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**D=10 Super Yang-Mills, D=11 Supergravity and the
Pure Spinor Superfield Formalism**

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Resumo

É bem conhecido como descrever as teorias de Super Yang-Mills (SYM) em $D = 10$ dimensões e Supergravidade (SG) em $D = 11$ dimensões no superespaço e via seus campos componentes. No entanto, uma nova versão desses modelos foi formulada nos finais da década de 2000, quando Martin Cederwall usando o formalismo de supercampo de espinor puro conseguiu construir uma *pure spinor* ação, que a diferença das anteriores abordagens, esta não precisa de impor *constraints* a mão, proporciona uma descrição completa de cada modelo (no sentido do formalismo BV) e as equações do movimento obtidas a partir do respectivo princípio de ação são supersimétricas. Neste trabalho iremos explicar toda a base necessária para entender a construção de tal formalismo. Para esse propósito, começaremos estudando a teoria SYM (abeliana) em $D = 10$ em suas formulações em componentes e no superespaço. Usaremos a ação da formulação on-shell para quantizar a teoria via o formalismo de Batalin-Vilkovisky (BV). Seguiremos para SG em $D = 11$ e estudaremos suas formulações em componentes e no superespaço. Então iremos mostrar que podemos obter o mesmo espectro físico de SYM em $D = 10$ (SG em $D = 11$) estudando a superpartícula em $D = 10$ ($D = 11$) na calibre do cone de luz. De forma a ter uma quantização covariante desses modelos, introduziremos a superpartícula de espinor puro em $D = 10$ ($D = 11$), a qual possui o operador BRST usual de espinor puro ($Q = \lambda D$). Verificar-se-á que a cohomologia desse operador coincidirá com a teoria SYM em $D=10$ (SG em $D=11$) linearizada depois de ser quantizada via o formalismo BV. Esse resultado introduzirá naturalmente a ideia de construir ações usando um supercampo de espinor puro. Finalmente, explicaremos como o formalismo de supercampo de espinor puro surge nesse contexto e como podemos usá-lo para construir ações manifestamente supersimétricas para SYM em $D=10$ e SG em $D=11$.

Palavras Chaves: Supersimetria, Super Yang-Mills, Supergravidade, Espinores puros.

Áreas do conhecimento: Supersimetria, Supergravidade, Supercordas.

Abstract

It is well known how to describe the $D = 10$ (SYM) Super Yang-Mills and $D = 11$ (SG) Supergravity theories on superspace and by component fields. However, a new version of these models was formulated in the late 2000, when Martin Cederwall using the pure spinor superfield formalism achieved to construct a pure spinor action for these theories, which unlike the previously mentioned approaches, this does not require to impose any constraint by hand, provides a full description of each model (in the BV sense) and the equations of motion coming from the corresponding action principle are supersymmetric. In this work we will explain all the background required to understand the construction of this action. For this purpose, we will start with the $D=10$ (abelian) SYM theory in its component and superspace formulations. We will use the action of the on-shell formulation to quantize the theory via the Batalin-Vilkovisky framework. We will move to $D=11$ supergravity and study its component and superspace formulations. Then we will show that we can obtain the same physical spectrum of $D = 10$ SYM ($D = 11$ SG) by studying the $D = 10$ ($D = 11$) superparticle in the light-cone gauge. In order to have a covariant quantization of these models, we will introduce the $D = 10$ ($D = 11$) pure spinor superparticle, which possesses the usual pure spinor BRST operator ($Q = \lambda D$). It will turn out that the cohomology of this operator will coincide with the linearized $D = 10$ SYM ($D = 11$ SG) theory after being quantized via BV-formalism. This result will introduce naturally the idea of constructing pure spinor actions. Finally, we will explain how the pure spinor superfield framework arises in this context and how we can use it to construct manifestly supersymmetric actions for $D = 10$ SYM and $D = 11$ SG.

Key words: Supersymmetry, Super Yang-Mills, Supergravity, Pure spinors.

Knowledge areas: Supersymmetry, Supergravity, Superstrings.

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Chapter 1

Introduction

For a long time we have been searching for *covariant* formulations¹ of maximally supersymmetric models². This problem was faced from different approaches: *first-quantized particle (string)* and *field theory*. The former presented the problem of mixture of first and second class constraints which could not be separated in a Lorentz covariant way as it can be seen in the Brink-Schwarz superparticle [14] or Green-Schwarz superstring [15]. The latter presented the difficulty to formulate a description via superfields (and so manifest supersymmetry³) for D=10 Super Yang-Mills (SYM) and D=11 Supergravity (SG) (which are the maximally supersymmetric theories of our interest). On the other hand it was already known how to construct component and superspace formulations for $D = 10$ SYM [1] [2] and $D = 11$ SG [26] [27], even though these ones were not manifestly supersymmetric⁴.

Pure spinors were known objects from some long time ago⁵. However, the discovery of the role of these objects⁶ in formulating manifestly supersymmetric action principles for maximally supersymmetric models was just realized recently from two lines of research. On the one hand, N. Berkovits achieved to quantize superstring covariantly by introducing an adequate set of ghost variables (which turned out to be pure spinor variables) [22]. On the other hand, attempts to find possible deformations of higher derivative terms in maximally supersymmetric models by studying the constraints on superspace revealed a cohomological structure of these deformations, which turned out to be the same as that of the pure spinor BRST operator [29], [35].

In the early 2000s it was already known the possibility to construct actions for the linearized versions (non-interacting theories) of SYM and SG by using a *pure spinor superfield* (which is an ordinary superfield depending on the ordinary superspace coordinates and also on a pure

¹This means formulations which manifestly exhibit the Lorentz symmetry and supersymmetry.

²This means 8 real supersymmetries for scalar multiplets, 16 for vector or tensor multiplets and 32 for supergravity multiplets.

³We are going to clarify the use of this term. From now on, *manifest supersymmetry* will mean anything which gives us manifestly supersymmetric equations of motion. This term will not have nothing to do with the (on-shell or off-shell) supersymmetry representation of the model.

⁴The component formulation does not take into account the θ -coordinate of superspace and the superspace formulation reproduces the right theory just after imposing some constraints by hand.

⁵Pure spinors were defined by E. Cartan [3] for even dimensions

⁶It is useful to mention that our definition of pure spinors sometimes will coincide with the Cartan's definition and sometimes not. For example in $D = 10$ our pure spinors will coincide with the Cartan's ones ($\lambda\gamma^m\lambda = 0$). Even we will use this definition of pure spinor ($\lambda\gamma^m\lambda = 0$) in $D = 11$ (odd dimension). Some cases where the Cartan's definition does not coincide with our definition are for example in $D = 3$ and $D = 6$ (Cartan's pure spinors are uninteresting for $D < 8$, however by using $\lambda\gamma^m\lambda = 0$ as our pure spinor constraint we can find interesting results in $D = 3$ [37] and $D = 6$ [38]).

spinor variable λ : $\Psi(x, \theta, \lambda)$) and a pure spinor BRST operator ($Q = \lambda D$, where D is the supersymmetric derivative in $D = 10$ or $D = 11$, respectively) [19], [21], [23]. In this approach, after defining what is called *ghost number* and assigning naturally the numerical values $+1$ for a pure spinor variable and 0 for matter fields, the corresponding physical theories are obtained as the comohology of the BRST operator (SYM at ghost number $+1$ and SG at ghost number $+3$), that is the equations of motion come from demanding that the corresponding pure spinor superfield is a BRST-closed state ($Q\Psi = 0$), and the gauge invariance arises from imposing that the variation of this superfield is BRST-exact ($\delta\Psi = Q\Lambda$, for some pure spinor superfield Λ). However, we do not just obtain the physical fields (at the ghost numbers mentioned) but also other fields (at different ghost numbers, for SYM in ghost numbers $0, 2, 3$ and for SG in ghost numbers $0, 1, 2, 4, 5, 6, 7$)⁷. To be precise, we obtain ghosts, antifields and ghost antifields with the corresponding equations of motion and gauge invariances as it would be dictated by the quantization via the Batalin-Vilkovisky (BV) framework [6] for SYM and SG. Therefore one single pure spinor superfield can contain the whole information about a (linearized) maximally supersymmetric theory by requiring to this superfield be an element of the cohomology of the pure spinor BRST operator. At this point, we can already see an curious fact. If we defined:

$$S = \int [dx] \langle \Psi Q \Psi \rangle \quad (1.1)$$

with a proper definition of $\langle \rangle$, we would obtain $Q\Psi = 0$ as the equation of motion of the corresponding theory⁸ and the gauge invariance $\delta\Psi = Q\Lambda$ could be guaranteed by the cohomology of the BRST operator Q and the definition of $\langle \rangle$ ⁹.

However, this approach presented some deficiencies because the non-manifestly supersymmetric nature of the equations of motion (for SYM, for instance, this happens because we just integrate over 5 θ 's and not over 16 θ 's) and also the fact of treating only with non-interacting theories. The solution to these problems gave rise to what is called *pure spinor superfield formalism*.

The first difficulty was solved by N. Berkovits by introducing the *non-minimal* version of the pure spinor formalism for $D = 10$ in the context of superstring scattering amplitudes [24]. This version contains the pure spinor λ^α (of the *minimal* pure spinor formalism, the framework mentioned above) and the *non-minimal* variables $\bar{\lambda}_\alpha$ and r_α which are constrained by the conditions $\bar{\lambda}\gamma^m\bar{\lambda} = 0$ and $\bar{\lambda}\gamma^m r = 0$. In order to maintain the results obtained with the *minimal formalism* was necessary to modify the (minimal) BRST operator to a new (non-minimal) BRST operator but at the same time holding the cohomology of the first one. This modification was:

$$Q_{n.m} = Q_m + \bar{w}r \quad (1.2)$$

where \bar{w}^α is the conjugate momentum to $\bar{\lambda}_\alpha$. In the context of superstring theory this modification allows us to use the tools of topological string theory to calculate scattering amplitudes after choosing an appropriate regularization factor (necessary to obtain well-defined results). It was precisely the measure obtained with this non-minimal version and the corresponding regularization factor which solved the problem of finding a well-defined measure in the construction

⁷Although a pure spinor superfield Ψ has arbitrarily high ghost number terms, it can be shown that when imposing that this one belongs to the Q -cohomology the field content becomes finite.

⁸To be precise this condition $Q\Psi = 0$ would be valid only for a finite number of components of $Q\Psi$ (for instance with 5 θ 's for SYM and 9 θ 's for SG). However as it will be explained in this work the other components of this equation (involving more than 5 θ 's for SYM and 9 θ 's for SG) do not affect the physical content of the theories.

⁹As we will explain later this bracket $\langle \rangle$ will be defined in such a way that picks out the top cohomology of each theory, which will not be BRST-exact implying so the gauge invariance desired.

of a manifestly supersymmetric action principle.

The second difficulty was solved by M. Cederwall in trying to find an systematic approach to construct full actions for maximally supersymmetric theories. Basically the idea was to find a BV-like formalism because of the field content obtained in this approach (fields, ghosts, antifields and ghost antifields) but defining carefully the *antibracket*, *operators* and *gauge fixing* [30], [31], [32], [33], [34], [35], [36]. So the problem was reduced to find a solution to the master equation¹⁰

$$(\mathcal{S}, \mathcal{S}) = 0 \quad (1.3)$$

where $(,)$ is the antibracket, and \mathcal{S} will be the *master action*.

With all of this machinery it was possible to construct pure spinor actions for SYM and SG.

For SYM, we obtain [31], [33]:

$$\mathcal{S} = \int [dZ] e^{-\lambda\bar{\lambda}-r\theta} \left(\frac{1}{2} \Psi Q \Psi + \frac{1}{3} \Psi \Psi \Psi \right) \quad (1.4)$$

where Ψ is in the adjoint representation of the gauge group.

For SG, we obtain [31]:

$$\mathcal{S} = \int [dZ] e^{-\lambda\bar{\lambda}-r\theta} \left[\frac{1}{2} \Psi Q \Psi + \frac{1}{6} (\lambda \gamma_{ab} \lambda) \left(1 - \frac{3}{2} T \Psi \right) \Psi R^a \Psi R^b \Psi \right] \quad (1.5)$$

where $[dZ]$ are the well-defined measures (depending on $[x, \theta, \lambda, \bar{\lambda}, r]$) mentioned above whose explicit form will be showed in this work. The linearized versions of SYM and SG are obtained by working just with the first term of each action ($\Psi Q \Psi$).

The structure of this work is as follows. In section 2 we will describe in a detailed way $N = 1$ $D = 10$ (abelian) SYM by showing explicitly the component and superspace formulations. In the former we will define the supersymmetry transformations and show the invariance of the corresponding component action. We will also show that the supersymmetry algebra is closed up to equations of motion. Then it will be presented the non-abelian case and discussed briefly. In the latter we will introduce a general discussion about differential geometry on superspace aiming to obtain the (super) Bianchi identities. After requiring to work in a flat superspace, we will obtain explicitly the Bianchi identities and we will solve them by using two kinds of constraints: *conventional* and *physical* constraints. These constraints can be summarized in the equation: $F_{\alpha\beta} = 0$. So, we will be just left with the lowest-dimensional superfield (A_α) which will contain the physical fields (gauge field and gaugino) in its corresponding θ -expansion. By using the Bianchi identities we will find that these physical fields satisfy the equations of motion and gauge invariances studied in the component formulation of $D = 10$ SYM.

In section 3 we will study the component and superspace formulations of $N = 1$ $D = 11$ SG. In this case we will present the equations of motion of the fields and the respective supersymmetry transformations under which the component action is invariant. In the superspace formulation we will define the *vielbein*, *spin connection*, *torsion*, *curvature* and a *3-form superfield* (C). It will be shown explicitly how to obtain the Bianchi identities and we will mention how to solve them. In this way, we will obtain the equations of motion for the Ricci tensor, gravitino and the 4-form field strength ($H = dC$), which will coincide with those studied in the component formulation.

In section 4 it will be introduced the pure spinor formalism by studying the Brink Schwarz superparticle. We will define the supersymmetry and κ transformations and show that these

¹⁰We are just talking about the classical field theory. To move on to the quantum level we must introduce what is called *gauge fixing*, but we will not study this in this work.

ones are invariances of the Brink-Schwarz action. Then, we will find that this system has constraints and calculate the corresponding constraint algebra. It will be seen that this system has first and second class constraints which cannot be separated in a Lorentz covariant manner, and so we will have to use a particular gauge, the light-cone gauge, to quantize this theory. In this way we will see that the first class constraints are the κ -symmetry generators and by using this local symmetry we will arrive at a free action with manifest $SO(8)$ symmetry which after quantizing it will give us the $D = 10$ SYM physical spectrum (a $SO(8)$ vector and a $SO(8)$ spinor). Next, we will study the pure spinor superparticle which unlike the previous description, this will provide us a covariant description of $D=10$ SYM. To this end it will be added a new set of fermionic variables and first class constraints (which will generate a corresponding gauge symmetry) to the Brink-Schwarz action in its manifestly invariant $SO(8)$ form. After gauge fixing we will write the corresponding gauge-fixed action and proceed to construct the respective BRST operator. It will be proved that the cohomology of this BRST operator is equivalent to that of the BRST operator $Q = \lambda^\alpha d_\alpha$ where λ^α is a pure spinor, and so we will obtain the pure spinor superparticle action, which will exhibit manifest supersymmetry and $SO(9, 1)$ symmetry. We will compute the Q -cohomology at zero momentum by showing a equivalence between the cohomologies corresponding to this operator Q and to an operator \hat{Q} which will be constructed by assuming that the 16 components of the pure spinor are independent (that is the spinor ghost field will be considered as a (non-pure) ordinary chiral spinor). In that way we will get not just the physical states of $D = 10$ SYM but also the ghosts, antifields and ghost antifields corresponding to a BV quantization of $D = 10$ SYM. Finally, we will write a *pure spinor superfield* and require it to be an element of the Q -cohomology, this will lead us to the equations of motion and gauge invariances which should be satisfied by the fields mentioned (matter fields, ghosts, antifields and ghost antifields). These ones will coincide with those obtained in a linearized (or abelian) version of $D = 10$ SYM (in the BV framework). Next we will do the same procedure for $D = 11$ SG, even though somewhat less detailed compared to the previous chapter. We will study the $D = 11$ superparticle by defining the supersymmetry and κ transformations, and it will be shown the invariance of the action under these ones. Next, we will find once again the constraints of this system and the corresponding constraint algebra. We will choose the light-cone gauge and obtain the known physical spectrum of $D = 11$ SG (a $SO(9)$ traceless symmetric tensor, a $SO(9)$ γ -traceless vectorspinor and a $SO(9)$ 3-form). In order to have a covariant description of $D = 11$ SG, we will present the $D = 11$ pseudo pure spinor superparticle¹¹. This action is constructed by analogy with $D = 10$ SYM. Again, we will calculate the Q -cohomology at zero momentum to figure out which is the field content of this theory. It will turn out that the field content is the same as that obtained by BV quantization of $D = 11$ SG (matter fields, antifields, ghosts, ghosts for ghosts, ghosts for ghosts for ghosts and their corresponding antifields). Finally we will construct a pure spinor superfield and after imposing to this be an element of the Q -cohomology (Q is the pure spinor BRST operator) we will obtain the equations of motion and gauge invariances of (linearized) $D = 11$ SG in the BV framework.

Finally in section 5 we will describe the *pure spinor superfield formalism* which will give us the interacting versions of the linearized models studied in the previous chapters. We will see that the BRST-exact condition satisfied by the pure spinor superfield in the previous chapters (that is, the condition which gave us the equations of motion) can be obtained from a manifestly supersymmetric action principle, if some adequate integration measure is defined. We will find this measure and introduce the pure spinor superfield framework by imitating the BV

¹¹The name *pseudo* is to emphasize the fact that the (Cartan's) pure spinors are defined in even dimensions. However, we will ignore this and maintain the pure spinor constraint as $\lambda\gamma^m\lambda = 0$ in $D = 11$ too.

framework. Finally we will construct the full actions for $D = 10$ SYM and $D = 11$ SG by solving the master equation. These actions will be manifestly supersymmetric and Lorentz covariant.

Chapter 2

D=10 Super Yang-Mills Theory

We will explain the component and superspace formulations of D=10 SYM in detail [1], [9], [39]. Furthermore, we will show how they are related to each other by using some suitable constraints (the conventional and dynamical constraints) on superfields.

2.1 Component formulation of D=10 SYM

It is known that we can have on-shell (or off-shell) representations of the supersymmetry algebra depending on if this one closes by using equations of motion (or not). In this section we will describe D=10 SYM by using on-shell fields. We will just focus on the abelian case, because it is easy to generalize the ideas to the non-abelian case.

Before of starting with our task, it is fine to say some words about the notation which will be used throughout of this chapter¹. We will work with the off-diagonal 16×16 symmetric matrices $\gamma_{\alpha\beta}^m$ (and $(\gamma^m)^{\alpha\beta}$) of the 32×32 matrix Γ^m , where Γ^m are the 32×32 matrices (satisfying $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}$). The charge conjugation matrix C (in D=10) is off-diagonal (and equivalent to Γ^0). These matrices $\gamma_{\alpha\beta}^m$ satisfy the following two useful properties: $\gamma_{\alpha\beta}^m (\gamma^n)^{\beta\delta} + \gamma_{\alpha\beta}^n (\gamma^m)^{\beta\delta} = 2\eta^{mn} \delta_\alpha^\delta$ and $\gamma_{\alpha(\beta}^m (\gamma_m)_{\lambda\delta)} = 0$. In addition to this, we will work with $D = 10$ Majorana-Weyl spinors ($\mathbf{32} = \mathbf{16} + \mathbf{16}'$). Weyl spinors will be denoted by χ^α ($\mathbf{16}$), and anti-Weyl spinors will be denoted by χ_α ($\mathbf{16}'$).

Let us start by counting the bosonic and fermionic physical degrees of freedom². A $SO(1,9)$ massless vector A^m has $D - 2$ degrees of freedom³. Therefore in particular in $D = 10$ we will have 8 bosonic degrees of freedom. A Weyl spinor has 16 real independent components⁴. If we impose the equation of motion on this one, we get 8 fermionic degrees of freedom, and so the number of bosonic degrees of freedom matches the number of fermionic degrees of freedom.

The SYM action in the component formulation is

$$S = \int d^{10}x \left[-\frac{1}{4} F^{mn} F_{mn} + \frac{i}{2} \chi \gamma^p \partial_p \chi \right] \quad (2.1.1)$$

¹A detailed discussion can be found in the appendix A.

²A general analysis is given in the appendix C.

³This can be seen by using the equations of motion and the gauge invariance in the momentum space: $p^2 A^m - (p \cdot A) p^m = 0$, but $p^2 = 0$ so we can choose a frame where $p^m = (p, 0, \dots, 0, p)$ and get $A^0 = A^{D-1}$. On the other hand, if we use the gauge invariance $\delta A^m = p^m \Lambda$, and we choose $\Lambda = -\frac{A^n}{p^n}$ with $p^n \neq 0$ then $A'^m = A^m + \delta A^m = (A^0, A^1, \dots, A^{n-1}, 0, A^{n+1}, \dots, A^0)$.

⁴We know that a Dirac spinor in $D = 10$ dimensions has $2^{10/2} = 32$ complex components, so we have 64 degrees of freedom. By imposing the Majorana condition, we get $\frac{64}{2} = 32$ independent components, and finally together with the Weyl condition we obtain $\frac{32}{2} = 16$ independent degrees of freedom.

