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Twistor-like constraints for the superparticle and the superstring

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ATA DA DEFESA PÚBLICA DA DISSERTAÇÃO DE MESTRADO DE LUIS ALBERTO YPANAQUE ROCHA, DISCENTE DO PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA, DO INSTITUTO DE FÍSICA TEÓRICA.

Aos 05 dias do mês de abril do ano de 2016, às 14:00 horas, no(a) Instituto de Física Teórica - UNESP, reuniu-se a Comissão Examinadora da Defesa Pública, composta pelos seguintes membros: Prof. Dr. NATHAN JACOB BERKOVITS - Orientador(a) do(a) Instituto de Física Teórica / UNESP, Prof. Dr. ANDREY YURYEVICH MIKHAYLOV do(a) Instituto de Física Teórica / UNESP, Prof. Dr. PEDRO VIEIRA do(a) Perimeter Institute, sob a presidência do primeiro, a fim de proceder a arguição pública da DISSERTAÇÃO DE MESTRADO de LUIS ALBERTO YPANAQUE ROCHA, intitulada **Twistor-like constraints for the superparticle and the superstring**. Após a exposição, o discente foi arguido oralmente pelos membros da Comissão Examinadora, tendo recebido o conceito final: Aprovado. Nada mais havendo, foi lavrada a presente ata, que após lida e aprovada, foi assinada pelos membros da Comissão Examinadora.

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Resumo

Neste trabalho nós estudamos ações para a superpartícula e a supercorda usando vínculos twistóricos. Escrevemos as cargas BRST para a superpartícula e a supercorda em dez dimensões, e a carga BRST para um modelo de partícula em quatro dimensões com a finalidade de fazer algumas comparações. Começamos mostrando como os twistors aparecem como objetos geométricos que parametrizam $SO(4)/U(2)$, e usamos uma generalização deste fato para dez dimensões e assim construir as teorías correspondentes. Então mostramos como obter as ações de espinores puros para a superpartícula e a supercorda em dez dimensões por meio de gauge-twisting e gauge-fixing da teoría inicial.

Palavras Chaves: Superpartícula; Supercorda; Twistors; Espinores Puros; Método BRST

Áreas do conhecimento: Teoría de Campos; Teoría de Cuerdas

Abstract

In this work we study actions for the superparticle and the superstring using twistor-like constraints. We write BRST charges for the superparticle and the superstring in ten dimensions, and the BRST charge for a particle model in four dimensions in order to make some comparisons. We start showing how twistors appear as geometrical objects parametrizing $SO(4)/U(2)$, and use a generalization of this fact to ten dimensions to construct the corresponding theories. We then show how to obtain the pure spinor superparticle and superstring actions in ten dimensions by gauge-twisting and gauge-fixing the initial theory.

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Chapter 1

Introduction

Superstring theory appeared in the 1970's as an attempt to introduce fermionic degrees of freedom into the previously invented bosonic string theory used to describe dual theories of strong interactions. It turned out that this new theory, initially formulated as a string theory with worldsheet supersymmetry (Ramond-Neveu-Schwarz (RNS) formalism) [1–3] and later as a target-space supersymmetric theory (Green-Schwarz (GS) formalism) [4–6], had some unexpected and pleasant consequences related to the fundamental questions physicists were trying to deal with at the time. The fact that general relativity arises as a necessity guided by symmetry-preservation-at-the-quantum-level requirements, and that Yang-Mills gauge theories naturally appear in those formalisms lead scientists into a journey of unification of several objects and ideas. One of them, supersymmetry, is crucial for the consistency of the theory.

The difficulty in the calculation of scattering amplitudes either in the RNS formalism (because of its lack of manifest target-space supersymmetry), or in the GS formalisms (because of the complications brought by choosing a specific light-cone gauge), motivated physicists to search for new formulations that could allow covariant quantization of the theory. The problems that appear when one tries to covariantly quantize the GS superstring can be seen already in the simpler case of the quantization of the superparticle. For example, the Brink-Schwarz superparticle [7] in ten dimensions consists of a system with a set of constraints $d_\alpha \approx 0, \alpha = 1, ...16$, half of them being first class and the other half being second class. The problem arises when one tries to quantize the system, since first and second class constraints cannot be treated in the same way, and there is no procedure to covariantly separate them as they are mixed in the conditions $d_\alpha \approx 0$ in an intricate manner. Several methods have been proposed to overcome this problem: working in harmonic superspace [8–10], writing twistor-like actions [11–16], or introducing other types of auxiliary variables and new constraints in order to obtain a full set of first class constraints of the system [17–21]. This last approach has encountered some difficulties, since the set of constraints turned out to be infinitely reducible [22,23], which makes the construction of the charge in the BRST treatment of the problem, very hard. There have been some attempts to solve this situation, too [24,25].

In 2000, Berkovits introduced a new formalism for the superstring which could be quantized in a manifestly covariant way [26]. The dynamical variables of that formalism were the ten dimensional superspace variables ($x^m, m = 0, 1, ..., 9$, $\theta^\alpha, \alpha = 1, ..., 16$) and a bosonic spinor λ^α satisfying the pure spinor condition, $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$, where $\gamma^m_{\alpha\beta}$ and $\gamma^{\alpha\beta}_m$ are 16×16 symmetric matrices which satisfy $\gamma^{(m}_{\alpha\beta} \gamma^{n)\beta\rho} = 2\eta^{mn} \delta^\rho_\alpha$. A similar treatment was found for the superparticle case [27,28]. The idea was to consider the physical states of the theory as elements of the cohomology of the BRST charge $Q = \lambda^\alpha d_\alpha$ where λ^α is again a pure spinor. Using this formalism it was possible to construct massless vertex operators and to compute scattering amplitudes in a manifestly ten-dimensional super-Poincar covariant way. However, the fact that λ^α enters into the theory as a constrained ghost is somewhat mysterious. There have been attempts to solve this problem considering a BRST double complex for the system [29,30]. It is also noticed that the pure spinor constraints are infinitely reducible which makes the construction of the BRST charge again, too complicated.

This dissertation is based on ref. [31] (see also [32,33]). In the second chapter, we give a brief review of the BRST method, following the discussion given in [34] and writing the essential formulas used later. We discuss how gauge symmetries can be seen as being generated by (first-class) constraints on the phase space of the system, and how can we pass from those local symmetries to a rigid symmetry if we extend the phase space to include ghost variables associated to the constraints. Once the charge of this rigid symmetry is calculated (just the lowest ghost-order terms) we can write gauge-fixed actions. We also generalize these results to the case of systems described by coordinates over a worldsheet or higher dimensional worldvolumes.

In chapter three we start talking about the topological particle and consider a complex split of the Euclidean coordinates. We will see that breaking $SO(4)$ in this specific way and trying to covariantize the theory introducing new appropriate dynamical variables, bring twistors into the discussion. For that reason we also introduce twistors as objects whose components are relevant in parametrizing the massless particle system. We write an extended action and the corresponding BRST charge for the rigid symmetry in the extended phase space.

In chapter four we consider the ten dimensional analogue to the previous system, write a twistor-like action for the superparticle and show how this can be gauge fixed so that we recover the pure spinor superparticle of Berkovits, we show how a ghost twisting is required in order to interpret in a correct way the matter content of the theory. Here it's worthwhile to comment about some differences in our procedure with respect to that of ref. [31]. First, we introduce the local scaling symmetry by means of the appropriate first class constraint and write the BRST charge in the complete extended phase space where we include the ghost associated with this constraint. We will see that new terms appear in the calculation. We also write the gauge-fixed action as the sum of a BRST-invariant kinetic term, and a BRST trivial term where the gauge-fixing fermion appears. Hence, this fermion will be written as a function of the variables of the extended phase space only, and not considering the Lagrange multipliers among its arguments (unless we work in the nonminimal sector). The Lagrange multipliers won't have BRST variations and it will be easy to see how we can fix their values in the gauge-fixed action. Another consequence of this approach is that the ghost-for-ghost ϕ appears as the ghost associated with a first stage generator due to a specific reducibility condition, and we don't need to write gauge-for-gauge transformations involving the Lagrange multipliers, δL^{α}.

Finally, in chapter five we generalize these results to the superstring, write an extended action and obtain the pure spinor superstring by gauge fixing.

Chapter 2

Preliminaries

2.1 First class constraints and gauge invariance

Consider a physical system* described by generalized coordinates $q^i(\tau)$ over a world-line parametrized by τ, where i is an index that includes both physical and possible auxiliary degrees of freedom, the equations of motion for the system being deducible from an action principle. If we assume these equations involve at most second-order τ derivatives, the Lagrangian-form action for the motion of a particle from τ_1 to τ_2 can be written in general as

$$S[q^i(\tau)] = \int_{\tau_1}^{\tau_2} d\tau \; L(q^i, \dot{q}^i) \tag{2.1}$$

The action principle says the variation of the action under arbitrary variations of the coordinates that vanish at τ_1 and τ_2 must vanish.

$$\delta S = \int_{\tau_1}^{\tau_2} d\tau \; \delta q^i \left(\frac{\partial L}{\partial q^i} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}^i} \right) + \left[\delta q^i \frac{\partial L}{\partial \dot{q}^i} \right]_{\tau_1}^{\tau_2} \tag{2.2}$$

Thus, from the principle just mentioned we obtain the equations

$$\frac{\partial L}{\partial q^i} = \frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}^i} \tag{2.3}$$

Also, we define the canonical conjugate momenta $p_i = \dfrac{\partial L}{\partial \dot{q}^i}$

Now, let $q_c^i(\tau)$, which we call a physical motion, be a solution of the equations of motion, and $p_{ci}(\tau)$ the respective momenta. If we perform an arbitrary variation around this solution, $q^i(\tau) = q_c^i(\tau) + \delta q^i(\tau)$, the variation of the action takes the form

$$\delta S[q_c] = \left[\delta q^i(\tau) p_{ci}(\tau) \right]_{\tau_1}^{\tau_2} \tag{2.4}$$

*This section is based on references [34, 35]

An infinitesimal symmetry of the action is defined as a continuous transformation, $\delta q^i(\tau)$, smoothly connected to the identity, such that the Lagrangian transforms to first order into a total τ derivative.

$$\delta L = \delta q^i \frac{\partial L}{\partial q^i} + \delta \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} = \frac{d}{d\tau} B(q^i, \dot{q}^i) \tag{2.5}$$

So $\delta S = [B]_{\tau_1}^{\tau_2}$, and we conclude that if δq^i is a symmetry transformation, then physical motions of the system satisfy

$$\left[\delta q^i p_{ci} - B_c\right]_{\tau_1}^{\tau_2} = 0 \tag{2.6}$$

where $B_c \equiv B(q_c^i, \dot{q}_c^i)$.

This is just a conservation law, since the ends of the τ interval are arbitrary.

$$\frac{d}{d\tau}(\delta q^i p_{ci} - B_c) = 0 \tag{2.7}$$

Suppose that the symmetry transformations are parametrized by k linearly independent parameters ϵ^a, $a = 1, ..., k$.

$$\delta q^i = R^i_{(0)a}\epsilon^a + R^i_{(1)a}\dot{\epsilon}^a + ... + R^i_{(N)a}\frac{d^N\epsilon^a}{d\tau^N} \equiv R^i[\epsilon] \tag{2.8}$$

for some finite N and $R^i_{(n)a} = R^i_{(n)a}(q^i, \dot{q}^i)$. B can be similarly expanded

$$B[\epsilon] = B_{(0)a}\epsilon^a + B_{(1)a}\dot{\epsilon}^a + ... + B_{(N)a}\frac{d^N\epsilon^a}{d\tau^N} \tag{2.9}$$

Defining the quantity

$$G[\epsilon] = p_{ci}R^i_c[\epsilon] - B_c[\epsilon] \tag{2.10}$$

$$= G_{(0)a}\epsilon^a + G_{(1)a}\dot{\epsilon}^a + ... + G_{(N)a}\frac{d^N\epsilon^a}{d\tau^N} \tag{2.11}$$

where $G_{(n)a} = p_{ci}R^i_{(n)ca} - B_{(n)ca}$, our conservation law implies

$$\frac{dG[\epsilon]}{d\tau} = \dot{G}_{(0)a}\epsilon^a + (G_{(0)a} + \dot{G}_{(1)a})\dot{\epsilon}^a + ... + (G_{(N-1)a} + \dot{G}_{(N)a})\frac{d^N\epsilon^a}{d\tau^N} + G_{(N)a}\frac{d^{N+1}\epsilon^a}{d\tau^{N+1}} = 0 \tag{2.12}$$

Now we focus on two especial cases. The first being the one where the symmetry exists only for constant ϵ; so we are talking about a rigid symmetry.

Then $\dfrac{d^n\epsilon^a}{d\tau^n} = 0$ for $n \geq 1$, and the conservation law implies $G_{(0)a}$ is a constant. This result is just Noether's fist theorem which says that rigid symmetries imply constants of motion.

In the second case we consider symmetries with arbitrary τ-dependent parameters $\epsilon^a(\tau)$. For these local (gauge) symmetries we deduce that the coefficients of the expansion of $G[\epsilon]$ satisfy the following equations

$$\dot{G}_{(0)a} = 0, \quad G_{(0)a} + \dot{G}_{(1)a} = 0, \quad G_{(N-1)a} + \dot{G}_{(N)a} = 0, \quad G_{(N)a} = 0 \tag{2.13}$$

which in turn imply $G_{(n)a} = 0$, for $n = 0, ..., N$. Thus, local worldline symmetries give a set of constraints relating coordinates and velocities on physical trajectories (recall that G is defined on-shell). Besides, since $\frac{dG}{d\tau} = 0$, constraints are preserved during the time evolution of the system. This is known as the Noether's second theorem.

Let us consider the action in Hamiltonian form

$$S[q,p] = \int_{\tau_1}^{\tau_2} d\tau \left[p_i \dot{q}^i - H(q,p)\right] \tag{2.14}$$

The action principle takes the form

$$\delta S = \int_{\tau_1}^{\tau_2} d\tau \left[\delta p_i \left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right) - \delta q^i \left(\dot{p}_i + \frac{\partial H}{\partial q^i}\right)\right] + \left[p_i \delta q^i\right]_{\tau_1}^{\tau_2} \tag{2.15}$$

And the resulting equations of motion are

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{2.16}$$

Let $F(q,p)$ and $G(q,p)$ be some functions on the phase space. We define the Poisson bracket between F and G as

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} \tag{2.17}$$

So the brackets between coordinates and momenta read $\{q^i, p_j\} = \delta^i_j$.

Immediately we can write the total τ-derivative of a function $G(q,p)$ as $\dot{G} = \{G, H\}$. Therefore, this function is a constant of motion if and only if $\{G, H\} = 0$ everywhere along the physical trajectory of the system in phase space. Now, denoting by $\eta(\tau)$ some infinitesimal parameter which can only depend on τ, define the following infinitesimal transformation of phase space variables.

$$\delta q^i = \eta\{q^i, G\} = \eta \frac{\partial G}{\partial p_i} \qquad \delta p_i = \eta\{p_i, G\} = -\eta \frac{\partial G}{\partial q^i} \tag{2.18}$$

It's easy to show that under this transformation any function $F(q,p)$ varies as

$$\delta F = \eta\{F, G\} \tag{2.19}$$

So $\delta H = 0$ and the variation of the action is

$$\begin{aligned}\delta S &= \int_{\tau_1}^{\tau_2} d\tau \left[\dot{q}^i \delta p_i - \dot{p}_i \delta q^i + \frac{d}{d\tau}(p_i \delta q^i)\right] \\ &= \left[\eta \frac{\partial G}{\partial p_i} p_i - \eta G\right]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \; \dot{\eta} G\end{aligned} \tag{2.20}$$

Then according to our definition this transformation is a symmetry of the action only if we impose the constraint $G(q,p) = 0$. We have just shown that constraints that are constants of motion generate local symmetries.

Suppose we have a system with some number of such symmetry-generating constraints $G_a(q,p) = 0$, $a = 1, ..., m$. Poisson brackets satisfies the Jacobi identity

$$\{\{A,B\},C\} + \{\{B,C\},A\} + \{\{C,A\},B\} = 0 \tag{2.21}$$

Therefore,

$$\{\{G_a, G_b\}, H\} = 0 \tag{2.22}$$

If $\{G_a\}$ is a complete set of generators, we must have that

$$\{G_a, G_b\} = P_{ab}(G) \tag{2.23}$$

where P_{ab} is a polynomial in the constraints, with constant coefficients, and anti-symmetric in a, b.

The constraint equations $G_a(q,p) = 0$ define a hypersurface in phase space to which all physical trajectories of the system are confined. That means we just need the constraints to commute with the Hamiltonian on this physical hypersurface. Then we obtain a more general structure of the Poisson bracket relations.

We write

$$\{H, G_a\} = W_a(G), \qquad W_a(0) = 0 \tag{2.24}$$

In this hypersurface we must have $P_{ab}(0) = 0$.

Constraints that satisfy the relations

$$\{H, G_a\} = W_a(G), \qquad \{G_a, G_b\} = P_{ab}(G) \tag{2.25}$$

with $W_a(0) = 0$ and $P_{ab}(0) = 0$, are called first class.

We then write the algebra of first class constraints as

$$\{G_a, G_b\} = C_{ab}{}^c G_c \tag{2.26}$$

and the Poisson brackets with the Hamiltonian as

$$\{H_0, G_a\} = V_a{}^b G_b \tag{2.27}$$

where both $C_{ab}{}^c$ and $V_a{}^b$ are functions defined on the phase space.

It turns out that physical trajectories remain the same if we add to the Hamiltonian an arbitrary polynomial in the constraints, with coefficients that may depend on the parameter τ.

$$H' = H + \rho_1^a G_a + \frac{1}{2}\rho_2^{ab} G_a G_b + ... \tag{2.28}$$

We use the symbol $\approx$ to say that two expressions are equal up to terms proportional to the constraints or to higher powers of them. So, a set of constraints is said to be a set of first class constraints if all its elements satisfy

$$\{H, G_a\} \approx 0, \qquad \{G_a, G_b\} \approx 0 \tag{2.29}$$

It could happen that the description of the system requires some other type of constraints called second class. These constraints have weakly non-vanishing Poisson brackets with at least one of the other constraints; hence, they don't generate gauge transformations of the dynamical variables. However, they can be absorbed into a redefinition of the brackets when one introduce Dirac brackets to describe the theory. We will always assume that we start with a complete set of first-class constraints only.

The extended Hamiltonian is defined as

$$H_E = H_0 + l^a G_a \tag{2.30}$$

where H_0 is the Hamiltonian without constraint terms, l^a are arbitrary functions of τ, and a runs over a complete set of first-class constraints.

H_E is, of course, a first-class function. The reason for introducing this H_E is that we want at first to describe a system whose motion is generated by a Hamiltonian which contains all arbitrary gauge functions of the theory. The evolution of gauge-invariant dynamical variables are determined in the same way either by H_0 or H_E. But this is not the case for other kinds of variable. We must use the extended Hamiltonian to account for all possible gauge freedom.

Also, the equations of motion derived by the extended action are not the same as those obtained by the original action principle. We will see that one can go from the set of equations in the extended case to the other set by gauge-fixing, that is, reducing degrees of freedom by fixing some of all of the auxiliary ones. The extended action principle comes from varying the action

$$S_E[q,p,l] = \int d\tau \left[p_i\dot{q}^i - H_E(q,p,l)\right] \tag{2.31}$$

with respect to both (q,p) and the Lagrange multipliers l^a. The variation with respect to l^a will put the constraints equal to zero as auxiliary equations of motion. So the transformation of the Lagrange multipliers must be specified in order to recover a symmetry of the action without imposing $G_a(q,p) = 0$.

The transformation of the original phase space variables is

$$\delta q^i = \eta^a\{q^i, G_a\} = \eta^a \frac{\partial G_a}{\partial p_i} \qquad \delta p_i = \eta^a\{p_i, G_a\} = -\eta^a \frac{\partial G_a}{\partial q^i} \tag{2.32}$$

and the action behaves like

$$\begin{aligned}
\delta S_E &= \left[\eta^a \frac{\partial G_a}{\partial p_i} p_i - \eta^a G_a\right]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \left[\dot{\eta}^a G_a - \delta H_0 - G_a \delta l^a - l^a \delta G_a\right] \\
&= [...]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \left[\dot{\eta}^a G_a - \eta^a\{H_0, G_a\} - G_a\delta l^a - l^a \eta^b \{G_a, G_b\}\right] \\
&= [...]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau G_a \left[\dot{\eta}^a - \eta^b V_b{}^a - \delta l^a + l^c \eta^b C_{bc}{}^a\right]
\end{aligned} \tag{2.33}$$

Then we obtain the appropriate transformation of the Lagrange multipliers:

$$\delta l^a = \dot{\eta}^a + l^c \eta^b C_{bc}{}^a - \eta^b V_b{}^a \tag{2.34}$$

Notice that we are enlarging the phase space by considering the Lagrange multipliers l^a as new coordinates, and by introducing the corresponding momenta which we denote by π_a obtaining,

$$(q^i, p_i, l^a, \pi_a) \tag{2.35}$$

and new Poisson brackets, $\{l^a, \pi_b\} = \delta^a_b$.

