{5\delta_{ij}}{3\eta^2}\delta(\vec{x}-\vec{y})\Big) \\ &- \frac{a_1\delta_{ij}}{\eta}\eta^{1-d}\phi[\partial^2_\eta\phi^\dagger(y),\phi(x)] \end{aligned} \tag{C.62}$$

Now[4],

$$\eta^{1-d}[\partial^2_\eta\phi^\dagger(y),\phi(x)] = \partial_\eta(\eta^{1-d}[\partial_\eta\phi^\dagger(y),\phi(x)]) - (1-d)\eta^{-d}[\partial_\eta\phi^\dagger(y),\phi(x)] \tag{C.63}$$

so the first term is

$$-a_1\eta^{1-d}[\phi\nabla_{(\eta}\nabla_i\nabla_{j)}\phi^\dagger(y),\phi(x)] = ia_1\phi_2(y)\Big(\partial_i\partial_j\delta(\vec{x}-\vec{y}) + \frac{(2+3d)}{3\eta^2}\delta_{ij}\delta(\vec{x}-\vec{y})\Big). \tag{C.64}$$

The second term is

$$\begin{aligned} a_2\eta^{1-d}[\nabla_{(\eta}\nabla_i\phi^\dagger\nabla_{j)}\phi(y),\phi(x)] &= \frac{a_2}{3}\eta^{1-d}[\nabla_\eta\nabla_i\phi^\dagger\nabla_j\phi(y) + \nabla_\eta\nabla_j\phi^\dagger\nabla_i\phi(y) \\ &+ \nabla_i\nabla_j\phi^\dagger\nabla_\eta\phi(y),\phi(x)] \\ &= \frac{a_2}{3}\eta^{1-d}\Big([(\partial_\eta\partial_i + \frac{1}{\eta}\partial_i)\phi^\dagger\nabla_j\phi(y),\phi(x)] \\ &+ [(\partial_\eta\partial_j + \frac{1}{\eta}\partial_j)\phi_1\nabla_\eta\phi_2(y),\phi_1(x)] \\ &+ [(\partial_i\partial_j + \frac{\delta_{ij}}{\eta}\partial_\eta)\phi^\dagger\nabla_\eta\phi(y),\phi(x)]\Big) \\ &= -i\frac{a_2}{3}\Big(\partial_i\delta(\vec{x}-\vec{y})\partial_j\phi(y) + \partial_j\delta(\vec{x}-\vec{y})\partial_i\phi(y) + \frac{\delta_{ij}}{\eta}\delta(\vec{x}-\vec{y})\partial_\eta\phi(y)\Big). \end{aligned}$$

[4] Here is important to have the term η^{1-d} inside the commutator in order to define $\pi(x) = \eta\sqrt{h}\partial_t\phi^\dagger(x)$ and then compute the commutator.

The term $(\phi^\dagger \leftrightarrow \phi)$ is also important here. It gives

$$\begin{aligned} -a_2\eta^{1-d}[\nabla_{(\eta}\nabla_i\phi\nabla_{j)}\phi^\dagger(y),\phi(x)] &= -\frac{a_2}{3}\eta^{1-d}[\nabla_j\nabla_i\phi\nabla_\eta\phi^\dagger(y),\phi(x)] \\ &= i\frac{a_2}{3}(\partial_i\partial_j+\delta_{ij}\frac{1}{\eta}\partial_\eta)\phi(y)\delta(\vec{x}-\vec{y}). \end{aligned} \tag{C.65}$$

So, the 2nd term is

$$\begin{aligned} a_2\eta^{1-d}[\nabla_{(\eta}\nabla_i\phi^\dagger\nabla_{j)}\phi(y),\phi(x)]-(\phi^\dagger\leftrightarrow\phi) &= -i\frac{a_2}{3}\Big(\partial_i\delta(\vec{x}-\vec{y})\partial_j\phi(y)+\partial_j\delta(\vec{x}-\vec{y})\partial_i\phi(y)\Big) \\ &+ i\frac{a_2}{3}\partial_i\partial_j\phi(y)\delta(\vec{x}-\vec{y}). \end{aligned} \tag{C.66}$$

The third term we need to be more careful to compute ,

$$\begin{aligned} \frac{a_3}{3}[\gamma_{ij}\nabla_\alpha\phi\nabla_\eta\nabla^\alpha\phi^\dagger(y),\phi(x)] &= \frac{a_3}{3}\eta^{1-d}[\gamma_{ij}(\nabla^\eta\phi\nabla_\eta\nabla_\eta\phi^\dagger(y)+\nabla^k\phi\nabla_\eta\nabla_k\phi^\dagger(y)),\phi(x)] \\ &= \frac{a_3}{3}\eta^{1-d}[\gamma_{ij}(\nabla^\eta\phi\nabla_\eta\nabla_\eta\phi^\dagger(y)+(\partial^k\phi(\partial_k\partial_\eta-\frac{1}{\eta}\partial_k)\phi^\dagger(y)),\phi(x)] \\ &= -\frac{a_3}{3}i\gamma_{ij}\partial_k\phi(y)\partial^k\delta(\vec{x}-\vec{y}) \\ &+ \frac{a_3}{3}\eta^{1-d}[\gamma_{ij}\nabla^\eta\phi\nabla_\eta\nabla_\eta\phi^\dagger(y),\phi(x)]. \end{aligned}$$

The last term needs more attention

$$\begin{aligned} \frac{a_3}{3}\eta^{1-d}[\gamma_{ij}\nabla^\eta\phi\nabla_\eta\nabla_\eta\phi^\dagger(y),\phi(x)] &= \frac{a_3}{3}\eta^{1-d}\gamma_{ij}\nabla^\eta\phi[(\partial_\eta\partial_\eta+\frac{1}{\eta}\partial_\eta)\phi^\dagger(y),\phi(x)] \\ &= -\frac{a_3}{3}\frac{i}{\eta}\gamma_{ij}\partial^\eta\phi(y)\delta(\vec{x}-\vec{y})+\frac{a_3}{3}\eta^{1-d}\gamma_{ij}\partial^\eta\phi_2[\partial_\eta\partial_\eta\phi^\dagger(y),\phi(x)] \end{aligned}$$

Again, using that

$$\eta^{1-d}[\partial^2_\eta\phi^\dagger(y),\phi(x)]=i\frac{(1-d)}{\eta}\delta(\vec{x}-\vec{y}),$$

we get

$$\begin{aligned} \frac{a_3}{3}\eta^{1-d}[\gamma_{ij}\nabla^\eta\phi\nabla_\eta\nabla_\eta\phi^\dagger(y),\phi(x)] &= \frac{a_3}{3}\frac{i}{\eta}\delta_{ij}\partial_\eta\phi(y)\delta(\vec{x}-\vec{y})-i\frac{a_3}{3}\frac{(1-d)}{\eta}\delta_{ij}\delta(\vec{x}-\vec{y})\partial_\eta\phi(y) \\ &= i\frac{a_3}{3}\frac{d}{\eta}\delta_{ij}\delta(\vec{x}-\vec{y})\partial_\eta\phi(y). \end{aligned}$$