The equations of motion can be obtained by using the Euler-Lagrange equations

$$\partial_m \left(\frac{\partial \mathcal{L}}{\partial (\partial_m A_n)} \right) = 0 \rightarrow \partial_m (\partial^m A^n - \partial^n A^m) = 0 \rightarrow \partial_m F^{mn} = 0 \quad (2.1.2)$$

$$\partial_m \left(\frac{\partial \mathcal{L}}{\partial (\partial_m \chi^\alpha)} \right) - \frac{\partial \mathcal{L}}{\partial \chi^\alpha} = 0 \rightarrow -\frac{i}{2} \partial_m \chi^\beta \gamma_{\alpha\beta}^m - \frac{i}{2} \gamma_{\alpha\beta}^m \partial_m \chi^\beta = 0 \rightarrow \gamma_{\alpha\beta}^m \partial_m \chi^\beta = 0 \quad (2.1.3)$$

Now, let us define the following fermionic global transformations

$$\delta_\epsilon A_m = i(\epsilon \gamma_m \chi) \quad (2.1.4)$$

$$\delta_\epsilon \chi^\alpha = \frac{1}{2} F_{mn} (\gamma^{mn})^\alpha{}_\beta \epsilon^\beta \quad (2.1.5)$$

where ϵ^α is a constant Majorana-Weyl spinor parameter. We will next show that the action 2.1.1 is invariant under these transformations by using the identities $\gamma^{mn} \gamma^p = \gamma^{mnp} + 2\eta^{p[n} \gamma^{m]}$ ⁵ and $\chi \gamma^p \gamma^{mn} \epsilon = \epsilon \gamma^{mn} \gamma^p \chi$ ⁶:

$$\begin{aligned} \delta S &= \int d^{10} x \left[-\frac{1}{2} F_{mn} (\delta F^{mn}) + \frac{i}{2} \delta \chi \gamma^m \partial_m \chi + \frac{i}{2} \chi \gamma^p \partial_p (\delta \chi) \right] \\ &= \int d^{10} x \left[-F_{mn} (\partial^m \delta A^n) + \frac{i}{2} \left(\frac{1}{2} \right) F_{mn} (\gamma^{mn})^\alpha{}_\lambda \epsilon^\lambda \gamma_{\alpha\beta}^p \partial_p \chi^\beta + \frac{i}{2} \chi^\alpha \gamma_{\alpha\beta}^p \left(\frac{1}{2} \right) \partial_p F_{mn} (\gamma^{mn})^\beta{}_\lambda \epsilon^\lambda \right] \\ &= i \int d^{10} x \left[-F_{mn} (\epsilon \partial^m \gamma^n \chi) + \frac{1}{4} F_{mn} (-\gamma^{mn})^\alpha{}_\lambda \epsilon^\lambda \gamma_{\alpha\beta}^p \partial_p \chi^\beta + \frac{1}{4} \partial_p F_{mn} \chi^\alpha \gamma_{\alpha\beta}^p (\gamma^{mn})^\beta{}_\lambda \epsilon^\lambda \right] \\ &= i \int d^{10} x \left[-F_{mn} (\epsilon \gamma^n \partial^m \chi) - \frac{1}{4} F_{mn} \epsilon \gamma^{mn} \gamma^p \partial_p \chi + \frac{1}{4} \partial_p F_{mn} \epsilon^\lambda (\gamma^{mn})^\beta{}_\lambda \gamma_{\beta\alpha}^p \chi^\alpha \right] \\ &= i \int d^{10} x \left[-F_{mn} (\epsilon \gamma^n \partial^m \chi) - \frac{1}{4} F_{mn} \epsilon \gamma^{mn} \gamma^p \partial_p \chi + \frac{1}{4} \partial_p (F_{mn} \epsilon \gamma^{mn} \gamma^p \chi) - \frac{1}{4} F_{mn} \epsilon \gamma^{mn} \gamma^p \partial_p \chi \right] \end{aligned}$$

where we used the antisymmetry of the 2-form $(\gamma^{mn})^\alpha{}_\lambda$ and the last term was integrated by

⁵We just have to expand the 3-form:

$$\begin{aligned} \gamma^{mpq} &= \frac{1}{6} (\gamma^m \gamma^p \gamma^q - \gamma^m \gamma^q \gamma^p + \gamma^p \gamma^q \gamma^m - \gamma^p \gamma^m \gamma^q + \gamma^q \gamma^m \gamma^p - \gamma^q \gamma^p \gamma^m) \\ \gamma^{mpq} &= \frac{1}{6} (2\gamma^{mp} \gamma^q - 2\eta^{qp} \gamma^m + \gamma^m \gamma^p \gamma^q + 2\eta^{qm} \gamma^p - \gamma^p \gamma^m \gamma^q + 2\eta^{qm} \gamma^p - \gamma^m \gamma^q \gamma^p - 2\eta^{qp} \gamma^m + \gamma^p \gamma^q \gamma^m) \\ \gamma^{mpq} &= \frac{1}{6} (4\gamma^{mp} \gamma^q - 2\eta^{qp} \gamma^m + 4\eta^{qm} \gamma^p - 2\eta^{qp} \gamma^m + \gamma^m \gamma^p \gamma^q - 2\eta^{qp} \gamma^m + 2\eta^{mq} \gamma^p - \gamma^p \gamma^m \gamma^q) \\ \gamma^{mpq} &= \frac{1}{6} (6\gamma^{mp} \gamma^q - 6\eta^{qp} \gamma^m + 6\eta^{qm} \gamma^p) \\ \gamma^{mpq} &= \gamma^{mp} \gamma^q - 2\eta^{q[p} \gamma^{m]} \end{aligned}$$

⁶We have to use the property $(\gamma^{mn})^\alpha{}_\beta = -(\gamma^{mn})_\beta{}^\alpha$:

$$\begin{aligned} \chi \gamma^p \gamma^{mn} \epsilon &= \chi^\alpha \gamma_{\alpha\beta}^p (\gamma^{mn})^\beta{}_\lambda \epsilon^\lambda \\ \chi \gamma^p \gamma^{mn} \epsilon &= -\epsilon^\lambda (-1) (\gamma^{mn})_\lambda{}^\beta \gamma_{\beta\alpha}^p \chi^\alpha \\ \chi \gamma^p \gamma^{mn} \epsilon &= \epsilon^\lambda (\gamma^{mn})_\lambda{}^\beta \gamma_{\beta\alpha}^p \chi^\alpha \\ \chi \gamma^p \gamma^{mn} \epsilon &= \epsilon \gamma^{mn} \gamma^p \chi \end{aligned}$$

parts. So we are left with:

$$\begin{aligned}
\delta S &= i \int d^{10}x [-F_{mn}(\epsilon\gamma^n \partial^m \chi) - \frac{1}{2}F_{mn}\epsilon\gamma^{mn}\gamma^p \partial_p \chi + \frac{1}{4}\partial_p(F_{mn}\epsilon\gamma^{mn}\gamma^p \chi)] \\
&= i \int d^{10}x [-F_{mn}(\epsilon\gamma^n \partial^m \chi) - \frac{1}{2}F_{mn}\epsilon\gamma^{mnp}\partial_p \chi - F_{mn}(\epsilon\gamma^m \partial^n \chi) + \frac{1}{4}\partial_p(F_{mn}\epsilon\gamma^{mn}\gamma^p \chi)] \\
&= i \int d^{10}x [-\frac{1}{2}\partial_p(F_{mn}\epsilon\gamma^{mnp}\chi) + \frac{1}{2}(\partial_p F_{mn})\epsilon\gamma^{mnp}\chi + \frac{1}{4}\partial_p(F_{mn}\epsilon\gamma^{mn}\gamma^p \chi)] \\
&= i \int d^{10}x [-\frac{1}{2}\partial_p(F_{mn}\epsilon\gamma^{mnp}\chi) + \frac{1}{4}\partial_p(F_{mn}\epsilon\gamma^{mn}\gamma^p \chi)] \\
&= 0
\end{aligned}$$

by using the Bianchi identities and assuming that the fields go to zero at infinity. Now, we will show that these transformations are supersymmetry transformations, that is to say the corresponding generator algebra will satisfy $\{Q_\beta, Q_\delta\}\Psi = -2i(\gamma^m)_{\beta\delta}\partial_m\Psi$, where $\Psi = (F_{mn}, \chi^\alpha)$. Let us start by computing the commutator of two arbitrary variations on A_m :

$$\begin{aligned}
\delta_2\delta_1 A_m &= i\delta_2(\epsilon_1\gamma_m\chi) \\
&= \frac{i}{2}\epsilon_1\gamma_m\gamma^{pq}\epsilon_2 F_{pq} \\
&= \frac{i}{2}\epsilon_1(\gamma_{mpq} + 2\eta_{m[p}\gamma_{q]})\epsilon_2 F^{pq} \\
&= \frac{i}{2}\epsilon_1\gamma_{mpq}\epsilon_2 F^{pq} + i\epsilon_1\gamma^q\epsilon_2 F_{mq}
\end{aligned}$$

and in the same way we get $\delta_1\delta_2 A_m = \frac{i}{2}\epsilon_2\gamma_{mpq}\epsilon_1 F^{pq} + i\epsilon_2\gamma^q\epsilon_1 F_{mq}$, therefore

$$\begin{aligned}
[\delta_1, \delta_2] A_m &= \delta_1\delta_2 A_m - \delta_2\delta_1 A_m \\
[\delta_1, \delta_2] A_m &= \frac{i}{2}\epsilon_2\gamma_{mpq}\epsilon_1 F^{pq} - i\epsilon_1\gamma^q\epsilon_2 F_{mq} - \frac{i}{2}\epsilon_2\gamma_{mpq}\epsilon_1 F^{pq} - i\epsilon_1\gamma^q\epsilon_2 F_{mq} \\
[\delta_1, \delta_2] A_m &= -2i(\epsilon_1\gamma^q\epsilon_2)F_{mq}
\end{aligned}$$

Now, we can calculate the same commutator on F_{mn} :

$$\begin{aligned}
\delta_1\delta_2(F_{mn}) &= \delta_1\delta_2(\partial_m A_n - \partial_n A_m) = \partial_m(\delta_1\delta_2 A_n) - \partial_n(\delta_1\delta_2 A_m) \\
\delta_2\delta_1(F_{mn}) &= \delta_2\delta_1(\partial_m A_n - \partial_n A_m) = \partial_m(\delta_2\delta_1 A_n) - \partial_n(\delta_2\delta_1 A_m) \\
\rightarrow [\delta_1, \delta_2] F_{mn} &= \partial_m([\delta_1, \delta_2] A_n) - \partial_n([\delta_1, \delta_2] A_m) \\
[\delta_1, \delta_2] F_{mn} &= -2i(\partial_m F_{np})\epsilon_1\gamma^p\epsilon_2 + 2i(\partial_n F_{mp})\epsilon_1\gamma^p\epsilon_2 \\
[\delta_1, \delta_2] F_{mn} &= 2i\epsilon_1\gamma^p\epsilon_2(-\partial_m F_{np} - \partial_n F_{pm}) \\
[\delta_1, \delta_2] F_{mn} &= 2i\epsilon_1\gamma^p\epsilon_2\partial_p F_{mn}
\end{aligned}$$

where in the last equality we used the Bianchi identity. Now if we denote the fermionic symmetry generators by Q_α , we can write the corresponding variations in terms of these ones: $\delta_1 F_{mn} = \epsilon_1^\alpha Q_\alpha F_{mn}$, so we have:

$$[\delta_1, \delta_2] F_{mn} = \delta_1\delta_2 F_{mn} - \delta_2\delta_1 F_{mn} \quad (2.1.6)$$

$$= -\epsilon_1^\alpha \epsilon_2^\beta Q_\alpha Q_\beta F_{mn} - (-\epsilon_2^\beta \epsilon_1^\alpha Q_\beta Q_\alpha) F_{mn} \quad (2.1.7)$$

$$[\delta_1, \delta_2] F_{mn} = -\epsilon_1^\alpha \epsilon_2^\beta \{Q_\alpha, Q_\beta\} F_{mn} \quad (2.1.8)$$

Hence, we arrive at the following result:

$$[\delta_1, \delta_2] F_{mn} = -\epsilon_1^\alpha \epsilon_2^\beta \{Q_\alpha, Q_\beta\} F_{mn} = 2i \epsilon_1^\alpha \epsilon_2^\beta \gamma_{\alpha\beta}^p \partial_p F_{mn} \quad (2.1.9)$$

$$\rightarrow \{Q_\alpha, Q_\beta\} F_{mn} = -2i (\gamma^p)_{\alpha\beta} \partial_p F_{mn} \quad (2.1.10)$$

Now let us find the same quantity for the spinor χ^α :

$$\begin{aligned} \delta_1 \delta_2 \chi^\alpha &= \frac{1}{2} \delta_1 (F_{mn} (\gamma^{mn})^\alpha{}_\lambda \epsilon_2^\lambda) \\ &= \frac{1}{2} \delta_1 (2 \partial_m A_n (\gamma^{mn})^\alpha{}_\lambda \epsilon_2^\lambda) \\ &= i \partial_m (\delta_1 A_n) (\gamma^{mn})^\alpha{}_\lambda \epsilon_2^\lambda \\ &= i \partial_m (\epsilon_1 \gamma_n \chi) (\gamma^{mn})^\alpha{}_\lambda \epsilon_2^\lambda \\ &= \frac{i}{2} (\epsilon_1 \gamma_n \partial_m \chi) (\gamma^m \gamma^n - \gamma^n \gamma^m)^\alpha{}_\lambda \epsilon_2^\lambda \\ &= \frac{i}{2} (\epsilon_1^\beta (\gamma_n)_{\beta\lambda} \partial_m \chi^\lambda) [(\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\delta} \epsilon_2^\delta - (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\delta} \epsilon_2^\delta] \\ &= -\frac{i}{2} \epsilon_1^\beta \epsilon_2^\delta [(\gamma_n)_{\beta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\delta} - (\gamma_n)_{\beta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\delta}] \partial_m \chi^\lambda \end{aligned}$$

Analogously, we can get the result of the action of these two variations acting in the reverse order:

$$\begin{aligned} \delta_2 \delta_1 \chi^\alpha &= \frac{i}{2} \epsilon_2 \gamma_n \partial_m \chi [(\gamma^m \gamma^n - \gamma^n \gamma^m) \epsilon_1]^\alpha \\ \delta_2 \delta_1 \chi^\alpha &= -\frac{i}{2} \epsilon_2^\delta \epsilon_1^\beta [(\gamma_n)_{\delta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\beta} - (\gamma_n)_{\delta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\beta}] \partial_m \chi^\lambda \end{aligned}$$

therefore, the commutator of the variations acting on χ^α is:

$$\begin{aligned} [\delta_1, \delta_2] \chi^\alpha &= -\frac{i}{2} \epsilon_1^\beta \epsilon_2^\delta [(\gamma_n)_{\beta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\delta} - (\gamma_n)_{\beta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\delta} \\ &\quad + (\gamma_n)_{\delta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\beta} - (\gamma_n)_{\delta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\beta}] \partial_m \chi^\lambda \end{aligned}$$

Now, we should conveniently group the terms inside the brackets to simplify this expression. Let us start with the first and third terms:

$$\begin{aligned} (\gamma_n)_{\beta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\delta} + (\gamma_n)_{\delta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\beta} &= (\gamma^m)^{\alpha\sigma} [(\gamma_n)_{\beta\lambda} (\gamma^n)_{\sigma\delta} + (\gamma_n)_{\delta\lambda} (\gamma^n)_{\sigma\beta}] \\ (\gamma_n)_{\beta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\delta} + (\gamma_n)_{\delta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\beta} &= -(\gamma^m)^{\alpha\sigma} (\gamma_n)_{\lambda\sigma} (\gamma^n)_{\beta\delta} \end{aligned}$$

where we used the well-known property of the gamma matrices: $(\gamma^n)_{\alpha(\beta} (\gamma_n)_{\lambda\delta)} = 0$, which is valid in $D = 3, 4, 6$ and 10 . Now, let us simplify the sum of the second and fourth terms:

$$\begin{aligned} -(\gamma_n)_{\beta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\delta} - (\gamma_n)_{\delta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\beta} &= -(\gamma_n)_{\beta\lambda} [2\eta^{mn} \delta_\delta^\alpha - (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\delta}] \\ &\quad - (\gamma_n)_{\delta\lambda} [2\eta^{mn} \delta_\beta^\alpha - (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\beta}] \\ -(\gamma_n)_{\beta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\delta} - (\gamma_n)_{\delta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\beta} &= -2\delta_\delta^\alpha (\gamma^m)_{\beta\lambda} - 2\delta_\beta^\alpha (\gamma^m)_{\delta\lambda} \\ &\quad + (\gamma_n)_{\beta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\delta} + (\gamma_n)_{\delta\lambda} (\gamma^m)^{\alpha\sigma} (\gamma^n)_{\sigma\beta} \\ -(\gamma_n)_{\beta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\delta} - (\gamma_n)_{\delta\lambda} (\gamma^n)^{\alpha\sigma} (\gamma^m)_{\sigma\beta} &= -2\delta_\delta^\alpha (\gamma^m)_{\beta\lambda} - 2\delta_\beta^\alpha (\gamma^m)_{\delta\lambda} \\ &\quad - (\gamma^m)^{\alpha\sigma} (\gamma_n)_{\lambda\sigma} (\gamma^n)_{\beta\delta} \end{aligned}$$

and so, we obtain the following result

$$[\delta_1, \delta_2] \chi^\alpha = -\frac{i}{2} \epsilon_1^\beta \epsilon_2^\delta [-2(\gamma^m)^{\alpha\sigma} (\gamma_n)_{\lambda\sigma} (\gamma^n)_{\beta\delta} - 2\delta_\delta^\alpha (\gamma^m)_{\beta\lambda} - 2\delta_\beta^\alpha (\gamma^m)_{\delta\lambda}] \partial_m \chi^\lambda \quad (2.1.11)$$

and by using the anticommutator of gamma matrices ($-2(\gamma^m)^{\alpha\sigma}(\gamma_n)_{\lambda\sigma}(\gamma^n)_{\beta\delta} = -4\delta_\lambda^\alpha(\gamma^m)_{\beta\delta} + 2(\gamma_n)^{\alpha\sigma}(\gamma^m)_{\lambda\sigma}(\gamma^n)_{\beta\delta}$) and the fact that we are considering on-shell fields ($(\gamma_{\alpha\beta}^m)\partial_m\chi^\beta = 0$), we get

$$[\delta_1, \delta_2]\chi^\alpha = 2i\epsilon_1^\beta\epsilon_2^\delta(\gamma^m)_{\beta\delta}\partial_m\chi^\alpha \quad (2.1.12)$$

Finally, if we remember that the variation can be written in terms of its generators ($\delta_1\chi^\alpha = \epsilon_1^\lambda Q_\lambda(\chi^\alpha)$), we can rewrite this result in a more convenient way:

$$\{Q_\beta, Q_\delta\}\chi^\alpha = -2i(\gamma^m)_{\beta\delta}\partial_m\chi^\alpha \quad (2.1.13)$$

Consequently, from the equations 2.1.10 and 2.1.13, we can see that the generators satisfy an on-shell supersymmetry algebra.

An interesting point here is that the action 2.1.1 is invariant under the transformations 2.1.4 - 2.1.5 for any value of D ; and only in the cases $D = 3, 4, 6$ and 10 , these transformations are supersymmetry transformations (in the sense that the corresponding generators satisfy the algebra $\{Q_\beta, Q_\delta\}\Psi = -2i(\gamma^m)_{\beta\delta}\partial_m\Psi$, where $\Psi = (F_{mn}, \chi^\alpha)$). Therefore, in our case $D = 10$, we have a realization of on-shell supersymmetry.

For the non-abelian case, we will have the usual replacement of ordinary derivatives ∂_m by covariant derivatives $\nabla_m = \partial_m + [, A_m]$ for Lie algebra valued fields (which is the case for χ^α and F_{mn}). The action is:

$$S = \int d^{10}x \text{Tr}[-\frac{1}{4}F^{mn}F_{mn} + \frac{i}{2}\chi\gamma^m\nabla_m\chi] \quad (2.1.14)$$

Let us calculate the equations of motion. We will work with the usual conventions: $\text{Tr}(T^a T^b) = \delta^{ab}$, where T^a is an hermitian generator of the corresponding Lie algebra satisfying $[T^a, T^b] = f^{abc}T^c$. We will expand the fields in terms of the hermitian generators as follows $F_{mn} = F_{mn}^a T^a$, $A_m = A_m^a T^a$ and $\chi^\beta = \chi^{\beta a} T^a$. So, the action is

$$S = \int d^{10}x [-\frac{1}{4}F^{mna}F_{mna}^a + \frac{i}{2}\chi^{\alpha a}(\gamma^m)_{\alpha\beta}\partial_m\chi^{\beta a} + \frac{i}{2}\chi^{\alpha c}(\gamma^n)_{\alpha\beta}f^{abc}\chi^{\beta a}A_m^b] \quad (2.1.15)$$

The equations of motion are:

$$\partial_m \left(\frac{\partial\mathcal{L}}{\partial(\partial_m\chi^{\beta a})} \right) - \frac{\partial\mathcal{L}}{\partial\chi^{\beta a}} = 0 \quad (2.1.16)$$

$$-\frac{i}{2}\gamma_{\alpha\beta}^m\partial_m\chi^{\alpha a} - \frac{i}{2}\gamma_{\alpha\beta}^m\partial_m\chi^{\alpha a} - \frac{i}{2}(\gamma_{\beta\alpha}^m f^{cba}\chi^{\alpha c}A_m^b - \chi^{\alpha b}\gamma_{\alpha\beta}^m f^{abc}A_m^c) = 0 \quad (2.1.17)$$

$$\gamma_{\beta\alpha}^m(\partial_m\chi^{\alpha a} + f^{abc}\chi^{\alpha b}A_m^c) = 0 \quad (2.1.18)$$

or in a more compact notation:

$$(\gamma^m)_{\alpha\beta}\nabla_m\chi^\beta = 0 \quad (2.1.19)$$

Now, for the gauge field:

$$\begin{aligned} \partial_m \left(\frac{\partial\mathcal{L}}{\partial(\partial_m A_n^a)} \right) - \frac{\partial\mathcal{L}}{\partial A_n^a} &= 0 \\ -\partial_m F^{mna} + f^{abc}(\partial^n A^{mb} - \partial^m A^{nb} - f^{bde}A^{nd}A^{me})A_m^c &= \frac{i}{2}\chi^{\alpha c}(\gamma^n)_{\alpha\beta}f^{bac}\chi^{\beta b} \\ -\partial_m F^{mna} + f^{abc}F^{nmb}A_m^c &= \frac{i}{2}(\gamma^n)_{\alpha\beta}f^{abc}\chi^{\beta a}\chi^{\alpha c} \\ -\partial_m F^{mna} - f^{abc}F^{mnb}A_m^c &= \frac{i}{2}\gamma_{\alpha\beta}^n\{\chi^{\beta b}, \chi^{\alpha c}\} \\ -\nabla_m F^{mn} &= \frac{i}{2}\gamma_{\alpha\beta}^n\{\chi^\beta, \chi^\alpha\} \end{aligned}$$

therefore,

$$\nabla_m F^{mn} = -\frac{i}{2} \gamma_{\alpha\beta}^n \{\chi^\beta, \chi^\alpha\} \quad (2.1.20)$$

The transformations 2.1.4 - 2.1.5 have the same form for the non-abelian case (the only difference being now that the fields are matrices). This action will be invariant under these transformations only for the cases $D = 3, 4, 6$ and 10 (because one can show that this variation is proportional to the term $(\gamma^m)_{\alpha(\lambda}(\gamma_m)_{\beta\delta)}$, which vanishes for the cases mentioned), this is different from the abelian case, where the action was invariant for any value of D . The on-shell closure of the generator algebra will be satisfied, as in the abelian case, for the same previous values of D . The on-shell supersymmetry algebra is:

$$\{\nabla_\alpha, \nabla_\beta\} = -2i(\gamma^m)_{\alpha\beta} \nabla_m \quad (2.1.21)$$

where ∇_α and ∇_m are the corresponding fermionic and bosonic covariant derivatives which will be defined in the next section.

2.2 Superspace formulation of D=10 SYM

In this section our goal is to describe the supersymmetry transformations as coordinate transformations on superspace and then constructing a gauge theory on it.

Construction of a gauge theory on superspace

Most of our analysis is a simple generalization of the case $D = 4$ [4]. However we have tried to explain this construction in a self-consistent way by following the same ideas developed in the (bosonic) construction of gauge theories displayed in [5].

We start with a supermanifold X , which in our case it will be a topological space where each open set can be parameterised by a set of coordinates $Z^M = (x^m, \theta^\alpha)$ with $m = 0, \dots, 9$, $\alpha = 1, \dots, 16$. The x^m are even-Grassmann numbers and the θ^α are odd-Grassmann numbers, and they are real.

Then, we introduce the tangent bundle TX which is the union of tangent spaces at all points on the manifold. Here, we have the so-called *coordinate basis* which is given by the set $\{\partial_M\} = \{(\partial_m, \partial_\alpha)\}$. In order to define the 1-forms, we take the dual of the tangent bundle TX , and we obtain the so-called *cotangent bundle*, T^*X . Here, we also have a coordinate basis which is simply the dual basis of the coordinate basis of TX . This basis will be denoted by dZ^M .

A section s of TX is a map $s : X \rightarrow TX$ which when composing with the projection map of the bundle gives the identity map. There is a similar definition for a section w of T^*X . Now, we can define a tensor of degree (r, s) as a section of the direct product $TX \otimes \dots \otimes TX \otimes T^*X \otimes \dots \otimes T^*X$ where there are r times TX and s times T^*X . If we also use the wedge product (\wedge), we can construct forms of arbitrary degree. We will denote the vector space of k -forms by $\Omega^k(X, \mathbb{R})$ and the direct sum of these ones $\bigoplus_{k \in \mathbb{Z}_0^+} \Omega^k(X, \mathbb{R})$ forms the space of all forms. So, a

k -form $p^{(k)}$ can be expanded in the coordinate basis:

$$p^{(k)} = \frac{1}{k!} dZ^{M_k} \dots dZ^{M_1} p_{M_1} \dots p_{M_k} \quad (2.2.1)$$

where we have dropped the wedge product symbol and we will do so from now on, understanding that the product among forms is always a wedge product. It is important to realise the position of the coefficients and indices in this expansion. This is so because we want to minimize the use of signs in our computations⁷.

Now, it is necessary to set some rules regarding the (wedge) product of two forms. Let dZ^M be a 1-form (in fact it is, because we already saw that it is an element of the coordinate basis of T^*X). We define the quantity $|M|$ in the following way:

$$\begin{aligned} |M| &= 1 \quad , \quad \text{if } dZ^M \text{ is anticommuting} \\ |M| &= 0 \quad , \quad \text{if } dZ^M \text{ is commuting} \end{aligned}$$

so, we have a natural generalization of the simple bosonic case in the following rule

$$dZ^M dZ^N = -(-1)^{|M||N|} dZ^N dZ^M \quad (2.2.2)$$

and, for a k -form $p^{(k)}$ and l -form $q^{(l)}$ we have:

$$p^{(k)} q^{(l)} = (-1)^{kl + |p||q|} q^{(l)} p^{(k)} \quad (2.2.3)$$

where k and l are the form degrees of the $p^{(k)}$ and $q^{(l)}$ forms, respectively⁸.

⁷A similar convention is adopted in $D = 4$ [4].

⁸We are following a natural generalization of the case $D = 4$ discussed in detail in [4].

In the ordinary case (no superisation) we usually introduce the exterior derivative as a map $d : \Omega^k(A, \mathbb{R}) \rightarrow \Omega^{k+1}(A, \mathbb{R})$ (where A is just a manifold), which in the flat space is given by $d = dx^\mu \partial_\mu$. In our case we will adopt the natural generalization: $d : \Omega^k(X, \mathbb{R}) \rightarrow \Omega^{k+1}(X, \mathbb{R})$ (where X is our supermanifold defined above), which is defined by $d = dZ^M \partial_M$. In addition to this, in order to reduce the quantity of extra signs which one can get when passing forms, we will define the action of the exterior derivative from the right:

$$dp^{(k)} = \frac{1}{k!} dZ^{M_k} \dots dZ^{M_1} dZ^N \partial_N p_{M_1 \dots M_k} \quad (2.2.4)$$

when acting on a k -form, $p^{(k)}$.