One important situation that could appear is that we may have an initial set of first-class constraints G_a which is complete, such that the equations $G_a = 0$ are not entirely independent and, for some reason, we don't want to drop any of the constraints in the set in order to make it independent. In the first case we say that the constraints are reducible or redundant. We write the index that runs over all first-class constraints as $a_0 = 1, 2, ..., m_0$. Then they will satisfy identities like $Z_{a_1}{}^{a_0} G_{a_0} = 0$, with possibly phase-space dependent coefficients $Z_{a_1}{}^{a_0}(q,p)$, where $a_1 = 1, ..., m_1$ and $m_1 < m_0$.

Then, there are local transformations of the gauge parameters that do not change the original gauge transformations.

$$\delta'\eta^{a_0} = \eta^{a_1} Z_{a_1}^{\ \ a_0} \quad \Rightarrow \quad \{\dots , \delta'\eta^{a_0} G_{a_0}\} = 0 \tag{2.36}$$

where η^{a_1} are the local parameters for these new transformations. These transformations may themselves be independent, the total number of independent generators being $m_0' = m_0 - m_1$. In such a case we say that this is a first-stage gauge theory. Nevertheless, it could happen that this set of new transformations is overcomplete, in the sense that too much generators have being removed in the process. Then there will be some identities like $Z_{a_2}^{\ \ a_1} Z_{a_1}^{\ \ a_0} = 0$, where $a_2 = 1, ..., m_2$, and the η^{a_1} parameters have also gauge freedom. So,

$$\delta''\eta^{a_1} = \eta^{a_2} Z_{a_2}^{\ \ a_1} \quad \Rightarrow \quad \delta''(\delta'\eta^{a_0}) = 0 \tag{2.37}$$

η^{a_2} are gauge parameters for the last set of transformations. Again, if these are independent we have $m_0' = m_0 - m_1 + m_2$, and we are in a second-stage theory. Otherwise, there will be more identities that originate new gauge freedom. This process can continue indefinitely or finish at some step. The $Z_{a_k}^{\ \ a_{k-1}}$'s are known as the reducibility functions of order k.

The previous discussion can be generalized to the case of systems parametrized by coordinates of a worldsheet or of higher dimensional worldvolumes [34, 36]. The parameters will be written as (τ, σ^A), $A = 1, ..., \Delta - 1$, where $\Delta \geq 2$. Indeed, one could take the indices characterizing the fields, the constraints, etc, as ranging over both a discrete set and the continuous set of all values taken by the σ-coordinates of the worldvolume, $i \to (i, \sigma^A)$, $a_0 \to (a_0, \sigma^A)$, etc; replacing sums by sums or integrals respectively and replacing ordinary derivatives with respect to dynamical variables by functional derivatives. However, there are some subtleties in this generalization, including the fact that we have explicitly considered only τ-gauge invariance. Hence, we change the approach in order to account for space-time locality in full.

We define a local functional $F[q, p]$ as the integral of functions of the dynamical variables and its derivatives (up to some finite order), over the σ subspace of the worldvolume.

$$F = \int d^{\Delta-1}\sigma \ f(q^i, p_i, \partial_A q^i, \partial_A p_i, ...) \tag{2.38}$$

Now, we say that a gauge theory in Hamiltonian form is local in spacetime if the following conditions hold:

- The extended action is local.

$$S_E[q,p,l] = \int d\tau \; d^{\Delta-1}\sigma \left[p_i \dot{q}^i - \mathscr{H}_0 - l^{a_0}\mathscr{G}_{a_0}\right] \tag{2.39}$$

  Here the Hamiltonian density $\mathscr{H}_0$ and the first-class constraints (at each point σ^A) in the set $\{\mathscr{G}_{a_0}, a_0 = 1, ..., m_0\}$ are functions of q, p and their σ-derivatives up to some finite order. We require the number of coordinates and constraints at each point σ^A to be finite.

- The Poisson brackets at equal τ are local in the sense that expressions like $\{\mathscr{F}(\sigma_1), \mathscr{G}(\sigma_2)\}$ involve the delta function $\delta(\sigma_1 - \sigma_2)$ and its derivatives up to some finite order, but no primitive of $\delta(\sigma_1 - \sigma_2)$. This is called local commutativity and ensures that no nonlocal terms appear as a result of taking the Poisson bracket between any pair of quantities.

- Identities between constraints that express reducibility take local form, so they can be written in general as

$$\zeta_{a_1}{}^{a_0}\mathscr{G}_{a_0} + \zeta_{a_1}{}^{a_0 A}\partial_A \mathscr{G}_{a_0} + ... + \zeta_{a_1}{}^{a_0 A_1 ... A_r}\partial_{A_1} ... \partial_{A_r}\mathscr{G}_{a_0} = 0 \tag{2.40}$$

  where $a_1 = 1, ..., m_1$; $m_1 < m_0$, r is finite and the coefficients may depend on phase space variables and their σ-derivatives up to a finite order.

There are two additional conditions that prove to be useful in the statements introduced later for these theories parametrized with coordinates of worldvolumes. They are called regularity conditions. The first is the required local completeness of the constraint functions. This means that any function of the phase space variables and their σ-derivatives that vanishes on the constraint surface is zero because of the constraint equations only, without having to invoke the boundary conditions (in the space σ). A similar condition is imposed on the reducibility identities. By this, it is meant that there is no local identity among $\mathscr{G}_{a_0}$, $\partial_A \mathscr{G}_{a_0}$, etc, that cannot be derived from the reducibility identities just by algebraic manipulations and differentiations (up to finite order). No boundary conditions must be used in the derivation.

Poisson brackets can be defined in such a way that

$$\{q^i(\sigma_1), p_j(\sigma_2)\} = \delta^i_j \, \delta(\sigma_1 - \sigma_2) \tag{2.41}$$

In general, for any two functions on the phase space, $\mathscr{F}$ and $\mathscr{G}$, we define

$$\{\mathscr{F}(\sigma_1), \mathscr{G}(\sigma_2)\} = \int d^{\Delta-1}\sigma \left[\frac{\delta\mathscr{F}(\sigma_1)}{\delta q^i(\sigma)}\frac{\delta\mathscr{G}(\sigma_2)}{\delta p_i(\sigma)} - \frac{\delta\mathscr{F}(\sigma_1)}{\delta p_i(\sigma)}\frac{\delta\mathscr{G}(\sigma_2)}{\delta q^i(\sigma)}\right] \tag{2.42}$$

where $\dfrac{\delta\mathscr{A}(\sigma)}{\delta\mathscr{B}(\sigma')}$ denotes a functional derivative.

If we have a set of constraints $\mathscr{G}_{a_0}(\sigma)$, they generate gauge transformations given by

$$\delta\mathscr{F}(\sigma) = \left\{\mathscr{F}(\sigma), \int d^{\Delta-1}\sigma'\, \eta^{a_0}(\sigma')\mathscr{G}_{a_0}(\sigma')\right\} \tag{2.43}$$

where $\mathscr{F}(\sigma)$ is a dynamical function and $\eta^{a_0}(\sigma)$ are just gauge parameters.

Under these transformations, the phase space variables change by

$$\begin{aligned}
\delta q^i(\sigma) &= \left\{q^i(\sigma), \int d^{\Delta-1}\sigma'\eta^{a_0}(\sigma')\mathscr{G}_{a_0}(\sigma')\right\} \\
&= \int d^{\Delta-1}\sigma'\eta^{a_0}(\sigma')\int d^{\Delta-1}\sigma_1\delta(\sigma-\sigma_1)\frac{\delta\mathscr{G}_{a_0}(\sigma')}{\delta p_i(\sigma_1)} \\
&= \int d^{\Delta-1}\sigma'\eta^{a_0}(\sigma')\frac{\delta\mathscr{G}_{a_0}(\sigma')}{\delta p_i(\sigma)}
\end{aligned} \tag{2.44}$$

$$\delta p_i(\sigma) = -\int d^{\Delta-1}\sigma'\eta^{a_0}(\sigma')\frac{\delta\mathscr{G}_{a_0}(\sigma')}{\delta q^i(\sigma)} \tag{2.45}$$

The extended action behaves like

$$\begin{aligned}
\delta S_E &= \int d\tau\; d^{\Delta-1}\sigma\left[\delta p_i\frac{\partial q^i}{\partial\tau} + p_i\frac{\partial}{\partial\tau}\delta q^i - \delta\mathscr{H}_0 - (\delta l^{a_0})\mathscr{G}_{a_0} - l^{a_0}\delta\mathscr{G}_{a_0}\right] \\
&= \int d\tau\; d^{\Delta-1}\sigma\left[\frac{\partial}{\partial\tau}\left(p_i\delta q^i\right) - \int d^{\Delta-1}\sigma'\eta^{a_0}(\sigma')\left(\frac{\partial q^i}{\partial\tau}(\sigma)\frac{\delta\mathscr{G}_{a_0}(\sigma')}{\delta q^i(\sigma)} + \frac{\partial p_i}{\partial\tau}(\sigma)\frac{\delta\mathscr{G}_{a_0}(\sigma')}{\delta p_i(\sigma)}\right)\right. \\
&\qquad\left. -\delta\mathscr{H}_0 - (\delta l^{a_0})\mathscr{G}_{a_0} - l^{a_0}\delta\mathscr{G}_{a_0}\right] \\
&= \left[\int d^{\Delta-1}\sigma p_i\delta q^i\right]_{\tau_1}^{\tau_2} - \int d\tau\; d^{\Delta-1}\sigma'\eta^{a_0}(\sigma')\frac{\partial}{\partial\tau}\mathscr{G}_{a_0}(\sigma') \\
&\quad - \int d\tau\; d^{\Delta-1}\sigma\left[\delta\mathscr{H}_0 + (\delta l^{a_0})\mathscr{G}_{a_0} + l^{a_0}\delta\mathscr{G}_{a_0}\right] \\
&= [...]_{\tau_1}^{\tau_2} + \int d\tau\; d^{\Delta-1}\sigma\left[\dot\eta^{a_0}\mathscr{G}_{a_0} - \delta\mathscr{H}_0 - (\delta l^{a_0})\mathscr{G}_{a_0} - l^{a_0}\delta\mathscr{G}_{a_0}\right]
\end{aligned} \tag{2.46}$$

If we assume we can always write

$$\{\mathscr{H}_0(\sigma), \mathscr{G}_{a_0}(\sigma')\} = V_{a_0}{}^{b_0}(\sigma,\sigma')\mathscr{G}_{b_0}(\sigma') \tag{2.47}$$

$$\{\mathscr{G}_{a_0}(\sigma), \mathscr{G}_{b_0}(\sigma')\} = C_{a_0b_0}{}^{c_0}(\sigma,\sigma')\mathscr{G}_{c_0}(\sigma') \tag{2.48}$$

the variation of the action becomes

$$\begin{aligned}
\delta S_E &= [...]_{\tau_1}^{\tau_2} + \int d\tau\; d^{\Delta-1}\sigma\mathscr{G}_{a_0}(\sigma)\left[\,\dot\eta^{a_0}(\sigma) - \delta l^{a_0}(\sigma)\right. \\
&\qquad\left. -\eta^{b_0}(\sigma)\int d^{\Delta-1}\sigma'\left(V_{b_0}{}^{a_0}(\sigma',\sigma) - l^{c_0}(\sigma')C_{b_0c_0}{}^{a_0}(\sigma',\sigma)\right)\right]
\end{aligned} \tag{2.49}$$

Then, the Lagrange multipliers transform as

$$\delta l^{a_0}(\sigma) = \dot{\eta}^{a_0}(\sigma) - \eta^{b_0}(\sigma) \int d^{\Delta-1}\sigma' \left(V_{b_0}{}^{a_0}(\sigma', \sigma) - l^{c_0}(\sigma') C_{b_0 c_0}{}^{a_0}(\sigma', \sigma) \right) \tag{2.50}$$

2.2 Extended phase space and BRST invariance

The main idea of the BRST method is to replace the local symmetry of a gauge theory by a global symmetry, working in an extended phase space which now include extra dynamical variables called ghosts [34, 35, 37].

We start with a system parametrized by a coordinate of a worldline. The first step is to include, for each constraint G_a, one ghost η^a of opposite statistics. Since the initial system was taken as having just bosonic constraints, ghosts will be fermionic. The conjugate momenta of these ghosts are denoted by $\mathscr{P}_a$. The coordinates of the extended phase space is $(q^i, p_i, \eta^a, \mathscr{P}_a)$ and the definition of the Poisson bracket can be generalized such that, between the ghosts we have $\{\mathscr{P}_a, \eta^b\} = \{\eta^b, \mathscr{P}_a\} = -\delta_a{}^b$. We can impose also that these new dynamical variables have vanishing Poisson brackets with the original coordinates and momenta of the theory.

With the appropriate definition, the algebraic properties of the generalized Poisson brackets can be written as

$$\begin{gathered} \{A, B\} = (-1)^{\varepsilon_A \varepsilon_B + 1}\{B, A\} \\ \{A, BC\} = \{A, B\}C + (-1)^{\varepsilon_A \varepsilon_B} B\{A, C\} \\ \{\{A, B\}, C\} + (-1)^{\varepsilon_A(\varepsilon_B + \varepsilon_C)}\{\{B, C\}, A\} + (-1)^{\varepsilon_C(\varepsilon_A + \varepsilon_B)}\{\{C, A\}, B\} = 0 \end{gathered} \tag{2.51}$$

where A, B and C are functions on the extended phase space; and ε_A is the Grassmann parity of the function A, that is, it's equal to 0 (mod2) if A is Grassmann even (bosonic) and to 1 (mod2) if A is Grassmann odd (fermionic). The third property is the generalized Jacobi identity.

It's easy to see that

$$AB = (-1)^{\varepsilon_A \varepsilon_B} BA, \qquad \varepsilon_{AB} = \varepsilon_A + \varepsilon_B \tag{2.52}$$

We introduce the ghost number of a dynamical function F denoted by $gh(F)$ as

$$\begin{gathered} gh(q^i) = gh(p_i) = 0, \quad gh(\eta^a) = 1, \quad gh(\mathscr{P}_a) = -1, \\ gh(AB) = gh(A) + gh(B) \end{gathered} \tag{2.53}$$

It turns out that any gauge symmetry allows for the construction of a global symmetry on the extended phase space just defined. This symmetry is known as the BRST symmetry. BRST transformations are such that its generator Q, $\delta_Q F = \{F, Q\}$, has the following properties:

- Q is real, Grassmann odd $\varepsilon(Q) = 1$, and has ghost number $gh(Q) = 1$.
- Q generates a nilpotent transformation.

  This means that $\delta_Q(\delta_Q F) = \{\{F, Q\}, Q\} = 0$, for any function F on the extended phase space. Using the Jacobi identity,

$$\{\{Q, Q\}, F\} + (-1)^{\varepsilon_F+1}\{\{Q, F\}, Q\} + \{\{F, Q\}, Q\} = 0 \tag{2.54}$$

$$\Rightarrow \quad \{\{Q, Q\}, F\} + 2\{\{F, Q\}, Q\} = 0 \tag{2.55}$$

  Since F is arbitray, Q satisfies $\{Q, Q\} = 0$

- If we express Q as a polynomial in the ghost momenta $\mathscr{P}_a$,

$$Q = \sum_{p\geq 0} Q^{(p)} = Q^{(0)} + \sum_{p\geq 1} U^{a_1\dots a_p}\mathscr{P}_{a_p}\dots\mathscr{P}_{a_1} \tag{2.56}$$

  where $Q^{(0)}$, and $U^{a_1\dots a_p}$ are dynamical functions independent of $\mathscr{P}_{a_1}$ then, at lowest order, Q reproduces the generator of the initial gauge symmetry with the gauge parameters replaced by the ghosts η^a. This means, $Q^{(0)} = \eta^a G_a$.

Thus, we can completely determine the BRST charge from the three conditions above. To have $gh(Q) = 1$ we need the coefficients $U^{a_1\dots a_p}$ to contain $p+1$ ghosts η^a. Then

$$U^{a_1\dots a_p} = \eta^{b_1}\dots\eta^{b_{p+1}} U^{a_1\dots a_p}_{b_{p+1}\dots b_1} \tag{2.57}$$

$U^{a_1\dots a_p}_{b_{p+1}\dots b_1}$ are functions only of the original phase space variables. They are called structure functions of order p. In these terms, the constraints G_a are the structure functions of order zero. Then, nilpotency of δ_Q implies

$$\{Q, Q\} = \{\eta^a G_a + \eta^{b_1}\eta^{b_2} U^a_{b_2 b_1}\mathscr{P}_a + \dots\ ,\ \eta^c G_c + \eta^{d_1}\eta^{d_2} U^c_{d_2 d_1}\mathscr{P}_c + \dots\} = 0 \tag{2.58}$$

Here we will only take care about terms up to second order in η^a.

$$\begin{aligned}
\{Q, Q\} &= \{\eta^a G_a, \eta^c G_c\} + 2\{\eta^a G_a, \eta^b\eta^c U^d_{cb}\mathscr{P}_d\} + \dots \\
&= -\eta^c\{\eta^a G_a, G_c\} + 2\{\eta^a, \eta^b\eta^c U^d_{cb}\mathscr{P}_d\} G_a + \dots \\
&= \eta^c\eta^a\{G_c, G_a\} + 2\eta^b\eta^c U^d_{cb}\{\eta^a, \mathscr{P}_d\} G_a + \dots \\
&= \eta^c\eta^a\left(C_{ca}{}^b + 2U^b_{ca}\right) G_b + O(\eta^3)
\end{aligned} \tag{2.59}$$

So, nilpotency of the transformation implies $U^b_{ca} = -\frac{1}{2}C_{ca}{}^b$, and

$$Q = \eta^a G_a - \frac{1}{2}\eta^b\eta^c C_{cb}{}^a \mathscr{P}_a + ... \tag{2.60}$$

Similarly, we can obtain higher order terms of the BRST charge. We have the BRST transformations

$$\delta_Q q^i = \{q^i, Q\}, \qquad \delta_Q p_i = \{p_i, Q\} \tag{2.61}$$

and $\delta_Q l^a$ is equal in form to the gauge transformation of l^a, with η^a appearing everywhere as a ghost.

In the reducible case we work in a bigger extended space [34,38,39]. We introduce ghosts η^{a_0} and ghost momenta $\mathscr{P}_{a_0}$ for each constraint G_{a_0}. Then, at first stage and lowest order in the ghost momenta, the gauge freedom of the ghost (originally a gauge parameter) can be obtained by introducing a first stage generator $G_{a_1} = Z_{a_1}{}^{a_0}\mathscr{P}_{a_0}$, because

$$\delta'\eta^{a_0} = \{\eta^{a_0}, \eta^{a_1}G_{a_1}\} = -\eta^{a_1}Z_{a_1}{}^{a_0} \tag{2.62}$$

where the sign is a matter of convention. We add new dynamical variables called ghosts for ghosts η^{a_1} and its respective ghost momenta $\mathscr{P}_{a_1}$. These new variables have opposite statistics from the ghosts η^{a_0}, $\mathscr{P}_{a_0}$, and are defined to have ghost number 2 and -2.

The generalized Poisson brackets are defined such that $\{\mathscr{P}_{a_1}, \eta^{b_1}\} = -\delta_{a_1}^{b_1}$.

The same procedure is repeated for higher stages of the theory, changing the statistics of the ghosts in each step.

In general for an rth-stage theory, we define the extended phase space as

$$(q^i, p_i, \eta^{a_0}, \mathscr{P}_{a_0}, \eta^{a_k}, \mathscr{P}_{a_k}) \tag{2.63}$$

where $k = 1, ..., r$ and the ghost number of the variables $(\eta^{a_k}, \mathscr{P}_{a_k})$ is $(k+1, -k-1)$. Here all the gauge symmetry in the initial system becomes a global symmetry in the extended phase space. Again, the BRST generator satisfies the following properties

- Q is real, Grassmann odd, and has ghost number 1.
- $\{Q, Q\} = 0$
- At the lowest order, Q appears as the generator of all the gauge symmetry in the system, with parameters replaced by ghosts, ghosts for ghosts, etc. That is,

$$Q = \eta^{a_0}G_{a_0} + \eta^{a_k}Z_{a_k}{}^{a_{k-1}}\mathscr{P}_{a_{k-1}} + ... \tag{2.64}$$

Let's define BRST-observables as the ghost number zero functions $A(q, p, \eta, \mathscr{P})$ that are BRST invariant,

$$\delta_Q A = \{A, Q\} = 0, \quad gh(A) = 0 \tag{2.65}$$

where two BRST-observables are equivalent if they differ by a trivial BRST-exact term, $A = B + \{C, Q\}$. The physical gauge invariant observable in the original phase space $A_0(q, p)$ will be the term in A that is independent of ghosts and ghost momenta. We say that A is a BRST-invariant extension of A_0. It can be shown that A is always determined by A_0 and the BRST-invariance condition recursively.

