IFT

Instituto de Física Teórica
Universidade Estadual Paulista

TESE DE DOUTORAMENTO IFT–T.000/07

Eletrodinâmica de Ordem Superior em $(2+1)D$

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15 de Junho de 2007

Aos meus pais e à minha esposa.

Agradecimentos

- Ao Prof. Antonio José Accioly, tanto pela orientação científica, como pelos seus conselhos como amigo.
- A todos os funcionários, professores e amigos do IFT-UNESP, especialmente ao Ricardo, Matheus e Urbano e os colegas da sala 05 do Galpão.
- Ao pessoal da ICE-UNIFEI, em especial o Prof. Claudino e pessoal da secretaria pela ajuda no período em que trabalhei lá como professor. Por esse mesmo motivo agradeço a ajuda dos colegas do IFUSP.
- Ao prof. Sérgio Novaes e pessoal do São Paulo Regional Analysis Center (SPRACE).
- A todos os meus amigos, em especial os colegas do IFUSP (Dimas, Julian, Pedro, Sandro e pessoal da Física Matemática).
- Aos meu pais, à quem devo minha formação e à minha esposa, que suportava o mundo enquanto eu me concentrava neste trabalho.
- À CAPES pelo apoio finaceiro.

Resumo

Apresentamos, neste trabalho, alguns de nossos resultados relativos à formulação do Eletromagnetismo de Podolsky em $(2+1)$ dimensões, dedicando especial atenção ao estudo do espalhamento na aproximação não-relativística. Analisamos a possibilidade de existência de estados ligados na equação de Schrödinger independente do tempo no caso do espalhamento de duas partículas bosônicas (com ou sem o termo topológico de Chern-Simons) e fermiônicas; neste último caso calculamos, também, correções a temperatura finita. Finalmente consideramos a aplicação da teoria à regularização da Eletrodinâmica Quântica, também em $(3+1)$ dimensões, computando explicitamente a função-β do grupo de renormalização.

Palavras Chaves:

Eletrodinâmica de Podolsky; Física em $(2+1)$ Dimensões; Potencial Não-Relativístico; Estados Ligados; Função-β do Grupo de Renormalização.
Áreas do conhecimento: 1.05.03.01-3

Abstract

We present in this work a number of results in Poldolsky's formulation of the (2+1) dimensional Electromagnetism. We focus our attention on the non-relativistic regime of scatterings. We analyse the possibility of existence of bound states for the time-independent Schrödinger equation, by considering the scattering of two bosonic (with and without the Chern-Simons term) and fermionic particles; in the latter case, we also compute finite-temperature corrections. We finally consider the application of the theory to Quantum Electrodynamics, also in $(3 + 1)$ dimensions, and we explicitly work out the renormalization group β function in this case.

Sumário

Introdução

A Eletrodinâmica de Podolsky, resumida na Lagrangeana abaixo :

$$\mathcal{L}_{Pod} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda}, \tag{1}$$

foi apresentada em 1942 por Podolsky e Schwed [56, 57, 58]. O termo que difere da Eletrodinâmica usual e que é invariante de "gauge" leva à existência de uma massa eletromagnética para uma carga puntiforme consistente com a Relatividade Especial.

Teorias utilizando derivadas de ordem superior foram (e são!) muito utilizadas no contexto da regularização. Por exemplo, [46] propôs a seguinte Lagrangeana

$$\mathcal{L} = \frac{-1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\alpha}{4\Delta^2}D^\sigma F^{\mu\nu}D_\sigma F_{\mu\nu} - \frac{\beta}{4\Delta^2}D^\sigma D_\sigma F^{\mu\nu}D_\rho D^\rho F_{\mu\nu}$$

para manter as teorias de "gauge" (no caso, Yang-Mills) com quebra espontânea de simetria finitas quando renormalizadas em um "loop".

Boris Podoslsky nasceu em Taganrog, Rússia, no dia 29 de Junho de 1896. Contava 14 anos quando emigrou do seu país em 1911. Mais tarde, obteve seu Bacharelado em Engenharia Elétrica e Mestrado em Matemática na University of Southern California. Seu PhD, sob a orientação de Paul Sophius Epstein, foi obtido no California Institute of Technology. De 1928 a 1930, foi membro do Conselho Nacional de Pesquisa deste mesmo instituto e da Universidade de Leipzig, e, mais tarde nomeado "National Research Associate".

Retornando à União Soviética, tornou-se Diretor de Física Teórica do Instituto Ucraniano Físico-Técnico em Kharkov. Lá, publicou vários artigos com V. Fock e P.A.M. Dirac, e concebeu a idéia, junto com Lev Landau, de produzir um livro-texto que enfatizasse mais os postulados teóricos que as leis experimentais, mas o projeto foi interrompido quando, então, retornou aos EUA em 1933 e trabalho no Instituto de Estudos Avançados em Prin-

Outra abordagem é o seu uso como teorias efetivas. Citando alguns casos, no problema de confinamento de cor, Baker *et al*, [12] propõem que a teoria de Yang-Mills em grande distâncias possa ser aproximada por

$$\mathcal{L}_{eff} = -\frac{-1}{4M^2} F_{\mu\nu} D^2 F^{\mu\nu},$$

e ainda Alekseev *et al* [9], com o propósito de descrever o comportamento infravermelho do propagador do glúon, propuseram a Lagrangeana

$$\mathcal{L}_{eff} = \frac{1}{4M^2} D_\rho^{ab} F_{\mu\nu}^b D_\rho^{ac} F_{\mu\nu}^c + \frac{g}{M^2} f^{abc} F_{\mu\nu}^a F_{\nu\rho}^b F_{\rho\mu}^c.$$

Existem ainda muitas outras motivações para o estudo de teorias com derivadas de ordem superior, como possíveis aplicações em sistemas dinâmicos em teoria de campos [22] e esquemas de regularização que preservam a supersimetria.

A utilidade da Eletrodinâmica de Podolsky não se restringe somente a estes contextos. De fato, um sério problema existente na Eletrodinâmica Clássica aparece na equação não-relativística, derivada por Abraham (1903) e Lorentz (1904)

$$\frac{4}{3}\frac{U(a)}{c^2}\frac{d\vec{v}}{dt} - \frac{2e^2}{3c^3}\frac{d^2\vec{v}}{dt^2} = \vec{F}_{ext}, \tag{2}$$

onde $\vec{v}$ é a velocidade da partícula e $U(a)$ sua energia eletrostática. O primeiro fator de (2), $(4/3)U/c^2$ pode ser identificado como uma massa eletromagnética. Sendo assim, o fator 4/3 torna-se um problema quando queremos concilia-lo com o valor $m = U/c^2$ previsto pela Relatividade Especial. Esse

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ceton (contudo, mais tarde, ele retomou esta idéia escrevendo junto com K. Kunz o livro "Fundamentals of Electrodynamics"(1969); Landau e E. Lifshitz seguiram esta linha e produziram "The Classical Theory of Fields"(1951)). Tornou-se conhecido então pelo seu trabalho com Albert Einstein e Nathan Rose, "Can Quantum-Mechanical Description of Physical Reality be Considered Complet?", sobre o paradoxo EPR. Em 1935 tornou-se professor assistente de Física Matemática na Universidade de Cincinnati. Em 1948 orientou Philip Schwed em sua tese "Topics in Field Theory", e posteriormente, assumiu a posição de professor titular em 1951. Seu último trabalho foi como professor de Física na faculdade da Universidade Xavier em 1961, onde permaneceu até sua morte. Seus interesses eram teoria quântica, Eletrodinâmica, relatividade e teoria da informação. Também participou ativamente na Academia de Ciência em Ohio em 1950 sendo eleito membro em 1957. Faleceu em 28 de Novembro de 1966 nos EUA.

problema foi solucionado em [31, 32], supondo-se que a Lagrangeana de Podolsky seja adotada; sendo assim, o termo adicional remove o fator 4/3 para a massa eletromagnética, e leva a massa correta.

Por outro lado, a Física em duas dimensões espaciais tem sido muito estudada atualmente, desde o primeiro trabalho sobre partículas de variados "spins" em $(2+1)$ dimensões [47], principalmente quanto as suas aplicações na Física do Estado Sólido, que revela muito aspectos interessantes. Por exemplo, para construir uma teoria eletromagnética consistente, é necessário que nenhuma quantidade tenha dependência na variável espacial z. Para eliminar qualquer dependência do momento nesta componente, devemos impor $E_z = 0$. Como a partícula movimenta-se no plano (x, y), a existência de qualquer uma das componentes do campo magnético neste plano iria gerar, via produto vetorial, uma força na direção z! Logo devemos ter $E_z = B_x = B_y = 0$.

Outra característica é que o campo magnético $B = \epsilon^{ij}\partial_i A_j = \partial_1 A_2 - \partial_2 A_1$ é um pseudo-escalar, em vez de um pseudo-vetor como em $(3+1)D$ (como $\vec{A}$ é um vetor bidimensional, o rotacional em duas dimensões produz um escalar). $\vec{E} = -\vec{\nabla} A_0 - \vec{A}$ é um vetor bidimensional, logo temos a seguinte forma para o tensor eletromagnético:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & B \\ E_y & -B & 0 \end{pmatrix}.$$

O aqui exposto, torna o estudo de teorias de "gauge", com derivadas superiores no plano não-trivial, mas ainda temos outras motivações.

A relevância do estudo de uma teoria de "gauge" $(2+1)$-dimensional para a descrição de supercondutores em altas temperaturas foi sugerida pela possibilidade de que a estrutura em camadas dos cristais de óxido de cobre indica que o movimento dos elétrons seja efetivamente bidimensional. Neste caso, a Lagrangeana efetiva $(2+1)$-dimensional pode ser obtida da Lagrangeana quadridimensional, impondo, além dos argumentos acima, que a carga tridimensional ao quadrado seja proporcional a [60, 43]

$$e_3^2 \propto \frac{\alpha}{\delta}, \tag{3}$$

onde $\alpha = \frac{4\pi}{137}$ é a constante de estrutura fina (igual à carga do elétron quadridimensional, no sistema natural de unidades) e δ a distância interplanar dos supercondutores. Logo, ela também não é mais uma constante adimensional (utilizando o mesmo sistema de unidades), mas seu quadrado tem dimensão de massa, de modo a manter a ação adimensional no espaço tridimensional.

A Física no plano ainda revela a existência de um novo tipo de teoria de "gauge", a teoria de *Maxwell-Chern-Simons*,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{s}{4}\epsilon^{\lambda\mu\nu}F_{\lambda\mu}A_{\nu},$$

permitindo também a existência de uma massa, desta vez topológica, s, invariante de "gauge". O termo de Chern-Simons pode ser induzido pelas correções radiativas de um campo fermiônico, e surge como uma das principais propostas para o efeito Hall quântico fracionário [45].

Percebemos que o estudo da Eletrodinâmica de Podolsky neste contexto já seria suficientemente rico, se a perspectiva de existência de estados ligados, em analogia ao caso de "anti-gravidade" na gravitação em $(2+1)D$ [1], não viesse a se somar a esse panorama, pois tal propriedade permitiria uma melhor compreensão do fenômeno de surgimento de "pares de Cooper" em um supercondutor.

Atualmente, a explicação para o surgimento destes pares é a seguinte: um elétron passando próximo da rede cristalina interage com os íons positivos da rede, que devido às suas propriedades elásticas, emitem um fônon. Logo que um segundo elétron de momento oposto passar próximo a esta região de deformação, onde a densidade de carga positiva passa a ser maior, pode receber este fônon e, conseqüentemente, o momento do primeiro elétron, resultando na interação atrativa entre eles, podendo ser maior que repulsão Coulombiana, formando um "par de Cooper". Para isso, as auto-funções espaciais devem ser simétricas sob a troca de índices, o que ocorre quando os "spins" são anti-paralelos.

Neste trabalho examinaremos o comportamento da Eletrodinâmica de Podolsky em $(2+1)D$ e, em especial, a existência de estados ligados que possibilitariam aplicações à Física da Estado Sólido ou Física Nuclear. A organização deste trabalho é a seguinte:

No capítulo I faremos uma primeira investigação utilizando a teoria de Podolsky no espalhamento de duas partículas escalares. O cálculo do potencial será feito sem apelar para as soluções da equação de Maxwell modificadas pelo termo de Podolsky [2], mas usando a aproximação não-relativística deste espalhamento.

A adição de um termo de Chern-Simons é um elemento chave para a explicação de partículas com estatística indefinida, os “ânions”. O Capítulo II será consagrado a este estudo. Este capítulo contém ainda uma comparação do número de estados ligados com, ou sem, o termo de Podolsky.

A aplicação da Eletrodinâmica de Podolsky ao espalhamento de dois elétrons é feita no Capítulo III, onde ainda estudaremos os efeitos da temperatura na forma deste potencial.

Estudaremos no Capítulo IV, a aplicação desta teoria para determinação da função-β, fazendo sempre um paralelo com o cálculo em $(3+1)D$. Esse estudo poderá ajudar-nos a compreender o comportamento da carga renormalizada (um dos ingredientes do potencial eletromagnético) em relação à massa, como também avaliar a influência da distância interplanar nesta quantidade, cujas conseqüências podem ter impacto direto nos nosso resultados.

As nossas conclusões serão apresentadas logo após o Capítulo IV.

O sistema de unidades *natural* é empregado, portanto $\hbar = c = k_B = 1$, onde k_B é a constante de Boltzmann. Neste sistema $1m = 5,068 \times 10^{-15}\, GeV^{-1}$, $1kg = 5,610 \times 10^{25} GeV$, $1s = 1,52 \times 10^{-6} GeV^{-1}$ e $1K = 8,617 \times 10^{-14} GeV$. A métrica $\eta_{\mu\nu}$ quando tratamos do espaço-tempo quadridimensional é tal que $\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1$, e zero para os outros casos. Em nossa notação, $\Box = \partial^\mu \partial_\mu$. A métrica tridimensional é dada por $(+,-,-)$.

Capítulo 1

Eletrodinâmica de Ordem Superior em (2+1) Dimensões: Partículas Escalares

Visando determinar a existência, ou não, de estados ligados na Eletrodinâmica de Podolsky, iremos iniciar nosso estudos com o sistema mais simples: o espalhamento de duas partículas escalares.

1.1 Primeira Aplicação: Espalhamento na QED Escalar

Iremos, agora, analisar a teoria de Podolsky, utilizando a teoria quântica de campos, usando como exemplo a Eletrodinâmica Escalar. Para determinar o propagador, nesta teoria, podemos fixar o "gauge" usando uma das duas condições:

$$\begin{aligned} \partial_\mu A^\mu &= 0 \\ (1+\varepsilon a^2 \Box)\partial_\mu A^\mu &= 0. \end{aligned}$$

Em [57], afirma-seque a condição $\partial_\mu A^\mu = 0$ é mais restritiva que a segunda e, por isso, deve ser utilizada. Escolhemos primeiramente utilizar a segunda equação com um parâmetro ε, de forma a transitar entre uma condição e outra, se isso fizer alguma diferença nos cálculos.

Expandimos a Lagrangeana (1), adicionando o termo de fixação de "gauge" em termos dos potenciais, para obter o propagador:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda} - \frac{1}{2\alpha}[(1+\varepsilon a^2\Box)\partial_\mu A^\mu]^2 \qquad (1.1)\\
&= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{a^2}{2}\partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu)\partial^\lambda(\partial_\mu A_\lambda - \partial_\lambda A_\mu)\\
&- \frac{1}{2\alpha}[1 + 2\varepsilon a^2\Box + a^4\varepsilon^2\Box^2]\partial_\nu A^\nu\partial_\mu A^\mu\\
&= -\frac{1}{2}(\partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu) + \frac{a^2}{2}(\partial_\lambda\partial^\mu A^\lambda\partial^\nu\partial_\mu A_\nu - \partial_\nu\partial^\mu A^\nu\partial^\lambda\partial_\lambda A_\mu -\\
&- \partial_\nu\partial^\nu A^\mu\partial^\lambda\partial_\mu A_\lambda + \partial_\nu\partial^\nu A^\mu\partial^\lambda\partial_\lambda A^\mu) - \frac{1}{2\alpha}[1 + 2a^2\varepsilon\Box + a^4\varepsilon^2\Box^2]\partial_\nu A^\nu\partial_\mu A^\mu.
\end{aligned}$$

Fazendo integrações por partes, e eliminando os termos de superfície:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}A^\mu(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu + \frac{1}{2\alpha}A^\mu[1 + 2a^2\varepsilon\Box + a^4\varepsilon^2\Box^2]\partial_\mu\partial_\nu A^\nu \qquad (1.2)\\
&+ \frac{a^2}{2}A^\mu(\Box\partial_\mu\partial_\nu - \Box\partial_\mu\partial_\nu - \Box\partial_\nu\partial_\mu + \eta_{\mu\nu}\Box^2)A^\nu,
\end{aligned}$$

podemos escrever, no espaço de momentos,

$$\begin{aligned}
\mathcal{L}' &= \frac{A'^\mu}{2}\left[-\eta_{\mu\nu}k^2 + k_\mu k_\nu - \frac{1}{\alpha}(1 - a^2\varepsilon k^2)^2 k_\mu k_\nu + a^2(-k^2 k_\nu k_\mu + \eta_{\mu\nu}k^4)\right]A'^\nu\\
&= \frac{A'^\mu}{2}\mathcal{D}_{\mu\nu}A'^\nu. \qquad (1.3)
\end{aligned}$$

Como veremos mais tarde, os termos proporcionais a ε contêm $k_\mu k_\nu$, logo não contribuem para o cálculo do potencial não-relativístico. Iremos então, de agora em diante, continuar os cálculos com $\varepsilon = 0$:

$$\mathcal{L} = \frac{A_\mu}{2}\left[\eta^{\mu\nu}\Box + \left(\frac{1}{\alpha} - 1\right)\partial^\nu\partial^\nu - a^2(\eta^{\mu\nu}\Box^2 - \Box\partial^\mu\partial^\nu)\right]A_\nu. \qquad (1.4)$$

O cálculo do propagador exige, então, que determinemos a solução da seguinte equação no espaço de momentos:

$$-\left[\eta^{\mu\nu}k^2 + \left(\frac{1}{\alpha} - 1\right)k^\nu k^\mu - a^2(\eta^{\mu\nu}k^4 - k^2 k^\mu k^\nu)\right]\mathcal{P}_{\mu\rho} = i\delta^\nu_\rho. \qquad (1.5)$$

O propagador será, assim, a solução, $\mathcal{P}^{\mu\rho}$, da equação $\mathcal{D}^{\mu\nu}\mathcal{P}_{\mu\rho} = i\delta^{\nu}_{\rho}$. Utilizando o operador de projeção $\mathcal{P}^{\mu\rho} = A\eta^{\mu\rho} + Bk^{\mu}k^{\rho}$, iremos extrair os coeficientes A e B. Feito isso, o propagador é escrito como:

$$\mathcal{P}^{\mu\rho} = \frac{-i}{k^2 - a^2k^4}\left[\eta^{\mu\rho} - \frac{(1-\alpha) + \alpha a^2k^2}{k^2}k^{\mu}k^{\rho}\right]. \tag{1.6}$$

Uma forma mais simplificada de obter este mesmo resultado (e utilizando bem menos cálculos) segue o método exemplificado em [6]. Os operadores utilizados neste método são, explicitamente:

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2};\ \omega_{\mu\nu} = \frac{k_{\mu}k_{\nu}}{k^2}, \tag{1.7}$$

com as seguintes propriedades:

$$\begin{aligned} \theta_{\mu\nu}\theta^{\nu\mu} &= \theta^{\mu}_{\mu} = D - 1; \\ \omega_{\mu\nu}\omega^{\mu\nu} &= 1; \\ \theta_{\mu\nu}\omega^{\mu\nu} &= 0; \\ k^{\mu}\theta_{\mu\nu} &= 0; \\ k^{\mu}\omega_{\mu\nu} &= k_{\nu}, \end{aligned} \tag{1.8}$$

e apresentam a seguinte tabela multiplicativa:

	θ^{ρ}_{ν}	ω^{ρ}_{ν}
$\theta_{\mu\rho}$	$\theta_{\mu\nu}$	0
$\omega_{\mu\rho}$	0	$\omega_{\mu\nu}$

Tabela 1.1: tabela de multiplicação dos operadores

Pela tabela acima, verificamos que estes operadores de projeção formam um conjunto completo. Ou seja, podemos expandir $\mathcal{D}_{\mu\nu} = A\theta_{\mu\nu} + B\omega_{\mu\nu}$, onde A e B são coeficientes a serem determinados. Da condição $\mathcal{D}_{\mu\nu}\mathcal{P}^{\mu\rho} = i\delta^{\rho}_{\nu}$, e da tabela 1.1:

$$\mathcal{D}_{\mu\nu}\mathcal{P}^{\mu\rho} = (A\theta_{\mu\nu} + B\omega_{\mu\nu})(A^{-1}\theta^{\mu\rho} + B^{-1}\omega^{\mu\rho}) = AA^{-1}\theta^{\rho}_{\nu} + BB^{-1}\omega^{\rho}_{\nu} = \delta^{\rho}_{\nu}, \tag{1.9}$$

Para que a última igualdade seja satisfeita, é necessário que $AA^{-1} = BB^{-1} = i$, o que nos permite escrever o propagador nesta nova base como sendo simplesmente:

$$\mathcal{P}^{\mu\rho} = i\left(\frac{1}{A}\theta^{\mu\rho} + \frac{1}{B}\omega^{\mu\rho}\right), \tag{1.10}$$

o que simplifica o trabalho de identificar os coeficientes A e B de $\mathcal{D}_{\mu\nu}$. Assim, podemos determinar, com relativa facilidade, o propagador para escolhas de "gauges" mais complexos. Por exemplo, inspecionando (1.4), encontramos para A e B os valores:

$$A = k^2 - a^2k^4;\ B = \frac{k^2}{\alpha}, \tag{1.11}$$

que reproduz (1.6). Outro exemplo: escolhendo $\varepsilon = 1$, em (1.2),

$$\begin{aligned} \mathcal{L} &= \frac{A_\mu}{2}[\eta^{\mu\nu}\Box - \partial^\nu\partial^\nu - a^2(\eta^{\mu\nu}\Box^2 - \Box\partial^\mu\partial^\nu) + \\ &+ \frac{1}{\alpha}(\partial^\mu\partial^\nu - a^2\Box\partial^\mu\partial^\nu - \frac{a^4}{2}\Box^2\partial^\mu\partial^\nu)]A_\nu, \end{aligned} \tag{1.12}$$

escrevendo no espaço de momento e substituindo os operadores $\theta_{\mu\nu}$ e $\omega_{\mu\nu}$, teremos:

$$\mathcal{D}^{\mu\nu} = (k^2 - a^2k^4)\theta^{\mu\nu} + \frac{1}{\alpha}(k^2 - a^2k^4 - a^4k^6)\omega^{\mu\nu}. \tag{1.13}$$

Logo, uma simples inspeção visual já nos permite afirmar que:

$$\mathcal{P}_{\mu\nu} = \frac{i\theta_{\mu\nu}}{k^2 - a^2k^4} + \frac{i\alpha\omega_{\mu\nu}}{k^2 - b^2k^4 - b^4k^6}. \tag{1.14}$$

1.2 O Espalhamento Escalar-Escalar e o Potencial Não-Relativístico

Com o propagador em mãos, podemos continuar a estudar a interação do campo escalar escalar complexo de carga Q com o campo de Podolsky, dada pela Lagrangeana:

$$\mathcal{L} = (D_\mu\phi)^* D^\mu\phi - M^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2, \tag{1.15}$$

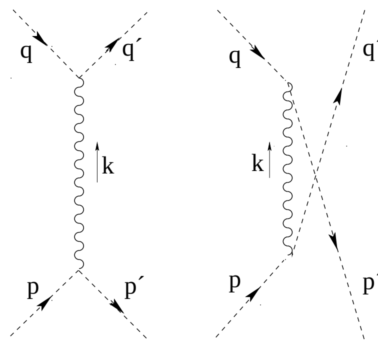


Figura 1.1: Espalhamento escalar-escalar.

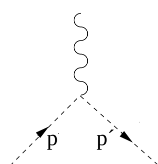
que via a prescrição do acoplamento minimal, $D_\mu = \partial_\mu + iQA_\mu$, permite a expansão dos dois primeiros termos à direita:

$$(D_\mu\phi)^* D^\mu\phi - M^2\phi^*\phi = \partial^\mu\phi^*\partial_\mu\phi - M^2\phi^*\phi + iQA_\mu(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) + Q^2A_\mu A^\mu\phi^*\phi, \tag{1.16}$$

onde o primeiro e o segundo termo a direita de (1.16) descrevem o propagador da linha escalar,

$$\frac{i}{p^2 - M^2},$$

o terceiro termo gera o vértice abaixo:



$$\rightarrow -iQ(p + p')^\mu$$

já escrito no espaço de momentos. O último termo não é importante em nível de árvore. Usando o propagador (1.6) e as regras de Feynman para a partícula escalar, desejamos determinar o potencial não-relativístico, partindo do espalhamento mostrado na figura (2.2).

O primeiro gráfico à esquerda da figura (2.2) fornece:

$$\begin{aligned} i\mathcal{M} &= -iQ(p+p')_\mu \times \frac{i}{k^2 - a^2k^4}\left[\eta^{\mu\nu} - \frac{(1-\alpha) + \alpha a^2k^2}{k^2}k^\mu k^\nu\right] \times -iQ(q+q')_\nu \\ &= \frac{-iQ^2}{k^2 - a^2k^4}\left[(p+p')\cdot(q+q') - \frac{k\cdot(q+q')(1-\alpha+\alpha a^2k^2)(p+p')\cdot k}{k^2}\right], \end{aligned} \tag{1.17}$$

utilizando conservação de momento em cada vértice e o fato das partículas escalares estarem na camada de massa, portanto $k = p - p' \Rightarrow k \cdot (p+p') = 0$, anulando todo o segundo termo na segunda linha. Logo, este gráfico fornece:

$$i\mathcal{M} = \frac{-iQ^2(p+p)(q+q')}{k^2 - a^2k^4}. \tag{1.18}$$

O segundo gráfico é trivial, fornecendo a mesma contribuição, entretanto não é necessário introduzir o fator 2, pois na mecânica quântica não-relativística (para onde queremos transitar) a função de onda já é simetrizada . Queremos determinar agora o limite não-relativístico de (1.18), onde $p^\mu \approx (M + O(\frac{|\vec{p}^2|}{M^2}), \vec{p})$, $k^\mu \approx (0, \vec{k})$:

$$\mathcal{M}_{NR} = \lim_{c\to\infty} \frac{-Q^2(p+p)(q+q')}{k^2 - a^2k^4} \approx \frac{Q^2 4M^2}{|\vec{k}|^2(|\vec{k}|^2 + \frac{1}{a^2})}, \tag{1.19}$$

e ignoramos no numerador os termos proporcionais aos momentos.

Na aproximação de Born , a seção de choque de partículas de massa m, sendo o momento relativo de uma das partículas a diferença entre o momento inicial e final no centro de massa, $\vec{k} = \vec{p} - \vec{p}\,'$, tal que $|\vec{k}| = \sqrt{E^2 - m^2}$, é dada por

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \left|\frac{m}{4\pi}\int U(r)e^{i\vec{k}\cdot\vec{r}}d^3\vec{r}\right|^2. \tag{1.20}$$

Mas, da teoria quântica de campos, usando a amplitude de espalhamento

$$\mathcal{M} = -iQ^2(2\pi)^3\delta(p_1' + p_2' - p_1 - p_2)\mathcal{T},$$

obtemos para a mesma seção de choque,

$$\left.\frac{d\sigma}{d\Omega}\right|_{cm} = \left|\frac{m^2Q^2}{4E^2(2\pi)^2}\mathcal{T}\right|^2.$$

Invertendo a transformada de Fourier (1.20), podemos calcular o potencial na aproximação não-relativística, usando a amplitude de espalhamento na primeira aproximação de Born, ou seja

$$U(\vec{r}) = \frac{1}{(2\pi)^3}\int \frac{\mathcal{M}_{NR}}{4m^2}e^{i\vec{k}\cdot\vec{r}}d^3k. \tag{1.21}$$

Com isso, podemos determinar o potencial não relativístico, usando

$$U(r) = \frac{1}{4M^2(2\pi)^{D-1}} \int d^{D-1}\vec{k}\, \mathcal{M}_{NR} \exp{(-i\vec{k}\cdot\vec{r})}. \tag{1.22}$$

Usando (1.19) em (1.22) e $D = 2$ temos

$$\begin{aligned} U(r) &= \frac{1}{4M^2}\frac{1}{(2\pi)^2}\int d^2\vec{k}\mathcal{M}_{NR}\exp{(-i\vec{k}\cdot\vec{r})} \\ &= \frac{1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\frac{Q^2}{k^2(k^2+\frac{1}{a^2})}e^{-ikr\,cos(\theta)}k\,d\theta\,dk \\ &= \frac{Q^2}{(2\pi)^2a^2}\int_0^\infty\frac{k\,dk}{k^2(k^2+\frac{1}{a^2})}\left[2\pi J_0(kr)\right] \\ &= \frac{Q^2}{2\pi a^2}\int_0^\infty J_0(kr)k\,dk\left(a^2\right)\left(\frac{1}{k^2}-\frac{1}{k^2+\frac{1}{a^2}}\right) \end{aligned} \tag{1.23}$$

(usamos a notação $k = |\vec{k}|$). Na penúltima linha de (1.23) o aparecimento da função de Bessel de ordem zero, $J_0(kr)$ é justificado, pois utilizamos integral tabelada [38]:

$$\int_0^{2\pi} e^{ikr\,cos(\theta)}d\theta = 2\pi J_0(kr), \tag{1.24}$$

e depois procedemos com uma decomposição em frações parciais. No entanto, a primeira integral advinda desta decomposição é divergente, problema que contornaremos fazendo o seguinte limite:

$$U(r) = \lim_{\epsilon\to 0}\frac{Q^2}{2\pi}\int_0^\infty J_0(kr)k\,dk\left(\frac{1}{k^2+\epsilon^2}-\frac{1}{k^2+\left(\frac{1}{a}\right)^2}\right). \tag{1.25}$$

Integrais como (1.25) também estão tabeladas, [38],

$$\int_0^\infty \frac{xJ_0(ax)}{x^2+\epsilon^2}dx = K_0(a\epsilon), \tag{1.26}$$

onde $K_0(a\epsilon)$ é a função modificada de Bessel de ordem zero (se $a > 0$ e $Re(\epsilon) > 0$). O limite assintótico quando $\epsilon \to 0$, $K_0(a\epsilon) \to -ln(a)$ é utilizado no cálculo da primeira integral, assim temos que a solução de (1.25) usando (1.26) é:

$$U(r) = \frac{-Q^2}{2\pi a^2}\left[ln\left(\frac{r}{r_0}\right) + K_0\left(\frac{1}{a}r\right)\right], \tag{1.27}$$

onde r_0é um regulador infravermelho. Esse potencial é negativo para qualquer r, portanto repulsivo.

1.2.1 Não-Existência de Estados Ligados

Usando (1.27) acrescida do termo devido à rotação, obtemos uma expressão para a energia potencial efetiva da partícula,

$$\begin{aligned} \mathcal{V}_{eff}(r,l) &= \frac{l^2}{mr^2} + U(r) & (1.28) \\ &= \frac{2l^2}{mr^2} - \frac{Q^2}{2\pi}\left[ln\,(r) + K_0\left(\frac{r}{a}\right)\right], & (1.29) \end{aligned}$$

onde l é o momento angular da partícula. Um potencial confinante só pode existir se ele tiver um mínimo, ou seja, devemos ter raízes (reais) para r quando derivamos (1.28) e igualamos a zero,

$$\begin{aligned} \frac{d\mathcal{V}_{eff}(r,l)}{dr} &= -\frac{2l^2}{mr^3} - \frac{Q^2}{2\pi}\left[\frac{1}{r} - \frac{1}{a}K_1\left(\frac{1}{a}r\right)\right] = 0 & (1.30) \\ &\Rightarrow \frac{2l^2}{mr^3} + \frac{Q^2}{2\pi r} = \frac{Q^2}{2\pi a}K_1\left(\frac{r}{a}\right). \end{aligned}$$

Suponha, por absurdo, que exista $\varepsilon > 0$ tal que

$$\frac{2l^2}{m\varepsilon^3} + \frac{Q^2}{2\pi\varepsilon} - \frac{Q^2}{2\pi a}K_1\left(\frac{\varepsilon}{a}\right) = 0. \tag{1.31}$$

Como $K_1(r)$ é uma função contínua em $\mathbb{R}_+$, teremos então

$$\lim_{r\to\varepsilon} \overbrace{\frac{Q^2}{2\pi a}K_1\left(\frac{r}{a}\right)}^{f(r)} = \frac{Q^2}{2\pi a}K_1\left(\frac{\epsilon}{a}\right) = \frac{2l^2}{m\varepsilon^3} + \frac{Q^2}{2\pi\varepsilon} = L. \tag{1.32}$$

Provaremos que este não é o caso. Usando a série de Frobenius

$$K_1(r) = \frac{1}{r} + \left[ln\left(\frac{r}{2}\right) - \frac{1}{2}\right]\sum_{k=0}^{\infty}\frac{r^{2k+1}}{2^{2k+1}\Gamma(k+2)k!} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{\sum_{k=1}^{n}\frac{2k+1}{k(k+1)}}{n!(n+1)!}\left(\frac{r}{2}\right)^{2n+1},$$

podemos ver, com $r > 0$, que $K_1(r) > \frac{1}{r} - 1$ (o gráfico desta duas funções mostra de maneira mais clara esta relação), fato que usaremos brevemente. Escolha de antemão

$$\epsilon = \frac{Q^2}{2\pi a} + \frac{2l^2}{m\varepsilon^3} > 0,$$

e $\delta > 0$. Façamos $r_\delta = \varepsilon + \frac{\delta}{2} > 0$. Então

$$\begin{aligned}
|f(r_\delta) - L| &= \left|\frac{Q^2}{2\pi a}K_1\left(\frac{r_\delta}{a}\right) - \frac{2l^2}{m\varepsilon^3} - \frac{Q^2}{2\pi\varepsilon}\right| > \left|\frac{Q^2(\varepsilon - r_\delta)}{2\pi r_\delta \varepsilon} - \frac{2l^2}{m\varepsilon^3} - \frac{Q^2}{2\pi a}\right| \\
&= \left|-\frac{Q^2\delta}{4\pi r_\delta\varepsilon} - \frac{2l^2}{m\varepsilon^3} - \frac{Q^2}{2\pi a}\right| = \underbrace{\frac{Q^2\delta}{4\pi r_\delta\varepsilon}}_{>0} + \underbrace{\frac{2l^2}{m\varepsilon^3} + \frac{Q^2}{2\pi a}}_{\epsilon} > \epsilon,
\end{aligned}$$

então seja qual for $\delta > 0$ pode-se achar r_δ tal que $0 < |r_\delta - \epsilon| < \delta$ e $|f(r_\delta) - L| > \epsilon$, ou seja $f(r)$ *não* tende à L, contrariando (1.32). Portanto não existe $\varepsilon \in \mathbb{R}_+$ que satisfaça (1.31).

1.2.2 Generalização para d Dimensões

Iremos, entretanto generalizar este resultado para d dimensões, para utilizá-la em outros casos. Usamos então (1.19) e (1.22), expressando em coordenadas polares:

$$\begin{aligned}
&\frac{1}{4M^2(2\pi)^{d-1}}\int d^{d-1}\vec{k}\ \mathcal{M}_{NR}\exp\left(-i\vec{k}\vec{r}\right) \qquad (1.33) \\
=\ &\frac{Q^2}{a^2(2\pi)^{d-1}}\int d^{d-1}\vec{k}\ \frac{\mathcal{D}_{NR}(k)\exp\left(-i\vec{k}\vec{r}\right)}{k^2(k^2+\frac{1}{a^2})} = \\
=\ &\frac{Q^2}{(2\pi)^{d-1}}\int d^{d-1}\vec{k}\mathcal{N}_{NR}(k)\ \exp\left(-i\vec{k}\vec{r}\right)\left[\frac{1}{k^2} - \frac{1}{k^2+\frac{1}{a^2}}\right] = \frac{Q^2}{(2\pi)^{d-1}}\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\cdots \\
\cdots\ &\int_0^{\pi}\int_0^{\infty}\mathcal{N}_{NR}(k)\exp\left(-i|\vec{k}||\vec{r}|\cos\theta_{d-2}\right)\left[\frac{1}{k^2} - \frac{1}{k^2+\frac{1}{a^2}}\right]|\vec{k}|^{d-1}\sin^{d-2}\theta_{d-2}d\theta_{d-2}\ldots \\
\ldots\ &\sin^2\theta_3 d\theta_3\sin\theta_2 d\theta_2 d\theta_1,
\end{aligned}$$

onde o $\cos\theta$ no argumento da exponencial é 1, no caso bidimensional, e $\mathcal{N}_{NR}(k)$ é o numerador de $\mathcal{M}_{NR}$. Queremos realizar primeiramente as as integrações nas variáveis angulares. É interessante verificar que podemos escrever em duas dimensões as coordenadas serão simplesmente

$$k_1 = |\vec{k}|cos\theta_1,\ k_2 = |\vec{k}|sen\theta_1, \qquad (1.34)$$

logo o Jacobiano da transformação de coordenadas é $J = k$. Em três dimensões, com:

$$k_1 = |\vec{k}|cos\theta_1,\ k_2 = |\vec{k}|sen\theta_1 cos\theta_2,\ k_3 = |\vec{k}|sen\theta_1 sen\theta_2 cos\theta_3, \qquad (1.35)$$

o Jacobiano será dado por $J = k^2 sen^2\theta_1$. Assim iremos generalizar então as coordenadas em d dimensões:

$$\begin{aligned}
k_1 &= |\vec{k}|cos\theta_1,\ k_2 = |\vec{k}|sen\theta_1 cos\theta_2,\ k_3 = |\vec{k}|sen\theta_1 sen\theta_2 cos\theta_3, \quad (1.36)\\
k_4 &= |\vec{k}|cos\theta_1 sen\theta_2 sen\theta_3 cos\theta_4,\ \ldots\\
k_j &= |\vec{k}|\left(\prod_{i=1}^{j-1} sen\theta_i\right)\cos\theta_j,\ \ldots\\
k_d &= |\vec{k}|\prod_{i=1}^{d-1} sen\theta_i. \quad (1.37)
\end{aligned}$$

Desta forma, podemos escrever o Jacobiano, como:

$$J = det\left(\frac{\partial(x_1, x_2, \ldots, x_d)}{\partial(r, \theta_1, \theta_2, \ldots, \theta_{d-1})}\right) = |\vec{k}|^{d-1}\prod_{i=1}^{d-1} sen^{i-1}\theta_i. \quad (1.38)$$

O elemento de volume será:

$$\prod_{i=1}^{d} = Jdk\prod_{i=1}^{d-1} d\theta_i = |\vec{k}|^{d-1}dkd\Omega_{d-1} = |\vec{k}|^{d-1}dk\prod_{i=1}^{d-1} sen^{i-1}\theta_i d\theta_i, \quad (1.39)$$

logo

$$\begin{aligned}
\int d^dk &= \int_0^\infty |\vec{k}|^{d-1}dk\int_0^{2\pi} d\theta_1\int_0^\pi sen\theta_2 d\theta_2\int_0^\pi sen^2\theta_3 d\theta_3\ldots\\
&\quad \ldots\int_0^\pi sen^{d-2}\theta_{d-1}d\theta_{d-1}. \quad (1.40)
\end{aligned}$$

A primeira integral à direita é simplesmente 2π. No nosso caso, estamos interessados na integral da forma abaixo:

$$\begin{aligned}
I &= \int_0^\infty f(|\vec{k}|)|\vec{k}|^{d-1}dk\int_0^{2\pi} d\theta_1\int_0^\pi sen\theta_2 d\theta_2\int_0^\pi sen^2\theta_3 d\theta_3\ldots\\
&\quad \ldots\int_0^\pi e^{i|\vec{k}||\vec{r}|\cos\theta_{d-1}} sen^{d-2}\theta_{d-1}d\theta_{d-1}. \quad (1.41)
\end{aligned}$$

A série iniciada pelas duas últimas integrais angulares na primeira linha de (1.41), pode ser simplificada se usarmos :

$$\int_0^\pi sen^m\theta d\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}(m+1)\right)}{\Gamma\left(\frac{1}{2}(m+2)\right)}, \quad (1.42)$$

e, assim, escrevemos todas as integrais angulares como o produto da primeira integral angular (que fornece 2π) por:

$$2\pi\frac{\sqrt{\pi}\Gamma(1)}{\Gamma(3/2)}\frac{\sqrt{\pi}\Gamma(3/2)}{\Gamma(2)}\frac{\sqrt{\pi}\Gamma(2)}{\Gamma(5/2)}\cdots\frac{\sqrt{\pi}\Gamma(\frac{d-2}{2})}{\Gamma(\frac{d-1}{2})}=2\pi^{\frac{d-1}{2}}\frac{\Gamma(1)}{\Gamma(\frac{d-1}{2})}=\frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}. \tag{1.43}$$

A última integral em (1.41) também pode ser determinada usando a definição integral da função de Bessel:

$$\int_0^{\pi} e^{i|\vec{k}||\vec{r}|cos\theta_{d-1}}sen^{d-2}\theta_{d-1}d\theta_{d-1}=\frac{2^{\frac{d-2}{2}}\Gamma(1/2)\Gamma(\frac{d-1}{2})}{(|\vec{k}||\vec{r}|)^{\frac{d-2}{2}}}J_{\frac{d-2}{2}}(|\vec{k}||\vec{r}|). \tag{1.44}$$

Usando (1.43), (1.44), $\Gamma(1/2)=\sqrt{\pi}$ em (1.41), obtemos:

$$I=\frac{2^{d/2}\pi^d}{|\vec{r}|^{\frac{d-2}{2}}}\int_0^{\infty} f(|\vec{k}|)J_{\frac{d-2}{2}}(|\vec{k}||\vec{r}|)|\vec{k}|^{d/2}dk. \tag{1.45}$$

A resolução do potencial não-relativístico reduz-se à solução de uma integral em k,

$$U(\vec{r})=\frac{Q^2}{(2\pi)^{\frac{d}{2}}|\vec{r}|^{\frac{d-2}{2}}}\int_0^{\infty}\left(\frac{1}{|\vec{k}|^2}-\frac{1}{|\vec{k}|^2+\frac{1}{a^2}}\right)\mathcal{N}_{NR}(k)\,|\vec{k}|^{d/2}J_{\frac{d-2}{2}}(|\vec{k}||\vec{r}|)dk, \tag{1.46}$$

Lembramos também que $d=n-1$, onde n é o número de dimensões do espaço-tempo. Para conferir se obtemos uma expressão correta, iremos calcular a integral no caso quadridimensional, onde $d=3$. Lembramos que:

$$J_{1/2}(x)=\left(\frac{2}{\pi x}\right)^{1/2}sen(x). \tag{1.47}$$

Logo, aplicamos (1.46):

$$\begin{aligned} U_4(\vec{r}) &= \frac{Q^2}{(2\pi)^{3/2}|\vec{r}|^{1/2}}\left[\int_0^{\infty}\frac{1}{|\vec{k}|^{1/2}}\left(\frac{2}{\pi|\vec{k}||\vec{r}|}\right)^{1/2}sen(|\vec{k}||\vec{r}|)dk-\right.\\ &- \left.\int_0^{\infty}\frac{|\vec{k}|^{3/2}}{|\vec{k}|^2+\frac{1}{a^2}}\left(\frac{2}{\pi|\vec{k}||\vec{r}|}\right)^{1/2}sen(|\vec{k}||\vec{r}|)dk\right] \qquad (1.48)\\ &= \frac{Q^2}{(2\pi)^{1/2}|\vec{r}|}\left[\int_0^{\infty}\frac{sen(|\vec{k}||\vec{r}|)}{\pi|\vec{k}|}dk-\int_0^{\infty}\frac{|\vec{k}|sen(|\vec{k}||\vec{r}|)}{\pi\left(|\vec{k}|^2+\frac{1}{a^2}\right)}dk\right], \end{aligned}$$

que coincide com equação (2.24) de [41]. O mesmo é válido quando $d = 2$.

Capítulo 2

Estados Ligados da Teoria Maxwell-Chern-Simons-Podolsky

Esse capítulo será devotado ao estudo da Eletrodinâmica de Podolsky (ou de Maxwell-Podolsky) em $(2+1)$ dimensões com um de Chern-Simons. Nossa expectativa é obter estados ligados a partir da equação de onda modificada.

Antes, iremos checar se a teoria resultante é consistente, principalmente quanto à unitariedade.

2.1 Análise da Unitariedade com a Teoria de Chern-Simons

Vamos partir primeiramente de um modelo mais conhecido, dado pela Lagrangeana de Proca-Chern-Simons (PCS):

$$\mathcal{L} = -\frac{c}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu^2}{2}A_\mu A^\mu - \frac{1}{2\lambda}(\partial_\mu A^\mu)^2 + \frac{s}{2}\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho. \tag{2.1}$$

Para encontrar o propagador é útil introduzir novamente os operadores tipo Barne-Rivers:

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}, \tag{2.2}$$

$$\begin{aligned}\omega_{\mu\nu} &= \frac{\partial_\mu\partial_\nu}{\Box},\\ S_{\mu\nu} &= \epsilon_{\mu\rho\nu}\partial^\rho,\end{aligned}$$

no espaço de coordenadas. Essencialmente, expandindo a parte quadrática da Lagrangeana, chegamos à equação descrita abaixo:

$$\mathcal{O} = x_1\theta_{\mu\nu} + x_2\omega_{\mu\nu} + x_3 S_{\mu\nu}.$$

Adaptando o algoritmo descrito em [8] , e usando as propriedade descritas na tabela (2.1), a equação inversa é:

$$\mathcal{O}^{-1} = y_1\theta^{\mu\nu} + y_2\omega^{\mu\nu} + y_3 S^{\mu\nu}, \tag{2.3}$$

onde

$$\begin{aligned}y_1 &= \frac{x_1}{x_1^2 + x_3^2\Box}\delta^3(x-y), \quad (2.4)\\ y_2 &= \frac{1}{x_2}\delta^3(x-y),\\ y_3 &= -\frac{x_3}{x_1^2 + x_3^2\Box}\delta^3(x-y).\end{aligned}$$

$\mathcal{O}^{-1}$ é somente o propagador no espaço tridimensional. Vamos aplicar este formalismo em (2.1), obtendo

$$\begin{aligned}\mathcal{O}^{-1} &= \left(\frac{c\Box + \mu^2}{c^2\Box^2 + (s^2 + 2c\mu^2)\Box + \mu^4}\theta^{\mu\nu} + \frac{\lambda}{\Box + \lambda\mu^2}\omega^{\mu\nu}\right. \quad (2.5)\\ &\quad \left.-\frac{s}{c^2\Box^2 + (s^2 + 2c\mu^2)\Box + \mu^4}S^{\mu\nu}\right)\delta^3(x-y).\end{aligned}$$

As massas físicas, correspondendo aos pólos em k^2 (ou alternativamente x) de (2.5), são facilmente encontradas resolvendo, já no espaço de momentos:

$$c^2x^2 - (2c\mu^2 + s^2)x + \mu^4 = 0, \tag{2.6}$$

obtendo duas soluções distintas,

$$x_{1,2} = \frac{\mu^2}{c} + \frac{s^2}{2c^2} \pm \frac{s}{c^2}\left(c\mu^2 + \frac{s^2}{4}\right)^{1/2}. \tag{2.7}$$

	θ^{ρ}_{ν}	ω^{ρ}_{ν}	S^{ρ}_{ν}
$\theta_{\mu\rho}$	$\theta_{\mu\nu}$	0	$S_{\mu\nu}$
$\omega_{\mu\rho}$	0	$\omega_{\mu\nu}$	0
$S_{\mu\rho}$	$S_{\mu\nu}$	0	$-\Box\theta_{\mu\nu}$

Tabela 2.1: Tabela de multiplicação de Barne-Rivers

O mesmo resultado foi obtido em [14].

A unitariedade requer que, saturando o propagador com correntes conservadas (j_{μ}), seus resíduos em k^2 seja sempre positivos [26] . Isso é baseado em um postulado sobre o valor esperado do vácuo de uma função de dois pontos ordenada temporalmente na representação espectral. Se o resultado é negativo, estados de norma negativa ("*ghosts*") aparecem já em nível de árvore. O propagador saturado é explicitamente representado por:

$$\mathcal{M} = j^{*\mu}(k)\langle T|A_{\mu}(-k)A_{\nu}(k)|T\rangle j^{\nu}(k). \tag{2.8}$$

Então, a condição de unitariedade impõe que $\mathrm{Res}\mathcal{M}|_{k^2=poles} \geq 0$.

Escrevendo a forma mais geral para corrente conservada em uma base apropriada, na representação de momentos, teremos

$$k^{\mu} = (k^0, \vec{k}), \tilde{k}^{\mu} = (k^0, -\vec{k}),\ \epsilon^{\mu} = (0, \vec{e});$$

onde $\vec{e}$ é um vetor unitário ortogonal a $\vec{k}$.

$$j^{\mu} = Ak^{\mu} + B\tilde{k}^{\mu} + C\epsilon^{\mu}. \tag{2.9}$$

A condição de conservação de corrente, $k_{\mu}j^{\mu} = 0$, permite-nos derivar uma relação entre A e B:

$$Ak^2 + B(k_0^2 + \vec{k}^2) = 0, \tag{2.10}$$

e

$$\omega_{\mu\nu}j^{\mu} = \omega_{\mu\nu}j^{\nu} = 0. \tag{2.11}$$

É fácil ver que:

$$\begin{aligned} j^{\mu}S_{\mu\nu}j^{\nu} &= 0, \\ j^{\mu}\theta_{\mu\nu}j^{\nu} &= j^{\mu}j_{\mu} = (B^2 - A^2)k^2 + C^2 \equiv F(k). \end{aligned} \tag{2.12}$$

Por (2.11) e (2.12), somente termos proporcionais a $\theta_{\mu\nu}$ em (2.5) sobrevivem quando saturamos o propagador com correntes conservadas; conhecendo as soluções (2.7) teremos como escrever $\mathcal{M}$,

$$\begin{aligned}\mathcal{M} &= -\left[\frac{1}{2c} - \frac{s}{4c\sqrt{c\mu^2+\frac{s^2}{4}}}\right]\frac{F(k)}{k^2-\left(\frac{\mu^2}{c}+\frac{s^2}{2c^2}+\frac{s}{c^2}\sqrt{c\mu^2+\frac{s^2}{4}}\right)} \\ &\quad -\left[\frac{1}{2c} + \frac{s}{4c^2\sqrt{c\mu^2+\frac{s^2}{4}}}\right]\frac{F(k)}{k^2-\left(\frac{\mu^2}{c}+\frac{s^2}{2c^2}-\frac{s}{c^2}\sqrt{c\mu^2+\frac{s^2}{4}}\right)}.\end{aligned} \tag{2.13}$$

Impondo que $\mathrm{Res}\mathcal{M}|_{k^2=x_1} \geq 0$, isto será somente possível se $\mu^2 \leq 0$, o que não é permitido em um modelo não taquiônico, ou se colocarmos um sinal errado no termo massivo em (2.1), ou seja não podemos obter unitariedade no nível de árvore.