We also have a product rule. Let $p^{(k)}$ and $q^{(l)}$ be a k -form and a l -form, respectively. So, we have:

$$d(p^{(k)} q^{(l)}) = p^{(k)} dq^{(l)} + (-1)^l dp^{(k)} q^{(l)} \quad (2.2.5)$$

where we get the sign $(-1)^l$ because of the right action of the exterior derivative.

It is not difficult to prove the nilpotency of this operator d :

$$\begin{aligned} d^2 p^{(k)} &= d^2 \left(\frac{1}{k!} dZ^{M_k} \dots dZ^{M_1} p_{M_1 \dots M_k} \right) \\ d^2 p^{(k)} &= \frac{1}{k!} d(dZ^{M_k} \dots dZ^{M_1} dZ^N \partial_N p_{M_1 \dots M_k}) \\ d^2 p^{(k)} &= \frac{1}{k!} dZ^{M_k} \dots dZ^{M_1} dZ^N dZ^M \partial_M \partial_N p_{M_1 \dots M_k} \\ d^2 p^{(k)} &= 0 \end{aligned}$$

since $dZ^N dZ^M = -(-1)^{|N||M|} dZ^M dZ^N$ and $\partial_M \partial_N = (-1)^{|N||M|} \partial_N \partial_M$.

Now, if we have a metric $g(p) : T_p X \times T_p X \rightarrow \mathbb{R}$ at each point p on the manifold X ($T_p X$ is the tangent space associated to the point p), we can write the metric in the coordinate basis: $g_{MN} = g(\partial_M, \partial_N)$. We can go to an orthonormal basis $E^A = E_A^M \partial_M$ ⁹ by requiring $g_{AB} = g(E_A, E_B) = E_A^M g_{MN} E_B^N = \begin{pmatrix} \eta_{mn} & 0 \\ 0 & \delta_{\alpha\beta} \end{pmatrix}$. This orthonormal basis is not unique as one can make a local rotation on the 10-vector E_m and the Majorana-Weyl spinor E_α without affecting the orthonormality condition.

As mentioned we will work with a flat superspace (because we just want a gauge theory on the superspace, namely SYM theory), that is with vanishing curvature. Therefore the connection w associated with the local Lorentz transformations mentioned above can be set to zero in a certain orthonormal basis of the tangent space. In addition, since we can make global Lorentz transformations and still have a vanishing curvature, we can use these transformations to choose a set of coordinates on the supermanifold where the tangent basis is the coordinate basis.

Let us define the following basis:

$$D_m = \partial_m \quad (2.2.6)$$

$$D_\alpha = \partial_\alpha + i(\gamma^m)_{\alpha\beta} \theta^\beta \partial_m \quad (2.2.7)$$

We will use it on the tangent bundle¹⁰. We can write this basis in terms of the coordinate basis: $D_A = D_A^M \partial_M$, where

$$D_A^M = \begin{pmatrix} \delta_n^m & 0 \\ +i(\gamma^m)_{\alpha\beta} \theta^\beta & \delta_\alpha^\beta \end{pmatrix} \quad (2.2.8)$$

⁹We will use letters from the beginning of the alphabet A, B, \dots to denote indices in the non-coordinate basis, and letters from the middle of the alphabet M, N, \dots to denote indices in the coordinate basis.

¹⁰Its usefulness will become clear later in this section.

If we define the 1-form basis by $E^A = dZ^M E_M^{A11}$, we find that

$$E_M^A = \begin{pmatrix} \delta_m^n & 0 \\ -i(\gamma^n)_{\beta\alpha}\theta^\alpha & \delta_\beta^\alpha \end{pmatrix} \quad (2.2.9)$$

Now, let us see the importance to work with this basis. Let us suppose we have a scalar field on superspace $\Phi(Z)$. Under the action of a supersymmetry transformation this field changes by $\delta\Phi(Z) = \epsilon^\alpha Q_\alpha \Phi(Z)$. On the other hand, if we make an arbitrary and infinitesimal change of coordinates on superspace $Z'^M = Z^M - \xi^M(Z)$, we get¹² $\Phi'(Z) = \Phi(Z) + \xi^M(Z)\partial_M\Phi(Z)$. Therefore, if we want to realize supersymmetry transformations as coordinates transformations on superspace, we must have: $\epsilon^\alpha Q_\alpha = \xi^M(Z)\partial_M$ (or what amounts the same $\delta Z^M = -\xi^M(Z) = -\epsilon^\alpha Q_\alpha Z^M$), with the generators Q_α satisfying the supersymmetry algebra $\{Q_\alpha, Q_\beta\} = -2i(\gamma^m)_{\alpha\beta}\partial_m$.

If we define $Q_\alpha = \partial_\alpha - i(\gamma^m)_{\alpha\beta}\theta^\beta\partial_m$, it is easy to show that these Q_α satisfy the defining property for supersymmetry generators:

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \{(\partial_\alpha - i(\gamma^m)_{\alpha\lambda}\theta^\lambda\partial_m), (\partial_\beta - i(\gamma^n)_{\beta\sigma}\theta^\sigma\partial_n)\} \\ &= \{\partial_\alpha, \partial_\beta\} - \{\partial_\alpha, i(\gamma^n)_{\beta\sigma}\theta^\sigma\partial_n\} - \{i(\gamma^m)_{\alpha\lambda}\theta^\lambda\partial_m, \partial_\beta\} - (\gamma^m)_{\alpha\lambda}\theta^\lambda(\gamma^n)_{\beta\sigma}\theta^\sigma[\partial_m, \partial_n] \\ &= -i\partial_\alpha((\gamma^n)_{\beta\sigma}\theta^\sigma\partial_n) - i(\gamma^n)_{\beta\sigma}\theta^\sigma\partial_n\partial_\alpha - i(\gamma^m)_{\alpha\lambda}\theta^\lambda\partial_m\partial_\beta - i\partial_\beta((\gamma^m)_{\alpha\lambda}\theta^\lambda\partial_m) \\ &= -i(\gamma^m)_{\alpha\beta}\partial_m - i(\gamma^m)_{\alpha\beta}\partial_m \\ &= -2i(\gamma^m)_{\alpha\beta}\partial_m \end{aligned}$$

Now, let us consider a vector field on superspace, $\Phi^M(Z)$. It is well known that this field satisfies the transformation rule $\Phi'^M(Z') = \frac{\partial Z'^M}{\partial Z^N}\Phi^N(Z)$. So, under a small change of coordinates $Z'^M = Z^M - \xi^M(Z)$, we will have:

$$\begin{aligned} \Phi'^N(Z') &= \Phi^M(Z)(\delta_M^N - \partial_M\xi^N) \\ \Phi'^N(Z) - \xi^M\partial_M\Phi'^N(Z) &= \Phi^N(Z) - (\partial_M\xi^N)\Phi^M(Z) \\ \Phi'^N(Z) &= \Phi^N(Z) - (\partial_M\xi^N)\Phi^M(Z) + \xi^M\partial_M\Phi^N(Z) \end{aligned}$$

at first order in $\xi^M(Z)$. We can see that in this case we have one additional term on the right hand side (compared to the scalar case). This term arises because the basis is also changing¹³. So, here is where the usefulness to work with the basis defined above becomes clear: This basis does not change when changing coordinate systems by a SUSY coordinate transformation ($\delta Z^M = -\epsilon^\alpha Q_\alpha Z^M$). Basically that property is a consequence of the following result:

$$\begin{aligned} \{D_\alpha, Q_\beta\} &= \{(\partial_\alpha + i(\gamma^m)_{\alpha\lambda}\theta^\lambda\partial_m), (\partial_\beta - i(\gamma^n)_{\beta\sigma}\theta^\sigma\partial_n)\} \\ &= \{\partial_\alpha, \partial_\beta\} - \{\partial_\alpha, i(\gamma^n)_{\beta\sigma}\theta^\sigma\partial_n\} + \{i(\gamma^m)_{\alpha\lambda}\theta^\lambda\partial_m, \partial_\beta\} + (\gamma^m)_{\alpha\lambda}\theta^\lambda(\gamma^n)_{\beta\sigma}\theta^\sigma[\partial_m, \partial_n] \\ &= -i(\gamma^m)_{\alpha\beta}\partial_m + i(\gamma^m)_{\alpha\beta}\partial_m \\ &= 0 \end{aligned}$$

¹¹In this way E_M^A is the inverse of E_A^M .

¹²An scalar field, by definition, obeys the relation $\Phi'(Z') = \Phi(Z)$, so under a small change $Z'^M = Z^M - \xi^M(Z)$, we are left with $\Phi'(Z') = \Phi(Z') + \xi^M(Z')\partial_M\Phi(Z')$ at first order in $\xi^M(Z)$.

¹³We can see this in the following way: The change of the coordinate basis is given by $\partial'_M = \frac{\partial Z^N}{\partial Z'^M}\partial_N$. It is this transformation rule which induces the transformation rule for the coordinates of the vector field. So, if we had a basis which does not change when making a certain coordinate transformation, we would get the same results as in the scalar case, i.e a single term representing just the change of coordinates.

Analogously it is easy to show that $[D_\alpha, Q_\beta] = 0$. Now, let us prove that $\delta D_A = 0$ under SUSY coordinate transformations. For that, let us see what the SUSY coordinate transformations are:

$$x'^m = x^n - \epsilon^\alpha \left(\frac{\partial}{\partial \theta^\alpha} - i(\gamma^m)_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^m} \right) x^n \quad (2.2.10)$$

$$x'^n = x^n + i\epsilon^\alpha (\gamma^n)_{\alpha\beta} \theta^\beta \quad (2.2.11)$$

and

$$\theta'^\beta = \theta^\beta - \epsilon^\beta \quad (2.2.12)$$

Now, let us see what happens with the basis D_α , when changing the coordinates by a SUSY coordinate transformation:

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial \theta^\alpha} + i(\gamma^n)_{\alpha\beta} \theta^\beta \frac{\partial}{\partial x^n} \\ D_\alpha &= \frac{\partial \theta'^\lambda}{\partial \theta^\alpha} \frac{\partial}{\partial \theta'^\lambda} + \frac{\partial x'^m}{\partial \theta^\alpha} \frac{\partial}{\partial x'^m} + i(\gamma^n)_{\alpha\beta} (\theta'^\beta + \epsilon^\beta) \left(\frac{\theta'^\sigma}{\partial x^n} \frac{\partial}{\partial \theta'^\sigma} + \frac{\partial x'^m}{\partial x^n} \frac{\partial}{\partial x'^m} \right) \\ D_\alpha &= \frac{\partial}{\partial \theta'^\alpha} - i\epsilon^\beta (\gamma^n)_{\alpha\beta} \frac{\partial}{\partial x'^n} + i(\gamma^n)_{\alpha\beta} \theta'^\beta \frac{\partial}{\partial x'^n} + i\epsilon^\beta (\gamma^n)_{\alpha\beta} \frac{\partial}{\partial x'^n} \\ D_\alpha &= \frac{\partial}{\partial \theta'^\alpha} + i(\gamma^n)_{\alpha\beta} \theta'^\beta \frac{\partial}{\partial x'^n} \\ \rightarrow D_\alpha &= D'_\alpha \end{aligned}$$

It is easy to see that $D_m = D'_m$. Therefore $\delta D_M = D'_M - D_M = 0$ for a SUSY coordinate transformation. So, if we write our quantities in this basis $D_M (= E_M)$, a coordinate transformation on superspace will imply a SUSY transformation on these quantities.

Now we are ready to introduce a gauge theory on superspace. Let G be the gauge group (Lie group). As usual we introduce a 1-form connection A in order to define covariant quantities, in particular the covariant derivative:

$$\nabla_m = \partial_m + A_m \quad (2.2.13)$$

$$\nabla_\alpha = D_\alpha + A_\alpha \quad (2.2.14)$$

We will convey in using an right action of the covariant derivative¹⁴ and A is Lie algebra valued¹⁵. The next step is to define the 2-form field strength as $F = dA + A \wedge A$ (and

¹⁴In a more compact notation this would be written as:

$$\nabla = d + \wedge A$$

¹⁵Of course if the covariant derivative acts on some field Φ which transforms in the adjoint representation of the gauge group, the corresponding covariant derivative takes the form:

$$\begin{aligned} \nabla_m \Phi &= \partial_m \Phi + [\Phi, A_m] \\ \nabla_\alpha \Phi &= D_\alpha \Phi + [\Phi, A_\alpha] \end{aligned}$$

$A = E^C A_C$), therefore:

$$F = dA + A \wedge A \quad (2.2.15)$$

$$\frac{1}{2} E^B E^C F_{CB} = d(E^B A_B) + E^A A_A (E^B A_B) \quad (2.2.16)$$

$$\frac{1}{2} E^B E^C F_{CB} = E^B E^C D_C A_B + d(E^B) A_B + (-1)^{AB} E^A E^B A_A A_B \quad (2.2.17)$$

$$\frac{1}{2} E^B E^C F_{CB} = E^B E^C D_C A_B + (-1)^{BC} E^B E^C A_B A_C + \frac{1}{2} E^C E^D T^B_{DC} A_B \quad (2.2.18)$$

$$\frac{1}{2} E^B E^C F_{CB} = E^B E^C D_C A_B + (-1)^{BC} E^B E^C A_B A_C + \frac{1}{2} E^B E^C T^D_{CB} A_D \quad (2.2.19)$$

where we used the property $A_A E^B = (-1)^{A.B+0.1} E^B A_A$ and defined the *torsion* by $dE^B = \frac{1}{2} E^C E^D T^B_{CD}$. Therefore

$$F_{CB} = 2D_{[C} A_{B]} + 2(-1)^{BC} A_{[B} A_{C]} + T^D_{CB} A_D \quad (2.2.20)$$

This leads us to the following explicit expressions for the field strength components:

$$F_{\alpha\beta} = D_\alpha A_\beta + D_\beta A_\alpha - \{A_\alpha, A_\beta\} - 2i(\gamma^m)_{\alpha\beta} A_m \quad (2.2.21)$$

$$F_{m\alpha} = \partial_m A_\alpha - D_\alpha A_m - [A_m, A_\alpha] \quad (2.2.22)$$

$$F_{mn} = \partial_m A_n - \partial_n A_m - [A_m, A_n] \quad (2.2.23)$$

where the only non-vanishing component of torsion is $T^m_{\alpha\beta} = 2i(\gamma^m_{\alpha\beta})^{16}$. These expressions can be written in a more compact form¹⁷

$$F_{\alpha\beta} = \{\nabla_\alpha, \nabla_\beta\} - 2i(\gamma^m)_{\alpha\beta} \nabla_m \quad (2.2.25)$$

$$F_{m\alpha} = [\nabla_m, \nabla_\alpha] \quad (2.2.26)$$

$$F_{mn} = [\nabla_m, \nabla_n] \quad (2.2.27)$$

¹⁶This can be computed from the definition of torsion:

$$\begin{aligned} T^B &= dE^B \\ \frac{1}{2} E^C E^D T^B_{DC} &= d(dZ^M E_M^B) \\ \frac{1}{2} E^C E^D T^B_{DC} &= dZ^M dE_M^B \\ \frac{1}{2} E^C E^D T^B_{DC} &= E^A E_A^M E^C D_C E_M^B \\ \frac{1}{2} E^A E^C T^B_{CA} &= (-1)^{C(A+M)} E^A E^C E_A^M D_C E_M^B \\ &\rightarrow T^B_{CA} = 2(-1)^{C(A+M)} E_{[A}^M D_C] E_M^B \end{aligned}$$

where E_A^M and E_M^A are given in the equations 2.2.8 and 2.2.9, respectively. In this way it can be shown that all of the components of the torsion vanish except that of dimension 0: $T^m_{\alpha\beta} = -2i(\gamma^m)_{\alpha\beta}$.

¹⁷These ones can be obtained by direct computation or from the definition of field strength [5], [9]:

$$\nabla^2 s = \nabla(E^A \nabla_A s) = E^A E^B \nabla_B \nabla_A s + dE^A \nabla_A s = F \wedge s \quad (2.2.24)$$

where s is a 0-form (scalar field). Therefore $F_{AB} = [\nabla_A, \nabla_B] + T^C_{BA} \nabla_C$.

Finally, we obtain the Bianchi identities as follows:

$$dF = d^2 A + A \wedge dA - dA \wedge A \quad (2.2.28)$$

$$dF = A \wedge (F - A \wedge A) - (F - A \wedge A) \wedge A \quad (2.2.29)$$

$$dF = A \wedge F - F \wedge A \quad (2.2.30)$$

$$dF + [F, A] = 0 \quad (2.2.31)$$

$$\rightarrow \nabla F = 0 \quad (2.2.32)$$

which in component form is

$$\begin{aligned} \nabla F &= \nabla \left(\frac{1}{2} E^A E^B F_{BA} \right) = 0 \\ \frac{1}{2} E^A E^B E^C \nabla_C F_{BA} + \frac{1}{2} E^A dE^B F_{BA} - \frac{1}{2} dE^A E^B F_{BA} &= 0 \\ \frac{1}{2} E^A E^B E^C \nabla_C F_{BA} + \frac{1}{4} E^A E^C E^D T_{DC}^B F_{BA} - \frac{1}{4} E^C E^D T_{DC}^A E^B F_{BA} &= 0 \\ \frac{1}{2} E^A E^C E^D \nabla_D F_{CA} + \frac{1}{4} E^A E^C E^D T_{DC}^B F_{BA} - \frac{1}{4} E^D E^C T_{CD}^B E^A F_{AB} &= 0 \\ \frac{1}{2} E^A E^C E^D \nabla_D F_{CA} + \frac{1}{4} E^A E^C E^D T_{DC}^B F_{BA} - (-1)^{A(B+C+D)} \frac{1}{4} E^D E^C E^A T_{CD}^B F_{AB} &= 0 \\ \frac{1}{2} E^A E^C E^D \nabla_D F_{CA} + \frac{1}{4} E^A E^C E^D T_{DC}^B F_{BA} - (-1)^{AB+CD+1} \frac{1}{4} E^A E^C E^D T_{CD}^B F_{AB} &= 0 \\ \frac{1}{2} E^A E^C E^D \nabla_D F_{CA} + \frac{1}{4} E^A E^C E^D T_{DC}^B F_{BA} - (-1)^{2AB+2CD+1} \frac{1}{4} E^A E^C E^D T_{DC}^B F_{BA} &= 0 \\ \frac{1}{2} E^A E^C E^D \nabla_D F_{CA} + \frac{1}{2} E^A E^C E^D T_{DC}^B F_{BA} &= 0 \\ \rightarrow \nabla_{[D} F_{CA]} + T_{[DC}^B F_{|B|A]} &= 0 \end{aligned}$$

Next we will solve these equations.

Description via super field-strength and super Bianchi identities

The Bianchi identity can be split into the following four equations:

$$\nabla_{(\alpha} F_{\beta\sigma)}(x, \theta) + 2i(\gamma^n)_{(\alpha\beta} F_{\sigma)n}(x, \theta) = 0 \quad (2.2.33)$$

$$\nabla_m F_{\alpha\beta}(x, \theta) + 2\nabla_{(\alpha} F_{\beta)m)}(x, \theta) - 2i(\gamma^n)_{\alpha\beta} F_{nm}(x, \theta) = 0 \quad (2.2.34)$$

$$2\nabla_{[m} F_{n]\alpha}(x, \theta) + \nabla_{\alpha} F_{mn}(x, \theta) = 0 \quad (2.2.35)$$

$$\nabla_{[m} F_{np]}(x, \theta) = 0 \quad (2.2.36)$$

We will work with the field strength $F_{AB} = (F_{mn}, F_{m\alpha}, F_{\alpha\beta})$ satisfying these four conditions. Now, in order to have the right spectrum and the equations of motion of D=10 SYM, we will impose two constraints on the field strength $F_{\alpha\beta}$, which are called *conventional* and *physical* constraints¹⁸. The first one will restrict the number of gauginos of the theory to one, and the

¹⁸Maybe it is necessary to say some more words about this procedure. We have found the Bianchi identities, which are conditions that F_{AB} must satisfy if this field strength can be written in terms of a certain gauge field A , $F = dA$. Usually, one uses the gauge invariance of the gauge field and the equations of motion to remove the unphysical degrees of freedom. Here, the gauge invariance is encoded in the field strength satisfying the Bianchi identities, so what is remaining is to use the corresponding equations of motion. But we do not have an action in this superspace formulation (from which one can derive the corresponding equations of motion). So, we have to impose some constraints on the field strength in order to satisfy the equations of motion trivially (one similar fact happens when you work with a self-dual field strength in ordinary YM theory, and the Bianchi identities imply the corresponding equations of motion). We will see that this procedure will give us the right spectrum and the equations of motion already studied in the previous section.

second one will complement to the first one to obtain D=10 SYM.

Conventional and dynamical constraints

Conventional constraint: This constraint is given by

$$(\gamma^m)^{\alpha\beta} F_{\alpha\beta}(x, \theta) = 0 \quad (2.2.37)$$

Let us explain the physical meaning of this equation. We can expand the symmetric bispinor $F_{\alpha\beta}(x, \theta)$ and the vector-spinor $F_{m\alpha}(x, \theta)$ in irreducible tensor representations:

$$F_{\alpha\beta}(x, \theta) = (\gamma^m)_{\alpha\beta} f_m(x, \theta) + (\gamma^{mnpqr})_{\alpha\beta} f_{mnpqr}(x, \theta) \quad (2.2.38)$$

$$F_{m\alpha}(x, \theta) = \tilde{f}_{m\alpha}(x, \theta) + (\gamma_m)_{\beta\alpha} \psi^\beta(x, \theta) \quad (2.2.39)$$

where in 2.2.38 we used the fact that the spinor indices have the same chirality, and in 2.2.39 the first term is a γ -traceless vector-spinor ($(\gamma^m)^{\alpha\beta} \tilde{f}_{m\alpha}(x, \theta) = 0$). Now, let us focus on the field $f_m(x, \theta)$ for a moment. We can expand it in powers of θ :

$$f_m(x, \theta) = f_m^{(0)}(x) + \theta^\alpha f_{m\alpha}^{(1)}(x) + \theta^\alpha \theta^\beta f_{m\alpha\beta}^{(2)}(x) + \dots \quad (2.2.40)$$

where the upper index over a term (in this case $f^{(i)}$) will represent the power of θ in which this term appears in the corresponding θ -expansion. We see a vector-spinor in this last expansion, namely, $f_{m\alpha}^{(1)}(x)$. This one can be also expanded in irreducible representations:

$$f_{m\alpha}^{(1)}(x) = \tilde{f}_{m\alpha}^{(1)}(x) + (\gamma_m)_{\alpha\beta} \lambda^\beta(x) \quad (2.2.41)$$

where, once again, the term $\tilde{f}_{m\alpha}^{(1)}(x)$ is a γ -traceless vector-spinor. The other term tells us that there exists a spinor field in this expansion, $\lambda^\beta(x)$ (whose physical dimension is the same as $f_{m\alpha}^{(1)}(x)$, that is $\frac{3}{2}$). However, we can easily see that from the expansion of $\psi^\beta(x, \theta)$ (in 2.2.39)

$$\psi^\beta(x, \theta) = (\psi^{(0)})^\beta(x) + \theta^\alpha (\psi^{(0)})_{\alpha}{}^\beta(x) + \dots \quad (2.2.42)$$

the presence of another spinor field in this description, namely, $(\psi^{(0)})^\beta(x)$ (whose physical dimension is the same as $F_{m\alpha}$, that is $\frac{3}{2}$). Because we know that we just have a single spinor field in D=10 SYM (whose physical dimension is $\frac{3}{2}$), we have to choose one of them and get rid of the other one. This is precisely what the conventional constraint does by fixing $f_m(x, \theta) = 0$, hence dropping the spinor field $\lambda^\beta(x)$.

Imposing only this constraint is not sufficient to have D=10 SYM. We must also impose the dynamical constraint.