For example, take the constraint functions G_{a_0}. Since we assumed they are all first class we know that $\delta G_{a_0} = \eta^{b_0}\{G_{a_0}, G_{b_0}\} \approx 0$ (here η is just a gauge parameter). Then G_{a_0} is gauge invariant for all a_0 and we can define its BRST-invariant extension, which is denoted by $\widetilde{G}_{a_0}$. Notice that, being $Q = \eta^{a_0}G_{a_0} + ...$ (now η is a ghost), we have

$$\{-\mathscr{P}_{a_0}, Q\} = -\{\mathscr{P}_{a_0}, \eta^{b_0}\}G_{b_0} + ... = G_{a_0} + ... \tag{2.66}$$

where in the last expresion the terms we didn't write contain ghosts.

This expression is obviously BRST-invariant and we define $\widetilde{G}_{a_0} = \{-\mathscr{P}_{a_0}, Q\}$. That assignment makes sense because, being $\widetilde{G}_{a_0}$ BRST-exact, it is identified with the zero operator, as it should be for the extension of a constraint.

Recall that $\{H_0, G_a\} \approx 0$. Then, H_0 is gauge invariant and will have a BRST-invariant extension which we simply call H, $\{H, Q\} = 0$. Now, we define the time evolution of a dynamical function F defined on the extended phase space as

$$\frac{dF}{d\tau} = \{F, H\} \tag{2.67}$$

It can be proved that the Poisson bracket structure is preserved when passing from gauge invariant observables in phase space to BRST-invariant ones in the extended phase space. From this results that if a dynamical quantity $F(q, p, \eta, \mathscr{P})$ is a BRST-invariant extension of an observable $F_0(q, p)$, then equation (2.67) is equivalent to the original equation of motion

$$\frac{dF_0}{d\tau} = \{F_0, H_0 + l^{a_0}G_{a_0}\} \approx \{F_0, H_0\} \tag{2.68}$$

If we give an appropriate definition of the time evolution of the ghosts in this extended phase space, then equations of motion are determined with no arbitrariness, in contrast to the case in the original gauge invariant formulation. In order to actually see this, take $\{H, Q\} = 0$. Then,

$$\{H, \eta^{a_0}G_{a_0} + ...\} = 0 \quad \Rightarrow \quad \{H, \eta^{a_0}\}G_{a_0} + \eta^{a_0}\{H, G_{a_0}\} + ... = 0 \tag{2.69}$$

$$\Rightarrow \quad \dot{\eta}^{a_0} = \{\eta^{a_0}, H\} = \eta^{b_0} V_{b_0}{}^{a_0} + ... \tag{2.70}$$

where the dots account for terms that contain at least one ghost momentum.

Similarly we can consider the Poisson bracket of H with the BRST-invariant extension $\widetilde{G}_{a_0}$ of G_{a_0}, taking care of the lowest order contribution which is just the bracket between gauge invariant dynamical functions

$$\{H, \widetilde{G}_{a_0}\} = \{H_0, G_{a_0}\} + ... \quad \Rightarrow \quad -\{H, \{\mathscr{P}_{a_0}, Q\}\} = V_{a_0}{}^{b_0} G_{b_0} + ... \tag{2.71}$$

Now using the generalized Jacobi identity for the variables $\mathscr{P}_{a_0}$, Q and H,

$$\{\{\mathscr{P}_{a_0}, Q\}, H\} - \{\{Q, H\}, \mathscr{P}_{a_0}\} + \{\{H, \mathscr{P}_{a_0}\}, Q\} = 0 \tag{2.72}$$

and $\{H, Q\} = 0$, we get

$$\{\{\mathscr{P}_{a_0}, H\}, Q\} = V_{a_0}{}^{b_0} G_{b_0} + ... \tag{2.73}$$

$$\Rightarrow \quad \{\{\mathscr{P}_{a_0}, H\}, \eta^{b_0} G_{b_0}\} + ... = V_{a_0}{}^{b_0} G_{b_0} + ... \tag{2.74}$$

$$\Rightarrow \quad \{\{\mathscr{P}_{a_0}, H\}, \eta^{b_0}\} G_{b_0} + ... = V_{a_0}{}^{b_0} G_{b_0} + ... \tag{2.75}$$

So we obtain

$$\dot{\mathscr{P}}_{a_0} = \{\mathscr{P}_{a_0}, H\} = -V_{a_0}{}^{b_0} \mathscr{P}_{b_0} + ... \tag{2.76}$$

where the terms we didn't write are at least of second order in the ghost momenta.

From the equations of motion of the ghosts one infers that if they vanish at initial τ_1, then they vanish at all τ, and the equations of motion for the variables in the original phase space become just the ones generated by the gauge invariant Hamiltonian H_0. We can think of this H_0 as the extended Hamiltonian in the fixed gauge $l^{a_0} = 0$,

$$H_0 = H_E(l^{a_0} = 0) \tag{2.77}$$

That's why equations defined by (2.67) are called gauge fixed equations of motion.

We set two BRST-observables equivalent if they differ by a BRST-exact term. Notice that this does not modify the dynamics of the BRST-invariant functions. We could replace the BRST-invariant Hamiltonian H by

$$H \longrightarrow H + \{\chi, Q\} \tag{2.78}$$

where χ is an arbitrary Grassmann (taken to have ghost number -1) odd function on the extended phase space. χ is called the gauge-fixing fermion, since the new

Hamiltonian describe in a generically different way the time evolution of non-BRST-invariant quantities, that means a new gauge-fixing at the level of the original phase space. If we want the extended Hamiltonian to be in the gauge $l^{a_0} = k^{a_0}$, we need to choose an appropriate gauge-fixing fermion, where

$$H + \{\chi, Q\} = H_0 + k^{a_0} G_{a_0} + ... \tag{2.79}$$

An evident choice is $\chi = k^{a_0} \mathscr{P}_{a_0} + ...$.

We can introduce a generalized action principle starting with the gauge-fixed action

$$S_\chi[q, p, \eta, \mathscr{P}] = \int d\tau \left[\dot{q}^i p_i + \sum_{k \geq 0} \dot{\eta}^{a_k} \mathscr{P}_{a_k} - H - \{\chi, Q\} \right] \tag{2.80}$$

and obtain all the equations of motion by varying the dynamical variables in this action.

More general gauge choices can be achieved; for example, to impose some gauge conditions $f^r = 0$, r running over a suitable index set, we could write the gauge-fixing fermion as $\chi = f^r \mathscr{P}_r$, $gh(\chi) = -1$. Considering $\{Q, \mathscr{P}_r\}$ as independent auxiliary fields, their equations of motion automatically impose the desired gauge conditions. These auxiliary fields are known as the Nakanishi-Lautrup fields.

We could also work in the phase space that contains l^a and their momenta π_a as additional coordinates. Since $\dot{l}^a$ is absent in the extended action, there are new constraints $\pi_a = 0$ which are also first class, $\{\pi_a, \pi_b\} = 0$. There will be fermionic ghost pairs associated to these constraints, denoted by $(\rho^a, \bar{C}_a)$ and called antighost momenta and antighost, respectively.

The new non-vanishing Poisson brackets are $\{\rho^a, \bar{C}_b\} = -\delta^a_b$.

We take $\bar{C}_a$ to be real and ρ^a to be imaginary, so that there is a new term in the BRST charge, $-i\rho^a \pi_a$. The original variables $(q^i, p_i, \eta^a, \mathscr{P}_a)$ are said to belong to the minimal sector and the new variables $(l^a, \pi_a, \rho^a, \bar{C}_a)$ to belong to the non-minimal sector. (Here we are restricting ourselves to the case of an irreducible set of constraints, $\{G_a\}$).

The total BRST charge is

$$Q_T = Q_{\text{Min}} + Q_{\text{Nonmin}} \tag{2.81}$$

$$Q_{\text{Min}} = \eta^a G_a + \sum_{p \geq 1} Q^{(p)}, \qquad Q_{\text{Nonmin}} = -i\rho^a \pi_a \tag{2.82}$$

It can be shown that it's equivalent to work with Q_{Min} in the extended phase space or with Q_T in the further extended phase space.

Thus the gauge fixed action is written as

$$S_\chi[q,p,\eta,\mathscr{P},l,\pi,\rho,\bar{C}] = \int d\tau \left[\dot{q}^i p_i + \dot{\eta}^a \mathscr{P}_a + \dot{l}^a \pi_a + \dot{\bar{C}}_a \rho^a - H - \widetilde{l^a G_a} - \{\chi, Q_T\}\right] \tag{2.83}$$

where $\widetilde{l^a G_a}$ is the BRST-invariant extension of $l^a G_a$.

Introducing the nonminimal sector in this way will allows us to choose more general kinds of gauges.

We can also generalize these results to the case of systems described by coordinates over a worldsheet or higher dimensional worldvolumes.

The BRST generator turns out to be a local functional

$$Q = \int d^{\Delta-1}\sigma \ \mathcal{Q}(\sigma) \tag{2.84}$$

Nilpotency of the BRST transformation implies

$$0 = \{Q,Q\} = \int d^{\Delta-1}\sigma \ d^{\Delta-1}\sigma' \ \{\mathcal{Q}(\sigma), \mathcal{Q}(\sigma')\} \tag{2.85}$$

So the lowest order terms must reproduce the gauge generators multiplied by the ghosts associated to the gauge parameters, or must be the first, second, or higher-order-stage generators, and we must require the quantity $\{\mathcal{Q}(\sigma), \mathcal{Q}(\sigma')\}$ to be zero up to derivatives of phase space functions.

If we assume that ${C_{a_0 b_0}}^{c_0}(\sigma', \sigma)$ has the form

$${C_{a_0 b_0}}^{c_0}(\sigma, \sigma') = \delta(\sigma - \sigma') {\mathscr{C}_{a_0 b_0}}^{c_0}(\sigma'), \tag{2.86}$$

then the first terms in $\mathcal{Q}$ turn out to be

$$\mathcal{Q}(\sigma) = \eta^{a_0} \mathscr{G}_{a_0} + \eta^{a_k} {\mathscr{Z}_{a_k}}^{a_{k-1}} \mathscr{P}_{a_{k-1}} - \frac{1}{2} \eta^{b_0} \eta^{c_0} {\mathscr{C}_{c_0 b_0}}^{a_0} \mathscr{P}_{a_0} + ... \tag{2.87}$$

where ${\mathscr{Z}_{a_k}}^{a_{k-1}}$'s are the reducibility functions of order k.

Finally, in order to quantize the system we let the dynamical variables of the extended phase space become linear operators acting on a Hilbert space $\mathscr{H}$ of state functions of the system. Then, our BRST charge becomes also a linear operator.

Upon quantization generalized Poisson brackets become graded commutators, so the nilpotency condition on the BRST transformation translates to

$$\{Q,Q\} = 2Q^2 = 0 \tag{2.88}$$

We also demand Q to be a hermitian operator and define BRST observables as linear operators which satisfy

$$\{A, Q\} = 0 \tag{2.89}$$

We require physical states ψ to be BRST invariant

$$Q\psi = 0 \tag{2.90}$$

We say that states satisfying $Q\psi = 0$ are BRST-closed, while states of the form $\psi = Q\phi$ are BRST-exact, and define physical states to be equivalence classes of BRST-closed states modulo BRST-exact ones (that is, elements of the quantum state cohomology). Similarly, a BRST-closed operator A satisfy $\{A, Q\} = 0$, and a BRST-exact operator B is of the form $B = \{C, Q\}$. We identify closed operators that differ by an exact one.

Chapter 3

The Superparticle in Four Dimensions

3.1 The topological particle

Consider a system described by a point particle moving in a D dimensional spacetime and whose trajectory is a worldline parametrized by τ, having the following gauge invariance

$$\delta x^m(\tau) = \varepsilon^m(\tau), \qquad m = 0, 1, ..., D-1 \tag{3.1}$$

where $\varepsilon^m(\tau)$ are arbitrary functions of τ. This is just invariance under general coordinate transformations and it is easy to see that the simplest Lagrangian that allows for the system to have that property is $L(x^m, \dot{x}^m) = 0$. This gives a vanishing action $S[x^m(\tau)] = 0$ and the system is known as the topological particle [40, 41].

The momenta are

$$P_m = \frac{\partial L}{\partial \dot{x}^m} = 0 \tag{3.2}$$

Then we have the constraints $P_m = 0$ and we inmediately obtain $H_0 = 0$. All constraints are first class since $\{P_m, P_n\} = 0$. Besides, $\{H_0, P_m\} = 0$. The gauge transformations read

$$\delta x^m = \{x^m, \varepsilon^n P_n\} = \varepsilon^n \delta^m_n = \varepsilon^m \tag{3.3}$$

$$\delta P^m = \{P^m, \varepsilon^n P_n\} = 0 \tag{3.4}$$

and the extended Hamiltonian and action,

$$H_E = H_0 + L^m P_m = L^m P_m, \qquad S_E = \int d\tau \, [P_m \dot{x}^m - L^m P_m] \tag{3.5}$$

where L^m are the Lagrange multipliers imposing the constraints as their equations of motion.

Now we define the extended phase space by introducing the ghosts θ^m associated with the constraints, and their corresponding momenta p_m,

$$(x^m, P_m, \theta^m, p_m) \tag{3.6}$$

Thus, in this space there is a global gauge invariance generated by the BRST charge

$$Q = \theta^m P_m \tag{3.7}$$

and, since $H_0 = 0$ is already BRST invariant (it can be set equal to its BRST invariant extension $H = H_0$), we write the gauge-fixed action as

$$S_\chi = \int d\tau \left[P_m \dot{x}^m - p_m \dot{\theta}^m - \{\chi, \theta^m P_m\} \right] \tag{3.8}$$

Consider the previous theory in a $D = 2d$ dimensional Euclidean spacetime.

If v^m is an $SO(2d)$ vector, we can define the complex coordinates

$$z^1 = v^{D-1} + iv^0, \qquad z^{\bar{1}} \equiv \bar{z}^1 = v^{D-1} - iv^0 \tag{3.9}$$

$$z^a = v^{2a-3} + iv^{2a-2}, \qquad z^{\bar{a}} \equiv \bar{z}^a = v^{2a-3} - iv^{2a-2}, \qquad a = 2, ..., d \tag{3.10}$$

and we say that $(z^a, z^{\bar{a}})$, $a = 1, 2, ..., d$ is a complex split of v^m.

Inverting the previous relations,

$$v^{D-1} = \frac{1}{2}(z^1 + z^{\bar{1}}), \qquad v^0 = \frac{1}{2i}(z^1 - z^{\bar{1}}) \tag{3.11}$$

$$v^{2a-3} = \frac{1}{2}(z^a + z^{\bar{a}}), \qquad v^{2a-2} = \frac{1}{2i}(z^a - z^{\bar{a}}), \qquad a = 2, ..., d \tag{3.12}$$

Similarly, for the vector u_m we write its complex split $(y_a, y_{\bar{a}})$ as

$$y_1 = u_{D-1} - iu_0, \qquad y_{\bar{1}} \equiv \bar{y}_1 = u_{D-1} + iu_0 \tag{3.13}$$

$$y_a = u_{2a-3} - iu_{2a-2}, \qquad y_{\bar{a}} \equiv \bar{y}_a = u_{2a-3} + iu_{2a-2}, \qquad a = 2, ..., d \tag{3.14}$$

Evaluating the $SO(2d)$ invariant expression

$$\begin{aligned} v^m u_m &= v^{D-1} u_{D-1} + v^0 u_0 + \sum_{a=2}^{D-2} v^a u_a \\ &= \frac{1}{4} \sum_{a=1}^{d} \left[(z^a + z^{\bar{a}})(y_a + y_{\bar{a}}) - (z^a - z^{\bar{a}})(y_{\bar{a}} - y_a) \right] \\ &= \tfrac{1}{2}(z^a y_a + z^{\bar{a}} y_{\bar{a}}) \end{aligned} \tag{3.15}$$

where in the last line we return to the convention of summing over repeated indices.

Denoting the complex splits of P_m, x^m and L^m as $(P_a, P_{\bar{a}})$, $(x^a, x^{\bar{a}})$ and $(L^a, L^{\bar{a}})$ respectively, we can write the action in a form which is not manifest invariant under

$SO(2d)$, although explicitly invariant under $U(d)$ transformations of the complex variables.

$$S_E = \frac{1}{2}\int d\tau\, [P_a\dot{x}^a + P_{\bar{a}}\dot{x}^{\bar{a}} - L^a P_a - L^{\bar{a}} P_{\bar{a}}] \tag{3.16}$$

In this action we read the constraints $P_a = 0$ and $P_{\bar{a}} = 0$; however, we could just impose one of the set of constraints obtaining

$$S_E = \frac{1}{2}\int d\tau\, [P_a\dot{x}^a + P_{\bar{a}}\dot{x}^{\bar{a}} - L^a P_a] \tag{3.17}$$

The resulting theory is not $SO(2d)$ invariant anymore. This can be expressed also as the fact that we chose a specific definition of the complex coordinates in the Euclidean space. It turns out that there are inequivalent ways to do this assignment, therefore this process does not preserve the initial symmetry of the system.

The extended phase space of this new system is $(x^a, P_a, x^{\bar{a}}, P_{\bar{a}}, \theta^a, p_a)$, the BRST generator, $Q = \theta^a P_a$, and the gauge-fixed action,

$$S_\chi = \frac{1}{2}\int d\tau \left[P_a\dot{x}^a + P_{\bar{a}}\dot{x}^{\bar{a}} - p_a\dot{\theta}^a - \{\chi, \theta^a P_a\}\right] \tag{3.18}$$

3.2 Twistor constraints

In 1967, Penrose introduced a new type of algebra for four-dimensional Minkowski spacetime, whose elements he called twistors [42]. Twistors can be seen as the 'spinors' associated with the conformal group of Minkowski spacetime, in a similar way as ordinary spinors are related to the Lorentz group [44–46].

Here, we will introduce twistors as objects whose components are spinors which determine the classical system of a particle with zero rest mass. This particle carries null momentum P_m, and angular momentum with respect to some choice of origin in spacetime M^{mn}, $m, n = 0, ..., 3$ (M^{mn} is antisymmetric).

In order to describe a zero rest mass, we must impose the following conditions on these quantities,

$$P_m P^m = 0 \tag{3.19}$$

$$S_m = sP_m, \qquad \text{for some } s \tag{3.20}$$

where $S_m = \frac{1}{2}\varepsilon_{mnpq}P^n M^{pq}$ is the Pauli-Lubanski vector (ε_{mnpq} is totally antisymmetric with $\varepsilon_{0123} = 1$).

Equation (3.19) can be solved in terms of a bosonic spinor $\lambda_{\dot\alpha}$ writing*

$$P_m = \frac{1}{2}\lambda_{\dot\alpha}\bar\sigma_m^{\dot\alpha\alpha}\bar\lambda_\alpha = \frac{1}{2}\bar\lambda^\alpha\sigma_{m\alpha\dot\alpha}\lambda^{\dot\alpha}, \qquad \text{where } \bar\lambda_\alpha = (\lambda_{\dot\alpha})^* \tag{3.21}$$

since

$$\begin{aligned}(\lambda\bar\sigma_m\bar\lambda)(\lambda\bar\sigma^m\bar\lambda) &= (\lambda\bar\sigma_m\bar\lambda)(\bar\lambda\sigma^m\lambda) &&= \lambda_{\dot\alpha}\bar\lambda_\alpha\bar\lambda^\beta\lambda^{\dot\beta}\bar\sigma_m^{\dot\alpha\alpha}\sigma^m_{\beta\dot\beta} \\ &= -2\delta^\alpha_\beta\delta^{\dot\alpha}_{\dot\beta}\lambda_{\dot\alpha}\bar\lambda_\alpha\bar\lambda^\beta\lambda^{\dot\beta} &&= -2\lambda_{\dot\alpha}\lambda^{\dot\alpha}\bar\lambda_\alpha\bar\lambda^\alpha \\ &= 0\end{aligned} \tag{3.22}$$

Now, notice that the quantity $\tilde M_{mn} \equiv \frac{1}{2}\varepsilon_{mnpq}M^{pq}$ is antisymmetric. Then we can write this antisymmetric tensor in terms of a symmetric complex bispinor $\mu^{\alpha\beta}$ as

$$\tilde M^{mn} = -\frac{i}{4}\sigma^m_{\alpha\dot\alpha}\sigma^n_{\beta\dot\beta}\mu^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} + \frac{i}{4}\sigma^m_{\beta\dot\beta}\sigma^n_{\alpha\dot\alpha}\bar\mu^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta} \tag{3.23}$$

where $\bar\mu^{\dot\alpha\dot\beta} = (\mu^{\alpha\beta})^*$.

This happens because

$$N^{mn} \equiv -i\sigma^m_{\alpha\dot\alpha}\sigma^n_{\beta\dot\beta}\mu^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} = -i\sigma^m_{\beta\dot\beta}\sigma^n_{\alpha\dot\alpha}\mu^{\beta\alpha}\varepsilon^{\dot\beta\dot\alpha} = i\sigma^n_{\alpha\dot\alpha}\sigma^m_{\beta\dot\beta}\mu^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} = -N^{nm} \tag{3.24}$$

Notice that the second term in (3.23) is necessary in order to keep M^{mn} real.