The $(\phi^\dagger \leftrightarrow \phi)$ also contribute in the third term. It gives

$$\begin{aligned} -\frac{a_3}{3}\eta^{1-d}[\gamma_{ij}\nabla_\eta\phi^\dagger\nabla_\eta\nabla^\eta\phi(y),\phi(x)] &= i\frac{a_3}{3}\gamma_{ij}\gamma^{\eta\eta}(\partial_\eta\partial_\eta-\Gamma^\eta_{\eta\eta}\partial_\eta)\phi(y)\delta(\vec{x}-\vec{y}) \\ &=-\ i\frac{a_3}{3}\delta_{ij}(\partial_\eta\partial_\eta+\frac{1}{\eta}\partial_\eta)\phi(y)\delta(\vec{x}-\vec{y}) \end{aligned}$$

So, the 3th term is

$$\begin{aligned} \frac{a_3}{3}[\gamma_{ij}\nabla_\alpha\phi\nabla_\eta\nabla^\alpha\phi^\dagger(y),\phi(x)]-(\phi^\dagger\leftrightarrow\phi) &= -i\frac{a_3}{3}\Big(\gamma_{ij}\partial_k\phi(y)\partial^k\delta(\vec{x}-\vec{y})-\frac{d}{\eta}\delta_{ij}\delta(\vec{x}-\vec{y})\partial_\eta\phi(y) \\ &+ \delta_{ij}(\partial_\eta\partial_\eta+\frac{1}{\eta}\partial_\eta)\phi(y)\delta(\vec{x}-\vec{y})\Big) \end{aligned} \tag{C.67}$$

The last term is only different than zero for the $(\phi^\dagger \leftrightarrow \phi)$, that's why we have a minus sign below

$$-\frac{a_4}{3}\eta^{1-d}[\gamma_{ij}\phi_2\nabla_\eta\phi^\dagger(y),\phi^\dagger(x)] = \frac{a_4}{3}i\gamma_{ij}\phi(y)\delta(\vec{x}-\vec{y}). \tag{C.68}$$

Finally we can put everything together :

$$\begin{aligned}
[Q^{(2)},\phi(x)] &= \int d^d y p^i p^j \eta^{1-d}[J_{\eta ij}(y),\phi(x)] \\
&= \int d^d y p^i p^j \Big[ia_1\phi(y)\Big(\partial_i\partial_j\delta(\vec{x}-\vec{y}) + \frac{(2+3d)}{3\eta^2}\delta_{ij}\delta(\vec{x}-\vec{y})\Big) \\
&- i\frac{a_2}{3}\Big(\partial_i\delta(\vec{x}-\vec{y})\partial_j\phi(y) + \partial_j\delta(\vec{x}-\vec{y})\partial_i\phi(y)\Big) + i\frac{a_2}{3}\partial_i\partial_j\phi(y)\delta(\vec{x}-\vec{y}). \\
&- i\frac{a_3}{3}\Big(\gamma_{ij}\partial_k\phi(y)\partial^k\delta(\vec{x}-\vec{y}) - \frac{d}{\eta}\delta_{ij}\delta(\vec{x}-\vec{y})\partial_\eta\phi(y) + \delta_{ij}(\partial_\eta\partial_\eta + \frac{1}{\eta}\partial_\eta)\phi(y)\delta(\vec{x}-\vec{y})\Big) \\
&+ i\frac{a_4}{3}\gamma_{ij}\phi(y)\delta(\vec{x}-\vec{y})\Big)
\end{aligned}$$

Then,

$$\begin{aligned}
[Q^{(2)},\phi(x)] &= i\Big[p^2\frac{a_1(2+3d)}{3\eta^2}\phi(x) + a_1 p^i p^j\partial_i\partial_j\phi(x) \\
&+ a_2 p^i p^j\partial_i\partial_j\phi(x) \\
&+ i\frac{a_3}{3}\Big(p^i p^j\gamma_{ij}\partial^k\partial_k\phi(y) + \frac{d}{\eta}p^2\partial_\eta\phi(y) - p^2(\partial_\eta\partial_\eta + \frac{1}{\eta}\partial_\eta)\phi(y)\Big) \\
&+ p^2\frac{a_4}{3}\frac{\ell^2}{\eta^2}\phi(x)\Big]
\end{aligned}$$

Or

$$\begin{aligned}
[Q^{(2)},\phi(x)] &= ip^2\Big(-\frac{a_3}{3}\partial_\eta^2 + \frac{a_3(d-1)}{3\eta}\partial_\eta + \frac{(a_1(2+3d)+a_4\ell^2)}{3\eta^2}\Big)\phi(x) \\
&+ (a_2+a_1)\partial_p^2\phi(x) + \frac{a_3}{3}p^2\partial_l\partial^l\phi(x)
\end{aligned} \tag{C.69}$$

Here we used that $\gamma_{ij} = \frac{\ell^2}{\eta^2}\delta_{ij}$, and $p^2 = \delta_{ij}p^i p^j$ and $\partial^l\partial_l = \delta^{lk}\partial_l\partial_k$. Inserting back the coefficients determined by the previous sections we obtain

$$[Q^{(2)},\phi(x)] = -i\frac{|\vec{p}|^2}{2(d+2)}\Big[-\partial_\eta^2 + \frac{(d-1)}{\eta}\partial_\eta - \frac{M^2\ell^2}{\eta^2} + p^2\delta^{ij}\partial_i\partial_j\Big]\phi(x) + i\partial_p^2\phi(x). \tag{C.70}$$

It turns out that the term in parenthesis above is precisely the Klein-Gordon operator for a massive scalar field in a de Sitter background. So, on-shell

$$[Q^{(2)},\phi(x)] = i\partial_p^2\phi(x). \tag{C.71}$$

C.4. Spin-4 current for a massive scalar field in d+1 dimensions

Just to give a taste we construct the spin-4 current

$$\begin{aligned} J_{\mu\nu\lambda\sigma} &= \Big[c_1\phi_1\nabla_{(\mu}\nabla_\nu\nabla_\lambda\nabla_{\sigma)}\phi_2 + c_2(1\leftrightarrow 2)\Big] + \Big[c_3\nabla_{(\mu}\phi_1\nabla_\nu\nabla_\lambda\nabla_{\sigma)}\phi_2 + c_4(1\leftrightarrow 2)\Big] \\ &+ c_5\nabla_{(\mu}\nabla_\nu\phi_1\nabla_\lambda\nabla_{\sigma)}\phi_2 + c_6\ell^{-4}\gamma_{(\mu\nu}\gamma_{\lambda\sigma)}\phi_1\phi_2 \\ &+ \ell^{-2}\Big(c_7\gamma_{(\mu\nu}\phi_1\nabla_\lambda\nabla_{\sigma)}\phi_2 + c_8\gamma_{(\mu\nu}\nabla_\lambda\phi_1\nabla_{\sigma)}\phi_2 + (1\leftrightarrow 2)\Big) \\ &+ \textit{contractions} \end{aligned}$$