Agora que ganhamos alguma experiência, podemos estender nossa análise para a Eletrodinâmica de Podolsky

$$\mathcal{L} = -\frac{c}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda} - \frac{1}{2\lambda}(\partial_\mu A^\mu)^2 + \frac{s}{2}\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho. \tag{2.14}$$

Usando o mesmo algoritmo obtemos o propagador

$$\begin{aligned}\mathcal{O}^{-1} &= \left(\frac{a^{-2}(\Box^2+ca^{-2}\Box)}{\Box^4+2a^{-2}c\Box^3+c^2a^{-4}\Box^2+s^2a^{-4}\Box}\theta^{\mu\nu} + \frac{\lambda}{\Box}\omega^{\mu\nu}\right. \\ &\quad \left. -\frac{sa^{-4}}{\Box^4+2a^{-2}c\Box^3+c^2a^{-4}\Box^2+s^2a^{-4}\Box}S^{\mu\nu}\right)\delta^3(x-y).\end{aligned} \tag{2.15}$$

Encontrar os pólos em k^2 de (2.15) segue a mesma seqüência de cálculo feita anteriormente. Antes, é conveniente fazer uma mudança de variáveis, $s^2 = \alpha a^{-2}$. Após alguma álgebra computacional, resolvemos

$$x(x^3 - 2a^{-2}cx^2 + c^2a^{-4}x - s^2a^{-4}) = 0, \tag{2.16}$$

e os resultados seguem, com $c = 1$:

$$x_0 = 0, \tag{2.17}$$

$$
\begin{aligned}
x_1 &= \frac{2}{3a^2} + \frac{2^{1/3}a^{-2}}{3(-2+27\alpha+3\sqrt{3}\sqrt{-4+27\alpha})^{1/3}} \\
&+ \frac{1}{3}\frac{a^{-2}(-2+27\alpha+3\sqrt{3}\sqrt{-4+27\alpha}))^{1/3}}{2^{1/3}}, \\
x_2 &= \frac{2}{3a^2} - \frac{1}{3}\frac{(1+i\sqrt{3})a^{-2}}{2^{2/3}(-2+27\alpha+3\sqrt{3}\sqrt{-4+27\alpha})^{1/3}} \\
&- \frac{1}{6}\frac{(1-i\sqrt{3})M^2(-2+27\alpha+3\sqrt{3}\sqrt{-4+27\alpha})^{1/3}}{2^{1/3}}, \\
x_3 &= \frac{2}{3a^2} - \frac{1}{3}\frac{(1-i\sqrt{3})a^{-2}}{2^{1/3}(-2+27\alpha+3\sqrt{3}\sqrt{-4+27\alpha})^{1/3}} \\
&- \frac{1}{6}\frac{(1+i\sqrt{3})a^{-2}(-2+27\alpha+3\sqrt{3}\sqrt{-4+27\alpha})^{1/3}}{2^{1/3}}.
\end{aligned}
$$

Se o discriminante polinomial de (2.16) for tal que $D < 0$, todas as raízes serão reais e distintas, impondo que:

$$
\begin{aligned}
D &= \frac{-b^6a^{-12}}{729} + \left(\frac{-b^3a^{-6}}{27} + \frac{s^2a^{-4}}{2}\right)^2 \\
&= \frac{-b^6a^{-12}}{729} + \left(\frac{-b^3a^{-6}}{27} + \frac{\alpha^2a^{-6}}{2}\right)^2. \qquad (2.18)
\end{aligned}
$$

Então para evitar duas massas complexas, o modelo requer que se tome a constante α menor que $\frac{4b^3}{27}$ (mas não-negativa). Mantendo α nesta faixa, aplicamos o método de Routh-Hurwitz para determinar quantas raízes terão partes reais positivas. Logo, da equação

$$x^3 + ax^2 + bx + c = 0,$$

construiremos a tabela da qual, extraindo a primeira coluna e contando o

1	b
a	c
$\frac{ab-c}{a}$	0
c	0

número de mudança de sinais, teremos o número de raízes positivas. Então, se $s^2 = \frac{4c^3a^{-2}}{27}$, a seqüência:

$$\left\{1, -2a^{-2}c, \frac{c^3a^{-6}\left(\frac{4}{27}-2\right)}{-2a^{-2}c}, \frac{-4c^3a^{-6}}{27}\right\}$$

deve ser $+, -, +, -$ para obtermos as desejadas três soluções reais e distintas. Por exemplo, se $\alpha = \frac{1}{27}$, teremos as seguintes soluções:

$$x_1 = 1.17736a^{-2},\ x_2 = 0.78243a^{-2},\ x_3 = 0.04020a^{-2}. \tag{2.19}$$

Após todo esse cuidado para que as soluções sejam reais, a expressão para $\mathcal{M}$ é

$$\begin{aligned}\mathcal{M} &= -\frac{(ca^{-4}-a^{-2}x_1)F(k)}{(x_1-x_2)(x_1-x_3)(k^2-x_1)} + \frac{(ca^{-4}-a^{-2}x_2)F(k)}{(x_1-x_2)(x_2-x_3)(k^2-x_2)} \\ &- \frac{(ca^{-4}-a^{-2}x_3)F(k)}{(x_1-x_3)(x_2-x_3)(k^2-x_3)}.\end{aligned} \tag{2.20}$$

Considere $x_1 > x_2$. Para obter Res $\mathcal{M}|_{k^2=x_1} \geq 0$, suponha: (i) $x_1 \geq a^{-2}$ e $x_3 < x_1$ ou (ii) $x_1 \leq a^{-2}$ e $x_3 > x_1$. Por outro lado, Res $\mathcal{M}|_{k^2=x_2} \geq 0$, requer : (iii) $a^{-2} \geq x_2$ e $x_2 > x_3$ ou (iv) $x_2 \geq a^{-2}$ e $x_2 < x_3$. Com qual par destas opções podemos obter Res $\mathcal{M}|_{k^2=x_3} \geq 0$? Claramente, o par de condições (ii)-(iv) and (ii)-(iii) viola nossa hipótese inicial, $x_1 > x_2$, e portanto são descartadas. O par (i)-(iii) impõe que $x_1 > a^{-2} > x_2$ e $x_1 > x_2 > x_3$, então $a^{-2} > x_3$, o que após analisarmos os sinais, não são suficientes para fazer Res $\mathcal{M}|_{k^2=x_3} \geq 0$. A mesma situação ocorre com (i)-(iv): $x_1 > x_2 > a^{-2}$ e $x_1 > x_3 > x_2$ (concluindo que $x_3 > a^{-2}$) produzindo Res $\mathcal{M}|_{k^2=x_3} < 0$. Retornando ao nosso exemplo numérico, se $\alpha = \frac{1}{27}$:

$$Res\ \mathcal{M}|_{k^2=x_1} = 0.39493,\ Res\ \mathcal{M}|_{k^2=x_2} = 0.74227,\ Res\ \mathcal{M}|_{k^2=x_3} = -1.13716 < 0.$$

Se $x_1 < x_2$, isso não irá modificar nosso cenário. As condições impostas por Res $\mathcal{M}|_{k^2=x_1} \geq 0$ tornam-se (i)$a^{-2} \geq x_1$ e $x_1 > x_3$ ou (ii) $x_1 \geq a^{-2}$ e $x_3 > x_1$, e Res $\mathcal{M}|_{k^2=x_12} \geq 0$ requerendo (iii) $a^{-2} \geq x_2$ e $x_3 > x_1$ ou (iv) $x_2 \geq a^{-2}$ e $x_2 > x_3$. O par (i)-(iii) e (ii)-(iii) são excluídos. Então, nem (i)-(iv) ou (ii)-(iv) podem tornar Res $\mathcal{M}|_{k^2=x_3} \geq 0$.

No entanto, a teoria pode ser utilizada com uma teoria efetiva somente. Esta será nossa abordagem neste trabalho.

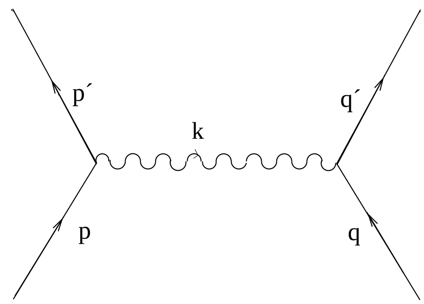


Figura 2.1: Espalhamento escalar-escalar.

2.2 Análise de Estados Ligados

Analisados os aspectos da unitariedade da teoria, partiremos para verificar se a teoria permite estados ligados ou não. A Lagrangeana:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda} - \frac{1}{2\lambda}(\partial_\mu A^\mu)^2 + \frac{s}{2}\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho, \quad (2.21)$$

permite que o propagador seja escrito no espaço de momentos como sendo:

$$\mathcal{P}_{\mu\nu} = -i\frac{k^2 - a^2k^4}{(k^2 - a^2k^4)^2 - s^2k^2}\tilde{\theta}_{\mu\nu} + i\frac{\lambda}{k^2}\tilde{\omega}_{\mu\nu} - \frac{s}{(k^2 - a^2k^4)^2 - s^2k^2}\tilde{S}_{\mu\nu}. \quad (2.22)$$

Com este propagador e o vértice para o campo escalar, iremos determinar o espalhamento dado pela figura (2.1).

Assim, o cálculo do primeiro gráfico irá fornecer:

$$\begin{aligned} \mathcal{M} &= (-iQ)^2(p+p')_\mu \left(-i\frac{k^2 - a^2k^4}{(k^2 - a^2k^4)^2 - s^2k^2}\tilde{\theta}_{\mu\nu} + i\frac{\lambda}{k^2}\tilde{\omega}_{\mu\nu}\right. \\ &- \left.\frac{s}{(k^2 - a^2k^4)^2 - s^2k^2}\tilde{S}_{\mu\nu}\right)(q+q')_\nu \\ &= \frac{iQ^2(k^2 - a^2k^4)(p+p')\cdot(q+q')}{(k^2 - a^2k^4)^2 - s^2k^2} + \frac{Q^2 s\epsilon_{\mu\rho\nu}(p+p')^\mu k^\rho (q+q')^\nu}{(k^2 - a^2k^4)^2 - s^2k^2}, \end{aligned}$$

e usaremos, agora, conservação de momento nos vértices, $p' = p - k$ ou $q' = q - k$,

$$\begin{aligned} \mathcal{M} &= \frac{iQ^2(k^2 - a^2k^4)(2p-k)\cdot(2q-k) + sQ^2\epsilon_{\mu\rho\nu}(2p-k)^\mu k^\rho (2q-k)^\nu}{(k^2 - a^2k^4)^2 - s^2k^2} \\ &= \frac{iQ^2(k^2 - a^2k^4)[4p\cdot q - 2k\cdot(p+q) + k^2] + sQ^2\epsilon_{\mu\rho\nu}p^\mu k^\rho q^\nu}{(k^2 - a^2k^4)^2 - s^2k^2}. \end{aligned} \quad (2.23)$$

O limite não-relativístico da expressão é tomado, como já discutido anteriormente, fazendo as substituições $p^0 = M - O(\frac{|\vec{p}|^2}{M^2})$, $k^0 = 0$ e $k^2 = -\vec{k}^2$, sempre mantendo termos somente até a ordem $\frac{|\vec{p}|}{M}$. Podemos reconhecer que $\epsilon_{\mu\rho\nu} p^\mu k^\rho q^\nu = k^\rho \epsilon_{\mu\rho\nu} q^\nu p^\mu = k^\rho (q \times p)_\rho = k \cdot (q \times p)$. Mais ainda, em 3 dimensões, no limite não-relativístico:

$$\begin{aligned} k \cdot (q \times p) &= k_0(p_2 q_1 - p_1 q_2) - k_1(p_0 q_2 - p_2 q_0) - k_2(p_1 q_0 - p_0 q_1) \\ &\approx M[k_1(p_2 - q_2) - k_2(p_1 - q_1)] = M\vec{k} \times (\vec{p} - \vec{q}), \end{aligned} \tag{2.24}$$

o que permite escrever (2.23), a expressão para $\mathcal{M} = -i\mathcal{M}_{NR}$, como:

$$\mathcal{M}_{NR} = \frac{4M^2 Q^2(\vec{k}^2 + a^2\vec{k}^4) - 4iMQ^2 s\vec{k} \times (\vec{p} - \vec{q})}{a^4|\vec{k}|^2(\vec{k}^6 + \frac{2}{a^2}\vec{k}^4 + \frac{\vec{k}^2}{a^4} + \frac{s^2}{a^4})}. \tag{2.25}$$

Seja $k^2 = x$. Um pouco de teoria sobre solução algébrica de equações cúbicas pode nos dizer algo sobre o termo entre parênteses no denominador da expressão (2.23). Dada uma equação do tipo:

$$x^3 + a_1 x^2 + a_2 x + c = 0, \tag{2.26}$$

se os coeficientes a_i são reais, e dada a definição:

$$\begin{aligned} Q &= \frac{3a_2 - a_1^2}{9} \\ R &= \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54}, \end{aligned} \tag{2.27}$$

o discriminante $D = Q^3 + R^3$ determina as características das soluções. Em especial, para termos todas as raízes reais e desiguais, devemos ter $D < 0$. Ainda neste caso especial as soluções podem ser expressas por:

$$\begin{aligned} x_1 &= \frac{-a_1}{3} + 2\sqrt{-Q}\cos\left(\frac{\theta}{3}\right) \\ x_2 &= \frac{-a_1}{3} + 2\sqrt{-Q}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) \\ x_3 &= \frac{-a_1}{3} + 2\sqrt{-Q}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right), \end{aligned} \tag{2.28}$$

onde $\cos\theta = R/\sqrt{-Q^3}$. Para que $D < 0$ teríamos como condição $a < \frac{2\sqrt{3}}{9s}$, o que pode ser conseguido fazendo:

$$a = \frac{2\sqrt{3}}{9\lambda s}, \tag{2.29}$$

onde o fator λ pode assumir qualquer valor maior que um. Assegurando deste modo que as soluções sejam reais (mas não necessariamente todas positivas) e distintas, podemos então partir para o cálculo do potencial não relativístico , substituindo (2.23) em (1.22). Podemos dividir esta integral em duas: uma com um termo proporcional a M^2 e a segunda com um termo proporcional a M.

Na primeira integral:

$$\int_0^\infty \int_0^{2\pi} \frac{1}{(2\pi)^2} \frac{Q^2(1+a^2k^2)e^{-i-i|\vec{k}||\vec{r}|\cos\theta}|\vec{k}|dkd\theta}{a^4(\vec{k}^6 + \frac{2}{a^2}\vec{k}^4 + \frac{\vec{k}^2}{a^4} + \frac{s^2}{a^4})}, \tag{2.30}$$

ou, equivalentemente, por frações parciais,

$$\int_0^\infty \int_0^{2\pi} \frac{Q^2}{(2\pi)^2 a^4} \left[\frac{(1+a^2x_1)}{(x_1-x_3)(x_1-x_2)(|\vec{k}|^2 - x_1)} + \frac{(1+a^2x_2)}{(x_2-x_3)(x_2-x_1)(|\vec{k}|^2 - x_2)} \right.$$
$$\left. + \quad \frac{1+a^2x_1}{(x_3-x_2)(x_3-x_1)(|\vec{k}|^2 - x_3)} \right] e^{-i|\vec{k}||\vec{r}|\cos\theta}|\vec{k}|dkd\theta. \tag{2.31}$$

A integral em θ é tabelada [38]:

$$\int_0^\infty \frac{Q^2}{(2\pi)a^4} \left[\frac{(1+a^2x_1)}{(x_1-x_3)(x_1-x_2)(|\vec{k}|^2 - x_1)} + \frac{(1+a^2x_2)}{(x_2-x_3)(x_2-x_1)(|\vec{k}|^2 - x_2)} \right.$$
$$\left. + \quad \frac{1+a^2x_1}{(x_3-x_2)(x_3-x_1)(|\vec{k}|^2 - x_3)} \right] J_0(|\vec{k}||\vec{r}|)|\vec{k}|dk, \tag{2.32}$$

onde $J_0(x)$ é a função de Bessel de ordem zero. A integral nos momentos também é tabelada,

$$\frac{Q^2}{(2\pi)a^4} \left[\frac{(1+a^2x_1)}{(x_1-x_3)(x_1-x_2)} K_0(\sqrt{|x_1|}|\vec{r}|) + \frac{(1+a^2x_2)}{(x_2-x_3)(x_2-x_1)} K_0(\sqrt{|x_2|}|\vec{r}|) \right.$$
$$\left. + \quad \frac{1+a^2x_1}{(x_3-x_2)(x_3-x_1)} K_0(\sqrt{|x_3|}|\vec{r}|) \right], \tag{2.33}$$

lembrando que $K_0(x)$ é a função de Bessel modificada.

A segunda integral, derivada do termo proporcional a M em (2.25), é

$$\int_0^\infty \int_0^{2\pi} \frac{i}{(2\pi)^2} \frac{-Q^2 s\vec{k} \times (\vec{p}-\vec{q})}{a^4 M|\vec{k}|^2(\vec{k}^6 + \frac{2}{a^2}\vec{k}^4 + \frac{\vec{k}^2}{a^4} + \frac{s^2}{a^4})} e^{-i|\vec{k}||\vec{r}|\cos\theta}|\vec{k}|dkd\theta, \quad (2.34)$$

onde mais uma vez iremos utilizar frações parciais para escrever:

$$\int_0^\infty \int_0^{2\pi} \frac{i}{M(2\pi)^2 a^4}[-Q^2 s\vec{k} \times (\vec{p}-\vec{q})] \left[-\frac{a^4}{s^2|\vec{k}|^2} + \frac{-(1+a^2x_1)^2}{s^2(x_1-x_3)(x_1-x_2)(|\vec{k}|^2-x_1)} \right.$$
$$\left. + \quad \frac{-(1+a^2x_2)^2}{s^2(x_3-x_2)(x_1-x_2)(|\vec{k}|^2-x_2)} + \frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)(|\vec{k}|^2-x_3)} \right] e^{-i|\vec{k}||\vec{r}|\cos\theta}|\vec{k}|dkd\theta, \quad (2.35)$$

e a integração angular segue como a anterior

$$\int_0^\infty \frac{1}{M(2\pi)} \frac{-Q^2 s\vec{k} \times (\vec{p}-\vec{q})}{a^4} \left[-\frac{a^4}{s^2|\vec{k}|^2} + \frac{-(1+a^2x_1)^2}{s^2(x_1-x_3)(x_1-x_2)(|\vec{k}|^2-x_1)} \right.$$
$$\left. + \quad \frac{-(1+a^2x_2)^2}{s^2(x_3-x_2)(x_1-x_2)(|\vec{k}|^2-x_2)} + \frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)(|\vec{k}|^2-x_3)} \right] J_0(|\vec{k}||\vec{r}|)|\vec{k}|dk. \quad (2.36)$$

Iniciamos, no segundo termo, fazendo a integração nos momentos do primeiro termo entre colchetes de (2.36). Relembrando a dinâmica do espalhamento, temos que, nas coordenadas do centro de massa, podemos expressar o momento relativo $\vec{t} = \frac{\vec{p}-\vec{q}}{2}$. Assim

$$\int_0^\infty \frac{2}{M(2\pi)}[Q^2\vec{k} \times \vec{t}] \frac{i}{s|\vec{k}|^2} J_0(|\vec{k}||\vec{r}|)|\vec{k}|dk, \quad (2.37)$$

que diverge no limite de $k \to 0$. Podemos introduzir um fator σ no denominador cujo limite $\sigma \to 0$ é realizado depois da integração, fazendo (2.37) tomar a seguinte forma,

$$\int_0^\infty \frac{2i}{M(2\pi)} \frac{Q^2\vec{k} \times \vec{t}}{s(|\vec{k}|^2 + \sigma^2)} J_0(|\vec{k}||\vec{r}|)|\vec{k}|dk. \quad (2.38)$$

Usaremos

$$\vec{k}\times\vec{t} = -\vec{t}\times\vec{k} = -\vec{t}\times\left(-i\vec{\nabla}\right) = i\vec{t}\times\left(\hat{r}\frac{\partial}{\partial r}\right) = i\vec{t}\times\left(\frac{\vec{r}}{|\vec{r}|}\frac{\partial}{\partial r}\right), \quad (2.39)$$

supondo que temos funções somente com dependência radial. Como $\frac{dK_0(\sigma|\vec{r}|)}{dr} = -\sigma K_1(\sigma|\vec{r}|)$, usamos este fato, junto com (2.39), para escrever a integral (2.37) como:

$$\int_0^\infty \frac{2i}{M(2\pi)}\frac{Q^2\vec{k}\times\vec{t}}{s(|\vec{k}|^2+\sigma^2)}J_0(|\vec{k}||\vec{r}|)|\vec{k}|dk = \frac{-2Q^2}{M(2\pi)s}\frac{\vec{r}\times\vec{t}}{|\vec{r}|}\sigma K_1(\sigma|\vec{r}|), \quad (2.40)$$

e assim já podemos identificar $\vec{L} = \vec{r}\times\vec{t}$ (lembre-se que $\vec{L}$ é um pseudo-escalar). Tomamos agora o limite $\sigma\to 0$:

$$\lim_{\sigma\to 0}\sigma K_1(\sigma|\vec{r}|) = \frac{1}{|\vec{r}|}, \quad (2.41)$$

aplicado à (2.38)

$$\frac{-Q^2}{M\pi s}\frac{\vec{L}}{|\vec{r}|^2}. \quad (2.42)$$

Trabalhamos, então, com os três últimos termos de (2.36) de forma análoga:

$$\begin{aligned}
&\int_0^\infty \frac{i}{M(2\pi)}\frac{-Q^2s\vec{k}\times(\vec{p}-\vec{q})}{a^4}\left[\frac{-(1+a^2x_1)^2}{s^2(x_1-x_3)(x_1-x_2)(|\vec{k}|^2-x_1)}\right. \quad (2.43)\\
+&\left.\frac{-(1+a^2x_2)^2}{s^2(x_3-x_2)(x_1-x_2)(|\vec{k}|^2-x_2)}+\frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)(|\vec{k}|^2-x_3)}\right]J_0(|\vec{k}||\vec{r}|)|\vec{k}|dk\\
=& -\frac{Q^2s\vec{L}}{a^4M\pi|\vec{r}|}\left[\frac{-(1+a^2x_1)^2}{s^2(x_1-x_3)(x_1-x_2)}\sqrt{|x_1|}K_1(\sqrt{|x_1|}|\vec{r}|)\right.\\
+&\left.\frac{-(1+a^2x_2)^2}{s^2(x_3-x_2)(x_1-x_2)}\sqrt{|x_2|}K_1(\sqrt{|x_2|}|\vec{r}|)+\frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)}\sqrt{|x_3|}K_1(\sqrt{|x_3|}|\vec{r}|)\right].
\end{aligned}$$

Logo, o potencial na aproximação não-relativística é expresso somando-se as partes (2.33), (2.42) e (2.43)

$$U(r) = \frac{Q^2}{(2\pi)a^4}\left[\frac{(1+a^2x_1)}{(x_1-x_3)(x_1-x_2)}K_0(\sqrt{|x_1|}|\vec{r}|)+\frac{(1+a^2x_2)}{(x_2-x_3)(x_2-x_1)}K_0(\sqrt{|x_2|}|\vec{r}|)\right.$$

$$
\begin{aligned}
&+ \quad \frac{1+a^2x_1}{(x_3-x_2)(x_3-x_1)}K_0(\sqrt{|x_3|}|\vec{r}|)\Bigg] \\
&- \quad \frac{Q^2 s\vec{L}}{a^4 M\pi|\vec{r}|}\left[\frac{-(1+a^2x_1)^2}{s^2(x_1-x_3)(x_1-x_2)}\sqrt{|x_1|}K_1(\sqrt{|x_1|}|\vec{r}|)\right. \\
&+ \quad \frac{-(1+a^2x_2)^2}{s^2(x_3-x_2)(x_1-x_2)}\sqrt{|x_2|}K_1(\sqrt{|x_2|}|\vec{r}|) \\
&+ \quad \left.\frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)}\sqrt{|x_3|}K_1(\sqrt{|x_3|}|\vec{r}|)+\frac{a^4}{s^2|\vec{r}|}\right].
\end{aligned}
$$

De posse do potencial (2.44), podemos utilizá-lo para encontrar as soluções da equação de Schrödinger independente do tempo

$$
\left(-\frac{\bigtriangledown^2}{2M}+U(\vec{r},\bigtriangledown)\right)\psi(\vec{r})=\left[-\frac{1}{2Mr}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)-\frac{1}{2Mr^2}\frac{\partial^2}{\partial^2\theta}+U(\vec{r},\vec{l})\right]\psi(\vec{r})=E\psi(\vec{r}),
\tag{2.44}
$$

e o operador de momento angular L é escrito como:

$$
L=-i\frac{\partial}{\partial\theta}. \tag{2.45}
$$

Convém fazer uma separação de variáveis $\psi(\vec{r})=R(r)\phi(\theta)$. Para a parte angular uma solução proposta é $\phi(\theta)=\frac{e^{il\theta}}{\sqrt{2\pi}}$. Substituindo em (2.44), obtemos a equação diferencial para a função radial:

$$
\left[-\frac{1}{2Mr}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{l^2}{2Mr^2}+U(\vec{r},l)\right]R(r)=ER(r), \tag{2.46}
$$

onde podemos ainda definir $u_E(r)=\sqrt{r}R(r)$, obtendo

$$
-\frac{1}{2M}\left(\frac{d^2}{d^2r}+\frac{1}{4r^2}\right)u_E+\mathcal{V}_{eff}u_E=Eu_E, \tag{2.47}
$$

com

$$
\mathcal{V}_{eff}=U(r,l)+\frac{l^2}{2Mr^2}.
$$

Solucionar analiticamente (2.47) com o potencial dado em (2.44) não parece possível em uma primeira análise. Partimos, então, para a solução numérica do problema usando métodos numéricos. Alguns resultados são mostrados na figura (2.2).

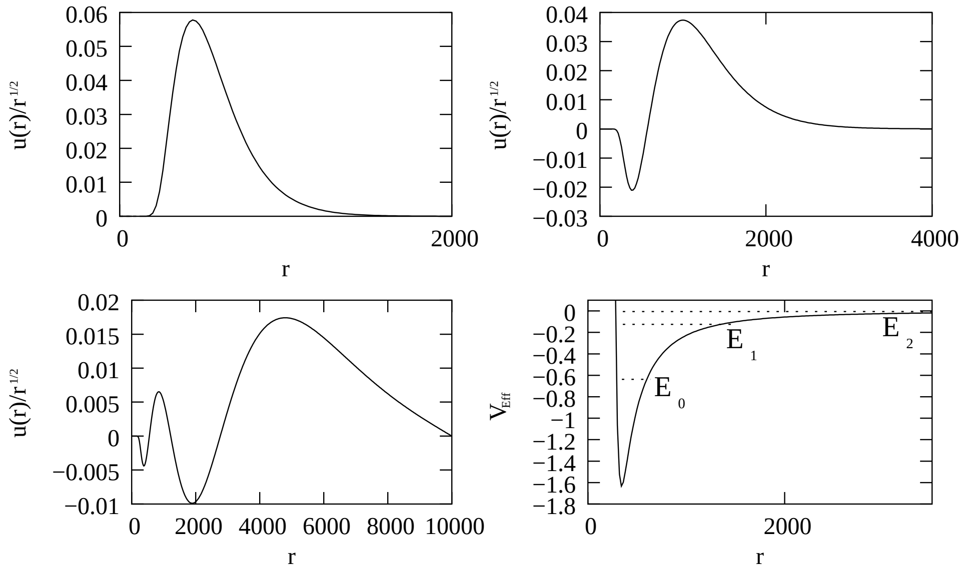


Figura 2.2: Primeiras funções de onda para $\lambda = 200, M = 140MeV, s = 0,003MeV$ e $l = 4$.

2.2.1 Solução Numérica do Problema de Auto-Estados

O método de Numerov [50] permite determinar a solução de equações diferenciais do tipo

$$u''(x) + q(x)u(x) = s(x), \tag{2.48}$$

o que é desejável para nosso problema, já que podemos transformar a equação de Schrödinger para a forma (2.48):

$$-\frac{1}{2m}\left(\frac{d^2}{dr^2} + \frac{1}{4r^2}\right)\psi(r) + \mathcal{V}_{eff}(r)\psi(r) = E\psi(r) \Rightarrow \tag{2.49}$$

$$\psi''(r) + \underbrace{\left\{\frac{1}{4r^2} + 2m[E - \mathcal{V}_{eff}(r)]\right\}}_{=q(r)}\psi(r) = \underbrace{0}_{=s(r)}.$$

O método consiste em aproximar a derivada segunda por

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} = u_n'' + \frac{h^2}{12}u_n'''' + \mathcal{O}(h^4), \tag{2.50}$$

com h tão pequeno quanto se queira. De (2.48) imediatamente segue que

$$\begin{aligned} u^{iv}(x_n) &= \frac{d^2}{dx^2}[-q(x)u(x)+s(x)]\Big|_{x=x_n} \qquad (2.51)\\ &= -\frac{(q(x)u(x))_{n+1}-2(q(x)u(x))_n+(q(x)u(x))_{n-1}}{h^2}\\ &+ \frac{s(x_{n+1})-2s(x_n)+s(x_{n+1})}{h^2}+\mathcal{O}(h^2). \end{aligned}$$

Substituindo (2.50) em (2.48), e após conveniente arranjo, obtemos

$$\left(1+\frac{h^2}{12}q(x_{n+1})\right)u_{n+1}-2\left(1-\frac{5h^2}{12}q(x_n)\right)+\left(1+\frac{h^2}{12}q(x_{n-1})\right)u_{n-1}=$$
$$\frac{h^2}{12}(s_{n+1}+10s_n+s_{n-1}). \qquad (2.52)$$

Resolvemos, então, a equação linear (2.52) para u_{n+1} ou u_{n-1} (usando o método da bissecção por exemplo). Podemos tomar como condições iniciais para o método $u(0)=0$ e $u(h)=A<<1$.

O problema de determinar os autovalores (veja, por exemplo, Koonin [44]) é o de encontrar as energias E para as quais (2.48) tem solução, com a condição inicial

$$\psi(x_{min})=\psi(x_{max})=0. \qquad (2.53)$$

Suponha que exista E que satisfaça estas condições. Existe um ponto x_r (usualmente um dos pontos onde E_0 é igual a energia potencial) para o qual

$$\begin{aligned} \psi_l(x_r) &= \psi_r(x_r); \qquad (2.54)\\ \psi_l'(x_r) &= \psi_r'(x_r)\\ \Rightarrow \frac{\psi_l'(x_r)}{\psi_l(x_r)} &= \frac{\psi_r'(x_r)}{\psi_r(x_r)}\\ \therefore F(E) &= \frac{[\psi_l(x_r+h)-\psi_r(x_r-h)]-[\psi_r(x_r+h)-\psi_r(x_r-h)]}{2h\psi(x_r)}=0, \end{aligned}$$

onde no último passo usamos como estimativa das derivadas a fórmula de três pontos.

O algoritmo para determinar um valor correto para E se resume em:

1. Escolha uma região para determinar a solução;

2. escolha uma escala para procurarmos E. O valor mais baixo desta escala será uma primeira tentativa, E_0. De posse deste valor podemos determinar x_r como mostra a figura (2.3), o ponto de inflexão;

3. determine $\psi_l(x)$ da esquerda para $x_r + h$ e $\psi_r(x)$ da direita para $x_r - h$, usando o método de Numerov;

4. reescalamos as soluções para assegurar $\psi_l(x_r) = \psi_r(x_r)$, através da multiplicação de $\psi_l(x_r)$ por $\psi_r(x_r)/\psi_l(x_r)$ depois de $x = x_r + h$

5. calculamos

$$F(E_0) = \frac{[\psi_l(x_r + h) - \psi_r(x_r - h)] - [\psi_r(x_r + h) - \psi_r(x_r - h)]}{2h\psi(x_r)}; \tag{2.55}$$

6. resolvemos (2.55) para obter E_0 tal que $F(E_0) = 0$, com uma dada tolerância. Se não for encontrada solução mude o valor de E_0 até chegar ao valor máximo da região .

2.3 Comparação do Número de Estados Ligados $N_0(U)$

Como a teoria de Maxwell-Chern-Simons permite a existência de estados ligados [48], queremos, então, verificar se existe vantagem em utilizarmos uma teoria mais complicada para obter estes estados. Uma boa estimativa seria comparar a quantidade de estados ligados.

2.3.1 Determinação dos Valores

Classicamente, o número de estados ligados $N_0(U)$ de um operador Hamiltoniano H é dado pelo número de células de tamanho h^d, aonde h é a constante de Planck, no espaço de fase aonde o hamiltoniano clássico

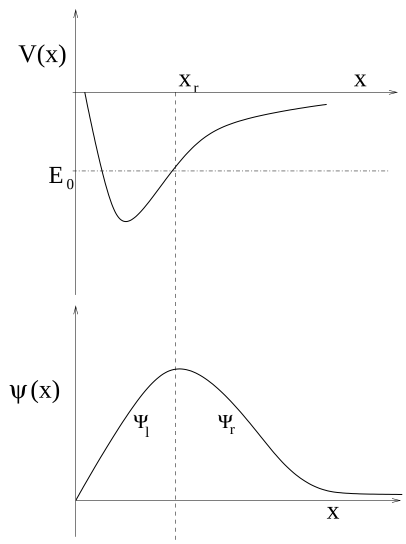


Figura 2.3: Potencial esquemático para a equação de Schrödinger em uma dimensão. Quando E_0 é um auto-valor a derivada é contínua no ponto de inflexão.

$H(x,p) = p^2 + U(x)$ é negativo. Seja $h = 2\pi$

$$N_0(U) \sim (2\pi)^{-d} \int\int_{H(x,p)\leq 0} dx\, dp = (2\pi)^{-d} \overbrace{\frac{\pi^{d/2}}{\Gamma(1+d/2)}}^{\substack{\text{Volume de uma bola} \\ \text{unitária em } \mathbb{R}^d}} \int_{\mathbb{R}^d} |U_-(x)|^{d/2} dx,$$

que é verdade no limite semi-clássico. Em 1961, M. Birman [20] e J. Schwinger [63], independentemente, determinaram que o número de estados ligados para $E \leq 0$ seria relacionado ao número de auto-estados do operador

$$K_E = |U|^{1/2}(-\nabla^2 + E)^{-1}|U|^{1/2}. \tag{2.56}$$

Pode-se provar que, se $U \in C_0^\infty(\mathbb{R}^d)$, $d = 2$, a norma do operador (2.56) diverge e qualquer potencial razoável, ou seja, que satisfaça às seguintes condições [17]:

$$\begin{aligned} &(i) \quad U \leq 0, \\ &(ii) \quad \int U < 0, \ \int |U| < \infty \\ &(iii) \quad \int\int U_+ < \infty, \int U_- = \infty \ \text{ aonde}\, U_\pm = sup(0, \pm U), \end{aligned} \tag{2.57}$$

possui minimamente um estado ligado.

Definida a função $N_E(U)$ que conta o número de estados ligados para qualquer $E \leq 0$, seja $U \leq W$, temos a seguinte propriedade:

$$N_E(W) \leq N_E(U). \tag{2.58}$$

No caso de potenciais centrais, o Hamiltoniano $H = -\nabla^2 + U(r)$ agora se transforma em uma família de Hamiltonianos radiais h_l, dados por

$$h_l = -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + \frac{\left(l + \frac{d-2}{d}\right)^2}{r^2} + U(r), \tag{2.59}$$

com $l = 0, 1, 2, \ldots$. Se Φ é uma auto-função de H, o problema de auto-valores se reduz a determinar o autovalor de h_l, introduzindo a função de onda

$$\Phi(r) = r^{\frac{1-d}{2}} u_l(r) Y_l(x/r),$$

e a degenerescência de cada nível é determinada pelo número H_l do polinômio harmônico de grau l em d dimensões

$$D_l = H_l - H_{l-2} \tag{2.60}$$

$$H_l = \frac{(d+l-1)!}{l!(d-1)!} \tag{2.61}$$

(se $d = 2$ e $l = 0$, claramente $D_l = 2$). O número total de estados ligados é relacionado com os valores de n_l por

$$N_0(U) = \sum_{l=0}^{\infty} D_l n_l(U). \tag{2.62}$$

Todas essa condições levam a uma estimativa dos valores de estados ligados dados por [36] :

$$n_l \leq \frac{1}{2l+d-2} \int_0^{\infty} U_-(r) r \, dr, \tag{2.63}$$

que é a condição de Bargmann [13], derivada de outro modo.

2.3.2 Eletrodinâmica de Podoslky-Chern-Simons no Plano e Estados Ligados

Consideramos um potencial da forma

$$U(r) = U_1(r) l + U_2(r). \tag{2.64}$$

(2.64) resume o potencial para duas partículas escalares, (2.44)

$$U(r) = -\frac{sQ^2}{\pi m a^4}\left[\frac{a^4}{s^2 r^2} + \frac{1}{r}\sum_j B_j \sqrt{|x_j|} K_1(\sqrt{|x_j|} r)\right] l + \frac{Q^2}{2\pi a^4}\sum_j A_j K_0(\sqrt{|x_j|} r), \tag{2.65}$$

onde $A_1 \equiv \frac{1+a^2x_1}{(x_1-x_2)(x_1-x_3)}$, $A_2 \equiv \frac{1+a^2x_2}{(x_2-x_1)(x_2-x_3)}$, $A_3 \equiv \frac{1+a^2x_3}{(x_3-x_1)(x_3-x_2)}$, $B_1 \equiv \frac{-(1+a^2x_1)^2}{s^2(x_1-x_2)(x_1-x_3)}$, $B_2 \equiv \frac{-(1+a^2x_2)^2}{s^2(x_2-x_1)(x_2-x_3)}$, $B_3 \equiv \frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)}$.

Para nossa análise, devemos extrair a parte negativa $U_-(r)$ de (2.65), o que é analiticamente complicado. Façamos uma majoração: como $U_-(r) = sup(0, -U(r)) \geq -|U(r)|$, usando (2.58)

$$N_0(-|U|) \geq N_0(U_-). \tag{2.66}$$

Ainda para que o potencial em (2.64) tenha um estado ligado as condições (2.57) devem valer; então, deve existir um valor l_{max} tal que (ii) naquele conjunto seja válida. Assim, trocando a soma em (2.62) por integrais, calculadas agora no limite superior l_{max} e usando a expressão (2.63), com $d = 2$, (2.66) e o fato que

$$\int |f(x)|dx \geq \left| \int f(x)dx \right|, \tag{2.67}$$

então

$$\int -|f(x)|dx \leq -\left| \int f(x)dx \right|, \tag{2.68}$$

teremos

$$\begin{aligned} N^{PCS}(l_{max}) &\leq \int_{l=1}^{l_{max}} \int_0^\infty \frac{2}{2l} U_-(r) r\, dr dl \leq - \int_{l=1}^{l_{max}} \int_0^\infty \frac{1}{l} |U(r)| r\, dr dl \\ &\leq - \int_{l=1}^{l_{max}} \frac{1}{l} \left| \int_0^\infty U(r) r\, dr \right| dl \leq \int_{l=1}^{l_{max}} \frac{1}{l} \int_0^\infty U(r) r\, dr dl \\ &\leq \left| \int_{l=1}^{l_{max}} \int_0^\infty \frac{1}{l} U(r) r\, dr dl \right| \end{aligned} \tag{2.69}$$

onde $D_l = 2$ em $d = 2$ (caso em que $l = 0$ é singular).

Em nossa análise, queremos a razão $N^{PCS}(l_{max})/N^{MCS}(l_{max})$, uma estimativa da razão do número de estados ligados na Eletrodinâmica de Podolsky-Chern-Simons e a de Maxwell-Chern-Simons, ou seja

$$N^{PCS}(l_{max})/N^{MCS}(l_{max}) \leq \frac{\left| \int_{l=1}^{l_{max}} \int_0^\infty \frac{1}{l} U^{PCS}(r) r\, dr dl \right|}{\left| \int_{l=1}^{l_{max}} \int_0^\infty \frac{1}{l} U^{MCS}(r) r\, dr dl \right|}. \tag{2.70}$$

Substituindo (2.64) em (2.69) teríamos

$$\begin{aligned} N^{PCS}(l_{max}) &\leq \left| \int_{r=0}^\infty \left(\int_{l=1}^{l_{max}} U_1(r) + \frac{U_2(r)}{l} \right) r\, dr \right| \qquad (2.71) \\ &= \left| \int_{r=0}^\infty r\, dr \left[U_1(r)(l_{max} - 1) + U_2(r) ln(l_{max}) \right] \right|. \end{aligned}$$

No caso da Eletrodinâmica de Podolsky-Chern-Simons, determinar l_{max} é resolver a inequação

$$-\frac{sQ^2 l_{max}}{\pi m a^4} \int_{r_0}^\infty \sum_j B_j \sqrt{|x_j|} K_1(\sqrt{|x_j|} r) dr \tag{2.72}$$

$$+\frac{Q^2}{2\pi a^4}\int_{r_0}^{\infty}\sum_j A_j K_0(\sqrt{|x_j|}r)r\,dr + l_{max}\left(\frac{l_{max}}{m}-\frac{Q^2}{\pi ms}\right)\int_{r_0}^{\infty}\frac{dr}{r} < 0$$

$$l_{max}\left(\frac{l_{max}}{m}-\frac{Q^2}{\pi ms}\right) ln(r)|_{r_0}^{\infty} + \frac{sQ^2 l_{max}}{\pi m a^4}\sum_j B_j K_0(\sqrt{|x_j|}r)\ |_{r_0}^{r_{max}}$$

$$-\frac{Q^2}{2\pi a^4}\sum_j \frac{A_j}{\sqrt{|x_j|}} r K_1(\sqrt{|x_j|}r)\ |_{r_0}^{r_{max}} < 0,$$

onde utilizamos as integrais tabeladas da função de Hankel

$$\int x^{p+1}H_p^{(1)}(x)dx = x^{p+1}H_{p+1}^{(1)}(x), \quad \int x^{-p+1}H_p^{(1)}(x)dx = -x^{-p+1}H_{p-1}^{(1)}(x),$$

e a definição da função de Bessel modificada de ordem ν

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix), \tag{2.73}$$

para obtermos

$$\int x\, K_0(\alpha x)dx = -\frac{x}{\alpha}K_1(\alpha x), \quad \int K_1(\alpha x)dx = -\frac{1}{\alpha}K_0(\alpha x).$$

Ainda faremos uso dos limites assintóticos

$$K_0(x) = \begin{cases} -ln(x) & \text{se } x \to 0 \\ \sqrt{\frac{\pi}{2x}}e^{-x} & \text{se } x \to \infty \end{cases}, \quad K_1(x) = \begin{cases} \frac{1}{x} & \text{se } x \to 0 \\ \sqrt{\frac{\pi}{2x}}e^{-x} & \text{se } x \to \infty \end{cases},$$

para escrever (2.72) como

$$l_{max}\left(\frac{l_{max}}{m}-\frac{Q^2}{\pi ms}\right) ln(r)|_{r_0}^{\infty} + \frac{sQ^2 l_{max}}{\pi m a^4}\sum_j B_j \ln(\sqrt{|x_j|}r_0)$$

$$+ \quad \frac{Q^2}{2\pi a^4}\sum_j \frac{A_j}{|x_j|} < 0. \tag{2.74}$$

O primeiro termo de (2.74) pode ser eliminado (com $l_{max} \neq 0$) se $l_{max} = \frac{Q^2}{\pi s}$, e assim não nos preocupamos com essa divergência; essa é a primeira limitação em nossa análise que será considerada na nossa comparação. Assim até aqui, dados os valores de a e s temos as soluções x_j de $x^3+\frac{2}{a^2}x^2+\frac{x}{a^4}+\frac{s^2}{a^4}=0$; com o valor de Q, a carga elétrica em $2+1$ dimensões, obtemos um valor de l_{max}. Para que não tenhamos uma indeterminação, ainda devemos limitar a coordenada radial tal que a análise será válida somente se $r_0 < r < r_{max}$. Usando o fato que $\sum_j B_j = 0$, os dois últimos termos de (2.74) força-nos ainda a limitar o valor da massa m da partícula analisada

$$\begin{aligned} & \frac{Q^2}{\pi m}\sum_j B_j \ln\left(r_0\sqrt{|x_j|}\right) < \frac{Q^2}{\pi m}\sum_j B_j \ln\left(\sqrt{|x_j|}\right) \qquad (2.75) \\ + \quad & \frac{Q^2}{\pi m}\ln\left(r_0\right)\sum_j B_j = -\frac{1}{2}\sum_j \frac{A_j}{|x_j|} \\ & \Rightarrow \frac{Q^2}{\pi m}\sum_j B_j \ln\left(\sqrt{|x_j|}\right) < -\frac{1}{2}\sum_j \frac{A_j}{|x_j|} \end{aligned}$$

a um valor que satisfaça (2.75).

De posse de l_{max}, podemos fazer a integral (2.71)

$$\begin{aligned} N^{PCS}(l_{max}) \quad \leq \quad & \left|\left(\frac{Q^2}{\pi s} - 1\right)\left[-\frac{Q^2}{\pi s m}\int_{r_0}^{r_{max}}\frac{dr}{r} - \frac{sQ^2}{\pi m a^4}\int_{r_0}^{r_{max}}\sum_j B_j\sqrt{|x_j|}K_1(\sqrt{|x_j|}r)dr\right]\right| \\ + \quad & \left|\frac{Q^2}{2\pi a^4}\ln\left(\frac{Q^2}{2\pi s}\right)\int_{r_0}^{r_{max}}\sum_j A_j K_0(\sqrt{|x_j|}r)r\,dr\right| \\ = \quad & \left|\left(1 - \frac{Q^2}{\pi s}\right)\left[\frac{Q^2}{\pi s m}ln(r)\,\Big|_{r_0}^{r_{max}} - \frac{sQ^2}{\pi m a^4}\sum_j B_j K_0(\sqrt{|x_j|}r)\,\Big|_{r_0}^{r_{max}}\right]\right. \\ - \quad & \left.\frac{Q^2}{2\pi a^4}ln\left(\frac{Q^2}{2\pi s}\right)\sum_j r\frac{A_j}{\sqrt{|x_j|}}K_1(\sqrt{|x_j|}r)\,\Big|_{r_0}^{r_{max}}\right| \\ \approx \quad & \left|\left(1 - \frac{Q^2}{\pi s}\right)\left[\frac{Q^2}{\pi s m}ln\left(\frac{r_{max}}{r_0}\right) - \frac{sQ^2}{\pi m a^4}\sum_j B_j ln(\sqrt{|x_j|})\right]\right. \\ + \quad & \left.\frac{Q^2}{2\pi a^4}ln\left(\frac{Q^2}{2\pi s}\right)\sum_j \frac{A_j}{|x_j|}\right|, \qquad (2.76) \end{aligned}$$

obtendo uma estimativa para o número de estados ligados nesta teoria.

2.3.3 Eletrodinâmica de Maxwell-Chern-Simons e seus Estados Ligados

A mesma análise é feita, agora usando o potencial

$$U(r) = -\frac{Q^2}{\pi m s r^2}\left(1 - srK_1(sr)\right) l + \frac{Q^2}{2\pi} K_0(sr), \tag{2.77}$$

ou seja, primeiramente determinamos l_{max} utilizando o potencial efetivo, tal que $\int U_{eff}(r,l)\, rdr < 0$, se $l < l_{max}$,

$$\begin{aligned} & l_{max}\left(\frac{l_{max}}{m} - \frac{Q^2}{\pi m s}\right) ln(r)\,|_{r_0}^{\infty} - \frac{Q^2 l_{max}}{\pi m s} K_0(sr)\,|_{r_0}^{\infty} - \frac{Q^2}{2\pi s} r K_1(sr)\,|_{r_0}^{\infty} < 0 \\ & \overset{r\to\infty}{\Longrightarrow} l_{max}\left(\frac{l_{max}}{m} - \frac{Q^2}{\pi m s}\right) ln(r)\,|_{r_0}^{\infty} - \frac{Q^2 l_{max}}{\pi m s} ln(sr_0) + \frac{Q^2}{2\pi s^2} < 0, \end{aligned} \tag{2.78}$$

utilizando as expressões assintóticas. Vemos que o primeiro termo de (2.78)contém a divergência logarítmica, que pode ser eliminada pela escolha de $l_{max} = \frac{Q^2}{\pi s}$. De (2.78) podemos obter o valor da massa m , mas para efeito de comparação com o caso anterior, a massa será aquela determinada em (2.75).

Nesse caso, (2.71) tem a forma abaixo:

$$\begin{aligned} N^{MCS}(l_{max}) \;\leq\; & \left| \int_{r_0}^{r_{max}} \frac{Q^2}{2\pi} ln\left(\frac{Q^2}{\pi s}\right) K_0(sr) r\, dr \right. \\ & \left. - \; \frac{Q^2}{\pi m}\left(\frac{Q^2}{\pi s} - 1\right)\left[\frac{1}{s}\int_{r_0}^{r_{max}} \frac{dr}{r} - \int_{r_0}^{r_{max}} K_1(sr) dr\right] \right| \\ \approx \; & \left| \frac{Q^2}{2\pi s^2} ln\left(\frac{Q^2}{\pi s}\right) - \frac{Q^2}{\pi m s}\left(\frac{Q^2}{\pi s} - 1\right) ln(sr_{max}) \right| . \end{aligned} \tag{2.79}$$

2.3.4 Comparação Entre os Dois Casos

A comparação foi feita resolvendo numericamente os valores obtidos em (2.76) e (2.79) para determinar a fração $N^{PCS}(l_{max})/N^{MCS}(l_{max})$.