We can insert equations (3.21) and (3.23) into relation (3.20). First,

$$N_{mn}P^n = -\frac{i}{2}\sigma_{m\alpha\dot\alpha}\sigma_{n\beta\dot\beta}\bar\sigma^{n\dot\gamma\gamma}\mu^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\lambda_{\dot\gamma}\bar\lambda_\gamma = i\sigma_{m\alpha\dot\alpha}\mu^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\lambda_{\dot\beta}\bar\lambda_\beta \tag{3.25}$$

and similarly with $\bar N^{mn} = (N^{mn})^*$.

Then (3.20) implies

$$i\sigma_{m\alpha\dot\alpha}(\mu^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} + \bar\mu^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta})\lambda_{\dot\beta}\bar\lambda_\beta = 2s\bar\lambda^\alpha\sigma_{m\alpha\dot\alpha}\lambda^{\dot\alpha} \tag{3.26}$$

Contracting both sides of this equation with $\bar\sigma^{m\dot\gamma\gamma}\bar\lambda_\gamma$ we get

$$\mu^{\alpha\beta}\bar\lambda_\alpha\bar\lambda_\beta = 0 \tag{3.27}$$

Solving for the bispinor,

$$\mu^{\alpha\beta} = i\mu^{(\alpha}\bar\lambda^{\beta)} \tag{3.28}$$

where μ^α is some other spinor.

*We will use the conventions given in [43], except for the fact that the antichiral spinor $\lambda_{\dot\alpha}$ is not written with a bar over its name, putting the bar on the chiral spinor instead, as in [44].

Thus, our system can be equivalently described by two spinors $\lambda_{\dot\alpha}$ and μ^α. The twistor will be the pair $(\mu^\alpha, \lambda_{\dot\alpha})$. These components satisfy

$$2s = \mu^\alpha \bar\lambda_\alpha + \lambda_{\dot\alpha}\bar\mu^{\dot\alpha} \tag{3.29}$$

For a spin-zero massless particle $\mu^\alpha \bar\lambda_\alpha + \lambda_{\dot\alpha}\bar\mu^{\dot\alpha} = 0$ and we can write

$$\mu_\alpha = x^m \sigma_{m\alpha\dot\alpha}\lambda^{\dot\alpha} \tag{3.30}$$

where x^m is any real vector.

To see how twistors arise in another context we could consider the topological particle discussed at the end of the previous chapter in four dimensional Minkowski spacetime. Since going from one signature to another just needs appropriate Wick rotations, we start with Euclidean spacetime.

The spacetime coordinates are x^m, $m = 0, 1, 2, 3$ and we define the complex split

$$z^1 = x^3 + ix^0, \qquad \bar z^1 = x^3 - ix^0 \tag{3.31}$$

$$z^2 = x^1 + ix^2, \qquad \bar z^2 = x^1 - ix^2 \tag{3.32}$$

The complex split of P_m will be denoted by $(\mathsf{P}_a, \mathsf{P}_{\bar a})$, and the topological particle was given by the extended action

$$S_E = \frac{1}{2}\int d\tau \Big[\mathsf{P}_a \dot z^a + \mathsf{P}_{\bar a}\dot z^{\bar a} - \mathsf{L}^a \mathsf{P}_a\Big] = \int d\tau \Big[P_m \dot x^m - \frac{1}{2}\mathsf{L}^a\mathsf{P}_a\Big] \tag{3.33}$$

So we have a system with two constraints

$$\mathsf{P}_1 = P_3 - iP_0 \approx 0, \qquad \mathsf{P}_2 = P_1 - iP_2 \approx 0 \tag{3.34}$$

that could be written equivalently as

$$\mathsf{G}_1 = \bar{\mathsf{P}}_1 + u\mathsf{P}_2 \approx 0, \qquad \mathsf{G}_2 = \bar{\mathsf{P}}_2 - u\mathsf{P}_1 \approx 0 \tag{3.35}$$

where u is an arbitrary complex quantity.

In terms of the components of the momentum vector, these last constraints read

$$\mathsf{G}_1 = P_3 + iP_0 + u(P_1 - iP_2) \tag{3.36}$$

$$\mathsf{G}_2 = P_1 + iP_2 - u(P_3 - iP_0) \tag{3.37}$$

or in matrix form,

$$\begin{pmatrix} \mathsf{G}_1 \\ \\ \mathsf{G}_2 \end{pmatrix} = \begin{pmatrix} P_3 + iP_0 & P_1 - iP_2 \\ \\ P_1 + iP_2 & -P_3 + iP_0 \end{pmatrix} \begin{pmatrix} 1 \\ \\ u \end{pmatrix} \tag{3.38}$$

Apparently we can write this equation in spinorial form by using four dimensional Euclidean sigma matrices $\sigma^m_{\alpha\dot{\alpha}}$, $m = 0, ..., 3$, $\alpha = 1, 2$, $\dot{\alpha} = \dot{1}, \dot{2}$

$$\sigma^0 = \begin{pmatrix} i & 0 \\ \\ 0 & i \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ \\ 1 & 0 \end{pmatrix} \tag{3.39}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ \\ 0 & -1 \end{pmatrix} \tag{3.40}$$

and defining the spinors (of different chirality),

$$G_\alpha = \begin{pmatrix} \mathsf{G}_1 \\ \\ \mathsf{G}_2 \end{pmatrix}, \qquad \xi^{\dot{\alpha}} = \begin{pmatrix} 1 \\ \\ u \end{pmatrix} \tag{3.41}$$

Then

$$G_\alpha = P_m \sigma^m_{\alpha\dot{\alpha}} \xi^{\dot{\alpha}} = P_m (\sigma^m \xi)_\alpha \tag{3.42}$$

Notice that $\xi^{\dot{\alpha}}$ is fixed but arbitrary. The fact that we are choosing a particular complex split of our variables is reflected here in the choice of u in our spinor, that is, the choice of particular element of the complex projective space $\mathbf{CP^1}$.

The action (3.33) can be written as

$$S_E = \int d\tau \Big[P_m \dot{x}^m - \frac{1}{2} L^\alpha P_m (\sigma^m \xi)_\alpha \Big] \tag{3.43}$$

This action is not $SO(4)$ invariant, but we would obtain a covariant theory by replacing the fixed spinor $\xi^{\dot{\alpha}}$ with a spinor dynamical variable $\lambda^{\dot{\alpha}}$, and adding the corresponding kinetic term to the action. We also include terms corresponding to the opposite chirality $\bar{\lambda}^\alpha = (\lambda^{\dot{\alpha}})^*$

$$S_E = \int d\tau \Big[P_m \dot{x}^m + \bar{\mu}_{\dot{\alpha}} \dot{\lambda}^{\dot{\alpha}} + \mu^\alpha \dot{\bar{\lambda}}_\alpha - \frac{1}{2} L^\alpha P_m (\sigma^m \lambda)_\alpha - \frac{1}{2} \bar{L}_{\dot{\alpha}} P_m (\bar{\sigma}^m \bar{\lambda})^{\dot{\alpha}} \Big] \tag{3.44}$$

where $\bar{\mu}_{\dot{\alpha}}, \mu^\alpha$ are the momenta conjugate to $\lambda^{\dot{\alpha}}$ and $\bar{\lambda}_\alpha$, respectively.

Moreover, since we are parametrizing $\mathbf{CP^1}$, we want the local scaling of the spinor to be a symmetry of the action. This symmetry will be generated by $\mu^\alpha\bar{\lambda}_\alpha - \bar{\mu}_{\dot{\alpha}}\lambda^{\dot{\alpha}}$. So the final extended action of the new theory is

$$S_E = \int d\tau \Big[P_m\dot{x}^m + \bar{\mu}_{\dot{\alpha}}\dot{\lambda}^{\dot{\alpha}} + \mu^\alpha\dot{\bar{\lambda}}_\alpha - \frac{1}{2}L^\alpha P_m(\sigma^m\lambda)_\alpha - \frac{1}{2}\bar{L}_{\dot{\alpha}}P_m(\bar{\sigma}^m\bar{\lambda})^{\dot{\alpha}} - A(\mu^\alpha\bar{\lambda}_\alpha - \bar{\mu}_{\dot{\alpha}}\lambda^{\dot{\alpha}})\Big] \tag{3.45}$$

where A is just a Lagrange multiplier.

So we have a system with the following constraints

$$G = \mu^\alpha\bar{\lambda}_\alpha - \bar{\mu}_{\dot{\alpha}}\lambda^{\dot{\alpha}} \approx 0 \tag{3.46}$$

$$G_\alpha = \frac{1}{2}P_m(\sigma^m\lambda)_\alpha \approx 0 \tag{3.47}$$

$$\bar{G}^{\dot{\alpha}} = \frac{1}{2}P_m(\bar{\sigma}^m\bar{\lambda})^{\dot{\alpha}} \approx 0 \tag{3.48}$$

obeying the algebra

$$\{G, G_\alpha\} = -\frac{1}{2}P_m\sigma^m_{\alpha\dot{\alpha}}\{\bar{\mu}_{\dot{\beta}}, \lambda^{\dot{\alpha}}\}\lambda^{\dot{\beta}} = G_\alpha \tag{3.49}$$

$$\{G, \bar{G}^{\dot{\alpha}}\} = \frac{1}{2}P_m\bar{\sigma}^{m\dot{\alpha}\alpha}\{\mu^\beta, \bar{\lambda}_\alpha\}\bar{\lambda}_\beta = -\bar{G}^{\dot{\alpha}} \tag{3.50}$$

The gauge transformations generated by these constraints are

$$\begin{aligned} \delta x^m &= \Big\{x^m, \zeta G + \theta^\alpha G_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{G}^{\dot{\alpha}}\Big\} \\ &= \frac{1}{2}\Big(\theta\sigma^m\lambda + \bar{\theta}\bar{\sigma}^m\bar{\lambda}\Big) \end{aligned} \tag{3.51}$$

$$\begin{aligned} \delta\bar{\mu}_{\dot{\alpha}} &= -\zeta\bar{\mu}_{\dot{\beta}}\{\bar{\mu}_{\dot{\alpha}}, \lambda^{\dot{\beta}}\} + \frac{1}{2}\theta^\beta P_m\sigma^m_{\beta\dot{\beta}}\{\bar{\mu}_{\dot{\alpha}}, \lambda^{\dot{\beta}}\} \\ &= \zeta\bar{\mu}_{\dot{\alpha}} - \frac{1}{2}P_m(\theta\sigma^m)_{\dot{\alpha}} \end{aligned} \tag{3.52}$$

$$\delta\mu^\alpha = -\zeta\mu^\alpha - \frac{1}{2}P_m(\bar{\theta}\bar{\sigma}^m)^\alpha \tag{3.53}$$

$$\delta P_m = 0, \qquad \delta\lambda^{\dot{\alpha}} = -\zeta\lambda^{\dot{\alpha}} \qquad \delta\bar{\lambda}_\alpha = \zeta\bar{\lambda}_\alpha \tag{3.54}$$

where ζ, θ^α and $\bar{\theta}_{\dot{\alpha}}$ are gauge parameters. We extend the phase space in order to include the ghosts and ghost momenta corresponding to the previous constraints. We call the ghosts the same name as the parameters of the gauge transformation given before ζ, θ^α and $\bar{\theta}_{\dot{\alpha}}$, and the we write the momenta ρ, p_α and $\bar{p}^{\dot{\alpha}}$ respectively. Besides, we take into account the reducibility condition

$$\bar{\lambda}^\alpha G_\alpha - \lambda_{\dot{\alpha}}\bar{G}^{\dot{\alpha}} = 0 \tag{3.55}$$

writing the first-stage generator $\bar{\lambda}^\alpha p_\alpha - \lambda_{\dot{\alpha}}\bar{p}^{\dot{\alpha}}$ and introducing the ghost for ghost ϕ and its associated conjugate momentum β.

The BRST is, according to equations (2.60) and (2.64),

$$\begin{aligned} Q &= \zeta(\mu^\alpha\bar{\lambda}_\alpha - \bar{\mu}_{\dot{\alpha}}\lambda^{\dot{\alpha}}) + \frac{1}{2}\theta^\alpha P_m(\sigma^m\lambda)_\alpha + \frac{1}{2}\bar{\theta}_{\dot{\alpha}}P_m(\bar{\sigma}^m\bar{\lambda})^{\dot{\alpha}} \\ &\quad +\phi(\bar{\lambda}^\alpha p_\alpha - \lambda_{\dot{\alpha}}\bar{p}^{\dot{\alpha}}) + \zeta\theta^\alpha p_\alpha - \zeta\bar{\theta}_{\dot{\alpha}}\bar{p}^{\dot{\alpha}} \end{aligned} \tag{3.56}$$

We perform a canonical transformation of the phase space variables in such a way that the charge transforms as

$$Q_{new} = Q + \{\zeta\rho, Q\} = \phi(\bar{\lambda}^\alpha p_\alpha - \lambda_{\dot{\alpha}}\bar{p}^{\dot{\alpha}}) + \frac{1}{2}\theta^\alpha P_m(\sigma^m\lambda)_\alpha + \frac{1}{2}\bar{\theta}_{\dot{\alpha}}P_m(\bar{\sigma}^m\bar{\lambda})^{\dot{\alpha}} \tag{3.57}$$

and we can gauge-fix the ghost-for-ghost $\phi = 1$. Then the BRST charge in this hypersurface reduces to

$$Q' = \bar{\lambda}^\alpha d_\alpha + \bar{q}_{\dot{\alpha}}\lambda^{\dot{\alpha}} \tag{3.58}$$

where

$$d_\alpha \equiv p_\alpha + \frac{1}{2}P_m(\sigma^m\bar{\theta})_\alpha \tag{3.59}$$

$$\bar{q}_{\dot{\alpha}} \equiv \bar{p}_{\dot{\alpha}} + \frac{1}{2}P_m(\theta\gamma^m)_{\dot{\alpha}} \tag{3.60}$$

The gauge-fixed action is

$$S_E = \int d\tau\Big[P_m\dot{x}^m + \bar{\mu}_{\dot{\alpha}}\dot{\lambda}^{\dot{\alpha}} + \mu^\alpha\dot{\bar{\lambda}}_\alpha + \dot{\theta}^\alpha p_\alpha + \dot{\bar{\theta}}_{\dot{\alpha}}\bar{p}^{\dot{\alpha}}\Big] \tag{3.61}$$

On the other hand, we could solve the constraints and obtain the solution

$$P_m = \bar{\lambda}\sigma_m\lambda, \qquad \bar{\mu}^{\dot{\alpha}} = x^m\bar{\sigma}_m^{\dot{\alpha}\alpha}\bar{\lambda}_\alpha, \qquad \mu_\alpha = x^m\sigma_{m\alpha\dot{\alpha}}\lambda^{\dot{\alpha}} \tag{3.62}$$

The expression for P_m is correct since $\bar{\lambda}\sigma_m\lambda = \lambda\bar{\sigma}_m\bar{\lambda}$ and

$$P_m\sigma^m_{\alpha\dot{\alpha}}\lambda^{\dot{\alpha}} = \lambda_{\dot{\beta}}\bar{\sigma}_m^{\dot{\beta}\beta}\bar{\lambda}_\beta\sigma^m_{\alpha\dot{\alpha}}\lambda^{\dot{\alpha}} = -2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}}\lambda_{\dot{\beta}}\bar{\lambda}_\beta\lambda^{\dot{\alpha}} = -2\bar{\lambda}_\alpha\lambda_{\dot{\alpha}}\lambda^{\dot{\alpha}} = 0 \tag{3.63}$$

and similarly $P_m\bar{\sigma}^{m\dot{\alpha}\alpha}\bar{\lambda}_\alpha = 0$.

The other two relations in (3.62) are more obvious to check.

Then the pair $(\mu^\alpha, \lambda_{\dot{\alpha}})$ can be considered as describing a twistor; we call the constraints $G_\alpha \approx 0$ and $\bar{G}^{\dot{\alpha}} \approx 0$, twistor constraints.

Chapter 4

The Superparticle in Ten Dimensions

4.1 Twistor-like action for the superparticle

The discussion about covariantizing the topological particle can be generalized to higher even dimensions. It is shown in [47] that projective pure spinors (pure spinors up to scale) in $D = 2d$ Euclidean dimensions parametrize complex structures on $\mathbf{R}^{2d}$ (see also [48]). This is equivalent to say that the space of projective pure spinors can be identified with $SO(2d)/U(d)$ and this allows us to rewrite the constraints $\mathsf{P}_a \approx 0$ into an equivalent form

$$P_m(\gamma^m \xi)_\alpha \approx 0 \tag{4.1}$$

In ten dimensions there is no matrix that can raise or lower indices, so we consider a theory with a pure spinor of a given chirality. The pure spinor condition here is

$$\xi^\alpha \gamma^m_{\alpha\beta} \xi^\beta = 0 \tag{4.2}$$

Now, instead of the fixed ξ^α we introduce the pure spinor λ^α as a dynamical variable (along with its conjugate momentum w_α), writing the appropriate kinetic term in the action and taking into account the scaling symmetry associated with that variable.

That is, we start with the extended action

$$S = \int d\tau \left[P_m \dot{x}^m + w_\alpha \dot{\lambda}^\alpha - A w_\alpha \lambda^\alpha - \frac{1}{2} L^\alpha P_m (\gamma^m \lambda)_\alpha - \frac{1}{2} \Lambda_m (\lambda \gamma^m \lambda) \right] \tag{4.3}$$

where A, L^α and Λ_m are Lagrange multipliers which impose the constraints

$$G = w_\alpha \lambda^\alpha \approx 0, \qquad G_\alpha = \frac{1}{2} P_m (\gamma^m \lambda)_\alpha \approx 0, \qquad G^m = \frac{1}{2} \lambda \gamma^m \lambda \approx 0 \tag{4.4}$$

These constraints are first class, satisfying the algebra

$$\{G, G_\alpha\} = -G_\alpha \tag{4.5}$$

$$\{G, G^m\} = -2G^m \tag{4.6}$$

and generate the following gauge transformations, with ζ, θ^α and c_m being the corresponding gauge parameters:

$$\begin{aligned}
\delta x^m &= \{x^m, \zeta G + \theta^\alpha G_\alpha + c_m G^m\} = \left\{x^m, \zeta w_\alpha \lambda^\alpha + \frac{1}{2}\theta^\alpha P_n(\gamma^n\lambda)_\alpha + \frac{1}{2}c_m(\lambda\gamma^m\lambda)\right\} \\
&= \frac{1}{2}\{x^m, P_n\}(\lambda\gamma^n\theta) = \frac{1}{2}\delta^m_n(\lambda\gamma^n\theta) = \frac{1}{2}\lambda\gamma^m\theta
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\delta w_\alpha &= \zeta w_\beta\{w_\alpha, \lambda^\beta\} + \frac{1}{2}P_m\theta^\beta\gamma^n_{\beta\rho}\{w_\alpha, \lambda^\rho\} + \frac{1}{2}c_m\gamma^m_{\beta\rho}\{w_\alpha, \lambda^\beta\lambda^\rho\} \\
&= -\zeta w_\beta\delta^\beta_\alpha - \frac{1}{2}P_m\theta^\beta\gamma^n_{\beta\rho}\delta^\rho_\alpha + \frac{1}{2}c_m\gamma^m_{\beta\rho}(\{w_\alpha, \lambda^\beta\}\lambda^\rho + \lambda^\beta\{w_\alpha, \lambda^\rho\}) \\
&= -\zeta w_\alpha - \frac{1}{2}P_m(\gamma^m\theta)_\alpha - c_m(\gamma^m\lambda)_\alpha
\end{aligned} \tag{4.8}$$

$$\delta P_m = 0, \qquad \delta\lambda^\alpha = \zeta\lambda^\alpha \tag{4.9}$$

The transformation of the Lagrange multipliers are obtain from eq. (2.34)

$$\delta L^\alpha = \dot{\theta}^\alpha + A\theta^\alpha - \zeta L^\alpha, \qquad \delta A = \dot{\zeta}, \qquad \delta\Lambda_m = \dot{c}_m + 2Ac_m - 2\zeta\Lambda_m \tag{4.10}$$

Now we extend the phase space to include the ghost variables ζ, θ^α, c_m and the corresponding ghost momenta ρ, p_α, b^m which are all fermionic.

Notice that the constraints are reducible since they satisfy

$$\lambda^\alpha G_\alpha - P_m G^m = 0 \tag{4.11}$$

and we have a first stage generator $\lambda^\alpha p_\alpha - P_m b^m$; then, we further extend the phase space to include one bosonic ghost-for-ghost coordinate, ϕ and the corresponding bosonic momentum, β.

One problem arises here since there are many more reducibility conditions involving the constraints G^m. It can be seen that many levels of reducibility appear, starting with $(\gamma_m\lambda)_\alpha G^m = 0$, and the construction of the BRST becomes too complicated, appearing ghost-for-ghosts at each level.