Dynamical constraint This constraint is given by

$$(\gamma^{mnpqr})^{\alpha\beta} F_{\alpha\beta}(x, \theta) = 0 \quad (2.2.43)$$

or in its weaker form

$$(\gamma^{mnpqr})_{\alpha\beta} (\gamma_{mnpqr})^{\lambda\sigma} F_{\lambda\sigma}(x, \theta) = 0 \quad (2.2.44)$$

This constraint eliminates the 5-form in the expansion of $F_{\alpha\beta}$. Therefore, the full action of the conventional and dynamical constraints can be summarized in the following condition:

$$F_{\alpha\beta}(x, \theta) = 0 \quad (2.2.45)$$

Hence, replacing this condition in the Bianchi identities, we are left with

$$(\gamma^n)_{(\alpha\beta}F_{\sigma)n}(x, \theta) = 0 \quad (2.2.46)$$

$$\nabla_{(\alpha}F_{\beta)m}(x, \theta) - i(\gamma^n)_{\alpha\beta}F_{nm}(x, \theta) = 0 \quad (2.2.47)$$

$$2\nabla_{[m}F_{n]\alpha}(x, \theta) + \nabla_{\alpha}F_{mn}(x, \theta) = 0 \quad (2.2.48)$$

$$\nabla_{[m}F_{np]}(x, \theta) = 0 \quad (2.2.49)$$

The equation 2.2.46 will put $\tilde{f}_{m\alpha}(x, \theta)$ to zero. Let us see how this works. We use our earlier expansion for $F_{m\alpha}$ (2.2.39) in the equation 2.2.46:

$$(\gamma^n)_{(\alpha\beta}F_{\sigma)n}(x, \theta) = 0 \quad (2.2.50)$$

$$(\gamma^n)_{(\alpha\beta}[\tilde{f}_{\sigma)n}(x, \theta) + (\gamma_n)_{\sigma\lambda}\psi^\lambda(x, \theta)] = 0 \quad (2.2.51)$$

$$(\gamma^n)_{(\alpha\beta}\tilde{f}_{\sigma)n}(x, \theta) + (\gamma^n)_{(\alpha\beta}(\gamma_n)_{\sigma\lambda}\psi^\lambda(x, \theta) = 0 \quad (2.2.52)$$

$$\text{by using } (\gamma^m)_{(\alpha\beta}(\gamma_m)_{\lambda)\delta} = 0 \rightarrow (\gamma^n)_{(\alpha\beta}\tilde{f}_{\sigma)n}(x, \theta) = 0 \quad (2.2.53)$$

$$\rightarrow \tilde{f}_{\sigma n}(x, \theta) = 0 \quad (2.2.54)$$

so we are left with

$$F_{m\alpha} = (\gamma_m)_{\alpha\beta}\psi^\beta \quad (2.2.55)$$

The equation 2.2.47 is symmetric in both spinor indices, so this can be expanded in terms of irreducible tensor representations (1-form and 5-form). Because these ones are linearly independent, it must be true that each coefficient in the expansion has to vanish, therefore:

$$(\gamma_s)^{\alpha\beta}(\nabla_{(\alpha}F_{\beta)m}(x, \theta) - i(\gamma^n)_{\alpha\beta}F_{nm}(x, \theta)) = 0 \quad (2.2.56)$$

$$(\gamma_{pqrst})^{\alpha\beta}(\nabla_{(\alpha}F_{\beta)m}(x, \theta) - i(\gamma^n)_{\alpha\beta}F_{nm}(x, \theta)) = 0 \quad (2.2.57)$$

Now, we will use the orthogonality property of the γ -matrices and the well-known fact that they satisfy¹⁹ $(\gamma^m)_{\alpha\beta}(\gamma^n)^{\beta\lambda} = \eta^{mn}\delta_\alpha^\lambda$. We also have to have in mind the symmetry properties of the components of the field strength ($F_{mn} = -F_{nm}$, $F_{m\alpha} = -F_{\alpha m}$, $F_{\alpha\beta} = F_{\beta\alpha}$). So, by using the equation 2.2.55, we obtain

$$-(\gamma_s)^{\alpha\beta}(\gamma_m)_{(\beta|\lambda|}\nabla_{\alpha)}\psi^\lambda(x, \theta) - i(\gamma_s)^{\alpha\beta}(\gamma^n)_{\alpha\beta}F_{nm}(x, \theta) = 0 \quad (2.2.58)$$

$$-(\gamma_s)^{\alpha\beta}(\gamma_m)_{\beta\lambda}\nabla_{\alpha}\psi^\lambda(x, \theta) - 16iF_{sm}(x, \theta) = 0 \quad (2.2.59)$$

$$-(\eta_{sm}\delta_\lambda^\alpha + (\gamma_{sm})^\alpha{}_\lambda)\nabla_{\alpha}\psi^\lambda(x, \theta) - 16iF_{sm}(x, \theta) = 0 \quad (2.2.60)$$

$$-(\gamma_{sm})^\alpha{}_\lambda\nabla_{\alpha}\psi^\lambda(x, \theta) - 16iF_{sm}(x, \theta) - \eta_{sm}\delta_\lambda^\alpha\nabla_{\alpha}\psi^\lambda(x, \theta) = 0 \quad (2.2.61)$$

The first two terms are antisymmetric in the s and m indices, and the last one is symmetric in those indices. Therefore

$$16iF_{sm}(x, \theta) = -(\gamma_{sm})^\alpha{}_\lambda\nabla_{\alpha}\psi^\lambda(x, \theta) \rightarrow F_{sm}(x, \theta) = \frac{i}{16}(\gamma_{sm})^\alpha{}_\lambda\nabla_{\alpha}\psi^\lambda(x, \theta) \quad (2.2.62)$$

$$\delta_\lambda^\alpha\nabla_{\alpha}\psi^\lambda(x, \theta) = 0 \quad (2.2.63)$$

Now, we will expand the bispinor $\nabla_{\alpha}\psi^\lambda(x, \theta)$ (where the spinor indices have opposite chirality):

$$\nabla_{\alpha}\psi^\lambda(x, \theta) = \delta_\alpha^\lambda C^{(0)}(x, \theta) + (\gamma_{mn})^\lambda{}_{\alpha}C^{[mn]}(x, \theta) + (\gamma_{mnpq})^\lambda{}_{\alpha}C^{[mnpq]}(x, \theta) \quad (2.2.64)$$

¹⁹Just to remember. We are using the following definition the symmetrization: $A_{m_1 m_2 \dots m_n} = \frac{1}{n!}(\sum_{\sigma_n} A_{m_{\sigma_1} m_{\sigma_2} \dots m_{\sigma_n}})$, where σ_n is the symmetric group of n elements.

So, the equation 2.2.63 tells us that $C^{(0)}(x, \theta) = 0$. By using the orthogonality property of the γ -matrices, the equation 2.2.62 gives us:

$$F_{sm}(x, \theta) = \frac{i}{16}(\gamma_{sm})^\alpha{}_\lambda(\gamma_{pq})^\lambda{}_\alpha C^{[pq]}(x, \theta) \quad (2.2.65)$$

$$F_{sm}(x, \theta) = \frac{i}{16}Tr(\gamma_{sm}\gamma_{pq})C^{[pq]}(x, \theta) \quad (2.2.66)$$

$$F_{sm}(x, \theta) = -2iC_{[sm]}(x, \theta) \quad (2.2.67)$$

Now, we replace these results in the equation 2.2.57 to obtain (using the orthogonality between the 1-form and the 5-form)

$$(\gamma_{pqrst})^{\alpha\beta}(\nabla_{(\alpha}F_{\beta)m}(x, \theta)) = 0 \quad (2.2.68)$$

$$(\gamma_{pqrst})^{\alpha\beta}(\gamma_m)_{(\beta|\lambda|}(\nabla_\alpha\psi^\lambda(x, \theta)) = 0 \quad (2.2.69)$$

$$(\gamma_{pqrst})^{\alpha\beta}(\gamma_m)_{\beta\lambda}(\nabla_\alpha\psi^\lambda(x, \theta)) = 0 \quad (2.2.70)$$

$$\frac{i}{2}Tr(\gamma_{pqrst}\gamma_m\gamma_{lk})F^{lk}(x, \theta) + Tr(\gamma_{pqrst}\gamma_m\gamma_{lkuv})C^{[lkuv]}(x, \theta) = 0 \quad (2.2.71)$$

$$Tr(\gamma_{pqrst}(\gamma_{mlkuv} + 4\eta_{m[l}\gamma_{kuv]}))C^{[lkuv]}(x, \theta) = 0 \quad (2.2.72)$$

$$16.5!\delta_{pqrst}^{mlkuv}C_{[lkuv]}(x, \theta) = 0 \quad (2.2.73)$$

$$\rightarrow C_{[qrst]}(x, \theta) = 0 \quad (2.2.74)$$

where we used the properties²⁰ $\gamma_m\gamma_{lk} = \gamma_{mlk} + 2\eta_{m[l}\gamma_{k]}$, $\gamma_m\gamma_{lkuv} = \gamma_{mlkuv} + 4\eta_{m[l}\gamma_{kuv]}$, and the orthogonality property $Tr(\gamma^{m_1\dots m_n}\gamma_{p_1\dots p_k}) = (-1)^{\frac{n(n-1)}{2}}\delta^{nk}16.n!\delta_{p_1\dots p_k}^{m_1\dots m_n}$. Therefore, we have the following result

$$\nabla_\alpha\psi^\lambda(x, \theta) = \frac{i}{2}(\gamma_{mn})^\lambda{}_\alpha F^{mn}(x, \theta) \quad (2.2.75)$$

The equation 2.2.48 will help us to understand what will happen when we expand our superfields $F^{mn}(x, \theta)$ and $\psi^\lambda(x, \theta)$ at higher θ -levels. So, let us do the corresponding θ -expansions:

$$\psi^\lambda(x, \theta) = \chi^\lambda(x) + \theta^\alpha(\psi^{(1)})_\alpha{}^\lambda(x) + \dots \quad (2.2.76)$$

$$F^{mn}(x, \theta) = F^{mn}(x) + \theta^\alpha(F^{(1)})_\alpha{}^{mn}(x) + \dots \quad (2.2.77)$$

where we made the replacement $(\psi^{(0)})^\lambda(x)$ by $\chi^\lambda(x)$. So, we can see that the zeroth θ -levels of each expansion contain the physical fields which we are looking for, namely the field-strength $F^{mn}(x)$ and the gaugino $\chi^\lambda(x)$. We can relate the θ -levels of the corresponding expansions by using the equation 2.2.75. For instance, the first θ -level of $\psi^\lambda(x, \theta)$ is related to the zeroth θ -level of $F^{mn}(x, \theta)$. With the same argument we can convince ourselves that the $(n+1)$ th θ -level of $\psi^\lambda(x, \theta)$ is related to the (n) th θ -level of $F^{mn}(x, \theta)$. So no new field will appear at higher θ -levels, these levels will just contain derivatives of the physical fields $\chi^\lambda(x)$ and $F^{mn}(x)$. Another way to see this is looking at the second θ -level of $\psi^\lambda(x, \theta)$. So, let us derive this superfield twice and then use the equation 2.2.75

$$\nabla_\alpha\nabla_\beta\psi^\lambda(x, \theta) = \frac{i}{2}(\gamma_{mn})^\lambda{}_\beta\nabla_\alpha F^{mn}(x, \theta) \quad (2.2.78)$$

and together with the equations 2.2.48 and 2.2.55

$$\nabla_\alpha F_{mn}(x, \theta) + 2\nabla_{[m}F_{n]\alpha}(x, \theta) = 0 \quad (2.2.79)$$

$$\nabla_\alpha F_{mn}(x, \theta) + 2\nabla_{[m}(\gamma_{n])\alpha\delta}\psi^\delta(x, \theta) = 0 \quad (2.2.80)$$

$$\nabla_\alpha F_{mn}(x, \theta) + 2(\gamma_{[n])\alpha\delta}|\nabla_m]\psi^\delta(x, \theta) = 0 \quad (2.2.81)$$

²⁰These identities can be proven by using the same steps used in the footnote of the page 4.

we obtain

$$\nabla_\alpha \nabla_\beta \psi^\lambda(x, \theta) = \frac{i}{2} (\gamma^{mn})^\lambda{}_\beta [-2(\gamma_{[n})_{|\alpha\delta|} \nabla_m] \psi^\delta(x, \theta)] \quad (2.2.82)$$

$$\nabla_\alpha \nabla_\beta \psi^\lambda(x, \theta) = -i (\gamma^{mn})^\lambda{}_\beta [(\gamma_n)_{\alpha\delta} \nabla_m \psi^\delta(x, \theta)] \quad (2.2.83)$$

which means that the second θ -level is just a derivative of the term appearing in the zeroth θ -level, which is the gaugino $\chi^\lambda(x)$, etc.

Finally the equation 2.2.49 at zeroth θ -level gives us the ordinary Bianchi identity for $F_{mn}(x)$:

$$\nabla_{[m} F_{np]}(x) = 0 \quad (2.2.84)$$

One can show that at higher θ -levels we will find expressions which become identities when we use the equation 2.2.84. Therefore, we have recovered the ordinary Bianchi identities.

So we have the right field content of $D = 10$ SYM and the desired ordinary Bianchi identities for the field strength $F^{mn}(x)$. Now, we will show that we also have the correct equations of motion.

We should use the full constraint²¹ $\{\nabla_\alpha, \nabla_\beta\} = 2i(\gamma^m)_{\alpha\beta} \nabla_m$:

$$\begin{aligned} -\frac{i}{2} \{\nabla_\alpha, \nabla_\beta\} \psi^\beta(x, \theta) &= (\gamma^m)_{\alpha\beta} \nabla_m \psi^\beta(x, \theta) \\ -(\gamma^{mn})^\beta{}_{(\alpha} [(\gamma_{|n})_{\beta\delta} \nabla_m] \psi^\delta(x, \theta)] &= (\gamma^m)_{\alpha\beta} \nabla_m \psi^\beta(x, \theta) \\ -\frac{1}{2} [(\gamma^{mn})^\beta{}_\alpha (\gamma_n)_{\beta\delta} \nabla_m \psi^\delta(x, \theta) + (\gamma^{mn})^\beta{}_\beta (\gamma_n)_{\alpha\delta} \nabla_m \psi^\delta(x, \theta)] &= (\gamma^m)_{\alpha\beta} \nabla_m \psi^\beta(x, \theta) \\ -\frac{1}{2} (\gamma_n \gamma^{mn})_{\delta\alpha} \nabla_m \psi^\delta(x, \theta) &= (\gamma^m)_{\alpha\beta} \nabla_m \psi^\beta(x, \theta) \\ -\frac{1}{4} (2\gamma^{mn} \gamma_n - 2\gamma_n \gamma^n \gamma^m)_{\delta\alpha} \nabla_m \psi^\delta(x, \theta) &= (\gamma^m)_{\alpha\beta} \nabla_m \psi^\beta(x, \theta) \\ \frac{7}{2} (\gamma^m)_{\alpha\beta} \nabla_m \psi^\beta(x, \theta) &= 0 \end{aligned}$$

where we used $Tr(\gamma^{mn}) = 0$ and $\gamma^n \gamma_n = 10$. So we obtain

$$(\gamma^m)_{\alpha\beta} \nabla_m \psi^\beta(x, \theta) = 0 \quad (2.2.85)$$

which at zeroth θ -level is the Dirac equation for the gaugino:

$$(\gamma^m)_{\alpha\beta} \nabla_m \chi^\beta(x) = 0 \quad (2.2.86)$$

But we can extract further information from the equation 2.2.85. In particular, we will see that

²¹Notice that this is consistent with the SUSY algebra $\{Q_\alpha, Q_\beta\} = -2i(\gamma^m)_{\alpha\beta} \nabla_m$.

we can obtain the equation of motion for the field strength $F^{mn}(x)$. Let us see how this works:

$$\begin{aligned}
0 &= \nabla_\sigma [(\gamma^m)_{\alpha\beta} \nabla_m \psi^\beta(x, \theta)] = (\gamma^m)_{\alpha\beta} \nabla_\sigma \nabla_m \psi^\beta(x, \theta) \\
0 &= (\gamma^m)_{\alpha\beta} [\nabla_\sigma, \nabla_m] \psi^\beta(x, \theta) + (\gamma^m)_{\alpha\beta} \nabla_m \nabla_\sigma \psi^\beta(x, \theta) \\
0 &= (\gamma^m)_{\alpha\beta} (F_{\sigma m}(x, \theta) \psi^\beta(x, \theta) + \psi^\beta(x, \theta) F_{\sigma m}(x, \theta)) + \frac{i}{2} (\gamma^m)_{\alpha\beta} (\gamma_{pq})^\beta{}_\sigma \nabla_m F^{pq}(x, \theta) \\
0 &= -(\gamma^m)_{\alpha\beta} (\gamma_m)_{\sigma\lambda} \{\psi^\lambda, \psi^\beta\}(x, \theta) + \frac{i}{2} (\gamma^m)_{\alpha\beta} (\gamma_{pq})^\beta{}_\sigma \nabla_m F^{pq}(x, \theta) \\
0 &= -(\gamma^m)_{\alpha\beta} (\gamma^n)^{\alpha\sigma} (\gamma_m)_{\sigma\lambda} \{\psi^\lambda, \psi^\beta\}(x, \theta) - \frac{i}{2} (\gamma^m)_{\alpha\beta} (\gamma^n)^{\alpha\sigma} (\gamma_{pq})^\beta{}_\sigma \nabla_m F^{pq}(x, \theta) \\
0 &= -(\gamma^m \gamma^n \gamma_m)_{\beta\lambda} \{\psi^\lambda, \psi^\beta\}(x, \theta) - \frac{i}{2} \text{Tr}(\gamma^m \gamma^n \gamma_{pq}) \nabla_m F^{pq}(x, \theta) \\
0 &= 8(\gamma^n)_{\beta\lambda} \{\psi^\lambda, \psi^\beta\}(x, \theta) - \frac{i}{2} \text{Tr}([\gamma^{mn} + \eta^{mn}] \gamma_{pq}) \nabla_m F^{pq}(x, \theta) \\
0 &= 8(\gamma^n)_{\beta\lambda} \{\psi^\lambda, \psi^\beta\}(x, \theta) - \frac{i}{2} \text{Tr}(\gamma^{mn} \gamma_{pq}) \nabla_m F^{pq}(x, \theta) \\
0 &= 8(\gamma^n)_{\beta\lambda} \{\psi^\lambda, \psi^\beta\}(x, \theta) + 16i \delta_{pq}^{mn} \nabla_m F^{pq}(x, \theta) \\
\rightarrow \nabla_m F^{mn}(x, \theta) &= \frac{i}{2} (\gamma^n)_{\beta\lambda} \{\psi^\lambda, \psi^\beta\}(x, \theta)
\end{aligned}$$

where in the third line we used the fact that $[\nabla_\sigma, \nabla_m] \psi^\beta(x, \theta) = [F_{\sigma m} \psi^\beta + \psi^\beta F_{\sigma m}](x, \theta)$ because it is acting on a Lie-algebra valued fermionic field, we also used the equations 2.2.55 and 2.2.75. In the fifth line we multiplied both sides by $(\gamma^n)^{\alpha\sigma}$. In the seventh and eighth lines we used the properties of the γ -matrices: $\gamma^m \gamma^n \gamma_m = -8\gamma^n$, $\gamma^m \gamma^n = \gamma^{mn} + \eta^{mn}$ and the corresponding orthogonality properties. So, at zeroth θ -level, we obtain

$$\nabla_m F^{mn}(x) = \frac{i}{2} (\gamma^n)_{\beta\lambda} \{\chi^\lambda, \chi^\beta\}(x) \quad (2.2.87)$$

that is, the equation of motion for the field strength F^{mn} ²².

Description via gauge connection and gauge invariance

The conventional constraint tells us how the superfields A_m and A_α relate to each other:

$$\begin{aligned}
(\gamma^p)^{\alpha\beta} F_{\alpha\beta}(x, \theta) &= 0 \\
(\gamma^p)^{\alpha\beta} (D_\alpha A_\beta + D_\beta A_\alpha - \{A_\alpha, A_\beta\} - 2i(\gamma^m)_{\alpha\beta} A_m)(x, \theta) &= 0 \\
(\gamma^p)^{\alpha\beta} (D_\alpha A_\beta - A_\alpha A_\beta - i(\gamma^m)_{\alpha\beta} A_m)(x, \theta) &= 0 \\
16i\eta^{pm} A_m(x, \theta) &= (\gamma^p)^{\alpha\beta} (D_\alpha A_\beta - A_\alpha A_\beta)(x, \theta)
\end{aligned}$$

so, we obtain

$$A^p(x, \theta) = -\frac{i}{16} (\gamma^p)^{\alpha\beta} (D_\alpha A_\beta(x, \theta) - A_\alpha(x, \theta) A_\beta(x, \theta)) \quad (2.2.88)$$

So, the superfield $A_p(x, \theta)$ is determined by the superfield $A_\alpha(x, \theta)$. Therefore, we will just work with $A_\alpha(x, \theta)$. The gauge invariance satisfied for the superconnection is:

$$\delta A_A(x, \theta) = \nabla_A \Lambda(x, \theta) \quad (2.2.89)$$

²²The extra sign in front of the anticommutator is due to the complex nature of χ^λ . This can be seen as follows: Because our conventions of the transpose of the product of two fermionic spinors, $F_{m\alpha}$ has to be imaginary, therefore χ^λ has to be imaginary.

where the index $A = (m, \alpha)$, and Λ is a gauge parameter (Lie-algebra valued scalar bosonic superfield). In particular, we have $\delta A_\alpha(x, \theta) = \nabla_\alpha \Lambda(x, \theta)$. So,

$$\delta A_\alpha(x, \theta) = D_\alpha \Lambda(x, \theta) + [\Lambda, A_\alpha](x, \theta) \quad (2.2.90)$$

$$\delta A_\alpha(x, \theta) = \partial_\alpha \Lambda(x, \theta) + i(\theta \gamma^m)_\alpha \partial_m \Lambda(x, \theta) + [\Lambda, A_\alpha](x, \theta) \quad (2.2.91)$$

We can expand the superfields in the following way

$$A_\alpha(x, \theta) = A_\alpha^{(0)}(x) + \theta^\beta A_{\alpha\beta}^{(1)}(x) + \theta^\beta \theta^\lambda A_{\alpha\beta\lambda}^{(2)} + \dots \quad (2.2.92)$$

$$\Lambda(x, \theta) = \Lambda^{(0)}(x) + \theta^\beta \Lambda_\beta^{(1)}(x) + \theta^\beta \theta^\lambda \Lambda_{\beta\lambda}^{(2)} + \dots \quad (2.2.93)$$

Replacing these expressions in the equation 2.2.91:

$$\begin{aligned} \delta A_\alpha^{(0)}(x) + \theta^\beta \delta A_{\alpha\beta}^{(1)}(x) + \theta^\beta \theta^\lambda \delta A_{\alpha\beta\lambda}^{(2)}(x) + \dots &= \Lambda_\alpha^{(1)}(x) + 2\theta^\lambda \Lambda_{\alpha\lambda}^{(2)}(x) + 3\theta^\beta \theta^\lambda \Lambda_{\alpha\beta\lambda}^{(3)} + i(\theta \gamma^m)_\alpha \partial_m \Lambda^{(0)}(x) \\ &+ i(\theta \gamma^m)_\alpha \theta^\beta \partial_m \Lambda_\beta^{(1)}(x) + i(\theta \gamma^m)_\alpha \theta^\beta \theta^\lambda \partial_m \Lambda_{\beta\lambda}^{(2)}(x) \\ &+ [\Lambda^{(0)}, A_\alpha^{(0)}](x) + \theta^\beta [\Lambda^{(0)}, A_{\alpha\beta}^{(1)}](x) + \theta^\beta \theta^\lambda [\Lambda^{(0)}, A_{\alpha\beta\lambda}^{(2)}](x) \\ &+ \theta^\beta \{\Lambda_\beta^{(1)}, A_\alpha^{(0)}\}(x) + \theta^\beta \theta^\lambda [\Lambda_\lambda^{(1)}, A_{\alpha\beta}^{(1)}](x) + \theta^\beta \theta^\lambda [\Lambda_{\beta\lambda}^{(2)}, A_\alpha^{(0)}](x) \\ &+ \dots \end{aligned}$$

so, we obtain conditions at each θ -level:

$$\delta A_\alpha^{(0)}(x) = \Lambda_\alpha^{(1)}(x) + [\Lambda^{(0)}, A_\alpha^{(0)}](x)$$

$$\delta A_{\alpha\beta}^{(1)}(x) = 2\Lambda_{\alpha\beta}^{(2)}(x) + i(\gamma^m)_{\alpha\beta} \partial_m \Lambda^{(0)}(x) + [\Lambda^{(0)}, A_{\alpha\beta}^{(1)}](x) + \{\Lambda_\beta^{(1)}, A_\alpha^{(0)}\}(x)$$

$$\delta A_{\alpha\beta\lambda}^{(2)}(x) = 3\Lambda_{\alpha\beta\lambda}^{(3)} + i(\gamma^m)_{\alpha\beta} \partial_m \Lambda_\lambda^{(1)}(x) + [\Lambda^{(0)}, A_{\alpha\beta\lambda}^{(2)}](x) + [\Lambda_\lambda^{(1)}, A_{\alpha\beta}^{(1)}](x) + [\Lambda_{\beta\lambda}^{(2)}, A_\alpha^{(0)}](x)$$

etc. We can see from the first of these equations that we can gauge away the component $A_\alpha^{(0)}(x)$ by choosing: $\Lambda_\alpha^{(1)}(x) = -A_\alpha^{(0)}(x) - [\Lambda^{(0)}, A_\alpha^{(0)}](x)$. From the second equation we can conclude that we can expand $A_{\alpha\beta}^{(1)}(x)$ in the corresponding tensor irreducible representations (1-form, 3-form and 5-form). However, because the antisymmetry of $\Lambda_{\alpha\beta}^{(2)}(x)$ in its indices α and β , this term is proportional to a 3-form. Therefore, we can gauge away the 3-form part of $A_{\alpha\beta}^{(1)}(x)$. But, it is also true that we can put the 5-form part of $A_{\alpha\beta}^{(1)}(x)$ to zero. This can be seen by using the dynamical constraint:

$$(\gamma^{mnpqr})^{\alpha\beta} (D_\alpha A_\beta - A_\alpha A_\beta - i\gamma_{\alpha\beta}^m A_m)(x) = 0$$

$$\text{by the orthogonality property} \rightarrow (\gamma^{mnpqr})^{\alpha\beta} (\partial_\alpha A_\beta + i(\theta \gamma^m)_\alpha \partial_m A_\beta - A_\alpha A_\beta)(x) = 0$$

$$\text{at zeroth } \theta\text{-level} \rightarrow (\gamma^{mnpqr})^{\alpha\beta} A_{\alpha\beta}^{(1)}(x) = 0$$

hence we are just left with the 1-form part of $A_{\alpha\beta}^{(1)}(x)$, which we will denote by $A_{\alpha\beta}^{(1)}(x) = i(\gamma^m)_{\alpha\beta} a_m(x)$. This $a_m(x)$ is the gauge potential of D=10 SYM, which can be seen as follows:

$$\text{Equation 2.2.88} \rightarrow A_m(x, \theta) = -\frac{i}{16} (\gamma_m)^{\alpha\beta} (D_\alpha A_\beta - A_\alpha A_\beta)(x, \theta) \quad (2.2.94)$$

$$\text{At the zeroth } \theta\text{-level} \rightarrow A_m(x, \theta)|_{\theta=0} = -\frac{i}{16} (\gamma_m)^{\alpha\beta} \partial_\alpha (i\theta^\lambda (\gamma^n)_{\beta\lambda} a_n(x)) \quad (2.2.95)$$

$$A_m(x, \theta)|_{\theta=0} = \frac{1}{16} (\gamma_m)^{\alpha\beta} (\gamma^n)_{\beta\alpha} a_n(x) \quad (2.2.96)$$

$$A_m(x, \theta)|_{\theta=0} = a_m(x) \quad (2.2.97)$$

satisfying the well-known gauge transformation:

$$\delta A_{\alpha\beta}^{(1)} = i(\gamma^m)_{\alpha\beta} \partial_m \Lambda^{(0)}(x) + [\Lambda^{(0)}, A_{\alpha\beta}^{(1)}](x) \quad (2.2.98)$$

$$i(\gamma^m)_{\alpha\beta} \delta a_m = i(\gamma^m)_{\alpha\beta} \partial_m \Lambda^{(0)}(x) + i(\gamma^m)_{\alpha\beta} [\Lambda^{(0)}, a_m](x) \quad (2.2.99)$$

$$\rightarrow \delta a_m(x) = \partial_m \Lambda^{(0)}(x) + [\Lambda^{(0)}, a_m](x) \quad (2.2.100)$$