Vejamos um caso específico: seja $Q^2 = 56,75\, MeV^{\frac{1}{2}}$, $a = 0,02\, MeV^{-1}$ e $s = 1,5\, MeV$. Depois de resolver a equação que determina os pólos do propagador e nos fornece os x_j (portanto A_j e B_j), determinamos, solucionando (2.75), qual seria a massa mínima para que uma partícula fosse ligada, e coletamos um valor menor, no caso

$$m = 0,18085 MeV, \tag{2.80}$$

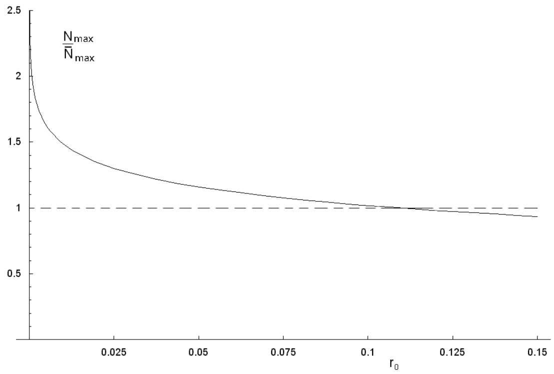


Figura 2.4: Razão $N^{PCS}/N^{MCS} \times r_0$.

e também fizemos $l_{max} = Q^2/(\pi s) \approx 12$, $r_{max} = 8 \times 10^{-6}\, MeV^{-1}$ e $r_0 = 1 \times 10^{-23}\, MeV^{-1}$. Com esse valores em mãos calculamos $N^{PCS}(l_{max})/N^{MCS}(l_{max}) = 4,48022$. Neste caso tivemos um aumento no número de estados ligados, respeitando nossas aproximações.

Mesmo levando em conta que este valor é somente uma estimativa (e somente deste caso específico), achamos que realmente o termo de Podolsky permite um caráter confinante maior que o da Eletrodinâmica de Maxwell-Chern-Simons (ou seja uma maior possibilidade de energias e portanto temperaturas) dentro de uma faixa de valores.

No entanto, nem sempre o potencial Podolsky-Chern-Simons oferece um maior número de estados ligados comparados com os valores fornecidos pela Eletrodinâmica Maxwell-Chern-Simons; supondo $r_{max} = 5 \times 10^{-3}\, MeV = 10\AA$, e variando somente r_0 (dado em ângstrons), teremos o gráfico de N^{PCS}/N^{MCS} dados pela figura (2.4), onde podemos observar um ganho abaixo de $0,1\AA$.

2.4 Possíveis Correções para o Propagador da Teoria

Para finalizar o capítulo vamos determinar a forma do propagador corrigido para a teoria Maxwell-Chern-Simons-Podolsky, devido à auto-energia do fóton. Queremos escrever o propagador usando os operadores tipo Barne-

Rivers:

$$P_{\mu\nu} = y_1\theta_{\mu\nu} + y_2\omega_{\mu\nu} + y_3 S_{\mu\nu},$$

a partir do seu inverso:

$$\mathcal{D}_{\mu\nu} = k_1\theta_{\mu\nu} + k_2\omega_{\mu\nu} + k_3 S_{\mu\nu},$$

com os coeficientes dados por:

$$k_1 = \frac{y_1}{y_1^2 + \Box y_3^2}\delta(x-y),\ k_2 = \frac{1}{y_2}\delta(x-y),\ \ k_3 = \frac{-y_3}{y_1^2 + \Box y_3^2}\delta(x-y).$$

A correção ao propagador é graficamente mostrado na figura (2.5):

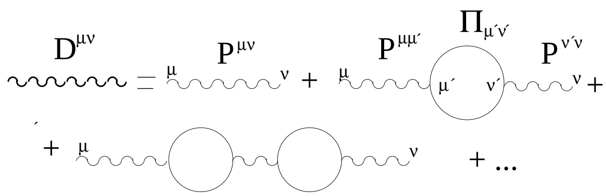


Figura 2.5: Correções ao propagador

O que é corresponde a uma série geométrica

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Assim

$$\begin{aligned} iP^{\mu\nu}(q) &= iP^{\mu\nu}(q) + iP^{\mu\mu'}(q)(-i\Pi_{\mu'\nu'}(q))iP^{\nu'\nu}(q) + \cdots \\ &= iP^{\mu\mu'}(q)[\delta^{\nu}_{\mu'} - (-i\Pi_{\mu'\nu'}(q))iP^{\nu'\nu}(q)]^{-1} \Rightarrow \\ -iD^{\mu\nu}(q) &= -i\mathcal{D}^{\mu\nu}(q) - (-i)\mathcal{D}^{\mu\mu'}(q)[-i\Pi_{\mu'\nu'}(q)]iP^{\nu'\nu}(q) \Rightarrow \\ D^{\mu\nu}(q) &= \mathcal{D}^{\mu\nu}(q) - \Pi^{\mu\nu}(q), \end{aligned} \tag{2.81}$$

logo temos

$$\mathcal{D}^{\mu\nu}_{corr}(q) = \mathcal{D}^{\mu\nu}(q) - \Pi^{\mu\nu}(q) = \mathcal{D}^{\mu\nu}(q) - \Pi(q)\theta^{\mu\nu}.$$

Portanto

$$\mathcal{D}^{\mu\nu}_{corr}(q) = \left[\frac{\overbrace{y_1 + \Pi(q)(y_1^2 + \Box y_3^2)}^{A}}{y_1^2 + \Box y_3^2} \theta^{\mu\nu} + \frac{\omega_{\mu\nu}}{y_2} - \frac{\overbrace{y_3}^{B}}{y_1^2 + \Box y_3^2} S_{\mu\nu} \right] \delta(x-y).$$

Dada a relação $\mathcal{D}^{\alpha\mu}_{corr} P_{\mu\beta} = \delta^{\alpha}_{\beta}$, invertemos usando novamente os operadores de Barne-Rivers, obtendo

$$P^{\mu\nu} = \left[\frac{A}{A^2 + B^2\Box}\theta^{\mu\nu} + \frac{1}{y_2}\omega^{\mu\nu} - \frac{B}{A^2 + B^2\Box} S^{\mu\nu} \right] \delta(x-y),$$

e após alguma álgebra chegamos à

$$\begin{aligned} P^{\mu\nu} &= \frac{ak^4 - k^2 - \Pi(k)}{a^2k^8 - 2ak^6 - 2ak^4\Pi(k) + k^4 + 2k^2\Pi(k) + \Pi(k)^2 - s^2k^2}\theta^{\mu\nu} + \frac{k^2}{\lambda_2}\omega^{\mu\nu} \\ &- \frac{s}{a^2k^8 - 2ak^6 - 2ak^4\Pi(k) + k^4 + 2k^2\Pi(k) + \Pi(k)^2 - s^2k^2} S^{\mu\nu}, \quad (2.82) \end{aligned}$$

que no limite de $a \to 0$ coincide com o obtido em [48].

No entanto, o que podemos observar é que se torna extremamente complicado e desnecessário trabalhar com o propagador acima num processo de espalhamento, já que podemos obter estados ligados usando apenas cálculos em árvore.

Esses resultados nos estimulam a verificar a existência de estados ligados entre partículas fermiônicas. É o que faremos no próximo capítulo.

Capítulo 3

Eletrodinâmica de Ordem Superior em $(2+1)D$: Partículas Fermiônicas

Neste capítulo, iremos analisar a existência de estados ligados através do estudo do espalhamento de dois férmions. Devemos ainda determinar a influência da temperatura na existência, ou não, destes estados.

Vale ressaltar alguns trabalhos feitos para determinar o potencial entre partículas fermiônicas em $(2+1)D$, tais como [34, 35, 21] e [16] .

3.1 Equação de Dirac e Soluções

As matrizes gama no plano apresentam as seguintes propriedades, conseqüências da álgebra de Clifford a que satisfazem :

$$
\begin{aligned}
\gamma^{\mu}\gamma^{\nu} &= \eta^{\mu\nu} - i\epsilon^{\mu\nu\rho}\gamma_{\rho}, \\
tr(\gamma^{\mu}\gamma^{\nu}) &= 2\eta^{\mu\nu}, \\
tr(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}) &= -2i\epsilon^{\mu\nu\rho}, \\
tr(\gamma^{\mu}\gamma^{\nu}\gamma^{\alpha}\gamma^{\beta}) &= 2(\eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\nu}\eta^{\alpha\beta}),
\end{aligned}
\tag{3.1}
$$

que podem ser escritas em uma representação 2×2, usando as matrizes de Pauli:

$$\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \gamma^2 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.2}$$

A equação de Dirac assume a forma usual

$$(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0, \tag{3.3}$$

que corresponde a soluções de energia positiva (portanto elétrons), $\Psi(x) = u(p)e^{-ipx}$. Então, (3.3) corresponde a:

$$(\gamma^\mu p_\mu - m)u(p) = 0. \tag{3.4}$$

As soluções de energia negativa, $\Phi(x) = v(p)e^{ipx}$, são soluções da equação que assume a forma:

$$(\gamma^\mu p_\mu + m)v(p) = 0. \tag{3.5}$$

Vamos nos concentrar no problema de determinar uma solução de (3.4). No referencial de repouso, $p = p_0 = (m, \vec{0})$. Logo, se

$$(\gamma^0 p_0 - m)u(p) = \left[\begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right]\begin{pmatrix} a \\ b \end{pmatrix} = 0,$$

impondo que $u(p_0)u^\dagger(p_0) = 1$, a solução é $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Para determinar a solução para qualquer momento p, ou aplicamos um "boost" à solução obtida, ou nos valemos do fato de que

$$(\not{p} + m)(\not{p} - m) = p^2 - m^2 = 0, \tag{3.6}$$

e as equações (3.4) e (3.5) sugerem que

$$\begin{aligned} u(p) &= C(m + \not{p})u(0) \\ v(p) &= C'(m - \not{p})v(0). \end{aligned} \tag{3.7}$$

Para determinar a constante C, iremos escrever $\bar{u}(p) = u^\dagger \gamma^0$ e impor as condições

$$\begin{aligned} \bar{u}(p)u(p) &= \bar{u}(0)u(0) = 1 \\ \bar{v}(p)v(p) &= \bar{v}(0)v(0) = 1. \end{aligned} \tag{3.8}$$

Então, usando explicitamente $p_\mu = (p_0, \vec{p}) = (E, \vec{p})$, finalmente achamos um valor para C:

$$
\begin{aligned}
\bar{u}(p)u(p) &= |C|^2\bar{u}(0)(m+\not{p})(m+\not{p})u(0) = |C|^2\bar{u}(0)(m^2+2m\not{p}+\overbrace{\not{p}^2}^{=m^2})u(0) \\
&= |C|^2\bar{u}(0)(2m^2+2m\not{p})u(0) = 2m|C|^2\bar{u}(0)(m+\not{p})u(0) \\
&= 2m|C|^2\bar{u}(0)[m+p_0\gamma^0-\cancelto{0}{\vec{\gamma}\cdot\vec{p}}]u(0) = 2m|C|^2(m+E) = 1 \\
&\therefore\ C = \frac{1}{\sqrt{2m(m+E)}},
\end{aligned}
$$

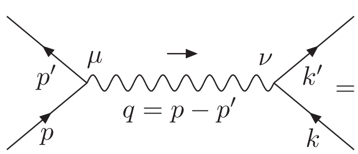
e, portanto,

$$
u(p) \;=\; \frac{\not{p}+m}{\sqrt{2m(m+E)}}u(0) = \frac{m+\gamma^0E-\gamma^ip_i}{\sqrt{2m(m+E)}}\begin{pmatrix}1\\0\end{pmatrix} \tag{3.9}
$$

na representação escolhida.

3.2 Espalhamento de Dois Férmions

Usando as regras de Feynman usuais temos o seguinte diagrama para o espalhamento de dois elétrons



$$
= i\mathcal{M} = \frac{(-ie)^2\bar{u}(p')\gamma^\mu u(p)}{a^2q^4-q^2}(-i)\left[\eta^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right]\bar{u}(k')\gamma^\nu u(k).
$$

Observe que não utilizamos o diagrama de troca: como queremos a aproximação não-relativística, fazendo uma transformada de Fourier, a parte da matriz S que garante a anti-simetria não precisa ser usada, pois esta garante somente a anti-simetria dos estados e não a forma da energia potencial e dessa forma recuperamos a noção clássica de trajetória.

Expandimos esta expressão

$$
i\mathcal{M} = -i\frac{\bar{u}(p')\gamma^\mu u(p)\bar{u}(k')\gamma_\mu u(k)}{a^2q^4-q^2}+i\frac{\bar{u}(p')\gamma^\mu(p'-p)_\mu u(p)\bar{u}(k')\gamma_\nu(p'-p)^\nu u(k)}{q^2(a^2q^4-q^2)}, \tag{3.10}
$$

onde o segundo termo de (3.10) é nulo, pois aplicando (3.4), teremos

$$\bar{u}(p')(\not{p} - \not{p}')u(p) = \bar{u}(p')[(\not{p} - m) - (\not{p}' - m)]u(p) = 0.$$

Trabalhamos, então, com as componentes do primeiro termo de (3.10) separadamente, usando explicitamente (3.2) e (3.9), com $p_\mu = (p_0, p^i)$:

$$\bar{u}(p')\gamma^\mu u(p) \quad = \quad \bar{u}(0)\frac{(m + \sigma^3 p'_0 - i\sigma^i p'_i)}{\sqrt{2m(m+p'_0)}}(\eta^{\mu 0}\sigma_3 + \eta^{\mu j} i\sigma_j)\frac{(m + \sigma^3 p_0 - i\sigma^k p_k)}{\sqrt{2m(m+p_0)}}u(0). \tag{3.11}$$

Iremos trabalhar o numerador desta expressão:

$$\begin{aligned}
& \bar{u}(0)[m^2\eta^{\mu 0}\sigma^3 + 2\,p'_0\,\eta^{\mu 0}m - im\eta^{\mu 0}\sigma^3\sigma^k p_k + im^2\eta^{\mu\,i}\sigma^i + 2\,i\sigma^3 p'_0\,\eta^{\mu\,i}\sigma^i m \\
+ \quad & m\eta^{\mu\,i}\sigma^i\sigma^k p_k + \sigma^3 {p'_0}^2\eta^{\mu 0} - ip'_0\,\eta^{\mu 0}\sigma^k p_k + i{p'_0}^2\eta^{\mu\,i}\sigma^i + \sigma^3 p'_0\,\eta^{\mu\,i}\sigma^i\sigma^k p_k \\
- \quad & i\sigma^j p'_j\,\eta^{\mu 0}\sigma^3 m - i\sigma^j p'_j\,\eta^{\mu 0}p'_0 - \sigma^j p'_j\,\eta^{\mu 0}\sigma^3\sigma^k p_k + \sigma^j p'_j\,\eta^{\mu\,i}\sigma^i m \\
+ \quad & \sigma^j p'_j\,\eta^{\mu\,i}\sigma^i\sigma^3 p'_0 - i\sigma^j p'_j\,\eta^{\mu\,i}\sigma^i\sigma^k p_k]u(0),
\end{aligned}$$

que, após alguma álgebra matricial (alguns elementos já possuem potências quadráticas no momento p^i e já serão desprezados), também eliminando os termos nulos, escreve-se como segue:

$$\bar{u}(0)[m^2\eta^{\mu 0}\sigma^3 + 2\,p'_0\,\eta^{\mu 0}m + m\eta^{\mu\,i}\sigma^i\sigma^k p_k + \sigma^j p'_j\,\eta^{\mu\,i}\sigma^i m + \sigma^3 {p'_0}^2\eta^{\mu 0}]u(0)$$

Sem perda de generalidade, podemos trabalhar no referencial centro de massa, utilizando o momento relativo $\vec{p}$,

$$\begin{aligned}
p_\mu &= (E, \vec{p}), \qquad k_\mu = (E, -\vec{p}) \\
p'_\mu &= (E, \vec{p}\,'), \qquad k'_\mu = (E, -\vec{p}\,'),
\end{aligned}$$

o que faremos daqui em diante.

Após simplificar a expressão (3.11), iremos retornar novamente ao cálculo de (3.10):

$$\begin{aligned}
\mathcal{N} \quad = \quad & \bar{u}(0)[{p'_0}^4 + 6\,{p'_0}^2 m^2 + m^4 + 4\,\sigma^3 {p'_0}^3 m - \sigma^3 {p'_0}^2\eta^{\mu 0}m\eta^{\mu\,r}\sigma^r\sigma^s p_s \\
- \quad & \sigma^3 {p'_0}^2\eta^{\mu 0}\sigma^t p'_t\,\eta^{\mu\,u}\sigma^u m + \sigma^j p'_j\,\eta^{\mu\,i}\sigma^i m\sigma^3 {p'_0}^2\eta^{\mu 0} \\
- \quad & \sigma^j p'_j\,\eta^{\mu\,i}\sigma^i m^2\sigma^t p'_t\,\eta^{\mu\,u}\sigma^u + \sigma^j p'_j\,\eta^{\mu\,i}\sigma^i m^3\eta^{\mu 0}\sigma^3 \\
+ \quad & 2\,\sigma^j p'_j\,\eta^{\mu\,i}\sigma^i m^2 p'_0\,\eta^{\mu 0} - m^2\eta^{\mu\,i}\sigma^i\sigma^k p_k\,\sigma^t p'_t\,\eta^{\mu\,u}\sigma^u
\end{aligned}$$

$$
\begin{aligned}
&- \quad \sigma^j p'_j\, \eta^{\mu\, i}\sigma^i m^2 \eta^{\mu\, r}\sigma^r\sigma^s p_s - 2\, p'_0\, \eta^{\mu 0} m^2 \sigma^t p'_t\, \eta^{\mu\, u}\sigma^u \\
&+ \quad m^3 \eta^{\mu\, i}\sigma^i\sigma^k p_k\, \eta^{\mu 0}\sigma^3 + 2\, m^2 \eta^{\mu\, i}\sigma^i\sigma^k p_k\, p'_0\, \eta^{\mu 0} \\
&- \quad m^2 \eta^{\mu\, i}\sigma^i\sigma^k p_k\, \eta^{\mu\, r}\sigma^r\sigma^s p_s - 2\, p'_0\, \eta^{\mu 0} m^2 \eta^{\mu\, r}\sigma^r\sigma^s p_s \\
&- \quad m^3 \eta^{\mu 0}\sigma^3 \eta^{\mu\, r}\sigma^r\sigma^s p_s - m^3 \eta^{\mu 0}\sigma^3\sigma^t p'_t\, \eta^{\mu\, u}\sigma^u \\
&+ \quad 4\, m^3\sigma^3 p'_0 + m\eta^{\mu\, i}\sigma^i\sigma^k p_k\, \sigma^3 {p'_0}^2 \eta^{\mu 0}]\frac{u(0)}{4m^2(m+E)^2},
\end{aligned}
$$

onde os termos proporcionais a η^{0i} são nulos, devido à métrica escolhida; logo:

$$
\begin{aligned}
\mathcal{N} \quad &= \quad \frac{\bar{u}(0)}{4m^2(m+E)^2}[{p'_0}^4 + 6{p'}^2_0 m^2 + m^4 + 4m\sigma^3 {p'}^3_0 + 4m^3\sigma^3 p'_0 \qquad (3.12) \\
&- \quad 2m^2(\sigma^k p_k)(\sigma^i p'_i) - m^2(\sigma^k p_k)(\sigma^i p_i) - m^2(\sigma^k p'_k)(\sigma^i p'_i)]u(0).
\end{aligned}
$$

Na aproximação não-relativística, desconsideramos termos de ordem maior que a linear nos momentos, pois $|\vec{p}|/m, |\vec{p}\,'|/m << 1$. Com isso, ignoramos a contribuição dos dois últimos termos de (3.12), com auxílio das relações

$$
\begin{aligned}
\bar{u}(0)\sigma^3 u(0) \quad &= \quad 1 \\
\bar{u}(0)(\sigma_i A^i)(\sigma_j B^j)u(0) \quad &= \quad \bar{u}(0)[A^1 B_1 + A^2 B_2 + i\sigma_3(A^1B^2 - A^2B^1)]u(0) \\
&= \quad \vec{A}\cdot\vec{B} + i\vec{A}\times\vec{B} \\
\bar{u}(0)\sigma^k(\sigma_i A^i)(\sigma_j B^j)\sigma_k u(0) \quad &= \quad \bar{u}(0)\sigma^k[\vec{A}\cdot\vec{B} + i\vec{A}\times\vec{B}]\sigma_k u(0) \\
&= \quad \vec{A}\cdot\vec{B} - i\vec{A}\times\vec{B}
\end{aligned}
$$

com produto vetorial definido em $(2+1)D$ tal que $\vec{A}\times\vec{B} = A_1B_2 - A_2B_1$.

Usamos agora a definição do momento linear trocado no espalhamento $\vec{q} = \vec{p} - \vec{p}\,' \Rightarrow \vec{p}\,' = \vec{p} - \vec{q}$ e também que, na aproximação não-relativística, $E = \sqrt{\vec{p}^2 + m^2} \approx m + \cancelto{0}{\frac{|\vec{p}|^2}{2m}} + \mathcal{O}(p^4)$, reescrevemos (3.12):

$$
\begin{aligned}
\mathcal{N}_{NR} \quad &\approx \quad \frac{1}{4m^2(m+E)^2}[16m^4 + 2im^2(\vec{q}\times\vec{p})] \\
&= \quad 1 + \frac{i}{8m^2}(\vec{q}\times\vec{p}),
\end{aligned}
$$

que, junto com a aproximação não-relativística do denominador de (3.10) (que é obtida com a simples substituição $q^2 = q^\mu q_\mu \approx -\vec{q}^2$, pois $q^0 \approx 0$), nos leva a

$$
i\mathcal{M}_{NR} = -i\frac{e^2}{a^2\vec{q}^4 + \vec{q}^2}\left(1 + \frac{i}{8m^2}\vec{q}\times\vec{p}\right). \qquad (3.13)
$$

3.2.1 Cálculo da Energia Potencial

Iremos aplicar (1.46) para o cálculo da energia potencial, com $Q = e$ e $\mathcal{N}_{NR}(q)$ dado em (3.13),

$$U(r) = \lim_{\zeta\to 0}\frac{e^2}{2\pi}\int_0^\infty J_0(|\vec{q}||\vec{r}|)|\vec{q}|\left[\frac{1}{\vec{q}^2+\zeta^2}-\frac{1}{\vec{q}^2+\frac{1}{a^2}}+\left(\frac{1}{\vec{q}^2+\zeta^2}-\frac{1}{\vec{q}^2+\frac{1}{a^2}}\right)\frac{i}{8m^2}\vec{q}\times\vec{p}\right]dq, \tag{3.14}$$

e, novamente, usamos a integral tabelada (1.26) e o limite assintótico de $K_0(x)$, $-ln(x)$, para chegarmos ao resultado das duas primeiras integrais de (3.14),

$$\lim_{\zeta\to 0}\frac{e^2}{2\pi}\int_0^\infty J_0(|\vec{q}||\vec{r}|)|\vec{q}|\left[\frac{1}{\vec{q}^2+\zeta^2}-\frac{1}{\vec{q}^2+\frac{1}{a^2}}\right]dq = -\frac{e^2}{2\pi}\left[ln(|\vec{r}|)+K_0\left(\frac{|\vec{r}|}{a}\right)\right], \tag{3.15}$$

A segunda integral segue como (2.38), utilizando a relação (2.39),

$$\vec{q}\times\vec{p} = -\vec{p}\times\left(-i\vec{\nabla}\right) = i\vec{p}\times\left(\frac{\vec{r}}{|\vec{r}|}\frac{\partial}{\partial r}\right), \tag{3.16}$$

e o limite (2.41), permitindo escrever esta integral como sendo

$$\begin{aligned}
&\lim_{\zeta\to 0}\frac{e^2}{2\pi}\frac{i}{8m^2}\int_0^\infty J_0(|\vec{q}||\vec{r}|)|\vec{q}|\left(\frac{1}{\vec{q}^2+\zeta^2}-\frac{1}{\vec{q}^2+\frac{1}{a^2}}\right)\vec{q}\times\vec{p}\,dq = \qquad (3.17)\\
&= \frac{ie^2}{16\pi m^2|\vec{r}|}\lim_{\zeta\to 0} i\overbrace{\vec{p}\times\vec{r}}^{-l}\frac{\partial}{\partial r}\int_0^\infty J_0(|\vec{q}||\vec{r}|)|\vec{q}|\left(\frac{1}{\vec{q}^2+\zeta^2}-\frac{1}{\vec{q}^2+\frac{1}{a^2}}\right)dq\\
&= \frac{e^2}{16\pi m^2|\vec{r}|}l\lim_{\zeta\to 0}\left[-K_1\left(\zeta|\vec{r}|\right)\zeta+\frac{K_1\left(\frac{|\vec{r}|}{a}\right)}{a}\right]\\
&= -\frac{e^2}{2\pi m^2 r^2}\left[1-\frac{|\vec{r}|}{a}K_1\left(\frac{|\vec{r}|}{a}\right)\right]l.
\end{aligned}$$

onde l é o momento angular da partícula.

Logo, o potencial é dado por (3.15)+(3.17), levando em conta esta observação, ou seja

$$U(r) = -\frac{e^2}{2\pi}\left[ln(|\vec{r}|)+K_0\left(\frac{|\vec{r}|}{a}\right)\right]-\frac{e^2}{16\pi m^2r^2}l\left[1-\frac{|\vec{r}|}{a}K_1\left(\frac{|\vec{r}|}{a}\right)\right] \tag{3.18}$$

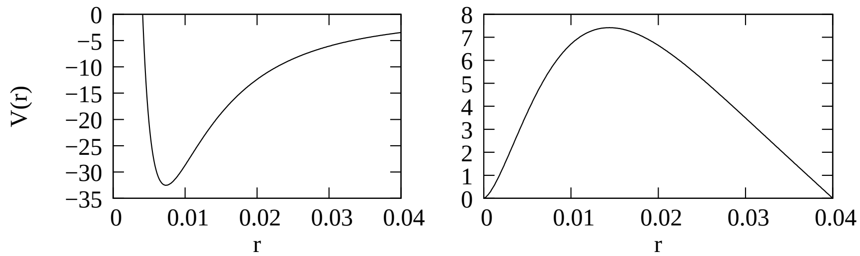


Figura 3.1: Estado ligado no espalhamento elétron-elétron. $e^2 = 100\, MeV$, $M = 0.511\, MeV$, $a = 0.006 MeV^{-1}$, $l = 1$ e a energia do auto-estado é de $-5,77\, MeV$.

Conhecendo o potencial ,começamos as simulações numéricas. Alguns resultados numéricos são mostrados na figura (3.1), evidenciando a possibilidade de existência de estados ligados nesta teoria, nas condições dadas. Observe que não há nenhuma restrição quanto ao spin da outra partícula envolvida, podendo ser paralelos.

3.3 Influência da Temperatura na Existência de Estados Ligados

Tendo obtido êxito na busca pelos estados ligados de duas partículas fermiônicas, o próximo passo seria determinar como a temperatura poderia afetar o sistema. Partimos, então, para o cálculo do potencial neste contexto. Todos os cálculos serão feitos no formalismo do tempo imaginário. Ainda, $T = \beta^{-1}$ e μ é o potencial químico.

3.3.1 Propagador Corrigido em $T \neq 0$

Supondo que podemos separar a auto-energia do fóton em $\Pi^{\mu\nu} = \Pi^{\mu\nu}_{vac} + \Pi^{\mu\nu}_{mat}$, respectivamente, a contribuição em $t = 0$ e na presença de matéria. Devido à existência de um termo induzido de Chern-Simons [29], teremos:

$$\Pi^{\mu\nu}_{vac} = \Pi(p)\tilde{\theta}^{\mu\nu} + \Pi'(p)\tilde{S}^{\mu\nu}, \tag{3.19}$$

que são nossos conhecidos operadores descritos em (2.1), no espaço de momentos.

Agora, iremos separar $\Pi^{\mu\nu}$ em suas partes transversais $P_T^{\mu\nu}$, longitudinais $P_L^{\mu\nu}$ e a proporcional ao termo típico de $(2+1)D$, $i\epsilon^{\mu\nu\alpha}p_\alpha$ [25, 24] (supondo invariância rotacional), à saber:

$$\Pi_{mat}^{\mu\nu} = GP_T^{\mu\nu} + FP_L^{\mu\nu} + H\tilde{S}^{\mu\nu}, \tag{3.20}$$

onde primeiramente, G, F e H são funções escalares de p^0 e $|\vec{p}|$. Ainda

$$P_T^{\mu\nu} = \tilde{\eta}^{\mu\nu} - \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2}, \quad P_L^{\mu\nu} = \frac{p^2}{\tilde{p}^2}\tilde{u}^\mu \tilde{u}^\nu = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - P_T^{\mu\nu}, \tag{3.21}$$

com

$$\tilde{\eta}^{\mu\nu} = \eta^{\mu\nu} - u^\mu u^\nu, \tilde{p}^\mu = p^\mu - (u\cdot p)u^\mu, \tilde{u}^\mu = u^\mu - \frac{u\cdot p}{p^2}p^\mu, \tag{3.22}$$

onde u^μ é a velocidade do fluido, satisfazendo $u^\mu u_\mu = 1$ (de fato no referencial de repouso deste sistema $u^\mu = (1,0,0)$). Dessa forma, teremos algumas propriedades importantes:

$$\begin{aligned} p_\mu P_T^{\mu\nu} &= p_\mu P_L^{\mu\nu} = 0, \\ \tilde{\theta}_\nu^\rho P_{\mu\rho}^T &= P_{\mu\nu}^T, \\ \tilde{\theta}_\nu^\rho P_{\mu\rho}^L &= P_{\mu\nu}^L, \\ \tilde{\omega}_\nu^\rho P_{\mu\rho}^T &= \tilde{\omega}_\nu^\rho P_{\mu\rho}^L = 0, \\ \tilde{S}^{\rho\nu} P_{\mu\rho}^T &= i\epsilon^{\rho\lambda\nu}p_\lambda P_{\mu\rho}^T = \tilde{S}_\mu^\nu, \\ \tilde{S}^{\rho\nu} P_{\mu\rho}^L &= 0. \end{aligned} \tag{3.23}$$

Também, no referencial de repouso ($u^\mu = (1,0,0)$), teremos

$$P_T^{00} = 0, \quad P_L^{00}(p^0 = 0) = 1, \tag{3.24}$$

por conseqüência, como $\tilde{S}^{00} = 0$, quando $\mu = \nu = 0$, (3.20) reduz-se a

$$\Pi_{mat}^{00}(p^0 = 0) = G(p^0 = 0)P_T^{00} + F(p^0 = 0)P_L^{00} = F(p^0 = 0) \tag{3.25}$$

tal que o limite $p^0 = 0$ é o chamado *limite estático* da teoria. Iremos trabalhar neste referencial (de repouso *do fluido*).

Relembrando (2.82), e usando (3.19) e (3.20),

$$\begin{aligned}D^{\mu\nu}(p) &= \mathcal{D}^{\mu\nu}(p) - \Pi^{\mu\nu}(p) = \mathcal{D}^{\mu\nu} - \Pi^{\mu\nu}_{vac} - \Pi^{\mu\nu}_{mat} \\ &= (p^2 - a^2p^4 - \Pi(p))\tilde{\theta}^{\mu\nu} - \frac{p^2}{\lambda}\tilde{\omega}^{\mu\nu} - GP_T^{\mu\nu} - FP_L^{\mu\nu} - (H + \Pi'(p))\tilde{S}^{\mu\nu},\end{aligned}$$

queremos determinar x_1, x_2 e x_3 do propagador corrigido $P^{\mu\nu}(p) = x_1\tilde{\theta}^{\mu\nu} + x_2\tilde{\omega}^{\mu\nu} + x_3\tilde{S}^{\mu\nu}$ usando a relação $P^{\mu\nu}(p)D_{\rho\nu} = \delta^\mu_\rho = \tilde{\omega}^\mu_\rho + \tilde{\theta}^\mu_\rho$, usando (3.23) nesta relação, teremos

$$\begin{aligned}\tilde{\omega}^{\mu\nu} + \tilde{\theta}^{\mu\nu} &= [x_1(p^2 - a^2p^4 - \Pi(p) - F) - x_3H]\tilde{\theta}^\mu_\rho + x_1(F - G)P^\mu_\rho(p) \\ &+ [x_3(p^2 - a^2p^4 - G) - x_1(H + \Pi'(p))]\tilde{S}^\mu_\rho - x_2\frac{p^2}{\lambda}\tilde{\omega}^\mu_\rho, \end{aligned} \tag{3.26}$$

de onde vem o sistema de equações (3.27)

$$\left\{\begin{array}{ccc} x_1(p^2 - a^2p^4 - \Pi(p) - F) - x_3H & = & 1 \\ x_1(F - G) & = & 0 \Rightarrow F = G \\ x_3(p^2 - a^2p^4 - G) - x_1(H + \Pi'(p)) & = & 0 \\ -x_2\frac{p^2}{\lambda} & = & 1 \end{array}\right. \tag{3.27}$$

cuja solução fornece os coeficientes x_1, x_2 e x_3 explicitando o nosso propagador:

$$\begin{aligned}P^{\mu\nu}(p) &= \frac{p^2 - a^2p^4 - F - \Pi(p)}{(p^2 - a^2p^4 - F - \Pi(p))^2 - (H + \Pi'(p))^2p^2}\tilde{\theta}^{\mu\nu} - \frac{\lambda}{p^2}\tilde{\omega}^{\mu\nu} \\ &+ \frac{H + \Pi'(p)}{(p^2 - a^2p^4 - F - \Pi(p))^2 - (H + \Pi'(p))^2p^2}\tilde{S}^{\mu\nu}.\end{aligned} \tag{3.28}$$

O propagador (3.28) leva em conta *todas* as correções de "loop"; no entanto, queremos somente a aproximação de um "loop" identificando os primeiros termos da série cuja soma até infinito fornece (3.28).

Para simplificar os cálculos, iremos chamar

$$\begin{aligned}X &= p^2 - a^2p^4 - F - \Pi(p), \\ Z &= [H + \Pi'(p)]p,\end{aligned}$$

nessa nova notação:

$$\begin{aligned}P^{\mu\nu}(p) &= \frac{X}{(X+Z)(X-Z)}\tilde{\theta}^{\mu\nu} - \frac{\lambda}{p^2}\tilde{\omega}^{\mu\nu} + \frac{Z}{p(X+Z)(X-Z)}\tilde{S}^{\mu\nu} \qquad (3.29) \\ &= \frac{1}{2}\left(\frac{1}{X+Z} + \frac{1}{X-Z}\right)\tilde{\theta}^{\mu\nu} - \frac{\lambda}{p^2}\tilde{\omega}^{\mu\nu} + \frac{1}{2p}\left(\frac{1}{X-Z} - \frac{1}{X+Z}\right)\tilde{S}^{\mu\nu},\end{aligned}$$

onde cada uma das frações que constituem (3.29) pode ser aproximada como

$$\begin{aligned}
\frac{1}{\underbrace{p^2-a^2p^4-F-\Pi(p)}_{X}\pm\underbrace{(H+\Pi'(p))p}_{Z}} &= \frac{1}{p^2-a^2p^4}\overbrace{\left(\frac{1}{1+\frac{\pm(H+\Pi'(p))p-F-\Pi(p)}{p^2-a^2p^4}}\right)}^{P.G.} \qquad (3.30)\\
&\approx \frac{1}{p^2-a^2p^4}\left(1-\frac{\pm(H+\Pi'(p))p-F-\Pi(p)}{p^2-a^2p^4}\right),
\end{aligned}$$

voltaremos mais tarde à (3.30).

O fato de (3.28) ser proporcional a $\tilde{S}^{\mu\nu}$ aplicado ao cálculo de (3.10) induz um termo inédito do tipo

$$\begin{aligned}
& i\left[\bar{u}(p')\gamma^\mu u(p)\right]\left[\bar{u}(k')\gamma^\nu u(k)\right]\epsilon_{\mu\nu\rho}q^\rho \approx i\left[\bar{u}(p')\gamma^2 u(p)\right]\left[\bar{u}(k')\gamma^0 u(k)\right]q_1 \qquad (3.31)\\
- & \ i\left[\bar{u}(p')\gamma^0 u(p)\right]\left[\bar{u}(k')\gamma^2 u(k)\right]q_1 + i\left[\bar{u}(p')\gamma^0 u(p)\right]\left[\bar{u}(k')\gamma^1 u(k)\right]q_2 - i\left[\bar{u}(p')\gamma^1 u(p)\right]\times\\
\times & \ \left[\bar{u}(k')\gamma^0 u(k)\right]q_2
\end{aligned}$$

pois $q^0\approx 0$. As expressões explícitas em componentes de $u(p)$ e $\bar{u}(p)$ são obtidas usando (3.9)

$$\begin{aligned}
u(p) &= \frac{m+p^0\sigma^3-i\sigma_i p^i}{\sqrt{2m(m+p^0)}}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2m(m+p^0)}}\begin{pmatrix} m+p^0 & -ip^1-p^2\\ ip_1-p_2 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\
&= \frac{1}{\sqrt{2m(m+p^0)}}\begin{pmatrix}p^0+m\\ -ip^1+p^2\end{pmatrix},\\
\bar{u}(p) &= u^\dagger(p)\gamma^0 = \frac{1}{\sqrt{2m(m+p^0)}}\begin{pmatrix}p^0+m & -ip^1+p^2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\
&= \frac{1}{\sqrt{2m(m+p^0)}}\begin{pmatrix}p^0+m & -ip^1-p^2\end{pmatrix}, \qquad (3.32)
\end{aligned}$$

transformam (3.31) na aproximação não-relativística em

$$\begin{aligned}
\approx & \ -i\left[\frac{k_1+k_1'+i\overbrace{(k_2-k_2')}^{q_2}}{2m}\right]q_2 + i\left[\frac{p_1+p_1'+i\overbrace{(p_2-p_2')}^{-q_2}}{2m}\right]q_2 \qquad (3.33)\\
- & \ i\left[\frac{p_2+p_2'-i\overbrace{(p_1-p_1')}^{-q_1}}{2m}\right]q_1 + i\left[\frac{k_2+k_2'-i\overbrace{(k_1-k_1')}^{q_1}}{2m}\right]q_1.
\end{aligned}$$

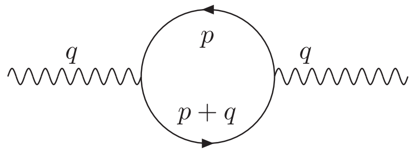


Figura 3.2: Correção de “loop”

No centro de massa, retendo apenas os termos lineares em p, teremos,

$$i\left[\bar{u}(p')\gamma^{\mu}u(p)\right]\left[\bar{u}(k')\gamma^{\nu}u(k)\right]\epsilon_{\mu\nu\rho}q^{\rho} \approx -\frac{2i\vec{q}\times\vec{p}}{m}. \tag{3.34}$$

3.3.2 Correções de “Loop” com $T = 0$

Adiantamos o resultado (4.11) para calcular estas contribuições :

$$\begin{aligned}
\Pi(q)\tilde{\theta}^{\mu\nu} &= \frac{2e^2}{(2\pi)^3}\int d^3p \int_0^1 dx \frac{2x(x-1)}{(p^2-\Delta)^2}(-\eta^{\mu\nu}q^2 + q^{\mu}q^{\nu}) \\
&= \frac{2e^2}{8\pi^3}\frac{i}{(-1)^2}\int d^3p_E \int_0^1 dx \frac{2x(x-1)}{(p_E^2+\Delta)^2}(-\eta^{\mu\nu}q^2 + q^{\mu}q^{\nu}) \\
&= \frac{ie^2}{4\pi^3}(4\pi)\int_0^1 2x(x-1)dx \int_0^{\infty} p_E^2 dp_E \frac{p_E^2}{(p_E^2+\Delta)^2}(-\eta^{\mu\nu}q^2 + q^{\mu}q^{\nu}) \\
&= \frac{ie^2}{\pi^2}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(2\right)}(-q^2\tilde{\theta}^{\mu\nu})\int_0^1 \frac{2x(x-1)}{\sqrt{\Delta}} \\
&= -\frac{ie^2}{4\pi}q^2\tilde{\theta}^{\mu\nu}\int_0^1 \frac{2x(x-1)}{\sqrt{q^2x(x-1)+m^2}}dx \Rightarrow \\
\Pi(q) &= -\frac{ie^2q^2}{4\pi}\left\{\frac{m}{q^2} + \left(\frac{m^2}{q^2} + \frac{1}{4|q|}\right)\left[arcsh\left(\frac{-q^2}{\sqrt{4mq^2-q^4}}\right)\right.\right. \\
&- \left.\left. arcsh\left(\frac{q^2}{\sqrt{4mq^2-q^4}}\right)\right]\right\} \approx -\frac{e^2im}{4\pi} - \frac{e^2imq}{4\pi},
\end{aligned} \tag{3.35}$$

onde, dada a condição $4m^2 > q^2$, que tornam as raízes do denominador da última integral reais, podemos escrever [38] a penúltima linha de (3.35). Na última linha usamos esta mesma condição para fazer a expansão em série de potências de q.

De modo análogo, de posse de (4.11) partimos para o cálculo de

$$
\begin{aligned}
\Pi'(q)\tilde{S}^{\mu\nu} &= \frac{2e^2}{(2\pi)^3}\int d^3p\int_0^1 dx\frac{im}{(p^2-\Delta)^2}\epsilon^{\mu\nu\alpha}q_\alpha \qquad (3.36)\\
&= \frac{2e^2}{8\pi^3}\frac{i}{(-1)^2}\int d^3p_E\int_0^1 dx\frac{im}{(p_E^2+\Delta)^2}\epsilon^{\mu\nu\alpha}q_\alpha\\
&= -\frac{2e^2m}{8\pi^3}(4\pi)\int_0^1 dx\int p_E^2dp_E\frac{\epsilon^{\mu\nu\alpha}q_\alpha}{(p_E^2+\Delta)^2}\\
&= -\frac{e^2m}{\pi^2}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(2\right)}\int_0^1 dx\frac{1}{\sqrt{\Delta}}\epsilon^{\mu\nu\alpha}q_\alpha\\
&= -\frac{e^2m}{4\pi}\epsilon^{\mu\nu\alpha}q_\alpha\int_0^1 dx\frac{1}{\sqrt{q^2x(x-1)+m^2}}\Rightarrow\\
\Pi'(q) &= \frac{-e^2m}{4\pi}\frac{1}{q}\left[arcsh\left(\frac{q}{\sqrt{4m^2-q^2}}\right)-arcsh\left(\frac{-q}{\sqrt{4m^2-q^2}}\right)\right]\\
&\approx -\frac{e^2}{4\pi}.
\end{aligned}
$$

Veremos, no capítulo IV, que, para preservar a identidade de Ward, o primeiro termo de (3.35) é anulado pelo regulador de Pauli-Villars. Passemos agora para às correções com $T\neq 0$.

3.3.3 Correções de "Loop" com $T \neq 0$

Começamos utilizando as regras de Feynman no espalhamento elétron-elétron [42], obtendo, com ajuda de (3.1) da segunda para a terceira linha de (3.37):

$$
\begin{aligned}
\Pi^{\mu\nu}_{mat} &= e^2T\sum_n\int\frac{d^2p}{(2\pi)^2}Tr\left[\gamma^\mu\frac{(\not{p}+m)}{p^2-m^2}\gamma^\nu\frac{(\not{p}+\not{q}+m)}{(p+q)^2-m^2}\right] \qquad (3.37)\\
&= e^2T\sum_n\int\frac{d^2p}{(2\pi)^2}\frac{Tr[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta p_\alpha(p+q)_\beta+mp_\alpha\gamma^\mu\gamma^\alpha\gamma^\nu+m(p+q)_\alpha\gamma^\mu\gamma^\nu\gamma^\alpha]}{(p^2-m^2)[(p+q)^2-m^2]}\\
&+ e^2T\sum_n\int\frac{d^2p}{(2\pi)^2}\frac{Tr[m^2\gamma^\mu\gamma^\nu]}{(p^2-m^2)[(p+q)^2-m^2]}\\
&= 2e^2T\sum_n\int\frac{d^2p}{(2\pi)^2}\left\{\frac{p^\mu(p+q)^\nu-\eta^{\mu\nu}[p\cdot(p+q)+m^2]+p^\nu(p+q)^\mu}{(p^2-m^2)[(p+q)^2-m^2]}\right. \qquad (3.38)
\end{aligned}
$$

$$
- \quad \frac{im\epsilon^{\mu\nu\alpha}q_\alpha}{(p^2-m^2)[(p+q)^2-m^2]}\Bigg\} . \tag{3.39}
$$

Observando (3.39) e comparando com (3.20), o coeficiente H surge naturalmente. O problema agora é calcular a soma sobre freqüências para férmions. Esta pode ser convertida em integrais de contorno, usando como artifício o fato da função $\frac{\beta}{2}tgh[\beta(p_0-\mu)/2]$ ter pólos em $p_0 = i\omega_n + \mu$, onde $\omega_n = (2n+1)\pi T$ e em seguida utilizando o teorema dos resíduos [19] resultando na soma desejada. Assim,

$$
\begin{aligned}
T\sum_n f(p_0 = i\omega_n+\mu) &= \frac{T}{2\pi i}\oint dp_0 f(p_0)\frac{\beta}{2}tgh\left(\frac{\beta}{2}(p_0-\mu)\right) \qquad (3.40)\\
&= \frac{1}{2\pi i}\int_{i\infty+\mu-\epsilon}^{-i\infty+\mu-\epsilon} dp_0 f(p_0)\left(-\frac{1}{2}+\frac{1}{e^{-\beta(p_0-\mu)}+1}\right)\\
&+ \frac{1}{2\pi i}\int_{-i\infty+\mu+\epsilon}^{i\infty+\mu+\epsilon} dp_0 f(p_0)\left(\frac{1}{2}-\frac{1}{e^{\beta(p_0-\mu)}+1}\right)\\
&= -\frac{1}{2\pi i}\int_{-i\infty+\mu-\epsilon}^{i\infty+\mu-\epsilon} dp_0 f(p_0)\frac{1}{e^{-\beta(p_0-\mu)}+1}\\
&- \frac{1}{2\pi i}\int_{-i\infty+\mu+\epsilon}^{i\infty+\mu+\epsilon} dp_0 f(p_0)\frac{1}{e^{\beta(p_0-\mu)}+1},
\end{aligned}
$$

onde as integrais (3.40) serão feitas no contorno mostrado na figura (3.3). Retornando à (3.39) usando (3.40),

$$
\begin{aligned}
H &= -2e^2T\sum_n\int\frac{d^2p}{(2\pi)^2}\frac{m}{(p_0^2-\vec{p}^2-m^2)[(q+p)_0^2-(\vec{p}+\vec{q})^2-m^2]} \qquad (3.41)\\
&= \frac{2e^2m}{2\pi i}\int_{-i\infty+\mu+\epsilon}^{i\infty+\mu+\epsilon} dp_0\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p_0^2-E_p^2)(p_0^2+E_p^2)(p_0^2-E_p^2+2p_0q^0-2\vec{p}\cdot\vec{q}+q^2)}\times\\
&\times \frac{1}{e^{\beta(p_0-\mu)}+1}\\
&+ \frac{2e^2m}{2\pi i}\int_{-i\infty+\mu-\epsilon}^{i\infty+\mu-\epsilon} dp_0\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p_0^2-E_p^2)(p_0^2+E_p^2)(p_0^2-E_p^2+2p_0q^0-2\vec{p}\cdot\vec{q}+q^2)}\times\\
&\times \frac{1}{e^{\beta(\mu-p_0)}+1}\\
&= \frac{e^2m}{2\pi^2}\frac{1}{2}\Bigg\{\int d^2p\frac{-1}{E_p(2E_pq_0-2\vec{p}\cdot\vec{q}+q^2)}\frac{1}{e^{\beta(E_p-\mu)}+1}
\end{aligned}
$$

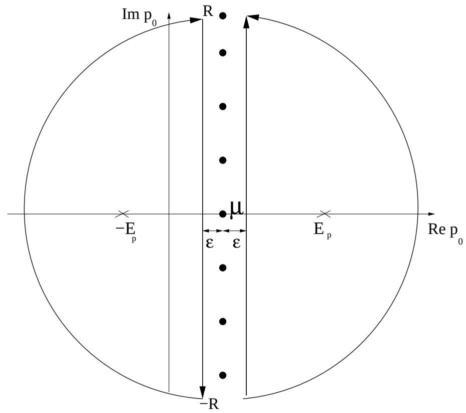


Figura 3.3: Contorno de integração no plano complexo, com $\epsilon \to 0$ e $R \to \infty$. Os pontos representam pólos da função $\frac{\beta}{2} tgh[\beta(p_0 - \mu)/2]$.

$$
\begin{aligned}
+ & \left.\int d^2p \frac{-1}{E_p(-2E_p q_o - 2\vec{p}\cdot\vec{q} + q^2)} \frac{1}{e^{\beta(E_p+\mu)}+1}\right\} \qquad (3.41.a) \\
= & -\frac{e^2 m}{4\pi^2}\int_0^\infty d|\vec{p}||\vec{p}| \int_{-\pi}^{\pi} \frac{d\theta}{E_p}\left[\frac{1}{-2E_p q_0 - 2|\vec{p}||\vec{q}|\cos(\theta) + q^2}\frac{1}{e^{\beta(E_p-\mu)}+1}\right. \\
+ & \left.\frac{1}{2E_p q_0 - 2|\vec{p}||\vec{q}|\cos(\theta) + q^2}\frac{1}{e^{\beta(E_p+\mu)}+1}\right], \text{ fazendo } q_0 = 0, \\
= & -\frac{e^2 m}{2\pi^2}\frac{1}{-\vec{q}^2}\int_0^\infty d|\vec{p}|\frac{|\vec{p}|}{E_p}\int_0^\pi \frac{d\theta}{1+\frac{2|\vec{p}|}{|\vec{q}|}\cos(\theta)}\left(\frac{1}{e^{\beta(E_p-\mu)}+1} + \frac{1}{e^{\beta(E_p+\mu)}+1}\right) \stackrel{4p^2<q^2}{\Longrightarrow} \\
= & \frac{e^2 m}{2\pi|\vec{q}|}\int_0^\infty \frac{d|\vec{p}||\vec{p}|}{\sqrt{\vec{p}^2+m^2}}(\vec{q}^2 - 4\vec{p}^2)^{-\frac{1}{2}}\left(\frac{1}{e^{\beta(E_p-\mu)}+1} + \frac{1}{e^{\beta(E_p+\mu)}+1}\right)
\end{aligned}
$$

onde $E_p^2 = \vec{p}^2 + m^2$, $q = q_0^2 - |\vec{q}|^2$. (3.41.a) é obtida usando o caminho de integração dado na figura (3.3) (a primeira tem pólo em E_p, a segunda em $-E_p$; note ainda o sinal negativo acompanhando o sentido de integração na primeira integral). Novamente, usamos [38]

$$
\int_0^\pi \frac{\cos(nx)}{1+a\cos(x)}dx = \frac{\pi}{\sqrt{1-a^2}}\left(\frac{\sqrt{1-a^2}-1}{a}\right)^n, \quad [a^2 < 1, n \geq 0]. \tag{3.42}
$$

A integral (3.42) é usada na última linha de (3.41). Vamos novamente

utilizar o fato de que $4p^2 < q^2 << m^2$ para expandir e fazer a integral (3.41)
,

$$\begin{aligned}
H(q_0=0) &\approx \frac{e^2 m}{2\pi\vec{q}^2}\int_0^\infty \frac{d|\vec{p}||\vec{p}|}{\sqrt{\vec{p}^2+m^2}}\left(1+2\frac{\vec{p}^2}{\vec{q}^2}\right)\left(\frac{1}{e^{\beta(E_p-\mu)}+1}+\frac{1}{e^{\beta(E_p+\mu)}+1}\right)\\
&= \frac{e^2 m}{2\pi\vec{q}^2}\left\{\frac{1-\frac{2m^2}{\vec{q}^2}}{\beta}[-2m\beta+\ln[(1+e^{\beta(m+\mu)})(1+e^{\beta(m-\mu)})]]\right.\\
&+ \frac{2}{3\beta^3}\left[-2m^3\beta^3+3m^2\beta^2\ln[(1+e^{\beta(m+\mu)})(1+e^{\beta(m-\mu)})]\right.\\
&+ 3m\beta\sum_{n=1}^\infty\frac{(-1)^n}{n^2}e^{\beta nm}\cosh(\beta n\mu)-6\sum_{n=1}^\infty\frac{(-1)^n}{n^3}\sinh(\beta n\mu)\\
&- \left.\left.6\sum_{n=1}^\infty\frac{(-1)^n}{n^3}e^{\beta nm}\cosh(\beta n\mu)\right]\right\}. \qquad (3.43)
\end{aligned}$$

Iremos nos concentrar no caso especial onde $\mu = 0$; ainda, como queremos verificar o comportamento da teoria com $\beta << 1$, é válida (mesmo que tenhamos que pagar com a perda do limite $T \to 0$) a expansão de (3.43) em β até primeira ordem, dada por

$$\begin{aligned}
H(q_0=0,\beta<<1,\mu=0) &\approx \frac{e^2 m}{48\vec{q}^4\pi\beta^3}[60m^3\beta^3+6\vec{q}^2m^2\beta^4+4m\beta(\pi^2\\
&- 6\beta^2\vec{q}^2)+6\vec{q}^2\beta^2\ln(4)-m^4\beta^4+72\zeta(3)],
\end{aligned} \qquad (3.44)$$

onde $\zeta(3) \approx 1.202057$ é a função Zeta de Riemann.