The strategy here is to exclude the generator $G^m = \lambda\gamma^m\lambda$ from the extended action, and to consider the dynamical variable λ^α as being a pure spinor from the beginning. There are some subtleties appearing now. The BRST transformation of the momenta w_α is nilpotent up to terms proportional to the transformation generated by G^m, and the Poisson bracket of the charge with itself vanishes up to terms proportional to the pure spinor condition. Besides, the reducibility condition now is just $\lambda^\alpha G_\alpha = 0$. However, now this relation becomes a source of an infinite tower of ghosts-for-ghosts due to the pure spinor condition; reducibility conditions start at the fisrt level with $(\gamma_{mn}\lambda)^\alpha G_\alpha = 0$. We will not take into account the infinite tower of ghosts in the following analysis and see what happens there.

To construct the BRST charge we write the extended phase space,

$$(x^m, P_m, \lambda^\alpha, w_\alpha, \zeta, \rho, \theta^\alpha, p_\alpha, \phi, \beta) \tag{4.12}$$

The ghost numbers of these variables are

$$(0, 0, 0, 0, 1, -1, 1, -1, 2, -2) \tag{4.13}$$

The BRST generator is, according to equations (2.60) and (2.64), $Q = Q_0 + Q'$, where

$$Q_0 = \zeta w_\alpha \lambda^\alpha + \frac{1}{2}\theta^\alpha P_m(\gamma^m\lambda)_\alpha + \phi\lambda^\alpha p_\alpha - \zeta\theta^\alpha p_\alpha \tag{4.14}$$

The terms in Q' are obtained by requiring nilpotency of the BRST transformation.

$$0 = \{Q, Q\} = \{Q_0, Q_0\} + \{Q', Q'\} + 2\{Q_0, Q'\} \tag{4.15}$$

Now,

$$\begin{aligned}
\{Q_0, Q_0\} &= \left\{\zeta w_\alpha \lambda^\alpha \ , \ \theta^\beta P_m(\gamma^m\lambda)_\beta + 2\phi\lambda^\beta p_\beta\right\} \\
&\quad + \left\{\theta^\alpha P_m(\gamma^m\lambda)_\alpha \ , \ \phi\lambda^\beta p_\beta - \zeta\theta^\beta p_\beta\right\} - 2\left\{\phi\lambda^\alpha p_\alpha \ , \ \zeta\theta^\beta p_\beta\right\} \\
&= \theta^\beta P_m \gamma^m_{\beta\delta}\zeta\left\{\lambda^\delta, w_\alpha\right\}\lambda_\alpha - 2\phi\zeta\left\{\lambda^\beta, w_\alpha\right\}\lambda^\alpha p_\beta \\
&\quad + \phi\lambda^\beta\left\{p_\beta, \theta^\alpha\right\}P_m(\gamma^m\lambda)_\alpha - \zeta\theta^\beta\left\{p_\beta, \theta^\alpha\right\}P_m(\gamma^m\lambda)_\alpha \\
&\quad + 2\zeta\phi\lambda^\beta\left\{\theta^\alpha, p_\beta\right\}p_\alpha \\
&= P_m(\theta\gamma^m\lambda)\zeta - 2\phi\zeta\lambda^\beta p_\beta - \phi P_m(\lambda\gamma^m\lambda) \\
&\quad + P_m\zeta(\theta\gamma^m\lambda) - 2\zeta\phi\lambda^\beta p_\beta \\
&= -4\phi\zeta\lambda^\beta p_\beta
\end{aligned} \tag{4.16}$$

So we must find a Q' such that

$$\{Q', Q'\} + 2\{Q_0, Q'\} = 4\phi\zeta\lambda^\beta p_\beta \tag{4.17}$$

Besides, Q' should contain terms with higher order in the ghosts or in the ghost momenta. Since, Q' must have ghost number 1, we have only some possibilities for the remaining terms. The next order terms will contain two ghosts and one antighost, with numbers $(1, 2, -2)$. Then we could have $\zeta\phi\beta$ or $...\theta^\alpha\phi\beta$. Let's try with the first one, i.e. $Q' = n\zeta\phi\beta$, where n is just a real number. In this case $\{Q', Q'\}$ is just zero, and eq. (4.17) becomes

$$n\{\phi\lambda^\alpha p_\alpha, \zeta\phi\beta\} = 2\phi\zeta\lambda^\alpha p_\alpha \tag{4.18}$$

Thus, $n=-2$ and the BRST charge reads

$$Q=\zeta w_\alpha\lambda^\alpha+\frac{1}{2}\theta^\alpha P_m(\gamma^m\lambda)_\alpha+\phi\lambda^\alpha p_\alpha-\zeta\theta^\alpha p_\alpha-2\zeta\phi\beta \tag{4.19}$$

The gauge invariant Hamiltonian is just $H_0=0$, so its BRST invariant extension can be set to zero too, $H=0$. Gauge-fixed actions can be obtained by choosing different gauge-fixing fermions χ, and writing

$$S_\chi=\int d\tau\left[P_m\dot{x}^m+w_\alpha\dot{\lambda}^\alpha+\dot{\zeta}\rho+\dot{\theta}^\alpha p_\alpha+\dot{\phi}\beta-\{\chi,Q\}\right] \tag{4.20}$$

Now we can evaluate de BRST transformation of all the variables in the extended phase space, using the notation $\delta_B F=\{F,Q\}$

$$\delta_B x^m=\frac{1}{2}\lambda\gamma^m\theta, \qquad \delta_B P_m=0 \tag{4.21a}$$

$$\delta_B\lambda^\alpha=\zeta\lambda^\alpha, \qquad \delta_B w_\alpha=-\zeta w_\alpha-\frac{1}{2}P_m(\gamma^m\theta)_\alpha-\phi p_\alpha \tag{4.21b}$$

$$\delta_B\zeta=0, \qquad \delta_B\rho=-w_\alpha\lambda^\alpha+\theta^\alpha p_\alpha+2\phi\beta=M \tag{4.21c}$$

$$\delta_B\theta^\alpha=-\phi\lambda^\alpha+\zeta\theta^\alpha, \qquad \delta_B p_\alpha=-\frac{1}{2}P_m(\gamma^m\lambda)_\alpha-\zeta p_\alpha=N_\alpha \tag{4.21d}$$

$$\delta_B\phi=-2\zeta\phi, \qquad \delta_B\beta=-\lambda^\alpha p_\alpha+2\zeta\beta=V \tag{4.21e}$$

where M, N_α and V are the Nakanishi-Lautrup auxiliary fields.
We could take the Lagrange multipliers A and L^α as additional coordinates of the system. Then we should include the corresponding conjugate momenta, which we will call r and π_α, respectively. Since neither $\dot{A}$ nor $\dot{L}^\alpha$ appear in the previous action, we must impose the constraints $r\approx 0$ and $\pi_\alpha\approx 0$ which are obviously first class.

The further extended action is

$$\begin{aligned} S \;=\; \int d\tau\Big[&P_m\dot{x}^m+w_\alpha\dot{\lambda}^\alpha+\dot{A}r+\dot{L}^\alpha\pi_\alpha \\ &-Aw_\alpha\lambda^\alpha-\frac{1}{2}L^\alpha P_m(\gamma^m\lambda)_\alpha-lr-l^\alpha\pi_\alpha\Big]\end{aligned} \tag{4.22}$$

where l and l^α are Lagrange multipliers which impose the appropriate constraints.

At this point, the extended phase space is given by the coordinates

$$(x^m,P_m,\lambda^\alpha,w_\alpha,A,r,L^\alpha,\pi_\alpha,\zeta,\rho,\theta^\alpha,p_\alpha,s,\bar{D},\rho^\alpha,\bar{C}_\alpha,\phi,\beta) \tag{4.23}$$

where s and $\bar{D}$ are respectively the antighost momentum and the antighost associated with the constraint $r\approx 0$; ρ^α and $\bar{C}_\alpha$ are those associated with $\pi_\alpha\approx 0$. Ghost numbers are

$$(0,0,0,0,0,0,0,0,1,-1,1,-1,1,-1,1,-1,2,-2) \tag{4.24}$$

A gauge-fixed action is obtained by means of some gauge-fixing fermion χ,

$$\begin{aligned} S_\chi &= \int d\tau \Big[P_m \dot{x}^m + w_\alpha \dot{\lambda}^\alpha + \dot{A} r + \dot{L}^\alpha \pi_\alpha + \dot{\zeta}\rho + \dot{\theta}^\alpha p_\alpha \\ &\qquad + \dot{\bar{D}} s + \dot{\bar{C}}_\alpha \rho^\alpha + \dot{\phi}\beta - \widetilde{AG} - \widetilde{L^\alpha G}_\alpha - \{\chi, Q_T\} \Big] \end{aligned} \tag{4.25}$$

where $\widetilde{AG}$ and $\widetilde{L^\alpha G}_\alpha$ are the (total) BRST-invariant extensions of AG and $L^\alpha G_\alpha$ respectively, and Q_T is the total BRST-charge which contains both the minimal and the non-minimal sectors,

$$Q_T = Q - isr - i\rho^\alpha \pi_\alpha \tag{4.26}$$

The total BRST charge is

$$Q_T = \zeta w_\alpha \lambda^\alpha + \frac{1}{2}\theta^\alpha P_m (\gamma^m \lambda)_\alpha + \phi \lambda^\alpha p_\alpha - \zeta \theta^\alpha p_\alpha - 2\zeta\phi\beta - isr - i\rho^\alpha \pi_\alpha \tag{4.27}$$

It's easy to see that $\{Q_T, Q_T\} = 0$.

Again, we can evaluate de BRST transformation of all the variables in the extended phase space, denoting $\delta_T F = \{F, Q_T\}$, we see that the transformation of the previous variables remain unchanged, and we have new transformations for the non-minimal sector.

$$\delta_T A = -is, \qquad \delta_T r = 0 \tag{4.28a}$$

$$\delta_T L^\alpha = -i\rho^\alpha, \qquad \delta_T \pi_\alpha = 0 \tag{4.28b}$$

$$\delta_T s = 0, \qquad \delta_T \bar{D} = ir \tag{4.28c}$$

$$\delta_T \rho^\alpha = 0, \qquad \delta_T \bar{C}_\alpha = i\pi_\alpha \tag{4.28d}$$

$$\tag{4.28e}$$

From the BRST transformation we can read the behaviour of each dynamical variable under local scale transformations, that is, under transformations generated by $G = w_\alpha \lambda^\alpha$. That's because we required the BRST-charge to generate the gauge transformations of the variables at the lowest ghost order. So we just have to look at the terms which contain ζ. Now, the actual form of the transformation must be generated by a bosonic quantity, so we take as a generator ηQ_T, where η is just a fermionic constant. Then, for a dynamical quantity F,

$$\delta F = \{F, \eta Q_T\} = (-1)^{\varepsilon_F} \eta \{F, Q_T\} = (-1)^{\varepsilon_F} \eta \delta_T F \tag{4.29}$$

So, for example, the bosonic variable λ^α behaves like $\delta\lambda^\alpha = \eta\zeta\lambda^\alpha$, while the fermionic θ^α behaves like $\delta\theta^\alpha = -\eta\zeta\theta^\alpha$. From this we easily conclude that under the finite local rescaling $\lambda^\alpha \to \Omega\lambda^\alpha$, the dynamical variable θ^α transforms as $\theta^\alpha \to \Omega^{-1}\theta^\alpha$. Similarly we obtain the scaling of the remaining variables.

So, when λ^α rescales like $\lambda^\alpha \to \Omega(\tau)\lambda^\alpha$, we obtain

$$w_\alpha \to \Omega^{-1} w_\alpha, \qquad \theta^\alpha \to \Omega^{-1}\theta^\alpha, \qquad p_\alpha \to \Omega p_\alpha \tag{4.30}$$

$$\phi \to \Omega^{-2}\phi, \qquad \beta \to \Omega^2 \beta \tag{4.31}$$

We also calculate the BRST-invariant extension of $AG + L^\alpha G_\alpha$. Since the extension of G is $\widetilde{G} = \{-\rho, Q_T\}$ and similarly $\widetilde{G}_\alpha = \{-p_\alpha, Q_T\}$(see eq.(2.66)), we try the expression

$$\begin{aligned} \{-\rho A - p_\alpha L^\alpha, Q_T\} &= \{-\rho, Q_T\}A - \rho\delta_T A + \{-p_\alpha, Q_T\}L^\alpha \\ &\quad - p_\alpha \delta_T L^\alpha \\ &= AG + L^\alpha G_\alpha + \Lambda_m G^m + ... \end{aligned} \tag{4.32}$$

This is the right choice since the dots contain terms of higher order in the ghosts or ghost momenta.

Thus, the gauge-fixed action becomes

$$\begin{aligned} S_\chi \;=\; \int d\tau \Big[& P_m \dot{x}^m + w_\alpha \dot{\lambda}^\alpha + \dot{A} r + \dot{L}^\alpha \pi_\alpha + \dot{\zeta}\rho + \dot{\theta}^\alpha p_\alpha \\ & + \dot{\bar{D}} s + \dot{\bar{C}}_\alpha \rho^\alpha + \dot{\phi}\beta + \{\rho A + p_\alpha L^\alpha, Q_T\} - \{\chi, Q_T\} \Big] \end{aligned} \tag{4.33}$$

4.2 Gauge fixing to the pure spinor superparticle

We can construct a dynamical function whose Poisson bracket with the dynamical variables of the extended phase space gives just the ghost number of these variables. We call it the ghost number function, and denote it by $\mathcal{N}$.

For instance, in the phase space $(x^m, P_m, \lambda^\alpha, w_\alpha, \zeta, \rho, \theta^\alpha, p_\alpha, \phi, \beta)$, we can define

$$\mathcal{N} = \zeta\rho + \theta^\alpha p_\alpha + 2\phi\beta \tag{4.34}$$

and we obtain

$$\{\zeta, \mathcal{N}\} = \zeta, \qquad \{\rho, \mathcal{N}\} = -\rho, \qquad \{\theta^\alpha, \mathcal{N}\} = \theta^\alpha, \qquad \{p_\alpha, \mathcal{N}\} = -p_\alpha \tag{4.35}$$

$$\{\phi, \mathcal{N}\} = 2\phi, \qquad \{\beta, \mathcal{N}\} = -2\beta \tag{4.36}$$

The Poisson bracket of the remaining variables with $\mathcal{N}$ gives just zero.

The condition $gh(Q) = 1$ implies that the BRST-variation of a dynamical variable must have ghost number equal to the ghost number of the variable plus one. That means, for any dynamical variable ψ of ghost number n_ψ,

$$\{\psi, \mathcal{N}\} = n_\psi \psi \quad \Rightarrow \quad \{\{\psi, Q\}, \mathcal{N}\} = (n_\psi + 1)\{\psi, Q\} \tag{4.37}$$

Jacobi identity for the variables ψ, Q and $\mathcal{N}$ reads

$$\{\{\psi, Q\}, \mathcal{N}\} + (-1)^{\varepsilon_\psi}\{\{Q, \mathcal{N}\}, \psi\} + \{\{\mathcal{N}, \psi\}, Q\} = 0 \tag{4.38}$$

$$\Rightarrow \quad \{\psi, Q\} + (-1)^{\varepsilon_\psi}\{\{Q, \mathcal{N}\}, \psi\} = 0 \tag{4.39}$$

$$\Rightarrow \quad \{\psi, Q - \{Q, \mathcal{N}\}\} = 0 \tag{4.40}$$

Since this holds for any ψ, we find

$$\{Q, \mathcal{N}\} = Q, \tag{4.41}$$

in agreement with $gh(Q) = 1$.

Now notice that the bosonic function $\Phi = w_\alpha \lambda^\alpha - \theta^\alpha p_\alpha - 2\phi\beta$ has Poisson bracket with Q equal to

$$\begin{aligned} \{Q, \Phi\} &= \left\{\phi\lambda^\alpha p_\alpha + \frac{1}{2}\theta^\alpha P_m(\gamma^m\lambda)_\alpha + \zeta\Phi \ , \ \Phi\right\} \\ &= \{\phi\lambda^\alpha p_\alpha \ , \ w_\beta\lambda^\beta - \theta^\beta p_\beta - 2\phi\beta\} \\ &\quad + \frac{1}{2}\{\theta^\alpha P_m(\gamma^m\lambda)_\alpha \ , \ w_\beta\lambda^\beta - \theta^\beta p_\beta\} \\ &= 0 \end{aligned} \tag{4.42}$$

$$\Rightarrow \quad \{Q, \Phi\} = 0$$

So, we can defined a new ghost number function, the so called twisted ghost number function, $\mathcal{N}' = \mathcal{N} + \Phi = \zeta\rho + w_\alpha\lambda^\alpha$, that satisfies

$$\{Q, \mathcal{N}'\} = Q \tag{4.43}$$

and assigns the following twisted ghost numbers to the variables of the phase space:

$$\{\zeta, \mathcal{N}'\} = \zeta, \qquad \{\rho, \mathcal{N}'\} = -\rho \tag{4.44}$$

$$\{\lambda^\alpha, \mathcal{N}'\} = \lambda^\alpha, \qquad \{w_\alpha, \mathcal{N}'\} = -w_\alpha \tag{4.45}$$

$$\{\theta^\alpha, \mathcal{N}'\} = 0, \qquad \{p_\alpha, \mathcal{N}'\} = 0 \tag{4.46}$$

$$\{\phi, \mathcal{N}'\} = 0, \qquad \{\beta, \mathcal{N}'\} = 0 \tag{4.47}$$

$$\{x^m, \mathcal{N}'\} = 0, \qquad \{P_m, \mathcal{N}'\} = 0 \tag{4.48}$$

So we see that after this twist, the matter and ghost content of the theory just exchange roles, except for ζ, ρ, and the usual space coordinates and momenta.

Now we make a change in the BRST charge induce by a canonical transformation of the extended phase space variables generated by $\mathcal{C} = \zeta\rho$.

$$Q_{new} = Q + \{\zeta\rho, Q\} = Q + \{\zeta\rho, \zeta\Phi\} = \phi\lambda^\alpha p_\alpha + \frac{1}{2}\theta^\alpha P_m(\gamma^m\lambda)_\alpha \tag{4.49}$$

And we write the gauge-fixed action

$$S_\chi = \int d\tau \left[P_m\dot{x}^m + w_\alpha\dot{\lambda}^\alpha + \dot{\zeta}\rho + \dot{\theta}^\alpha p_\alpha + \dot{\phi}\beta - \{\chi, Q_{new}\}\right] \tag{4.50}$$

The BRST transformation of the variables is

$$\delta_B x^m = \frac{1}{2}\lambda\gamma^m\theta, \qquad \delta_B P_m = 0 \tag{4.51a}$$

$$\delta_B\lambda^\alpha = 0, \qquad \delta_B w_\alpha = -\frac{1}{2}P_m(\gamma^m\theta)_\alpha - \phi p_\alpha \tag{4.51b}$$

$$\delta_B\zeta = 0, \qquad \delta_B\rho = -w_\alpha\lambda^\alpha + \theta^\alpha p_\alpha + 2\phi\beta = M \tag{4.51c}$$

$$\delta_B\theta^\alpha = -\phi\lambda^\alpha, \qquad \delta_B p_\alpha = -\frac{1}{2}P_m(\gamma^m\lambda)_\alpha = N_\alpha \tag{4.51d}$$

$$\delta_B\phi = 0, \qquad \delta_B\beta = -\lambda^\alpha p_\alpha = V \tag{4.51e}$$

Now we gauge fix $\phi = 1$ by using the gauge fixing fermion $\chi = (\phi - 1)\rho$. Then $\{\chi, Q\} = (\phi - 1)M$ and the gauge-fixed action takes the form

$$S_{fixed} = \int d\tau \left[P_m\dot{x}^m + w_\alpha\dot{\lambda}^\alpha + \dot{\zeta}\rho + \dot{\theta}^\alpha p_\alpha + \dot{\phi}\beta - (\phi - 1)M\right] \tag{4.52}$$

Solving the equation of motion for ρ and for the Nakanishi-Lautrup auxiliary field M, we get

$$S'_{fixed(\phi=1)} = \int d\tau \left[P_m\dot{x}^m + w_\alpha\dot{\lambda}^\alpha - p_\alpha\dot{\theta}^\alpha\right] \tag{4.53}$$

The BRST transformations in the hypersurface $\phi = 1$ are

$$\delta x^m = \frac{1}{2}\lambda\gamma^m\theta, \qquad \delta P_m = 0 \tag{4.54a}$$

$$\delta\lambda^\alpha = 0, \qquad \delta w_\alpha = -p_\alpha - \frac{1}{2}P_m(\gamma^m\theta)_\alpha \tag{4.54b}$$

$$\delta\theta^\alpha = -\lambda^\alpha, \qquad \delta p_\alpha = -\frac{1}{2}P_m(\gamma^m\lambda)_\alpha \tag{4.54c}$$

(4.54d)

These transformations are nilpotent (except for w_α as already explained in the previous section) and are generated by the ghost number 1 charge

$$Q_{pure\ spinor} = \lambda^\alpha p_\alpha + \frac{1}{2}\theta^\alpha P_m (\gamma^m \lambda)_\alpha \tag{4.55}$$

This is the BRST charge for the pure spinor superparticle introduced by Berkovits in [27].