The full current is

$$\begin{aligned} J_{\mu\nu\lambda\sigma} &= \Big[c_1\phi_1\nabla_{(\mu}\nabla_\nu\nabla_\lambda\nabla_{\sigma)}\phi_2 + c_2(1\leftrightarrow 2)\Big] + \Big[c_3\nabla_{(\mu}\phi_1\nabla_\nu\nabla_\lambda\nabla_{\sigma)}\phi_2 + c_4(1\leftrightarrow 2)\Big] \\ &+ c_5\nabla_{(\mu}\nabla_\nu\phi_1\nabla_\lambda\nabla_{\sigma)}\phi_2 + c_6(\ell^{-4}+M^4)\gamma_{(\mu\nu}\gamma_{\lambda\sigma)}\phi_1\phi_2 \\ &+ \ell^{-2}\Big(c_7\gamma_{(\mu\nu}\phi_1\nabla_\lambda\nabla_{\sigma)}\phi_2 + c_8\gamma_{(\mu\nu}\nabla_\lambda\phi_1\nabla_{\sigma)}\phi_2 + (1\leftrightarrow 2)\Big) \\ &+ \Big[c_{10}\phi_1\gamma_{(\mu\nu}\nabla^\alpha\nabla_\lambda\nabla_{\sigma)}\nabla_\alpha\phi_2 + c_{11}\phi_1\gamma_{(\mu\nu}\nabla_\lambda\nabla^\alpha\nabla_{\sigma)}\nabla_\alpha\phi_2 \\ &+ c_{12}\phi_1\gamma_{(\mu\nu}\nabla_\lambda\nabla_{\sigma)}\Box\phi_2 + c_{13}\phi_1\gamma_{(\mu\nu}\Box\nabla_\lambda\nabla_{\sigma)}\phi_2 + (1\leftrightarrow 2)\Big] \\ &+ \Big[c_{17}\gamma_{(\mu\nu}\nabla_\lambda\phi_1\nabla^\alpha\nabla_{\sigma)}\nabla_\alpha\phi_2 + c_{18}\gamma_{(\mu\nu}\nabla_\lambda\phi_1\nabla_{\sigma)}\Box\phi_2 + c_{19}\gamma_{(\mu\nu}\nabla_\lambda\phi_1\Box\nabla_{\sigma)}\phi_2 + (1\leftrightarrow 2)\Big] \\ &+ \Big[c_{23}\gamma_{(\mu\nu}\nabla^\alpha\phi_1\nabla_\alpha\nabla_\lambda\nabla_{\sigma)}\phi_2 + c_{24}\gamma_{(\mu\nu}\nabla^\alpha\phi_1\nabla_\lambda\nabla_{\sigma)}\nabla_\alpha\phi_2 + (1\leftrightarrow 2)\Big] \\ &+ c_{26}\gamma_{(\mu\nu}\nabla^\alpha\nabla_\lambda\phi_1\nabla_\alpha\nabla_{\sigma)}\phi_2 + c_{27}\nabla^\alpha\nabla^\beta\phi_1\nabla_\alpha\nabla_\beta\phi_2 \\ &+ \gamma_{(\mu\nu}\gamma_{\lambda\sigma)}\Big[c_{28}\nabla^\alpha\phi_1\Box\nabla_\alpha\phi_2 + c_{29}M^2\nabla^\alpha\phi_1\nabla_\alpha\phi_2 + c_{30}\phi_1\nabla^\alpha\Box\nabla_\alpha\phi_2 + (1\leftrightarrow 2)\Big]. \end{aligned} \tag{C.72}$$

D. Higgs potential and the plateau-like region

Here we explicitly construct the models mentioned in chapter D.4. We start with models that coincide with the Higgs potential for small values of the scalar field ($\varphi << M_{pl}$) and by imposing a $SO(1,1)$ symmetry at large field values we get a plateau-like potential. We explain in detail why the very well known models with a plateau-like behavior at large field values usually predict a small value for the tensor to scalar ratio. In the end of this appendix an analysis of two non-minimally coupled scalars field with the curvature tensor is made.

D.1. The Starobinsky action rewritten with an auxiliary field

We rewrite the Starobinsky model:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R+\beta R^2) \tag{D.1}$$

linearizing it with an auxiliary field,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}[R(1+2\beta\alpha)-\beta\alpha^2] \tag{D.2}$$

The α field can be viewed as a Lagrange multiplier because the equation of motion for it is a constraint equation (no kinetic term for α):

$$\frac{\delta S}{\delta\alpha} = 0 \Rightarrow \alpha - R = 0 \tag{D.3}$$

Putting this constraint back in action (**??**) we get (D.1).
After a conformal transformation,

$$g^E_{\mu\nu} = [1+2\beta\alpha]g_{\mu\nu} = \Omega^{-2}g_{\mu\nu} \tag{D.4}$$

The Ricci tensor transforms as

$$R[g_{\mu\nu}] = [1+2\beta\alpha](R_{\mu\nu}[g^E_{\mu\nu}] - 6g^{\mu\nu}\nabla^E_\mu\nabla^E_\nu\ln(\sqrt{1+2\beta\alpha}) - 6g^{\mu\nu}_E\nabla_\mu\ln\Omega\nabla_\nu\ln\Omega). \tag{D.5}$$

So,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[R - \frac{6\beta^2\partial^\mu\alpha\partial_\mu\alpha}{(1+2\beta\alpha)^2} - \frac{\beta\alpha^2}{(1+2\beta\alpha)^2}\Big] \tag{D.6}$$

Changing variables in order to get a canonical kinetic term for the scalar field, $\varphi = \sqrt{\frac{3}{2}}\ln(1+2\beta\alpha) \Rightarrow \alpha = \frac{e^{\sqrt{2/3}\varphi}-1}{2\beta}$. Thus,

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g^E}\Big[R^E - \partial^\mu\varphi\partial_\mu\varphi - \frac{1}{4\beta}(1-e^{-\sqrt{2/3}.\varphi})^2\Big]. \tag{D.7}$$

D.2. Potentials that glue Higgs with a plateau potential

As we saw in Chapter 4 a general class of potentials that can glue the Higgs potential for low field values and a plateau at high field values is given by,

$$V = \lambda[f(\phi/\chi)\phi^2 - g(\phi/\chi)\chi^2]^2. \tag{D.8}$$

Let us consider two explicitly examples given by formulas (4.45) and (4.37).
Example 1:

$$V_1 = \lambda\left\{\phi^2 - \left[\omega^2 + (1-\omega^2)\sin\left(\frac{\pi}{2}\left(\frac{\phi}{\chi}\right)^n\right)\right]\chi^2\right\}^2. \tag{D.9}$$