Usando (3.25), podemos a partir de (3.39) obter o coeficiente F de (3.20) no caso em que $q_0 = 0$:

$$\begin{aligned}
F(q_0=0) &= \Pi^{00}(q_0=0) = 2e^2T\sum_n\int\frac{d^2p}{(2\pi)^2}\frac{\vec{p}\cdot(\vec{p}+\vec{q})+p_0(q+p)^0+m^2}{(p_0^2-\vec{p}^2-m^2)[(q_0+p_0)^2-(\vec{q}+\vec{p})^2-m^2]}\\
&= -\frac{2e^2}{2\pi i}\int_{-i\infty+\mu+\epsilon}^{i\infty+\mu+\epsilon}dp_0\int\frac{d^2p}{(2\pi)^2}\frac{\vec{q}\cdot\vec{p}+p_0q^0+p_0^2+E_p^2}{(p_0^2-E_p^2)[p_0^2-E_p^2+2p_0q^0-2\vec{p}\cdot\vec{q}+q^2]}\times\\
&\times \frac{1}{e^{\beta(p_0-\mu)}+1}\\
&- \frac{2e^2}{2\pi i}\int_{-i\infty+\mu-\epsilon}^{i\infty+\mu-\epsilon}dp_0\int\frac{d^2p}{(2\pi)^2}\frac{\vec{q}\cdot\vec{p}+p_0q^0+p_0^2+E_p^2}{(p_0^2-E_p^2)[p_0^2-E_p^2+2p_0q^0-2\vec{p}\cdot\vec{q}+q^2]}\times
\end{aligned}$$

$$
\begin{aligned}
&\times \quad \frac{1}{e^{\beta(\mu-p_0)}+1} \\
&= \quad \frac{e^2}{(2\pi)^2}\int_0^\infty |\vec{p}|d|\vec{p}|\int_{-\pi}^{\pi} d\theta\left[\frac{|\vec{p}||\vec{q}|\cos(\theta)+2E_p^2+E_pq_0}{E_p(2E_pq_0-2|\vec{p}||\vec{q}|\cos(\theta)+q^2)}\frac{1}{e^{\beta(E_p-\mu)}+1}\right. \\
&+ \quad \left.\frac{|\vec{p}||\vec{q}|\cos(\theta)+2E_p^2-E_pq_0}{E_p(-2E_pq_0-2|\vec{p}||\vec{q}|\cos(\theta)+q^2)}\frac{1}{e^{\beta(E_p+\mu)}+1}\right] \quad \text{, de novo } q_0=0, \\
&= \quad \frac{e^2}{2\pi^2\vec{q}^2}\int_0^\infty |\vec{p}|d|\vec{p}|\int_0^{\pi}\frac{d\theta}{\sqrt{\vec{p}^2+m^2}}\frac{|\vec{p}||\vec{q}|\cos(\theta)+2(\vec{p}^2+m^2)}{1+2\frac{|\vec{p}|}{|\vec{q}|}\cos(\theta)}\times \\
&\times \quad \left[\frac{1}{e^{\beta\left(\sqrt{\vec{p}^2+m^2}-\mu\right)}+1}+\frac{1}{e^{\beta\left(\sqrt{\vec{p}^2+m^2}+\mu\right)}+1}\right] \\
&= \quad \frac{e^2}{4\pi\vec{q}^2}\int_0^\infty\frac{|\vec{p}|d|\vec{p}|}{\sqrt{\vec{p}^2+m^2}}\frac{1}{\sqrt{1-4\frac{\vec{p}^2}{\vec{q}^2}}}\left[4(\vec{p}^2+m^2)+\vec{q}^2\left(1-\sqrt{1-4\frac{\vec{p}^2}{\vec{q}^2}}\right)\right]\times \\
&\times \quad \left[\frac{1}{e^{\beta\left(\sqrt{\vec{p}^2+m^2}-\mu\right)}+1}+\frac{1}{e^{\beta\left(\sqrt{\vec{p}^2+m^2}+\mu\right)}+1}\right],
\end{aligned}
\tag{3.45}
$$

novamente expandimos o termo $\left(1-4\frac{\vec{p}^2}{\vec{q}^2}\right)^{-\frac{1}{2}}$ em (3.45); tomando $m>\mu$, teremos:

$$
\begin{aligned}
F(q_0=0) \quad &\approx \quad \frac{e^2}{2\pi\vec{q}^2}\left\{\frac{(\vec{q}^2-2m^2)}{2\beta}\left[\ln\left[\left(1+e^{\beta(m+\mu)}\right)\left(1+e^{\beta(m-\mu)}\right)\right]\right.\right. \qquad (3.46)\\
&- \quad 2m\beta] - \frac{1}{\beta^3\vec{q}^2}\left[-2m^3\beta^3+3m^2\beta^2\ln\left[\left(1+e^{\beta(m+\mu)}\right)\left(1+e^{\beta(m-\mu)}\right)\right]\right. \\
&+ \quad 3m\beta\sum_{n=1}^{\infty}(-1)^n\frac{e^{\beta nm}}{n^2}\cosh(\beta n\mu)-6\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sinh(\beta n\mu) \\
&- \quad \left.6\sum_{n=1}^{\infty}(-1)^n\frac{e^{\beta nm}}{n^3}\cosh(\beta n\mu)\right]+\frac{\vec{q}^2}{\beta}\left[\beta m-\frac{1}{2}\ln\left[\left(1+e^{\beta(m+\mu)}\right)\left(1+e^{\beta(m-\mu)}\right)\right]\right]\Bigg\},
\end{aligned}
$$

que, após expandir em β, obtemos em primeira ordem,

$$
\begin{aligned}
F(q_0=0,\beta<<1,\mu=0) \quad &\approx \quad \frac{e^2}{32\vec{q}^4\pi\beta^3}\left[4(3+4\vec{q}^2)m^3\beta^3+4m\beta\pi^2\right. \\
&+ \quad \left.(11-4\vec{q}^2)m^4\beta^4-4(\vec{q}^2-3)m^2\beta^2\ln(256)+72\zeta(3)\right].
\end{aligned}
\tag{3.47}
$$

3.3.4 Cálculo do Potencial com $T \neq 0$

Observando (3.35), (3.36), (3.44) e (3.47) , iremos aplicar o limite não-relativístico ($q_0 \approx 0, q^2 \to -\vec{q}^2$) nessas expressões. Dessa forma, podemos classificar de acordo com a dependência em $\vec{q}$ em (3.30):

$$
\begin{aligned}
\frac{1}{X \pm Z|\vec{q}|} &\approx \frac{-1}{\vec{q}^2 + a^2\vec{q}^4}\left[1 + \frac{\pm\overbrace{\frac{g(\beta)\vec{q}^2 + h(\beta)}{\vec{q}^4}}^{H}|\vec{q}| \pm \overbrace{\frac{d(\beta)\vec{q}^4}{\vec{q}^4}}^{\Pi'(q)}|\vec{q}| - \overbrace{\frac{a(\beta)\vec{q}^2 + b(\beta)}{\vec{q}^4}}^{F} - \overbrace{\frac{f(\beta)\vec{q}^5}{\vec{q}^4}}^{\Pi(q)}}{\vec{q}^2 + a^2\vec{q}^4}\right] \\
&\approx \frac{-1}{\vec{q}^2 + a^2\vec{q}^4}\left[1 + \frac{-b(\beta) \pm h(\beta)|\vec{q}|}{\vec{q}^4(\vec{q}^2 + a^2\vec{q}^4)}\right], \qquad (3.48)
\end{aligned}
$$

onde retemos os termos até primeira ordem em $|\vec{q}|$, de forma que os termos de (3.47) sem dependência em $|\vec{q}|$ definem $b(\beta)$, como o mesmo tipo de termos em (3.44) definem $h(\beta)$.

Utilizando (3.48) em (3.29), o propagador corrigido na aproximação não-relativística e estática, pode ser escrito após alguma álgebra como sendo:

$$
P^{\mu\nu}(q) \approx \frac{1}{q^2 - a^2q^4}\tilde{\theta}^{\mu\nu} - \frac{\lambda}{q^2}\tilde{\omega}^{\mu\nu} + \frac{b(\beta)}{q^4(q^2 - a^2q^4)^2}\tilde{\theta}^{\mu\nu} + \frac{h(\beta)}{q^4(q^2 - a^2q^4)^2}\tilde{S}^{\mu\nu} \qquad (3.49)
$$

Os dois primeiros termos de (3.49) já foram discutidos em (3.10) resultando no potencial (3.18).

Somos agora obrigados a encarar a integral (1.46), usando a aproximação não-relativística aplicada ao espalhamento em questão (em verdade, o uso direto de (1.22) levaria à uma integral divergente; ao invés disto, usaremos (1.46) como forma de *regularização* [39]). Usando, então, (3.49) e (3.34):

$$
\begin{aligned}
U(\vec{r}, \beta) &\approx (3.18) + \frac{e^2}{(2\pi)^{\frac{d}{2}}|\vec{r}|^{\frac{d-2}{2}}}\left[\int_0^\infty \frac{b(\beta)|\vec{q}|^{d/2}}{\vec{q}^8(1 + a^2\vec{q}^2)^2} J_{\frac{d-2}{2}}(|\vec{q}||\vec{r}|)dq\right. \\
&- \left.\int_0^\infty \frac{2i\vec{q} \times \vec{p}}{m}\frac{h(\beta)|\vec{q}|^{d/2}}{\vec{q}^8(1 + a^2\vec{q}^2)^2} J_{\frac{d-2}{2}}(|\vec{q}||\vec{r}|)dq\right]. \qquad (3.50)
\end{aligned}
$$

Concentremo-nos no cálculo da primeira integral de (3.50): o denominador comum a estas integrais pode ser decomposto em frações parciais:

$$
\frac{1}{\vec{q}^8(1 + a^2\vec{q}^2)^2} = \frac{ia^8}{2}\left(\frac{1}{a|\vec{q}| + i} - \frac{1}{a|\vec{q}| - i}\right) - \frac{a^6}{\vec{q}^2} + \frac{a^4}{\vec{q}^4} - \frac{a^2}{\vec{q}^6} + \frac{1}{\vec{q}^8}, \qquad (3.51)
$$

assim, os quatro últimos termos a direita de (3.51) permitem (mediante uma substituição de variáveis $|\vec{q}||\vec{r}| \rightarrow k$) generalizar, com $n = 2, 4, 6$ ou 8, parte da primeira integral de (3.50) ao cálculo de integrais do tipo [64]

$$\begin{aligned}
I_1(|\vec{r}|) &= |\vec{r}|^{n-\frac{d}{2}-1}\int_0^\infty \frac{k^{\frac{d}{2}}}{k^n} J_{\frac{d}{2}-1}(k)dk = |\vec{r}|^{n-\frac{d}{2}-1}\int_0^\infty \frac{J_{\frac{d}{2}-1}(k)}{k^{\frac{d}{2}-1-(d-n)+1}}dk \\
&= -\frac{2i\sin(d-n)\pi\Gamma\left(\frac{d-n}{2}\right)}{2^{n-\frac{d}{2}}\Gamma\left(\frac{n}{2}\right)}|\vec{r}|^{n-\frac{d}{2}-1} \quad \text{, fazendo } d = 2+\epsilon \qquad (3.52) \\
&= \lim_{\epsilon\rightarrow 0}\frac{-2i\sin(2+\epsilon-n)\pi\Gamma\left(1-\frac{n}{2}+\frac{\epsilon}{2}\right)}{2^{n-1-\frac{\epsilon}{2}}\Gamma\left(\frac{n}{2}\right)}|\vec{r}|^{n-\frac{\epsilon}{2}-3} \\
&\approx \lim_{\epsilon\rightarrow 0}\frac{-2i\pi\epsilon\cos[(2-n)\pi]\Gamma\left(1-\frac{n}{2}\right)}{2^{n-1}\left(1-\frac{\epsilon}{2}\ln(2)\right)\Gamma\left(\frac{n}{2}\right)}|\vec{r}|^{n-3}\left(1-\frac{\epsilon}{2}\ln(r)\right) = 0,
\end{aligned}$$

onde usamos

$$\begin{aligned}
\sin(2+\epsilon-n)\pi &= \sin(\epsilon\pi)\cos[(2-n)\pi] + \cos(\epsilon\pi)\cancelto{0}{\sin[(2-n)\pi]} \\
&\approx \pi\epsilon\cos[(2-n)\pi], \qquad (3.53) \\
\Gamma\left(z+\frac{\epsilon}{2}\right) &\approx \Gamma(z) + \frac{\epsilon}{2}\frac{\Psi(z)}{\Gamma(z)}, \\
r^{-\frac{\epsilon}{2}} &= e^{-\frac{\epsilon}{2}\ln(r)} \approx 1-\frac{\epsilon}{2}\ln(r), \quad 2^{-\frac{\epsilon}{2}} \approx 1-\frac{\epsilon}{2}\ln(2)
\end{aligned}$$

com $\Psi(z) = \frac{d}{dz}\ln\Gamma(z)$. Por sua vez, os dois primeiros termos do lado direito de (3.51) permitem o cálculo do seguinte termo da primeira integral de (3.50) (observe que as duas integrais podem ser obtidas fazendo $\rho = \frac{i}{a}$ ou $\rho = -\frac{i}{a}$, respectivamente),

$$\begin{aligned}
I_2(|\vec{r}|) &= \int_0^\infty \frac{|\vec{q}|^{\frac{d}{2}} J_{\frac{d}{2}-1}(|\vec{q}||\vec{r}|)}{a|\vec{q}|+i}d|\vec{q}| = \int_0^\infty \frac{|\vec{q}|^{\frac{d}{2}+1-1} J_{\frac{d}{2}-1}(|\vec{q}||\vec{r}|)}{a(|\vec{q}|+\rho)}d|\vec{q}| \qquad (3.54) \\
&= \frac{\pi\rho^{\frac{d}{2}}}{a\sin(\pi d)\Gamma(1)}\left[-\sum_{m=0}^{\infty}\frac{\left(\frac{r\rho}{2}\right)^{m-\frac{d}{2}}\Gamma(m+1)\sin\left[(d-m)\frac{\pi}{2}\right]}{m!\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{3-d-m}{2}\right)}\right. \qquad (3.54.a) \\
&+ \left.\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{r\rho}{2}\right)^{\frac{d}{2}-1+2m}\Gamma(2m+d)}{m!\Gamma\left(\frac{d}{2}+m\right)\Gamma(2m+d)}\right]. \qquad (3.54.b)
\end{aligned}$$

A soma em (3.54.a) é melhor vista se a decompormos na soma de inteiros pares ($m = 2l$) e ímpares ($m = 2l+1$), $l = 0, 1, 2, \ldots$,

$$(3.54.a) \;=\; \sum_{l=0}^{\infty}\left[-\frac{\left(\frac{r\rho}{2}\right)^{2l+1-\frac{d}{2}}\Gamma(2l+2)(-1)^l\cos\left(\frac{\pi d}{2}\right)}{(2l+1)!\Gamma(l+1)\Gamma\left(l+2-\frac{d}{2}\right)}+\frac{\left(\frac{r\rho}{2}\right)^{2l-\frac{d}{2}}\Gamma(2l+1)(-1)^l\sin\left(\frac{\pi d}{2}\right)}{(2l)!\Gamma\left(l+\frac{1}{2}\right)\Gamma\left(l+\frac{3}{2}-\frac{d}{2}\right)}\right],$$

usando $d+\epsilon$ na primeira expressão entre parênteses de (3.54.a),

$$-\sum_{l=0}^{\infty}\frac{(-1)^l\left(\frac{r\rho}{2}\right)^{2l}\left(\frac{r\rho}{2}\right)^{-\frac{\epsilon}{2}}\cancel{\Gamma(2l+2)}\cos\left[(2+\epsilon)\frac{\pi}{2}\right]}{\cancel{\Gamma(2l+2)}\Gamma(l+1)\Gamma\left(l+1+\frac{\epsilon}{2}\right)} \approx -\sum_{l=0}^{\infty}\frac{(-1)^l\left(\frac{r\rho}{2}\right)^{2l}\left(\frac{r\rho}{2}\right)^{-\frac{\epsilon}{2}}(-1)}{\Gamma(l+1)^2}$$
$$= \left(\frac{r\rho}{2}\right)^{-\frac{\epsilon}{2}}\left[J_0(r\rho)+\mathcal{O}(\epsilon)\right], \quad (3.55)$$

onde usamos, além de [59] ,

$$\frac{1}{\Gamma\left(l+1+\frac{\epsilon}{2}\right)} \approx \frac{1}{\Gamma(l+1)} - \frac{\epsilon}{2}\frac{\Psi(l+1)}{\Gamma(l+1)}, \qquad (3.56)$$

com $\mathcal{O}(\epsilon)$ uma função par em relação ao seu argumento. Usando $\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$, além de $2l! = \Gamma(2l+1)$, a segunda expressão entre parênteses de (3.54.a) pode ser escrita como

$$\sum_{l=0}^{\infty}\frac{\left(\frac{r\rho}{2}\right)^{2l-1+\frac{\epsilon}{2}}2^{2l}\cancel{\Gamma(l+1)}(-1)^l\sin\left[(2+\epsilon)\frac{\pi}{2}\right]}{\Gamma\left(l+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}-\frac{\epsilon}{2}\right)\cancel{\Gamma(l+1)}} \approx -\frac{\epsilon\pi}{2}\sum_{l=0}^{\infty}\frac{\left(\frac{r\rho}{2}\right)^{2l}(-1)^l\left(\frac{r\rho}{2}\right)^{\frac{\epsilon}{2}-1}}{\Gamma\left(l+\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)}$$
$$\overset{l=k+1}{\Longrightarrow} \; -\frac{\epsilon\pi}{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(\frac{r\rho}{2}\right)^{\frac{\epsilon}{2}-1}+\frac{\epsilon\pi}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{r\rho}{2}\right)^{2k}(-1)^k\left(\frac{r\rho}{2}\right)^{\frac{\epsilon}{2}+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}$$
$$= \; -\frac{\epsilon}{2}\left(\frac{r\rho}{2}\right)^{\frac{\epsilon}{2}-1}+\frac{\epsilon\pi}{r\rho}\left(\frac{r\rho}{2}\right)^{1+\frac{\epsilon}{2}}H_0(r\rho)+\mathcal{O}(\epsilon^2), \qquad (3.57)$$

onde $H_0(z)$ é a função de Struve de ordem zero.

Voltamos nossa atenção à (3.54.b), fazendo a substituição usual, $d \to 2+\epsilon$:

$$(3.54.b) \;=\; \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{r\rho}{2}\right)^{\frac{d}{2}-1+2m}2^{2m+d-1}\Gamma\left(m+\frac{d}{2}\right)\Gamma\left(m+\frac{d}{2}+\frac{1}{2}\right)}{m!2^{2m+d-1}\Gamma\left(m+\frac{d}{2}\right)\Gamma\left(m+\frac{d}{2}+\frac{1}{2}\right)}$$
$$= \left(\frac{r\rho}{2}\right)^{\frac{\epsilon}{2}}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{r\rho}{2}\right)^{2m}}{m!\Gamma\left(m+1+\frac{\epsilon}{2}\right)}$$

$$
\begin{aligned}
&\approx \left(\frac{r\rho}{2}\right)^{\frac{\epsilon}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{r\rho}{2}\right)^{2m}}{m!\Gamma(m+1)} \left[1 - \frac{\epsilon}{2}\Psi\left(m + \frac{1}{2}\right)\right] \\
&= \left(\frac{r\rho}{2}\right)^{\frac{\epsilon}{2}} J_0(r\rho) + \mathcal{O}(\epsilon),
\end{aligned} \tag{3.58}
$$

onde novamente $\mathcal{O}(\epsilon)$ é par.

Primeiramente reunimos (3.55), (3.57) e (3.58) em (3.54), aplicando a última linha de (3.53), de forma a compor

$$
\begin{aligned}
I_2(|\vec{r}|) &= \lim_{\epsilon\to 0} \frac{\pi \rho^{\frac{2+\epsilon}{2}}}{a \sin[\pi(2+\epsilon)]\Gamma(1)} \left\{ -\left(\frac{r\rho}{2}\right)^{-\frac{\epsilon}{2}} [J_0(r\rho) + \mathcal{O}(\epsilon)] + \frac{\epsilon}{2}\left(\frac{r\rho}{2}\right)^{\frac{\epsilon}{2}-1} \right. \\
&- \left. \frac{\epsilon\pi}{r\rho}\left(\frac{r\rho}{2}\right)^{1+\frac{\epsilon}{2}} H_0(r\rho) + \mathcal{O}(\epsilon^2) + \left(\frac{r\rho}{2}\right)^{\frac{\epsilon}{2}} J_0(r\rho) + \mathcal{O}(\epsilon) \right\} \\
&= \frac{\rho}{a}\left[\ln\left(\frac{r\rho}{2}\right) J_0(r\rho) + \frac{1}{r\rho} - \frac{\pi}{2} H_0(r\rho)\right],
\end{aligned} \tag{3.59}
$$

então, fazendo $\rho = \pm\frac{i}{a}$, e nos valendo do seguinte conjunto de propriedades:

$$
\begin{aligned}
\ln\left(\frac{ir}{2a}\right) &= \ln\left(\frac{r}{2a} e^{i\frac{\pi}{2}}\right) = \ln\left(\frac{r}{2a}\right) + i\frac{\pi}{2}, \\
H_\nu(z e^{m\pi i}) &= e^{m(\nu+1)\pi i} H_\nu(z) \Rightarrow H_0\left(-\frac{ir}{a}\right) = H_0\left(e^{i\pi}\frac{ir}{a}\right) = -H_0\left(\frac{ir}{a}\right), \\
J_\nu(z e^{m\pi i}) &= e^{m\nu\pi i} J_\nu(z) \Rightarrow J_0\left(-\frac{ir}{a}\right) = J_0\left(\frac{ir}{a}\right), \\
I_\nu(z) &= e^{-\frac{1}{2}\nu\pi i} J_\nu(z e^{\frac{\pi i}{2}}),
\end{aligned}
$$

onde $I_\nu(z)$ é função de Bessel modificada de ordem ν. Assim, teremos

$$
\begin{aligned}
&\frac{ia^8}{2}\int_0^\infty |\vec{q}|^{\frac{d}{2}} J_{\frac{d}{2}-1}(|\vec{q}||\vec{r}|)\left(\frac{1}{a|\vec{q}|+i} - \frac{1}{|\vec{q}|-i}\right) dq = \frac{i^2 a^8}{2a^2}\left\{ \ln\left(\frac{ir}{2a}\right) J_0\left(\frac{ir}{a}\right) \right. \\
&- \left. \frac{\pi}{2} H_0\left(\frac{ir}{a}\right) + \frac{a}{ir} + \ln\left(-\frac{ir}{2a}\right) J_0\left(\frac{-ir}{a}\right) - \frac{a}{ir} - \frac{\pi}{2} H_0\left(\frac{-ir}{a}\right) \right\} \\
&= -\frac{a^6}{2} J_0\left(\frac{ir}{a}\right)\left[\ln\left(\frac{r}{2a}\right) + i\frac{\pi}{2} + \ln\left(\frac{r}{2a}\right) - i\frac{\pi}{2}\right] = -a^6 J_0\left(\frac{ri}{a}\right) \ln\left(\frac{r}{2a}\right) \\
&= -a^6 I_0\left(\frac{r}{a}\right) \ln\left(\frac{r}{2a}\right).
\end{aligned} \tag{3.60}
$$

Como a contribuição (3.52) é nula, usando o resultado (3.60) podemos dizer que a primeira parte do potencial $U(|\vec{r}|,\beta)$, apresentado em (3.50) pode ser escrita como

$$U_1(\vec{r},\beta) = \lim_{\epsilon\to 0} \frac{-e^2\, a^6 b(\beta)}{(2\pi)^{1+\frac{\epsilon}{2}}|\vec{r}|^{\frac{\epsilon}{2}}} I_0\left(\frac{r}{a}\right)\ln\left(\frac{r}{2a}\right) = \frac{-e^2\, a^6 b(\beta)}{2\pi} I_0\left(\frac{r}{a}\right)\ln\left(\frac{r}{2a}\right), \tag{3.61}$$

e para o cálculo da segunda parte de (3.50) de novo nos valeremos de (2.39) , sabendo que $I_0'(z) = I_1(z)$,

$$\begin{aligned} U_2(\vec{r},\beta) &= \frac{2i}{m}\frac{i\overbrace{\vec{p}\times\vec{r}}^{\vec{l}}}{|\vec{r}|}\frac{\partial}{\partial r}\left[\lim_{\epsilon\to 0}\frac{e^2\, a^6 h(\beta)}{(2\pi)^{1+\frac{\epsilon}{2}}|\vec{r}|^{\frac{\epsilon}{2}}} I_0\left(\frac{r}{a}\right)\ln\left(\frac{r}{2a}\right)\right] \qquad (3.62) \\ &= -\frac{e^2 a^6}{m\pi r}h(\beta)\left[\frac{I_1\left(\frac{r}{a}\right)}{a}\ln\left(\frac{r}{2a}\right) + \frac{I_0\left(\frac{r}{a}\right)}{r}\right]\vec{l}. \end{aligned}$$

O momento orbital l será usado como um operador de agora em diante.

Juntando todos os pedaços, (3.18), (3.44), (3.47), (3.61) e (3.62), chegamos finalmente ao resultado desejado ($\beta << 1, \mu = 0$):

$$\begin{aligned} U(r,\beta) &= \frac{-e^4\, a^6}{64\pi^2\beta^3}\left[12m^3\beta^3 + 4m\beta\pi^2 + 11m^4\beta^4 + 12m^2\beta^2\ln(256)\right. \qquad (3.63) \\ &+ \left. 72\zeta(3)\right] I_0\left(\frac{r}{a}\right)\ln\left(\frac{r}{2a}\right) - \frac{e^4 a^6}{48\pi^2 r\beta^3}[60m^3\beta^3 + 4m\beta\pi^2 \\ &- m^4\beta^4 + 72\zeta(3)]\left[\frac{I_1\left(\frac{r}{a}\right)}{a}\ln\left(\frac{r}{2a}\right) + \frac{I_0\left(\frac{r}{a}\right)}{r}\right] l - \frac{e^2}{2\pi}\left[\ln(r)\right. \\ &+ \left. K_0\left(\frac{r}{a}\right)\right] - \frac{e^2}{16\pi m^2 r^2} l\left[1 - \frac{r}{a}K_1\left(\frac{r}{a}\right)\right]. \end{aligned}$$

3.3.5 Algumas Considerações

Analisando (3.63), observamos que, se $a << 1$, o termo de correção devido à temperatura não influencia os cálculos (veja uma estimativa no apêndice A), se o potencial for suficientemente comportado; de fato, é o que ocorre. Usando os mesmos dados de entrada explicitados na figura (3.1), ocorre a existência de um estado ligado com auto-energia de $-5,79\, MeV$, quando $\beta = 0.00625$ (valor de β típico de supercondutores de alta temperatura), uma correção de $0,003\%$ sem a temperatura.

Capítulo 4

Cálculo da Função-β na Eletrodinâmica de Podolsky

Reconsideramos a Lagrangeana de Dirac-Podolsky, à qual acrescentamos o termo de fixação de "gauge":

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda} + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\Psi}(i\not{D} - m)\Psi, \tag{4.1}$$

com Q e m, designando, respectivamente, a carga e a massa do elétron. É nossa intenção obter uma teoria com um número finito de gráficos divergentes, os quais iremos regularizar usando o esquema de Pauli-Villars, que mantém invariância de "gauge".

É importante, para o cálculo do potencial em $(2+1)$ dimensões, determinarmos o comportamento da carga elétrica renormalizada (lembre-se que estamos trabalhando no sentido de uma teoria efetiva, cuja carga depende do valor da carga elétrica em $(3+1)D$, dividida pela distância interplanar). Faremos isso ao determinar $\beta(m)$ no final deste capítulo.

O cálculo da função-β é útil também para determinar o potencial via solução da equação de Callan-Symanzik,

$$\left[M\frac{\partial}{\partial M} + \beta(Q_R)\frac{\partial}{\partial Q_R}\right]U(q, M, Q_R) = 0, \tag{4.2}$$

onde M é a escala de renormalização, Q_R a carga renormalizada e q o momento. Não solucionaremos (4.2) neste trabalho, reservando esta tarefa para um estudo posterior.

Um outro propósito deste trabalho é verificar que a teoria de Podolsky pode também ser utilizada no contexto da regularização da Eletrodinâmica em $(3+1)D$.

4.1 Correções em 1-loop

4.1.1 Auto-Energia do Fóton em $d = 4$

A 1-loop, a auto-energia do fóton é divergente. Esta é escrita, no "gauge" de Landau, cmo seque abaixo:

$$\Pi^{\mu\nu}_{(a)} = -4Q^2 \int \frac{d^4k}{(2\pi)^4} \frac{-\eta^{\mu\nu}(k^2 - p\cdot k - m^2) + k^\mu(k-p)^\nu + k^\nu(k-p)^\mu}{(k^2-m^2)[(k-p)^2-m^2]}.$$

Partindo do cálculo de $\Pi^{\mu\nu}_{(a)}$, iremos regularizar as divergências ultravioleta calculando a parte que viola a identidade de Ward, e depois igualando-a a zero, utilizando campos auxiliares e uma conveniente escolha de constantes, como veremos a seguir. Usando a simplificação do denominador dada por Feynman, podemos escrever:

$$\begin{aligned} D &= k^2x - 2p\cdot kx + p^2x - m^2x + k^2(1-x) - m^2(1-x) \\ &= k^2 - 2p\cdot kx + p^2x - m^2, \end{aligned}$$

onde agora deixaremos somente termos quadráticos em k, através da substituição $k \to k + px$, tornando o denominador:

$$k^2 - \overbrace{[p^2x(x-1) + m^2]}^{\Delta},$$

e alterando o numerador da expressão $\Pi^{\mu\nu}_{(a)}$,

$$\begin{aligned} \Pi^{\mu\nu}_{(a)} &= 4Q^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{-\eta^{\mu\nu}\left[\frac{k^2}{2} + p^2x(x-1) - m^2\right] + 2p^\mu p^\nu x(x-1)}{(k^2-\Delta)^2} \\ &= 4Q^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \left[\frac{2x(x-1)}{(k^2-\Delta)^2}(-\eta^{\mu\nu}p^2 + p^\mu p^\nu) - \eta^{\mu\nu}\frac{k^2/2-\Delta}{(k^2-\Delta)^2}\right], \end{aligned} \tag{4.3}$$

onde na última linha de (4.3) separamos a parte proporcional a $p^2\eta^{\mu\nu} - p^\mu p^\nu$, que satisfaz à identidade de Ward, da parte que *não* respeita esta identidade. Fazendo uma rotação de Wick, tal que $k^2 \to -k_E^2$ e $d^4k \to id^4k_E$, teremos

$$\Pi^{\mu\nu}_{(a)} = \frac{4Q^2}{(2\pi)^4} \overbrace{(2\pi^2)}^{\text{ângulo sólido}} \quad i\int_0^1 dx \int \frac{d(k_E^2)}{2} k_E^2 \left[\frac{2x(x-1)}{(k_E^2+\Delta)^2}(p^2\eta^{\mu\nu} - p^\mu p^\nu) - \eta^{\mu\nu} \frac{-\frac{k_E^2}{2} - \Delta}{(k_E^2+\Delta)^2} \right].$$

Iremos calcular primeiro o termo que viola a identidade de Ward

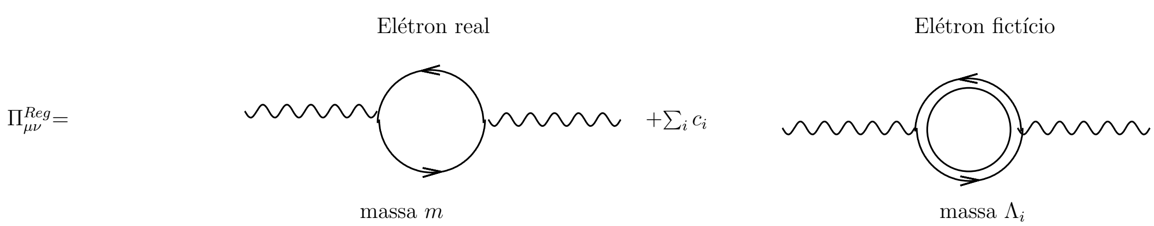
$$\frac{-4iQ^2}{(2\pi)^4}\eta^{\mu\nu} \int_0^1 dx (2\pi^2)\frac{1}{2}\int_0^\infty k_E^2 d(k_E^2) \frac{k_E^2/2 + \Delta}{(k_E^2+\Delta)^2} \tag{4.4}$$

$$= \frac{-2iQ^2}{(2\pi)^4}\eta^{\mu\nu} \int_0^1 dx (2\pi^2) \int_0^\infty d\overbrace{\rho}^{k_E^2} \frac{\overbrace{(\rho+\Delta-\Delta)}^{k_E^2}(\rho+\Delta+\Delta)/2}{(\rho+\Delta)^2}$$

$$= \frac{-2i\pi^2Q^2}{(2\pi)^4}\eta^{\mu\nu} \int_0^1 dx \int_0^\infty d\rho \left[1 - \frac{\Delta^2}{(\rho+\Delta)^2}\right] = \frac{-2i\pi^2Q^2}{(2\pi)^4}\eta^{\mu\nu} \int_0^1 dx \int_0^\infty d\rho\, [1-\Delta]$$

$$= \frac{-2i\pi^2Q^2}{(2\pi)^4} \left[\int_0^\infty d\rho - m^2 + \frac{p^2}{6}\right]$$

Podemos eliminá-la, incorporando a contribuição de partículas fictícias de Pauli-Villars, de forma que:



Assim (4.4) deverá ser expresso como

$$\frac{-2i\pi^2Q^2}{(2\pi)^4} \left[\int_0^\infty d\rho \left(1 + \sum_i c_i\right) - m^2 - \sum_i c_i\Lambda_i^2 + \frac{p^2}{6}\left(1 + \sum_i c_i\right)\right]. \tag{4.5}$$

Se impusermos

$$1 + \sum_i c_i = 0 \tag{4.6}$$

$$\sum_i c_i \Lambda_i^2 = -m^2,$$

automaticamente eliminamos a parte que viola a identidade de Ward. Retornamos ao cálculo da auto-energia, levando em conta as restrições dadas pelo nosso regulador

$$
\begin{aligned}
\Pi_{\mu\nu}^{(a)} &= -\frac{4iQ^2}{(2\pi)^4}(2\pi^2)(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\int_0^1 2x(x-1)dx \times \\
&\times \int_0^\infty \frac{\rho}{2} d\overbrace{\rho}^{\rho=\rho+\Delta-\Delta}\left[\frac{1}{(\rho+\Delta)^2} + \sum_i c_i \int_0^\infty \frac{\rho}{2} d\overbrace{\rho}^{\rho=\rho+\Delta_i-\Delta_i}\frac{1}{(\rho+\Delta_i)^2}\right] \\
&= \lim_{\rho_{max}\to\infty} -\frac{iQ^2}{2\pi^2}(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\int_0^1 x(x-1)\left[\ln\left(\frac{\rho_{max}+\Delta}{\Delta}\right)\right. \\
&+ \left.\sum_i c_i \ln\left(\frac{\rho_{max}+\Delta_i}{\Delta_i}\right)\right] \\
&= \lim_{\rho_{max}\to\infty} -\frac{iQ^2}{2\pi^2}(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\int_0^1 x(x-1)\left\{\left[ln(\rho_{max}+\Delta)\right.\right. \\
&+ \left.\left.\sum_i c_i ln(\rho_{max}+\Delta_i)\right] - \left[ln(\Delta) + \sum_i c_i ln(\Delta_i)\right]\right\}.
\end{aligned} \tag{4.7}
$$

Da segunda linha para a terceira, temos que considerar que

$$\int_0^\infty \frac{\Delta_i}{(\rho+\Delta_i)^2} = 1,$$

e $\sum_i c_i = -1$. Retornando à última integral, notemos que

$$\lim_{\rho_{max}\to\infty} \ln(\rho_{max}+\Delta) = \ln(\rho_{max}),$$

e usando novamente $\sum_i c_i = -1$, anulamos a divergência logarítmica na penúltima linha.

Resta, ainda, a divergência do último termo. Para continuar o cálculo de (4.7), definimos:

$$\sum_i c_i \ln(\Delta_i) \stackrel{\Lambda_i >> m^2}{\longrightarrow} \sum_i c_i \ln(\Lambda_i^2) \equiv -\ln(\tilde{\Lambda}^2). \tag{4.8}$$

Esse fator $\tilde{\Lambda}^2$ irá definir a escala de nossa divergência. Por exemplo, considere dois campos fermiônicos. Resolvendo (4.6), obteremos

$$c_1 = \frac{m^2 - \Lambda_2^2}{\Lambda_2^2 - \Lambda_1^2}, \quad c_2 = \frac{\Lambda_1^2 - m^2}{\Lambda_2^2 - \Lambda_1^2}.$$

Agora se $\Lambda_2^2 = 2\Lambda_1^2$, teremos, então, em (4.8) que $\tilde{\Lambda} = \frac{\Lambda_1^2}{2}$. Com isso, chegamos, ao final de nosso cálculos, com $\tilde{\Lambda} \to \infty$, uma divergência.

Posto isto, retornamos a (4.7)

$$
\begin{aligned}
\Pi^{(a)}_{\mu\nu} &= -\frac{iQ^2}{2\pi^2}(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\int_0^1 \overbrace{x(x-1)}^{dv}\,\overbrace{\ln\left(\frac{\tilde{\Lambda}^2}{m^2 - p^2x(1-x)}\right)}^{u} dx \quad (4.9)\\
&= -\frac{iQ^2}{2\pi^2}(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\ln\left(\frac{\tilde{\Lambda}^2}{m^2 - p^2x(1-x)}\right)\Bigg|_0^1\\
&+ \frac{iQ^2}{2\pi^2}(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\int_0^1\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\frac{m^2 - p^2x(1-x)}{\tilde{\Lambda}^2}(1-2x)dx\\
&= \frac{iQ^2}{12\pi^2}(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\ln\left(\frac{\tilde{\Lambda}^2}{m^2}\right) = \frac{iQ^2}{6\pi^2}(\eta_{\mu\nu}p^2 - p_\mu p_\nu)\ln\left(\tilde{\Lambda}\right)\\
&+ \text{ termos independentes de } \tilde{\Lambda},
\end{aligned}
$$

e o resultado fica finito dentro desta escala.

Assim, a contribuição para a correção do propagador do fóton (vamos chamá-la de Z_3) é dada pelo termo (4.9),

$$
Z_3 = 1 + \frac{iQ^2}{6\pi^2}\ln(\tilde{\Lambda}) + \ldots, \quad (4.10)
$$

(nossa correção se restringe a 1-loop).

4.1.2 Auto-Energia do Fóton em $d = 3$

O cálculo de $\Pi^{(a)}_{\mu\nu}$ será diferente de (4.9), devido aos traços dados em (3.1). De fato,

$$
\begin{aligned}
\Pi^{\mu\nu}_{(a)} &= -(-iQ)^2\int\frac{d^3k}{(2\pi)^3}tr\left[\gamma^\mu\frac{1}{\not{k} - m}\gamma^\nu\frac{1}{\not{k} - \not{p} - m}\right] = Q^2\int\frac{d^3k}{(2\pi)^3}tr\left[\gamma^\mu\frac{(\not{k} + m)}{k^2 - m^2}\times\right.\\
&\times \left.\gamma^\nu\frac{(\not{k} - \not{p} + m)}{(k-p)^2 - m^2}\right] = Q^2\int\frac{d^3k}{(2\pi)^3}\frac{tr(\gamma^\mu\gamma_\alpha\gamma^\nu\gamma_\beta)k^\alpha(k-p)_\beta + tr(\gamma^\mu\gamma_\alpha\gamma^\nu)k^\alpha}{(k^2 - m^2)[(k-p)^2 - m^2]}\\
&+ \frac{m\,tr(\gamma^\mu\gamma^\nu\gamma^\beta)(k-p)_\beta + m^2\,tr(\gamma^\mu\gamma^\nu)}{(k^2 - m^2)[(k-p)^2 - m^2]}.
\end{aligned}
$$

Utilizando (3.1) e lembrando que queremos manter somente termos quadráticos no denominador, a simplificação de Feynman com a substituição $k \to k + px$, nos fornece

$$\begin{aligned}
\Pi^{(a)}_{\mu\nu} &= \frac{-2Q^2}{(2\pi)^3}\int d^3k \int_0^1 dx \frac{\eta_{\mu\nu}[k^2 - m^2 + p^2x(x-1)] + 2x(1-x)p_\mu p_\nu - 2\overbrace{k_\mu k_\nu}^{\frac{k^2\eta_{\mu\nu}}{3}}}{(k^2-\Delta)^2} \qquad (4.11)\\
&- \frac{im\epsilon_{\mu\nu\alpha}p^\alpha}{(k^2-\Delta)^2}\\
&= \frac{2Q^2}{(2\pi)^3}\int d^3k \int_0^1 dx \left[\frac{2x(x-1)}{(k^2-\Delta)^2}(-\eta_{\mu\nu}p^2 + p_\mu p_\nu) - \eta_{\mu\nu}\frac{k^2/3 - \Delta}{(k^2-\Delta)^2} + \frac{im\epsilon_{\mu\nu\alpha}p^\alpha}{(k^2-\Delta)^2}\right]
\end{aligned}$$

(potências ímpares no numerador foram eliminadas pela simetria do integrando). Novamente, utilizamos o regulador de Pauli-Villars, mesmo que (4.11) seja finita, eliminando o segundo termo de (4.11), pois este viola a identidade de Ward. Passemos antes às correções de vértice em $(3+1)D$

4.1.3 Correções de Vértice

O gráfico de correção de vértice também deve ser calculado, como mostrado na figura (4.1). Não é necessário neste caso, mas já visando uma aplicação à teoria não-Abeliana (onde o trabalho algébrico é maior), faremos os cálculos [61] utilizando os operadores de projeção do momento magnético. Seja u_1 e $\bar{u}_2$ dois espinores que descrevem os estados finais (quadrimomento p_2) e iniciais (quadrimomento p_1) do elétron. A amplitude dos processo de espalhamento é ilustrada na figura (4.1). O fator intermediário, $M_\mu(p_1, p_2)$, entre estes dois estados, pode ser decomposto somente usando invariância de Lorentz, na seguinte forma:

$$\bar{u}_2 M_\mu u_1 = u_2\left[F_1(t)\gamma_\mu - \frac{i}{4m}F_2(t)(\gamma_\mu \not{\Delta} - \not{\Delta}\gamma_\mu) - \frac{i}{m}F_3(t)\Delta_\mu\right]u_1, \quad (4.12)$$

onde $\Delta = p_1 - p_2$ e $t = -(p_1 - p_2)^2$. Na tentativa de determinar os fatores de forma , $F_i(t)$, verificamos *a priori* que $F_3(t)$ é nulo devido a conservação da corrente eletromagnética. Essa mesma decomposição (4.12) é satisfeita se trocarmos os espinores por $-i\not{p}_1 + m$ e $-i\not{p}_2 + m$, com $p_1^2 = p_2^2 = m^2$,

$$(-i\not{p}_1 + m)\left[F_1(t)\gamma_\mu - \frac{i}{4m}F_2(t)(\gamma_\mu \not{\Delta} - \not{\Delta}\gamma_\mu) - \frac{i}{m}F_3(t)\Delta_\mu\right](-i\not{p}_2 + m). \quad (4.13)$$

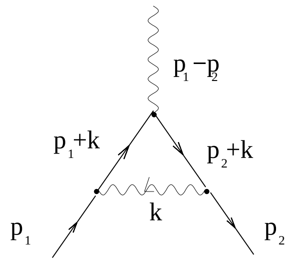


Figura 4.1: Correção de vértice

Multiplicando ambos os lados pelas estruturas vetoriais em mãos, no caso, $P_\mu = (p_1+p_2)_\mu$ e γ_μ e tomando os traços, chegamos a um sistema de equações para os fatores de forma

$$\left\{ \begin{array}{rcl} tr\left[iP^\mu(-i\not{p}_1+m)M_\mu(-i\not{p}_2+m)\right] &=& m(t-4m^2)\left(4F_i(t)+\frac{t}{m^2}F_2(t)\right) \\ tr\left[\gamma_\mu(-i\not{p}_1+m)M_\mu(-i\not{p}_2+m)\right] &=& 4(t+2m^2)F_1(t)+6tF_2(t), \end{array} \right. \quad (4.14)$$

cuja relação pode ser resolvida em um sistema para $F_1(t)$ e $F_2(t)$. É aqui que entra em ação, tanto no cálculo dos traços, quanto na resolução do sistema (4.14), o programa desenvolvido usando o pacote GiNaC [15]. O gráfico da figura (4.1) permite escrever

$$\begin{aligned} N^\mu_{(a)} &= \int \frac{d^4k}{(2\pi)^4}\bar{u}(p_2)(-iQ\gamma^\nu)\frac{\eta_{\nu\rho}k^2-k_\nu k_\rho}{k^2}\frac{i}{a^2\left(k^4-\frac{k^2}{a^2}\right)}\frac{i(\not{k}+\not{p}_2+m)}{(k+p_2)^2-m^2}(-iQ\gamma^\mu)\times \\ &\times \frac{i(\not{p}_1+\not{k}+m)}{(k+p_1)^2-m^2}(-iQ\gamma^\rho)u(p_1) \qquad (4.15) \\ &= \frac{iQ^3}{a^2(2\pi)^4}\int d^4k\bar{u}(p_2)\overbrace{\frac{\gamma^\nu(\eta_{\nu\rho}k^2-k_\rho k_\nu)(\not{k}+\not{p}_2+m)\gamma^\mu(\not{k}+\not{p}_1+m)\gamma^\rho}{k^2\left(k^2-\frac{1}{a^2}\right)\left[(k+p_2)^2-m^2\right]\left[(k+p_1)^2-m^2\right]k^2}}^{M^\mu}u(p_1). \end{aligned}$$

Usando a parametrização de Feynman, o denominador de (4.15) é simplificado

$$\begin{aligned} D &= k^2x+k^2y-\frac{1}{a^2}x+zk^2+2p_2\cdot kz++p_2^2z-m^2z+k^2w \qquad (4.16) \\ &+ 2p_1\cdot kw+p_1^2w-m^2w+k^2v = k^2-\frac{1}{a^2}x-m^2(1-x-y-v) \\ &+ 2k\cdot p_2z+2p_1\cdot k(1-x-y-z-v)+p_1^2(1-x-y-z-v)+p_2^2z, \end{aligned}$$

e podemos eliminar os termos $k \cdot p_i$, fazendo uma transformação de variáveis adequada,

$$k \rightarrow k' = k - p_1(1-x-y-v-z) - p_2 z. \qquad (4.17)$$

Reescrevendo (4.16), obtemos:

$$\begin{aligned} D &= k^2 - p_1^2(1-x-y-v-z) + p_2^2(1-z)z - m^2(1-x-y-v) - 2p_1 \cdot p_2 z(1-x \\ &- y-z-v) + p_1^2(1-x-y-v-z) - \frac{1}{a^2}x \\ &= k^2 - \Delta. \end{aligned}$$

Ou seja, (4.15) é expressa mais simplesmente como abaixo:

$$N^\mu_{(a)} = 5! \frac{iQ^3}{a^2(2\pi)^4} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \int_0^{1-x-y-z} dv \bar{u}(p_2) \frac{M^\mu(k \rightarrow k')}{(k^2-\Delta)^5} u(p_1). \qquad (4.18)$$

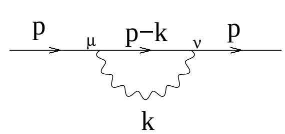
Substituindo (4.17) na expressão M^μ, determinamos $F_1(t)$ e $F_2(t)$ correspondentes (são mostradas aqui as maiores potências)

$$\begin{aligned} F_1(t) &= (-17 - 4\frac{t}{m^2})k^4 + \ldots, \qquad (4.19) \\ F_2(t) &= (-20 - 4\frac{t}{m^2})k^4 + \ldots. \end{aligned}$$

Assim, comparando (4.19) e (4.18) quanto às potências, vemos que este valor é finito e, portanto, não contribui para o cálculo da função-β (o mesmo é válido para $d = 3$).