Chapter 5

The Superstring

5.1 Twistor-like action for the superstring

Now we try to generalize the previous discussion to the case of the superstring. We consider as dynamical variables of the system the spacetime coordinates $x^m(\tau,\sigma)$ and their conjugate momenta $P_m(\tau,\sigma)$, as well as the pair of pure spinors, λ^α, $\hat{\lambda}^{\hat{\alpha}}$ and the their momenta w_α, $\hat{w}_{\hat{\alpha}}$. The τ-dependence of these variables will be implicitly assumed always.

The natural generalization of the twistor-like constraints of the superparticle is given by [31],

$$G_\alpha = \frac{1}{2}(P^m + \partial_\sigma x^m)(\gamma_m\lambda)_\alpha \approx 0 \tag{5.1}$$

$$\hat{G}_{\hat{\alpha}} = \frac{1}{2}(P^m - \partial_\sigma x^m)(\gamma_m\hat{\lambda})_{\hat{\alpha}} \approx 0 \tag{5.2}$$

We also include de constraints

$$G = w_\alpha\lambda^\alpha \approx 0, \qquad \hat{G} = \hat{w}_{\hat{\alpha}}\hat{\lambda}^{\hat{\alpha}} \approx 0 \tag{5.3}$$

We evaluate the Poisson brackets of these constraints and obtain

$$\{G(\sigma_1), G_\alpha(\sigma_2)\} = -\delta(\sigma_1-\sigma_2)G_\alpha(\sigma_2) \tag{5.4}$$

$$\{\hat{G}(\sigma_1), \hat{G}_{\hat{\alpha}}(\sigma_2)\} = -\delta(\sigma_1-\sigma_2)\hat{G}_{\hat{\alpha}}(\sigma_2) \tag{5.5}$$

as well as

$$\begin{aligned}
\{G_\alpha(\sigma_1), G_\beta(\sigma_2)\} &= \frac{1}{4}\left(\{P^m(\sigma_1), \partial_{\sigma_2}x_n(\sigma_2)\}\right. \\
&\qquad \left. + \{\partial_{\sigma_1}x^m(\sigma_1), P_n(\sigma_2)\}\right)(\gamma_m\lambda)_\alpha(\sigma_1)(\gamma^n\lambda)_\beta(\sigma_2) \\
&= -\frac{1}{4}\left(\partial_{\sigma_2}\delta(\sigma_1-\sigma_2) - \partial_{\sigma_1}\delta(\sigma_1-\sigma_2)\right)(\gamma_m\lambda)_\alpha(\sigma_1)(\gamma^m\lambda)_\beta(\sigma_2) \\
&= \frac{1}{4}\delta(\sigma_1-\sigma_2)(\gamma_m\lambda)_{[\alpha}(\gamma^m\partial_{\sigma_1}\lambda)_{\beta]}(\sigma_1)
\end{aligned} \tag{5.6}$$

where we have considered that eventually we will integrate over the σ-variables, so the previous result holds up to σ-derivatives.

Similarly,

$$\{\hat{G}_{\hat{\alpha}}(\sigma_1), \hat{G}_{\hat{\beta}}(\sigma_2)\} = -\frac{1}{4}\delta(\sigma_1 - \sigma_2)(\gamma_m\hat{\lambda})_{[\hat{\alpha}}(\gamma^m\partial_{\sigma_1}\hat{\lambda})_{\hat{\beta}]}(\sigma_1) \tag{5.7}$$

$$\{G_\alpha(\sigma_1), \hat{G}_{\hat{\beta}}(\sigma_2)\} = 0 \tag{5.8}$$

If these were the only constraints of the system, we notice that G_α and $\hat{G}_{\hat{\alpha}}$ would be second class constraints. In order to describe a theory with first class constraints only, we must introduce some more constraints to the system.

$$F^\alpha = \partial_\sigma\lambda^\alpha \approx 0, \qquad \hat{F}^{\hat{\alpha}} = \partial_\sigma\hat{\lambda}^{\hat{\alpha}} \approx 0 \tag{5.9}$$

Thus, the constraints $G, G_\alpha, F^\alpha, \hat{G}, \hat{G}_{\hat{\alpha}}$ and $\hat{F}^{\hat{\alpha}}$ are first class and satisfy the Poisson algebra

$$\{G(\sigma_1), G_\alpha(\sigma_2)\} = -\delta(\sigma_1 - \sigma_2)G_\alpha(\sigma_1) \tag{5.10}$$

$$\{\hat{G}(\sigma_1), \hat{G}_{\hat{\alpha}}(\sigma_2)\} = -\delta(\sigma_1 - \sigma_2)\hat{G}_{\hat{\alpha}}(\sigma_1) \tag{5.11}$$

$$\{G(\sigma_1), F^\alpha(\sigma_2)\} = -\delta(\sigma_1 - \sigma_2)F^\alpha(\sigma_1) \tag{5.12}$$

$$\{\hat{G}(\sigma_1), \hat{F}^{\hat{\alpha}}(\sigma_2)\} = -\delta(\sigma_1 - \sigma_2)\hat{F}^{\hat{\alpha}}(\sigma_1) \tag{5.13}$$

$$\{G_\alpha(\sigma_1), G_\beta(\sigma_2)\} = \frac{1}{4}\delta(\sigma_1 - \sigma_2)(\gamma_m\lambda)_{[\alpha}\gamma^m_{\beta]\delta}F^\delta(\sigma_1) \tag{5.14}$$

$$\{\hat{G}_{\hat{\alpha}}(\sigma_1), \hat{G}_{\hat{\beta}}(\sigma_2)\} = -\frac{1}{4}\delta(\sigma_1 - \sigma_2)(\gamma_m\hat{\lambda})_{[\hat{\alpha}}\gamma^m_{\hat{\beta}]\hat{\delta}}F^{\hat{\delta}}(\sigma_1) \tag{5.15}$$

First class constraints generate gauge transformations.
Let us denote

$$\delta X(\sigma) = \left\{X(\sigma)\ ,\ \int d\sigma' \left(\zeta G + \theta^\alpha G_\alpha + \eta_\alpha F^\alpha + \hat{\zeta}\hat{G} + \hat{\theta}^{\hat{\alpha}}\hat{G}_{\hat{\alpha}} + \hat{\eta}_{\hat{\alpha}}\hat{F}^{\hat{\alpha}}\right)(\sigma')\right\} \tag{5.16}$$

where $\zeta, \theta^\alpha, \eta_\alpha, \hat{\zeta}, \hat{\theta}^{\hat{\alpha}}, \hat{\eta}_{\hat{\alpha}}$ are the corresponding gauge parameters.

Then

$$\begin{aligned}
\delta x^m &= \frac{1}{2}\int d\sigma'\{x^m(\sigma), P_n(\sigma')\}\left(\theta\gamma^n\lambda + \hat{\theta}\gamma^n\hat{\lambda}\right)(\sigma') \\
&= \frac{1}{2}\left(\lambda\gamma^m\theta + \hat{\lambda}\gamma^m\hat{\theta}\right) \\
\delta P_m &= \frac{1}{2}\int d\sigma'\{P_m(\sigma), \partial_{\sigma'}x^n(\sigma')\}\left(\theta\gamma^n\lambda - \hat{\theta}\gamma^n\hat{\lambda}\right)(\sigma') \\
&= \frac{1}{2}\partial_\sigma\left(\lambda\gamma^m\theta - \hat{\lambda}\gamma^m\hat{\theta}\right)
\end{aligned} \tag{5.17}$$

$$\delta\lambda^\alpha = \int d\sigma'\zeta(\sigma')\{\lambda^\alpha(\sigma), w_\beta(\sigma')\}\lambda^\beta(\sigma') = \zeta\lambda^\alpha, \qquad \delta\hat{\lambda}^{\hat{\alpha}} = \hat{\zeta}\hat{\lambda}^{\hat{\alpha}} \tag{5.18}$$

Similarly we obtain

$$\delta w_\alpha = -\zeta w_\alpha - \frac{1}{2}\left(P_m + \partial_\sigma x_m\right)(\gamma^m\theta)_\alpha + \partial_\sigma\eta_\alpha \tag{5.19}$$

$$\delta\hat{w}_{\hat{\alpha}} = -\hat{\zeta}\hat{w}_{\hat{\alpha}} - \frac{1}{2}\left(P_m - \partial_\sigma x_m\right)(\gamma^m\hat{\theta})_{\hat{\alpha}} + \partial_\sigma\hat{\eta}_{\hat{\alpha}} \tag{5.20}$$

The extended action is

$$\begin{aligned} S \;=\; \int d\tau d\sigma \Big[& P_m\dot{x}^m + w_\alpha\dot{\lambda}^\alpha + \hat{w}_{\hat{\alpha}}\dot{\hat{\lambda}}^{\hat{\alpha}} - Aw_\alpha\lambda^\alpha - \frac{1}{2}L^\alpha(P_m + \partial_\sigma x_m)(\gamma^m\lambda)_\alpha \\ & -B_\alpha\partial_\sigma\lambda^\alpha - \hat{A}\hat{w}_{\hat{\alpha}}\hat{\lambda}^{\hat{\alpha}} - \frac{1}{2}\hat{L}^{\hat{\alpha}}(P_m - \partial_\sigma x_m)(\gamma^m\hat{\lambda})_{\hat{\alpha}} - \hat{B}_{\hat{\alpha}}\partial_\sigma\hat{\lambda}^{\hat{\alpha}} \Big] \end{aligned} \tag{5.21}$$

And the gauge transformation of the Lagrange multipliers will be

$$\delta L^\alpha = \dot{\theta}^\alpha + A\theta^\alpha - \zeta L^\alpha, \qquad \delta\hat{L}^{\hat{\alpha}} = \dot{\hat{\theta}}^{\hat{\alpha}} + \hat{A}\hat{\theta}^{\hat{\alpha}} - \hat{\zeta}\hat{L}^{\hat{\alpha}} \tag{5.22}$$

$$\delta A = \dot{\zeta}, \qquad \delta\hat{A} = \dot{\hat{\zeta}} \tag{5.23}$$

$$\delta B_\alpha = \dot{\eta}_\alpha + \frac{1}{4}\theta^\beta L^\delta(\gamma_m\lambda)_{[\beta}\gamma^m_{\delta]\alpha}, \qquad \delta\hat{B}_{\hat{\alpha}} = \dot{\hat{\eta}}_{\hat{\alpha}} - \frac{1}{4}\hat{\theta}^{\hat{\beta}}\hat{L}^{\hat{\delta}}(\gamma_m\hat{\lambda})_{[\hat{\beta}}\gamma^m_{\hat{\delta}]\hat{\alpha}} \tag{5.24}$$

We extended the phase space by introducing the ghosts associated with the respective constraints. We introduce the pair (ghost, ghost momentum), (ζ, ρ) for the constraint G, the pair $(\theta^\alpha, p_\alpha)$ for G_α, and the pair $(\eta_\alpha, \mathscr{P}^\alpha)$ for F^α. Similarly we add the variables $(\hat{\zeta}, \hat{\rho}, \hat{\theta}^{\hat{\alpha}}, \hat{p}_{\hat{\alpha}}, \hat{\eta}_{\hat{\alpha}}, \hat{\mathscr{P}}^{\hat{\alpha}})$.

Notice that there appear reducibility conditions,

$$\lambda^\alpha G_\alpha = 0, \qquad \hat{\lambda}^{\hat{\alpha}}\hat{G}_{\hat{\alpha}} = 0 \tag{5.25}$$

$$(\lambda\gamma^m)_\alpha F^\alpha = 0, \qquad (\hat{\lambda}\gamma^m)_{\hat{\alpha}}\hat{F}^{\hat{\alpha}} = 0 \tag{5.26}$$

which give the first stage generators $\lambda^\alpha p_\alpha$, $\hat{\lambda}^{\hat{\alpha}}\hat{p}_{\hat{\alpha}}$, $(\lambda\gamma^m)_\alpha\mathscr{P}^\alpha$ and $(\hat{\lambda}\gamma^m)_{\hat{\alpha}}\hat{\mathscr{P}}^{\hat{\alpha}}$. So we extend further the phase space to include the ghost-for-ghosts ϕ, $\hat{\phi}$, ϖ_m, $\hat{\varpi}_m$, and the ghost-for-ghost momenta β, $\hat{\beta}$, ϱ^m, $\hat{\varrho}^m$.

We are not done yet, although. Notice that there are higher order reducibility functions, coming from relation (5.26). The coefficientes of the linear combination of constraints satisfy themselves the identities

$$(\lambda\gamma_m)_\beta(\lambda\gamma^m)_\alpha = 0, \qquad (\hat{\lambda}\gamma_m)_{\hat{\beta}}(\hat{\lambda}\gamma^m)_{\hat{\alpha}} = 0 \tag{5.27}$$

There are second-stage generators associated with these conditions, $(\lambda\gamma_m)_\beta\varrho^m$ and $(\hat\lambda\gamma_m)_{\hat\beta}\hat\varrho^m$, and we introduce fermionic ghost-for-ghost-for-ghosts s^α, $\hat s^{\hat\alpha}$ and their corresponding fermionic momenta r_α, $\hat r_{\hat\alpha}$.

Again, the coefficients of the previous conditions satisfy the identities

$$\lambda^\beta(\lambda\gamma_m)_\beta = 0, \qquad \hat\lambda^{\hat\beta}(\hat\lambda\gamma_m)_{\hat\beta} = 0 \tag{5.28}$$

so we have third-stage generators $\lambda^\alpha r_\alpha$, $\hat\lambda^{\hat\alpha}\hat r_{\hat\alpha}$, and we extend further the phase space to include third-stage bosonic ghosts ϖ, $\hat\varpi$ and their conjugate momenta ϱ, $\hat\varrho$. Here we also have an infinite tower of ghosts related to the reducibility levels starting with the relation $(\gamma_{mn}\lambda)^\alpha G_\alpha = 0$. As in the superparticle case we work as if the theory had first-stage reducibility only.

The BRST generator is $Q = \displaystyle\int d\sigma \mathcal{Q}(\sigma)$, where

$$\begin{aligned}
\mathcal{Q}(\sigma) \;=\;& \zeta w_\alpha\lambda^\alpha + \frac{1}{2}\theta^\alpha(P^m + \partial_\sigma x^m)(\gamma_m\lambda)_\alpha + \eta_\alpha\partial_\sigma\lambda^\alpha + \phi\lambda^\alpha p_\alpha + \varpi_m(\lambda\gamma^m)_\alpha\mathscr{P}^\alpha \\
&+s^\alpha(\lambda\gamma_m)_\alpha\varrho^m + \varpi\lambda^\alpha r_\alpha - \zeta\theta^\alpha p_\alpha - \zeta\eta_\alpha\mathscr{P}^\alpha - \frac{1}{8}\theta^\beta\theta^\alpha(\gamma_m\lambda)_{[\alpha}\gamma^m_{\beta]\delta}\mathscr{P}^\delta \\
&+\hat\zeta\hat w_{\hat\alpha}\hat\lambda^{\hat\alpha} + \frac{1}{2}\hat\theta^{\hat\alpha}(P^m - \partial_\sigma x^m)(\gamma_m\hat\lambda)_{\hat\alpha} + \hat\eta_{\hat\alpha}\partial_\sigma\hat\lambda^{\hat\alpha} + \hat\phi\hat\lambda^{\hat\alpha}\hat p_{\hat\alpha} + \hat\varpi_m(\hat\lambda\gamma^m)_{\hat\alpha}\hat{\mathscr{P}}^{\hat\alpha} \\
&+\hat s^{\hat\alpha}(\hat\lambda\gamma_m)_{\hat\alpha}\hat\varrho^m + \hat\varpi\hat\lambda^{\hat\alpha}\hat r_{\hat\alpha} - \hat\zeta\hat\theta^{\hat\alpha}\hat p_{\hat\alpha} - \hat\zeta\hat\eta_{\hat\alpha}\hat{\mathscr{P}}^{\hat\alpha} + \frac{1}{8}\hat\theta^{\hat\beta}\hat\theta^{\hat\alpha}(\gamma_m\hat\lambda)_{[\hat\alpha}\gamma^m_{\hat\beta]\hat\delta}\hat{\mathscr{P}}^{\hat\delta} + ...
\end{aligned} \tag{5.29}$$

Note that

$$\theta^\beta\theta^\alpha(\gamma_m\lambda)_{[\alpha}\gamma^m_{\beta]\delta}\mathscr{P}^\delta = 2(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta) \tag{5.30}$$

Then $\mathcal{Q}$ is

$$\mathcal{Q}(\sigma) = \mathcal{Q}_0 + \hat{\mathcal{Q}}_0 + \tilde{\mathcal{Q}} \tag{5.31}$$

where

$$\begin{aligned}
\mathcal{Q}_0(\sigma) \;=\;& \zeta w_\alpha\lambda^\alpha + \frac{1}{2}(P^m + \partial_\sigma x^m)(\lambda\gamma_m\theta) + \eta_\alpha\partial_\sigma\lambda^\alpha + \phi\lambda^\alpha p_\alpha + \varpi_m(\lambda\gamma^m\mathscr{P}) \\
&+(\lambda\gamma_m s)\varrho^m + \varpi\lambda^\alpha r_\alpha - \zeta\theta^\alpha p_\alpha - \zeta\eta_\alpha\mathscr{P}^\alpha - \frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta) \\
\hat{\mathcal{Q}}_0(\sigma) \;=\;& \hat\zeta\hat w_{\hat\alpha}\hat\lambda^{\hat\alpha} + \frac{1}{2}(P^m - \partial_\sigma x^m)(\hat\lambda\gamma_m\hat\theta) + \hat\eta_{\hat\alpha}\partial_\sigma\hat\lambda^{\hat\alpha} + \hat\phi\hat\lambda^{\hat\alpha}\hat p_{\hat\alpha} + \hat\varpi_m(\hat\lambda\gamma^m\hat{\mathscr{P}}) \\
&+(\hat\lambda\gamma_m\hat s)\hat\varrho^m + \hat\varpi\hat\lambda^{\hat\alpha}\hat r_{\hat\alpha} - \hat\zeta\hat\theta^{\hat\alpha}\hat p_{\hat\alpha} - \hat\zeta\hat\eta_{\hat\alpha}\hat{\mathscr{P}}^{\hat\alpha} + \frac{1}{4}(\hat\lambda\gamma_m\hat\theta)(\hat{\mathscr{P}}\gamma^m\hat\theta)
\end{aligned} \tag{5.32}$$

and we have to find a $\tilde{\mathcal{Q}}$ such that the nilpotency of the BRST transformation is guaranteed. Thus

$$\begin{aligned}\int d\sigma d\sigma'\{\mathcal{Q}(\sigma),\mathcal{Q}(\sigma')\} &= \int d\sigma d\sigma'\Big[\{\mathcal{Q}_0(\sigma),\mathcal{Q}_0(\sigma')\}+\{\hat{\mathcal{Q}}_0(\sigma),\hat{\mathcal{Q}}_0(\sigma')\}\\ &\quad+\{\tilde{\mathcal{Q}}(\sigma),\tilde{\mathcal{Q}}(\sigma')\}+2\{\mathcal{Q}_0(\sigma),\hat{\mathcal{Q}}_0(\sigma')\}\\ &\quad+2\{\mathcal{Q}_0(\sigma),\tilde{\mathcal{Q}}(\sigma')\}+2\{\hat{\mathcal{Q}}_0(\sigma),\tilde{\mathcal{Q}}(\sigma')\}\Big]\end{aligned} \tag{5.33}$$

must be zero.

Now we show that $\{\mathcal{Q}_0(\sigma),\hat{\mathcal{Q}}_0(\sigma')\}$ is just a Dirac delta times the partial σ-derivative of some expression. It's easy to see that the only non-trivial brackets come from terms which have P_m or x^m inside them. So,

$$\begin{aligned}\{\mathcal{Q}_0(\sigma),\hat{\mathcal{Q}}_0(\sigma')\} &= \frac{1}{4}\{(P^m+\partial_\sigma x^m)(\sigma),(P^n-\partial_{\sigma'}x^n)(\sigma')\}(\lambda\gamma_m\theta)(\sigma)(\hat{\lambda}\gamma_n\hat{\theta})(\sigma')\\ &= \frac{1}{4}\left(\partial_{\sigma'}\delta(\sigma-\sigma')+\partial_\sigma\delta(\sigma-\sigma')\right)(\lambda\gamma_m\theta)(\sigma)(\hat{\lambda}\gamma^m\hat{\theta})(\sigma')\\ &= -\frac{1}{4}\delta(\sigma-\sigma')\partial_\sigma\Big[(\lambda\gamma_m\theta)(\hat{\lambda}\gamma^m\hat{\theta})(\sigma)\Big]\end{aligned} \tag{5.34}$$

Therefore, this term does not contribute in $\{Q,Q\}$.