Now, let us look at the second θ -level of the expansion of $A_\alpha(x, \theta)$. For that, let us use once again the dynamical constraint:

$$(\gamma^{mnpqr})^{\alpha\beta} (\partial_\alpha A_\beta + i(\theta\gamma^m)_\alpha \partial_m A_\beta - A_\alpha A_\beta)(x, \theta) = 0 \quad (2.2.101)$$

$$\text{At the first } \theta\text{-level } \rightarrow (\gamma^{mnpqr})^{\alpha\beta} \theta^\lambda A_{\alpha\beta\lambda}^{(2)}(x) = 0 \quad (2.2.102)$$

$$A_{\alpha\beta\lambda}^{(2)}(x) \text{ is antisymmetric in } \beta, \lambda \rightarrow A_{\alpha\beta\lambda}^{(2)}(x) = (\gamma^{mnp})_{\beta\lambda} f_{mnp\alpha}(x) \quad (2.2.103)$$

where $f_{mnp\alpha}$ is antisymmetric in m, n, p . If we remember the following identity²³:

$$\gamma_m \gamma^{rstuv} \gamma^m = 0 \quad (2.2.104)$$

so, if we choose $f_{mnp\alpha} = (\gamma_{mnp})_{\alpha\sigma} \xi^\sigma$, $A_{\alpha\beta\lambda}^{(2)}(x)$ takes the form:

$$A_{\alpha\beta\lambda}^{(2)}(x) = (\gamma^{mnp})_{\beta\lambda} (\gamma_{mnp})_{\alpha\sigma} \xi^\sigma \quad (2.2.105)$$

and the equation 2.2.102 is trivially satisfied:

$$(\gamma^{rstuv})^{\alpha\beta} \theta^\lambda (\gamma^{mnp})_{\beta\lambda} (\gamma_{mnp})_{\alpha\sigma} \xi^\sigma \rightarrow -\theta \gamma^{mnp} \gamma^{rstuv} \gamma_{mnp} \xi = 0 \quad (2.2.106)$$

We can also look at the first θ -level in the expansion of $A_m(x, \theta)$ from the conventional constraint at the first θ -level:

$$\theta^\lambda A_{r\lambda}^{(1)}(x) = -\frac{i}{16} (\gamma_r)^{\alpha\beta} (2\theta^\lambda A_{\alpha\beta\lambda}^{(2)})(x) \quad (2.2.107)$$

$$\theta^\lambda A_{r\lambda}^{(1)}(x) = -\frac{i}{8} (\gamma_r)^{\alpha\beta} \theta^\lambda (\gamma^{mnp})_{\beta\lambda} (\gamma_{mnp})_{\alpha\sigma} \xi^\sigma(x) \quad (2.2.108)$$

$$\theta^\lambda A_{r\lambda}^{(1)}(x) = \frac{i}{8} \theta \gamma^{mnp} \gamma_r \gamma_{mnp} \xi(x) \quad (2.2.109)$$

$$\theta^\lambda A_{r\lambda}^{(1)}(x) = \frac{-48i \cdot 3!}{8} \theta^\lambda (\gamma_r)_{\lambda\sigma} \xi^\sigma(x) \quad (2.2.110)$$

$$\rightarrow A_{r\lambda}^{(1)}(x) = -36i (\gamma_r)_{\lambda\sigma} \xi^\sigma(x) \quad (2.2.111)$$

so, if we define $\xi^\sigma(x) = -\frac{1}{36} \chi^\sigma(x)$, we get:

$$A_m(x, \theta) = a_m(x) + i\theta \gamma_m \chi + \dots \quad (2.2.112)$$

$$A_\alpha(x, \theta) = i(\gamma^m \theta)_\alpha a_m - \frac{1}{36} (\theta \gamma^{mnp} \theta) (\gamma_{mnp} \chi)_\alpha + \dots \quad (2.2.113)$$

Analogously with somewhat effort we can calculate the gauge transformation of the gaugino χ^σ by using the equation for the variation of $A_{\alpha\beta\lambda}^{(2)}$. The result is:

$$\delta \chi^\sigma(x) = [\Lambda^{(0)}, \chi^\sigma](x) \quad (2.2.114)$$

²³One easy way to see this is to take particular values for the indices of the 5-form: $\gamma_m \gamma^{12345} \gamma^m$. Now, this quantity will be +1 for $m = 1, 2, 3, 4, 5$ and -1 for $m = 0, 6, 7, 8, 9$. Therefore we will get: $\gamma_m \gamma^{12345} \gamma^m = 5 \cdot (+1) + 5 \cdot (-1) = 0$. It is not difficult to see that in the general case we will obtain the same result.

where w is a gauge parameter. Therefore this description gives us the physical fields and the gauge invariances which these ones must satisfy. We can continue this analysis and try to figure out the exact form of the next terms, however this is a mess and we will omit it. In addition to this, it can be shown that one obtains the SUSY coordinate transformations which turn out to coincide with the SUSY transformations defined in the equations 2.1.4 and 2.1.5.

So, for example the SUSY transformation for the gaugino (6.2) can be obtained from the following equation:

$$\delta\psi^\alpha(x, \theta) = \epsilon^\beta Q_\beta \psi^\alpha(x, \theta) \quad (2.2.115)$$

where ψ^α is given in the equation 2.2.76, $Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\gamma^m)_{\alpha\beta} \frac{\partial}{\partial x^m}$ and ϵ^α is the SUSY parameter. We can use the equations 2.2.75 and 2.2.77 in order to determine $(\psi^{(1)})_\alpha{}^\lambda(x)$:

$$\nabla_\alpha \psi^\lambda(x, \theta) = \frac{i}{2} (\gamma_{mn})^\lambda{}_\alpha F^{mn}(x, \theta) \quad (2.2.116)$$

$$\text{at zeroth } \theta\text{-level} \rightarrow (\psi^{(1)})_\alpha{}^\lambda(x) = \frac{i}{2} (\gamma_{mn})^\lambda{}_\alpha F^{mn}(x) \quad (2.2.117)$$

Therefore,

$$\psi^\lambda(x, \theta) = \chi^\lambda(x) + \frac{i}{2} (\gamma_{mn}\theta)^\lambda F^{mn}(x) + \dots \quad (2.2.118)$$

and the SUSY transformation for the gaugino will be

$$\delta\chi^\lambda(x) = \frac{i}{2} \epsilon^\beta (\gamma_{mn})^\lambda{}_\beta F^{mn}(x) \quad (2.2.119)$$

$$\delta\chi^\lambda(x) = \frac{i}{2} F^{mn}(x) (\gamma_{mn})^\lambda{}_\beta \epsilon^\beta \quad (2.2.120)$$

and if we remember that in the description of $D = 10$ SYM via (super) field strength and Bianchi identities, the gaugino field was a imaginary field, we can write the real version (by using a field redefinition $\chi^\lambda \rightarrow i\chi^\lambda$):

$$\delta\chi^\lambda(x) = \frac{1}{2} F^{mn}(x) (\gamma_{mn})^\lambda{}_\beta \epsilon^\beta \quad (2.2.121)$$

which is the usual SUSY transformation for the gaugino showed in the equation 2.1.5. In the same way it can be showed that we obtain the usual SUSY transformation for the field strength $F_{mn}(x)$ as that one obtained from the equation 2.1.4. We will omit this calculation.

2.3 Batalin-Vilkovisky quantization of D=10 SYM

We will quantize D=10 SYM by using the Batalin-Vilkovisky framework [6]. We will not do an exhaustive description²⁴ of this formalism but just describe the relevant steps for our purposes. The recipe is:

1. Introduce a ghost for each gauge symmetry.
2. Introduce an antifield for each field.
3. Solving the master equation.
4. Gauge fix the master action.

We will denote by Φ^I the matter fields and ghosts (ϕ^i, c^A) , and by Φ_I^* the antifields for the matter fields and ghosts (ϕ_i^*, c_A^*) . An antifield will have the opposite Grassmann parity to its corresponding field. The ghost number of an antifield is given by $\text{gh}(\Phi_I^*) = -1 - \text{gh}(\Phi^I)$. Let us define the antibracket for two functionals $A(\Phi^I, \Phi_I^*)$ and $B(\Phi^I, \Phi_I^*)$, in the following way:

$$(A, B) = \frac{\delta_R A}{\delta \Phi^I} \frac{\delta_L B}{\delta \Phi_I^*} - \frac{\delta_R A}{\delta \Phi_I^*} \frac{\delta_L B}{\delta \Phi^I} \quad (2.3.1)$$

where L and R refer to the standard left and right derivatives, respectively²⁵. The classical master equation is defined by:

$$(\mathcal{S}, \mathcal{S}) = 0 \quad (2.3.3)$$

A solution \mathcal{S} of the master equation will be called master action. This will be required to be bosonic, so the equation 2.3.3 takes the following form:

$$0 = (\mathcal{S}, \mathcal{S}) = \frac{\delta_R \mathcal{S}}{\delta \Phi^I} \frac{\delta_L \mathcal{S}}{\delta \Phi_I^*} - \frac{\delta_R \mathcal{S}}{\delta \Phi_I^*} \frac{\delta_L \mathcal{S}}{\delta \Phi^I} \quad (2.3.4)$$

$$\mathcal{S} \text{ is bosonic} \rightarrow 0 = 2 \frac{\delta_L \mathcal{S}}{\delta \Phi^I} \frac{\delta_L \mathcal{S}}{\delta \Phi_I^*} \quad (2.3.5)$$

$$\rightarrow \frac{\delta_L \mathcal{S}}{\delta \Phi^I} \frac{\delta_L \mathcal{S}}{\delta \Phi_I^*} = 0 \quad (2.3.6)$$

where we used the fact that $\frac{\delta_R \mathcal{S}}{\delta(\text{boson})} = \frac{\delta_L \mathcal{S}}{\delta(\text{boson})}$, $\frac{\delta_R \mathcal{S}}{\delta(\text{fermion})} = -\frac{\delta_L \mathcal{S}}{\delta(\text{fermion})}$. Now, we can expand \mathcal{S} in terms of the number of antifields:

$$\mathcal{S} = S + S_1 + S_2 + \dots \quad (2.3.7)$$

S_i denoting a term in the master action containing i antifields. The master equation must hold at each antifield level, therefore:

$$\frac{\delta_L S}{\delta \Phi^I} \frac{\delta_L S}{\delta \Phi_I^*} = 0 \quad (2.3.8)$$

$$\frac{\delta_L S}{\delta \Phi^I} \frac{\delta_L S_1}{\delta \Phi_I^*} = 0 \quad (2.3.9)$$

$$\frac{\delta_L S}{\delta \Phi^I} \frac{\delta_L S_2}{\delta \Phi_I^*} + \frac{\delta_L S_1}{\delta \Phi^I} \frac{\delta_L S_1}{\delta \Phi_I^*} = 0 \quad (2.3.10)$$

²⁴A good short review can be found in [7].

²⁵The left and right derivatives are related by the equality:

$$\frac{\delta_R A}{\delta(\text{field})} = (-1)^{\epsilon_{\text{field}}(\epsilon_A+1)} \frac{\delta_L A}{\delta(\text{field})} \quad (2.3.2)$$

where ϵ denotes the Grassmann parity.

etc. We must impose some boundary conditions on the possible solutions to the master equation (like for instance, $\mathcal{S}(\Phi^I, \Phi_I^* = 0) = S$, etc). These solutions will be called proper solutions. In addition to this, we should remember that the action must have ghost number zero. It turns out that²⁶ $S_1 = \phi_i^* R_A^i[\phi] c^A$ satisfies the required conditions. Furthermore, it can be shown that $S_2 = c_A^* T_{BC}^A c^B c^C$ is also an allowed term, where T_{BC}^A are structure constants of the corresponding gauge algebra. For a closed algebra, which is our case, that is all. Therefore, our master action is:

$$\mathcal{S} = S + \phi^* R_A^i[\phi] c^A + c_A^* T_{BC}^A c^B c^C + b_A^* h^A \quad (2.3.12)$$

We have added a couple of fields, the antifield for the antighost b^A , and a corresponding auxiliary field h^A . It is easy to see that this term will not affect the master equation.

Now, we can construct the generalized BRST operator: $sO = (O, \mathcal{S})$. The nilpotency of this operator is a consequence of the fact that $(\mathcal{S}, \mathcal{S}) = 0$. Therefore

$$s\Phi^I = (\Phi^I, \mathcal{S}) = \frac{\delta_L \mathcal{S}}{\delta \Phi_I^*} \quad (2.3.13)$$

$$s\Phi_I^* = (\Phi_I^*, \mathcal{S}) = -\frac{\delta_L \mathcal{S}}{\delta \Phi^I} \quad (2.3.14)$$

Now, we are ready to write the master action for D=10 SYM. Firstly, let us show explicitly the closure of the gauge algebra:

$$\begin{aligned} \delta F^{mna} = f^{abc} \alpha^b F^{mnc} &\rightarrow \delta_1 \delta_2 F^{mna} = f^{abc} f^{cde} \alpha_2^b \alpha_1^d F^{mne} \\ \delta_2 \delta_1 F^{mna} &= f^{abc} f^{cde} \alpha_1^b \alpha_2^d F^{mne} \\ \rightarrow [\delta_1, \delta_2] F^{mna} &= (f^{adc} f^{cbe} - f^{abc} f^{cde}) \alpha_1^b \alpha_2^d F^{mne} \\ [\delta_1, \delta_2] F^{mna} &= (f^{cad} f^{cbe} + f^{cab} f^{ced}) \alpha_1^b \alpha_2^d F^{mne} \\ [\delta_1, \delta_2] F^{mna} &= f^{ace} f^{cbd} \alpha_1^b \alpha_2^d F^{mne} \\ [\delta_1, \delta_2] F^{mna} &= f^{ace} \alpha_3^c F^{mne} \\ [\delta_1, \delta_2] F^{mna} &= \delta_3 F^{mna} \end{aligned}$$

where $\alpha_3^c = f^{abd} \alpha_1^b \alpha_2^d$ is the parameter of the resulting gauge transformation. Because the gauge transformation for the spinor field χ^β has the same rule of transformation, we can apply the same arguments to get a similar result. Therefore the gauge algebra is closed, and we can apply the formula 2.3.12 to obtain the corresponding master action. The first step is to find out who $R_A^i[\phi]$ is. So, let us calculate the matrix $R_A^i[\phi]$:

$$\delta A_m^a(x) = \nabla_m^{ac} \alpha^c(x) \quad (2.3.15)$$

$$\delta A_m^a(x) = (\delta^{ac} \partial_m + f^{abc} A_m^b(x)) \alpha^c(x) \quad (2.3.16)$$

$$\delta A_m^a(x) = \int d^{10}y (-\delta^{ac} \partial_m (\delta(y-x)) + f^{abc} A_m^b(y) \delta(y-x)) \alpha^c(y) \quad (2.3.17)$$

$$\rightarrow R_m^{ac}(y, x) = [-\delta^{ac} \partial_m + f^{abc} A_m^b(y)] \delta(y-x) \quad (2.3.18)$$

²⁶We are using the De Witt notation. So, for example, for a gauge transformation we have:

$$\delta_\alpha \phi^i = \delta \phi^i(x) = \int dy R[\phi](x, y) \alpha(y) = R[\phi]_A^i \alpha^A \quad (2.3.11)$$

and, also we have

$$\delta\chi^{\sigma a}(x) = f^{abc}\alpha^b(x)\chi^{\sigma c}(x) \quad (2.3.19)$$

$$\delta\chi^{\sigma a}(x) = \int d^{10}y f^{abc}\alpha^b(y)\chi^{\sigma c}(y)\delta(y-x)\alpha^a(y) \quad (2.3.20)$$

$$\rightarrow R^{\sigma ac}(y, x) = f^{abc}(y)\chi^{\sigma c}(y)\delta(y-x) \quad (2.3.21)$$

Using the equation 2.3.12, it is easy to see that the master action is:

$$S = \int d^{10}x Tr[-\frac{1}{4}F^{mn}F_{mn} + \frac{i}{2}\chi\gamma^m\nabla_m\chi + ia_m^*\nabla^m c - i\chi_\alpha^*\{c, \chi^\alpha\} - ic^*cc] \quad (2.3.22)$$

where the factors of i were added in order the whole expression to be real. The corresponding equations of motion are:

$$\nabla^m F_{mn} - \frac{i}{2}(\gamma_m)_{\alpha\beta}\{\chi^\alpha, \chi^\beta\} - i\{a_m^*, c\} = 0 \quad (2.3.23)$$

$$(\gamma^m)_{\alpha\beta}\nabla_m\chi^\beta + [\chi_\alpha^*, c] = 0 \quad (2.3.24)$$

$$\nabla^m a_m^* - [\chi^\alpha, \chi_\alpha^*] - [c, c^*] = 0 \quad (2.3.25)$$

$$\nabla_m c = 0 \quad (2.3.26)$$

$$\{c, \chi^\alpha\} = 0 \quad (2.3.27)$$

$$cc = 0 \rightarrow (f^{abc}c^b c^c = 0) \quad (2.3.28)$$

which in the abelian case become in the following simple equations:

$$\partial^m F_{mn} = 0 \quad (2.3.29)$$

$$(\gamma^m)_{\alpha\beta}\partial_m\chi^\beta = 0 \quad (2.3.30)$$

$$\partial^m a_m^* = 0 \quad (2.3.31)$$

$$\partial_m c = 0 \quad (2.3.32)$$

These last equations will appear when we quantize the N=1 D=10 superparticle using the pure spinor formalism.

To finish our analysis we should try to figure how the gauge invariances arise in this formalism. For that, it is necessary to realize that the master action has gauge symmetries²⁷ which means that we cannot use it to quantize the theory by using the path integral formalism.

²⁷This follows from the master equation in its form 2.3.6:

$$\begin{aligned} \frac{\delta_L}{\delta\Phi^K} \left(\frac{\delta_L\mathcal{S}}{\delta\Phi^I} \frac{\delta_L\mathcal{S}}{\delta\Phi_I^*} \right) &= 0 \\ \left(\frac{\delta_L^2\mathcal{S}}{\delta\Phi^K\delta\Phi^I} \right) \frac{\delta_L\mathcal{S}}{\delta\Phi_I^*} + (-1)^{IK} \frac{\delta_L\mathcal{S}}{\delta\Phi^I} \left(\frac{\delta_L^2\mathcal{S}}{\delta\Phi^K\delta\Phi_I^*} \right) &= 0 \end{aligned}$$

where the extra sign takes into account the Grassmannian nature of the variables. Now, if we realize that the Grassmanian properties of Φ^I and Φ_I^* are opposite, we obtain:

$$\begin{aligned} \left(\frac{\delta_L^2\mathcal{S}}{\delta\Phi^K\delta\Phi^I} \right) \frac{\delta_L\mathcal{S}}{\delta\Phi_I^*} + (-1)^{2IK} \left(\frac{\delta_L^2\mathcal{S}}{\delta\Phi^K\delta\Phi_I^*} \right) \frac{\delta_L\mathcal{S}}{\delta\Phi^I} &= 0 \\ \left(\frac{\delta_L^2\mathcal{S}}{\delta\Phi^K\delta\Phi^I} \right) \frac{\delta_L\mathcal{S}}{\delta\Phi_I^*} + \left(\frac{\delta_L^2\mathcal{S}}{\delta\Phi^K\delta\Phi_I^*} \right) \frac{\delta_L\mathcal{S}}{\delta\Phi^I} &= 0 \end{aligned}$$

We should gauge fix the action. Let us see how this works. After using the Jacobi identity,²⁸ one can show that $\mathcal{S}' = \mathcal{S} + (\delta F, \mathcal{S})$ is also a solution of the master equation, where δF is an infinitesimal fermionic functional of ghost number -1:

$$\begin{aligned} (\mathcal{S} + (\delta F, \mathcal{S}), \mathcal{S} + (\delta F, \mathcal{S})) &= (\mathcal{S}', \mathcal{S}') \\ (\mathcal{S}, \mathcal{S}) + (\mathcal{S}, (\delta F, \mathcal{S})) + ((\delta F, \mathcal{S}), \mathcal{S}) &= (\mathcal{S}', \mathcal{S}') \\ \text{by using } (A, B) = -(-1)^{(|A+1|)(|B+1|)}(B, A) \rightarrow (\mathcal{S}, \mathcal{S}) + 2(\mathcal{S}, (\delta F, \mathcal{S})) &= (\mathcal{S}', \mathcal{S}') \\ \text{by using } ((\mathcal{S}, \delta F), \mathcal{S}) = -\frac{1}{2}(\delta F, (\mathcal{S}, \mathcal{S})) \rightarrow 0 &= (\mathcal{S}', \mathcal{S}') \end{aligned}$$

We will choose this function to depend on the fields (and not antifields): $\delta F = -\epsilon\Psi[\Phi]$, where ϵ is an infinitesimal parameter. Therefore $\delta\mathcal{S} = -(\epsilon\Psi, \mathcal{S}) = -\epsilon\frac{\delta_L\Psi}{\delta\Phi^I}\frac{\delta_R\mathcal{S}}{\delta\Phi_I^*}$. This is equivalent to claim that $\mathcal{S}'[\Phi, \Phi^*] = \mathcal{S}[\Phi, \Phi^* - \epsilon\frac{\delta_L\Psi}{\delta\Phi}]$ also satisfies the master equation. The corresponding finite transformation is $\Phi_I'^* = \Phi_I^* - \frac{\delta_L\Psi}{\delta\Phi^I}$. So our gauge fixed action will be constructed by requiring that $\Phi_I'^* = \Phi_I^* - \frac{\delta_L\Psi}{\delta\Phi^I} = 0 \rightarrow \Phi_I^* = \frac{\delta_L\Psi}{\delta\Phi^I}$. Our resulting gauge fixed action:

$$S_\Psi = \mathcal{S}\left[\Phi, \frac{\delta_L\Psi}{\delta\Phi}\right] \quad (2.3.34)$$

will be invariant under the BRST-transformations:

$$s\Phi^I = (\Phi^I, \mathcal{S})|_{\Phi^* = \frac{\delta_L\Psi}{\delta\Phi}} \quad (2.3.35)$$

So let us deduce the gauge transformations of the $D = 10$ SYM fields. The starting point will be the equation:

$$\delta_\sigma\Phi^I = \sigma^K \frac{\delta_L^2\mathcal{S}}{\delta\Phi^K\delta\Phi_I^*} \quad (2.3.36)$$

The idea is to choose adequately the gauge parameter σ^K and to find the respective gauge transformations for each field. We will just show the relevant gauge transformations for our purposes:

1. If we choose σ^K to have just a single component in the c^m direction, we obtain for the

so the action will be invariant under the following gauge transformations:

$$\begin{aligned} \delta_\sigma\Phi_I^* &= \sigma^K \frac{\delta_L^2\mathcal{S}}{\delta\Phi^K\delta\Phi_I^*} \\ \delta_\sigma\Phi^I &= \sigma^K \frac{\delta_L^2\mathcal{S}}{\delta\Phi^K\delta\Phi_I^*} \end{aligned}$$

A similar result can be obtained by deriving the master equation with respect to Φ_I^* .