4.1.4 Auto-Energia do Elétron

O próximo gráfico é relacionado à auto-energia do elétron (em $d = 4$), ou



seja,

$$\Sigma = (-iQ^2) \int \frac{d^4k}{(2\pi)^4} \frac{i}{a^2k^4 - k^2} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\nu \qquad (4.20)$$

$$
\begin{aligned}
&= Q^2 \int \frac{d^4k}{(2\pi)^4} \frac{\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}{a^2k^4 - k^2} \gamma^\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2} \gamma^\nu \\
&= \frac{Q^2}{(2\pi)^4} \int d^4k \frac{1}{a^2k^4 - k^2} \left\{ \frac{\gamma^\nu \gamma^\alpha (p-k)_\alpha \gamma_\nu + m\gamma^\nu\gamma_\nu}{(p-k)^2 - m^2} - \frac{k_\mu k_\nu \gamma^\nu \gamma^\alpha (p-k)_\alpha \gamma^\nu + mk^\nu k^\mu \gamma_\mu \gamma_\nu}{k^2[(p-k)^2 - m^2]} \right\} \\
&= Q^2 \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{-2(\not{p} - \not{k}) + 4m}{a^2k^2(k^2 - \frac{1}{a^2})[(k-p)^2 - m^2]} - \frac{k^2 - p\cdot k + i\sigma^{\mu\nu} k_\mu (p-k)_\nu \not{k}}{a^2k^2\left(k^2 - \frac{1}{a^2}\right)[(k-p)^2 - m^2]k^2} \right\}.
\end{aligned}
$$

Usando novamente os parâmetros de Feynman para a resolução da última integral em (4.20), o denominador é escrito como sendo:

$$
\begin{aligned}
D &= k^2x + k^2y + k^2z - \frac{1}{a^2}y - 2p\cdot kz + p^2z - m^2z \qquad (4.21) \\
&= k^2 - 2p\cdot kz - \frac{1}{a^2}(1-x-z) + p^2z - m^2z.
\end{aligned}
$$

Podemos remover de (4.21) os termos cruzados, através da substituição $k \to k + pz$, o que afetaria o denominador:

$$
D = k^2 - \left[-p^2z + \frac{1}{a^2}(1-x-z) + m^2z\right] = k^2 - \Delta, \qquad (4.22)
$$

e o numerador

$$
\begin{aligned}
N &= [k^2 + p\cdot k(2z-1) + p^2z(z-1)](\not{k} + \not{p}z) - mk^2 \qquad (4.23) \\
&\to k^2\left(\frac{3z}{2} - \frac{1}{4}\right)\not{p} - mk^2 + \not{p}p^2z^2(z-1) - mp^2z^2,
\end{aligned}
$$

onde na segunda linha de (4.23) já ignoramos os termos pares que não contribuem devido à simetria da integral. Juntando (4.21), (4.23) em (4.20),

$$
\begin{aligned}
\Sigma_{(b)} &= \frac{3!Q^2}{a^2(2\pi)^4} \int d^4k \int_0^1 dx \int_0^{1-x} dz\, x \frac{N}{(k^2 - \Delta)^4} \xrightarrow{k \to ik_E} \\
&= \frac{iQ^2}{a^2(4\pi)^2} \int_0^1 xdx \int_0^{1-x} dz \left\{ \frac{\left(\frac{3z}{2} - \frac{1}{4}\right)\not{p} - m}{\left[\frac{1}{a^2}(1-x-z) + m^2z - p^2z\right]^2} \right. \\
&+ \left. \frac{2[p^2z^2(z-1)\not{p} - mp^2z^2]}{\left[\frac{1}{a^2}(1-x-z) + m^2z - p^2z\right]^3} \right\},
\end{aligned}
$$

temos um resultado claramente convergente no ultravioleta! Voltemos à primeira integral em (4.20), e repitamos todo o procedimento, ou seja, a parametrização de Feynman

$$\begin{aligned} D &= k^2x - k^2y - 2p\cdot ky + p^2y - m^2y - \frac{1}{a^2}x + k^2z \\ &= k^2 - 2p\cdot ky + p^2y - m^2y - \frac{1}{a^2}x, \end{aligned} \tag{4.24}$$

e a substituição

$$D = k^2 + p^2y(1-y) - m^2y - \frac{1}{a^2}x = k^2 - \Delta, \tag{4.25}$$

onde $\Delta = m^2y + \frac{1}{a^2}x - p^2y(1-y)$. Essa transformação afeta o numerador. De fato,

$$N = 2\not{p}y + 4m. \tag{4.26}$$

Assim sendo, a integral se reduz a

$$\Sigma_{(a)} = i\frac{Q^2}{a^2(2\pi)^4}\int d^4k\int_0^1 dx\int_0^{1-x} dy\frac{2\not{p}y + 4m}{(k^2-\Delta)^3}, \tag{4.27}$$

o que, após uma rotação de Wick, torna-se

$$\Sigma_{(a)} = \frac{Q^2}{2a^2(2\pi)^4}\int_0^1 dx\int_0^{1-x} dy\frac{2\not{p}y + 4m}{m^2y + \frac{1}{a^2}x - p^2y(1-y)}, \tag{4.28}$$

portanto, novamente convergente no ultravioleta e, assim, não nos interessa no contexto desse cálculo. Ainda, se ele é convergente em $d = 4$, certamente será convergente em $d = 3$.

4.2 Contra-termos na Eletrodinâmica de Podolsky

Dada a Lagrangeana original,

$$\mathcal{L} = -\frac{1}{4}\left(F^{(0)}_{\mu\nu}\right)^2 + a^2\partial_\mu F^{(0)\,\mu\nu}\partial_\lambda F^{(0)\,\lambda}_{\nu} + \bar{\Psi^{(0)}}(i\not{D} - m^{(0)})\Psi^{(0)}, \tag{4.29}$$

redefinimos os campos

$$\Psi^{(0)} = Z_2^{1/2}\Psi_R, \quad A_\mu^{(0)} = Z_3^{1/2}A_R^\mu,$$

Escolhemos não empregar qualquer renormalização na constante a^2, pois, além de eventualmente querermos que $a^2 \to 0$, podemos argumentar que esse fator não precisa ser renormalizado.

Dessa forma:

$$Q_R = Z_3^{1/2}Q_{(0)}. \tag{4.30}$$

Logo, usando a definição da função-β e (4.30), teremos

$$\beta \;=\; \tilde{\Lambda}\frac{\partial Q_R}{\partial\tilde{\lambda}} = Q_{(0)}\tilde{\Lambda}\frac{\partial}{\partial\tilde{\Lambda}}Z_3^{1/2} = \frac{1}{2}Q_{(0)}Z_3^{-\frac{1}{2}}\tilde{\Lambda}\frac{\partial}{\partial\tilde{\Lambda}}Z_3^{1/2} = \frac{1}{2}Q_{(0)}Z_3^{1/2}\tilde{\Lambda}\frac{\partial}{\partial\tilde{\Lambda}}\ln(Z_1),$$

mas, trabalhando na ordem mais baixa da constante de acoplamento Q, poderemos substituir as funções pelas suas expansões,

$$\begin{aligned}
Z_3 &= 1 + \delta Z_3 + \ldots \\
Z_3^{1/2} &= 1 + \frac{1}{2}\delta Z_3 + \ldots \\
\ln(Z_3) &= \delta Z_3 + \ldots,
\end{aligned}$$

e usando a forma explícita de Z_1, (4.10),

$$\beta = \frac{1}{2}Q_{(0)}Z_3^{1/2}\tilde{\Lambda}\frac{\partial}{\partial\tilde{\Lambda}}\ln(Z_3) \approx \frac{1}{2}Q_{(0)}\frac{\partial\delta Z_3}{\partial(\ln\tilde{\Lambda})} = \frac{Q_{(0)}^3}{12\pi^2}; \tag{4.31}$$

como as outras contribuições são todas nulas, a função coincide com o cálculo correspondente da Eletrodinâmica usual, em 4 dimensões.

4.3 Cálculo da Função-$\beta(m)$ em $d = 3$

O termo que viola a identidade de Ward em (4.11) é:

$$\begin{aligned}
-\;& \frac{2Q^2}{(2\pi)^3}\eta^{\mu\nu}\int d^3k\int_0^1 dx\frac{k^3/3 - \Delta}{(k^2-\Delta)^2} = \frac{2iQ^2}{(2\pi)^3}(4\pi)\eta^{\mu\nu}\int_0^1\int_0^\infty k_E^2\frac{k_E^2/3+\Delta}{(k_E^2+\Delta)^2}dk_E \\
=\;& \frac{iQ^2}{2\pi^2}\eta^{\mu\nu}\int_0^1 dx\left[-\Delta^{\frac{1}{2}}\frac{\pi}{2} + \Delta^{\frac{1}{2}}\frac{3\pi}{4}\right] = \frac{iQ^2}{8\pi}\eta^{\mu\nu}\int_0^1 dx\Delta^{\frac{1}{2}} \\
=\;& \frac{iQ^2}{8\pi}\eta^{\mu\nu}\frac{4mp + (4m^2-p^2)\ln\left(\frac{2m+p}{2m-p}\right)}{8p}.
\end{aligned} \tag{4.32}$$

Para que este termo seja eliminado com a ajuda de um regulador de Pauli-Villars de massa Λ_i,

$$\frac{iQ^2}{64\pi p}\left[4p\left(m+\sum_i c_i\Lambda_i\right)+4m^2\ln\left(\frac{2m+p}{2m-p}\right)+4\sum_i c_i\Lambda_i^2\ln\left(\frac{2\Lambda_i+p}{2\Lambda_i-p}\right)\right.$$
$$\left.-\quad p^2\left(\ln\left(\frac{2m+p}{2m-p}\right)+\sum_i c_i\ln\left(\frac{2\Lambda_i+p}{2\Lambda_i-p}\right)\right)\right], \tag{4.33}$$

devemos ter

$$\left\{\begin{array}{ccc} \sum_i c_i\Lambda_i & = & -m \\ \sum_i c_i\Lambda_i^2\ln\left(\frac{2\Lambda_i+p}{2\Lambda_i-p}\right) & = & -m^2\ln\left(\frac{2m+p}{2m-p}\right) \\ \sum_i c_i\ln\left(\frac{2\Lambda_i+p}{2\Lambda_i-p}\right) & = & -\ln\left(\frac{2m+p}{2m-p}\right) \end{array}\right. . \tag{4.34}$$

Multiplicamos a última equação de (4.34) e subtraímos o resultado da penúltima, para simplificar mais o sistema de equações:

$$\left\{\begin{array}{ccc} \sum_i c_i\Lambda_i & = & -m \\ \sum_i c_i(\Lambda_i^2-m^2)\ln\left(\frac{2\Lambda_i+p}{2\Lambda_i-p}\right) & = & 0 \end{array}\right. . \tag{4.35}$$

Seja

$$\begin{aligned} c_2 &= \frac{m(\Lambda_1^2-m^2)}{\Lambda_2(\Lambda_1^2-m^2)+\frac{\Lambda_1(m^2-\Lambda_2^2)\ln\left(\frac{2\Lambda_2+p}{2\Lambda_2-p}\right)}{\ln\left(\frac{2\Lambda_1+p}{2\Lambda_1-p}\right)}} \\ c_1 &= -\frac{m+c_2\Lambda_2}{\Lambda_1}. \end{aligned} \tag{4.36}$$

Considerando

$$\Lambda_1^2=m^2+\Lambda^2, \quad \Lambda_2^2=m^2+2\Lambda^2, \tag{4.37}$$

teremos

$$\lim_{\Lambda\to\infty} c_2=0 \quad \text{e portanto} \quad \lim_{\Lambda\to\infty} c_1=0, \tag{4.38}$$

(usamos L'Hôpital para tratar a indeterminação no segundo termo do denominador de c_2).

A conseqüência direta desta escolha, observando que

$$\lim_{\Lambda\to\infty} arcsh\left(\frac{p}{\sqrt{4\Lambda_i - p^2}}\right) = 0,$$

é a eliminação dos termos indesejados, e o fato de que, quando as massas Λ_i das partículas fictícias tenderem ao infinito, um dos termos de $\Pi^{(a)}_{\mu\nu}$ (3.35) desaparece:

$$\begin{aligned}\Pi(p) &= -\frac{iQ^2p^2}{4\pi}\left\{\cancel{\frac{m}{p^2}} + \left(\frac{m^2}{p^2}+\frac{1}{4p}\right)\left[arcsh\left(\frac{-p}{\sqrt{4m-p^2}}\right)\right.\right. \qquad (4.39)\\ &- \left. arcsh\left(\frac{p}{\sqrt{4m-p^2}}\right)\right] + \cancel{\sum_i c_i\frac{\Lambda_i}{p^2}} + \sum_i c_i\left(\frac{\Lambda_i^2}{p^2}+\frac{1}{4p}\right)\left[arcsh\left(\frac{-p}{\sqrt{4\Lambda_i-p^2}}\right)\right.\\ &- \left.\left. arcsh\left(\frac{p}{\sqrt{4\Lambda_i-p^2}}\right)\right]\right\} \xrightarrow{\Lambda\to\infty}\\ &= -\frac{iQ^2p^2}{4\pi}\left(\frac{m^2}{p^2}+\frac{1}{4p}\right)\left[arcsh\left(\frac{-p}{\sqrt{4m-p^2}}\right) - arcsh\left(\frac{p}{\sqrt{4m-p^2}}\right)\right],\end{aligned}$$

com as condições impostas pelo regulador, pois todos os últimos termo são nulos. $\Pi'(p)$ (3.36) não sofre alteração após este limite. Desta forma, teremos somente termos finitos de correção de carga a 1-loop.

No entanto, não estamos trabalhando com o valor direto da carga tridimensional, mas, ao invés, escolhemos usar a carga elétrica em $(3+1)D$ dividida pela distância interplanar, e vimos que a primeira apresenta correção. Vejamos como esta escolha influencia nossos resultados.

Como em $d < 4$ o "cutoff" pode ser infinito, o valor da constante Q_R depende do único parâmetro dimensional restante, a massa m, a qual também controla o valor de $Q_{(0)}$ para que, em $d \neq 4$, a ação seja adimensional. Em $d = 4 - 2\epsilon$, (4.30) é escrita desta forma

$$Q_R(m) = Z_3^{1/2} m^{-\epsilon} Q_{(0)}, \qquad (4.40)$$

onde $Q_{(0)}$ é a carga não-renormalizada em $d = 4 - 2\epsilon$. Definamos, então a função-β em d dimensões, β_d como sendo uma função dada por

$$\beta_d(m) = \frac{\partial}{\partial(\ln(m))}Q_R(m) = m\frac{\partial}{\partial m}\left(m^{-\epsilon}Z_3^{1/2}Q_{(0)}\right) \qquad (4.41)$$

$$
\begin{aligned}
&= -\epsilon Q_R + m^{1-\epsilon}\frac{\partial Z_3^{1/2}}{\partial m}Q_{(0)} = -\epsilon Q_R + m^{-\epsilon}\frac{1}{2}Z_3^{-1/2}\left(m\frac{\partial Q_R}{\partial m}\right)\frac{\partial Z_3}{\partial Q_R}Q_{(0)} \\
&= -\epsilon Q_R + \frac{1}{2}m^{-\epsilon}Z_3^{1/2}Q_0\beta_d\frac{1}{Z_3}\frac{\partial Z_3}{\partial Q_R} = -\epsilon Q_R + \frac{Q_R}{2}\frac{\partial \ln(Z_3)}{\partial Q_R}\beta_d \therefore \\
\beta_d(m) &= \frac{-\epsilon Q_R}{1-\frac{Q_R}{2}\frac{\partial \ln(Z_3)}{\partial Q_R}} = -\epsilon Q_R\left(1+\sum_{i=1}^{\infty}\frac{z_i(Q_R)}{\epsilon^i}\right),
\end{aligned}
$$

já que as constantes de renormalização, como Z_3, devem depender de ϵ, pois as integrais envolvidas para a determinação destas envolvem a dimensão do espaço-tempo em que estão sendo feitos os cálculos. As potências de ϵ também devem ser pólos destas constantes, pois quando o "cutoff" tende a infinito, essas funções divergem para $\epsilon \to 0$.

Contudo, neste limite, β_d deve ser finita, então somente o coeficiente z_1 não é diferente de zero. Além disso, $\beta_d \to \beta_4$, o valor da constante β em quatro dimensões, logo

$$
\beta_4 = \lim_{\epsilon\to 0} -\epsilon Q_R\left(1+\frac{z_1}{\epsilon}\right) = -Q_R z_1 \Rightarrow z_1 = -\frac{\beta_4}{Q_R}. \tag{4.42}
$$

Segue-se, já que a função β é a mesma da QED usual, ou seja,

$$
\beta_d = -\epsilon Q_R + \beta_4 \Rightarrow \beta_3 = -\frac{Q_{(0),3}}{2\sqrt{m}} + \frac{Q^3_{(0),4}}{12\pi^2}, \tag{4.43}
$$

onde $Q_{(0),d}$ é a carga não-renomalizada em d dimensões.

Relembrando (3) da Introdução,

$$
Q_{(0),3} \propto \frac{Q_{(0),4}}{\delta}, \tag{4.44}
$$

onde δ é a distância interplanar, aplicado em (4.43) e usando a definição $\beta(m) = m\frac{\partial Q_R}{\partial m}$,

$$
Q_R = \frac{3Q_{(0),4}}{4\delta m^{\frac{5}{2}}} + \frac{Q^3_{(0),4}}{12\pi^2}\ln(m) + constante, \tag{4.45}
$$

podemos ter uma idéia qualitativa do comportamento da função-$\beta(m)$ nesta dimensão em relação à massa. Como a carga renormalizada permanece positiva (o que poderia mudar o comportamento do potencial), isso garante que, variando δ, nossos resultados permanecem qualitativamente iguais.

Conclusões

Ao longo deste trabalho, investigamos o comportamento da Eletrodinâmica de Podolsky em $(2+1)$ dimensões, dando ênfase à determinação de estados ligados, utilizando para isto a aproximação não-relativística no caso do espalhamento de duas partículas, ao invés de procurar diretamente a solução das equações de campo; em seguida, utilizamos métodos numéricos para determinar os auto-estados e auto-energias, visando possíveis aplicações ao fenômeno da supercondutividade em altas temperaturas ou mesmo em modelos nucleares. Tendo em conta os problemas de unitariedade da teoria não a encaramos como uma teoria fundamental da Natureza, mas sim como uma teoria efetiva.

No capítulo I fizemos nossa primeira incursão em busca destes resultados; o potencial encontrado para a interação de duas partículas escalares mostrou-se essencialmente repulsivo. Na seqüência, a generalização deste cálculo para o espaço d-dimensional não teve como objetivo somente reduzi-lo a quadraturas; de fato sua utilização como esquema de regularização do potencial corrigido em 1-loop, obtido no capítulo II, nos permitiu obter um resultado finito, já que mesmo a introdução de um "cutoff" não foi suficiente para este fim. A idéia de regularizar um potencial eletromagnético não é inédita (vide o potencial devido a um fio infinito com uma densidade de carga uniforme), mas o que fizemos foi utilizar a expressão obtida como um tipo de regularização dimensional.

Demonstramos, no capítulo II, que a teoria de Maxwell-Chern-Simons-Podolsky também deveria ser encarada como uma teoria efetiva (ou seja usada dentro de uma escala de energia, ou distâncias específicas), devido aos problemas que enfrentamos com a falta de unitariedade em gráficos de árvore (ou seja, não há conservação de probabilidade). Neste âmbito, comparamos o número de estados ligados desta teoria, com ou sem o termo de Podolsky, obtendo através de uma estimativa bons resultados (mas, infelizmente, em distâncias inferiores aos $10^4 \mathring{A}$, separação típica dos elétrons de um par de Cooper num sólido). Esse fato foi verificado na prática durante a análise numérica: com o termo de Podolsky, conseguimos um número maior de auto-funções de onda e auto-energias associadas, com os mesmos valores de entrada do que na ausência deste. Como o número de auto-energias possíveis é maior, se associarmos ao movimento de translação do par uma energia de $k_B T/2$, ao mesmo tempo associamos uma maior possibilidade de temperaturas favoráveis à formação destes pares.

O espalhamento de dois férmions, desenvolvido no capítulo III, forneceu um potencial confinante, sem a necessidade de adicionar um termo de Chern-Simons, e os cálculos subseqüentes demonstraram que o efeito da temperatura não é suficiente para fazê-los desaparecer quando $a << 1$ (já que a correção é proporcional a a^6, o que a torna desprezível). Esses cálculos foram feitos propositalmente nas condições em que o momento do "loop" é menor que o momento trocado entre as partículas (para conservar a aproximação não-relativística), e expandimos, considerando altas temperaturas. Uma abordagem alternativa àquela feita neste trabalho seria a de procurar por transições de fase do sistema à temperatura finita como em [18].

Uma surpresa foi o fato dos "spins" das partículas envolvidas no processo de espalhamento não serem necessariamente anti-paralelos para que haja um estado estável de ligação entre dois elétrons (contudo, este comportamento é exibido no estado supercondutor de alguns materiais, como, por exemplo, o Sr_2RuO_4, o que também pode ocorrer no vácuo). Este último fato seria simplesmente inconcebível do ponto de vista do eletromagnetismo usual. Ou seja, não existe dependência de uma rede cristalina para ocorrência do fenômeno. Esse fato inédito poderia fornecer novas perspectivas no estudo de confinamento em modelos nucleares, onde existe uma linha de pesquisa testando a analogia destes sistemas com o fenômeno da supercondutividade.

Estendemos, no capítulo IV, ainda mais a nossa proposta, desta vez utilizando a Eletrodinâmica de Podolsky junto com a regularização de Pauli-Villars, seja em $(3+1)$ ou $(2+1)$ dimensões (não esquecendo o caráter efetivo da última), para obtermos resultados finitos na Eletrodinâmica Quântica, obtendo um valor de β em $(3+1)D$ igual ao apresentado em outros esquemas de regularização.

A vantagem desta proposta seria que a regularização de Pauli-Villars viria tornar finita uma certa quantidade de gráficos divergentes, enquanto os restantes seriam finitos por simples contagem de potências, devido ao propagador da Eletrodinâmica de Podolsky. No nosso caso, por exemplo, tanto as correções de vértices, quanto a auto-energia do elétron, tornaram-se finitas. Determinamos ainda o comportamento da carga renormalizada e sua influência nos nossos valores numéricos quando fazemos correções de "loop".

Acreditamos que alguns rumos interessantes poderiam ser tomados utilizando estes resultados. Um estudo numérico mais extenso dos possíveis estados ligados elétron-elétron e suas aplicações em materiais supercondutores ou teorias efetivas para confinamento nuclear, mereciam ser feitos. Ainda a introdução de uma derivada não-covariante na Lagrangeana de Podolsky cobriria uma lacuna existente na literatura atual.

Apêndice A

Uma pergunta interessante é: qual seria o valor exato da constante a? Por falta de experimentos para responder a este tipo de indagação no âmbito da Eletrodinâmica Planar, iremos recorrer a um dedicado originalmente para determinação da massa do fóton e utilizar seus resultados experimentais em nossa estimativa neste apêndice.

Estimativa Clássica para a Constante da Eletrodinâmica de Podolsky

Retornando a (1), suas equações de campo são:

$$\begin{aligned} (1+a^2\Box)\partial_\mu F^{\mu\nu} &= j^\nu \\ (1+a^2\Box)\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) &= j^\nu, \end{aligned}$$

ou usando as funções de Green,

$$\nabla^2(1-a^2\nabla^2)G(\vec{r},\vec{r'}) = \delta(\vec{r'}-\vec{r}), \tag{46}$$

onde $\partial_\mu A^\mu = 0$, e os campos são estáticos. Para obter a função de Green, temos as relações:

$$G(\vec{r}-\vec{r'}) = \frac{1}{(2\pi)^3}\int d\vec{k}G(\vec{k})e^{-i\vec{k}\cdot(\vec{r}-\vec{r'})};\ \ \delta(\vec{r}-\vec{r'}) = \frac{1}{(2\pi)^3}\int d\vec{k}e^{-i\vec{k}\cdot(\vec{r}-\vec{r'})}. \tag{47}$$

Substituindo em (46),

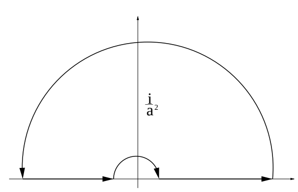
$$\frac{1}{(2\pi)^3}\int(-\vec{k}^2 - a^2\vec{k}^4)G(\vec{k})e^{-i\vec{k}\cdot(\vec{r}-\vec{r'})}d\vec{k} \ \ = \ \ \frac{1}{(2\pi)^3}\int e^{-i\vec{k}\cdot(\vec{r}-\vec{r'})}d\vec{k} \Rightarrow$$

$$G(\vec{k}) = \frac{-1}{\vec{k}^2 + a^2\vec{k}^4}.$$

Integrando

$$\begin{aligned}
G(\vec{r}-\vec{r}') &= \frac{1}{(2\pi)^3}\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} d\vec{k}\vec{k}^2\, sen(\theta)d\theta\, d\phi \frac{e^{-i|\vec{k}||\vec{r}|cos(\theta)}}{\vec{k}^2 + a^2\vec{k}^4} \\
&= \frac{-2}{(2\pi)^3}(2\pi)\int_0^{\infty} \frac{sen(|\vec{k}||\vec{r}|)}{|\vec{k}||\vec{r}|}\frac{\vec{k}^2 d\vec{k}}{\vec{k}^2 + a^2\vec{k}^4} \qquad (48) \\
&= \frac{-1}{|\vec{r}|(2\pi)^2}2\left[\int_0^{\infty}\frac{sen(|\vec{k}||\vec{r}|)}{|\vec{k}|}dk - \int_0^{\infty}\frac{|\vec{k}|sen(|\vec{k}||\vec{r}|)}{(\vec{k}^2 + 1/a^2)}dk\right] \\
&= \frac{-1}{|\vec{r}|(2\pi)^2}2Im\left[\int_0^{\infty}\frac{e^{i|\vec{k}||\vec{r}|}}{|\vec{k}|}dk - \int_0^{\infty}\frac{|\vec{k}|e^{i|\vec{k}||\vec{r}|}}{(\vec{k}^2 + 1/a^2)}dk\right] \\
&= \frac{-1}{4\pi^2 a^2|\vec{r}|}Im\left(i\pi - Res\left.\frac{|\vec{k}|e^{i|\vec{k}||\vec{r}|}}{(k+\frac{i}{a})}\right|_{k=\frac{i}{a}}\right) \\
&= \frac{-1}{4\pi^2|\vec{r}|}Im\left[i\pi\left(1 - e^{-r/a}\right)\right] \\
&= \frac{1}{4\pi|\vec{r}|}Im\left[i\pi e^{\frac{-|\vec{r}|}{a}}senh\left(\frac{r}{a}\right)\right] = \frac{-e^{\frac{-|\vec{r}|}{a}}}{4\pi|\vec{r}|}senh\left(\frac{|\vec{r}|}{a}\right),
\end{aligned}$$

onde a integral foi realizada sobre o contorno abaixo:



Iremos calcular a função de Green para uma esfera carregada com potencial U. Em coordenadas esféricas a função de Green assume a forma

$$G(\vec{r}-\vec{r}') = \frac{-1}{4\pi}\frac{1 - e^{-\frac{1}{a}(r^2 + r'^2 - 2rr'cos(\gamma))^{1/2}}}{(r^2 + r'^2 - 2rr'cos(\gamma))^{1/2}}.$$

Seja, então, $r' = b$ o raio da esfera. Da figura (2), vemos que $cos(\gamma) = cos(\theta)cos(\theta') + sen(\theta)sen(\theta')cos(\phi - \phi')$. Assim, no eixo z, $cos(\gamma) = cos(\theta')$.

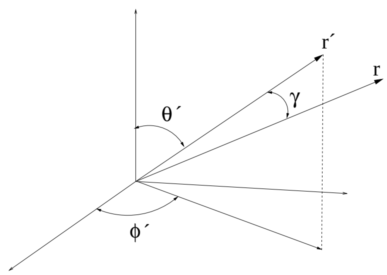


Figura 2: Sistema de coordenadas utilizado

Logo, integrando sobre a área, $dA = b^2 d\Omega$, onde $d\Omega$ é o ângulo sólido, $d\Omega = d\phi' d(cos(\theta')$, obtemos,

$$
\begin{aligned}
U(\vec{r}) &= \int_S G(\vec{r}, \vec{r'})\rho(\vec{r'})d^3\vec{r} \\
&= \frac{-\rho(b)}{4\pi}\int_{\phi'=0}^{2\pi}\int_{cos(\theta')=-1}^{1}\frac{1-e^{-\frac{1}{a}(r^2+b^2-2rbcos(\gamma))^{1/2}}}{(r^2+b^2-2rb\,cos(\gamma))^{1/2}}\rho(b)b^2 d\Omega \\
&= \frac{1}{4\pi}\int_0^{\pi}\int_{-1}^{1} -2\frac{1-e^{-\frac{1}{a}(r^2+b^2-2rbcos(\theta'))^{1/2}}}{(r^2+b^2-2rb\,cos(\theta'))^{1/2}} b^2 d\phi' d(cos(\theta')) \\
&= \frac{2\rho(b)}{4\pi}\int_{u=|r+b|}^{|r-b|}\frac{1-e^{\frac{-u}{a}}}{u}\left(\frac{b^2}{br}\right)u\,du = \frac{2\pi b\rho(b)}{4\pi r}\left[u + ae^{-u/a}\right]_{|r+b|}^{|r-b|} \\
&= \frac{b}{2r}\rho(b)\left[|r-b| - |r+b| + a(e^{-\frac{|r-b|}{a}} - e^{-\frac{|r+b|}{a}})\right], \qquad (49)
\end{aligned}
$$

Iremos aplicar este resultado no caso de cascas concêntricas.

Em $r = b$, com uma das esfera com potencial fixo U, (49) teremos

$$\rho(b) = \frac{2U}{-2b + a(1 - e^{-2b/a})}.$$

Nossa solução torna-se

$$
U(\vec{r}) = \begin{cases} \frac{Ub}{r[-2b+a(1-e^{-2b/a})]}\left[-2b + ae^{\frac{-(r-b)}{a}} - ae^{\frac{-(r+b)}{a}}\right] & r \geq b \\ \frac{Ub}{r[-2b+a(1-e^{-2b/a})]}\left[-2r + ae^{\frac{-(b-r)}{a}} - ae^{\frac{-(b+r)}{a}}\right] & r \leq b \end{cases}.
$$

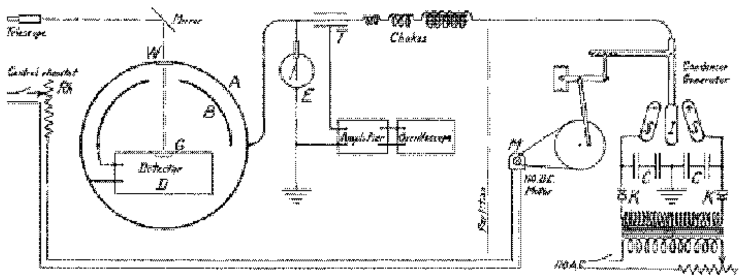


Figura 3: Experimento de S. J. Plimpton e W. E. Lawton

Assim, uma casca esférica com $r = c$, dentro de uma outra casca esférica , $r = b$, carregada com potencial U, permite escrever a razão $\Delta U/U$, literalmente, a diferença de potencial entre uma esfera concêntrica no interior de uma esfera carregada com potencial fixo como sendo:

$$\frac{\Delta U}{U} = \frac{U(b) - U(c)}{U(b)} = 1 - \frac{b\left[-2c + a\left(e^{-\frac{b-c}{a}} - e^{-\frac{b+c}{a}}\right)\right]}{c\left[-2b + a(1 - e^{-2b/a})\right]}. \tag{50}$$

Solucionamos numericamente (50), com o intuito de determinar a constante a com base em alguns dados experimentais. Usando $U = 3000\,V$, $b = 0.762\,m$, $c = 0.6096\,m$, o experimento resultou em $\Delta V = 10^{-6}\,V$ [55]. Então, com esses dados, obtemos para a constante a, usando o método da secante com 17 passos, o valor numérico de $5,86058 \times 10^{-15}\,m$.

Em [32], uma estimativa para o valor de a é dada utilizando a equação de Abraham-Lorentz, derivada no contexto da Eletrodinâmica de Podolsky. Analisando a expressão, foi determinado que, para garantir conservação de energia, é necessário que a seja menor que metade do raio clássico do elétron, que é de $2.817 \times 10^{-15}\,m$. Outra estimativa foi feita em [23], onde se obteve $a = 9.96738 \times 10^{-15} m$.

Apêndice B

International Journal of Theoretical Physics, Vol. 44, No. 8, August 2005 (© 2005)
DOI: 10.1007/s10773-005-4050-1

Algorithm for Probing the Unitarity of Topologically Massive Models

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Received December 18, 2004; accepted March 9, 2005

An uncomplicated and easy handling prescription that converts the task of checking the unitarity of massive, topologically massive, models into a straightforward algebraic exercise, is developed. The algorithm is used to test the unitarity of both topologically massive higher-derivative electromagnetism (TMHDE) and topologically massive higher-derivative gravity (TMHDG). The novel and amazing features of these effective field models are also discussed.

KEY WORDS: topologically massive models; unitarity; effective field models.

1. INTRODUCTION

The momentous discovery that there are dynamics possible for gauge theories in an odd number of space–time dimension that are not open to those in an even number, allowed the construction of field models endowed with novel and amazing properties. In three dimensions, for instance, the addition of a topologically massive Chern-Simons term to the fundamental Lagrangian for a gauge field gives rise to a gauge-invariant theory (Deser *et al.*, 1988a,b). Indeed, this term has a coupling that scales like a mass, but unlike the ways in which gauge fields are usually given a mass, no gauge symmetry is broken, although parity is. Of course, the addition of the esoteric Chern-Simons term is certainly not the unique mass-generating mechanism for gauge fields. We can also utilize for this purpose the well-known Proca/Fierz-Pauli, or the more sophisticated higher-derivative electromagnetic/higher-derivative gravitational, terms. In this vein, it would be interesting to analyze the new physics that emerges from the models obtained by enlarging Maxwell (Einstein)–Chern-Simons theory through the Proca (Fierz-Pauli), or higher-derivative electromagnetic (gravitational), terms. Our aim here is to study the three-term models with higher derivatives. Interesting enough,

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0020-7748/05/0800-1123/0

these models are gauge invariant; besides, they possess rather unusual and exciting properties. In fact, as we shall see, in the context of the electromagnetic models, an attractive interaction between equal charge scalar bosons can occurs which leads to an amazing planar electrodynamics: scalar pairs can condense into bound states; while in the framework of the gravitational systems, unlike what happens within the context of the odorless and insipid three-dimensional general relativity, there exists both attractive and repulsive gravity. We can also have a null gravitational interaction, such as in three-dimensional gravity that is trivial outside the sources.

We present an algorithm for probing the unitarity of massive, topologically massive, models (MTM) in Section 2, which is quite simple to use. This procedure converts the hard task of checking the unitary of the MTM in a trivial algebraic exercise. It is utilized to test the unitarity of both topologically massive higher-derivative electromagnetism (TMHDE) and topologically massive higher-derivative gravity (TMHDG), in Section 3. The novel and amazing features of the electromagnetic models are discussed in Section 4, while those of the gravitational ones are analyzed in Section 5. We conclude in Section 6 with some discussions and comments. We use natural units throughout.

2. ALGORITHM FOR PROBING THE UNITARITY OF MASSIVE, TOPOLOGICALLY MASSIVE, MODELS

To probe the tree unitarity of the massive, topologically massive, models, we will make use of the procedure that consists basically in saturating the propagator with external conserved currents, compatible with the symmetries of the system. The unitarity of the models depends on the sign of the residues of the saturated propagator (SP)—the unitarity is ensured if the residue at each simple pole of the SP is positive (propagating modes) or zero (non-propagating modes). Note that we are using the loose expression "the residue's sign is equal to zero" as synonymous with "the residue is equal to zero."

The idea here is to construct a simple algorithm for analyzing the unitarity of the massive, topologically massive, models, using the procedure we have just outlined. We begin by building the prescription for the massive, topologically massive, electromagnetic models (MTME); next we construct the algorithm for the massive, topologically massive, gravitational models (MTMG).

2.1. Algorithm for Analyzing the Unitarity of the MTME

The saturated propagator related to the MTME, can be written as

$$\mathrm{SP}_{\mathrm{MTME}} = J^{\mu} \left(O^{-1}_{\mathrm{MTME}}\right)_{\mu\nu} J^{\nu}, \tag{1}$$

where J and O^{-1} are, respectively, the conserved current and the propagator concerning the specific massive, topologically massive, electromagnetic model which we are interested in probing the unitarity. Our next step is to obtain the propagator associated with the model at hand. Consider, in this direction, the Lagrangian for the MTME, namely $\mathcal{L}_{\text{MTME}} = \mathcal{L}_{\text{E}} + \epsilon\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{T}}$, where $\mathcal{L}_{\text{E}}$ is the Lagrangian associated with the electromagnetic part of the model, $\mathcal{L}_{\text{gf}}$ is a gauge-fixing Lagrangian, ϵ is a parameter equal to $+1$, if $\mathcal{L}_{\text{E}}$ is gauge invariant, or 0, if $\mathcal{L}_{\text{E}}$ is not gauge invariant, and $\mathcal{L}_{\text{T}} \equiv \frac{s}{2}\varepsilon_{\mu\nu\rho}A^{\mu}\partial^{\nu}A^{\rho}$ is the Chern-Simons term, with A^{μ} being the three-dimensional vector potential and $s > 0$ the topological mass. This Lagrangian, of course, can be written as $\mathcal{L}_{\text{MTME}} = \frac{1}{2}A^{\mu}O_{\mu\nu}A^{\nu}$. Now, it is important for the success of the method that we can find a basis for expanding the wave operator and, consequently, the propagator, such that when one contracts their basis vectors with JJ, the greatest possible number of cancellations may be obtained. The basis $\{\theta, \omega, S\}$, for instance, where $\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{\partial_{\mu}\partial_{\nu}}{\Box}$ and $\omega_{\mu\nu} = \frac{\partial_{\mu}\partial_{\nu}}{\Box}$ are, respectively, the usual transverse and longitudinal vector projector operators, $S_{\mu\nu} \equiv \varepsilon_{\mu\rho\nu}\partial^{\rho}$ is the operator associated with the topological term, and $\eta_{\mu\nu}$ is the Minkowski metric, does the job since $J\omega J = JSJ = 0$. The algebra obeyed by these operators is displayed in Table I. Our signature conventions are $(+, -, -)$, $\varepsilon^{012} = +1 = \varepsilon_{012}$.

Expanding O in the basis $\{\theta, \omega, S\}$, yields $O = a\theta + b\omega + cS$. With the help of Table I, we promptly obtain

$$O_{\text{MTME}}^{-1} = \frac{a}{a^2 + c^2\Box}\theta + \frac{1}{b}\omega - \frac{c}{a^2 + c^2\Box}S. \tag{2}$$

Inserting Equation (2) into Equation (1), we get

$$\text{SP}_{\text{MTME}} = \frac{a}{a^2 + c^2\Box}J^{\mu}J_{\mu}. \tag{3}$$

Note that only the θ-component of O_{MTME}^{-1} contributes to the calculation of SP_{MTME}.

Before going further, we need a lemma.

Lemma 1. *If $m \geq 0$ is the mass of a generic physical particle associated with the MTME and k is the corresponding momentum exchanged, then $J_{\mu}J^{\mu}|_{k^2=m^2} < 0$.*

Table I. Multiplicative Table for the Operators θ, ω, and S

	θ	ω	S
θ	θ	0	S
ω	0	ω	0
S	S	0	$-\Box\theta$

Note. The operators are supposed to be in the ordering "row times column."

Proof: To begin with, let us expand the current in a suitable basis. The set of independent vectors in momentum space,

$$k^\mu \equiv (k^0, \mathbf{k}), \qquad \tilde{k}^\mu \equiv (k^0, -\mathbf{k}), \qquad \varepsilon^\mu \equiv (0, \vec{\epsilon}), \tag{4}$$

where $\vec{\epsilon}$ is a unit vector orthogonal to $\mathbf{k}$, serves our purpose. Using this basis, $J^\mu(k)$ takes the form

$$J^\mu = Ak^\mu + B\tilde{k}^\mu + C\varepsilon^\mu.$$

On the other hand, the current conservation gives the constraint $A(k_0^2 - \mathbf{k^2}) - B(k_0^2 + \mathbf{k^2}) = 0$, which allows to conclude that $A^2 > B^2$. Now, it is trivial to see that $J_\mu J^\mu = k^2(B^2 - A^2) - C^2$. Consequently, $J_\mu J^\mu|_{k^2=m^2} < 0$. □

We are now ready to present the algorithm for probing the unitarity of the MTME.

Algorithm 1. *Calculate the θ-component of the propagator in the basis $\{\theta, \omega, S\}$ which, for short, we shall designate as f_θ. Next, determine the signs of the residues at each simple pole of f_θ. If all the signs are ≤ 0, the model is unitary; if at least one of the signs is positive, the system is non-unitary.*

2.2. Algorithm for Analyzing the Unitarity of the MTMG

The Lagrangian for the MTMG can be written as $\mathcal{L}_{\text{MTMG}} = \mathcal{L}_{\text{G}} + \epsilon\mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{T}}$, where $\mathcal{L}_{\text{G}}$ is the Lagrangian concerning the gravitational part of the theory, and $\mathcal{L}_{\text{T}} \equiv \frac{1}{\mu}\varepsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}(\partial_\mu\Gamma^\sigma{}_{\rho\nu} + \frac{2}{3}\Gamma^\sigma{}_{\mu\beta}\Gamma^\beta{}_{\nu\rho})$ is the Chern-Simons Lagrangian, with $\mu > 0$ being a dimensionless parameter, whereas the corresponding SP is given by

$$\text{SP}_{\text{MTMG}} = T^{\mu\nu}\left(O^{-1}_{\text{MTMG}}\right)_{\mu\nu,\,\rho\sigma} T^{\rho\sigma}, \tag{5}$$

where $T^{\mu\nu}$ is the conserved current which, obviously, is symmetric in the indices μ and ν. Our conventions are $R^\alpha{}_{\beta\gamma\delta} = -\partial_\delta\Gamma^\alpha{}_{\beta\gamma} + \cdots$, $R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}$, $R = g^{\mu\nu}R_{\mu\nu}$, where $g_{\mu\nu}$ is the metric tensor, and signature $(+, -, -)$. To calculate the SP_{MTMG}, we need to know the propagator beforehand. This can be done by linearizing $\mathcal{L}_{\text{MTMG}}$. Setting $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where κ is a constant that in four dimensions is equal to $\sqrt{32\pi G}$, with G being Newton's constant, we can rewrite the linearized Larangian as $\mathcal{L}^{(\text{lin})}_{\text{MTMG}} = \frac{1}{2}h_{\mu\nu}O^{\mu\nu,\,\rho\sigma}h_{\rho\sigma}$.It is extremely convenient to expand O in the basis $\{P^1, P^2, P^0, \overline{P}^0, \overline{\overline{P}}^0, P\}$, where $P^1, P^2, P^0, \overline{P}^0$, and $\overline{\overline{P}}^0$, are the usual three-dimensional Barnes–Rivers operators (Antoniadis and Tomboulis, 1986; Nieuwenhuizen, 1973; Rivers, 1964; Stelle, 1977), namely,

$$P^1_{\mu\nu,\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}),$$

Table II. Multiplicative Operator Algebra Fulfilled by P^1, P^2, P^0, $\overline{P}^0$, $\overline{\overline{P}}^0$, and P

	P^1	P^2	P^0	$\overline{P}^0$	$\overline{\overline{P}}^0$	P
P^1	P^1	0	0	0	0	0
P^2	0	P^2	0	0	0	P
P^0	0	0	P^0	0	$P^{\theta\omega}$	0
$\overline{P}^0$	0	0	0	$\overline{P}^0$	$P^{\omega\theta}$	0
$\overline{\overline{P}}^0$	0	0	$P^{\omega\theta}$	$P^{\theta\omega}$	$2(P^0 + \overline{P}^0)$	0
P	0	P	0	0	0	$-\Box^3 P^2$

Note. Here $P^{\theta\omega}{}_{\mu\nu\,,\,\rho\sigma} \equiv \theta_{\mu\nu}\omega_{\rho\sigma}$ and $P^{\omega\theta}{}_{\mu\nu\,,\,\rho\sigma} \equiv \omega_{\mu\nu}\theta_{\rho\sigma}$.

$$P^2_{\mu\nu,\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho} - \theta_{\mu\nu}\theta_{\rho\sigma}),$$

$$P^0_{\mu\nu,\rho\sigma} = \frac{1}{2}\theta_{\mu\nu}\theta_{\rho\sigma}, \qquad \overline{P}^0_{\mu\nu,\rho\sigma} = \omega_{\mu\nu}\omega_{\rho\sigma},$$

$$\overline{\overline{P}}^0_{\mu\nu,\,\rho\sigma} = \theta_{\mu\nu}\,\omega_{\rho\sigma} + \omega_{\mu\nu}\,\theta_{\rho\sigma},$$

and P is the operator associated with the linearized Chern-Simons term, i.e.,

$$P_{\mu\nu,\,\rho\sigma} \equiv \frac{\Box\partial^\lambda}{4}[\epsilon_{\mu\lambda\rho}\,\theta_{\nu\sigma} + \epsilon_{\mu\lambda\sigma}\,\theta_{\nu\rho} + \epsilon_{\nu\lambda\rho}\,\theta_{\mu\sigma} + \epsilon_{\nu\lambda\sigma}\,\theta_{\mu\rho}],$$

since $TP^1T = T\overline{P}^0T = T\overline{\overline{P}}^0T = TPT = 0$. The corresponding multiplicative table is displayed in Table II. The expansion of O in the basis $\{P^1, P^2, P^0, \overline{P}^0, \overline{\overline{P}}^0, P\}$ is greatly facilitated if use is made of the following tensorial identities:

$$\frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \equiv I_{\mu\nu,\,\rho\sigma} = \left[P^1 + P^2 + P^0 + \overline{P}^0\right]_{\mu\nu,\,\rho\sigma},$$

$$\eta_{\mu\nu}\eta_{\rho\sigma} = \left[2P^0 + \overline{P}^0 + \overline{\overline{P}}^0\right]_{\mu\nu,\,\rho\sigma}, \;\; \frac{1}{\Box^2}(\partial_\mu\partial_\nu\partial_\rho\partial_\sigma) = \overline{P}^0{}_{\mu\nu,\,\rho\sigma},$$

$$\frac{1}{\Box}(\eta_{\mu\rho}\partial_\nu\partial_\sigma + \eta_{\mu\sigma}\partial_\nu\partial_\rho + \eta_{\nu\rho}\partial_\mu\partial_\sigma + \eta_{\nu\sigma}\partial_\mu\partial_\rho) = \left[2P^1 + 4\overline{P}^0\right]_{\mu\nu,\,\rho\sigma},$$

$$\frac{1}{\Box}(\eta_{\mu\nu}\partial_\rho\partial_\sigma + \eta_{\rho\sigma}\partial_\mu\partial_\nu) = \left[\overline{\overline{P}}^0 + 2\overline{P}^0\right]_{\mu\nu,\,\rho\sigma}.$$

Expanding O in the basis $\{P^1, P^2, P^0, \overline{P}^0, \overline{\overline{P}}^0, P\}$, we obtain $O = x_1P^1 + x_2P^2 + x_0P^0 + \overline{x}_0\overline{P}^0 + \overline{\overline{x}}_0\overline{\overline{P}}^0 + pP$. With the help of Table II, we find that the propagator for MTMG is given by

$$O_{\text{MTMG}}^{-1} = \frac{P^1}{x_1} + \frac{x_2 P^2}{x_2^2 - p^2 k^6} + \frac{\overline{x}_0 P^0}{x_0 \overline{x}_0 - 2\overline{\overline{x}}_0^2} + \frac{x_0 \overline{P}^0}{x_0 \overline{x}_0 - 2\overline{\overline{x}}_0^2}$$
$$- \frac{\overline{\overline{x}}_0 \overline{\overline{P}}^0}{x_0 \overline{x}_0 - 2\overline{\overline{x}}_0^2} - \frac{pP}{x_2^2 - p^2 k^6}. \tag{6}$$

Now, substituting Equation (6) into Equation (5), and taking the identities,

$$P^2{}_{\mu\nu,\,\rho\sigma} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} - \left[P^1 + \frac{1}{2}\overline{P}^0 - \frac{1}{2}\overline{\overline{P}}^0\right]_{\mu\nu,\,\rho\sigma},$$

$$P^0{}_{\mu\nu,\,\rho\sigma} = \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma} - \frac{1}{2}\left[\overline{P}^0 + \overline{\overline{P}}^0\right]_{\mu\nu,\,\rho\sigma},$$

into account, yields

$$\text{SP}_{\text{MTMG}} = \left[T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right]\frac{x_2}{x_2^2 - p^2 k^6} + \frac{1}{2}T^2 \frac{\overline{x}_0}{x_0\overline{x}_0 - 2\overline{\overline{x}}_0^2}. \tag{7}$$

We call attention to the fact that $f_{P^2} \equiv x_2/(x_2^2 - p^2 k^6)$ and $f_{P^0} \equiv \overline{x}_0/(x_0\overline{x}_0 - 2\overline{\overline{x}}_0^2)$ are, in this order, the components P^2 and P^0 of O_{MTMG}^{-1} in the basis $\{P^1, P^2, P^0, \overline{P}^0, \overline{\overline{P}}^0, P\}$.