Then we evaluate $\int d\sigma d\sigma'\{\mathcal{Q}_0(\sigma),\mathcal{Q}_0(\sigma')\}$.

$$\begin{aligned}= \int d\sigma d\sigma'\Big[&\Big\{\zeta w_\alpha\lambda^\alpha(\sigma)\ ,\ \Big((P^m+\partial_{\sigma'}x^m)(\lambda\gamma_m\theta)+2\eta_\beta\partial_{\sigma'}\lambda^\beta+2\phi\lambda^\beta p_\beta\\ &\qquad+2\varpi(\lambda\gamma^m\mathscr{P})-\frac{1}{2}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta)\Big)(\sigma')\Big\}\\ &+\frac{1}{2}\Big\{(P^m+\partial_\sigma x^m)(\lambda\gamma_m\theta)(\sigma)\ ,\ \Big(\frac{1}{2}(P^n+\partial_{\sigma'}x^n)(\lambda\gamma_n\theta)\\ &\qquad+2\phi\lambda^\beta p_\beta-2\zeta\theta^\beta p_\beta\Big)(\sigma')\Big\}\\ &+\Big\{\eta_\alpha\partial_\sigma\lambda^\alpha(\sigma)\ ,\ \Big(2\varpi_m(\lambda\gamma^m\mathscr{P})-2\zeta\eta_\beta\mathscr{P}^\beta-\frac{1}{2}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta)\Big)(\sigma')\Big\}\\ &-\Big\{\phi\lambda^\alpha p_\alpha(\sigma)\ ,\ \Big(2\zeta\theta^\beta p_\beta+\frac{1}{2}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta)\Big)(\sigma')\Big\}\\ &-2\Big\{\varpi_m(\lambda\gamma^m\mathscr{P})(\sigma),\zeta\eta_\alpha\mathscr{P}^\alpha\Big\}(\sigma')\\ &+\frac{1}{2}\Big\{\zeta\theta^\alpha p_\alpha(\sigma)\ ,\ (\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta)(\sigma')\Big\}\\ &+\frac{1}{2}\Big\{\zeta\eta_\alpha\mathscr{P}^\alpha,(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta)\Big\}\Big]\end{aligned} \tag{5.35}$$

$$
\begin{aligned}
= \int d\sigma d\sigma' \Big[& \zeta\lambda^\alpha(\sigma)\delta(\sigma-\sigma')\Big(-(P^m + \partial_{\sigma'}x^m)(\gamma_m\theta)_\alpha - 2\phi p_\alpha - 2\varpi_m(\gamma^m\mathscr{P})_\alpha \\
& + \frac{1}{2}(\gamma_m\theta)_\alpha(\mathscr{P}\gamma^m\theta)\Big)(\sigma') - 2\zeta\lambda^\alpha(\sigma)\partial_{\sigma'}\delta(\sigma-\sigma')\eta_\alpha(\sigma') \\
& - \frac{1}{4}(\lambda\gamma_m\theta)(\sigma)\Big(\partial_{\sigma'}\delta(\sigma-\sigma') - \partial_\sigma\delta(\sigma-\sigma')\Big)(\lambda\gamma_m\theta)(\sigma') \\
& - (P^m + \partial_\sigma x^m)(\lambda\gamma_m)_\alpha(\sigma)\delta(\sigma-\sigma')(\phi\lambda^\alpha - \zeta\theta^\alpha)(\sigma') \\
& - \partial_\sigma\lambda^\alpha(\sigma)\delta(\sigma-\sigma')\Big(2\varpi_m(\lambda\gamma^m)_\alpha - 2\zeta\eta_\alpha + \frac{1}{2}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha\Big)(\sigma') \\
& - 2\phi\lambda^\alpha(\sigma)\delta(\sigma-\sigma')\zeta p_\alpha(\sigma')
\end{aligned} \tag{5.36}
$$

$$
\begin{aligned}
& + \frac{1}{2}\phi\lambda^\alpha(\sigma)\{p_\alpha(\sigma), \theta^\beta\theta^\delta(\sigma')\}(\lambda\gamma_m)_\beta(\mathscr{P}\gamma^m)_\delta(\sigma') \\
& - 2\varpi_m(\lambda\gamma^m)_\alpha(\sigma)\delta(\sigma-\sigma')\zeta\mathscr{P}^\alpha(\sigma') \\
& - \frac{1}{2}\zeta\theta^\alpha(\sigma)\{p_\alpha(\sigma), \theta^\beta\theta^\delta(\sigma')\}(\lambda\gamma_m)_\beta(\mathscr{P}\gamma^m)_\delta(\sigma') \\
& - \frac{1}{2}\zeta\mathscr{P}^\alpha\delta(\sigma-\sigma')(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha\Big]
\end{aligned} \tag{5.37}
$$

$$
= \int d\sigma\Big[- 4\phi\zeta\lambda^\alpha p_\alpha - 4\zeta\varpi_m(\lambda\gamma^m\mathscr{P}) + 2\zeta\partial_\sigma(\lambda^\alpha\eta_\alpha)\Big] \tag{5.38}
$$

where we have used $\{p_\alpha(\sigma), \theta^\beta\theta^\delta(\sigma')\} = -\delta(\sigma-\sigma')\delta_\alpha^{[\beta}\theta^{\delta]}(\sigma')$ and the pure spinor properties

$$
\lambda\gamma^m\partial_\sigma\lambda = 0, \qquad (\lambda\gamma_m)_\alpha(\lambda\gamma^m)_\beta = 0 \tag{5.39}
$$

in order to simplify some of the terms in the second expression.

Analogously we obtain $\int d\sigma d\sigma'\{\hat{\mathcal{Q}}_0(\sigma), \hat{\mathcal{Q}}_0(\sigma')\}$

$$
\int d\sigma\Big[- 4\hat{\phi}\hat{\zeta}\hat{\lambda}^{\hat{\alpha}}\hat{p}_{\hat{\alpha}} - 4\hat{\zeta}\hat{\varpi}_m(\hat{\lambda}\gamma^m\hat{\mathscr{P}}) + 2\hat{\zeta}\partial_\sigma(\hat{\lambda}^{\hat{\alpha}}\hat{\eta}_{\hat{\alpha}})\Big] \tag{5.40}
$$

We found a term like $-4\phi\zeta\lambda^\alpha p_\alpha$ when we studied the twistor-like superparticle. This term contains a ghost-for-ghost and a first-stage generator and we had to include in the BRST charge a term like $-2\zeta\phi\beta$ to make the transformation nilpotent. So we expect the charge to contain that term along with the one corresponding to the new first stage-generator that appeared in the superstring case.

Then we write

$$
\tilde{\mathcal{Q}} = -2\zeta\phi\beta - 2\zeta\varpi_m\varrho^m - 2\hat{\zeta}\hat{\phi}\hat{\beta} - 2\hat{\zeta}\hat{\varpi}_m\hat{\varrho}^m + \mathcal{Q}' \tag{5.41}
$$

In fact

$$\begin{aligned}\int d\sigma d\sigma'\{(\mathcal{Q}_0+\hat{\mathcal{Q}}_0)(\sigma),\tilde{\mathcal{Q}}(\sigma')\} &= \int d\sigma\big[2\phi\zeta\lambda^\alpha p_\alpha+2\zeta\varpi_m(\lambda\gamma^m\mathscr{P}) \\ &\quad +2\hat{\phi}\hat{\zeta}\hat{\lambda}^{\hat{\alpha}}\hat{p}_{\hat{\alpha}}+2\hat{\zeta}\hat{\varpi}_m(\hat{\lambda}\gamma^m\hat{\mathscr{P}})\big] \\ &\quad +\int d\sigma d\sigma'\{(\mathcal{Q}_0+\hat{\mathcal{Q}}_0)(\sigma),\mathcal{Q}'(\sigma')\}\end{aligned} \tag{5.42}$$

and

$$\begin{aligned}\int d\sigma d\sigma'\{\tilde{\mathcal{Q}}(\sigma),\tilde{\mathcal{Q}}(\sigma')\} &= \int d\sigma d\sigma'\{\mathcal{Q}'(\sigma),\mathcal{Q}'(\sigma')\} \\ &\quad -4\int d\sigma d\sigma'\big\{(\zeta\phi\beta+\zeta\varpi_m\varrho^m+\hat{\zeta}\hat{\phi}\hat{\beta}+\hat{\zeta}\hat{\varpi}_m\hat{\varrho}^m)(\sigma),\mathcal{Q}'(\sigma')\big\}\end{aligned} \tag{5.43}$$

Thus, nilpotency of the transformation implies

$$\begin{aligned} 0 &= \int d\sigma\Big[2\zeta\partial_\sigma(\lambda^\alpha\eta_\alpha)+2\hat{\zeta}\partial_\sigma(\hat{\lambda}^{\hat{\alpha}}\hat{\eta}_{\hat{\alpha}})\Big] \\ &\quad +\int d\sigma d\sigma'\Big[\Big\{(\mathcal{Q}'+2\mathcal{Q}_0+2\hat{\mathcal{Q}}_0)(\sigma),\mathcal{Q}'(\sigma')\Big\} \\ &\qquad -4\Big\{\big(\zeta\phi\beta+\zeta\varpi_m\varrho^m+\hat{\zeta}\hat{\phi}\hat{\beta}+\hat{\zeta}\hat{\varpi}_m\hat{\varrho}^m\big)(\sigma),\mathcal{Q}'(\sigma')\Big\}\Big]\end{aligned} \tag{5.44}$$

If we suppose $\partial_\sigma\zeta=0$, we're done. Pick $Q'=0$.

Then $\mathcal{Q}$ is

$$\mathcal{Q}(\sigma)=\mathcal{Q}_1+\hat{\mathcal{Q}}_1 \tag{5.45}$$

where

$$\begin{aligned}\mathcal{Q}_1(\sigma) &= \zeta w_\alpha\lambda^\alpha+\frac{1}{2}(P^m+\partial_\sigma x^m)(\lambda\gamma_m\theta)+\eta_\alpha\partial_\sigma\lambda^\alpha+\phi\lambda^\alpha p_\alpha \\ &\quad +\varpi_m(\lambda\gamma^m\mathscr{P})-\zeta\theta^\alpha p_\alpha-\zeta\eta_\alpha\mathscr{P}^\alpha-\frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta) \\ &\quad -2\zeta\phi\beta-2\zeta\varpi_m\varrho^m \\ \hat{\mathcal{Q}}_1(\sigma) &= \hat{\zeta}\hat{w}_{\hat{\alpha}}\hat{\lambda}^{\hat{\alpha}}+\frac{1}{2}(P^m-\partial_\sigma x^m)(\hat{\lambda}\gamma_m\hat{\theta})+\hat{\eta}_{\hat{\alpha}}\partial_\sigma\hat{\lambda}^{\hat{\alpha}}+\hat{\phi}\hat{\lambda}^{\hat{\alpha}}\hat{p}_{\hat{\alpha}} \\ &\quad +\hat{\varpi}_m(\hat{\lambda}\gamma^m\hat{\mathscr{P}})-\hat{\zeta}\hat{\theta}^{\hat{\alpha}}\hat{p}_{\hat{\alpha}}-\hat{\zeta}\hat{\eta}_{\hat{\alpha}}\hat{\mathscr{P}}^{\hat{\alpha}}+\frac{1}{4}(\hat{\lambda}\gamma_m\hat{\theta})(\hat{\mathscr{P}}\gamma^m\hat{\theta}) \\ &\quad -2\hat{\zeta}\hat{\phi}\hat{\beta}-2\hat{\zeta}\hat{\varpi}_m\hat{\varrho}^m\end{aligned} \tag{5.46}$$

From now on we will write only terms concerning the left moving sector of the superstring, the results for the right moving sector are analogous, but we have to take care of a few signs.

The extended phase space is

$$(x^m, P_m, \lambda^\alpha, w_\alpha, \zeta, \rho, \theta^\alpha, p_\alpha, \eta_\alpha, \mathscr{P}^\alpha, \phi, \beta, \varpi_m, \varrho^m) \tag{5.47}$$

and a gauge fixed action is constructed by means of a gauge fixing fermion χ.

$$S_\chi = \int d\tau d\sigma \Big[P_m \dot{x}^m + w_\alpha \dot{\lambda}^\alpha + \dot{\zeta}\rho + \dot{\theta}^\alpha p_\alpha + \dot{\eta}_\alpha \mathscr{P}^\alpha + \dot{\phi}\beta + \dot{\varpi}_m \varrho^m - \{\chi, Q\} \Big] \tag{5.48}$$

The BRST transformations of the phase space variables are

$$\delta_B x^m = \frac{1}{2}\lambda\gamma^m\theta, \qquad \delta_B P_m = \frac{1}{2}\partial_\sigma(\lambda\gamma_m\theta) \tag{5.49a}$$

$$\delta_B \lambda^\alpha = \zeta\lambda^\alpha, \qquad \delta_B w_\alpha = -\zeta w_\alpha - \frac{1}{2}(P_m + \partial_\sigma x^m)(\gamma^m\theta)_\alpha + \partial_\sigma \eta_\alpha \tag{5.49b}$$

$$- \phi p_\alpha - \varpi_m(\gamma^m \mathscr{P})_\alpha + \frac{1}{4}(\gamma_m\theta)_\alpha(\mathscr{P}\gamma^m\theta) \tag{5.49c}$$

$$\delta_B \zeta = 0, \qquad \delta_B \rho = -w_\alpha\lambda^\alpha + \theta^\alpha p_\alpha + \eta_\alpha \mathscr{P}^\alpha + 2\phi\beta + 2\varpi_m\varrho^m = M \tag{5.49d}$$

$$\delta_B \theta^\alpha = -\phi\lambda^\alpha + \zeta\theta^\alpha, \qquad \delta_B p_\alpha = -\frac{1}{2}(P_m + \partial_\sigma x^m)(\gamma^m\lambda)_\alpha - \zeta p_\alpha \tag{5.49e}$$

$$+ \frac{1}{4}(\gamma_m\lambda)_\alpha(\mathscr{P}\gamma^m\theta) + \frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m)_\alpha = N_\alpha \tag{5.49f}$$

$$\delta_B \eta_\alpha = -\varpi_m(\gamma^m\lambda)_\alpha + \zeta\eta_\alpha, \qquad \delta_B \mathscr{P}^\alpha = -\partial_\sigma\lambda^\alpha - \zeta\mathscr{P}^\alpha \tag{5.49g}$$

$$- \frac{1}{4}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha \tag{5.49h}$$

$$\delta_B \phi = -2\zeta\phi, \qquad \delta_B \beta = -\lambda^\alpha p_\alpha + 2\zeta\beta = V \tag{5.49i}$$

$$\delta_B \varpi_m = -2\zeta\varpi_m, \qquad \delta_B \varrho^m = -\lambda\gamma^m\mathscr{P} + 2\zeta\varrho^m \tag{5.49j}$$

Including the nonminimal sector the further extended action reads

$$\begin{aligned} S \;=\; \int d\tau d\sigma \Big[& P_m \dot{x}^m + w_\alpha \dot{\lambda}^\alpha + \dot{A} r + \dot{L}^\alpha \pi_\alpha + \dot{B}_\alpha \pi^\alpha_B - A w_\alpha \lambda^\alpha \\ & -\tfrac{1}{2} L^\alpha (P_m + \partial_\sigma x_m)(\gamma^m\lambda)_\alpha - B_\alpha \partial_\sigma \lambda^\alpha - l r - l^\alpha \pi_\alpha - l^B_\alpha \pi^\alpha_B \Big] \end{aligned} \tag{5.50}$$

and the phase space includes now also the antighosts, and the antighost momenta corresponding to the zero momentum constraints.

$$(s, \bar{D}, \rho^\alpha, \bar{C}_\alpha, \rho^B_\alpha, \bar{C}^\alpha_B) \tag{5.51}$$

The total BRST generator is

$$Q_T = Q - i \int d\sigma \big(s r + \rho^\alpha \pi_\alpha + \rho^B_\alpha \pi^\alpha_B \big) \tag{5.52}$$

A gauge fix action is written as

$$\begin{aligned} S_\chi &= \int d\tau d\sigma \Big[P_m \dot{x}^m + w_\alpha \dot{\lambda}^\alpha + \dot{A} r + \dot{L}^\alpha \pi_\alpha + \dot{B}_\alpha \pi^\alpha_B + \dot{\zeta}\rho \\ &\qquad + \dot{\theta}^\alpha p_\alpha + \dot{\eta}_\alpha \mathscr{P}^\alpha + \dot{\bar{D}} s + \dot{\bar{C}}_\alpha \rho^\alpha + \dot{\bar{C}}^\alpha_B \rho^B_\alpha + \dot{\phi}\beta \\ &\qquad + \dot{\varpi}_m \varrho^m + \{\rho A + p_\alpha L^\alpha + \mathscr{P}^\alpha B_\alpha, Q_T\} - \{\chi, Q_T\} \Big] \end{aligned} \tag{5.53}$$

5.2 Gauge fixing to the pure spinor superstring

As usual we first define the ghost number function

$$\mathcal{N} = \zeta\rho + \theta^\alpha p_\alpha + \eta_\alpha \mathscr{P}^\alpha + 2\phi\beta + 2\varpi_m \varrho^m \tag{5.54}$$

and we write the BRST charge as follows

$$\begin{aligned} \mathcal{Q} &= +\frac{1}{2}(P^m + \partial_\sigma x^m)(\lambda\gamma_m\theta) + \eta_\alpha \partial_\sigma \lambda^\alpha \\ &\quad + \phi\lambda^\alpha p_\alpha + \varpi_m(\lambda\gamma^m \mathscr{P}) - \frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta) \\ &\quad + \zeta(w_\alpha\lambda^\alpha - \theta^\alpha p_\alpha - \eta_\alpha \mathscr{P}^\alpha - 2\phi\beta - 2\varpi_m\varrho^m) \end{aligned} \tag{5.55}$$

The twisted ghost number is $\mathcal{N}' = \mathcal{N} + \Phi = \zeta\rho + w_\alpha\lambda^\alpha$, where

$$\Phi = w_\alpha\lambda^\alpha - \theta^\alpha p_\alpha - \eta_\alpha \mathscr{P}^\alpha - 2\phi\beta - 2\varpi_m\varrho^m \tag{5.56}$$

so the only variables with non vanishing twisted ghost number are $\zeta, \rho, \lambda^\alpha$ and w_α. A new BRST charge is induced by a canonical transformation of the extended phase space generated by $\mathcal{C} = \int d\sigma \zeta(\sigma)\rho(\sigma)$

$$\begin{aligned} \mathcal{Q}_{new}(\sigma) &= \mathcal{Q}(\sigma) + \int d\sigma' \Big\{ \zeta(\sigma')\rho(\sigma'), \mathcal{Q}(\sigma) \Big\} \\ &= \frac{1}{2}(P^m + \partial_\sigma x^m)(\lambda\gamma_m\theta) + \eta_\alpha \partial_\sigma \lambda^\alpha \\ &\quad + \phi\lambda^\alpha p_\alpha + \varpi_m(\lambda\gamma^m \mathscr{P}) - \frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta) \end{aligned} \tag{5.57}$$

which generate new BRST transformations

$$\delta_B x^m = \frac{1}{2}\lambda\gamma^m\theta, \qquad \delta_B P_m = \frac{1}{2}\partial_\sigma(\lambda\gamma_m\theta) \tag{5.58a}$$

$$\delta_B\lambda^\alpha = 0, \qquad \delta_B w_\alpha = -\frac{1}{2}(P_m + \partial_\sigma x^m)(\gamma^m\theta)_\alpha + \partial_\sigma\eta_\alpha \tag{5.58b}$$

$$- \phi p_\alpha - \varpi_m(\gamma^m\mathscr{P})_\alpha + \frac{1}{4}(\gamma_m\theta)_\alpha(\mathscr{P}\gamma^m\theta) \tag{5.58c}$$

$$\delta_B\zeta = 0, \qquad \delta_B\rho = -w_\alpha\lambda^\alpha + \theta^\alpha p_\alpha + \eta_\alpha\mathscr{P}^\alpha + 2\phi\beta + 2\varpi_m\varrho^m = M \tag{5.58d}$$

$$\delta_B\theta^\alpha = -\phi\lambda^\alpha, \qquad \delta_B p_\alpha = -\frac{1}{2}(P_m + \partial_\sigma x^m)(\gamma^m\lambda)_\alpha \tag{5.58e}$$

$$+ \frac{1}{4}(\gamma_m\lambda)_\alpha(\mathscr{P}\gamma^m\theta) + \frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m)_\alpha = N_\alpha \tag{5.58f}$$

$$\delta_B\eta_\alpha = -\varpi_m(\gamma^m\lambda)_\alpha, \qquad \delta_B\mathscr{P}^\alpha = -\partial_\sigma\lambda^\alpha \tag{5.58g}$$

$$- \frac{1}{4}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha \tag{5.58h}$$

$$\delta_B\phi = 0, \qquad \delta_B\beta = -\lambda^\alpha p_\alpha = V \tag{5.58i}$$

$$\delta_B\varpi_m = 0, \qquad \delta_B\varrho^m = -\lambda\gamma^m\mathscr{P} \tag{5.58j}$$

Actually, to obtain the pure spinor superstring we must perform two extra canonical transformations of the extended phase space of our system.