²⁸The Jacobi identity for the antibracket is:

$$(A, (B, C)) + (-1)^{(|B+1|)(|A+1|)}(B, (C, A)) + (-1)^{(|C+1|)(|B+1|)}(C, (A, B)) = 0 \quad (2.3.33)$$

where $|A|$ is the Grassmann parity of the object A .

gauge variation of A^m :

$$\begin{aligned}
\delta_{\sigma_c} A^m(x) &= \int d^{10}y \sigma_c^e(y) \frac{\delta_L^2 \mathcal{S}}{\delta c^e(y) \delta a_m^*(x)} \\
\delta_{\sigma_c} A^m(x) &= \int d^{10}y \sigma_c^e(y) \frac{\delta_L}{\delta c^e(y)} (i \nabla^m c(x)) \\
\delta_{\sigma_c} A^m(x) &= \int d^{10}y \sigma_c^e(y) \frac{\delta_L}{\delta c^e(y)} (i(\partial^m c^a(x) + f^{abc} \sigma_c^b(x) A^{mc}(x) T^a)) \\
\delta_{\sigma_c} A^m(x) &= i \int d^{10}y (\sigma_c^e(y) \partial^m (\delta(y-x)) T^e + f^{abc} \delta(y-x) \sigma_c^b(y) A^{mc}(x) T^a) \\
\delta_{\sigma_c} A^m(x) &= i(\partial^m (\sigma_c^e(x) T^e) + f^{abc} \sigma_c^b(x) A^{mc}(x) T^a) \\
\delta_{\sigma_c} A^m(x) &= i \nabla^m \sigma_c(x)
\end{aligned}$$

which for the abelian case it gives us

$$\delta_{\sigma_c} A^m(x) = i \partial^m \sigma_c(x) \quad (2.3.37)$$

If we compute the gauge variations for the other fields we will obtain $\delta\phi = i[\phi, A]$, that is the usual gauge transformations. However, as we said we will not show these transformations because they will not be relevant for our future analysis.

2. If we choose σ^K to have a single component in the a^n direction, we obtain for the gauge variation of a^{*m} :

$$\begin{aligned}
\delta_{\sigma_A} a_m^{*d} &= \int d^{10}y \sigma_A^{ne}(y) \frac{\delta_L^2 \mathcal{S}}{\delta A^{ne}(y) \delta A^{md}(x)} \\
\delta_{\sigma_A} a_m^{*d} &= \int d^{10}y \sigma_A^{ne}(y) \frac{\delta_L}{\delta A^{ne}(y)} (f^{dbc} (\partial_m A_p^b - \partial_p A_m^b - f^{bah} A_m^a A_p^h) A^{pc} \\
&\quad + \frac{i}{2} \chi^{\alpha c} (\gamma_m)_{\alpha\beta} f^{bdc} \chi^{\beta b} + i a_m^{*a} f^{abd} c^b)(x) \\
\delta_{\sigma_A} a_m^{*d} &= \int d^{10}y \sigma_A^{ne}(y) [(g_{pn} f^{dec} \partial_m (\delta(y-x)) A^{pc} - g_{mn} f^{dec} \partial_p (\delta(y-x)) A^{pc} + f^{dbe} \delta(y-x) (\partial_m A_n^b \\
&\quad - \partial_n A_m^b)(x) - f^{dbc} \delta(y-x) (\delta_{mn} f^{beh} A_p^h A_m^c + f^{bhe} A_m^h A_n^c + \delta^{ec} f^{bah} A_m^a A_n^h)(x)] \\
\delta_{\sigma_A} a_m^{*d} &= f^{dbe} \sigma_A^{ne} F_{mn}^b + f^{dbe} \partial_p \sigma_{mA}^e A_p^b - f^{dbe} \partial_m \sigma_A^{ne} A_n^b - \sigma_A^{ne} f^{dbc} f^{cae} A_m^a A_n^b \\
&\quad - \sigma_{mA}^e f^{dbc} f^{ceg} A^{pg} A_p^b
\end{aligned}$$

Multiplying both sides by T^d :

$$\delta_{\sigma_A} a_m^* = [\sigma_A^n, F_{mn}] + [\nabla_m \sigma_{nA}, A^n] - [\nabla_n \sigma_{mA}, A^n] \quad (2.3.38)$$

which for the abelian case it gives us

$$\delta_{\sigma_A} a_m^* = 0 \quad (2.3.39)$$

By following this reasoning we can calculate the gauge variations for the remaining fields, and we obtain (for the abelian case):

$$\delta \chi^*_{\alpha} = i(\gamma^m)_{\alpha\beta} \partial_m \rho^{\beta} \quad (2.3.40)$$

$$\delta c^* = -i \partial^m v_m \quad (2.3.41)$$

where ρ^{β} , v_m are arbitrary gauge parameters. These gauge transformations will appear later when we study the pure spinor versions of the $D = 10$ superparticle.

Chapter 3

D=11 supergravity

In this chapter we will study the component and superspace formulations of the D=11 supergravity (SUGRA) theory. We will follow the same structure developed in the previous chapter of this dissertation.

3.1 Component formulation of D=11 supergravity

Basically this section will try to describe the action proposed by E. Cremmer, B. Julia and J. Scherk in [26] for $N = 1$ $D = 11$ Supergravity¹. We will start by writing a certain action which will turn out to be invariant under certain supersymmetry transformations (on-shell). Let us mention some words about the notation which will be used throughout this chapter. We will denote curved (Einstein) indices by m, n, \dots and flat (Lorentz) indices by a, b, \dots . The Clifford algebra in the tangent space is $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$ (we have an analog expression for the curved case, with η^{ab} replaced by g^{mn}). The Γ^a matrices are 32×32 symmetric matrices (Majorana representation). We will denote $SO(10, 1)$ spinor indices by μ, ν, \dots and work with Majorana spinors ψ^μ (32). In $D = 11$ there is a charge conjugation matrix $C^{\alpha\beta}$ which is an antisymmetric metric tensor, and this (and its corresponding inverse $C_{\alpha\beta}^{-1}$) will be used to raise and lower spinor indices.

The field content of D=11 SUGRA is the following: The vielbein e_m^a , a Majorana vectorspinor ψ_m^μ and a three-form A_{mnp} . This can be justified by counting the on-shell degrees of freedom. In the appendix C we develop a general discussion about the on-shell and off-shell degrees of freedom of several fields. We will just apply the results obtained there. So, the vielbein field has $\frac{(11-1)(11-2)}{2} - 1 = 44$ physical degrees of freedom, the three-form field has $\frac{(11-2)!}{(11-2-3)!3!} = 84$ physical degrees of freedom and the Majorana spin $\frac{3}{2}$ field has $\frac{1}{2} \cdot 32 \cdot (11 - 3) = 128$ physical degrees of freedom. Therefore, we can see a match between the bosonic and fermionic physical degrees of freedom².

¹Our conventions for the metric and the normalization of the graviton and gravitino fields will be different from those ones used in the original paper.

²There is another way to see that this field content is the desired one. One starts with this field content and by dimensional reduction (from D=11 to D=10), we get to the field content of type IIA closed superstring theory (at zero mass level) [11] [12]. We will omit the details of this second approach.

The action for D=11 SUGRA is

$$\begin{aligned}
S &= \frac{1}{2\kappa^2} \int d^{11}x \{ \det(e_m^a) [R(w) - 2i\bar{\psi}_m \Gamma^{mnp} \nabla_n (\frac{w + \hat{w}}{2}) \psi_p - \frac{1}{48} F_{mnpq} F^{mnpq} \\
&+ \frac{i}{96} (\bar{\psi}_m \Gamma^{mnpqrs} \psi_n + 12\bar{\psi}^p \Gamma^{qr} \psi^s) (F_{pqrs} + \hat{F}_{pqrs}) \} \\
&+ \frac{1}{6 \cdot 3! (4!)^2} \epsilon^{m_1 m_2 m_3 m_4 n_1 n_2 n_3 n_4 p q s} F_{m_1 m_2 m_3 m_4} F_{n_1 n_2 n_3 n_4} A_{p q s} \}
\end{aligned}$$

where the covariant derivative is given by $\nabla_m(w)\psi_n = \partial_m \psi_n + \frac{1}{4} w_{mab} \Gamma^{ab} \psi_n$, the Ricci scalar is written in terms of the spin connection w_{mab} : $R_{mn}{}^{ab} = \partial_m w_n{}^{ab} - \partial_n w_m{}^{ab} + w_m{}^{ac} w_n{}^{cb} - w_n{}^{ac} w_m{}^{cb}$, the field strength F_{mnpq} is defined as $F_{mnpq} = 4\partial_{[m} A_{npq]}$. As usual, we will have a contorsion tensor which will give us information about the difference between a free-torsion spin connection $w_{mab}^{(0)}$ (which is obtained from the condition $T_{mn}{}^a = \nabla_{[m} e_n{}^a] = 0$) and a non free-torsion spin connection (w_{mab}):

$$w_{mab} = w_{mab}^{(0)} + K_{mab} \quad (3.1.1)$$

and the contorsion tensor is

$$K_{mab} = \frac{i}{4} [\bar{\psi}_p \Gamma_{mab}{}^{pq} \psi_q - 2(\bar{\psi}_m \Gamma_b \psi_a - \bar{\psi}_m \Gamma_a \psi_b + \bar{\psi}_b \Gamma_m \psi_a)] \quad (3.1.2)$$

and $w_{mab}^{(0)}$ is defined by

$$w_{mab}^{(0)} = e_m{}^n (\Omega_{abn} - \Omega_{bna} - \Omega_{nab}) \quad (3.1.3)$$

and $\Omega_{abn} = \frac{1}{2} (e_a{}^p e_b{}^q - e_b{}^p e_a{}^q) \partial_q e_{pm}$. In addition to these definitions, we have written \hat{w}_{mab} and \hat{F}_{mnpq} , which are defined by the following expressions

$$\hat{w}_{mab} = w_{mab} - \frac{i}{4} \bar{\psi}_p \Gamma_{mab}{}^{pq} \psi_q \quad (3.1.4)$$

$$\hat{F}_{mnpq} = F_{mnpq} - 3\bar{\psi}_{[m} \Gamma_{np} \psi_{q]} \quad (3.1.5)$$

these tensors receive the name of *supercovariant tensors*, because they transform without derivatives of the transformation parameter (ϵ) under the following supersymmetry transformations:

$$\delta e_m^a = i\bar{\epsilon} \Gamma^a \psi_m \quad (3.1.6)$$

$$\delta \psi_m = \nabla_m(\hat{w})\epsilon - \frac{1}{12 \cdot 4!} (\Gamma^{pqrs}{}_m + 8\Gamma^{qrs} \delta_m^p) \epsilon \hat{F}_{pqrs} \equiv \hat{\nabla}_m \epsilon \quad (3.1.7)$$

$$\delta A_{mnp} = 3i\bar{\epsilon} \Gamma_{[mn} \psi_{p]} \quad (3.1.8)$$

These transformation laws are constructed in a systematic way in [26]. Once again, the supersymmetry algebra is closed only up to equations of motion what means that we will just have on-shell supersymmetry. These equations of motion can be deduced in the standard way, the result is

$$R_{mn} - \frac{1}{2} g_{mn} R - \frac{1}{12} (F_{pqr}^m F^{npqr} - \frac{1}{8} g^{mn} F_{pqr} F^{pqr}) = 0 \quad (3.1.9)$$

$$\Gamma^{mnp} \hat{\nabla}_n \psi_p = 0 \quad (3.1.10)$$

$$\partial_m (\sqrt{-g} F^{mm_1 m_2 m_3}) + \frac{1}{2(4!)^2} \epsilon^{m_1 m_2 m_3 \dots m_{11}} F_{m_4 m_5 m_6 m_7} F_{m_8 m_9 m_{10} m_{11}} = 0 \quad (3.1.11)$$

which at linear order they become

$$R_{mn} - \frac{1}{2}g_{mn}R = 0 \tag{3.1.12}$$

$$\Gamma^{mnp}\partial_n\psi_p = 0 \tag{3.1.13}$$

$$\partial_m F^{mm_1m_2m_3} = 0 \tag{3.1.14}$$

These are the equations which will be useful in our analysis of the $D = 11$ semi pure spinor superparticle.

3.2 Superspace formulation of D=11 Supergravity

We will follow the work developed in [27]. Let us define the following 1-forms:

$$E^A = dZ^M E_M^A \quad (3.2.1)$$

$$\Omega_A^B = dZ^M \Omega_{MA}^B \quad (3.2.2)$$

where E_M^A is the vielbein and Ω_{MA}^B is the spin connection. We are denoting Einstein (curved) indices in superspace by M, N, \dots , and Lorentz (flat) indices in superspace by A, B, \dots . Now, we will make the Lorentzian assumption³ which is:

$$\Omega_{ab} = -\Omega_{ba} \quad , \quad \Omega_{\alpha\beta} = 0 \quad (3.2.3)$$

$$\Omega_\alpha^\beta = \frac{1}{4}(\Gamma^{ab})_\alpha^\beta \Omega_{ab} \quad , \quad \Omega_{a\beta} = 0 \quad (3.2.4)$$

Now, we can define a torsion:

$$T^A = DE^A = dE^A + E^B \Omega_B^A \quad (3.2.5)$$

which has to be a 2-form, therefore it can be written in the following form

$$T^A = \frac{1}{2} E^C E^B T_{BC}^A \quad (3.2.6)$$

Also, we can define a curvature as follows:

$$R_A^B = d\Omega_A^B + \Omega_A^C \Omega_C^B \quad (3.2.7)$$

and this one will be also a 2-form, so

$$R_A^B = \frac{1}{2} E^D E^C R_{CDA}^B \quad (3.2.8)$$

The factors $\frac{1}{2}$ in the equations 3.2.6 and 3.2.8 are just a convention, they take into account the double counting when expanding each expression. From the equation 3.2.7 we can see that R_A^B has the same symmetry properties as Ω_A^{B4}

We can find the Bianchi identities. For this end, we should calculate the following quantities:

$$\begin{aligned} DT^A &= d(dE^A + E^B \Omega_B^A) + (dE^B + E^C \Omega_C^B) \Omega_B^A \\ &= E^B d\Omega_B^A - dE^B \Omega_B^A + dE^B \Omega_B^A + E^B \Omega_B^C \Omega_C^A \\ &= E^B (d\Omega_B^A + \Omega_B^C \Omega_C^A) \end{aligned}$$

³This assumption reproduces the usual property of the bosonic covariant derivative expressed in terms of the spin connection acting on a spinor (showed in [13]): $D_m \psi_\alpha = \partial_m \psi_\alpha + \frac{1}{4} \Omega_{mab} (\Gamma^{ab})_\alpha^\beta \psi_\beta$. The only difference is the factor $\frac{1}{4}$ in the present case, but this is just a matter of convention.

⁴For example,

$$R_{ba} = d\Omega_{ba} + \Omega_b^c \Omega_{ca} \quad (3.2.9)$$

$$= -\Omega_{ab} + \Omega_b^\alpha \Omega_{\alpha a} + \Omega_b^d \Omega_{da} \quad (3.2.10)$$

$$= -d\Omega_{ab} - \Omega_a^d \Omega_{db} \quad (3.2.11)$$

$$= -(d\Omega_{ab} + \Omega_a^c \Omega_{cb}) \quad (3.2.12)$$

$$= -R_{ab} \quad (3.2.13)$$

etc.

where we used that Ω_B^A is a 1-form and the right action of the exterior derivative. Therefore,

$$DT^A = E^B R_B^A \quad (3.2.14)$$

Also,

$$\begin{aligned} DR_A^B &= dR_A^B + R_A^C \Omega_C^B - \Omega_A^C R_C^B \\ &= \Omega_A^C d\Omega_C^B - d\Omega_A^C \Omega_C^B + (d\Omega_A^C + \Omega_A^D \Omega_D^C) \Omega_C^B \\ &\quad - (-1)^{C^2+|B||C|+|A||C|+|A||B|} (d\Omega_C^B + \Omega_C^D \Omega_D^B) \Omega_A^C \\ &= \Omega_A^C d\Omega_C^B - d\Omega_A^C \Omega_C^B + d\Omega_A^C \Omega_C^B + \Omega_A^D \Omega_D^C \Omega_C^B \\ &\quad - (-1)^{C^2+|B||C|+|A||C|+|A||B|} (d\Omega_C^B \Omega_A^C) - (-1)^{D^2+|B||D|+|A||D|+|A||B|} (\Omega_D^C \Omega_C^B \Omega_A^D) \\ &= \Omega_A^C d\Omega_C^B - d\Omega_A^C \Omega_C^B + d\Omega_A^C \Omega_C^B + \Omega_A^D \Omega_D^C \Omega_C^B - \Omega_A^C d\Omega_C^B - \Omega_A^D \Omega_D^C \Omega_C^B \\ &= 0 \end{aligned}$$

where we used the fact that $d\Omega_C^B \Omega_A^C = (-1)^{2.1+C^2+|B||C|+|A||C|+|A||B|} \Omega_A^C d\Omega_C^B = \Omega_A^C d\Omega_C^B$, etc. Therefore,

$$DR_A^B = 0 \quad (3.2.15)$$

The equations 3.2.14 and 3.2.15 are called *Bianchi identities*. As it is mentioned in [27], for the Lorentz gauge group the equation 3.2.15 is satisfied if 3.2.14 is. We can expand these identities in components:

$$\begin{aligned} DT^A &= \frac{1}{2} E^C E^B E^F D_F T_{BC}^A + \frac{1}{2} D(E^B E^C) T_{BC}^A \\ &= \frac{1}{2} E^C E^B E^F D_F T_{BC}^A + \frac{1}{2} E^C E^F D_F E^B T_{BC}^A - \frac{1}{2} E^F D_F E^C E^B T_{BC}^A \\ &= \frac{1}{2} E^C E^B E^F D_F T_{BC}^A + \frac{1}{2} E^C T^B T_{BC}^A - \frac{1}{2} T^C E^B T_{BC}^A \\ &= \frac{1}{2} E^C E^B E^F D_F T_{BC}^A + \frac{1}{4} E^C E^F E^G T_{GF}^B T_{BC}^A - (-1)^{|B||C|} \frac{1}{4} E^C E^F E^G T_{GF}^B T_{CB}^A \\ &= \frac{1}{2} E^C E^B E^F D_F T_{BC}^A + \frac{1}{4} E^C E^F E^G T_{GF}^B T_{BC}^A - (-1)^{2|B||C|+1} \frac{1}{4} E^C E^F E^G T_{GF}^B T_{BC}^A \\ &= \frac{1}{2} E^C E^B E^F D_F T_{BC}^A + \frac{1}{2} E^C E^F E^G T_{GF}^B T_{BC}^A \end{aligned}$$

and by using $R_B^A = \frac{1}{2} E^D E^C R_{CDB}^A$:

$$\begin{aligned} E^B \left(\frac{1}{2} E^D E^C R_{CDB}^A \right) &= \frac{1}{2} E^C E^B E^F D_F T_{BC}^A + \frac{1}{2} E^C E^F E^G T_{GF}^B T_{BC}^A \\ E^B E^D E^C R_{CDB}^A &= E^C E^B E^F D_F T_{BC}^A + E^C E^B E^F T_{FB}^D T_{DC}^A \\ E^C E^B E^F R_{FBC}^A &= E^C E^B E^F D_F T_{BC}^A + E^C E^B E^F T_{FB}^D T_{DC}^A \\ \rightarrow \sum_{(FBC)} R_{FBC}^A &= \sum_{(FBC)} D_F T_{BC}^A + \sum_{(FBC)} T_{FB}^D T_{DC}^A \end{aligned}$$

where (FBC) denotes a graded cyclic sum in the indices F, B, C . Relabelling indices, we obtain:

$$\sum_{(ABC)} [R_{ABC}^D - D_A T_{BC}^D - T_{AB}^E T_{EC}^D] = 0 \quad (3.2.16)$$

Also,

$$\begin{aligned}
DR_A^B &= 0 \\
\frac{1}{2}E^DE^C DR_{CDA}^B + \frac{1}{2}D(E^BE^C)R_{CDA}^B &= 0 \\
\frac{1}{2}E^DE^CE^F D_F R_{CDA}^B + \frac{1}{2}E^D DE^C E_{CDA}^B - \frac{1}{2}DE^DE^C R_{CDA}^B &= 0 \\
\frac{1}{2}E^DE^CE^F D_F R_{CDA}^B + \frac{1}{2}E^D T^C R_{CDA}^B - (-1)^{|C||D|} \frac{1}{2}E^C T^D R_{CDA}^B &= 0 \\
\frac{1}{2}E^DE^CE^F D_F R_{CDA}^B + \frac{1}{2}E^D T^C R_{CDA}^B - (-1)^{2|C||D|+1} \frac{1}{2}E^D T^C R_{CDA}^B &= 0 \\
\frac{1}{2}E^DE^CE^F D_F R_{CDA}^B + E^D T^C R_{CDA}^B &= 0 \\
\frac{1}{2}E^DE^CE^F D_F R_{CDA}^B + \frac{1}{2}E^D E^F E^G T_{GF}^C R_{CDA}^B &= 0 \\
\frac{1}{2}E^DE^CE^F D_F R_{CDA}^B + \frac{1}{2}E^D E^C E^F T_{FC}^G R_{GDA}^B &= 0 \\
\rightarrow \sum_{(FCD)} [D_F R_{CDA}^B + T_{FC}^G R_{GDA}^B] &= 0
\end{aligned}$$

where (FCD) means graded cyclic sum in the indices F, C, D . Relabelling indices, we obtain:

$$\sum_{(ABC)} [D_A R_{BCD}^E + T_{AB}^F R_{FCD}^E] = 0 \quad (3.2.17)$$

In these proofs we have used the fact that E^A is an 1-form, $\phi_p \psi_q = (-1)^{pq+|p||q|} \psi_q \phi_p$ (where the lower index indicates the form degree and $|p| = 0$ if the form is bosonic, and $|p| = 1$ if the form is fermionic, as it was discussed in chapter two) and $R_{ABCD} = (-1)^{|A||B|+1} R_{BACD}$, etc. Now, we are going to introduce a new 3-form superfield X , which will contain the bosonic 3-form X_{mnp} . So, we define it in the following way:

$$X = \frac{1}{3!} E^C E^B E^A X_{ABC} \quad (3.2.18)$$

with gauge invariance

$$\delta X = dY \quad (3.2.19)$$

where $Y = \frac{1}{2} E^B E^A Y_{AB}$ is a 2-form. We can define the corresponding field strength, which will be denoted by H :

$$H = dX \quad (3.2.20)$$

and, because it is a 4-form, we can write it as follows

$$H = \frac{1}{4!} E^D E^C E^B E^A H_{ABCD} \quad (3.2.21)$$

From the equation 3.2.20 one obtains the identity

$$dH = 0 \quad (3.2.22)$$

which can be expressed in components

$$\begin{aligned}
dH &= 0 \\
E^D E^C E^B E^A dH_{ABCD} + d(E^D E^C E^B E^A) H_{ABCD} &= 0 \\
E^D E^C E^B E^A E^F D_F H_{ABCD} + (E^D E^C E^B dE^A - d(E^D E^C E^B) E^A) H_{ABCD} &= 0 \\
E^D E^C E^B E^A E^F D_F H_{ABCD} + (E^D E^C E^B dE^A - E^D E^C dE^B E^A \\
&\quad + d(E^D E^C) E^B E^A) H_{ABCD} = 0 \\
E^D E^C E^B E^A E^F D_F H_{ABCD} + (E^D E^C E^B dE^A - E^D E^C dE^B E^A + E^D dE^C E^B E^A \\
&\quad - dE^D E^C E^B E^A) H_{ABCD} = 0 \\
E^D E^C E^B E^A E^F D_F H_{ABCD} + \frac{1}{2} E^D E^C E^B E^F E^G T_{GF}{}^A H_{ABCD} - \frac{1}{2} E^D E^C E^F E^G T_{GF}{}^B E^A H_{ABCD} \\
&\quad + \frac{1}{2} E^D E^F E^G T_{GF}{}^C E^B E^A H_{ABCD} - \frac{1}{2} E^F E^G T_{GF}{}^D E^C E^B E^A H_{ABCD} = 0 \\
E^D E^C E^B E^A E^F D_F H_{ABCD} + \frac{1}{2} E^D E^C E^B E^F E^G T_{GF}{}^A H_{ABCD} \\
&\quad - (-1)^{2|A||B|+|B||F|+|G||B|+1} \frac{1}{2} E^D E^C E^F E^G E^B T_{GF}{}^A H_{ABCD} \\
&\quad + (-1)^{|G||A|+|G||B|+|B||C|+|A||C|+|A||F|+|F||B|} \frac{1}{2} E^D E^F E^G E^B E^A T_{GF}{}^C H_{ABCD} \\
+ (-1)^{2|C||D|+|G||D|+2|F||D|} \frac{1}{2} E^D E^F E^G E^B E^A (-1)^{|B||C|+|C||A|+|G||A|+|F||A|+|G||B|+|F||B|} T_{GF}{}^C H_{ABCD} &= 0 \\
E^D E^C E^B E^A E^F D_F H_{ABCD} + \frac{1}{2} E^D E^C E^B E^F E^G T_{GF}{}^A H_{ABCD} + \frac{1}{2} E^D E^C E^B E^F E^G T_{GF}{}^A H_{ABCD} \\
&\quad + (-1)^{|G||A|+|F||A|+|G||B|+|F||B|} \frac{1}{2} E^D E^F E^G E^B E^A T_{GF}{}^C H_{CABD} \\
&\quad + (-1)^{|G||A|+|F||A|+|G||B|+|F||B|} \frac{1}{2} E^D E^F E^G E^B E^A T_{GF}{}^C H_{CABD} = 0 \\
E^D E^C E^B E^A E^F D_F H_{ABCD} + E^D E^C E^B E^F E^G T_{GF}{}^A H_{ABCD} \\
&\quad (-1)^{|G||A|+|F||A|+|G||B|+|F||B|} E^D E^F E^G E^B E^A T_{GF}{}^C H_{CABD} = 0 \\
E^E E^D E^C E^B E^A D_A H_{BCDE} + E^E E^D E^C E^B E^A T_{AB}{}^F H_{FCDE} \\
&\quad (-1)^{|A||B|+|D||B|+|A||E|+|D||E|} E^C E^D E^A E^E E^B T_{AD}{}^F H_{FBEC} = 0 \\
E^E E^D E^C E^B E^A D_A H_{BCDE} + E^E E^D E^C E^B E^A T_{AB}{}^F H_{FCDE} \\
&\quad (-1)^{|D||B|+|D||C|+|E||C|+1} E^E E^D E^C E^B E^A T_{AD}{}^F H_{FBEC} = 0
\end{aligned}$$

therefore

$$\sum_{(ABCDE)} [D_A H_{BCDE} + T_{AB}{}^F H_{FCDE}] - \sum_{(ADBEC)} (-1)^{|D||B|+|D||C|+|E||C|} T_{AD}{}^F H_{FBEC} = 0 \quad (3.2.23)$$

where we have used once again that E^A is an 1-form, $\phi_p \psi_q = (-1)^{pq+|p||q|} \psi_q \phi_p$ (where the lower index indicates the form degree and $|p| = 0$ if the form is bosonic, and $|p| = 1$ if the form is fermionic, as it was discussed in chapter two) and $H_{ABCD} = (-1)^{|A||B|+1} H_{BACD}$, etc.