The lemma that follows clears up the question of the sign of $T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2$ at the physical poles; it is also very useful for checking the presence of massless spin-2 non-propagating excitations in the models we are analyzing.

Lemma 2. *If $m \geq 0$ is the mass of a generical physical particle associated with the MTMG and k is the corresponding momentum exchanged, then $[T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2]_{k^2=m^2} > 0$ and $[T^{\mu\nu}T_{\mu\nu} - T^2]_{k^2=0} = 0$.*

Proof: Using Equation (4), we can write the symmetric current tensor as follows

$$T^{\mu\nu} = Ak^\mu k^\nu + B\tilde{k}^\mu\tilde{k}^\nu + C\varepsilon^\mu\varepsilon^\nu + Dk^{(\mu}\tilde{k}^{\nu)} + Ek^{(\mu}\varepsilon^{\nu)} + F\tilde{k}^{(\mu}\varepsilon^{\nu)}.$$

The current conservation gives the following constraints for the coefficients A, B, D, E, and F:

$$Ak^2 + \frac{D}{2}\left(k_0^2 + \mathbf{k^2}\right) = 0, \tag{8}$$

$$B\left(k_0^2 + \mathbf{k^2}\right) + \frac{D}{2}k^2 = 0, \tag{9}$$

$$Ek^2 + F\left(k_0^2 + \mathbf{k^2}\right) = 0. \tag{10}$$

From Equations (8) and (9), we get $Ak^4 = B(k_0^2 + \mathbf{k^2})^2$, while Equation (10) implies $E^2 > F^2$. On the other hand, saturating the indices of $T^{\mu\nu}$ with momenta k_μ, we arrive at a consistent relation for the coefficients A, B, and D:

$$Ak^4 + B\left(k_0^2 + \mathbf{k^2}\right)^2 + Dk^2\left(k_0^2 + \mathbf{k^2}\right) = 0.$$

After a lengthy but otherwise straightforward calculation using the earlier equations, we obtain

$$T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2 = \left[\frac{k^2(A-B)}{\sqrt{2}} - \frac{C}{\sqrt{2}}\right]^2 + \frac{k^2}{2}(E^2 - F^2), \qquad (11)$$

$$T^{\mu\nu}T_{\mu\nu} - T^2 = k^2\left[\frac{1}{2}(E^2 - F^2) - 2C(A-B)\right]. \qquad (12)$$

Therefore, $[T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2]_{k^2=m^2} > 0$ and $[T^{\mu\nu}T_{\mu\nu} - T^2]_{k^2=0} = 0$. □

We remark that $T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2$ is always greater than zero for any physical particle; in addition, $T^{\mu\nu}T_{\mu\nu} - T^2$ is zero for massless spin-2 non-propagating modes.

We are ready now to enunciate the algorithm for testing the unitarity of the MTMG.

Algorithm 2. *Compute* $\mathrm{SP}_{\mathrm{MTMG}}$ *using Equation (7) and then find the signs of the residues at each simple pole of* $\mathrm{SP}_{\mathrm{MTMG}}$ *with the help of the Lemma 2. If all the signs are* ≥ 0*, the model is unitary; however, if at least one of the signs is negative, the system is non-unitary.*

3. CHECKING THE UNITARITY OF TMHDE AND TMHDG

We introduce here the two three-term systems we want to test the unitarity, i.e., TMHDE and TMHDG, and afterwards we study their unitarity.

3.1. The Models

The Lagrangian for TMHDE is the sum of Maxwell, higher-derivative (Podolsky and Schwed, 1948), gauge-fixing (Lorentz-gauge), and Chern-Simons, terms, i.e.,

$$\mathcal{L}_{\mathrm{TMHDE}} = -\frac{F_{\mu\nu}F^{\mu\nu}}{4} + \frac{l^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda} - \frac{1}{2\lambda}(\partial_\nu A^\nu)^2 + \frac{s}{2}\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho. \qquad (13)$$

Here, $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the usual electromagnetic tensor field, and l is a cutoff. The corresponding propagator is given by

$$O^{-1}_{\text{TMHDE}} = \frac{l^2k^4 + k^2}{(l^2k^4 + k^2)^2 - s^2k^2}\,\theta - \frac{\lambda}{k^2}\,\omega - \frac{s}{(l^2k^4 + k^2)^2 - s^2k^2}\,S. \tag{14}$$

The Lagrangian related to TMHDG, in turn, is given by

$$\begin{aligned}\mathcal{L}_{\text{TMHDG}} = \sqrt{g}\left(-\frac{2R}{\kappa^2} + \frac{\alpha}{2}R^2 + \frac{\beta}{2}R^2_{\mu\nu}\right) \\ + \frac{1}{\mu}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\rho}\right),\end{aligned} \tag{15}$$

where α and β are suitable constants with dimension L. For the sake of simplicity, the gauge-fixing term was omitted. Linearizing Equation (15) and adding to the result the gauge-fixing term $\mathcal{L}_{\text{gf}} = \frac{1}{2\lambda}(h_{\mu\nu},{}^{\nu} - \frac{1}{2}h_{,\,\mu})^2$ (de Donder gauge), we find that the propagator concerning TMHDG takes the form

$$\begin{aligned}O^{-1}_{\text{TMHDG}} = \frac{1}{\Box\big[-1 + b\big(\frac{3}{2} + 4c\big)\Box\big]}\overline{\overline{P}}^{0} + \frac{2\lambda}{k^2}P^1 + \frac{1}{\Box\big[-1 + b\big(\frac{3}{2} + 4c\big)\Box\big]}P^0 \\ + \frac{4M}{\Box[M^2b^2\Box^2 + 4(bM^2 + 1)\Box + 4M^2]}P \\ \times \frac{2M^2(2 + b\Box)}{\Box[M^2b^2\Box^2 + 4(bM^2 + 1)\Box + 4M^2]}P^2 \\ + \left[-\frac{4\lambda}{\Box} + \frac{2}{\Box\big[-1 + b\big(\frac{3}{2} + 4c\big)\Box\big]}\right]\overline{P}^{0},\end{aligned} \tag{16}$$

where $b \equiv \frac{\beta\kappa^2}{2}$, $c \equiv \frac{\alpha}{\beta}$, and $M \equiv \frac{\mu}{\kappa^2}$.

3.2. Testing the Unitarity of TMHDE

The calculations that are needed for checking the unitarity of TMHDE are somewhat complicated because this model represents in general three massive excitations. Since the θ-component of the propagator concerning TMHDE can be written as $f_\theta = \frac{M^2(x-M^2)}{x^3-2M^2x^2+M^4x-M^4s^2}$, where $M \equiv \frac{1}{l}$, we have to analyze the nature, as well as the signs, of the roots of the cubic equation $x^3 + a_2x^2 + a_1x + a_0 = 0$, where $a_2 \equiv -2M^2$, $a_1 \equiv M^4$, and $a_0 \equiv -M^4s^2$. Taking into account that we are only interested in those roots that are both real and unequal, we require $D < 0$, where $D \equiv Q^3 + R^2$, with Q and R being, in this order, equal to $\frac{3a_1-a_2^2}{9}$ and $\frac{9a_1a_2-27a_0-2a_2^3}{54}$, is the polynomial discriminant. Performing the computations we get $D = M^8s^2[\frac{s^2}{4} - \frac{M^2}{27}]$, implying that only and if only $s^2 < \frac{4M^2}{27}$ will the roots be real and distinct. Our next step is to verify whether or not these roots are positive.

This can be accomplished by building the Routh–Hurwitz array (Uspensky, 1948), namely,

$$\begin{array}{cc} 1 & M^4 \\ -2M^2 & -M^4 s^2 \\ M^2\left(M^2 - \frac{s^2}{2}\right) & 0 \\ -M^4 s^2 & 0 \end{array}$$

Noting that there are three signs changes in the first column of the array given earlier, we conclude that all the three roots are positive. In summary, if $s^2 < \frac{4m^2}{27}$, TMHDE is a model with acceptable values for the masses. Denoting these roots as x_1, x_2, and x_3, and assuming without any loss of generality that $x_1 > x_2 > x_3$, we get

$$f_\theta = \frac{M^2(x_1 - M^2)}{(x_1 - x_2)(x_1 - x_3)} \frac{1}{x - x_1} + \frac{M^2(x_2 - M^2)}{(x_2 - x_1)(x_2 - x_3)} \frac{1}{x - x_2} + \frac{M^2(x_3 - M^2)}{(x_3 - x_1)(x_3 - x_2)} \frac{1}{x - x_3}.$$

Hence, TMHDE with will be unitary if the conditions $x_1 - M^2 < 0,\ x_2 - M^2 > 0,$ and $x_3 - M^2 < 0$ hold simultaneously. Obviously, this will never occur, which allows us to conclude that TMHDE is non-unitary.

Should we expect intuitively that TMHDE faced unitary problems? The answer is affirmative. In fact, setting $s = 0$, for instance, in its Lagrangian, we recover the Lagrangian for the usual Podolsky electromagnetism which is non-unitary (Podolsky and Schwed, 1948). Nonetheless, Podolsky–Chern-Simons (PCS) planar electromagnetism with $s^2 < \frac{4M^2}{27}$, despite being haunted by ghosts, has normal massive modes. Note that the existence of these well-behaved excitations is subordinated to the condition $s < \frac{2M}{\sqrt{27}}$, which really encourages us to regard this system as an effective field model. We shall discuss their astonishing properties in Section 4.

3.3. Testing the Unitarity of TMHDG

The SP concerning TMHDG can be written as

$$\begin{aligned} \mathrm{SP}_{\mathrm{TMHDG}} = {} & \frac{M^2 b}{2} \frac{\left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right)}{k^2 - M_1^2} \frac{-1 + \sqrt{1 + 2bM^2}}{\sqrt{1 + 2bM^2}[1 + bM^2 - \sqrt{1 + 2bM^2}]} \\ & + \frac{M^2 b}{2} \frac{\left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right)}{k^2 - M_2^2} \frac{1 + \sqrt{1 + 2bM^2}}{\sqrt{1 + 2bM^2}[1 + bM^2 + \sqrt{1 - +bM^2}]} \\ & + - \frac{T^{\mu\nu}T_{\mu\nu} - T^2}{k^2} - \frac{\frac{1}{2}T^2}{(k^2 - m^2)}, \end{aligned} \tag{17}$$

where

$$M_1^2 \equiv \left(\frac{2}{b^2M^2}\right)[1 + bM^2 - \sqrt{1 + 2bM^2}],$$

$$M_2^2 \equiv \left(\frac{2}{b^2M^2}\right)[1 + bM^2 + \sqrt{1 + 2bM^2}],$$

$$m^2 \equiv -\frac{1}{b(3/2 + 4c)}.$$

It is interesting to note that $M_1^2 \to M^2$, and $M_2^2 \to +\infty$, as $b \to 0$, implying that when $\alpha, \beta \to 0$, Equation (17) reduces to

$$\mathrm{SP} = \left(T^{\mu\mu}T_{\mu\nu} - \frac{1}{2}T^2\right)\frac{1}{k^2 - M^2} + \left(T^{\mu\mu}T_{\mu\nu} - T^2\right)\frac{1}{k^2}, \tag{18}$$

which is the expression for the SP related to Maxwell–Chern-Simons theory (MCS). Using Equation (18), we promptly obtain

$$\mathrm{Res(SP)}|_{k^2=M^2} > 0, \qquad \mathrm{Res(SP)}|_{k^2=0} = 0,$$

which means that MCS is unitary. Thence, we have reobtained, in a trivial way, a well-known result (Deser *et al.*, 1988a,b).

We are now ready to analyze the excitations and mass counts concerning TMHDG. To avoid needless repetitions, we restrict ourselves to presenting a summary of the main results in Table III. The systems that do not appear in this table are tachyonic, i.e., unphysical. As intuitively expected, TMHDG is non-unitary. Indeed, if the topologically massive term is removed, TMHDG reduces to three-dimensional higher-derivative gravity—an effectively multimass model of the fourth-derivative order with interesting properties of its own (Accioly *et al.*, 2001a,b,c,)—which is non-unitary. Nonetheless, TMHDG is in general non-tachyonic, which means that under circumstances it may be viewed as an effective

Table III. Unitarity Analysis of Topologically Massive Higher-Derivative Gravity

b	$\frac{3}{2} + 4c$	Excitations and mass counts	Tachyons	Unitarity
>0	<0	2 massive spin-2 normal particles, 1 massless spin-2 non-propagating particle, 1 massive spin-0 ghost	No one	Non-unitary
$\frac{-1}{2M^2} < b < 0$	>0	1 massive spin-2 normal particle, 1 massless spin-2 non-propagating particle, 1 massive spin-2 ghost, 1 massive spin-0 ghost	No one	Non-unitary

field model. We shall investigate, in passing, the novel and amazing features of this effective system in Section 5.

4. ATTRACTIVE INTERACTION BETWEEN EQUAL CHARGE BOSONS IN THE FRAMEWORK OF MAXWELL–CHERN–SIMONS ELECTRODYNAMICS

In order to avoid extremely long calculations, we investigate here Maxwell–Chern–Simons electrodynamics (MCSE) instead of Podolsky–Chern–Simons electrodynamics (PCSE). Certainly, the two models share similar characteristics. In other words, the exciting features of PCSE are also present, *mutatis mutandis*, in MCSE. Accordingly, let us analyze the interaction between equal charge bosons in the context of the MCSE coupled to a charged-scalar field. To do that we need to compute, first of all, the effective non-relativistic potential for the interaction of two charged-scalar bosons. Now, non-relativistic quantum mechanics tells us that in the first Born approximation the cross section for the scattering of two indistinguishable massive particles, in the center-of-mass frame (CoM), is given by $\frac{d\sigma}{d\Omega} = |\frac{m}{4\pi}\int e^{-i\mathbf{p}'\cdot\mathbf{r}}V(r)e^{i\mathbf{p}\cdot\mathbf{r}}d^2\,\mathbf{r}|^2$, where $\mathbf{p}$ ($\mathbf{p}'$) is the initial (final) momentum of one of the particles in the CoM. In terms of the transfer momentum, $\mathbf{k} \equiv \mathbf{p}' - \mathbf{p}$, it reads

$$\frac{d\sigma}{d\Omega} = \left|\frac{m}{4\pi}\int V(r)e^{i\mathbf{k}\cdot\mathbf{r}}d^{D-1}\,\mathbf{r}\right|^2. \tag{19}$$

On the other hand, from quantum field theory we know that the cross section, in the CoM, for the scattering of two identical massive scalars bosons by an electromagnetic field, can be written as $\frac{d\sigma}{d\Omega} = |\frac{1}{16\pi E}\,\mathcal{M}|^2$, where E is the initial energy of one of the bosons and $\mathcal{M}$ is the Feynman amplitude for the process at hand, which in the non-relativistic limit (N.R.) reduces to

$$\frac{d\sigma}{d\Omega} = \left|\frac{1}{16\pi m}\,\mathcal{M}_{\text{N.R.}}\right|^2. \tag{20}$$

From Equations (19) and (20) we come to the conclusion that the expression that enables us to compute the effective non-relativistic potential has the form

$$V(r) = \frac{1}{4m^2}\frac{1}{(2\pi)^2}\int d^2\,\mathbf{k}\,\mathcal{M}_{\text{N.R.}}\,e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{21}$$

which clearly shows how the potential from quantum mechanics and the Feynman amplitude obtained via quantum field theory are related to each other. Now, in the Lorentz gauge the MCSE coupled to a charged-scalar field is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{s}{2}\varepsilon_{\mu\nu\rho}A^{\mu}\partial^{\nu}A^{\rho} - \frac{1}{2\lambda}(\partial_{\nu}A^{\nu})^2 + (D_{\mu}\phi)^*D^{\mu}\phi - m^2\phi^*\phi, \tag{22}$$

where $D_\mu \equiv \partial_\mu + iqA_\mu$. Therefore, the interaction Lagrangian to order Q for the process $S + S \longrightarrow S + S$, where S denotes a spinless boson of mass m and charge Q, is $\mathcal{L}_{\text{int}} = iQA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)$, implying that the elementary vertice is given by

$$\Gamma^\mu_\phi(p, p') = -Q(p + p')^\mu,$$

where p (p') is the momentum of the incoming (outgoing) scalar boson. As a consequence, the Feynman amplitude for the interaction of two charged spinless bosons of equal mass is

$$\mathcal{M} = \Gamma^\mu_\phi(p, p')O^{-1}_{\mu\nu}\Gamma^\nu_\phi(q, q') \tag{23}$$

where

$$O^{-1} = -\frac{\theta}{k^2 - s^2} - \frac{\lambda\omega}{k^2} - \frac{sS}{k^4 - s^2k^2}.$$

In the non-relativistic limit, the Feynman amplitude for the process under consideration assumes the form

$$\mathcal{M}_{\text{N.R.}} = \left[\frac{4Q^2m^2}{\mathbf{k}^2 + s^2} + \frac{8ismQ^2\ \mathbf{k} \wedge \mathbf{P}}{\mathbf{k}^2(\mathbf{k}^2 + s^2)}\right],$$

where $\mathbf{P} \equiv \frac{1}{2}(\mathbf{p} - \mathbf{q})$ is the relative momentum of the incoming charged-scalar bosons in the CoM.

It follows that the effective non-relativistic potential is given by

$$V(r) = -\frac{Q^2}{m\pi s}\left[\frac{1}{r^2} - \frac{sK_1(sr)}{r}\right]\mathbf{L} + \frac{Q^2}{2\pi s}K_0(sr), \tag{24}$$

where $\mathbf{L} \equiv \mathbf{r} \wedge \mathbf{P}$ is the orbital angular momentum, and K is the modified Bessel function. Let us then investigate whether or not this potential can bind a pair of identical charged-scalar bosons. In this case, the corresponding time-independent Schrödinger equation can be written as

$$\begin{aligned}
\mathcal{H}_l\mathcal{R}_{nl} &= -\frac{1}{m}\left(\frac{d^2}{dr^2}\mathcal{R}_{nl} + \frac{1}{r}\frac{d}{dr}\mathcal{R}_{nl}\right) + V^{\text{eff}}_l\mathcal{R}_{nl} \\
&= E_{nl}\mathcal{R}_{nl}, \\
V^{\text{eff}}_l &\equiv \frac{l^2}{mr^2} + V(r) \\
&= \frac{l^2}{mr^2} - \frac{Q^2}{m\pi s}\left[\frac{1}{r^2} - \frac{sK_1(sr)}{r}\right]\mathbf{L} + \frac{Q^2}{2\pi s}K_0(sr),
\end{aligned} \tag{25}$$

where $\mathcal{R}_{nl}$ is the nth normalizable eigenfunction of the radial Hamiltonian $\mathcal{H}_l$ whose corresponding eigenvalue is E_{nl} and V^{eff}_l is the lth partial wave effective potential. Note that V^{eff}_l behaves as $\frac{l^2}{mr^2}$ at the origin and as $\frac{l}{m}[\,l - \frac{Q^2s}{\pi s}]\frac{1}{r}$

asymptotically. On the other hand,

$$\frac{d}{dr}V_l^{\text{eff}} = -\frac{2l}{m}\left[l - \frac{Q^2}{\pi s}\right]\frac{1}{r^3} - \frac{Q^2 sl}{m\pi}\frac{1}{r}K_0(sr) - \left[\frac{Q^2 2l}{m\pi r^2} + \frac{Q^2 s}{2\pi}\right]K_1(sr)$$

Assuming, without any loss of generality, that $l > 0$, it is trivial to see that, if $l > \frac{Q^2}{\pi s}$, the potential is strictly decreasing, which precludes the existence of bound states. The remaining possibility is $l < \frac{Q^2}{\pi s}$. In this interval V_l^{eff} approaches $+\infty$ at the origin and 0^- for $r \to +\infty$, which is indicative of a local minimum. Consequently, the existence of charged-scalar boson bound states is subordinated to the condition $0 < l < \frac{Q^2}{\pi s}$. In terms of the dimensionless parameters $y \equiv sr$, $\alpha \equiv \frac{Q^2}{\pi s}$, $\beta \equiv \frac{m}{s}$, and $\tilde{E}_{nl} \equiv \frac{mE_{nl}}{s^2}$, Equation (25) reads

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy}\right]\mathcal{R}_{nl} + \left[\tilde{E}_{nl} - \tilde{V}_l^{\text{eff}}\right]\mathcal{R}_{nl} = 0, \tag{26}$$

with

$$\tilde{V}_l^{\text{eff}} \equiv -\frac{l(\alpha - l)}{y^2} + \frac{\alpha\beta}{2}K_0(y) - \frac{\alpha l}{y}K_1(y).$$

Of course, Equation (26) cannot be solved analytically; nevertheless, it can be solved numerically. To accomplish this, we rewrite the radial function as $\mathcal{R}_{nl} \equiv \frac{u_{nl}}{\sqrt{y}}$. As a consequence, Equation (26) takes the form

$$\left[\frac{d^2}{dy^2} + \frac{1}{4y^2}\right]u_{nl} + \left[\tilde{E}_{nl} - \tilde{V}_l^{\text{eff}}\right]u_{nl}. \tag{27}$$

Using the Numerov algorithm (Numerov, 1924), we have solved Equation (27) numerically for several values of the parameters α, β, and l. In Fig. 1 we present our numerical results for the potential in the specific case of $l = 6$. The corresponding ground-state energy is -1.68×10^{-8} MeV. The graphic shown in Fig. 1 exhibits the generic features of the potential, although it has been composed using particular values of the parameters α, β, and l.

In conclusion we may say that since "Cooper pairs" exist in the framework of MCSE, they also exist, as a consequence, in the context of PCSE. A detailed study of the potential, as well as the eigenvalue structure, for the PCSE coupled with a charged-scalar field, will be published elsewhere (Accioly and Dias, 2004a).

5. GRAVITY, ANTIGRAVITY, AND GRAVITATIONAL SHIELDING IN THE CONTEXT OF THREE-DIMENSIONAL GENERAL RELATIVITY WITH HIGHER DERIVATIVES

For reasons similar to those discussed in Section 4, we consider here the astonishing features of higher-derivative gravity instead of TMHDG. Let us then

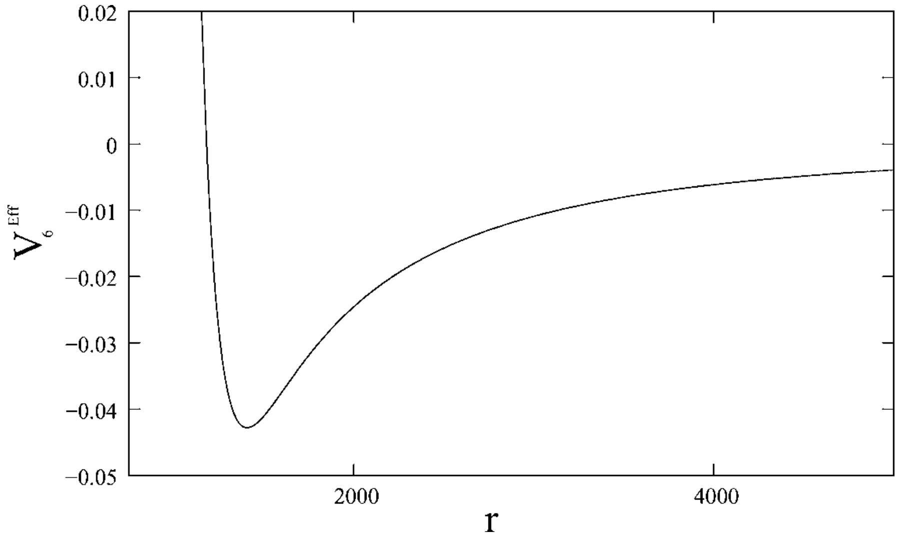

Fig. 1. Attractive effective non-relativistic potential corresponding to the eigenvalue $l = 6$. Here $[V_6^{\text{eff}}] = \text{eV}$, $[r] = \text{MeV}^{-1}$, $\alpha = 7.6$, and $\beta = 7000$.

compute the effective non-relativistic potential for the interaction of two identical massive bosons of zero spin via a graviton exchange. The expression for the potential is

$$V(r) = \frac{1}{4m^2}\frac{1}{(2\pi)^2}\int d^2\,\mathbf{k}\,\mathcal{M}_{\text{N.R.}}\,e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{28}$$

where m is the mass of one of the bosons. Now, the interaction Lagrangian for the process we are analyzing is

$$\mathcal{L}_{\text{int}} = -\frac{\kappa h^{\mu\nu}}{2}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2\right)\right],$$

implying that the elementary vertice can be written as

$$\Gamma^{\phi}_{\mu\nu}(p, p') = \frac{1}{2}\kappa\left[p_\mu p'_\nu + p_\nu p'_\mu - \eta_{\mu\nu}\left(p.p' + m^2\right)\right], \tag{29}$$

where the momenta are supposed to be incoming. The expression for the non-relativistic Feynman amplitude is, in turn, given by

$$\mathcal{M}_{\text{N.R.}} = -\frac{1}{2}\frac{\kappa^2 m^4 m_1^2}{\mathbf{k^2}\left(\mathbf{k^2} + m_1^2\right)} + \frac{1}{2}\frac{\kappa^2 m^4 m_0^2}{\mathbf{k^2}\left(\mathbf{k^2} + m_0^2\right)}, \tag{30}$$

where $m_0^2 \equiv \frac{1}{\kappa^2[\frac{3}{4}\beta + 2\alpha]}$ and $m_1^2 \equiv -\frac{4}{\kappa^2\beta}$ are supposed to be positive in order to avoid the presence of tachyons in the dynamical field. Performing the appropriate integrations using Equations (28) and (30), we obtain the effective non-relativistic potential, namely,

$$V(r) = 2Gm^2\left[K_0(m_1 r) - K_0(m_0 r)\right]. \tag{31}$$

Note that $V(r)$ behaves as $2Gm^2 \ln(\frac{m_0}{m_1})$ at the origin and as

$$2Gm^2 \left[\sqrt{\frac{\pi}{2m_1 r}}\, e^{-m_1 r} - \sqrt{\frac{\pi}{2m_0 r}}\, e^{-m_0 r} \right]$$

asymptotically. Note that this potential is extremely well behaved: it is finite at the origin and zero at infinity. On the other hand, the derivative of the potential with respect to r is given by

$$\frac{dV}{dr} = 2Gm^2 \left[-m_1 K_1(m_1 r) + m_0 K_1(m_0 r) \right],$$

implying that it is everywhere attractive if $m_0 > m_1$, is repulsive if $m_1 > m_0$, and vanishes if $m_1 = m_0$. If we appeal to the usual tools of Einstein's geometrical theory, we arrive at the same conclusions. In fact, in the weak field approximation the gravitational acceleration, $\gamma^l = \frac{dv^l}{dt}$, of a slowly moving particle is given by $\gamma^l = -\kappa[\partial_t h_0^l - \frac{1}{2}\partial^l h_{00}]$, which for time-independent fields reduces to $\gamma^l = \frac{\kappa}{2}\partial^l h_{00}$. Now, taking into account that $h_{00} = \frac{2V}{m\kappa}$, we obtain

$$\gamma^l = 2mG\frac{x^l}{r} \left[m_0 K_1(r m_0) - m_1 K_1\,(m_1 r) \right].$$

Therefore, the gravitational force exerted on the particle,

$$F^l = 2Gm^2 \frac{x^l}{r} \left[m_0 K_1(r m_0) - m_1 K_1\,(m_1 r) \right],$$

is everywhere attractive if $m_0 > m_1$, is repulsive if $m_1 > m_0$ (antigravity), and vanishes if $m_1 = m_0$ (gravitational shielding). It is remarkable that this force does not exist in general relativity. It is peculiar to both higher-derivative gravity and TMHDG (Accioly and Dias, 2005).

In Fig. 2 it is shown a schematic picture of the effective non-relativistic potential for the three situations described earlier, i.e., $m_0 > m_1$, $m_1 > m_0$, and $m_1 = m_0$.

6. DISCUSSIONS AND COMMENTS

According to a somewhat obscure unitarity lore it is expected that the operation of augmenting a non-topological massive gravity model through the topological term would transform the non-unitary systems into unitary ones and preserve the unitarity of the originally unitary models. This false idea is, perhaps, responsible for the claims in the literature concerning the pseudo-unitarity of both topologically massive Fierz-Pauli gravity (TMFPG) (Pinheiro *et al.*, 1997a,b) and TMHDG (Pinheiro *et al.*, 1997c). The authors of these works wrongly state that these models are unitary. As far as TMPFG is concerned, it was shown recently that this system with the Einstein's term with the "wrong sign" is forbidden,

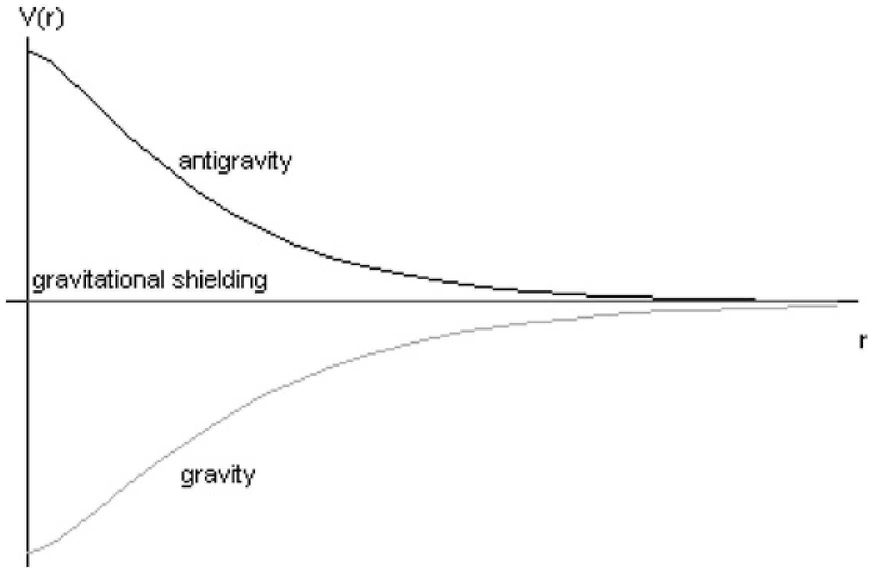

Fig. 2. Gravity, antigravity and gravitational shielding in the framework of three-dimensional Einstein's gravity with higher derivatives.

while the model with the usual sign has acceptable mass ranges but faces ghosts problems (Deser and Tekin, 2002). On the other hand, the non-unitarity problem of TMHDG was recently rehearsed (Accioly, 2003, 2004) and carefully tackled (Accioly and Dias, 2004b). In truth, we may say that we will never be ale to construct an unitary, massive, topologically massive, gravitational model. Indeed, the fancy way Einstein–Chern-Simons theory is built, i.e., with the Einstein's term with the opposite sign, precludes the existence of ghost-free, massive, topologically massive, gravitational models (Accioly and Dias, 2005). It is worth mentioning that these idiosyncrasies do not occur in the framework of massive, topologically massive, electromagnetic models because the Maxwell sign's term concerning MCS theory is the same as that of the usual Maxwell's theory.

Nonetheless, the massive topologically massive models with higher derivatives may be utilized under certain circumstance as effective field models, i.e., as low-energy approximations to more fundamental theories that, quoting Weinberg (1995), "may not be field theories at all." The physics associated with these models is not only intriguing, but also fascinating. Certainly it deserves to be much better known.

ACKNOWLEDGMENTS

We are very grateful to Prof. S. Deser for calling our attention to the article "Massive, Topologically Massive, Models" (Deser and Tekin, 2002). A. Accioly thanks CNPq-Brazil for partial support, while M. Dias thanks CAPES-Brazil for full support.

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Modern Physics Letters A
Vol. 19, No. 11 (2004) 817–826

UNAVOIDABLE CONFLICT BETWEEN MASSIVE GRAVITY MODELS AND MASSIVE TOPOLOGICAL TERMS

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Received 16 December 2003
Revised 20 January 2004

Massive gravity models in $(2+1)$ dimensions, such as those obtained by adding to Einstein's gravity the usual Fierz–Pauli, or the more complicated Ricci scalar squared (R^2), terms, are tree level unitary. Interesting enough these seemingly harmless systems have their unitarity spoiled when they are augmented by a Chern–Simons term. Furthermore, if the massive topological term is added to $R + R^2_{\mu\nu}$ gravity, or to $R + R^2_{\mu\nu} + R^2$ gravity (higher-derivative gravity), which are nonunitary at the tree level, the resulting models remain nonunitary. Therefore, unlike the common belief, as well as the claims in the literature, the coexistence between three-dimensional massive gravity models and massive topological terms is conflicting.

Keywords: Massive gravity models; topological terms; tree unitarity.

The remarkable properties of topological tensor gauge theories in $(2+1)$ dimensions are by now not only well-appreciated but also well-understood. The linearized versions of these models describe single massive but gauge-invariant excitations.[1] Nonetheless, according to a somewhat obscure tree unitarity lore it is expected that the operation of augmenting a nontopological massive gravity model through the topological term would transform the nonunitary systems into unitary ones and preserve the tree unitarity of the originally unitary models. In truth, this addition does more harm than good. Indeed, innocuous avowedly tree unitary models, such as Fierz–Pauli gravity or the more sophisticated $R + R^2$ gravity, become nonunitary after the topological addition, while admittedly nonunitary systems, such as $R + R^2_{\mu\nu}$ gravity or higher-derivative gravity $(R + R^2 + R^2_{\mu\nu})$ remain stubbornly nonunitary after the topological enlargement.

Our aim here is to discuss the incompatibility between massive gravity models and massive topological terms. The analysis comprises massive topological gravity systems that are focus of the controversy in the literature: topological Fierz–Pauli gravity and topological higher-derivative gravity — they are wrongly considered as tree unitary models[2–4] — and topological $R + R^2$ and $R + R^2_{\mu\nu}$ gravity. We will show that these topological models are without exception nonunitary at the tree level.

To probe the tree unitarity of the models we make use of the method that consists of saturating the propagator with external conserved currents, $T^{\mu\nu}$, compatible with the symmetries of the theory. The unitarity analysis is based on the residues of the saturated propagator (SP): the tree unitarity is ensured if the residue at each pole of the SP is positive. Note that the SP is nothing but the current–current amplitude in momentum space.

Natural units are used throughout. Our signature is $(+,-,-)$. The Riemann and Ricci tensors are defined respectively as $R^{\rho}{}_{\lambda\mu\nu} = -\partial_\nu \Gamma^{\rho}{}_{\lambda\mu} + \partial_\mu \Gamma^{\rho}{}_{\lambda\nu} - \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\rho}{}_{\sigma\nu} + \Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\rho}{}_{\sigma\mu}$ and $R_{\mu\nu} = R^{\rho}{}_{\mu\nu\rho}$.

We consider first topological Fierz–Pauli gravity (TFPG). The Lagrangian related to this model is the sum of Einstein, standard Fierz–Pauli and Chern–Simons, terms, namely,

$$\mathcal{L} = a\frac{2}{\bar{\kappa}^2}\sqrt{g}R - \frac{m^2}{2}({h_{\mu\nu}}^2 - h^2) + \frac{1}{\mu}\varepsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu \Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\rho}\right), \qquad (1)$$

at quadratic order in $\bar{\kappa}$, where $\bar{\kappa}^2$ is a suitable constant that in four dimensions is equal to $24\pi G$, with G being Newton's constant.[5] Here $g_{\mu\nu} \equiv \eta_{\mu\nu} + \bar{\kappa} h_{\mu\nu}$, $h \equiv \eta_{\mu\nu}h^{\mu\nu}$, and a is a convenient parameter that can take the values $+1$ (Einstein's term with the usual sign) or -1 (Einstein's term with the "wrong" sign), so that this is the most general such model. From now on indices are raised (lowered) with $\eta^{\mu\nu}$ ($\eta_{\mu\nu}$).

To compute the SP we have to find beforehand the propagator, which involves much algebra. However, the calculations are greatly simplified if we appeal to a set of operators made up by the usual three-dimensional Barnes–Rivers operators,[6] i.e.

$$P^1_{\mu\nu,\,\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho})\,,$$

$$P^2_{\mu\nu,\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho} - \theta_{\mu\nu}\theta_{\rho\sigma})\,,$$

$$P^0_{\mu\nu,\rho\sigma} = \frac{1}{2}\theta_{\mu\nu}\theta_{\rho\sigma}\,, \qquad \bar{P}^0_{\mu\nu,\rho\sigma} = \omega_{\mu\nu}\omega_{\rho\sigma}\,,$$

$$\bar{\bar{P}}^0_{\mu\nu,\rho\sigma} = \theta_{\mu\nu}\omega_{\rho\sigma} + \omega_{\mu\nu}\theta_{\rho\sigma}\,,$$

where $\theta_{\mu\nu}$ and $\omega_{\mu\nu}$ are the well-known transverse and longitudinal vector projector operators $\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}$, $\omega_{\mu\nu} = \frac{\partial_\mu\partial_\nu}{\Box}$, and the operator

$$P_{\mu\nu,\rho\sigma} \equiv \frac{\Box\partial^\lambda}{4}[\varepsilon_{\mu\lambda\rho}\theta_{\nu\sigma} + \varepsilon_{\mu\lambda\sigma}\theta_{\nu\rho} + \varepsilon_{\nu\lambda\rho}\theta_{\mu\sigma} + \varepsilon_{\nu\lambda\sigma}\theta_{\mu\rho}]\,,$$

which has its origin in the linearization of the Chern–Simons term, i.e.

$$\mathcal{L}_{\text{C.S.lin}} = \frac{1}{2}\frac{1}{M}h^{\mu\nu}P_{\mu\nu,\rho\sigma}h^{\rho\sigma}\,,$$

where $M \equiv \mu/\bar{\kappa}^2$. The corresponding multiplicative table is displayed in Table 1.

Table 1. Multiplicative operator algebra fulfilled by P^1, P^2, P^0, $\bar{P}^0$, $\bar{\bar{P}}^0$ and P. Here $P^{\theta\omega}{}_{\mu\nu,\rho\sigma} \equiv \theta_{\mu\nu}\omega_{\rho\sigma}$ and $P^{\omega\theta}{}_{\mu\nu,\rho\sigma} \equiv \omega_{\mu\nu}\theta_{\rho\sigma}$.

	P^1	P^2	P^0	$\bar{P}^0$	$\bar{\bar{P}}^0$	P
P^1	P^1	0	0	0	0	0
P^2	0	P^2	0	0	0	P
P^0	0	0	P^0	0	$P^{\theta\omega}$	0
$\bar{P}^0$	0	0	0	$\bar{P}^0$	$P^{\omega\theta}$	0
$\bar{\bar{P}}^0$	0	0	$P^{\omega\theta}$	$P^{\theta\omega}$	$2(P^0+\bar{P}^0)$	0
P	0	P	0	0	0	$-\Box^3 P^2$

Adapting to $(2+1)$ dimensions, with the help of the data from Table 1, the algorithm for calculating the propagator related to four-dimensional gravity theories[7] and using the resulting prescription, we find that the propagator for TFPG assumes the form

$$\begin{aligned} O^{-1} = &-\frac{1}{m^2}P^1 - \frac{M^2(m^2+a\Box)}{\Box^3 + M^2a^2\Box^2 + 2am^2M^2\Box + M^2m^4}P^2 - \frac{m^2+a\Box}{2m^4}\bar{P}^0 \\ &+ \frac{1}{2m^2}\bar{\bar{P}}^0 - \frac{M}{\Box^3 + M^2a^2\Box^2 + 2am^2M^2\Box + M^2m^4}P\,. \end{aligned}$$

Consequently, the saturated propagator is given by

$$SP_{\text{TFPG}} = T^{\mu\nu}O^{-1}_{\mu\nu,\rho\sigma}T^{\rho\sigma}\,.$$

Performing the computations, we promptly obtain in momentum space

$$SP_{\text{TFPG}} = \left[T^{\mu\nu}(k)T_{\mu\nu}(k) - \frac{1}{2}T^2(k)\right]\frac{M^2(m^2-k^2a)}{k^6 - M^2a^2k^4 + 2am^2M^2k^2 - M^2m^4}\,.$$

At first sight it seems that we should start our analysis by setting $a=-1$ since in the limit $m^2=0$ (1) reduces to pure massive topological gravity (MTG) — a theory that requires $a=-1$ to be ghost-free.[1] Nevertheless, a straightforward numerical computation shows that among the roots of the equation

$$x^3 - a^2\lambda m^2x^2 + 2a\lambda m^4x - \lambda m^6 = 0\,, \tag{2}$$

where $x \equiv k^2$, $a=-1$ and $\lambda \equiv \left(\frac{M}{m}\right)^2$, there are always two complex roots for any positive λ value. Therefore, this model is unphysical and must be rejected.

Now we shall concentrate on the system with $a=+1$. In this case the roots of (2) can be classified as

$\lambda < 27/4$:	one real root and two complex ones,
$\lambda = 27/4$:	three real roots: $x_1 = x_2 = 4x_3 = 4M^2/9\,,$
$\lambda > 27/4$:	three distinct positive real roots.

Accordingly, the viability of the theory requires $\lambda > 27/4$, which implies that the SP_{TFPG} may be written as

$$SP_{\text{TFPG}} = X_1 + X_2 + X_3\,,$$

where

$$X_1 = \frac{M^2(m^2 - x_1)F(k)}{(x_1 - x_2)(x_1 - x_3)(k^2 - x_1)}\,, \qquad X_2 = \frac{M^2(m^2 - x_2)F(k)}{(x_2 - x_1)(x_2 - x_3)(k^2 - x_2)}\,,$$

$$X_3 = \frac{M^2(m^2 - x_3)F(k)}{(x_3 - x_1)(x_3 - x_2)(k^2 - x_3)}$$

with $F(k) \equiv T^{\mu\nu}(k)T_{\mu\nu}(k) - \frac{1}{2}T^2(k)$. We are assuming without any loss of generality that $x_1 > x_2 > x_3$.

Let us then find a suitable basis for expanding the sources. The class of independent vectors in momentum space,

$$k^\mu \equiv (k^0, \mathbf{k})\,, \qquad \tilde{k}^\mu \equiv (k^0, -\mathbf{k})\,, \qquad \varepsilon^\mu \equiv (0, \boldsymbol{\varepsilon})\,,$$

where $\boldsymbol{\varepsilon}$ is a unit vector orthogonal to $\mathbf{k}$, does the job. In this basis the symmetric current tensor assumes the form

$$T^{\mu\nu} = Ak^\mu k^\nu + B\tilde{k}^\mu\tilde{k}^\nu + C\varepsilon^\mu\varepsilon^\nu + Dk^{(\mu}\tilde{k}^{\nu)} + Ek^{(\mu}\varepsilon^{\nu)} + F\tilde{k}^{(\mu}\varepsilon^{\nu)}\,.$$

As a consequence,

$$F(k) = \frac{1}{2}[(A - B)k^2]^2 + \frac{C^2}{2} + \frac{k^2}{2}(E^2 - F^2) - (A - B)k^2 C\,.$$

Assuming, as usual, that $T \geq 0$, we get $C \leq 0$, implying $F(k) > 0$.

If $x_1 < m^2$, Res $SP_{\text{TFPG}}|_{k^2=x_1} > 0$, Res $SP_{\text{TFPG}}|_{k^2=x_2} < 0$ and Res $SP_{\text{TFPG}}|_{k^2=x_3} > 0$. The theory is causal and has two spin-2 physical particles of masses equal to x_1 and x_3, respectively, and one spin-2 ghost of mass x_2. On the other hand, if $x_1 > m^2$, the model is causal as well and has at least one spin-2 ghost of mass x_1. These models are thus nonunitary at the tree level due to the presence of the ghosts. Note that if we set $M^2 = \infty$, we recover pure Fierz–Pauli gravity (FPG). In this case $SP_{\text{FPG}} = \frac{F(k)}{k^2 - m^2}$ and Res $SP_{\text{FPG}}|_{k^2=m^2} > 0$. Since the residue of the SP is positive, FPG is tree unitary — a well-known result. Therefore, the topological term is responsible for breaking down the tree unitarity of the harmless FPG.

We discuss in the following the claims in the literature[2,3] concerning the tree unitarity of TFPG with $a = +1$ and $\lambda > 27/4$. The authors of Refs. 2 and 3 simply affirm that the model in hand is tree unitary; however, no explicit proof is presented to give support to their statement. For clarity's sake, we quote from Refs. 2 and 3, in this order: "It can be checked that the condition $\lambda > 27/4$ must be fulfilled in order that poles corresponding to tachyons and ghosts be suppressed from the spectrum." "It is checked that tachyons and ghosts are excluded from the spectrum whenever $\lambda > 27/4$." We have shown in detail that this is not true. Actually, the authors of these works have made a mistake as far as the analysis of the sign of the residues of SP_{TFPG} is concerned, which led them to conclude incorrectly that TFPG is unitary at the tree level.

We consider now topological higher-derivative gravity (THDG), whose Lagrangian has the form

$$\mathcal{L} = \sqrt{-g}\left(a\frac{2R}{\kappa^2} + \frac{\alpha}{2}R^2 + \frac{\beta}{2}R^2_{\mu\nu}\right) + \frac{1}{\mu}\varepsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\rho}\right), \qquad (3)$$

where κ^2 is a constant that in four dimensions is equal to $32\pi G$. Here α and β are suitable dimensional constants. Before going on we must answer a crucial question: What is the use of augmenting pure three-dimensional gravity through the quadratic terms R^2 and $R^2_{\mu\nu}$? The answer is quite straightforward: The quadratic terms convert Einstein's gravity, that is trivial from the classical viewpoint, into a nontrivial model; furthermore, within the quantum scheme, higher-derivative gravity (HDG), unlike pure Einstein's gravity, has propagating degrees of freedom. In other words, the net effect of adding quadratic or topological terms to pure Einstein's gravity is just the same: to produce a nontrivial gravity model with gravitons. HDG models have interesting properties of their own:

(i) The general solution of its linearized version[8] great resembles, *mutatis mutantis*, the four-dimensional metric of a straight U(1) gauge cosmic string in the context of linearized four-dimensional HDG.[9]
(ii) Contrary to what happens with the Newtonian potential — which has a logarithmic singularity at the origin and is unbounded at infinity — HDG's nonrelativistic potential is extremely well-behaved: it is finite at the origin and zero at infinity.[10]
(iii) Both antigravity and gravitational shielding can coexist without conflict with HDG.[11]
(iv) The gravitational deflection angle associated to HDG is always less than that related to pure gravity.[8]

Despite these nice properties, HDG possesses a ghost pole in the tree propagator which renders it nonunitary within the standard pertubation scheme. However, according to the already mentioned tree unitarity lore, it is naively expected that if we augment HDG via the Chern–Simons term we would arrive at a tree unitary theory.[4] Our main objective in what follows is to expose the fallacy of this conjecture.

To find the propagator concerning THDG, we linearize (3) and add to the result the gauge-fixing Lagrangian, $\mathcal{L}_{\text{g.f.}} = \frac{1}{2\lambda}(h_{\mu\nu}{}^{,\nu} - h_{,\mu})^2$, that corresponds to the de Donder gauge. The propagator is given by

$$\begin{aligned}\mathcal{O}^{-1} &= \frac{2\lambda}{k^2}P^1 - \frac{2M^2(2a - b\Box)}{\Box[M^2b^2\Box^2 - 4(abM^2-1)\Box + 4M^2a^2]}P^2 \\ &\quad + \frac{1}{\Box[a + b(\frac{3}{2}+4c)\Box]}P^0 + \left[-\frac{4\lambda}{\Box} + \frac{2}{\Box[a+b(\frac{3}{2}+4c)\Box]}\right]\bar{P}^0 \\ &\quad + \frac{1}{\Box[a+b(\frac{3}{2}+4c)\Box]}\bar{\bar{P}}^0 + \frac{4M}{\Box[M^2b^2\Box^2 - 4(abM^2-1)\Box + 4M^2a^2]}P\,,\end{aligned}$$

where $b \equiv \beta\kappa^2/2$ and $c = \alpha/\beta$.