Take the generator $\mathcal{C}' = \displaystyle\int d\sigma\ \eta_\alpha(\sigma)\partial_\sigma\theta^\alpha(\sigma)$.

$$\begin{aligned}
\mathcal{Q}'_{new}(\sigma) &= \mathcal{Q}_{new}(\sigma) + \int d\sigma'\Big\{\eta_\alpha(\sigma')\partial_{\sigma'}\theta^\alpha(\sigma'), \mathcal{Q}_{new}(\sigma)\Big\} \\
&= \frac{1}{2}(P^m + \partial_\sigma x^m)(\lambda\gamma_m\theta) + \eta_\alpha\partial_\sigma\lambda^\alpha \\
&\quad + \phi\lambda^\alpha p_\alpha + \varpi_m(\lambda\gamma^m\mathscr{P}) - \frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta) \\
&\quad + \int d\sigma'\Big\{\eta_\alpha(\sigma')\partial_{\sigma'}\theta^\alpha(\sigma'), \phi\lambda^\alpha p_\alpha(\sigma)\Big\} \\
&\quad + \int d\sigma'\Big\{\eta_\alpha(\sigma')\partial_{\sigma'}\theta^\alpha(\sigma'), \Big(\varpi_m(\lambda\gamma^m\mathscr{P}) - \frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta)\Big)(\sigma)\Big\}
\end{aligned} \tag{5.59}$$

Now, the last but one line gives

$$\int d\sigma'\eta_\alpha(\sigma')\Big(- \partial_{\sigma'}\delta(\sigma - \sigma')\delta^\alpha_\beta\Big)\phi\lambda^\beta(\sigma) = \phi(\partial_\sigma\eta_\alpha)\lambda^\alpha(\sigma) \tag{5.60}$$

and the last one gives

$$\begin{aligned}
&\int d\sigma'\Big\{\eta_\alpha(\sigma')\partial_{\sigma'}\theta^\alpha(\sigma'), \Big(\varpi_m(\lambda\gamma^m\mathscr{P}) - \frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta)\Big)(\sigma)\Big\} \\
&\qquad = \Big(\varpi_m(\lambda\gamma^m\partial_\sigma\theta^\alpha) - \frac{1}{4}(\lambda\gamma_m\theta)(\partial_\sigma\theta^\alpha\gamma^m\theta)\Big)(\sigma)
\end{aligned} \tag{5.61}$$

Then

$$\begin{aligned}
\mathcal{Q}'_{new} &= \phi\lambda^\alpha p_\alpha + \frac{1}{2}(P^m + \partial_\sigma x^m)(\lambda\gamma_m\theta) + \eta_\alpha\partial_\sigma\lambda^\alpha \\
&\quad +\varpi_m(\lambda\gamma^m\mathscr{P}) - \frac{1}{4}(\lambda\gamma_m\theta)(\mathscr{P}\gamma^m\theta) \\
&\quad +\phi(\partial_\sigma\eta_\alpha)\lambda^\alpha + \varpi_m(\lambda\gamma^m\partial_\sigma\theta^\alpha) - \frac{1}{4}(\lambda\gamma_m\theta)(\partial_\sigma\theta^\alpha\gamma^m\theta)
\end{aligned} \tag{5.62}$$

If we take one more generator of canonical transformations

$$\mathcal{C}'' = \int d\sigma\eta_\alpha\mathscr{P}^\alpha \tag{5.63}$$

We arrive at the BRST charge density

$$\begin{aligned}
\mathcal{Q}''_{new} &= \phi\lambda^\alpha p_\alpha + \frac{1}{2}(P^m + \partial_\sigma x^m)(\lambda\gamma_m\theta) + \partial_\sigma(\phi\eta_\alpha\lambda^\alpha) \\
&\quad +\varpi_m(\lambda\gamma^m\partial_\sigma\theta^\alpha) - \frac{1}{4}(\lambda\gamma_m\theta)(\partial_\sigma\theta^\alpha\gamma^m\theta)
\end{aligned} \tag{5.64}$$

Thus, we obtain the BRST charge

$$Q''_{new} = \int d\sigma\Big[\phi\lambda^\alpha p_\alpha + \frac{1}{2}(P^m + \partial_\sigma x^m)(\lambda\gamma_m\theta) + \varpi_m(\lambda\gamma^m\partial_\sigma\theta) - \frac{1}{4}(\lambda\gamma_m\theta)(\partial_\sigma\theta\gamma^m\theta)\Big] \tag{5.65}$$

We gauge fix $\phi = 1, \varpi_m = 0$, and the gauge fixed action reads

$$S_\chi = \int d\tau d\sigma\Big[P_m\dot{x}^m + w_\alpha\dot{\lambda}^\alpha - p_\alpha\dot{\theta}^\alpha\Big] \tag{5.66}$$

with BRST charge

$$Q_{pure\ spinor} = \int d\sigma\Big[\lambda^\alpha p_\alpha + \frac{1}{2}(P^m + \partial_\sigma x^m)(\lambda\gamma_m\theta) - \frac{1}{4}(\lambda\gamma_m\theta)(\partial_\sigma\theta\gamma^m\theta)\Big] \tag{5.67}$$

We have to calculate the BRST transformation of the variable p_α explicitly,

$$\begin{aligned}
\delta p_\alpha(\sigma) &= \{p_\alpha(\sigma), Q''_{new}\} \\
&= -\frac{1}{2}(P^m + \partial_\sigma x^m)(\gamma_m\lambda)_\alpha(\sigma) - \frac{1}{4}\int d\sigma'\Big\{p_\alpha(\sigma), (\lambda\gamma_m\theta)(\partial_{\sigma'}\theta\gamma^m\theta)(\sigma')\Big\}
\end{aligned} \tag{5.68}$$

Evaluating the Poisson bracket $\Big\{p_\alpha(\sigma), (\lambda\gamma_m\theta)(\partial_\sigma\theta\gamma^m\theta)(\sigma')\Big\}$

$$\begin{aligned}
&= -\delta(\sigma-\sigma')(\lambda\gamma_m)_\alpha(\partial_{\sigma'}\theta\gamma^m\theta)(\sigma') + \Big[\partial_{\sigma'}\delta(\sigma-\sigma')\Big](\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha(\sigma') \\
&\quad -\delta(\sigma-\sigma')(\lambda\gamma_m\theta)(\partial_{\sigma'}\theta\gamma^m)_\alpha(\sigma') \\
&= -\delta(\sigma-\sigma')\Big[(\lambda\gamma_m)_\alpha(\partial_{\sigma'}\theta\gamma^m\theta) + \partial_{\sigma'}(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha \\
&\quad -2(\gamma_m\partial_{\sigma'}\theta)_\alpha(\lambda\gamma^m\theta)\Big] + \partial'_\sigma\Big[...\Big]
\end{aligned} \tag{5.69}$$

So the BRST transformation of p_α is

$$\begin{aligned}\delta p_\alpha(\sigma) &= -\frac{1}{2}(P^m+\partial_\sigma x^m)(\gamma_m\lambda)_\alpha+\frac{1}{4}\Big[(\lambda\gamma^m)_\alpha(\partial_\sigma\theta\gamma_m\theta)\\ &\qquad +\partial_\sigma(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha-2(\gamma_m\partial_\sigma\theta)_\alpha(\lambda\gamma^m\theta)\Big]\\ &= -\frac{1}{2}(P^m+\partial_\sigma x^m)(\gamma_m\lambda)_\alpha-\frac{1}{4}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial_\sigma\lambda)\\ &\qquad +\frac{1}{4}\Big[\partial_\sigma\theta^\beta(\lambda\gamma_m)_{(\alpha}(\gamma^m\theta)_{\beta)}-2(\gamma_m\partial_\sigma\theta)_\alpha(\lambda\gamma^m\theta)\Big]\\ &= -\frac{1}{2}(P^m+\partial_\sigma x^m)(\gamma_m\lambda)_\alpha-\frac{1}{4}(\gamma_m\theta)_\alpha(\theta\gamma^m\partial_\sigma\lambda)-\frac{3}{4}(\gamma_m\partial_\sigma\theta)_\alpha(\lambda\gamma^m\theta)\end{aligned} \tag{5.70}$$

In the last line we have used the identity of the gamma matrices in 10 dimensions, $\gamma_{m\alpha(\delta}\gamma^m_{\beta\rho)}=0$.

So we can write the BRST transformation of the variables after some canonical transformations of the phase space and gauge fixing of the ghost-for-ghosts.

$$\delta_B x^m=\frac{1}{2}\lambda\gamma^m\theta, \qquad \delta_B P_m=\frac{1}{2}\partial_\sigma(\lambda\gamma_m\theta) \tag{5.71a}$$

$$\delta_B\lambda^\alpha=0, \qquad \delta_B w_\alpha=-\frac{1}{2}(P^m+\partial_\sigma x^m)(\gamma_m\theta)_\alpha-p_\alpha+\frac{1}{4}(\gamma_m\theta)_\alpha(\partial_\sigma\theta\gamma^m\theta) \tag{5.71b}$$

$$\delta_B\theta^\alpha=-\lambda^\alpha, \qquad \delta_B p_\alpha=-\frac{1}{2}(P^m+\partial_\sigma x^m)(\gamma_m\lambda)_\alpha \tag{5.71c}$$

$$-\frac{1}{4}(\gamma_m\theta)_\alpha(\theta\gamma^m\partial_\sigma\lambda)-\frac{3}{4}(\gamma_m\partial_\sigma\theta)_\alpha(\lambda\gamma^m\theta) \tag{5.71d}$$

Including both right-moving and left-moving sectors the action turns out to be

$$S_\chi=\int d\tau d\sigma\Big[P_m\dot{x}^m+w_\alpha\dot{\lambda}^\alpha+\hat{w}_{\hat{\alpha}}\dot{\hat{\lambda}}^{\hat{\alpha}}-p_\alpha\dot{\theta}^\alpha-\hat{p}_{\hat{\alpha}}\dot{\hat{\theta}}^{\hat{\alpha}}\Big] \tag{5.72}$$

with BRST charge

$$Q_{pure\ spinor}=\int d\sigma\Big[\lambda^\alpha d_\alpha+\hat{\lambda}^{\hat{\alpha}}\hat{d}_{\hat{\alpha}}\Big] \tag{5.73}$$

where

$$d_\alpha\equiv p_\alpha+\frac{1}{2}(P^m+\partial_\sigma x^m)(\gamma_m\theta)_\alpha-\frac{1}{4}(\gamma_m\theta)_\alpha(\partial_\sigma\theta\gamma^m\theta) \tag{5.74}$$

$$\hat{d}_{\hat{\alpha}}\equiv\hat{p}_{\hat{\alpha}}+\frac{1}{2}(P^m-\partial_\sigma x^m)(\gamma_m\hat{\theta})_{\hat{\alpha}}+\frac{1}{4}(\gamma_m\hat{\theta})_{\hat{\alpha}}(\partial_\sigma\hat{\theta}\gamma^m\hat{\theta}) \tag{5.75}$$

Chapter 6

Conclusion

We have obtained the pure spinor superparticle action in ten dimensions by gauge fixing an action with twistor-like first-class constraints. These constraints appeared in the superparticle case when we introduced the coset $SO(10)/U(5)$ into the discussion by defining a complex splitting of the Euclidean space, breaking $SO(10)$ invariance and then trying to covariantize again the theory in a different way. The fact that projective pure spinors parametrized the mentioned coset allowed us to introduce them as dynamical variables along with the usual space variables, that is, pure spinors were initially part of the matter content of theory. We considered the four- and ten-dimensional particle cases. All spinors in four dimensions are pure spinors and we saw that twistors appeared in the discussion. Although the resulting gauge fixed action didn't describe the superparticle in four dimensions, because we've got a non-trivial BRST cohomoly different from that of the superparticle; the BRST charge for our theory was not supersymmetric. Notice also that in four dimensions the ghost-for-ghost ϕ didn't scale in contrast with the ten dimensional case, where it scaled as $\phi \to \Omega^{-2}\phi$.

It was natural to include the pure spinor condition in ten dimensions as constraints of the system; however, the infinite reducibility of those constraints prevented us to write a simple BRST charge. Even if we do not consider the pure spinor condition as a constraint and use $\lambda\gamma^m\lambda = 0$ everywhere, the first-class constraints G_α turn out to be infinitely reducible. Nevertheless, sensible results were obtained not taking into account the infinite tower of ghost-for-ghosts appearing there and just working as in a first-stage reducible theory. We wrote appropriate BRST charges for the twistor-like superstring and superparticle and showed how those charges were related to the charges in [31] by similarity transformations.

In order to arrive at the pure spinor superparticle or superstring, the gauge-fixing of the ghost-for-ghost ϕ was necessary. This was achieved by ghost-twisting, that is, we defined a new ghost number function which still assigned ghost number one to the BRST charge, but exchanged the roles of some matter and ghost content of the theory. The initial pure spinor variables became ghosts and the initial ghosts associated to the twistor-like constraints became the fermionic part of the usual ten dimensional superspace (x^m, θ^α).

It was noticed that in the specific case of the superstring, the generalization of the superparticle twistor-like constraints were not first class by themselves, so we had to add the constraints $\partial_\sigma \lambda^\alpha \approx 0$. It remains to be checked that worldsheet reparametrization invariance can be made manifest and the new constraints are consistent with it. The new constraints had also two branches of reducibility, one branch produced an infinite tower of ghost-for-ghosts and the other gave ghost-for-ghosts up to third-stage reducibility. Again, sensible results are obtained even considering the theory as a first-stage theory. It would be interesting to study the problem of the infinite reducibility of the first class constraints we started with.

It remains to be obtained the Green-Schwarz superparticle and superstring using a different choice of gauge-fixing. The BRST procedure in the form used in this work allows the required type of gauge fixing by considering a nonminimal sector which includes the Lagrange multipliers (and the corresponding momenta) as variables of the extended phase space. There are some subtleties, however, since we couldn't find a gauge-fixing fermion respecting the symmetries of w_α due to the pure spinor condition, $w_\alpha = \Lambda_m(\gamma^m\lambda)$. This problem does not appear in the procedure followed by Berkovits in [31].

References

[1] P. Ramond, *Dual theory for free fermions*, Phys. Rev. D **3**(10) (1971) 2415,

[2] A. Neveu and J. H. Schwarz, *Factorizable dual model of pions*, Nucl. Phys. B **31** (1971) 86-112,

[3] S. Mandelstam, *Manifestly dual formulation of the Ramond-model*, Phys. Lett. B **46** (1973) 447-451

[4] M. B. Green and J. H. Schwarz, *Supersymmetrical dual string theory*, Nucl. Phys. B **181** (1981) 502-530

[5] M. B. Green and J. H. Schwarz, *Supersymmetrical string theories*, Phys. Lett. B **109** (1982) 444-448

[6] M. B. Green and J. H. Schwarz, *Covariant description of superstrings*, Phys. Lett. B **136** (1984) 367-370

[7] L. Brink and J. H. Schwarz, *Quantum superspace*, Phys. Lett. B **100**(4) (1981) 310-312

[8] E. Sokatchev, *Harmonic superparticle*, Class. Quantum Grav. **4** (1987) 237-246

[9] E. R. Nissimov andS. J. Pacheva, *Quantization of N=1,2 superparticle with irreducible constraints*, Phys. Lett. B **189**(1,2) (1987) 57-62

[10] E. R. Nissimov andS. J. Pacheva, *Manifestly super-Poincar-covariant quantization of the Green -Schwarz superstring*, Phys. Lett. B **202**(3) (1988) 325-332

[11] A. Ferber, *Supertwistors and conformal supersymmetry*, Nucl. Phys. B **132** (1978) 55-64

[12] T. Shirafuji, *Lagrangian mechanics of massless particles with spin*, Progress of Theoretical Physics **70**(1) (1983) 18-35

[13] A. K. H. Bengtsson, I. Bengtsson, M. Cederwall, and N. Linden, *Particles, superparticles, and twistors*, Phys. Rev. D **36**(6) (1987) 1766-1772

[14] Y. Eisenberg, S. Solomon, *The twistor geometry of the covariantly quantized Brink-Schwarz superparticle*, Nucl. Phys. B **309** (1988) 709-732

[15] D. P. Sorokin, V. I. Tkach, and D. V. Volkov, *Superparticles, twistors and Siegel symmetry*, Mod. Phys. Lett. A **4**(10) (1989) 901-908

[16] N. Berkovits, *A supertwistor description of the massless superparticle in ten-dimensional superspace*, Phys. Lett. B **247**(1) (1990) 45-49

[17] L. Brink, M. Henneaux, C. Teitelboim, *Covariant hamiltonian formulation of the superparticle*, Nucl. Phys. B **293** (1987) 505-540

[18] E. Dur, *BRST quantization of the superparticle*, Phys. Lett. B **215**(4) (1988) 657-662

[19] W. Siegel, *The superparticle revisited*, Phys. Lett. B **203**(1,2) (1988) 79-85

[20] A. Mikovic, et al. *Covariantly quantizable superparticles*, Phys. Lett. B **235**(1,2) (1990) 106-112

[21] F. Essler, et al. *Covariant quantization of the first-ilk superparticle*, Nucl. Phys. B **364** (1991) 67-84

[22] S. Aoyama, et al. *Covariant BRST quantization of the four-dimensional superparticle*, Phys. Lett. B **216**(1,2) (1989) 133-136

[23] U. Lindström, et al. *Lorentz-covariant quantization of the superparticle*, Phys. Lett. B **224**(3) (1989) 285-287

[24] A. A. Deriglazov, A. V. Galajinsky, *Covariant supplementation scheme for infinitely reducible first class constraints*, Phys. Lett. B **381** (1996) 105-112

[25] S. Bellucci, A. Galajinsky, *Curing the infinite ghost tower in 4D Siegel superparticle*, JHEP **2000**(07) (2000) 010, [hep-th/0002071]

[26] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, JHEP **0004** (2000) 018, [hep-th/0001035]

[27] N. Berkovits, *Covariant quantization of the superparticle using pure spinors*, JHEP **2001**(09) (2001) 016, [hep-th/0105050]

[28] N. Berkovits, *ICTP lectures on covariant quantization of the superstring*, arXiv preprint hep-th/0209059 (2002)

[29] M. Chesterman, *Ghost constraints and the covariant quantization of the superparticle in ten dimensions*, JHEP **2004**(02) (2004) 011, [hep-th/0212261]

[30] M. Chesterman, *A simple BRST system with quadratically constrained ghosts*, arXiv preprint hep-th/0803.4191 (2009)

[31] N. Berkovits, *Origin of the pure spinor and Green-Schwarz formalisms*, arXiv preprint hep-th/1503.03080 (2015)

[32] N. Berkovits, *Pure spinors, twistors, and emergent supersymmetry*, JHEP **2012**(12) (2012) 1-10 [hep-th/1105.1147]

[33] N. Berkovits, *Twistor origin of the superstring*, JHEP **2015**(3) (2015) 1-16, [hep-th/1409.2510]

[34] M. Henneaux, and C. Teitelboim, *Quantization of gauge systems* (Princeton university press, 1992)

[35] J. W. van Holten, *Aspects of BRST quantization*, Topology and geometry in physics (Springer Berlin Heidelberg, 2005) 99-166 [hep-th/0201124]

[36] M. Henneaux, *Spacetime locality of the BRST formalism*, Communications in Mathematical Physics **140**(1) (1991) 1-13

[37] M. Henneaux, *Hamiltonian form of the path integral for theories with a gauge freedom*, Phys. Rep. **126**(1) (1985) 1-66

[38] I. A. Batalin, and G. A. Vilkovisky, *Gauge algebra and quantization*, Phys. Lett. B **102**(1) (1981) 27-31

[39] I. A. Batalin, and E. S. Fradkin, *A generalized canonical formalism and quantization of reducible gauge theories*, Phys. Lett. B **122**(2) (1983) 157-164

[40] N. Kawamoto, Y. Watabiki, *Topological particle field theory, general coordinate invariance and generalized Chern-Simons actions*, Commun. Math. Phys. **148** (1992) 169-176

[41] A. Rogers, *The topological particle and Morse theory*, Class. Quantum Grav.**17**(18) (2000) 3703

[42] R. Penrose, *Twistor algebra*, J. Math. Phys. **8**(2) (1967) 345

[43] I. L. Buchbinder, and S. M. Kuzenko, *Ideas and methods of supersymmetry and supergravity*, IOP, 1995

[44] R. Penrose, and M. A. MacCallum, *Twistor theory: an approach to the quantization of fields and space-time*, Phys. Rep. **6**(4) (1973), 241-315

[45] L. P. Hughston, *Twistors and particles*, Twistors and particles **97** (1979)

[46] S. A. Huggett, and K. P. Tod, *An introduction to twistor theory, Vol. 4*, Cambridge University Press, 1994

[47] N. Berkovits, and S. A. Cherkis, *Higher-dimensional twistor transforms using pure spinors*, JHEP **2004**(12) (2005) 049

[48] N. Nekrasov, *Lectures on curved beta-gamma systems, pure spinors, and anomalies*, arXiv preprint hep-th/0511008 (2005)