This potential glue the Higgs and Inflationary potentials. To see it explicitly we choose

$$\phi = \sqrt{6}M_{pl}\sinh(\varphi/\sqrt{6}M_{pl}), \tag{D.10}$$
$$\chi = \sqrt{6}M_{pl}\cosh(\varphi/\sqrt{6}M_{pl}), \tag{D.11}$$

and take the limit $\varphi/M_{pl} << 1$. In this case,

$$\sin\left(\frac{\pi}{2}(\frac{\varphi}{\sqrt{6}M_{pl}})^n\right) \simeq \frac{\pi}{2}\left(\frac{\varphi}{\sqrt{6}M_{pl}}\right)^n \tag{D.12}$$

and, $\phi \simeq \varphi$, $\chi = \sqrt{6}M_{pl}$. So

$$V_1 = \lambda\left\{\varphi^2 - 6\omega^2 M_{pl}^2\right\}^2, \tag{D.13}$$

which recover the Higgs potential for $n > 2$.
On the other way for the gauge fixed choices,

$$\phi = \sqrt{6}\sinh(\varphi/\sqrt{6}M_{pl}), \tag{D.14}$$
$$\chi = \sqrt{6}\cosh(\varphi/\sqrt{6}M_{pl}), \tag{D.15}$$

implies (in the $\varphi \to \infty$ limit) that

$$\sin\left(\frac{\pi}{2}\left(\frac{\phi}{\chi}\right)^n\right) \to 1. \tag{D.16}$$

Thus,

$$V_1 = \lambda(\phi^2 - \chi^2)^2 = 36\lambda. \tag{D.17}$$

This gives us a cosmological constant term (a flat region of the potential).
The next natural step is to compute the slow-roll parameters. For it, we use that,

$$\tanh^n(\varphi/\sqrt{6}) \longrightarrow 1 - 2ne^{\frac{-2\varphi}{\sqrt{6}}}, \tag{D.18}$$
$$\sin(\frac{\pi}{2} - x) = \cos x, \tag{D.19}$$

then

$$\sin\left(\frac{\pi}{2} - n\pi e^{-2\varphi/\sqrt{6}M_{pl}}\right) \simeq 1 - \frac{n^2\pi^2}{2}e^{-4\varphi/\sqrt{6}M_{pl}}. \tag{D.20}$$

This results inside (D.9) gives

$$\begin{aligned}
V_1 &= 36\lambda\Big\{-1+\frac{e^{2\varphi/\sqrt{6}M_{pl}}+2}{4}(1-\omega^2)[1-1+\frac{n^2\pi^2}{2}e^{-4\varphi/\sqrt{6}M_{pl}}]\Big\}^2 \\
&= 36\lambda\Big\{-1+\frac{n^2\pi^2e^{-2\varphi/\sqrt{6}M_{pl}}}{8}\Big\}^2 \\
&= 36\lambda\Big\{1-\frac{n^2\pi^2e^{-2\varphi/\sqrt{6}M_{pl}}}{4}\Big\}.
\end{aligned} \tag{D.21}$$

The slow-roll parameters are now straightfoward to compute and are given by,

$$\epsilon \simeq \frac{n^4\pi^4}{48}e^{-4\varphi/\sqrt{6}M_{Pl}} \to \epsilon << 1, \tag{D.22}$$

$$\eta \simeq -\frac{n^2\pi^2}{6}e^{-2\varphi/\sqrt{6}M_{Pl}} \to \epsilon << |\eta| << 1. \tag{D.23}$$

Thus,

$$n_s - 1 \simeq 2\eta \simeq -\frac{1}{3}n^2\pi^2e^{-2\varphi/\sqrt{6}M_{pl}}, \tag{D.24}$$

$$r \simeq \frac{1}{3}n^4\pi^4e^{-4\varphi/\sqrt{6}M_{pl}} \simeq 3(n_s-1)^2. \tag{D.25}$$

The number of e-folds is,

$$N_e = -\int_{\varphi_0/M_{pl}}^{\varphi_f/M_{pl}} \frac{d\phi/M_{pl}}{\sqrt{2\epsilon}} \simeq \frac{6}{n^2\pi^2}e^{2\varphi_0/\sqrt{6}M_{pl}} \tag{D.26}$$

where we use again, φ_f, such that, $e^{2\varphi_f/\sqrt{6}M_{pl}} \sim 1$, at the end of inflation. So,

$$\frac{\varphi_0}{M_{pl}} = \frac{\sqrt{6}}{2}\ln\left(\frac{n^2\pi^2N_e}{6}\right). \tag{D.27}$$

If, $\varphi_0 = \phi_{CMB}$, then, $n_s - 1 = -\frac{1}{3}n^2\pi^2e^{-2\varphi_0/\sqrt{6}M_{pl}}$, which implies,

$$\frac{\varphi_0}{M_{pl}} = \frac{\sqrt{6}}{2}\ln\left(\frac{n^2\pi^2}{3(1-n_s)}\right). \tag{D.28}$$

From (D.27) and (D.28) we get,

$$N_e = \frac{2}{1-n_s}. \tag{D.29}$$

Also the running of the scalar tilt is

$$\frac{dn_s}{d\ln k} = -14\epsilon\eta + 2\eta\epsilon^2 + 2\xi^2 \simeq 2\xi^2, \tag{D.30}$$

which gives,

$$\frac{dn_s}{d\ln k} \simeq \frac{(n_s-1)^2}{2} \simeq 8\times 10^{-4}. \tag{D.31}$$

Finally, the λ parameter can be computed using

$$\frac{H_{inf}^2}{8\pi^2 \epsilon M_{pl}^2} \simeq 2.4 \times 10^{-4} \tag{D.32}$$

which implies,

$$\lambda = \frac{\pi^2}{2N_e^2} \times 2.4 \times 10^{-9}. \tag{D.33}$$

Here we see explicitly how a measurement of the tensor to scalar ratio implies the energy scale for inflation.[1]