In momentum space, the associated SP assumes the form:

$$\begin{aligned}SP_{\text{THDG}} &= \left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right)\left[-\frac{1+\sqrt{1-2abM^2}}{2a\sqrt{1-2abM^2}(k^2-M_1^2)}\right] \\ &\quad + \left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right)\left[-\frac{-1+\sqrt{1-2abM^2}}{2a\sqrt{1-2abM^2}(k^2-M_2^2)}\right] \\ &\quad + \frac{T^{\mu\nu}T_{\mu\nu} - T^2}{ak^2} + \frac{\frac{1}{2}T^2}{a(k^2-m^2)}\,, \end{aligned} \tag{4}$$

where

$$\begin{aligned} M_1^2 &\equiv \left(\frac{2}{b^2M^2}\right)[1 - abM^2 - \sqrt{1-2abM^2}]\,, \\ M_2^2 &\equiv \left(\frac{2}{b^2M^2}\right)[1 - abM^2 + \sqrt{1-2abM^2}]\,, \\ m^2 &\equiv \frac{a}{b(\frac{3}{2}+4c)}\,. \end{aligned}$$

We are now ready to analyze the excitations and mass counts for generic signs and values of the parameters and for both allowed signs of a. To begin with, we set $a = -1$. The absence of tachyons in the dynamical field requires $b > 0$ and $\frac{3}{2} + 4c < 0$ or $-\frac{1}{2} < bM^2 < 0$ and $\frac{3}{2} + 4c > 0$. The former leads to Res $SP_{\text{THDG}}|_{k^2=M_1^2} > 0$, Res $SP_{\text{THDG}}|_{k^2=M_2^2} > 0$, Res $SP_{\text{THDG}}|_{k^2=0} = 0$ and Res $SP_{\text{THDG}}|_{k^2=m^2} < 0$, while the latter results in Res $SP_{\text{THDG}}|_{k^2=M_1^2} > 0$, Res $SP_{\text{THDG}}|_{k^2=M_2^2} < 0$, Res $SP_{\text{THDG}}|_{k^2=0} = 0$ and Res $SP_{\text{THDG}}|_{k^2=m^2} < 0$. The particle content related to the first situation is two massive spin-2 physical particles, one massless spin-2 nonpropagating particle and one massive spin-0 ghost, whereas that concerning the second one is one massive spin-2 physical particle, one massive spin-2 ghost, one nonpropagating graviton and one massive spin-0 ghost. Therefore, unlike the claim in the literature,[4] THDG with Einstein's term and the "wrong" sign is tree nonunitary. Note that the authors of Ref. 4 state that "the spin-0 sector displays a massless pole along with massive poles" as well as that "the massive gravitons propagate as in the pure Einstein–Chern–Simons model: negative-norm

Table 2. Unitarity analysis of topological higher-derivative gravity.

a	b	$\frac{3}{2}+4c$	excitations and mass counts	tachyons	unitarity
-1	>0	<0	2 massive spin-2 normal particles 1 massless spin-2 nonpropagating particle 1 massive spin-0 ghost	no one	nonunitary at the tree level
-1	$\frac{-1}{2M^2}<b<0$	>0	1 massive spin-2 normal particle 1 massless spin-2 nonpropagating particle 1 massive spin-2 ghost 1 massive spin-0 ghost	no one	nonunitary at the tree level
$+1$	<0	<0	2 massive spin-2 ghosts 1 massless spin-2 nonpropagating particle 1 massive spin-0 normal particle	no one	nonunitary at the tree level
$+1$	$0<b<\frac{1}{2M^2}$	>0	1 massive spin-2 normal particle 1 massless spin-2 nonpropagating particle 1 massive spin-2 ghost 1 massive spin-0 normal particle	no one	nonunitary at the tree level

states do not appear that spoil the spectrum, which does not affect the unitarity", pure and simple. A quick glimpse at Table 2 is sufficient to convince any one of the wrongness of these affirmations. In truth, the authors of this work made a mistake in examining both the excitations and mass counts of THDG with $a=-1$.

Could it be that if we have chosen $a=+1$ we would have arrived at a unitary system? The response to this question is negative. Indeed, assuming (i) $b<0$ and $\frac{3}{2}+4c<0$ or (ii) $0<bM^2<\frac{1}{2}$ and $\frac{3}{2}+4c>0$ in order to get rid of the tachyons, we come to the conclusion that (i) Res $SP_{\text{THDG}}|_{k^2=M_1^2}<0$, Res $SP_{\text{THDG}}|_{k^2=M_2^2}<0$, Res $SP_{\text{THDG}}|_{k^2=0}=0$ and Res $SP_{\text{THDG}}|_{k^2=m^2}>0$, and (ii) Res $SP_{\text{THDG}}|_{k^2=M_1^2}<0$, Res $SP_{\text{THDG}}|_{k^2=M_2^2}>0$, Res $SP_{\text{THDG}}|_{k^2=0}=0$ and Res $SP_{\text{THDG}}|_{k^2=m^2}>0$, implying that the model has (i) two massive spin-2 ghosts, one massless spin-2 nonpropagating particle and one massive spin-0 physical particle or (ii) one massive spin-2 ghost, one massive spin-2 physical particle, one nonpropagating graviton and one massive spin-0 physical particle. These systems, as the preceding ones, are also nonunitary at the tree level. The above is summarized in Table 2. The remaining systems are tachyonic.

We discuss now the tree unitarity of the models obtained from THDG by judiciously chosing the parameters α, β and M^2 as well as the signs of a.

• *Pure massive topological gravity* ($\alpha = \beta = 0$)

$$SP = -\frac{T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2}{a(k^2 - a^2M^2)} + \frac{T^{\mu\nu}T_{\mu\nu} - T^2}{ak^2}$$

$a = -1$: One massive spin-2 physical particle and one nonpropagating graviton; the model is nontachyonic and tree unitary.
$a = +1$: One massive spin-2 ghost and one nonpropagating graviton; the system is nontachyonic and nonunitary at the tree level.

Comment: Pure massive topological gravity is unitary if and only if the sign of the Einstein's term is chosen to be -1. This result was obtained in Ref. 1 using a quite different approach.

• *Pure* $R + R^2_{\mu\nu}$ *gravity* ($\alpha = 0$, $M^2 = \infty$)

$$SP = -\frac{T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2}{a(k^2 + \frac{2a}{b})} + \frac{T^{\mu\nu}T_{\mu\nu} - T^2}{ak^2} + \frac{\frac{1}{2}T^2}{a(k^2 - \frac{3}{2}b)}$$

$a = -1$ and $b > 0$: One spin-2 physical particle of mass $m_2 = \sqrt{2/b}$, one nonpropagating graviton and one spin-0 ghost of mass $m_0 = \sqrt{3b/2}$; the model is nontachyonic and tree nonunitary.

The remaining models are unphysical because they have complex masses.

Comment: Pure $R + R^2_{\mu\nu}$ gravity possesses no tachyons if and only if $a = -1$ (Einstein's term with the "wrong" sign) and $b > 0$. Nonetheless, the model has a massive scalar ghost which renders it nonunitary at the tree level.

• *Topological* $R + R^2_{\mu\nu}$ *gravity* ($\alpha = 0$)

$$\begin{aligned} SP = {} & \left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right)\left[-\frac{1 + \sqrt{1 - 2abM^2}}{2a\sqrt{1 - 2abM^2}(k^2 - M_1^2)}\right] \\ & + \left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right)\left[-\frac{-1 + \sqrt{1 - 2abM^2}}{2a\sqrt{1 - 2abM^2}(k^2 - M_2^2)}\right] \\ & + \frac{T^{\mu\nu}T_{\mu\nu} - T^2}{ak^2} + \frac{\frac{1}{2}T^2}{a(k^2 - \frac{3}{2}b)} \,. \end{aligned}$$

$a = -1$ and $b > 0$: Two spin-2 physical particles of masses equal to M_1 and M_2, one propagating graviton and one spin-0 ghost of mass $m_0 = \sqrt{3b/2}$; the system is nontachionic and tree nonunitary.

$a = +1$ and $0 < b < \frac{1}{2M^2}$: One spin-2 ghost of mass M_1, one spin-2 physical particle of mass M_2, one nonpropagating graviton and one scalar physical particle of mass $m_0 = \sqrt{3b/2}$; the model is nontachyonic and tree nonunitary. The other models are tachyonic.

Comment: The topological term does not cure the nonunitarity of pure $R + R^2_{\mu\nu}$ gravity.

- *Pure $R + R^2$ gravity* ($\beta = 0, M^2 = \infty$)

$$SP = \frac{T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2}{ak^2} + \frac{\frac{1}{2}T^2}{a(k^2 - \frac{\alpha\kappa^2}{2})} \,.$$

$a = +1$ and $\alpha > 0$: One scalar physical particle of mass $m_0 = \sqrt{\alpha\kappa^2/2}$ and one nonpropagating graviton; the model is nontachyonic and tree unitary.

$a = -1$ and $\alpha > 0$: One scalar ghost of mass $m_0 = \sqrt{\alpha\kappa^2/2}$ and one nonpropagating graviton; the system is nontachyonic and tree nonunitary.

The remaining systems are tachyonic.

Comment: $R + R^2$ gravity is tree unitary if and only if the sign of the Einstein's term is the conventional one.

- *Topological $R + R^2$ gravity* ($\beta = 0$)

$$SP = -\frac{T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2}{a(k^2 - a^2M^2)} + \frac{T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2}{ak^2} + \frac{\frac{1}{2}T^2}{a(k^2 - \frac{\alpha\kappa^2}{2})} \,.$$

$a = -1$ and $\alpha > 0$: One spin-2 physical particle of mass M, one nonpropagating graviton and a scalar ghost of mass $m_0 = \sqrt{\alpha\kappa^2/2}$; the system is nontachyonic and nonunitary at the tree level.

$a = +1$ and $\alpha > 0$: One spin-2 ghost of mass M and a massive scalar physical particle of mass $m_0 = \sqrt{\alpha\kappa^2/2}$; the model is nontachyonic and tree nonunitary. The other models are unphysical due to the complex masses.

Comment: The topological term spoils the tree unitarity of the innocuous pure $R + R^2$ gravity.

- *Higher-derivative gravity* ($M^2 = \infty$)

$$SP = -\frac{T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2}{a(k^2 + \frac{2a}{b})} + \frac{T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2}{ak^2} + \frac{\frac{1}{2}T^2}{a(k^2 - m^2)} \,.$$

$a = +1$, $b < 0$ and $3/2 + 4c < 0$: One spin-2 ghost of mass $m_2 = \sqrt{-2/b}$, one nonpropagating graviton and one scalar physical particle of mass m; the system is nontachyonic and nonunitary at the tree level.

$a = -1$, $b > 0$ and $3/2 + 4c > 0$: One spin-2 physical particle of mass $m_2 = \sqrt{2/b}$ and one scalar ghost of mass m; the model is nontachyonic and tree nonunitary.

The remaining models are tachyonic.

Comment: Higher-derivative gravity is nonunitary for both choices of the sign of the Einstein's term.

To conclude we remark that the enlargement of massive gravitational models via the topological term is a complete nonsense. On the one hand, it does not cure the nonunitarity of massive nonunitary systems (HDG, $R + R^2_{\mu\nu}$ gravity). On the other hand, it spoils the unitarity of originally unitary masive models (FPG, $R+R^2$ gravity).

Acknowledgments

A.A. thanks CNPq-Brazil for partial support while M.D. is very grateful to CAPES-Brazil for full support.

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International Journal of Modern Physics A
Vol. 21, No. 3 (2006) 559–573

World Scientific
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IS IT PHYSICALLY SOUND TO ADD A TOPOLOGICALLY MASSIVE TERM TO THREE-DIMENSIONAL MASSIVE ELECTROMAGNETIC OR GRAVITATIONAL MODELS?

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Received 19 May 2005

The addition of a topologically massive term to an admittedly nonunitary three-dimensional massive model, be it an electromagnetic system or a gravitational one, does not cure its nonunitarity. What about the enlargement of avowedly unitary massive models by way of a topologically massive term? The electromagnetic models remain unitary after the topological augmentation but, surprisingly enough, the gravitational ones have their unitarity spoiled. Here we analyze these issues and present the explanation why unitary massive gravitational models, unlike unitary massive electromagnetic ones, cannot coexist from the viewpoint of unitarity with topologically massive terms. We also discuss the novel features of the three-term effective field models that are gauge-invariant.

Keywords: Topologically massive models; massive electromagnetic models; massive gravitational models; unitarity; effective field models.

PACS numbers: 11.10.Kk, 04.60.Kz, 11.10.St, 12.20.Ds

1. Introduction

In the last two decades much attention has been devoted to the study of the remarkable properties of gauge theories in $(2 + 1)$ dimensions.[1,2] Certainly, it would not be an exaggeration to claim that by now these properties are not only well-appreciated but also well-understood. Therefore, it should be natural, at least from a naive point of view, to expect that the addition of a Chern–Simons term to massive electromagnetic or gravitational models would produce systems endowed with properties that, in principle, should be as exciting as those concerning the well-known theories of Maxwell–Chern–Simons or Einstein–Chern–Simons. Our aim here is to analyze these massive, topologically massive, models. Since, currently, there are two distinct nontopological mass-generating mechanisms for gauge fields: adding the well-known Proca/Fierz–Pauli, or the more sophisticated higher-derivative electromagnetic/higher-derivative gravitational, terms, our analysis will comprise topologically massive Proca electromagnetism (TMPE), topologically massive higher-derivative electromagnetism (TMHDE), which is also known as Podolsky–Chern–Simons planar electromagnetism,[3] topologically massive Fierz–Pauli gravity (TMFPG), and topologically massive higher-derivative gravity (TMHDG). These systems will be examined for both possible sign choices of the Maxwell/Einstein Lagrangian, as well as in its absence, which implies that they are the most general such models. Both TMPE and TMFPG are not gauge-invariant due to the presence of an explicit mass term, but the three-term models with higher-derivatives are gauge-invariant.

We are particularly interested in two issues that are somewhat correlated:

(i) The compatibility — from the point of view of the unitarity — between massive electromagnetic or gravitational models and topologically massive terms.
(ii) The exciting physics resulting from the utilization of the gauge-invariant three-term systems as effective field models. We remark that it was recently shown that boson–boson bound states do exist in the framework of three-dimensional higher-derivative electromagnetism augmented by a topological Chern–Simons term.[3]

To probe the unitarity of the massive, topologically massive models, we will make use of an uncomplicated and easily handling algorithm that converts the task of checking the unitarity, which in general demands much work, into a straightforward algebraic exercise.[4] The prescription consists basically in saturating the propagator with external conserved currents, compatible with the symmetries of the system, and in examining afterwards the residues of the saturated propagator (SP) at each of their simple poles. We use natural units throughout.

2. Massive, Topologically Massive, Electromagnetic Models

The Lagrangian for TMPE is the sum of Maxwell, standard Proca mass, Chern–Simons terms, namely,

$$\mathcal{L}_{\text{TMPE}} = -a\frac{F_{\mu\nu}F^{\mu\nu}}{4} + \frac{1}{2}m^2 A^\mu A_\mu + \frac{s}{2}\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho\,, \tag{1}$$

while the Lagrangian for TMHDE is the sum of Maxwell, higher-derivative,[5] gauge-fixing (Lorentz-gauge), and Chern–Simons terms, i.e.

$$\mathcal{L}_{\text{TMHDE}} = -a\frac{F_{\mu\nu}F^{\mu\nu}}{4} + \frac{l^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda} - \frac{1}{2\lambda}(\partial_\nu A^\nu)^2 + \frac{s}{2}\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho\,. \tag{2}$$

Here, $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the usual electromagnetic tensor field, l is a cutoff, $s > 0$ is the topological mass, and a is a convenient parameter that can take the values $+1$ (Maxwell's term with the standard sign), -1 (Maxwell's term with the "wrong sign"), or 0 (absence of the Maxwell's term). The corresponding propagators are given by

$$\mathcal{P}_{\mathrm{TMPE}} = -\frac{ak^2 - m^2}{(ak^2 - m^2)^2 - s^2k^2}\,\theta + \frac{1}{m^2}\,\omega - \frac{s}{(ak^2 - m^2)^2 - s^2k^2}\,S\,, \tag{3}$$

$$\mathcal{P}_{\mathrm{TMHDE}} = \frac{l^2k^4 - ak^2}{(l^2k^4 - ak^2)^2 - s^2k^2}\,\theta - \frac{\lambda}{k^2}\,\omega - \frac{s}{(l^2k^4 - ak^2)^2 - s^2k^2}\,S\,, \tag{4}$$

where $\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}$ and $\omega_{\mu\nu} = \frac{\partial_\mu\partial_\nu}{\Box}$ are, respectively, the usual transverse and longitudinal vector projector operators, $S_{\mu\nu} \equiv \varepsilon_{\mu\rho\nu}\partial^\rho$ is the operator associated with the topological term, and $\eta_{\mu\nu}$ is the Minkowski metric. Our signature conventions are $(+,-,-)$, $\varepsilon^{012} = +1 = \varepsilon_{012}$. Now, the algorithm from Ref. 4 says that all we have to do in order to check the unitarity of a massive, topologically massive, electromagnetic model is to verify whether the residues at each simple pole of the θ-component of the propagator in the basis $\{\theta, \omega, S\}$ which, for short, we will designate as f_θ, are ≤ 0. We use this recipe in the following to test the unitarity of TMPE and TMHDE, in this order.

2.1. *Checking the unitarity of TMPE*

We start our analysis by setting the parameter a in Eq. (1) equal to $+1$ because it must be positive both in the Proca ($s = 0$) and Maxwell–Chern–Simons ($m = 0$) limits.[2] Now, the θ-component of the propagator in the basis $\{\theta, \omega, S\}$ is, according to Eq. (3), equal to $f_\theta = \frac{m^2-k^2}{(k^2-m_+^2)(k^2-m_-^2)}$, where $m_\pm = \frac{1}{2}\left[\sqrt{4m^2+s^2} \pm s\right]$. Therefore, the model has two degrees of freedom with masses m_+ and m_-, which is precisely what Deser and Tekin have found using a rather different approach.[6] Our result is also in agreement with another works existing in the literature.[7] On the other hand, it is trivial to show that both $\operatorname{Res} f_\theta|_{k^2=m_+^2}$ and $\operatorname{Res} f_\theta|_{k^2=m_-^2}$ are less than zero. Thence, TMPE with the Maxwell's term with the usual sign is unitary. Choosing $a = -1$, we see that if $s^2 > 4m^2$, then $f_\theta = \frac{m^2+k^2}{(k^2-m_+^2)(k^2-m_-^2)}$, with $m_\pm = \frac{1}{2}\left[s \pm \sqrt{s^2 - 4m^2}\right]$. A straightforward calculation allows us to conclude that $\operatorname{Res} f_\theta|_{k^2=m_+^2} > 0$, and $\operatorname{Res} f_\theta|_{k^2=m_-^2} < 0$, implying that, if $s^2 > 4m^2$, TMPE with Maxwell's term with the wrong sign is nonunitary. It is worth mentioning that this system, despite having acceptable values for the masses, faces ghost problems. Of course, if $s^2 < 4m^2$ the two roots of $x^2 + x(2m^2 - s^2) + m^4 = 0$, where $k^2 \equiv x$, are imaginary; note also that for $s^2 = 4m^2$ the two hitherto complex roots coalesce and the masses are simply $m_+ = m_- = \frac{s}{2}$: these models are never viable. We focus, at last, on the case $a = 0$ (absence of the Maxwell's term). This model was analyzed long ago by Deser and Jackiw, who came to the conclusion that setting $a = 0$ yields just another version of the Maxwell–Chern–Simons theory and so it is equivalent to choosing $m^2 = 0$.[8]

2.1.1. *Discussion*

Note that Proca electromagnetism ($s = 0$) with $a = -1$ contains tachyons; however, TMPE with $a = -1$ and $s^2 > 4m^2$ is plagued by ghosts but not by tachyons: the particle content of the model is one nontachyonic spin-1 ghost of mass m_+ and one massive spin-1 normal particle of mass m_-. Thence, a field theory built from this model would not be satisfactory from the point of view of their fundamentals. It could regarded, perhaps, as an effective field theory, i.e. a low-energy approximation to a more fundamental theory. Nonetheless, the condition $s > 2m$ is greatly discouraging as far as the possibility of applying this kind of model, for instance, to some condensed matter systems where one deals, in general, with low-energy excitations. Interesting enough, only the model with $a = +1$ may be viewed as physically sound. Why is this so? Because the aforementioned system has a Lagrangian that reproduces the Lagrangians of well-behaved physical models when the appropriate limits are taken. Indeed, if $s = 0$, we recover Proca electromagnetism; on the other hand, setting $m = 0$, we obtain Chern–Simons electromagnetism. It is remarkable that we also arrive at a nice physical model by removing the Maxwell's term: the system with $a = 0$ and the "self-dual" model of Ref. 8 are equivalent.

2.2. *Checking the unitarity of TMHDE*

Based on the above informations, we have every reason to begin the unitarity analysis of TMHDE by setting $a = +1$ in Eq. (2). The calculations are now more complicated because, unlike the previous model, this represents in general three massive excitations rather than two massive ones. Since the θ-component of the propagator concerning TMHDE with $a = +1$ can be written as $f_\theta = \frac{M^2(x-M^2)}{x^3-2M^2x^2+M^4x-M^4s^2}$, where $M \equiv \frac{1}{l}$, we have to analyze the nature, as well as the signs, of the roots of the cubic equation $x^3 + a_2x^2 + a_1x + a_0 = 0$, where $a_2 \equiv -2M^2$, $a_1 \equiv M^4$, and $a_0 \equiv -M^4s^2$. Taking into account that we are only interested in those roots that are both real and unequal, we require $D < 0$, where $D \equiv Q^3 + R^2$, with Q and R being, in this order, equal to $\frac{3a_1-a_2^2}{9}$ and $\frac{9a_1a_2-27a_0-2a_2^3}{54}$, is the polynomial discriminant. Performing the computations we get $D = M^8s^2\left[\frac{s^2}{4} - \frac{M^2}{27}\right]$, implying that only and if only $s^2 < \frac{4M^2}{27}$ will the roots be real and unequal. Our next step is to verify whether or not these roots are positive. This can be accomplished by building the Routh–Hurwitz array,[9] namely,

$$
\begin{array}{cc}
1 & M^4 \\
-2M^2 & -M^4s^2 \\
M^2\left(M^2 - \dfrac{s^2}{2}\right) & 0 \\
-M^4s^2 & 0
\end{array}\,.
$$

Noting that there are three signs changes in the first column of the array above, we conclude that all the three roots are positive. In summary, if $s^2 < \frac{4m^2}{27}$, TMHDE with $a = +1$ is a model with acceptable values for the masses. Denoting these roots as x_1, x_2, and x_3, and assuming without any loss of generality that $x_1 > x_2 > x_3$, we get

$$f_\theta = \frac{M^2(x_1 - M^2)}{(x_1 - x_2)(x_1 - x_3)} \frac{1}{x - x_1} + \frac{M^2(x_2 - M^2)}{(x_2 - x_1)(x_2 - x_3)} \frac{1}{x - x_2} + \frac{M^2(x_3 - M^2)}{(x_3 - x_1)(x_3 - x_2)} \frac{1}{x - x_3} .$$

Hence, TMHDE with $a = +1$ will be unitary if the conditions $x_1 - M^2 < 0$, $x_2 - M^2 > 0$, and $x_3 - M^2 < 0$ hold simultaneously. Obviously, this will never occur, which allows us to conclude that TMHDE with the Maxwell's term with the standard sign is nonunitary. If $a = -1$, $f_\theta = \frac{M^2(x+M^2)}{x^3+2M^2x^2+M^4x-M^4s^2}$. Since the polynomial discriminant, $D = M^8 s^2 \left[\frac{s^2}{4} + \frac{M^2}{27}\right]$, for the cubic equation $x^3+2M^2x^2+M^4x - M^4s^2 = 0$ is greater than zero, one of the roots of the equation is real and the other two are complex conjugates, which means that the system with $a = -1$ is forbidden. To finish our analysis we set $a = 0$ in Eq. (2). In this case $f_\theta = \frac{xM^2}{x^3-M^4s^2}$, and the polynomial discriminant related to $x^3 - M^4 s^2 = 0$ is greater than zero. This model, as the previous one, is also forbidden.

2.2.1. *Discussion*

Should we expect intuitively that TMHDE with $a = +1$ faced unitary problems? The answer is affirmative. In fact, setting $s = 0$, for instance, in its Lagrangian, we recover the Lagrangian for the usual Podolsky electromagnetism which is nonunitary. Nonetheless, Podolsky–Chern–Simons (PCS) planar electromagnetism with $a = +1$ and $s^2 < \frac{4M^2}{27}$, despite being haunted by ghosts, has normal massive modes. Since the existence of these well-behaved excitations is subordinated to the condition $s < \frac{2M}{\sqrt{27}}$, we are really encouraged to regard this system as an effective field model. It is quite remarkable that the coupling of PCS planar electrodynamics with a charged scalar field, produces an attractive interaction between equal charge bosons. To see this we need to know beforehand the expression of the effective nonrelativistic potential for the interaction of two charged-bosons in the center-of-mass frame. A somewhat involved computation, yields

$$V(r) = -\frac{sQ^2}{\pi m l^4}\left[\frac{l^4}{s^2}\frac{1}{r^2} + \frac{1}{r}\sum_j B_j \sqrt{|x_j|} K_1\left(\sqrt{|x_j|} r\right)\right]\mathbf{L} + \frac{Q^2}{2\pi l^4}\left[\sum_j A_j K_0\left(\sqrt{|x_j|} r\right)\right], \tag{5}$$

where $A_1 \equiv \frac{1+l^2x_1}{(x_1-x_2)(x_1-x_3)}$, $A_2 \equiv \frac{1+l^2x_2}{(x_2-x_1)(x_2-x_3)}$, $A_3 \equiv \frac{1+l^2x_3}{(x_3-x_1)(x_3-x_2)}$, $B_1 \equiv \frac{-(1+l^2x_1)^2}{s^2(x_1-x_2)(x_1-x_3)}$, $B_2 \equiv \frac{-(1+l^2x_2)^2}{s^2(x_2-x_1)(x_2-x_3)}$, and $B_3 \equiv \frac{-(1+l^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)}$, x_1, x_2, and x_3 are the roots of the equation $x^3+\frac{2x^2}{l^2}+\frac{x}{l^4}+\frac{s^2}{l^4}=0$, $\mathbf{L}$ is the angular momentum, K is the modified Bessel function, and Q and m are, respectively, the charge and the mass of the scalar boson. On the other hand, the radial Schrödinger equation associated with this potential is given by

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right]\mathcal{R}_{n\bar{l}}+m[E_{n\bar{l}}-V_{\bar{l}}^{\mathrm{eff}}]\mathcal{R}_{n\bar{l}}=0\,, \tag{6}$$

where

$$V_{\bar{l}}^{\mathrm{eff}}(r)=-\frac{sQ^2}{\pi m l^4}\left[\frac{l^4}{s^2}\frac{1}{r^2}+\frac{1}{r}\sum_j B_j\sqrt{|x_j|}K_1\left(\sqrt{|x_j|}r\right)\right]\bar{l}$$
$$+\frac{Q^2}{2\pi l^4}\left[\sum_j A_jK_0\left(\sqrt{|x_j|}r\right)\right]+\frac{\bar{l}^2}{mr^2}\,.$$

Here $\bar{l}$ denotes the eigenvalues of the operator $\mathbf{L}$. In terms of the dimensionless parameters $y\equiv sr$, $\alpha\equiv\frac{Q^2}{\pi s}$, $b_j\equiv\frac{s^2}{l^4}B_j$, $X_j\equiv\frac{|x_j|}{s}$, $\beta\equiv\frac{m}{s}$, $a_j\equiv\frac{A_j}{l^4}$, and $\tilde{E}_{n\bar{l}}\equiv\frac{mE_{n\bar{l}}}{s^2}$, Eq. (6) reads

$$\left[\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}\right]\mathcal{R}_{n\bar{l}}+[\tilde{E}_{n\bar{l}}-\tilde{V}_{\bar{l}}^{\mathrm{eff}}]\mathcal{R}_{n\bar{l}}=0\,,$$

with

$$\tilde{V}_{\bar{l}}^{\mathrm{eff}}\equiv-\frac{\bar{l}(\alpha-\bar{l})}{y^2}+\frac{\alpha\beta}{2}\sum_j a_jK_0(X_jy)-\frac{\alpha\bar{l}}{y}\sum_j b_jX_jK_1(X_jy)\,.$$

We call attention to the fact that $\tilde{V}_{\bar{l}}^{\mathrm{eff}}$ behaves as $\frac{\bar{l}^2}{y^2}$ at the origin and as $\frac{\bar{l}(\bar{l}-\alpha)}{y^2}$ asymptotically. Now, in four dimensions, the anomalous factor of $\frac{4}{3}$ in the Abraham–Lorentz model for the electron does not show up if $l>\frac{1}{2}r_e$, where r_e denotes the classical radius of the electron.[10] Therefore, we assume $l\ll 1$. In this limit the derivative of the potential with respect to y reduces to

$$\frac{d}{dy}\tilde{V}_{\bar{l}}^{\mathrm{eff}}\sim\frac{2\bar{l}(\alpha-\bar{l})}{y^3}-\left[\frac{2\alpha\bar{l}}{y^2}+\frac{\alpha\beta}{2}\right]K_1(y)-\frac{\alpha\bar{l}}{y}K_0(y)\,.$$

Supposing $\bar{l}>0$, without any loss of generality, we promptly see that, if $\bar{l}>\alpha$, the potential is strictly decreasing. The remaining possibility is $\bar{l}<\alpha$. In this interval $\tilde{V}_{\bar{l}}^{\mathrm{eff}}$ approaches $+\infty$ at the origin and 0^- for $y\to+\infty$, which is indicative of a local minimum. Consequently, the existence of the attractive potential is subordinated to the conditions $a\ll 1$ and $0<\bar{l}<\alpha$. One can show that the effective potential with $l\ll 1$ and $0<\bar{l}<\alpha$ can bind a pair of identical charged-scalar bosons.[3] Accordingly, the addition of the topologically massive term to Podolsky's electromagnetism with $a = +1$ — an admittedly nonunitary model — did not cure its nonunitary problem; nonetheless, the condition for the resulting three-term model to be free of tachyons gives rise to a constraint between the topological and Podolsky masses which is responsible for a scalar attractive interaction.

3. Massive, Topologically Massive, Gravitational Models (MTMG)

TMFPG is defined by the Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{TMFPG}} &= a\frac{2}{\bar{\kappa}^2}\sqrt{g}\,R - \frac{m^2}{2}({h_{\mu\nu}}^2 - h^2) \\ &\quad + \frac{1}{\mu}\epsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma{}_{\rho\nu} + \frac{2}{3}\Gamma^\sigma{}_{\mu\beta}\Gamma^\beta{}_{\nu\rho}\right),\end{aligned} \tag{7}$$

at quadratic order in $\bar{\kappa}$, where $\bar{\kappa}^2$ is a suitable constant that in four dimensions is equal to $24\pi G$, with G being Newton's constant.[11] Here $g_{\mu\nu} \equiv \eta_{\mu\nu} + \bar{\kappa}h_{\mu\nu}$, $\mu > 0$ is a dimensionless parameter, and $h \equiv \eta_{\mu\nu}h^{\mu\nu}$. Indices are raised (lowered) with $\eta^{\mu\nu}$ ($\eta_{\mu\nu}$). The Lagrangian related to TMHDG, in turn, is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{TMHDG}} &= \sqrt{g}\left(a\frac{2R}{\kappa^2} + \frac{\alpha}{2}R^2 + \frac{\beta}{2}R^2_{\mu\nu}\right) \\ &\quad + \frac{1}{\mu}\epsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma{}_{\rho\nu} + \frac{2}{3}\Gamma^\sigma{}_{\mu\beta}\Gamma^\beta{}_{\nu\rho}\right),\end{aligned} \tag{8}$$

where α and β are suitable constants with dimension L. For the sake of simplicity, the gauge-fixing term was omitted. Note that the parameter a appearing in Eqs. (7) and (8) allows for choosing the Einstein sign's term or even removing it.

The propagator related to TMFPG is

$$\begin{aligned}\mathcal{P}_{\mathrm{TMFPG}} &= -\frac{1}{m^2}P^1 - \frac{M^2(m^2 + a\Box)}{\Box^3 + M^2a^2\Box^2 + 2am^2M^2\Box + M^2m^4}P^2 \\ &\quad - \frac{M}{\Box^3 + M^2a^2\Box^2 + 2am^2M^2\Box + M^2m^4}P \\ &\quad - \frac{m^2 + a\Box}{2m^4}\bar{P}^0 + \frac{1}{2m^2}\bar{\bar{P}}^0,\end{aligned} \tag{9}$$

where $M \equiv \mu/\bar{\kappa}^2$. On the other hand, linearizing Eq. (8) and adding to the result the gauge-fixing term $\mathcal{L}_{\mathrm{gf}} = \frac{1}{2\lambda}\left({h_{\mu\nu}}^{,\nu} - \frac{1}{2}h_{,\mu}\right)^2$ (de Donder gauge), we find that the propagator concerning TMHDG takes the form

$$\begin{aligned}\mathcal{P}_{\mathrm{TMHDG}} &= \frac{1}{\Box[a + b(\frac{3}{2} + 4c)\Box]}\bar{\bar{P}}^0 + \frac{2\lambda}{k^2}P^1 + \frac{1}{\Box\left[a + b\left(\frac{3}{2} + 4c\right)\Box\right]}P^0 \\ &\quad + \frac{4M}{\Box[M^2b^2\Box^2 - 4(abM^2 - 1)\Box + 4M^2a^2]}P\end{aligned}$$

$$
- \frac{2M^2(2a - b\Box)}{\Box[M^2b^2\Box^2 - 4(abM^2 - 1)\Box + 4M^2a^2]} P^2
$$

$$
+ \left[-\frac{4\lambda}{\Box} + \frac{2}{\Box\left[a + b\left(\frac{3}{2} + 4c\right)\Box\right]} \right] \bar{P}^0 , \tag{10}
$$

where $b \equiv \frac{\beta\kappa^2}{2}$, $c \equiv \frac{\alpha}{\beta}$, and $M \equiv \frac{\mu}{\kappa^2}$. Our conventions are $R^{\alpha}{}_{\beta\gamma\delta} = -\partial_\delta \Gamma^{\alpha}{}_{\beta\gamma} + \cdots$, $R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha}$, $R = g^{\mu\nu}R_{\mu\nu}$, where $g_{\mu\nu}$ is the metric tensor, and signature $(+,-,-)$. The propagators were calculated using the basis

$$
\{P^1, P^2, P^0, \bar{P}^0, \bar{\bar{P}}^0, P\} ,
$$

where P^1, P^2, P^0, $\bar{P}^0$, and $\bar{\bar{P}}^0$, are the usual three-dimensional Barnes–Rivers operators,[12] namely,

$$
\begin{aligned}
P^1_{\mu\nu,\rho\sigma} &= \frac{1}{2}\left(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}\right) , \\
P^2_{\mu\nu,\rho\sigma} &= \frac{1}{2}\left(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho} - \theta_{\mu\nu}\theta_{\rho\sigma}\right) , \\
P^0_{\mu\nu,\rho\sigma} &= \frac{1}{2}\theta_{\mu\nu}\theta_{\rho\sigma} , \\
\bar{P}^0_{\mu\nu,\rho\sigma} &= \omega_{\mu\nu}\omega_{\rho\sigma} , \\
\bar{\bar{P}}^0_{\mu\nu,\rho\sigma} &= \theta_{\mu\nu}\omega_{\rho\sigma} + \omega_{\mu\nu}\theta_{\rho\sigma} ,
\end{aligned}
$$

and P is the operator associated with the linearized Chern–Simons term, i.e.

$$
P_{\mu\nu,\rho\sigma} \equiv \frac{\Box\partial^\lambda}{4}\left[\epsilon_{\mu\lambda\rho}\theta_{\nu\sigma} + \epsilon_{\mu\lambda\sigma}\theta_{\nu\rho} + \epsilon_{\nu\lambda\rho}\theta_{\mu\sigma} + \epsilon_{\nu\lambda\sigma}\theta_{\mu\rho}\right] .
$$

According to Ref. 4, the saturated propagator concerning the MTMG, is given by

$$
SP_{\mathrm{MTMG}} = \left[T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right] f_{P^2} + \frac{1}{2}T^2 f_{P^0} , \tag{11}
$$

where $T^{\mu\nu}$ is the external conserved current that, obviously, is symmetric in the indices μ and ν, $T \equiv \eta_{\mu\nu}T^{\mu\nu}$, and f_{P^2} and f_{P^0} are, respectively, the components P^2 and P^0 of the propagator in the basis $\{P^1, P^2, P^0, \bar{P}^0, \bar{\bar{P}}^0, P\}$. Therefore, to find out whether or not the gravitational model is unitary, we must compute SP_{MTMG} using Eq. (11) and determine afterwards the the residue at each simple pole of SP_{MTMG}: if all the residues are ≥ 0, the model is unitary; however, if at least one of them is negative, the system is nonunitary. The unitarity analysis is greatly facilitated if we take into account that $\left[T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right]_{k^2=m^2} > 0$ and $[T^{\mu\nu}T_{\mu\nu} - T^2]_{k^2=0} = 0$, where $m \geq 0$ is the mass of a generic physical particle associated with the MTMG, and k is the corresponding momentum exchanged. Using this prescription, we check in the following the unitarity of TMFPG and TMHDE, in this order.

3.1. *Checking the unitarity of TMFPG*

To begin with, we set $a = -1$ in Eq. (8) because we want to recover the Einstein–Chern–Simons Lagrangian in the $m = 0$ limit — topologically massive gravity is a theory that requires $a = -1$ to be ghost-free.[2] The corresponding saturated propagator is given by

$$\mathrm{SP}_{\mathrm{TMFPG}} = \left[T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right]\frac{M^2(m^2 + k^2)}{k^6 - M^2k^4 - 2m^2M^2k^2 - M^2m^4}\,.$$

Our next step is to study the roots of the cubic equation $x^3 - M^2x^2 - 2m^2M^2x^2 - M^2m^4 = 0$. Since the discriminant, $D = M^4m^6\left[\frac{M^2}{27} + \frac{m^2}{4}\right]$, related to this equation is greater than zero, the model at hand is unphysical and must be rejected. Consequently, we turn our attention to the system with $a = +1$. Now, we have to consider the roots of the equation $x^3 - M^2x^2 + 2m^2M^2x^2 - M^2m^4 = 0$, whose polynomial discriminant can be written as $D = M^4m^6\left[\frac{m^2}{4} - \frac{M^2}{27}\right]$. Therefore, if $\frac{m^2}{M^2} < \frac{4}{27}$, our equation has three distinct real roots. The corresponding Routh–Hurwitz array is

$$\begin{array}{cc} 1 & 2m^2M^2 \\ -M^2 & -M^2m^4 \\ m^2(2M^2 - m^2) & 0 \\ -M^2m^4 & 0 \end{array}\,.$$

Accordingly, the system with $a = +1$ and $\frac{m^2}{M^2} < \frac{4}{27}$ has acceptable values for the masses. Proceeding just as we have done for TMHDE with $a = +1$ and $s^2 < \frac{4M^2}{27}$, we promptly obtain

$$\begin{aligned} \mathrm{SP}_{\mathrm{TMFPG}} = {} & \frac{F(k)(m^2 - x_1)}{(x_1 - x_2)(x_1 - x_3)}\frac{1}{k^2 - x_1} \\ & + \frac{F(k)(m^2 - x_2)}{(x_2 - x_1)(x_2 - x_3)}\frac{1}{k^2 - x_2} \\ & + \frac{F(k)(m^2 - x_3)}{(x_3 - x_1)(x_3 - x_2)}\frac{1}{k^2 - x_3}\,, \end{aligned}$$

where $F(k) \equiv \left\{T^{\mu\nu}(k)T_{\mu\nu}(k) - \frac{1}{2}[T(k)]^2\right\}M^2$. From the above, we clearly see that this model will be unitary if $m^2 > x_1$, $m^2 < x_2$, and $m^2 > x_3$. We thus come to the conclusion that TMFPG with $a = +1$ and $\frac{m^2}{M^2} < \frac{4}{27}$ is nonunitary, which means that the topological term is responsible for breaking down the unitarity of the harmless Fierz–Pauli gravity. If $a = 0$, the discriminant associated with the equation $x^3 - M^2m^4 = 0$ is greater than zero, which implies that this model is physically unsound. We remark that our conclusions are in complete agreement with those of Ref. 6 where a quite different approach to the unitarity problem was employed.

3.1.1. *Discussion*

The above results points to an important and at the same time interesting question: why can unitary massive electromagnetic models coexist in peace with topologically massive terms, whereas unitary massive gravitational ones cannot? The root of the problem lies in the rather odd way Einstein–Chern–Simons theory is constructed: the presence of the ghosts in the dynamical field is avoided by choosing the Einstein's term with the wrong sign.[2] This is trivial to show. Indeed, writing the Einstein–Chern–Simons Lagrangian as

$$\mathcal{L} = a\sqrt{g}\frac{2R}{\kappa^2} + \frac{1}{\mu}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\rho}\right), \tag{12}$$

with $a = \pm 1$, we promptly see that the corresponding saturated propagator is given by

$$\mathrm{SP} = -\frac{1}{a}\left(T^{\mu\mu}T_{\mu\nu} - \frac{1}{2}T^2\right)\frac{1}{k^2 - M^2} - \frac{1}{a}\left(T^{\mu\mu}T_{\mu\nu} - T^2\right)\frac{1}{k^2}\,. \tag{13}$$

Thus, to render the theory unitary we are obliged to set $a = -1$ in Eq. (13). Note that as far as these three-term systems are concerned, we are always in a dilemma: which value should we assign to a, -1 or $+1$? If we single out $a = -1$, for instance, we recover Einstein–Chern–Simons theory when the nontopological massive term is removed; however, in the absence of the topological term, we do not get a nice physical theory because now the Einstein's term has the wrong sign. On the other hand, if we pick out $a = +1$, we do not recover Einstein–Chern–Simons theory when the nontopological massive term is removed. In other words, due to the unusual Einstein sign's term in the Lagrangian concerning Einstein–Chern–Simons theory, the augmented systems do not reduce to well-behaved physical models in the suitable limits. Note that these idiosyncrasies do not occur in the framework of massive, topologically massive, electromagnetic models because the Maxwell sign's term concerning Maxwell–Chern–Simons theory is the same as that of the usual Maxwell's theory.

3.2. *Checking the unitarity of TMHDG*

Assuming $a \neq 0$, the SP concerning TMHDG can be written as

$$\begin{aligned}\mathrm{SP}_{\mathrm{TMHDG}} = {} & \frac{M^2 b}{2}\,\frac{\left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right)}{k^2 - M_1^2}\,\frac{-1 + \sqrt{1 - 2abM^2}}{\sqrt{1 - 2abM^2}[1 - abM^2 - \sqrt{1 - 2abM^2}]}\\ & + \frac{M^2 b}{2}\,\frac{\left(T^{\mu\nu}T_{\mu\nu} - \frac{1}{2}T^2\right)}{k^2 - M_2^2}\,\frac{1 + \sqrt{1 - 2abM^2}}{\sqrt{1 - 2abM^2}[1 - abM^2 + \sqrt{1 - 2abM^2}]}\\ & + \frac{T^{\mu\nu}T_{\mu\nu} - T^2}{ak^2} + \frac{\frac{1}{2}T^2}{a(k^2 - m^2)}\,,\end{aligned} \tag{14}$$

Table 1. Unitarity analysis of the topologically massive higher-derivative gravity models.

a	b	$\frac{3}{2}+4c$	Excitations and mass counts	Tachyons	Unitarity
-1	>0	<0	2 massive spin-2 normal particles 1 massless spin-2 nonpropagating particle 1 massive spin-0 ghost	no one	nonunitary
-1	$\frac{-1}{2M^2}<b<0$	>0	1 massive spin-2 normal particle 1 massless spin-2 nonpropagating particle 1 massive spin-2 ghost 1 massive spin-0 ghost	no one	nonunitary
$+1$	<0	<0	2 massive spin-2 ghosts 1 massless spin-2 nonpropagating particle 1 massive spin-0 normal particle	no one	nonunitary
$+1$	$0<b<\frac{1}{2M^2}$	>0	1 massive spin-2 normal particle 1 massless spin-2 nonpropagating particle 1 massive spin-2 ghost 1 massive spin-0 normal particle	no one	nonunitary

where

$$M_1^2 \equiv \left(\frac{2}{b^2M^2}\right)[1-abM^2-\sqrt{1-2abM^2}]\,,$$

$$M_2^2 \equiv \left(\frac{2}{b^2M^2}\right)\left[1-abM^2+\sqrt{1-2abM^2}\,\right],$$

$$m^2 \equiv \frac{a}{b(3/2+4c)}\,.$$

It is interesting to note that $M_1^2 \to M^2$, and $M_2^2 \to +\infty$, as $b \to 0$, which implies that Eq. (14) reduces to Eq. (13) when $\alpha, \beta \to 0$, as expected. We are now ready to analyze the excitations and mass counts concerning TMHDG for both allowed signs of a. To avoid needless repetitions, we restrict ourselves to presenting a summary of the main results in Table 1. The systems that do not appear in this table are tachyonic, i.e. unphysical.

In conclusion, we consider TMHDG with $a=0$. In this case,

$$\mathrm{SP}_{\mathrm{TMDHDG}} = \frac{M^2b}{2}\left[-\frac{T^{\mu\nu}T_{\mu\nu}-\frac{1}{2}T^2}{k^2}+\frac{T^{\mu\nu}T_{\mu\nu}-\frac{1}{2}T^2}{k^2-\frac{4}{M^2b^2}}\right]+\frac{1}{b\left(\frac{3}{2}+4c\right)}\frac{\frac{1}{2}T^2}{k^4}\,.$$

The pole of order two at $k^2=0$ indicates that these models are unphysical.

3.2.1. *Discussion*

As intuitively expected, TMHDG is nonunitary for $a = \pm 1$; nonetheless, these models are in general nontachyonic which means that under certain circumstances they may be viewed as effective field models. Our aim here is to investigate, in passing, the novel features of one of these nonunitary gauge-invariant three-term effective field models. To be more specific, we fix our attention on the first model of Table 1, i.e. TMHDG with $a = -1$, $b > 0$, and $\frac{3}{2} + 4c < 0$ (Ref. 13). We have chosen the $a = -1$ system because it reduces, in the absence of the topologically massive term, to higher-derivative gravity with $a = -1$ — an effectively multimass model of the fourth-derivative order with interesting properties of its own.[14] Now, it can be shown that the effective nonrelativistic potential for the interaction of two scalars bosons in the framework of TMHDG with $a = -1$, $b > 0$, and $\frac{3}{2} + 4c < 0$, is given by[15]

$$V(r) = 2\bar{m}^2\bar{G}\left[K_0(rm) - \frac{K_0(rM_+)}{1 + \frac{bM_+^2}{2}} - \frac{K_0(rM_-)}{1 + \frac{bM_-^2}{2}}\right], \tag{15}$$

where $\bar{m}$ is the mass of one of the neutral bosons, $\bar{G} \equiv \frac{\kappa^2}{32\pi}$, and

$$M_\pm = \frac{1}{bM}\left[\sqrt{1 + 2bM^2} \pm 1\right].$$

Note that $V(r)$ behaves as $2\bar{m}^2\bar{G}\ln\left(\frac{M_+^{1+\frac{bM_+^2}{2}} M_-^{1+\frac{bM_-^2}{2}}}{m}\right)$ at the origin and as

$$2\bar{m}^2\bar{G}\left[\sqrt{\frac{\pi}{2rm}}\,e^{-rm} - \frac{1}{1 + \frac{bM_+^2}{2}}\sqrt{\frac{\pi}{2rM_+}}\,e^{-rM_+} - \frac{1}{1 + \frac{bM_-^2}{2}}\sqrt{\frac{\pi}{2rM_-}}\,e^{-rM_-}\right]$$

asymptotically. Accordingly, $V(r)$ is finite at the origin and zero at infinity. The derivative of this potential with respect to r is in turn given by

$$\frac{dV}{dr} = 2\bar{m}^2\bar{G}\left[-mK_1(rm) + \frac{M_+}{1 + \frac{bM_+^2}{2}}K_1(rM_+) + \frac{M_-}{1 + \frac{bM_-^2}{2}}K_1(rM_-)\right]. \tag{16}$$

On the other hand, it was shown recently that in four dimensions the propagation of photons in the context of higher-derivative gravity (HDG) is dispersive.[16] In other words, gravitational rainbows and semiclassical HDG can coexist without conflict. On the basis of the fact that the rainbow effect is currently undetectable, it is possible to show that $|\beta| \leq 10^{60}$ (Ref. 17). How reliable is this result? The aforementioned constraint is of the same order as that obtained by testing the gravitational inverse-square law in the submillimeter regime.[18] Thence, we assume $b \gg 1$. As a consequence, Eq. (16) reduces to

$$\frac{dV}{dr} \sim 2\bar{m}^2\bar{G}\left[-mK_1(rm) + \sqrt{\frac{2}{b}}K_1\left(r\sqrt{\frac{2}{b}}\right)\right],$$

implying that the potential $V(r)$, which in this approximation may be expressed as

$$V(r) \sim 2\bar{m}^2\bar{G}\left[K_0(rm) - K_0\left(r\sqrt{\frac{2}{b}}\right)\right], \tag{17}$$

is everywhere attractive if $\sqrt{\frac{2}{b}} > m$, is repulsive if $m > \sqrt{\frac{2}{b}}$, and vanishes if $m = \sqrt{\frac{2}{b}}$. If we appeal to the usual tools of Einstein's geometrical theory, we arrive at the same conclusions. In fact, in the weak field approximation the gravitational acceleration , $\gamma^l = \frac{dv^l}{dt}$, of a slowly moving test particle is given by $\gamma^l = -\kappa\left[\frac{\partial}{\partial t}h^l_0 - \frac{1}{2}\frac{\partial}{\partial l}h_{00}\right]$, which for time-independent fields reduces to $\gamma^l = \frac{\kappa}{2}\frac{\partial}{\partial l}h_{00}$. Now, taking into account that $h_{00} = \frac{2V}{\bar{m}\kappa}$, we obtain

$$\gamma^l = 2\bar{m}\bar{G}\frac{x^l}{r}\left[-mK_1(rm) + \sqrt{\frac{2}{b}}K_1\left(r\sqrt{\frac{2}{b}}\right)\right].$$

Therefore, the gravitational force exerted on the particle,

$$F^l = 2\bar{m}^2\bar{G}\frac{x^l}{r}\left[-mK_1(rm) + \sqrt{\frac{2}{b}}K_1\left(r\sqrt{\frac{2}{b}}\right)\right],$$

is everywhere attractive if $\sqrt{\frac{2}{b}} > m$, is repulsive if $m > \sqrt{\frac{2}{b}}$ (antigravity), and vanishes if $m = \sqrt{\frac{2}{b}}$ (gravitational shielding). It is remarkable that this force does not exist in general relativity. It is peculiar to topologically massive higher-derivative gravity.