A complete treatment of $D = 11$ SG can be found in [29] where it is showed explicitly how the *conventional* and *physical* constraints work for $D = 11$ SG, analogously as we did for $D = 10$ SYM. Because this analysis is a mess we will omit the details and just describe how the physical fields and its corresponding equations of motion arise in this superspace formulation. So we will just follow the dimensional argument cited in [27]. The bosonic and fermionic coordinates will

have mass dimensions 0 and $-\frac{1}{2}$, respectively. Because we will focus on the mass shell, $T_{AB}{}^C$ and H_{ABCD} can contain only constants, the field strengths for the 3-form potential X_{mnr} , the gravitino and the graviton, whose mass dimensions are -1 , $-\frac{3}{2}$ and -2 , respectively; and also the equations of motion for the 3-form, the gravitino and the graviton whose mass dimensions are -2 , $-\frac{3}{2}$ and -2 , respectively. Therefore, up to a normalization, we have:

$$T_{\alpha\beta}{}^c = -i(\Gamma_{\alpha\beta}^c) \quad , \quad \text{mass.dim: } 0 \quad (3.2.24)$$

$$T_{\alpha\beta}{}^\gamma = 0 \quad , \quad \text{mass.dim: } -\frac{1}{2} \quad (3.2.25)$$

$$T_{ab}{}^c = 0 \quad , \quad \text{mass.dim: } -\frac{1}{2} \quad (3.2.26)$$

$$(3.2.27)$$

and

$$H_{\alpha abc} = 0 \quad , \quad \text{mass.dim: } -\frac{1}{2} \quad (3.2.28)$$

$$H_{\alpha\beta\gamma c} = 0 \quad , \quad \text{mass.dim: } \frac{1}{2} \quad (3.2.29)$$

$$H_{\alpha\beta\gamma\delta} = 0 \quad , \quad \text{mass.dim: } 1 \quad (3.2.30)$$

$$H_{\alpha\beta cd} = i(\Gamma_{cd})_{\alpha\beta} \quad , \quad \text{mass.dim: } 0 \quad (3.2.31)$$

Furthermore even though $T_{ab}{}^c$ has dimension -1 , this can be set to zero by choosing a particular connection. Therefore

$$T_{ab}{}^c = 0 \quad (3.2.32)$$

So let us solve the Bianchi identities in their forms 3.2.17 and 3.2.23 by using these assumptions. Utilizing the $R_{\alpha\beta cd}$ and $R_{\alpha\beta\gamma\delta}$ identities from 3.2.17 and the $D_\alpha H_{\beta cde}$ identity from 3.2.23 one gets for $T_{a\beta}{}^\gamma$

$$T_{a\beta}{}^\gamma = \frac{1}{36} H_{abcd} (\Gamma^{abcd})_{\beta}{}^\gamma + \frac{1}{288} (\Gamma_{abcde})_{\beta}{}^\gamma H^{bcde} \quad (3.2.33)$$

whose mass dimension is -1 . Now the $R_{abc}{}^d$ and $R_{ab\gamma}{}^\delta$ identities from 3.2.17 and the $D_\alpha H_{bcde}$ from 3.2.23 give us the following results:

$$T_{ab}{}^\alpha = \frac{1}{42} i(\Gamma^{cd})_\alpha{}^\beta D_\beta H_{abcd} \quad (3.2.34)$$

$$D_\alpha H_{bcde} = -\frac{1}{7} (\Gamma_{[bc}\Gamma^{fg})_\alpha{}^\beta D_\beta H_{de]fg} \quad (3.2.35)$$

$$(\Gamma^{abc})_{\alpha\beta} T_{bc}{}^\beta = 0 \quad (3.2.36)$$

where $T_{ab}{}^\alpha$ has mass dimension $-\frac{3}{2}$. Clearly the equation 3.2.36 is the spin- $\frac{3}{2}$ equation of motion (where $T_{bc}{}^\beta$ will contain the field strength of the gravitino $\nabla_b \psi_c^\beta$). Finally by using the identity 3.2.17 we obtain for $R_{ab\gamma\delta}$:

$$R_{ab\gamma\delta} = D_a T_{b\gamma\delta} - D_b T_{a\gamma\delta} + D_\gamma T_{ab\delta} + T_{a\gamma}{}^\epsilon T_{b\epsilon\delta} - T_{b\gamma}{}^\epsilon T_{a\epsilon\delta} \quad (3.2.37)$$

This equation will allow us to find the equations of motion for the graviton and the 3-form potential. Let us see how this works. If we contract this equation with $(\Gamma_{cd})^{\gamma\delta}$ and use the expressions found for $T_{ab\gamma}$ and $T_{ab\alpha}$ in the equations 3.2.33 and 3.2.34, we find

$$R_{ab} - \frac{1}{2} \eta_{ab} R = -\frac{1}{48} (4H_{acde} H_b{}^{cde} - \frac{1}{2} \eta_{ab} H^{cdef} H_{cdef}) \quad (3.2.38)$$

where $R_{ab} = \eta^{cd} R_{abcd}$ and $R = \eta^{ab} R_{ab}$. This is the equation of motion for the spin-2 field, the graviton. On the other hand, if we contract the equation 3.2.37 with $(\Gamma_c)^{\gamma\delta}$, we obtain

$$D^a H_{abcd} = -\frac{1}{1728} \epsilon_{bcde_1 \dots e_4 f_1 \dots f_4} H^{e_1 \dots e_4} H^{f_1 \dots f_4} \quad (3.2.39)$$

which is the equation for the field strength of the 3-form. Therefore we have obtained the field content and the equations of motion of $D = 11$ SG. One interesting thing here is that with our assumptions we have expressed the curvature, the non-vanishing components of torsion and H_{ABCD} in terms of the superfield H_{abcd} , whose independent components are $H_{abcd}|_{\theta=0}$, the part of the field strength of the gravitino which does not vanish by the equation of motion and the Weyl tensor. In order to make the equations of motion in the component and superspace descriptions coincide we should just work with a different normalization in our assumptions deduced by dimensional arguments.

If we want to work with a linearized version of $D = 11$ SG, we just have to ignore the quadratic terms in the equations of motion. So the linearized equations of motion are:

$$\Gamma_{\alpha\beta}^{abc} \partial_b \psi_c^\beta = 0 \quad (3.2.40)$$

$$R_{ab} - \frac{1}{2} \eta_{ab} R = 0 \quad (3.2.41)$$

$$\partial^a H_{abcd} = 0 \quad (3.2.42)$$

These are precisely the equations which will be obtained by studying the $D = 11$ semi pure spinor superparticle.

Chapter 4

Pure spinor Formalism

In this chapter we will introduce the pure spinor formalism by studying the superparticle. We will focus on the cases D=10 and D=11.

4.1 N=1 D=10 superparticle (Brink-Schwarz superparticle)

In order to see the advantages of the pure spinor formalism we will start describing the Brink-Schwarz superparticle (D=10) [14]. The Brink-Schwarz action is:

$$S = \int d\tau (\Pi^m P_m + e P^m P_m) \quad (4.1.1)$$

where $\Pi^m = \dot{X}^m - i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \theta^\beta$, P^m is the conjugate momentum of the coordinate X^m , e is a Lagrange multiplier (which constrains our study to the massless case)¹.

This action is clearly reparameterization-invariant², but is also invariant under the following transformations [16] [19] [20] [21]:

$$\begin{aligned} \text{SUSY transformations} &\rightarrow \delta\theta^\alpha = \epsilon^\alpha, \delta X^m = i\theta^\alpha \gamma_{\alpha\beta}^m \epsilon^\beta, \delta P_m = \delta e = 0 \\ \kappa \text{ (local) transformations} &\rightarrow \delta\theta^\alpha = P^m \gamma_m^{\alpha\beta} \kappa_\beta, \delta X^m = -i\theta^\alpha \gamma_{\alpha\beta}^m \delta\theta^\beta, \delta P_m = 0, \delta e = 2i\dot{\theta}^\beta \kappa_\beta \end{aligned}$$

Let us start showing the invariance of the action by SUSY transformations:

$$\delta S = \int d\tau (\delta\Pi^m P_m + \Pi^m \delta P_m) \quad (4.1.2)$$

$$\delta S = \int d\tau (\delta\dot{X}^m - i\delta\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \theta^\beta - i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \delta\theta^\beta) P_m \quad (4.1.3)$$

$$\delta S = \int d\tau (i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \epsilon^\beta - i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \epsilon^\beta) P_m \quad (4.1.4)$$

$$\delta S = 0 \quad (4.1.5)$$

¹We are using the same conventions considered in the previous chapter, namely, Majorana-Weyl spinors, 16×16 real symmetric gamma matrices, the SUSY generators satisfying $\{Q_\alpha, Q_\beta\} = -2i\gamma_{\alpha\beta}^m \partial_m$, etc.

²Under an arbitrary reparameterization $\tau \rightarrow \tau'$, we have $(\Pi')^m \rightarrow \frac{d\tau}{d\tau'} (\Pi^m)$ and $e' \rightarrow \frac{d\tau}{d\tau'} e$. These changes are canceled by the transformation of the measure: $d\tau' \rightarrow \frac{d\tau'}{d\tau} d\tau$. An instructive analysis of the superparticle symmetries can be found in [10].

and now, by κ -transformations³:

$$\delta S = \int d\tau (\delta \Pi^m P_m + \delta e P^m P_m) \quad (4.1.6)$$

$$\delta S = \int d\tau (-2i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \delta\theta^\beta P_m + 2i\dot{\theta}^\beta \kappa_\beta P^m P_m) \quad (4.1.7)$$

$$\delta S = \int d\tau (-2i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m P^n (\gamma_n)^{\beta\lambda} \kappa_\lambda P_m + 2i\dot{\theta}^\beta \kappa_\beta P^m P_m) \quad (4.1.8)$$

$$\delta S = \int d\tau (-2i\eta_m^n \delta_\alpha^\lambda \dot{\theta}^\alpha \kappa_\lambda) P^n P_m + 2i\dot{\theta}^\beta \kappa_\beta P^m P_m) \quad (4.1.9)$$

$$\delta S = \int d\tau (-2i\dot{\theta}^\alpha \kappa_\alpha P^m P_m + 2i\dot{\theta}^\beta \kappa_\beta P^m P_m) \quad (4.1.10)$$

$$\delta S = 0 \quad (4.1.11)$$

Now, we can calculate the conjugate momentum to θ^α :

$$p_\alpha = \frac{\partial L}{\partial \dot{\theta}^\alpha} = -i\gamma_{\alpha\beta}^m \theta^\beta P_m \quad (4.1.12)$$

therefore we have a system with constraints [17] which are given by

$$d_\alpha = p_\alpha + i\gamma_{\alpha\beta}^m \theta^\beta P_m \quad (4.1.13)$$

and if we remember that⁴ $\{\theta^\alpha, p_\beta\}_{P.B} = i\delta_\beta^\alpha$, we obtain:

$$\{d_\alpha, d_\beta\} = \{p_\alpha + i\gamma_{\alpha\lambda}^m \theta^\lambda P_m, p_\beta + i\gamma_{\beta\sigma}^n \theta^\sigma P_n\} \quad (4.1.14)$$

$$\{d_\alpha, d_\beta\} = \{p_\alpha, p_\beta\} + i\gamma_{\beta\sigma}^n P_n \{p_\alpha, \theta^\sigma\} + i\gamma_{\alpha\lambda}^m P_m \{\theta^\lambda, p_\beta\} - \gamma_{\alpha\lambda}^m \gamma_{\beta\sigma}^n P_m P_n \{\theta^\lambda, \theta^\sigma\} \quad (4.1.15)$$

$$\{d_\alpha, d_\beta\} = -\gamma_{\beta\sigma}^n P_n \delta_\alpha^\sigma - \gamma_{\alpha\lambda}^m P_m \delta_\beta^\lambda \quad (4.1.16)$$

$$\{d_\alpha, d_\beta\} = -2(\gamma^m)_{\alpha\beta} P_m \quad (4.1.17)$$

where we should understand $\{, \}$ as being a Poisson bracket. Because $P^2 = 0$ we will have 8 first-class constraints and 8 second-class constraints⁵. If we define $K^\alpha = -iP^m \gamma_m^{\alpha\beta} d_\beta$, we can show that these ones are nothing but the first-class constraints generating the κ -symmetry:

$$\{K^\alpha, K^\beta\} = \{-iP^m \gamma_m^{\alpha\lambda} d_\lambda, -iP^n \gamma_n^{\beta\sigma} d_\sigma\} \quad (4.1.18)$$

$$\{K^\alpha, K^\beta\} = -P^m P^n \gamma_m^{\alpha\lambda} \gamma_n^{\beta\sigma} (-2\gamma_{\lambda\sigma}^p P_p) \quad (4.1.19)$$

$$\{K^\alpha, K^\beta\} = 2P^m P^n P^p (\gamma_m \gamma_p \gamma_n)^{\alpha\beta} \quad (4.1.20)$$

³Let us calculate the κ variation of Π^m :

$$\delta \Pi^m = \delta \dot{X}^m - i\delta \dot{\theta}^\alpha \gamma_{\alpha\beta}^m \theta^\beta - i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \delta \theta^\beta$$

$$\delta \Pi^m = -i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \delta \theta^\beta - i\theta^\alpha \gamma_{\alpha\beta}^m \delta \dot{\theta}^\beta - i\delta \dot{\theta}^\alpha \gamma_{\alpha\beta}^m \theta^\beta - i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \delta \theta^\beta$$

$$\delta \Pi^m = -2i\dot{\theta}^\alpha \gamma_{\alpha\beta}^m \delta \theta^\beta$$

⁴This i is put in front of the δ_β^α because our p_α will be pure imaginary (in order to have a real action). We will always use this convention, unless otherwise specified.

⁵We can see this by choosing a frame where, for instance, $P = (P, 0, \dots, 0, P)$. Now, we can define the light-cone coordinates and γ matrices: $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^9)$, $\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9)$. It is clear that $P^- = 0$, hence the equation 4.1.17 takes the form: $\{d_\alpha, d_\beta\} = 2(\gamma^-)_{\alpha\beta} P^+ \propto \begin{pmatrix} 1_{8 \times 8} & 0_{8 \times 8} \\ 0_{8 \times 8} & 0_{8 \times 8} \end{pmatrix}$.

It is easy to see that this quantity vanishes⁶. Therefore $\{K_\alpha, K_\beta\} = 0$. Now, let us see how these ones generate the κ -transformations:

$$\delta\theta^\alpha = \{K^\lambda, \theta^\alpha\} \kappa_\lambda \quad (4.1.21)$$

$$\delta\theta^\alpha = \{-iP^m \gamma_m^{\lambda\sigma} d_\sigma, \theta^\alpha\} \kappa_\lambda \quad (4.1.22)$$

$$\delta\theta^\alpha = -iP^m \gamma_m^{\lambda\sigma} \{p_\sigma, \theta^\alpha\} \kappa_\lambda \quad (4.1.23)$$

$$\delta\theta^\alpha = P^m \gamma_m^{\lambda\alpha} \kappa_\lambda \quad (4.1.24)$$

which is the κ -transformation for θ^α defined above. For the case of the coordinates X^m , we have⁷:

$$\delta X^m = i\{D^\lambda, X^m\} \kappa_\lambda \quad (4.1.25)$$

$$\delta X^m = i\{-iP^n \gamma_n^{\lambda\beta} d_\beta, X^m\} \kappa_\lambda \quad (4.1.26)$$

$$\delta X^m = P^n \gamma_n^{\lambda\beta} \{i\gamma_{\beta\sigma}^q \theta^\sigma P_q, X^m\} \kappa_\lambda \quad (4.1.27)$$

$$\delta X^m = -iP^n \gamma_n^{\lambda\beta} \gamma_{\beta\sigma}^m \theta^\sigma \kappa_\lambda \quad (4.1.28)$$

$$\delta X^m = -i\theta^\sigma \gamma_{\sigma\beta}^m \delta\theta^\beta \quad (4.1.29)$$

which is the corresponding κ -transformation for X^m . Let us go back to the equation 4.1.17. There is no simple way to covariantly separate out the second-class constraints. However, it is simple to find the physical spectrum by choosing an specific non-covariant gauge: *the semi-light-cone gauge* ($P^+ \neq 0$). The light-cone coordinates are defined by:

$$P^\pm = \frac{1}{\sqrt{2}}(P^0 \pm P^9) \quad (4.1.30)$$

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^9) \quad (4.1.31)$$

$$\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9) \quad (4.1.32)$$

we are working in a frame where $P^m = (P, 0, \dots, 0, P)$. From these definitions we can easily obtain the following results:

$$(\gamma^+)_{\alpha\beta} (\gamma^+)^{\beta\delta} = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^9) \cdot \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^9) = \frac{1}{2}((\gamma^0)^2 + \gamma^9)^2 = \frac{1}{2}(-1 + 1) = 0 \quad (4.1.33)$$

$$P^- = \frac{1}{\sqrt{2}}(P - P) = 0 \quad (4.1.34)$$

So the κ -transformation for θ^α in terms of the light-cone coordinates is: $\delta\theta^\alpha = P^m \gamma_m^{\alpha\beta} \kappa_\beta = P^+ \gamma_+^{\alpha\beta} \kappa_\beta + P^- \gamma_-^{\alpha\beta} \kappa_\beta + P^i \gamma_i^{\alpha\beta} \kappa_\beta = -P^+ (\gamma^-)^{\alpha\beta} \kappa_\beta$, because $\eta_{+-} = -1$ and in our frame $P^i = P_i = 0$. If we choose $\kappa_\beta = -\frac{1}{2P^+} (\gamma_{\beta\lambda}^+ \theta^\lambda) \rightarrow \delta\theta^\alpha = \frac{1}{2} [(\gamma^-)^{\alpha\beta} (\gamma^+)_{\beta\lambda} \theta^\lambda]$. Furthermore, we can express the $SO(9, 1)$ Majorana-Weyl spinor θ^α in terms of its $SO(8)$ components as follows:

$$\theta^\alpha = \begin{pmatrix} \theta^a \\ \theta^{\dot{a}} \end{pmatrix} \quad (4.1.35)$$

⁶We should use the γ -matrices identities:

$$\begin{aligned} P^m P^n P^p (\gamma_m \gamma_p \gamma_n)^{\alpha\beta} &= P^m P^n P^p (2\eta_{np} \gamma_m - \gamma_m \gamma_n \gamma_p)^{\alpha\beta} \\ P^2 = 0 \rightarrow P^m P^n P^p (\gamma_m \gamma_p \gamma_n)^{\alpha\beta} &= -P^m P^p P^n (\gamma_m \gamma_p \gamma_n)^{\alpha\beta} \\ P_m, P_n \text{ commute} \rightarrow P^m P^n P^p (\gamma_m \gamma_p \gamma_n)^{\alpha\beta} &= 0 \end{aligned}$$

⁷We use i in front of the Poisson bracket in order to have a real quantity.

where a, \dot{a} are spinorial indices of $SO(8)$. Therefore we can write this spinor θ^α in the form⁸: $\theta^\alpha = -\frac{1}{2}(\gamma^+\gamma^-\theta)^\alpha - \frac{1}{2}(\gamma^-\gamma^+\theta)^\alpha$. So under a κ -transformation we can always obtain a θ^α satisfying $(\gamma^+\theta)_\alpha = \gamma_{\alpha\beta}^+\theta^\beta = 0$:

$$\theta'^\alpha = \theta^\alpha + \delta\theta^\alpha \quad (4.1.36)$$

$$\theta'^\alpha = -\frac{1}{2}(\gamma^+\gamma^-\theta)^\alpha - \frac{1}{2}(\gamma^-\gamma^+\theta)^\alpha + \frac{1}{2}(\gamma^-\gamma^+\theta)^\alpha \quad (4.1.37)$$

$$\theta'^\alpha = -\frac{1}{2}(\gamma^+\gamma^-\theta)^\alpha \quad (4.1.38)$$

which clearly satisfies the condition $(\gamma^+\theta')^\alpha = 0$ ⁹. From now on we will work with θ^α satisfying this condition. The action takes the following form

$$S = \int d\tau \{(\dot{X}^m - i\dot{\theta}\gamma^m\theta)P_m + eP^m P_m\} \quad (4.1.40)$$

$$S = \int d\tau \{\dot{X}^m P_m - i(-\dot{\theta}\gamma^+\theta P^- - \dot{\theta}\gamma^-\theta P^+ + \dot{\theta}\gamma^i\theta P^i) + eP^m P_m\} \quad (4.1.41)$$

$$S = \int d\tau (\dot{X}^m P_m + i\dot{\theta}\gamma^-\theta P^+ + eP^m P_m) \quad (4.1.42)$$

and if we define $S_a = \sqrt{\sqrt{2}P^+}(\gamma^-\theta)_a$, we get

$$S = \int d\tau (\dot{X}^m P_m - \frac{i}{2}\dot{S}^a S_a + eP^m P_m) \quad (4.1.43)$$

The conjugate momentum to S_a is: $p_a = \frac{dL}{dS^a} = -\frac{i}{2}S_a$. By imposing that $\{S_a, p^b\} = -i\delta_a^b$, we obtain that the constraints $\tilde{d}_a = p_a + \frac{i}{2}S_a$ satisfy the following Poisson bracket:

$$\{\tilde{d}_a, \tilde{d}_b\} = \{p_a + \frac{i}{2}S_a, p_b + \frac{i}{2}S_b\} \quad (4.1.44)$$

$$\{\tilde{d}_a, \tilde{d}_b\} = \frac{i}{2}\{p_a, S_b\} + \frac{i}{2}\{S_a, p_b\} \quad (4.1.45)$$

$$\{\tilde{d}_a, \tilde{d}_b\} = \delta_{ab} \quad (4.1.46)$$

⁸By using our conventions:

$$\begin{aligned} \theta^\alpha &= -\frac{1}{2}(\gamma^+)^{\alpha\beta}(\gamma^-)_{\beta\lambda}\theta^\lambda - \frac{1}{2}(\gamma^-)^{\alpha\beta}(\gamma^+)_{\beta\lambda}\theta^\lambda \\ \theta^\alpha &= -\frac{1}{2}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \theta^a \\ \theta^{\dot{a}} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} \theta^a \\ \theta^{\dot{a}} \end{pmatrix} \\ \rightarrow \theta^\alpha &= \begin{pmatrix} \theta^a \\ \theta^{\dot{a}} \end{pmatrix} \end{aligned}$$

⁹We just have to use the equation 4.1.33:

$$(\gamma^+\theta')^\alpha = -\frac{1}{2}(\gamma^+)_{\alpha\lambda}(\gamma^+)^{\lambda\delta}(\gamma^-)_{\delta\sigma}\theta^\sigma = 0 \quad (4.1.39)$$

This allows us to calculate the constraint matrix $C_{ab} = \delta_{ab}$ and its inverse $(C^{-1})^{ab} = \delta^{ab}$. Therefore the corresponding Dirac bracket is:

$$\{S_a, S_b\}_D = \{S_a, S_b\}_P - \sum_{e,f} \{S_a, \tilde{d}_e\}_P (C^{-1})^{ef} \{\tilde{d}_f, S_b\}_P \quad (4.1.47)$$

$$\{S_a, S_b\}_D = 0 - \sum_{e,f} (-i\delta_{de})(\delta^{ef})(-i\delta_{fb}) \quad (4.1.48)$$

$$\{S_a, S_b\}_D = \delta_{ab} \quad (4.1.49)$$

If we remember the property of the $SO(8)$ Pauli matrices σ_{ab}^j :

$$\sigma_{ac}^i \sigma_{bc}^j + \sigma_{bc}^i \sigma_{ac}^j = 2\delta_{ab} \delta^{ij} \quad (4.1.50)$$

we realize that with the following definitions:

$$S^a \psi_j(x) = \frac{1}{\sqrt{2}} \sigma_j^{ab} \psi_b(x) \quad (4.1.51)$$

$$S_a \psi_b(x) = \frac{1}{\sqrt{2}} \sigma_{ab}^j \psi_j(x) \quad (4.1.52)$$

we reproduce the equation 4.1.49¹⁰. So the physical spectrum is a $SO(8)$ vector $\psi_j(x)$ and a $SO(8)$ spinor $\psi_i(x)$, which is the same spectrum of $D = 10$ SYM.