Example 2: A two parameter generalization of (4.37)

$$V_2 = \lambda \Big\{ \phi^2 - \Big[\omega^2 \Big[1 - \Big(\frac{\phi}{\chi}\Big)^m \Big] + \Big(\frac{\phi}{\chi}\Big)^n \Big] \chi^2 \Big\}^2, \tag{D.34}$$

also glues the Higgs and cosmological constant limits. In conformal gauge $\chi = \sqrt{6} M_{pl}$ the potential becomes

$$V_2 = \lambda \Big\{ \phi^2 - \Big[\omega^2 \Big[1 - \Big(\frac{\phi}{\sqrt{6}M_{pl}}\Big)^m \Big] + \Big(\frac{\phi}{\sqrt{6}M_{pl}}\Big)^n \Big] 6M_{pl}^2 \Big\}^2, \tag{D.35}$$

and in the $\phi/M_{pl} << 1$ case we have

$$V_2 = \lambda(\phi^2 - 6\omega^2 M_{pl}^2)^2, \tag{D.36}$$

if $m, n > 2$. It is also trivial to see that it gives the cosmological constant in the $\varphi \to \infty$ limit. To compute the slow-roll parameters we use again

$$\phi = \sqrt{6} M_{pl} \sinh(\varphi/\sqrt{6} M_{pl}), \tag{D.37}$$
$$\chi = \sqrt{6} M_{pl} \cosh(\varphi/\sqrt{6} M_{pl}), \tag{D.38}$$

and the $\varphi \to \infty$ limit. Then

$$\begin{aligned} V_2 = 36\lambda \Big\{ & - 1 + \cosh^2(\varphi/\sqrt{6}M_{pl}) - \Big(\omega^2 \Big[1 - \tanh^m(\varphi/\sqrt{6}M_{pl}) \Big] \\ & + \tanh^n(\varphi/\sqrt{6}M_{pl}) \Big) \cosh^2(\varphi/\sqrt{6}M_{pl}) \Big\}^2. \end{aligned} \tag{D.39}$$

together with,

$$\begin{aligned} \cosh^2(\varphi/\sqrt{6}) &\longrightarrow \frac{e^{\frac{2\varphi}{\sqrt{6}}} + 2}{4}; \\ \tanh^n(\varphi/\sqrt{6}) &\longrightarrow 1 - 2n e^{\frac{-2\phi}{\sqrt{6}}} + 2n^2 e^{\frac{-4\phi}{\sqrt{6}}}. \end{aligned} \tag{D.40}$$

implies,

$$V_0 \simeq 36\lambda M_{pl}^4 \frac{(n-2)}{2} \Big[\frac{n-2}{2} - n(n-2) e^{\frac{-2\varphi}{\sqrt{6}M_{pl}}} \Big]. \tag{D.41}$$

The same result as (4.37) (see (4.41)). So as we claimed, this two parameter generalization gives nothing new.

[1] The measurement of $r = 16\epsilon$ would give us ϵ and we can plug it into the last equation in order to find λ.

D.3. How to obtain an arbitrary tensor to scalar ratio with plateau-like potentials?

It is well known that plateau-like potentials give rise to very small value for the tensor to scalar ratio. Based on the two examples above, we realize some pattern in the calculations and show explicitly here how to go around and propose plateau-like potentials but with an arbitrary value for the r parameter.

The two examples above, after taking the large field limit ($\varphi \to \infty$), are of the form

$$V = A(1 + Ce^{-D\phi}), \tag{D.42}$$

where $D > 0$.

$$\epsilon \simeq \frac{M_{pl}^2}{2} C^2 D^2 e^{-2D\phi}, \qquad \eta \simeq M_{pl}^2 C D^2 e^{-D\phi}. \tag{D.43}$$

So, $\epsilon << \eta$, and

$$n_s - 1 \simeq 2\eta \simeq 2M_{pl}^2 C D^2 e^{-D\phi}, \tag{D.44}$$

$$r \simeq 16\epsilon \simeq 8M_{pl}^2 C^2 D^2 e^{-2D\phi}. \tag{D.45}$$

So, we find a general relation between r and n_s parameters,

$$r \simeq \frac{2}{M_{pl}^2 D^2}(n_s - 1)^2. \tag{D.46}$$

The two examples in the first section of this appendix have, $D = 2/\sqrt{6}$, so it is natural that they have the same relation. Here we also see that any attempt to obtain an arbitrary value for the tensor to scalar ratio with a plateau-like potential implies a potential that has a different value for D. As can be seem in chapter 4, all natural plateau-like potentials at high field values give $D = 2/\sqrt{6}$, so non natural potentials are necessary[2]. Similarly, for the number of e-folds we have,

$$N_e = -\int_{\phi_0/M_{pl}}^{\phi_f/M_{pl}} \frac{d\phi/M_{pl}}{\sqrt{2\epsilon}} \simeq \frac{1}{M_{pl}^2 C D^2} e^{D\phi_0} \tag{D.47}$$

if $e^{D\phi_f} \sim 1$ at the end of inflation. Thus,

$$\phi_0 = \frac{1}{D} \ln(|C| D^2 N_e M_{pl}^2), \tag{D.48}$$

where we used $|C|$ because we extract the square root of C^2. If, $\phi_0 = \phi_{CMB}$, then, $n_s - 1 = 2M_{pl}^2 C D^2 e^{D\phi_0}$, which implies,

$$\phi_0 = \frac{1}{D} \ln\left(\frac{2|C|D^2 M_{pl}^2}{(1 - n_s)}\right). \tag{D.49}$$

From (D.48) and (D.49) we obtain,

$$N_e = \frac{2}{1 - n_s}. \tag{D.50}$$

[2] The case of non-natural potentials seemed to be preferred by the BICEP2 data. After the release of the data for the dust obtained by the Planck satellite the natural potentials are again in agreement with the more recent data.

Equation (D.50) is exactly the results obtained for the potentials (4.37) and (4.45). It is interesting that it does not depend of any of the parameters A, C or D. For,

$$\frac{dn_s}{d\ln k} = -14\epsilon\eta + 2\eta\epsilon^2 + 2\xi^2 \simeq 2\xi^2, \tag{D.51}$$

where,

$$\xi^2 \equiv M_{pl}^4 \frac{V'V'''}{V^2} \tag{D.52}$$

we have,

$$\xi^2 \equiv M_{pl}^4 C^2 D^4 e^{-2D\phi}. \tag{D.53}$$

Since, $\frac{(n_s-1)}{2CD^2M_{pl}^2} \simeq e^{-D\phi}$,

$$\frac{dn_s}{d\ln k} \simeq \frac{(n_s-1)^2}{2} \simeq 8\times 10^{-4}. \tag{D.54}$$

It also does not depend of A, C or D and match the results of the examples. The λ could be different because,

$$\frac{H_{inf}^2}{8\pi^2\epsilon M_{pl}^2} \simeq 2.4\times 10^{-4} \tag{D.55}$$

and using,

$$\begin{aligned} V &= A[1+C]e^{-D\phi} \simeq \lambda A', \qquad \epsilon \simeq M_{pl}^2 C D^2 e^{-D\phi} \\ N_e &= \frac{1}{M_{pl}^2 C} e^{D\phi}, \qquad H_{inf}^2 = V/3M_{pl}^2, \end{aligned} \tag{D.56}$$

we obtain,

$$\lambda = \frac{12\pi^2 M_{pl}^2}{A' D^2 N_e^2} \times 2.4\times 10^{-9}. \tag{D.57}$$

It is easy to check that the result (D.57) recovers equation (D.33)[3]. It also becomes clear why they differ from each other.