4. Final Remarks

We have shown that topologically massive terms cannot be used as a panacea for curing the nonunitarity of massive electromagnetic/gravitational models. In truth, the addition of a Chern–Simons term to a massive electromagnetic/gravitational model is physically sound only and if only the resulting three-term system reduces to well-behaved physical models in the suitable limits. A direct consequence of this fact is that we will never be able to construct an unitary, massive, topologically massive, gravitational model. Indeed, the fancy way Einstein–Chern–Simons theory is built, i.e. with the Einstein's term with the opposite sign, precludes the existence of ghost-free, massive, topologically massive, gravitational models. Therefore, from a conceptual point of view, the addition of a topologically massive term to a massive gravitational model is a complete nonsense: on the one hand, it does not cure the nonunitarity of the original model; on the other hand, it spoils the unitarity of admittedly unitary models. An interesting and elucidatory example is furnished by $R + R^2$ gravity, which is defined by the Lagrangian $\mathcal{L} = \left[a\frac{2R}{\kappa^2} + \frac{\alpha}{2}R^2\right]\sqrt{g}$, with $a = \pm 1$. If $\alpha > 0$, this theory is nontachyonic regardless of the sign of a; in addition, it is unitary if $a = +1$, and nonunitary if $a = -1$. Incidentally, $R + R^2$ gravity with $a = +1$ is the only known gravity theory with higher-derivatives that is unitary. However, topologically massive $R+R^2$ gravity is nonunitary for both possible sign choices of a.[19] Yet, a new and surprising physics emerges when we analyze the three-term effective field models that are both gauge-invariant and nonunitary. In the framework of the electromagnetic models, an attractive interaction between equal charge particles can be produced that leads to an unusual planar dynamics: scalar pairs can condense into bound states. In the framework of the gravity systems, in turn, unlike what occurs in the context of the insipid and odorless three-dimensional Einstein's general relativity, we have a gravitational interaction that can be both attractive and repulsive. We can also have a null gravitational interaction, such as in three-dimensional gravity that is trivial outside the sources. Certainly, these effective field models deserve to be both much better known and further investigated.

Acknowledgments

We are very grateful to Prof. S. Deser for calling our attention to Ref. 6. A. Accioly thanks CNPq-Brazil for partial support while M. Dias is indebted to CAPES-Brazil for full support.

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Boson-boson effective nonrelativistic potential for higher-derivative electromagnetic theories in D dimensions

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(Received 1 July 2004; published 15 November 2004)

The problem of computing the effective nonrelativistic potential U_D for the interaction of charged-scalar bosons, within the context of D-dimensional electromagnetism with a cutoff, is reduced to quadratures. It is shown that U_3 cannot bind a pair of identical charged-scalar bosons; nevertheless, numerical calculations indicate that boson-boson bound states do exist in the framework of three-dimensional higher-derivative electromagnetism augmented by a topological Chern-Simons term.

DOI: 10.1103/PhysRevD.70.107705 PACS numbers: 11.10.St, 11.10.Kk, 12.20.Ds

We consider in this Brief Report the problem of determining the effective charged-scalar-boson—charged-scalar-boson low energy potential U_D arising from D-dimensional electromagnetism with a cutoff a. The Lagrangian concerning this theory can be written as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda}, \tag{1}$$

where $F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$ is the usual electromagnetic tensor field. Lagrangian (1) is gauge and Lorentz invariant; in addition, it leads to local field equations which are linear in the field quantities. At distances much larger than the cutoff, the fields described by Eq. (1) become essentially equivalent to the Maxwell fields.

We are motivated by two quite similar developments: In the first, we investigate whether U_3 can form "Cooper pairs."

Our second topic is related to three-dimensional Podolsky-Chern-Simons theory. Based on the interesting discussions from Jackiw [1] about the consistency of the nonrelativistic limit of certain relativistically invariant quantum field theories, it can be shown that the Chern-Simons term alone is unable to form boson-boson bound states [2]. Nonetheless, numerical calculations indicate that the Podolsky term provides a stabilizing mechanism allowing for the existence of Cooper pairs.

We use natural units throughout; our signature is $(+, -, -, \cdots, -)$.

Nonrelativistic quantum mechanics tells us that in the first Born approximation the cross section for the scattering of two indistinguishable massive particles, in the center-of-mass frame (CoM), is given by $\frac{d\sigma}{d\Omega} = |\frac{m}{4\pi} \times \int e^{-i\mathbf{p}'\cdot\mathbf{r}}V(r)e^{i\mathbf{p}\cdot\mathbf{r}}d^{D-1}\mathbf{r}|^2$, where $\mathbf{p}$ ($\mathbf{p}'$) is the initial (final) momentum of one of the particles in the CoM. In terms of the transfer momentum, $\mathbf{k} \equiv \mathbf{p}' - \mathbf{p}$, it reads

$$\frac{d\sigma}{d\Omega} = \left|\frac{m}{4\pi}\int V(r)e^{i\mathbf{k}\cdot\mathbf{r}}d^{D-1}\mathbf{r}\right|^2. \tag{2}$$

On the other hand, from quantum field theory we know that the cross section, in the CoM, for the scattering of two identical massive scalar bosons by an electromagnetic field, can be written as $\frac{d\sigma}{d\Omega} = |\frac{1}{16\pi E}\mathcal{M}|^2$, where E is the initial energy of one of the bosons and $\mathcal{M}$ is the Feynman amplitude for the process at hand, which in the nonrelativistic limit (NR) reduces to

$$\frac{d\sigma}{d\Omega} = \left|\frac{1}{16\pi m}\mathcal{M}_{\text{NR}}\right|^2. \tag{3}$$

From Eqs. (2) and (3), we come to the conclusion that the expression that enables us to compute the D-dimensional effective nonrelativistic potential has the form

$$V(r) = \frac{1}{4m^2}\frac{1}{(2\pi)^{D-1}}\int d^{D-1}\mathbf{k}\,\mathcal{M}_{\text{NR}}e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{4}$$

which clearly shows how the potential from quantum mechanics and the Feynman amplitude obtained via quantum field theory are related to each other.

Now, in the Lorentz gauge Podolsky's scalar QED is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\alpha F_{\mu\alpha} - \frac{1}{2\lambda}(\partial_\nu A^\nu)^2 + (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi, \tag{5}$$

where $D_\mu \equiv \partial_\mu + iQA_\mu$. Therefore, the interaction Lagrangian to order Q for the process $S + S \longrightarrow S + S$, where S denotes a spinless boson of mass m and charge Q, is $\mathcal{L}_{\text{int}} = iqQA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)$. The Feynman rule for the elementary vertex is shown in Fig. 1. Accordingly, the Feynman amplitude for the interaction of two charged spinless bosons of equal mass via a Podolskian photon exchange (see Fig. 2) is

$$\mathcal{M} = V^\mu(p, p')D_{\mu\nu}(k)V^\nu(q, q'), \tag{6}$$

where $D_{\mu\nu}(k)$ designates the Podolskian photon propagator.

We propose now an algorithm for computing the propagator for electromagnetic theories with higher deriva-

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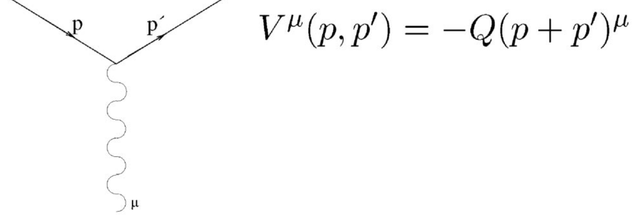

FIG. 1. The relevant vertex for boson-boson interaction.

tives, based on the usual transverse and longitudinal vector projector operators, namely, $\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box}$, $\omega_{\mu\nu} = \frac{\partial_\mu \partial_\nu}{\Box}$, which satisfy the relations $\theta_{\mu\rho}\theta^\rho_\nu = \theta_{\mu\nu}$, $\omega_{\mu\rho}\omega^\rho_\nu = \omega_{\mu\nu}$, $\theta_{\mu\rho}\omega^\rho_\nu = 0$, where $\eta_{\mu\nu}$ is the Minkowski metric. The set of operators $\{\theta, \omega\}$ is a complete set of projector operators for rank-one tensors. Indeed, they are idempotent, mutually orthogonal, and satisfy the completeness relation $[\theta + \omega]_{\mu\nu} = \eta_{\mu\nu} \equiv I_{\mu\nu}$.

Let $\bar{\mathcal{L}}$ be the Lagrangian for electromagnetism with higher derivatives. Since $\bar{\mathcal{L}}$ is a gauge-invariant Lagrangian, we add to it a gauge-fixing Lagrangian $\mathcal{L}_{gf}$, which implies that $\mathcal{L} \equiv \bar{\mathcal{L}} + \mathcal{L}_{gf}$ can be written as $\mathcal{L} = \frac{1}{2} A^\mu O_{\mu\nu} A^\nu$. Expanding O in the basis $\{\theta, \omega\}$ yields $O = x_1 \theta + x_2 \omega$. Accordingly, $O^{-1} = y_1 \theta + y_2 \omega$, where O^{-1} is the propagator and y_1 and y_2 are parameters to be determined. Now, taking into account that $OO^{-1} = I$, we promptly obtain

$$O^{-1} = \frac{1}{x_1}\theta + \frac{1}{x_2}\omega,$$

where we are supposing that both x_1 and x_2 are nonvanishing. Note that the procedure we have just outlined is quite straightforward: On the one hand, it reduces the work of calculating the propagator to a trivial algebraic exercise; on the other hand, it greatly simplifies calculations involving the contraction of conserved currents ($\partial_\nu J^\nu = 0$) with the propagator since in this case the alluded contraction simply gives

$$O^{-1}_{\mu\nu} J^\mu = \frac{J_\nu}{x_1}. \tag{7}$$

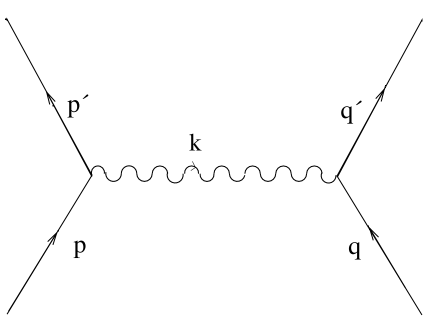

FIG. 2. One-Podolskian photon-exchange contribution to the scattering of two identical massive charged bosons.

From the above we find that the propagator for Podolsky's electrodynamics in the Lorentz gauge assumes the form

$$D_{\mu\nu}(k) = \frac{M^2}{k^2(k^2 - M^2)}\theta_{\mu\nu} - \frac{\lambda}{k^2}\omega_{\mu\nu}, \tag{8}$$

where $M^2 \equiv 1/a^2$.

From Eqs. (6)–(8), we get immediately $\mathcal{M} = \frac{M^2 Q^2 (2p-k)(2q+k)}{k^2(k^2-M^2)}$, which implies

$$\mathcal{M}_{\rm NR} = \frac{4m^2 M^2 Q^2}{\mathbf{k}^2(\mathbf{k}^2 + M^2)}. \tag{9}$$

Inserting Eq. (9) into Eq. (4), we obtain

$$V(r) = \int_0^\infty f(|\mathbf{k}|)|\mathbf{k}|^{n-1} d|\mathbf{k}| \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2 d\theta_2 \int_0^\pi \times \sin^2\theta_3 d\theta_3 \ldots \int_0^\pi e^{-i|\mathbf{k}|r\cos\theta_{n-1}} \sin^{n-2}\theta_{n-1} d\theta_{n-1},$$

where $2 < n \equiv D - 1$ and $f(|\mathbf{k}|) \equiv \frac{Q^2}{(2\pi)^n}(\frac{1}{\mathbf{k}^2} - \frac{1}{\mathbf{k}^2 - M^2})$. Now, taking into account that

$$\int_0^\pi \sin^m\theta d\theta = \frac{\sqrt{\pi}\Gamma(\frac{m-1}{2})}{\Gamma(\frac{m+2}{2})},$$

$$\int_0^\pi e^{-i|\mathbf{k}|r\cos\theta_{n-1}} \sin^{n-2}\theta_{n-1} d\theta_{n-1} = \frac{2^{(n-2)/2}\Gamma(\frac{1}{2})}{(|\mathbf{k}|r)^{(n-2)/2}} \Gamma\Big(\frac{n-1}{2}\Big) J_{(n-2)/2}(|\mathbf{k}|r),$$

where J denotes the Bessel function, we arrive at the following expression for the potential:

$$U_D(r) = \frac{Q}{(2\pi)^{(D-1)/2} r^{(D-3)/2}} \int_0^\infty \Big(\frac{1}{\mathbf{k}^2} - \frac{1}{\mathbf{k}^2 - M^2}\Big) \times |\mathbf{k}|^{(D-1)/2} J_{(D-3)/2}(|\mathbf{k}|r) d|\mathbf{k}|, \tag{10}$$

where $U_D(r) \equiv \frac{V(r)}{Q}$ and $D > 3$.

On the other hand, it is trivial to show that U_3 can be evaluated from the expression

$$U_3(r) = \frac{Q}{2\pi} \int_0^\infty \Big(\frac{1}{\mathbf{k}^2} - \frac{1}{\mathbf{k}^2 - M^2}\Big)|\mathbf{k}| J_0(|\mathbf{k}|r) d|\mathbf{k}|,$$

which allows us to conclude that Eq. (10) can also be applied to the case $D = 3$. Hence, the problem of computing the effective nonrelativistic potential for $D > 2$ was reduced to quadratures.

If Eq. (10) is correct, it must reproduce the Podolskian potential in 3 + 1 dimensions. Performing the computation for $D = 4$, we get

$$U_4(r) = \frac{Q}{4\pi} \frac{1 - e^{-(r/a)}}{r},$$

which is just the same result as that obtained in Podolsky's electromagnetic theory.

For $D = 3$, Eq. (10) yields

$$U_3(r) = -\frac{Q}{2\pi}\left[\ln\frac{r}{r_0} + K_0(Mr)\right], \tag{11}$$

where r_0 is an infrared regulator and K is the modified Bessel function.

We discuss now the existence of boson-boson bound states in the context of planar quadratic electromagnetism. The corresponding time-independent Schrödinger equation can be written as

$$\begin{aligned}\mathcal{H}_l\mathcal{R}_{nl} &= -\frac{1}{m}\left(\frac{d^2}{dr^2}\mathcal{R}_{nl} + \frac{1}{r}\frac{d}{dr}\mathcal{R}_{nl}\right) + V_l^{\text{eff}}\mathcal{R}_{nl} \\ &= E_{nl}\mathcal{R}_{nl},\end{aligned}$$

$$\begin{aligned}V_l^{\text{eff}} &\equiv \frac{l^2}{mr^2} + QU_3(r) \\ &= \frac{l^2}{mr^2} - \frac{Q^2}{2\pi}\left[\ln\frac{r}{r_0} + K_0(Mr)\right],\end{aligned}$$

where $\mathcal{R}_{nl}$ is the nth normalizable eigenfunction of the radial Hamiltonian $\mathcal{H}_l$ whose corresponding eigenvalue is E_{nl} and V_l^{eff} is the lth partial wave effective potential. On the other hand,

$$\frac{d}{dr}V_l^{\text{eff}} = -\frac{2l^2}{m}\frac{1}{r^3} - \frac{Q^2}{2\pi}\frac{1}{r} + \frac{Q^2M}{2\pi}K_1(Mr),$$

which allows us to conclude that $\frac{d}{dr}V_l^{\text{eff}} < 0$ in the interval $0 < r < \infty$, implying that V_l^{eff} is strictly decreasing in this interval. Consequently, in the framework of planar quadratic electromagnetism, no bound state concerning the two charged-scalar bosons system exists.

Since boson-boson bound states do not show up in Podolsky planar electromagnetism, we investigate here whether the effective boson-boson low energy potential related to Podolsky-Chern-Simons (PCS) planar theory can bind a pair of identical charged-scalar bosons. The Lagrangian for PCS scalar QED, in the Lorentz gauge, can be written as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\alpha F_{\mu\alpha} - \frac{1}{2\lambda}(\partial_\nu A^\nu)^2 \\ &+ (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi + \frac{s}{2}\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho,\end{aligned} \tag{12}$$

where $s > 0$ is the topological mass.

In the basis $\{\theta, \omega, S\}$, where $S_{\mu\nu} \equiv \varepsilon_{\mu\rho\nu}\partial^\rho$, the propagator assumes the form

$$\begin{aligned}O^{-1} = &\frac{(a^2k^4 - k^2)\theta}{(a^2k^4 - k^2)^2 - s^2k^2} - \frac{\lambda\omega}{k^2} \\ &- \frac{sS}{(a^2k^4 - k^2)^2 - s^2k^2}.\end{aligned}$$

Now, in the nonrelativistic limit the Feynman amplitude for the process shown in Fig. 2 reduces to

$$\mathcal{M}_{\text{NR}} = \left[\frac{(a^2\mathbf{k}^4 + \mathbf{k}^2)4Q^2m}{(a^2\mathbf{k}^4 + \mathbf{k}^2)^2 + s^2\mathbf{k}^2} + \frac{8ismQ^2\mathbf{k}\wedge\mathbf{P}}{(a^2\mathbf{k}^4 + \mathbf{k}^2)^2 + s^2\mathbf{k}^2}\right],$$

where $\mathbf{P} \equiv \frac{1}{2}(\mathbf{p} - \mathbf{q})$ is the relative momentum of the incoming charged-scalar bosons in the CoM. On the other hand, it is trivial to show that if $a < (2\sqrt{3})/(9s)$ the equation

$$x^3 + \frac{2x^2}{a^2} + \frac{x}{a^4} + \frac{s^2}{a^4} = 0, \tag{13}$$

where $x \equiv \mathbf{k}^2$ has three distinct negative real roots. In this case $\mathcal{M}_{\text{NR}}$ can be rewritten as

$$\begin{aligned}\mathcal{M}_{\text{NR}} = &\frac{8ismQ^2\mathbf{k}\wedge\mathbf{P}}{a^4}\left[\sum_{j=1}^{3}\frac{B_j}{\mathbf{k}^2 - x_j} + \frac{a^4}{s^2\mathbf{k}^2}\right] + \frac{4Q^2m^2}{a^4} \\ &\times\sum_{j=1}^{3}\frac{A_j}{\mathbf{k}^2 - x_j},\end{aligned}$$

where x_1, x_2, and x_3 are the roots of Eq. (13) and $A_1 \equiv \frac{1+a^2x_1}{(x_1-x_2)(x_1-x_3)}$, $A_2 \equiv \frac{1+a^2x_2}{(x_2-x_1)(x_2-x_3)}$, $A_3 \equiv \frac{1+a^2x_3}{(x_3-x_1)(x_3-x_2)}$, $B_1 \equiv \frac{-(1+a^2x_1)^2}{s^2(x_1-x_2)(x_1-x_3)}$, $B_2 \equiv \frac{-(1+a^2x_2)^2}{s^2(x_2-x_1)(x_2-x_3)}$, and $B_3 \equiv \frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)}$.

It follows that the effective nonrelativistic potential can be calculated from the expression

$$\begin{aligned}U_3(r) = &\frac{isQ}{\pi m a^4}\left[\frac{a^4}{s^2}\lim_{\sigma\to 0}\int_0^\infty\frac{(\mathbf{k}\wedge\mathbf{P})J_0(|\mathbf{k}|r)|\mathbf{k}|d|\mathbf{k}|}{\mathbf{k}^2 + \sigma^2}\right. \\ &\left.+\sum_j\int_0^\infty\frac{(\mathbf{k}\wedge\mathbf{P})B_jJ_0(|\mathbf{k}|r)|\mathbf{k}|d|\mathbf{k}|}{\mathbf{k}^2 - x_j}\right] \\ &+\frac{Q}{2\pi a^4}\sum_j\int_0^\infty\frac{A_j}{\mathbf{k}^2 - x_j}J_0(|\mathbf{k}|r)|\mathbf{k}|d|\mathbf{k}|.\end{aligned}$$

Performing the computations, we obtain

$$\begin{aligned}U_3(r) = &-\frac{sQ}{\pi m a^4}\left[\frac{a^4}{s^2}\frac{1}{r^2} + \frac{1}{r}\sum_j B_j\sqrt{|x_j|}K_1(\sqrt{|x_j|}r)\right]\mathbf{L} \\ &+\frac{Q}{2\pi a^4}\left[\sum_j A_jK_0(\sqrt{|x_j|}r)\right],\end{aligned} \tag{14}$$

where $\mathbf{L} \equiv \mathbf{r}\wedge\mathbf{P}$ is the orbital angular momentum.

Using Jackiw's arguments [1], one can show that the topological term alone is unable to bind the charged scalar bosons [2].

We return now to the problem of probing whether "Cooper pairs" exist in the framework of PCS scalar QED. In this case, the radial Schrödinger equation is

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right]\mathcal{R}_{nl} + m[E_{nl} - V_l^{\text{eff}}]\mathcal{R}_{nl} = 0, \tag{15}$$

where

$$V_l^{\rm eff}(r) = -\frac{sQ}{\pi m a^4}\left[\frac{a}{s^2}\frac{1}{r^2} + \frac{1}{r}\sum_j B_j\sqrt{|x_j|}K_1(\sqrt{|x_j|}r)\right]l + \frac{Q^2}{2\pi a^4}\left[\sum_j A_j K_0(\sqrt{|x_j|}r)\right] + \frac{l^2}{mr^2}.$$

Employing the dimensionless parameters $y \equiv sr$, $\alpha \equiv \frac{Q^2}{\pi s}$, $b_j \equiv \frac{s^2}{a^4}B_j$, $X_j \equiv \frac{|x_j|}{s}$, $\beta \equiv \frac{m}{s}$, $a_j \equiv \frac{A_j}{a^4}$, and $\tilde{E}_{nl} \equiv \frac{mE_{nl}}{s^2}$, we can rewrite Eq. (15) as

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy}\right]\mathcal{R}_{nl} + [\tilde{E}_{nl} - \tilde{V}_l^{\rm eff}]\mathcal{R}_{nl} = 0, \qquad (16)$$

with

$$\tilde{V}_l^{\rm eff} \equiv -\frac{l(\alpha - l)}{y^2} + \frac{\alpha\beta}{2}\sum_j a_j K_0(X_j y) - \frac{\alpha l}{y}\sum_j b_j X_j K_1(X_j y).$$

Note that $\tilde{V}_l^{\rm eff}$ behaves as $\frac{l^2}{y^2}$ at the origin and as $\frac{l(l-\alpha)}{y^2}$ asymptotically. On the other hand, the derivative of this potential with respect to y is given by

$$\frac{d}{dy}\tilde{V}_l^{\rm eff} = \frac{2l(\alpha - l)}{y^3} + \alpha\sum_j\left[\frac{2l}{y^2}b_j - \frac{\beta a_j}{2}\right]X_j K_1(X_j y) + \frac{\alpha l}{y}\sum_j b_j X_j^2 K_0(X_j y).$$

In order to find out whether or not boson-boson bound states could be formed, we shall analyze how $\frac{d}{dy}\tilde{V}_l^{\rm eff}$ behaves for small values of the cutoff a. Indeed, only if $a \lll 1$ will the well-recognized properties of QED_3 be preserved. In this limit, we get

$$\frac{d}{dy}\tilde{V}_l^{\rm eff} \sim \frac{2l(\alpha - l)}{y^3} - \left[\frac{2\alpha l}{y^2} + \frac{\alpha\beta}{2}\right]K_1(y) - \frac{\alpha l}{y}K_0(y).$$

We assume from now on $a \lll 1$ and $l > 0$, without any loss of generality. It is trivial to see that, if $l > \alpha$, the potential is strictly decreasing, which precludes the existence of bound states. The remaining possibility is $l < \alpha$. In this interval $\tilde{V}_l^{\rm eff}$ approaches $+\infty$ at the origin and 0^- for $y \to +\infty$, which is indicative of a local minimum. Therefore, the existence of Cooper pairs is subordinated to the conditions $a \lll 1$ and $0 < l < \alpha$.

Of course, it is impossible to solve Eq. (16) analytically; however, it can be solved numerically. To do that, we rewrite beforehand the radial function as $\mathcal{R}_{nl} \equiv u_{nl}/\sqrt{y}$. As a consequence, Eq. (16) takes the form

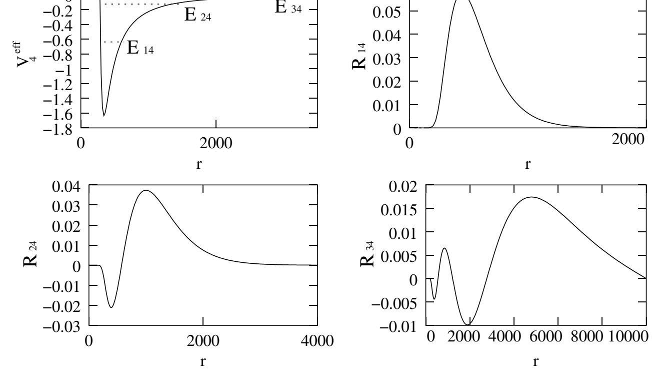

FIG. 3. $V_4^{\rm eff}$ with the lowest three allowed energies and the corresponding energy eigenfunctions. Here $[V_4^{\rm eff}] = \text{eV}$, $[r] = \text{MeV}^{-1}$, $\alpha = 8$, $\beta = 2000$, and $a = 0.009\,52\ \text{MeV}^{-1}$.

$$\left[\frac{d^2}{dy^2} + \frac{1}{4y^2}\right]u_{nl} + [\tilde{E}_{nl} - \tilde{V}_l^{\rm eff}]u_{nl} = 0. \qquad (17)$$

Using the Numerov algorithm [3], we solved Eq. (17) numerically for several values of the parameters α, β, and l, keeping the cutoff a fixed. The latter was chosen equal to $\frac{2}{3}r_e = 9.520\,33 \times 10^{-3}\ \text{MeV}^{-1}$, where r_e is the four-dimensional classical radius of the electron. It is worth mentioning that the anomalous factor of $\frac{4}{3}$ in the inertia related to the Abraham-Lorentz model for the electron does not show up if $a > \frac{1}{2}r_e$ [4,5].

In Fig. 3 we present our numerical results for the potential and the corresponding radial eigenfunctions concerning the first three bound states in the specific case of $l = 4$. The associated energies are $E_{14} = -6.375\,01 \times 10^{-7}$ MeV, $E_{24} = -1.2536 \times 10^{-7}$ MeV, $E_{34} = -5.224\,41 \times 10^{-9}$ MeV.

The graphics shown in Fig. 3 exhibit, in a sense, the generic features of the potential and of the radial eigenfunctions, although they have been composed using particular values of the parameters α, β, l, and a. A detailed study of the modifications of the effective potential induced by radioactive corrections, as well as the corresponding alterations to the eigenvalue structure, will be published elsewhere [2].

A. A. thanks CNPq-Brazil and M. D. thanks CAPES-Brazil for financial support.

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PoS OF SCIENCE

Boson-Boson Bound States in Higher-Derivative Electromagnetism Augmented by a Chern-Simons Term

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The Chern-Simons term alone is unable to form "scalar Cooper pairs". Nonetheless, there exist charged-scalar-boson — charged-scalar-boson bound states in the framework of Maxwell-Chern-Simons theory. Numerical calculations indicate that there are also Cooper pairs within the context of three-dimensional electromagnetism with a cutoff (three-dimensional electromagnetism with higher derivatives) supplemented by a Chern-Simons term. We show that it is always possible to find an interval where the models with higher derivatives have a total number of bound states greater than those with lower derivatives.

Fifth International Conference on Mathematical Methods in Physics — IC2006
April 24-28 2006
Centro Brasilerio de Pesquisas Fisicas, Rio de Janeiro, Brazil

*Speaker.

PoS(IC2006)018

1. Introduction

It is well-known that "scalar Cooper pairs" do not occur in the framework of pure Chern-Simons theory [1]. Nonetheless, charged-scalar-boson — charged-scalar-boson bound states do exist in the framework of Mawell-Chern-Simons theory [2]. Interesting enough, numerical calculations show that there are also scalar Cooper pairs within the context of three-dimensional electromagnetism with a cutoff [3, 4] (three-dimensional electromagnetism with higher derivatives) enlarged by a Chern-Simons term. The electromagnetic part of this model is defined by the Lagrangian

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\lambda F_{\mu\lambda},$$

where $F_{\mu\nu} \equiv \partial_\nu A_\mu - \partial_\mu A_\nu$ is the usual electromagnetic tensor field, and a is a cutoff. This Lagrangian is, of course, gauge and Lorentz invariant; in addition it leads to local field equations which are linear in the field quantities. Moreover, at distances much larger than the cutoff, the fields described by it become essentially equivalent to the Maxwell fields. The classical and quantum formalism for the constrained Hamiltonian related to the singular higher-order Lagrangian in (2 +1) dimensions mentioned above were constructed by Greco *et al.* [5], and afterward the canonical and the path-integral quantization were performed [5, 6]. The latter was accomplished by extending the Faddeev-Senjanovic method [7]. The massive spin-1 part of the electromagnetic field unluckily has negative energy, which implies that three-dimensional electromagnetism with a cutoff is nonunitary due to the presence of a ghost state with negative norm. On the other hand, the breakdown of causality may perhaps only occur on a microscopic scale if the parameter a is small enough to make the massive field only important on distance scales near the Planck length $\sim 10^{-33}$ cm. It is therefore not unlikely that a higher-derivative model can represent an *effective* theory of electromagnetism at more familiar lengths.

It is worth noticing that recently it was shown that in the framework of four-dimensional electromagnetism with a cutoff, the electromagnetic mass of a point charge occurs in the equation of motion in a form consistent with special relativity; furthermore, the exact equation of motion does not exhibit runaway solutions or non-causal behavior, when the cutoff is larger than half of the classical radius of the electron [8, 9].

Our main objective here is to analyze whether or not higher-derivative terms may be used as a mechanism for increasing the number of charged-scalar-boson — charged-scalar-boson bound states (scalar Cooper pairs) related to the usual three-dimensional Maxwell-Chern-Simons theory.

The paper is organized as follows. In section 2, we discuss the occurrence of bound states is both Maxwell-Chern-Simons and higher-derivative models. A rough estimate of the number of bound states in the context of the preceding models is made in section 3. Finally, the concluding remarks are presented in section 4.

2. Scalar Cooper Pairs

To begin with, we derive an expression that allows us to calculate the three-dimensional nonrelativistic potential for the interaction of two identical charged scalar bosons via a photon exchange.

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This expression is used afterward to find the bound states for both Maxwell-Chern-Simons and higher-derivative models.

2.1 Charged-Scalar-Boson — Charged-Scalar-Boson Low energy Potential or the Marriage of Nonrelativistic Quantum Mechanics and Quantum Field Theory in the Nonrelativistic Limit

Nonrelativistic quantum mechanics tells us that in the first Born approximation the cross section for the scattering of two indistinguishable massive particles, in the center-of-mass frame (CoM), is given by

$$\frac{d\sigma}{d\Omega} = \left| \frac{m}{4\pi} \int e^{-i\mathbf{p}'\cdot\,\mathbf{r}} V(r) e^{i\mathbf{p}\cdot\,\mathbf{r}} d^2\,\mathbf{r} \right|^2,$$

where $\mathbf{p}$ $(\mathbf{p}')$ is the initial (final) momentum of one of the particles in the CoM.

In terms of the transfer momentum, $\mathbf{k} \equiv \mathbf{p}' - \mathbf{p}$, it reads

$$\frac{d\sigma}{d\Omega} = \left| \frac{m}{4\pi} \int V(r) e^{i\mathbf{k}\cdot\,\mathbf{r}} d^2\,\mathbf{r} \right|^2. \tag{2.1}$$

On the other hand, from quantum field theory we know that the cross section, in the CoM, for the scattering of two identical charged scalars bosons by an electromagnetic field, can be written as

$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{16\pi E}\,\mathscr{M} \right|^2,$$

where E is the initial energy of one of the bosons and $\mathscr{M}$ is the Feynman amplitude for the process at hand, which in the nonrelativistic limit $(\mathrm{N}.R.)$ reduces to

$$\frac{d\sigma}{d\Omega} = \left| \frac{1}{16\pi m}\,\mathscr{M}_{\mathrm{N.R.}} \right|^2. \tag{2.2}$$

From Eqs. (2.1) and (2.2) we come to the conclusion that the expression that enables us to compute the three-dimensional effective nonrelativistic potential has the form

$$V(r) = \frac{1}{4m^2} \frac{1}{(2\pi)^2} \int d^2\,\mathbf{k}\,\mathscr{M}_{\mathrm{N.R.}}\, e^{-i\mathbf{k}\cdot\,\mathbf{r}}, \tag{2.3}$$

which clearly shows how the potential from quantum mechanics and the Feynman amplitude obtained via quantum field theory are related to each other.

2.2 Bound States for Maxwell-Chern-Simons Model

In the Lorentz gauge the Maxwell-Chern-Simons electromagnetism coupled to a charged-scalar field is described by the Lagrangian

$$\begin{aligned} \mathscr{L} \;=\; & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{s}{2} \varepsilon_{\mu\nu\rho} A^{\mu} \partial^{\nu} A^{\rho} - \frac{1}{2\lambda} (\partial_{\nu} A^{\nu})^2 \\ & + (D_{\mu}\phi)^{*} D^{\mu}\phi - m^2 \phi^{*}\phi, \end{aligned} \tag{2.4}$$

where $D_\mu \equiv \partial_\mu + iqA_\mu$.

It follows that the nonrelativistic potential is given by

$$V(r) = -\frac{Q^2}{m\pi s}\left[\frac{1}{r^2} - \frac{sK_1(sr)}{r}\right]\mathbf{L} + \frac{Q^2}{2\pi s}K_0(sr), \tag{2.5}$$

where $\mathbf{L} \equiv \mathbf{r} \wedge \mathbf{P}$ is the orbital angular momentum, and K is the modified Bessel function.

Let us then investigate whether or not this potential can bind a pair of identical charged-scalar bosons. In this case, the corresponding time-independent Schrödinger equation can be written as

$$\begin{aligned} \mathscr{H}_l \mathscr{R}_{nl} &= -\frac{1}{m}\left(\frac{d^2}{dr^2}\mathscr{R}_{nl} + \frac{1}{r}\frac{d}{dr}\mathscr{R}_{nl}\right) + V_l^{\text{eff}}\mathscr{R}_{nl} \\ &= E_{nl}\mathscr{R}_{nl}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} V_l^{\text{eff}} &\equiv \frac{l^2}{mr^2} + V(r) \\ &= \frac{l^2}{mr^2} - \frac{Q^2}{m\pi s}\left[\frac{1}{r^2} - \frac{sK_1(sr)}{r}\right]\mathbf{L} + \frac{Q^2}{2\pi s}K_0(sr), \end{aligned}$$

where $\mathscr{R}_{nl}$ is the nth normalizable eigenfunction of the radial Hamiltonian $\mathscr{H}_l$ whose corresponding eigenvalue is E_{nl} and V_l^{eff} is the lth partial wave effective potential. Note that V_l^{eff} behaves as $\frac{l^2}{mr^2}$ at the origin and as $\frac{l}{m}\left[l - \frac{Q^2 s}{\pi s}\right]\frac{1}{r}$ asymptotically.

On the other hand,

$$\begin{aligned} \frac{d}{dr}V_l^{\text{eff}} &= -\frac{2l}{m}\left[l - \frac{Q^2}{\pi s}\right]\frac{1}{r^3} - \frac{Q^2 sl}{m\pi}\frac{1}{r}K_0(sr) \\ &\quad - \left[\frac{Q^2 2l}{m\pi r^2} + \frac{Q^2 s}{2\pi}\right]K_1(sr). \end{aligned}$$

Assuming, without any loss of generality, that $l > 0$, it is trivial to see that, if $l > \frac{Q^2}{\pi s}$, the potential is strictly decreasing, which precludes the existence of bound states. The remaining possibility is $l < \frac{Q^2}{\pi s}$. In this interval V_l^{eff} approaches $+\infty$ at the origin and 0^- for $r \to +\infty$, which is indicative of a local minimum. Consequently, the existence of charged-scalar-boson — charged-scalar-boson bound states is subordinated to the condition $0 < l < \frac{Q^2}{\pi s}$.

In terms of the dimensionless parameters $y \equiv sr$, $\alpha \equiv \frac{Q^2}{\pi s}$, $\beta \equiv \frac{m}{s}$, and $\tilde{E}_{nl} \equiv \frac{mE_{nl}}{s^2}$, Eq. (2.6) reads

$$\left[\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy}\right]\mathscr{R}_{nl} + \left[\tilde{E}_{nl} - \tilde{V}_l^{\text{eff}}\right]\mathscr{R}_{nl} = 0, \tag{2.7}$$

with

$$\tilde{V}_l^{\text{eff}} \equiv -\frac{l(\alpha - l)}{y^2} + \frac{\alpha\beta}{2} K_0(y) - \frac{\alpha l}{y} K_1(y).$$

Of course, Eq. (2.7) cannot be solved analytically; nevertheless, it can be solved numerically. To accomplish this, we rewrite the radial function as $\mathscr{R}_{nl} \equiv \frac{u_{nl}}{\sqrt{y}}$. As a consequence, Eq. (2.7) takes the form

$$\left[\frac{d^2}{dy^2} + \frac{1}{4y^2}\right] u_{nl} + \left[\tilde{E}_{nl} - \tilde{V}_l^{\text{eff}}\right] u_{nl}. \tag{2.8}$$

Using the Numerov algorithm [10], we have solved Eq. (2.8) numerically for several values of the parameters α, β, and l. In Fig. 1 we present our numerical results for the potential in the specific case of $l = 6$. The corresponding ground-state energy is -1.68×10^{-8} MeV. The graphic shown in Fig. 1 exhibits the generic features of the potential, although it has been composed using particular values of the parameters α, β, and l.

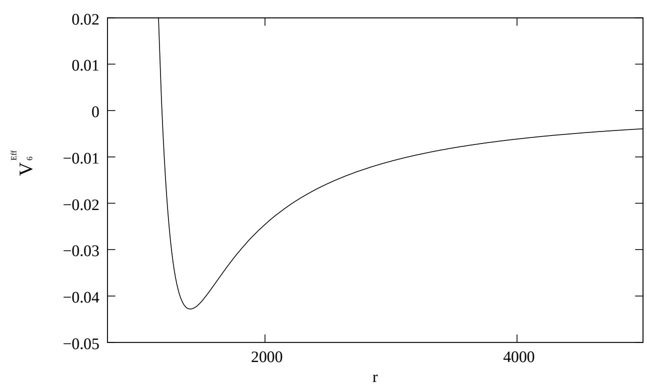

Figure 1: Attractive effective nonrelativistic potential corresponding to the eigenvalue $l = 6$. Here $[V_6^{eff}] = eV$, $[r] = MeV^{-1}$, $\alpha = 7.6$, and $\beta = 7000$.

2.3 Bound States for the Model with Higher Derivatives

2.3.1 The Lagrangian

$$\begin{aligned}\mathscr{L} \;=\; & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{a^2}{2}\partial_\nu F^{\mu\nu}\partial^\alpha F_{\mu\alpha} - \frac{1}{2\lambda}(\partial_\nu A^\nu)^2 \\ & + (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi + \frac{s}{2}\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho ,\end{aligned} \tag{2.9}$$

where $s > 0$ is the topological mass.

2.3.2 The Nonrelativistic Potential

$$V(r) = -\frac{sQ}{\pi m a^4}\left[\frac{a^4}{s^2}\frac{1}{r^2} + \frac{1}{r}\sum_j B_j\sqrt{|x_j|}\,K_1(\sqrt{|x_j|}\,r)\right] l$$
$$+\frac{Q}{2\pi a^4}\left[\sum_j A_j K_0(\sqrt{|x_j|}\,r)\right], \tag{2.10}$$

where

$$A_1 \equiv \frac{1+a^2x_1}{(x_1-x_2)(x_1-x_3)}\,, A_2 \equiv \frac{1+a^2x_2}{(x_2-x_1)(x_2-x_3)}\,, A_3 \equiv \frac{1+a^2x_3}{(x_3-x_1)(x_3-x_2)},$$

$$B_1 \equiv \frac{-(1+a^2x_1)^2}{s^2(x_1-x_2)(x_1-x_3)}\,, B_2 \equiv \frac{-(1+a^2x_2)^2}{s^2(x_2-x_1)(x_2-x_3)}\,, B_3 \equiv \frac{-(1+a^2x_3)^2}{s^2(x_3-x_1)(x_3-x_2)},$$

and x_1, x_2, and x_3 are the roots of the equation

$$x^3 + \frac{2x^2}{a^2} + \frac{x}{a^4} + \frac{s^2}{a^4} = 0. \tag{2.11}$$

We are supposing $a < \frac{2\sqrt{3}}{9s}$, which implies that Eq. (2.11) has three distinct negative real roots.

2.3.3 Bound States

Employing the dimension parameters $y \equiv sr$, $\alpha \equiv \frac{Q^2}{\pi s}$, $\beta \equiv \frac{m}{s}$, $X_j \equiv \frac{|x_j|}{s}$, $b_j \equiv \frac{s^2}{a^4}B_j$, and $a_j \equiv \frac{A_j}{a^4}$, the effective nonrelativistic potential assumes the form

$$\tilde{V}_l^{\text{eff}} \equiv -\frac{l(\alpha-l)}{y^2} + \frac{\alpha\beta}{2}\sum_j a_j K_0(X_j y) - \frac{\alpha l}{y}\sum_j b_j X_j^2 K_1(X_j y). \tag{2.12}$$

However, only if $a \ll 1$ will the well-recognized properties of QED$_3$ be preserved. In this case it can be shown that the existence of scalar Cooper pairs is subordinated to the condition $0 < l < \alpha$, where we have assumed $l > 0$, without any loss of generality.

In Fig. 2 we present our numerical results for the potential and the corresponding radial eigenfunctions concerning the first three bound states in the specific case of $l = 4$. The associated energies are $E_{14} = -6.4\times 10^{-7}$ MeV, $E_{24} = -1.3\times 10^{-7}$ MeV, $E_{34} = -5.2\times 10^{-9}$ MeV. These graphics exhibit, in a sense, the generic features of the potential, although they have been composed using particular values of the parameters $\alpha, \beta,\ l,$ and a.

3. A Rough Estimate of the Number of Bound States

We derive here approximate expressions for the maximal number of bound states related to both higher-derivative and Maxwell-Chern-Simons models.

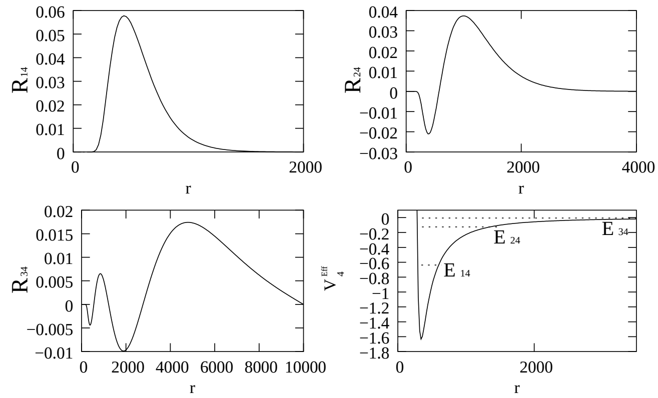

Figure 2: V_4^{eff} with the lowest three allowed energies and the corresponding energy eigenfunctions. Here $\left[V_4^{\text{eff}}\right] = eV$, $[r] = MeV^{-1}$, $\alpha = 8$, $\beta = 2000$, and $a = 0.00952\, MeV^{-1}$.

3.1 Bargmann's Bound in Two Dimensions

It was shown by Bargmann [11, 12] that for a central potential the number of bound states in two dimensions is given by

$$N_0(V) = \sum_0^\infty 2n_l(V), \tag{3.1}$$

where

$$n_l(V) \leq \frac{1}{2l}\int_0^\infty V_-(r) r dr.$$

Here $V_-(r) = sup(V(r), 0)$. However, as far as our potentials are concerned, the analytical computation of $V_-(r)$ is rather involved. But, since $V_-(r) = sup(V(r), 0) \geq -|V(r)|$, we get that $N_0(-|V|) \geq N_0(V)$. Since we only want to make a rough estimate of the total number of bound states we replace the sum in Eq. (3.1) by an integral. Noting that for $l = 0$ the bound is divergent reflecting the fact that a negative potential always has a bound state in two dimensions, we arrive at the expression

$$N(l_{\max}) \leq \left| \int_{l=1}^{l_{max}} \int_0^\infty \frac{V(r) r}{l}\, drdl \right|, \tag{3.2}$$

where $l_{\max}$ is the maximal angular momentum. This inequality, unlike Bargmann's one, is especially suitable for our purposes.

3.2 Finding $l_{\max}$

3.2.1 The Model with Higher Derivatives

In order to find $l_{\max}$ for this model, we have to solve the inequality

$$
\begin{aligned}
& l_{\max}\left(\frac{l_{\max}}{m}-\frac{Q}{\pi m s}\right)\ln(r)|_{r_0}^{\infty} \\
& +\frac{sQl_{\max}}{\pi m a^4}\sum_j B_j K_0(\sqrt{|x_j|}\, r)|_{r_0}^{\infty} \\
& -\frac{Q}{2\pi a^4}\sum_j \frac{A_j}{\sqrt{|x_j|}} r K_1(\sqrt{|x_j|}\, r)|_{r_0}^{\infty} < 0.
\end{aligned} \tag{3.3}
$$

The radial variable was limited to the interval $r_0 < r < \infty$ to avoid the usual infrared divergences.

From (3.3) we obtain the constraints

$$
l_{\max} = \frac{Q}{\pi s}, \tag{3.4}
$$

$$
\frac{Q}{\pi m}\sum_j B_j ln(\sqrt{|x_j|}) < -\frac{1}{2}\sum_j \frac{A_j}{|x_j|}. \tag{3.5}
$$

3.2.2 The Maxwell-Chern-Simons Model

In this case the constraint on $l_{\max}$ is the same as in **3.2.1**. We have also a constraint on m; however, since we want to compare **3.2.1** with **3.2.2**, we shall use here the constraint on the mass found in **3.2.1**.

3.3 An Estimate of the Number of Bound States

3.3.1 The Model with Higher Derivatives

$$
N(l_{\max}) \leq F \,,
$$

where

$$
\begin{aligned}
F \approx & \left| \left(1-\frac{Q}{\pi s}\right)\left[\frac{Q}{\pi s m} ln\left(\frac{r_{max}}{r_0}\right) - \frac{sQ}{\pi m a^4}\sum_j B_j ln(\sqrt{|x_j|})\right] \right. \\
& \left. + \; \frac{Q}{2\pi a^4} ln\left(\frac{Q}{2\pi s}\right)\sum_j \frac{A_j}{|x_j|} \right| .
\end{aligned}
$$

3.3.2 The Maxwell-Chern-Simons Model

$$
\bar{N}(l_{\max}) \leq G \,,
$$

where,

$$
G \approx \left| \frac{Q}{2\pi s^2} ln\left(\frac{Q}{\pi s}\right) - \frac{Q}{\pi m s}\left(\frac{Q}{\pi s}-1\right) ln(s r_{max}) \right| .
$$

Note that in both **3.3.1** and **3.3.2** we have assumed that $r \leq r_{\max}$ in order to avoid that $\ln r$ blows up at infinity.

4. Concluding Remarks

If we choose, for instance, $\frac{Q}{s} = 38MeV^{-1}$, we promptly obtain $l_{\max} \approx 12$. On the other hand, taking $a = 0.02\ MeV^{-1}$, we see that $m \approx 0.18\ MeV$, satisfies the constraint (3.5). For r (in MeV) varying in the range $1 \times 10^{-23} < r < 8 \times 10^{-6}$, we get $N_{\max} \approx 4\overline{N}_{\max}$.

At first sight, it seems that the models with higher-derivatives will have a total number of bound states greater than that of the Chern-Simons model. However, this is a misleading conclusion. Indeed, if we fix $r_{\max}$, say, equal to $5 \times 10^{-3} MeV$ (10 *Ansgtrom*), and vary r_0 keeping the values of $\frac{Q}{s}, l_{\max}$, and m, equal to those of the example above, as it is shown in Fig. 3, we see that if $0 < r_0 < 0.1$, higher derivatives win the game; now, if $r_0 = 0.1$, the game ends in a tie, and , finally, if $r_0 > 0.1$, higher derivatives lose the Cup.

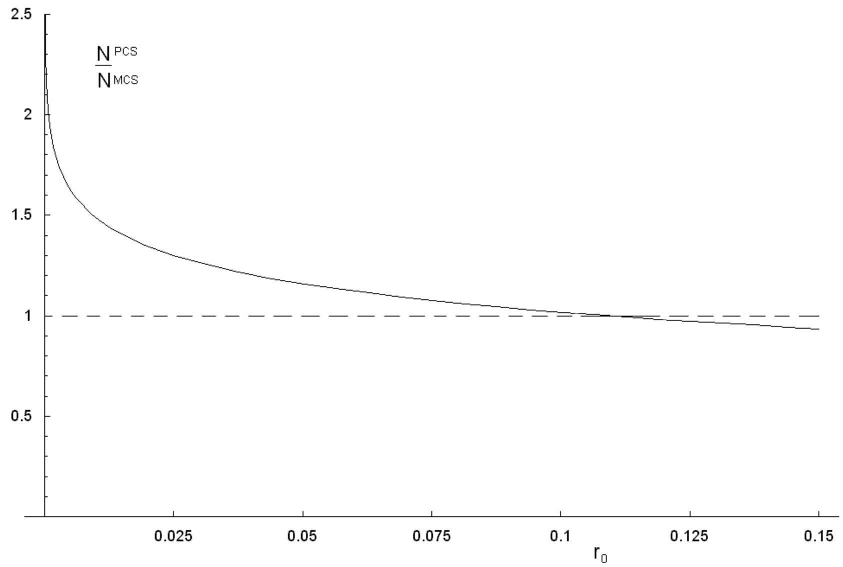

Figure 3: $N_{\max}/\overline{N}_{\max}$ versus r_0. Here $[r_0] = Ansgtrom$, and $r_{\max} = 10\ Ansgtrom$.

In conclusion, we may say that our rough calculations seem to indicate that it is possible to find an interval $I \subseteq [r_0, r_{\max}]$ where higher derivatives win the Cup.

Acknowledgments

A. Accioly is very grateful to FAPERJ-Brazil and CNPq-Brazil for partial support. M. Dias is very indebted to CAPES-Brazil for full support.

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