D.4. A more general class of potentials

We can also think in a more general way to deform the potential and still have Weyl symmetry. The way we do it is by the following potential,

$$V = \lambda(1 + F(\phi/\chi))[f(\phi/\chi)\phi^2 - g(\phi/\chi)\chi^2]^2. \tag{D.58}$$

The simplest example of it is to choose is a power law function, $F(\phi/\chi) = (\phi/\chi)^p$ together with, $f(x) = 1$, and, $g(x) = \omega^2 + (1-\omega^2)x^n$, for $n > 2$. So,

$$V = \lambda(1 + (\phi/\chi)^p)\left[\phi^2 - \omega^2\chi^2 - (1-\omega^2)\left(\frac{\phi}{\chi}\right)^n \chi^2\right]^2. \tag{D.59}$$

[3]Here A' is the number that multiplies λ in the potential we want to use. For example, in equation (D.21) $A' = 36$.

It is easy to see that it recover the Higgs potential in the, $\varphi/M_{pl} << 1$, limit after fixing the gauge,

$$\phi = \sqrt{6}M_{pl}\sinh(\varphi/\sqrt{6}M_{pl}), \tag{D.60}$$
$$\chi = \sqrt{6}M_{pl}\cosh(\varphi/\sqrt{6}M_{pl}). \tag{D.61}$$

In fact,

$$\begin{aligned} V_2 &= 36\lambda M_{pl}^4(1+\tanh^p(\varphi/\sqrt{6}M_{pl}))\Big\{\sinh^2(\varphi/\sqrt{6}M_{pl}) \\ &- \Big[\omega^2 + (1-\omega^2)\tanh^n(\varphi/\sqrt{6}M_{pl})\Big]\cosh^2(\varphi/\sqrt{6}M_{pl})\Big\}^2 \\ &= \lambda(1+(\varphi/\sqrt{6}M_{pl})^p)\Big\{\varphi^2 - \Big[\omega^2 M_{pl}^2 + (1-\omega^2)(\varphi/\sqrt{6}M_{pl})^n M_{pl}^2\Big](6+\varphi^2/M_{pl}^2)\Big\} \\ &= \lambda(\varphi^2 - 6\omega^2 M_{pl}^2)^2, \end{aligned} \tag{D.62}$$

where we use, $\varphi/M_{pl} << 1$, and considered, $n > 2$.

Now, in the limit $\varphi \to \infty$, we get

$$V = \lambda\Big[\Phi^2 - \chi^2\Big]^2 = 36M_{pl}^4\lambda. \tag{D.63}$$

So the scalar field mimics a cosmological constant.

In the region where we have a plateau, the potential is

$$\begin{aligned} V &= \lambda(1+1-2pe^{-2\varphi/\sqrt{6}M_{Pl}})36M_{Pl}^4\frac{n-2}{2}\left[\frac{n-2}{2} - n(n-2)e^{-2\varphi/\sqrt{6}M_{Pl}}\right] \\ &- 36\lambda M_{Pl}^4(n-2)\left[\frac{n-2}{2} - n(n-2)e^{-2\varphi/\sqrt{6}M_{Pl}}\right] - 36\lambda p M_{Pl}^4\frac{(n-2)^2}{2}e^{-2\varphi/\sqrt{6}M_{Pl}} \\ &= 36\lambda M^4(n-2)^2\left[\frac{1}{2} - [n+p/2]e^{-2\varphi/\sqrt{6}M_{Pl}}\right], \end{aligned} \tag{D.64}$$

which also is of the form, $V = A[1+Ce^{-D\phi}]$, with $D = 2/\sqrt{6}$. So, even this non trivial deformation of the potential will give a small tensor to scalar ratio.

D.5. Non-minimal conformal coupling and attractor behavior

First consider the action

$$\mathcal{L}\sqrt{-g}\Big\{\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{\chi^2}{12}R(g) - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{\phi^2}{12}R(g) - \frac{1}{36}[f(\phi/\chi)\phi^2 - g(\phi/\chi)\chi^2)]^2\Big\}. \tag{D.65}$$

It is invariant under the transformations,

$$\tilde{g}_{\mu\nu} = e^{-2\theta}g_{\mu\nu}, \quad \tilde{\chi} = e^{\theta}\chi, \quad \tilde{\phi} = e^{-\theta}\phi. \tag{D.66}$$

When f and g are constant and equal we also have $SO(1,1)$ symmetry.

Let's check it. First remember how the Ricci tensor changes under the transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$,

$$\begin{aligned} \tilde{R} = \Omega^{-2}[R &- 2(D-1)g^{\mu\nu}\nabla_\mu\nabla_\nu\ln\Omega \\ &- (D-2)(D-1)g^{\mu\nu}\nabla_\mu\ln\Omega\nabla_\nu\ln\Omega]. \end{aligned} \tag{D.67}$$

In our case $\Omega = e^{-\theta}$ and $D = 4$, so

$$\tilde{R} = e^{2\theta}[R + 6g^{\mu\nu}\nabla_\mu\nabla_\nu\theta + 6g^{\mu\nu}\nabla_\mu\theta\nabla_\nu.\theta] \tag{D.68}$$

Then,

$$\frac{\tilde{\chi}^2}{12}\tilde{R} = e^{4\theta}\frac{\chi^2}{12}(R + 6\partial^\mu\partial_\mu\theta - 6\partial_\mu\theta\partial^\mu\theta). \tag{D.69}$$

The kinetic term changes as

$$\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\chi\partial^\mu\chi = \frac{1}{2}e^{4\theta}[\partial^\mu\theta\partial_\mu\theta\chi^2 + \partial_\mu\chi\partial^\mu\chi + 2\partial_\mu\chi\partial^\mu\theta\chi]. \tag{D.70}$$

The determinant changes easily

$$\sqrt{-\tilde{g}} = e^{-4\theta}\sqrt{-g}. \tag{D.71}$$

And finally, the potential changes to

$$\frac{1}{36}[f(\tilde{\phi}/\tilde{\chi})\tilde{\phi}^2 - g(\tilde{\phi}/\tilde{\chi})\tilde{\chi}^2)]^2 = \frac{e^{4\theta}}{36}[f(\phi/\chi)\phi^2 - g(\phi/\chi)\chi^2)]^2. \tag{D.72}$$

The last thing to note is that the term $\frac{\chi^2}{2}\partial^\mu\partial_\mu\theta = -\chi\partial^\mu\chi\partial_\mu\theta$ after an integration by parts, so the second term of (D.69) cancels the third term of (D.70) in the final expression for the Lagrangian (D.65). An analogous thinking can be made for ϕ field and we check the invariance of the action under the transformations (D.66).

Next, if $f = g = \lambda$, with λ constant, we can rewrite the action as

$$\mathcal{L} = \sqrt{-g}\left[\frac{\eta^{ab}}{2}\partial_\mu\Phi_a\partial^\mu\Phi_b + \frac{\eta^{ab}}{12}\Phi_a\Phi_b R(g) - \frac{\lambda}{36}(-\eta^{ab}\Phi_a\Phi_b)^2\right] \tag{D.73}$$

where $\Phi \equiv (\phi, \chi)$ and $\eta^{\mu\nu} = diag(-1, 1)$, which is clearly invariant under $SO(1,1)$ transformations.

The requirement to have a plateau in the large field values puts constraints in the functions f and g. In the $\chi^2 - \phi^2 = 6$ gauge ($\chi = \sqrt{6}\cosh\varphi/\sqrt{6}$, $\phi = \sqrt{6}\sinh\varphi/\sqrt{6}$) we immediately see a restriction on functions f and g (which were obeyed in our examples), that is, $f(\tanh\varphi) \longrightarrow \sqrt{\lambda}$ (constant), and $g(\tanh\varphi) \longrightarrow \sqrt{\lambda})$ (the same constant as f) when $\varphi \to \infty$ (limit for inflation). In fact, with this restriction the lagrangian density becomes

$$\mathcal{L} = \sqrt{-g}\left[\frac{R}{2} - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \lambda\right]. \tag{D.74}$$

Note also that the $SO(1,1)$ symmetry is restored asymptotically if the f and g functions satisfies the above constraints.

It will be instructive to consider the $\chi(x) = \sqrt{6}$ gauge in the Lagrangian (D.65) also, in order to discuss the non-minimal coupling later. In this gauge (D.65) becomes

$$\mathcal{L} = \sqrt{-g_J}\left\{\frac{1}{2}\left(1 - \frac{\phi^2}{6}\right)R(g_J) - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - [f(\phi/\sqrt{6})\phi^2/6 - g(\phi/\sqrt{6}))]^2\right\} \tag{D.75}$$

The above action is said to be in the Jordan frame. By a change of variables we can bring the above action to the Einstein frame. In fact, if $g^{\mu\nu} = (1 - \phi^2/6)^{-1}g_J^{\mu\nu}$ the ϕ coupling with R disappear. To see it, remember that

$$\begin{aligned}\tilde{R} = \Omega^{-2}[R &- 2(D-1)g^{\mu\nu}\nabla_\mu\nabla_\nu\ln\Omega \\ &- (D-2)(D-1)g^{\mu\nu}\nabla_\mu\ln\Omega\nabla_\nu\ln\Omega],\end{aligned} \tag{D.76}$$

but now $\Omega^2 = (1-\phi^2/6)^{-1}$ and $D = 4$, so

$$\begin{aligned}\mathcal{L} &= \sqrt{-g}\Big\{\frac{1}{2}R(g) - \frac{3}{4}g^{\mu\nu}\nabla_\mu(1-\phi^2/6)\nabla_\nu(1-\phi^2/6) - (1-\phi^2/6)^{-2}\frac{1}{2}\partial^\mu\phi\partial_\mu\phi \\ &- (1-\phi^2/6)^{-2}[f(\phi/\sqrt{6})\phi^2/6 - g(\phi/\sqrt{6}))]^2\Big\},\end{aligned} \tag{D.77}$$

now use

$$\nabla_\nu \ln(1-\phi^2/6) = \frac{-\phi\nabla_\nu\phi}{3(1-\phi^2/6)} \tag{D.78}$$

and

$$\begin{aligned}\mathcal{L} &= \sqrt{-g}\Big\{\frac{1}{2}R(g) - \frac{\phi^2}{12}\frac{\nabla^\nu\phi\nabla_\nu\phi}{(1-\phi^2/6)^2} - \frac{(1-\phi^2/6)\nabla^\mu\phi\nabla_\mu\phi}{2(1-\phi^2/6)^2} \\ &- (1-\phi^2/6)^{-2}[f(\phi/\sqrt{6})\phi^2/6 - g(\phi/\sqrt{6}))]^2\Big\},\end{aligned} \tag{D.79}$$

so

$$\begin{aligned}\mathcal{L} &= \sqrt{-g}\Big\{\frac{1}{2}R(g) - \frac{1}{2}\frac{\nabla^\nu\phi\nabla_\nu\phi}{(1-\phi^2/6)^2} \\ &- (1-\phi^2/6)^{-2}[f(\phi/\sqrt{6})\phi^2/6 - g(\phi/\sqrt{6}))]^2\Big\}.\end{aligned} \tag{D.80}$$

Thus, by defining a canonically normalized field by φ such that

$$\frac{d\varphi}{d\phi} = \frac{1}{(1-\phi^2/6)} \quad \Rightarrow \quad \phi = \tanh(\varphi/\sqrt{6}), \tag{D.81}$$

we get,

$$\mathcal{L} = \sqrt{-g}\Big\{\frac{R(g)}{2} - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - V(\phi(\varphi))\Big\}, \tag{D.82}$$

where,

$$V(\phi(\varphi)) = \frac{[f(\tanh\varphi/\sqrt{6})\frac{\tanh^2\varphi/\sqrt{6}}{6} - g(\tanh\varphi/\sqrt{6})]^2}{(1-\frac{\tanh^2\varphi/\sqrt{6}}{6})^2}. \tag{D.83}$$

Supposing that when $x \to \infty$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) \equiv \sqrt{H(x)} = \sqrt{\lambda}$

$$\begin{aligned}V(\phi(\varphi)) &\to H(\tanh\varphi/\sqrt{6})\Big\{\Big(1-\frac{\tanh^2\varphi/\sqrt{6}}{6}\Big)^2\Big(1-\frac{\tanh^2\varphi/\sqrt{6}}{6}\Big)^{-2}\Big\} \\ &= \lambda.\end{aligned} \tag{D.84}$$

Now we consider a generalization of the Jordan frame changing the conformal coupling $\xi = -1/6$ by a general $\xi < 0$ parameter. In this case the Lagrangian density becomes

$$\mathcal{L} = \sqrt{-g_J}\Big\{\frac{R(g_J)}{2}(1+\xi\phi^2) - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - [f(\sqrt{|\xi|}\phi)\xi\phi^2 + g(\sqrt{|\xi|}\phi))]^2\Big\} \tag{D.85}$$

D. Higgs potential and the plateau-like region

By a change of variables $g^{\mu\nu} = (1+\xi\phi^2)^{-1} g_J^{\mu\nu}$, we get

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}\Big\{\frac{R(g)}{2} - \frac{1}{2}\Big[\frac{1+\xi\phi^2+6\xi^2}{(1+\xi\phi^2)^2}\Big]\nabla^\nu\phi\nabla_\nu\phi \\ &- (1+\xi\phi^2)^{-2}[f(\phi/\sqrt{6})\phi^2/6 - g(\phi/\sqrt{6})]^2\Big\} \end{aligned} \tag{D.86}$$

and by defining a canonical normalized field

$$\frac{d\varphi}{d\phi} = \frac{\sqrt{1+\xi\phi^2+6\xi^2\phi^2}}{(1+\xi\phi^2)} \tag{D.87}$$

$$\mathcal{L} = \sqrt{-g}\Big\{\frac{R(g)}{2} - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - H(\sqrt{|\xi|}\phi(\varphi))\Big\}. \tag{D.88}$$

In the boundary of moduli space $M_{pl}^2 - \xi\phi^2 << M_{pl}^2$ then we can rewrite (D.87) as

$$\frac{d\varphi}{d\phi} = \frac{\sqrt{6}|\xi|\phi}{(1+\xi\phi^2)}. \tag{D.89}$$

The solution of this equation is easily obtained and gives

$$\phi^2 = \frac{1}{|\xi|}(1-e^{-\sqrt{2/3}|\varphi|}), \qquad -\infty < \varphi < \infty. \tag{D.90}$$

To see one example, consider the simple case where $f(\sqrt{|\xi|}\phi) = \lambda_1\xi\phi^2$ and $g(\sqrt{|\xi|}\phi) = \lambda_2\xi\phi^2$. In this case

$$V(\sqrt{|\xi|}\phi(\varphi)) = \frac{\lambda_2^2\xi^2\phi^4[1+\frac{\lambda_1}{\lambda_2}\xi\phi^2)]^2}{(1+\xi\phi^2)^2}. \tag{D.91}$$

We see that for the case $\lambda_1 = \lambda_2$ it reduces to a $V \sim \phi^4$ potential already treated in [20].

Using (D.90) inside (D.91) we get

$$\begin{aligned} V(\varphi) &\simeq \frac{\Big[\lambda_2^2(1-e^{-\sqrt{2/3}|\varphi|})^2 + 2\lambda_1\lambda_2(1-e^{-\sqrt{2/3}|\varphi|})^3 + \lambda_1^2(1-e^{-\sqrt{2/3}|\varphi|})^4\Big]}{2} \\ &\simeq \frac{\Big[\lambda_2^2(1-2e^{-\sqrt{2/3}|\varphi|}) + 2\lambda_1\lambda_2(1-3e^{-\sqrt{2/3}|\varphi|}) + \lambda_1^2(1-4e^{-\sqrt{2/3}|\varphi|})\Big]}{2} \\ &\simeq \frac{1}{2}\Big[(\lambda_1+\lambda_2)^2 - (4\lambda_1^2 + 2\lambda_2^2 + 6\lambda_1\lambda_2)e^{-\sqrt{2/3}|\varphi|}\Big] \end{aligned} \tag{D.92}$$

This is again of the form $V = A[1+Ce^{-D\phi}]$ that we already discussed for any values of the λ's.

E. Stochastic region

A detailed justification for the extra term in the equations of motion in the stochastic region is presented. Some details of the computations of the stochastic region for the chaotic model is also shown.

E.1. Equation of motion with a stochastic correction

When one takes into account the back-reaction of sub-Hubble quantum fluctuations, it gives a contribution to the classical equations of motion known as *stochastic term*. This term enters in the classical equation of motion as a white noise and it is basically the superposition of the sub-Hubble modes.

The idea is the following: take the classical equations of motion for a scalar field in a FLRW Universe

$$-\Box\varphi + 3H\dot{\varphi} + V'(\varphi) = 0. \tag{E.1}$$

If we decompose the scalar field into Fourier modes, the equation of motion becomes

$$\partial_t^2\varphi_k + 3H\partial_t\varphi_k - \frac{k^2}{a^2}\varphi_k + V'(\varphi_k) = 0. \tag{E.2}$$

It is important to note that if the derivative of the potential is linear in φ_k (which is the case for chaotic inflation $V \propto \varphi^2$), the solution for each Fourier mode will be independent of each other. Also, the field is spread everywhere, but we know that the super-Hubble modes ($k < H^{-1}$) have different dynamics than the sub-Hubble ($k > H^{-1}$) ones. In order to take into account this fact, we split the field as

$$\varphi = \varphi_c + \varphi_q \tag{E.3}$$

where φ_c is the superposition of the super-Hubble modes and φ_q is the superposition of the sub-Hubble modes.

Due to the dynamics of the Universe, the sub-Hubble modes can become super-Hubble and vice-versa. Then we should take into account in the dynamics of the super-Hubble modes and vice-versa. In the inflationary era, the dynamics of the super-Hubble modes are

$$-\Box\varphi_c + 3H\dot{\varphi}_c + V'(\varphi_c) + F_{stc} = 0. \tag{E.4}$$

This stochastic force is the stochastic term we mentioned before. This is a white noise for the classical equation of motion coming from stretched wave length modes.

Now, we show how to justify the coefficient of equation (5.1). If a field is slowly-rolling in a region where stochastic effects are important, the equation of motion for this field becomes

$$3H\dot{\varphi}_0 \approx -V' + F_{st}. \tag{E.5}$$

The stochastic force can be estimated in the regime where $F_{st} \gg F_{cl}$. In this regime we can integrate the above formula in a Hubble time,

$$3H \int_{\varphi_i}^{\varphi_e} d\varphi \approx \int_t^{t+H^{-1}} dt F_{st} \tag{E.6}$$

where we used the fact that in an almost exponential expanding universe the Hubble radius doesn't change significantly. Now, in a Hubble time, the field rolls $\Delta\varphi_{qu} \approx \frac{H}{2\pi}$, so

$$\begin{aligned} 3H\Delta\varphi_{qu} &\approx \int_t^{t+H^{-1}} F_{st} dt \\ F_{st} &\approx \frac{3H^3}{2\pi}. \end{aligned} \tag{E.7}$$

assuming that the stochastic force doesn't change significantly in a Hubble time.

E.2. Region where chaotic model becomes chaotic

For the chaotic model, $V = \frac{m^2\varphi^2}{2}$, the region where stochastic effects become dominant can be determined by

$$\begin{aligned} \frac{3}{2\pi} H^3 &> m^2\varphi \\ \left(\frac{V}{3m_{pl}^2}\right)^{3/2} &> \frac{2\pi}{3} m^2\varphi \\ \left(\frac{m^2\varphi^2}{6m_{pl}^2}\right)^{3/2} &> \frac{2\pi}{3} m^2\varphi \\ \left(\frac{m\varphi}{\sqrt{6}m_{pl}}\right)^3 &> \frac{2\pi}{3} m^2\varphi \\ \left(\frac{\varphi}{m_pl}\right)^2 &> \frac{2\pi}{3}(\sqrt{6})^3 \frac{m_{pl}}{m} \\ \left|\frac{\varphi}{m_pl}\right| &> \sqrt{4\pi}(6)^{1/4}\left(\frac{m_{pl}}{m}\right)^{1/2}. \end{aligned} \tag{E.8}$$

where from the first line to the second we use Friedmann's equation.

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