4.2 N=1 D=10 Pure spinor superparticle

In order to use the ideas mentioned in the previous section, we will follow the structure of the pure spinor superparticle discussion developed at (references from ICTP lectures). So, we will introduce a new set of variables $(\theta^\alpha, p_\alpha)$ (these ones have nothing to do with the used ones in the previous chapter) and introduce a new symmetry coming from the following first-class constraints:

$$\hat{d}_\alpha = d_\alpha + \frac{1}{\sqrt{\sqrt{2}P^+}} (\gamma_m \gamma^+ S)_\alpha P^m \quad (4.2.1)$$

where $\{S_a, S_b\} = \delta_{ab}$ and we define $d_\alpha = p_\alpha + i\gamma_{\alpha\beta}^m \theta^\beta P_m$. By using the usual condition $\{\theta^\alpha, p_\beta\} = i\delta_\beta^\alpha$ we show that $\{d_\alpha, d_\beta\} = -2(\gamma^m)_{\alpha\beta} P_m$. Let us check that these ones are indeed first-class

¹⁰We should use the suitable identities of the $SO(8)$ Pauli matrices:

$$\begin{aligned} \{S_a, S^b\} \psi_j(x) &= S_a S^b \psi_j(x) + S^b S_a \psi_j(x) \\ \{S_a, S^b\} \psi_j(x) &= \frac{1}{\sqrt{2}} S_a (\sigma_j)^{bb} \psi_b + \frac{1}{\sqrt{2}} S^b (\sigma_{ab})^j \psi^b \\ \{S_a, S^b\} \psi_j(x) &= \frac{1}{2} \sigma_j^{bb} (\sigma^i)_{ab} \psi_i(x) + \frac{1}{2} (\sigma_j)_{ab} (\sigma^i)^{bb} \psi_i(x) \\ \{S_a, S^b\} \psi_j(x) &= \delta^{ij} \delta_a^b \psi_i(x) \\ \{S_a, S^b\} \psi_j(x) &= \delta_a^b \psi_j(x) \end{aligned}$$

and an analog procedure for the fermion $\psi_i(x)$.

constraints:

$$\{\hat{d}_\alpha, \hat{d}_\beta\} = \left\{ \left(d_\alpha + \frac{1}{\sqrt{\sqrt{2}P^+}} (\gamma_m \gamma^+ S)_\alpha P^m \right), \left(d_\beta + \frac{1}{\sqrt{\sqrt{2}P^+}} (\gamma_n \gamma^+ S)_\beta P^n \right) \right\} \quad (4.2.2)$$

$$\{\hat{d}_\alpha, \hat{d}_\beta\} = \{d_\alpha, d_\beta\} + \frac{1}{\sqrt{2}P^+} P^m P^n (\gamma_m)_{\alpha\delta} (\gamma_n)_{\beta\sigma} (\gamma^+)^{\delta a} (\gamma^+)^{\sigma b} \{S_a, S_b\} \quad (4.2.3)$$

$$\{\hat{d}_\alpha, \hat{d}_\beta\} = -2(\gamma^m)_{\alpha\beta} P_m + \frac{1}{\sqrt{2}P^+} P^m P^n (\gamma_m)_{\alpha\delta} (\gamma_n)_{\beta\sigma} (\gamma^+)^{\delta a} (\gamma^+)^{\sigma a} \quad (4.2.4)$$

$$\{\hat{d}_\alpha, \hat{d}_\beta\} = -2(\gamma^m)_{\alpha\beta} P_m + \frac{1}{P^+} P_m P_n (\gamma^m \gamma^+ \gamma^n)_{\alpha\beta} \quad (4.2.5)$$

$$\{\hat{d}_\alpha, \hat{d}_\beta\} = -2(\gamma^m)_{\alpha\beta} P_m + \frac{1}{P^+} P_m P_n (2\eta^{+n} \gamma^m - \gamma^m \gamma^n \gamma^+)_{\alpha\beta} \quad (4.2.6)$$

$$\{\hat{d}_\alpha, \hat{d}_\beta\} = -2(\gamma^m)_{\alpha\beta} P_m + 2(\gamma^m)_{\alpha\beta} P_m - \frac{1}{P^+} P_m P_n (\gamma^m \gamma^n \gamma^+)_{\alpha\beta} \quad (4.2.7)$$

$$\{\hat{d}_\alpha, \hat{d}_\beta\} = -\frac{1}{P^+} P^m P_m (\gamma^+)_{\alpha\beta} \quad (4.2.8)$$

where we used the identities $(\gamma^+)^{\delta a} (\gamma^+)^{\sigma a} = \sqrt{2}(\gamma^+)^{\delta\sigma}$ and $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$. So the modified Brink-Schwarz action will be:

$$S = \int d\tau (\dot{X}^m P_m - \frac{i}{2} \dot{S}^a S_a + e P^m P_m + \theta^\alpha p_\alpha + f^\alpha \hat{d}_\alpha) \quad (4.2.9)$$

where we added the usual kinetic term for the variables $(\theta^\alpha, p_\alpha)$ and the last term takes into account the constraint through the fermionic Lagrange multiplier f^α . Now, we will gauge fix this action and the BRST procedure tells us that we should introduce new (ghosts) variables of opposite statistics to the corresponding symmetry generators. So, when we fix $e = -\frac{1}{2}$ and $f^\alpha = 0$, we get the fermionic ghost c (with its antighost b) and the bosonic ghost $\hat{\lambda}^\alpha$ (with its antighost \hat{w}_α), respectively:

$$S = \int d\tau (\dot{X}^m P_m - \frac{i}{2} \dot{S}^a S_a - \frac{1}{2} P^m P_m + \theta^\alpha p_\alpha + \dot{c}b + \hat{\lambda}^\alpha \hat{w}_\alpha) \quad (4.2.10)$$

and the BRST operator is

$$\hat{Q} = \hat{\lambda}^\alpha \hat{d}_\alpha + c P^m P_m + \frac{1}{2P^+} (\hat{\lambda} \gamma^+ \hat{\lambda}) b \quad (4.2.11)$$

Now we will show that the cohomology of this BRST operator \hat{Q} is equivalent to the cohomology of a BRST operator of the form $Q = \lambda^\alpha d_\alpha$, where λ^α is a pure spinor ($\lambda \gamma^m \lambda = 0$)¹¹. We will show this claim in two steps. First, we show that the \hat{Q} -cohomology is equivalent to Q' -cohomology, where $Q' = \lambda'^\alpha \hat{d}_\alpha$, and $\lambda' \gamma^+ \lambda' = 0$. Finally, we will prove that the Q' -cohomology is equivalent to the Q -cohomology.

Let us start by defining the following operator $Q_0 = \lambda_0^\alpha \hat{d}_\alpha$ ¹². Now, let V be a state such that $Q_0 V = (\lambda_0 \gamma^+ \lambda_0) W$ for some W ¹³. By using the equation 4.2.1, we get $(Q_0)^2 = -\frac{1}{2P^+} P^m P_m \lambda_0 \gamma^+ \lambda_0$. So, we conclude that $(Q_0)^2 V = -\frac{1}{2P^+} P^m P_m \lambda_0 \gamma^+ \lambda_0 V = (\lambda_0 \gamma^+ \lambda_0) Q_0 W \rightarrow Q_0 W = -\frac{1}{2P^+} P^m P_m V$.

¹¹Note that $\hat{\lambda}^\alpha$ is an unconstrained bosonic spinor.

¹² Q_0 becomes the first term of \hat{Q} or Q' when λ_0 is $\hat{\lambda}$ or λ' , respectively.

¹³It is clear that because the property satisfied by λ' , V is annihilated by Q' .

Therefore the state $\hat{V} = V - 2P^+cW$ will be annihilated by \hat{Q} :

$$\hat{Q}\hat{V} = \hat{Q}(V - 2P^+cW) \quad (4.2.12)$$

$$\hat{Q}\hat{V} = \hat{Q}V - 2P^+(\hat{Q}c)W + 2P^+c(\hat{Q}W) \quad (4.2.13)$$

$$\hat{Q}\hat{V} = (\hat{\lambda}\gamma^+\hat{\lambda})W + cP^mP_mV - 2P^+(\frac{1}{2P^+})(\hat{\lambda}\gamma^+\hat{\lambda})W + 2P^+c(-\frac{1}{2P^+})P^mP_mV \quad (4.2.14)$$

$$\hat{Q}\hat{V} = (\hat{\lambda}\gamma^+\hat{\lambda})W + cP^mP_mV - (\hat{\lambda}\gamma^+\hat{\lambda})W - cP^mP_mV \quad (4.2.15)$$

$$\hat{Q}\hat{V} = 0 \quad (4.2.16)$$

where in the third line we used the results obtained for Q_0 for the particular case when $\lambda_0^\alpha = \hat{\lambda}^\alpha$, in that way we obtained the first term of \hat{Q} when acting on V . In addition to this, we assumed that b annihilates physical states ($bV = 0$). Now, let us show that if a state V is BRST-trivial (in the Q' -cohomology), we can find a state $\hat{V} = V - 2P^+cW$ which is also BRST-trivial (in the \hat{Q} -cohomology). Let us see how this works. Let V be a state which satisfies $V = Q_0\Omega + (\lambda_0\gamma^+\lambda_0)Y$, for some Y ¹⁴. So, we have

$$\hat{Q}(\Omega + 2P^+cY) = \hat{Q}\Omega + 2P^+(\hat{Q}c)Y - 2P^+c(\hat{Q}Y)$$

$$\hat{Q}(\Omega + 2P^+cY) = V - (\hat{\lambda}\gamma^+\hat{\lambda})Y + cP^mP_m\Omega + 2P^+(\frac{1}{2P^+})\hat{\lambda}\gamma^+\hat{\lambda}Y - 2P^+c(W + \frac{1}{2P^+}P^mP_m\Omega)$$

where in the second line we used the fact that b annihilates Ω and a result satisfied by \hat{Q} which follows from the properties satisfied by Q_0 ¹⁵, just replacing λ_0 by $\hat{\lambda}$ (this term is just the first term of \hat{Q} , but the other two terms will vanish because $c^2 = 0$ and b annihilates Y). Therefore we obtain

$$\begin{aligned} \hat{Q}(\Omega + 2P^+cY) &= V - (\hat{\lambda}\gamma^+\hat{\lambda})Y + cP^mP_m\Omega + (\hat{\lambda}\gamma^+\hat{\lambda})Y - 2P^+cW - cP^mP_m\Omega \\ \hat{Q}(\Omega + 2P^+cY) &= V - 2P^+cW \\ \hat{Q}(\Omega + 2P^+cY) &= \hat{V} \end{aligned}$$

Therefore we have proven that for each state V in the Q' -cohomology, we can find a state \hat{V} in the \hat{Q} -cohomology. If we reverse the arguments given above we can show that any state in the \hat{Q} -cohomology corresponds to a state in the Q' -cohomology.

Furthermore it can be shown that the Q' -cohomology is equivalent to the Q -cohomology. To this end, we should use the property of λ' , $(\lambda'\gamma^+\lambda') = 0$ ¹⁶. This leads us to conclude that $(\gamma^+\lambda')_{\dot{a}}$ is a null $SO(8)$ antichiral spinor: $(\gamma^+\lambda')_{\dot{a}}(\gamma^+\lambda')_{\dot{a}} = \gamma_{\dot{a}\beta}^+\lambda'^{\beta}\gamma_{\dot{a}\sigma}^+\lambda'^{\sigma} = \gamma_{\dot{a}\beta}^+\gamma_{\dot{a}\sigma}^+\lambda'^{\beta}\lambda'^{\sigma} =$

¹⁴Once again, if $\lambda_0^\alpha = \lambda^\alpha$, we have that V is Q' -exact. If $\lambda_0^\alpha = \hat{\lambda}^\alpha$, we have that the first term of \hat{Q} is equal to $V - (\hat{\lambda}\gamma^+\hat{\lambda})Y$.

¹⁵From the definition of V , we get:

$$\begin{aligned} Q_0V &= (Q_0)^2\Omega + (\lambda_0\gamma^+\lambda_0)Q_0Y \\ (\lambda_0\gamma^+\lambda_0)W &= -\frac{1}{2P^+}P^mP_m(\lambda_0\gamma^+\lambda_0)\Omega + (\lambda_0\gamma^+\lambda_0)Q_0Y \\ W + \frac{1}{2P^+}P^mP_m\Omega &= Q_0Y \end{aligned}$$

¹⁶This property can be seen in this other useful way:

$$\lambda'^\alpha\gamma_{\alpha\beta}^+\lambda'^\beta = 0 \rightarrow (\lambda^a \quad \lambda^{\dot{a}}) \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} \lambda^a \\ \lambda^{\dot{a}} \end{pmatrix} = 0 \rightarrow \lambda^{\dot{a}}\lambda^{\dot{a}} = 0 \quad (4.2.17)$$

$-\sqrt{2}\gamma_{\beta\sigma}^+\lambda'^\beta\lambda'^\sigma = -\sqrt{2}(\lambda'\gamma^+\lambda') = 0$. So there exists an $U(4)$ subgroup of $SO(8)$ under which this null spinor is invariant (up to a phase). By using this $U(4)$, a general chiral $SO(8)$ spinor χ_a can be described by the (4) fundamental and ($\bar{4}$) antifundamental representations of $U(4)$. Hence the chiral spinors $(\gamma^-\lambda')_a$, $(\gamma^+d)_a$, S_a can be expressed in terms of these representations:

$$(\gamma^-\lambda')_a \rightarrow \{(\gamma^-\lambda')_A, (\gamma^-\lambda')_{\bar{A}}\} \quad (4.2.18)$$

$$(\gamma^+d)_a \rightarrow \{(\gamma^+d)_A, (\gamma^+d)_{\bar{A}}\} \quad (4.2.19)$$

$$S_a \rightarrow \{S_A, S_{\bar{A}}\} \quad (4.2.20)$$

where $A, \bar{A} = 1, \dots, 4$. If we make the following shift on S_A :

$$S_A \rightarrow S_A - i \frac{\sqrt{\sqrt{2}}}{4\sqrt{2P^+}} (\gamma^+d)_A \quad (4.2.21)$$

the Q' operator will change by a similarity transformation:

$$Q' \rightarrow e^{-i[-iS_{\bar{A}}(\gamma^+d)_A(\frac{\sqrt{\sqrt{2}}}{4\sqrt{2P^+}})]} Q' e^{i[-iS_{\bar{A}}(\gamma^+d)_A(\frac{\sqrt{\sqrt{2}}}{4\sqrt{2P^+}})]} \quad (4.2.22)$$

In order to simplify the notation we will define the following constant quantity $K = -i \frac{\sqrt{\sqrt{2}}}{4\sqrt{2P^+}}$. Now, we have to use the well-known formula:

$$e^{-iS} X e^{iS} = X + i[X, S] + \frac{1}{2}[[X, S], S] + \dots \quad (4.2.23)$$

In our case $X = Q' = \lambda'^\alpha \hat{d}_\alpha$ and $S = K S_{\bar{A}}(\gamma^+d)_A$. So, let us compute the first commutator:

$$[Q', S_{\bar{A}}d_A] = \left[\lambda'^\alpha d_\alpha + \frac{1}{\sqrt{\sqrt{2}P^+}} \lambda'^\alpha (\gamma^m \gamma^+ S)_\alpha P_m, S_{\bar{A}}d_A \right] \quad (4.2.24)$$

$$[Q', S_{\bar{A}}d_A] = \left[\lambda'^\alpha d_\alpha - \frac{1}{\sqrt{\sqrt{2}P^+}} \lambda'^\alpha (\gamma^- \gamma^+ S)_\alpha P^+, S_{\bar{A}}d_A \right] \quad (4.2.25)$$

$$[Q', S_{\bar{A}}d_A] = \left[\lambda'^\alpha d_\alpha + \frac{2\sqrt{P^+}}{\sqrt{\sqrt{2}}} \lambda'^a S^a, S_{\bar{A}}d_A \right] \quad (4.2.26)$$

where we used the fact that $\lambda'^\alpha \gamma_{\alpha\beta}^- \gamma^{+\beta b} S_b = \lambda'^a (-\sqrt{2})(\sqrt{2}) \delta_a^b S_b = -2\lambda'^a S^a$, because in $SO(8)$ we can raise or lower indices by using the deltas δ_{ab} , δ^{ab} and δ_a^b . Therefore,

$$[Q', S_{\bar{A}}d_A] = [\lambda'^\alpha d_\alpha, S_{\bar{A}}d_A] + \frac{2\sqrt{P^+}}{\sqrt{\sqrt{2}}} [\lambda'^a S^a, S_{\bar{A}}d_A] \quad (4.2.27)$$

$$[Q', S_{\bar{A}}d_A] = -\lambda'^\alpha S_{\bar{A}}\{d_\alpha, d_A\} + \frac{2\sqrt{P^+}}{\sqrt{\sqrt{2}}} \lambda'^a \{S_a, S_{\bar{A}}\} d_A \quad (4.2.28)$$

Now we should figure out which is the algebra satisfied by the $SO(4)$ components of d_α and S_a . For this end, we define explicitly the corresponding $SU(4)$ components of these quantities:

$$S_A = \frac{1}{\sqrt{2}}(S_{2a} + iS_{2a+1}) \quad (4.2.29)$$

$$S_{\bar{A}} = \frac{1}{\sqrt{2}}(S_{2a} - iS_{2a-1}) \quad (4.2.30)$$

$$d_A = \frac{1}{\sqrt{2}}(d_{2a} + id_{2a+1}) \quad (4.2.31)$$

$$d_{\bar{A}} = \frac{1}{\sqrt{2}}(d_{2a} - id_{2a-1}) \quad (4.2.32)$$

and by using the previous algebras found for these variables, we get

$$\{S_A, S_{\bar{A}}\} = 2\eta_{A\bar{A}} \quad (4.2.33)$$

$$\{d_A, d_{\bar{A}}\} = -4\sqrt{2}\eta_{A\bar{A}}P^+ \quad (4.2.34)$$

where $\eta_{A\bar{A}}$ has the numerical value 1¹⁷ and the other (anti)commutators vanish. Hence, we obtain

$$-iK\sqrt{2}[Q', S_{\bar{A}}d_A] = -i\sqrt{2}\left(-i\frac{\sqrt{\sqrt{2}}}{4\sqrt{2P^+}}\right)(4\sqrt{2}\lambda'_A S_{\bar{A}}P^+ + \frac{4\sqrt{P^+}}{\sqrt{\sqrt{2}}}\lambda'_{\bar{A}}d_A) \quad (4.2.35)$$

$$-iK\sqrt{2}[Q', S_{\bar{A}}d_A] = -\sqrt{2\sqrt{2}P^+}\lambda'_A S_{\bar{A}} - \lambda'_{\bar{A}}d_A \quad (4.2.36)$$

Let us expand Q' :

$$Q' = \lambda'^\alpha d_\alpha + \frac{1}{\sqrt{\sqrt{2}P^+}}\lambda'\gamma^m\gamma^+SP_m \quad (4.2.37)$$

$$Q' = \lambda'^\alpha d_\alpha + \frac{2}{\sqrt{\sqrt{2}P^+}}\lambda'^a S^a P^+ \quad (4.2.38)$$

$$Q' = \lambda'^\alpha d_\alpha + \frac{2\sqrt{P^+}}{\sqrt{\sqrt{2}}}\lambda'^a S^a \quad (4.2.39)$$

$$Q' = \lambda'^\alpha d_\alpha + \frac{2\sqrt{\sqrt{2}}\sqrt{P^+}}{\sqrt{2}}\lambda'^a S^a \quad (4.2.40)$$

$$Q' = \lambda'^\alpha d_\alpha + \sqrt{2\sqrt{2}P^+}\lambda'^a S^a \quad (4.2.41)$$

$$Q' = \lambda'_a d_{\dot{a}} + \lambda'_{\bar{A}}d_A + \lambda'_A d_{\bar{A}} + \sqrt{2\sqrt{2}P^+}\lambda'_{\bar{A}}S_A + \sqrt{2\sqrt{2}P^+}\lambda'_A S_{\bar{A}} \quad (4.2.42)$$

So, after using the equations 4.2.36 and 4.2.42, we find that

$$Q' \rightarrow \lambda'_a d_{\dot{a}} + \lambda'_A d_{\bar{A}} + \sqrt{2\sqrt{2}P^+}\lambda'_{\bar{A}}S_{\bar{A}} + \frac{1}{2}[[Q', S], S] + \dots \quad (4.2.43)$$

However since $S = KS_{\bar{A}}(\gamma^+d)_A$ and $[Q', S] = -i(\sqrt{2\sqrt{2}P^+}\lambda'_{\bar{A}}S_{\bar{A}} + \lambda'_{\bar{A}}d_A)$ (equation 4.2.36), we conclude that¹⁸:

$$[[Q', S], S] = 0 \quad (4.2.44)$$

and so all of the other terms (which were represented by ...) vanish. Therefore, we have arrived at the following result:

$$Q' \rightarrow \lambda'_a d_{\dot{a}} + \lambda'_A d_{\bar{A}} + \sqrt{2\sqrt{2}P^+}\lambda'_{\bar{A}}S_{\bar{A}} \quad (4.2.45)$$

where $\lambda'^{\dot{a}}$ is a null spinor ($\lambda^{\dot{a}}\lambda_{\dot{a}} = 0$). If we define a spinor $\lambda^\alpha = [\lambda_{\dot{a}}, \lambda_A, \lambda_{\bar{A}}] = [\lambda'_{\dot{a}}, \lambda'_A, 0]$, the previous expression can be written as

$$Q' \rightarrow \lambda^\alpha d_\alpha + \sqrt{2\sqrt{2}P^+}\lambda'_{\bar{A}}S_{\bar{A}} \quad (4.2.46)$$

¹⁷It is not difficult to see that with our definitions, the $SU(4)$ metric components are $\eta_{AB} = \eta_{\bar{A}\bar{B}} = 0$ and $\eta_{A\bar{B}} = \eta_{\bar{A}B} = 1$.

¹⁸Because $\{S_{\bar{A}}, S_B\} = \{d_A, d_B\} = 0$.

Now by using the quartet argument [18] it is clear that the Q' -cohomology is equivalent to the Q -cohomology¹⁹:

$$Q' \rightarrow Q = \lambda^\alpha d_\alpha \quad (4.2.47)$$

where λ^α is a pure spinor.

Therefore we have proved that the modified Brink-Schwarz superparticle with action 4.2.10 is equivalent to the theory with the following action

$$S = \int d\tau (\dot{X}^m P_m - \frac{1}{2} P^m P_m + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha w_\alpha) \quad (4.2.48)$$

together with the BRST operator $Q = \lambda^\alpha d_\alpha$, where $\lambda \gamma^m \lambda = 0$. This action is clearly manifestly Lorentz covariant. This is the action for the pure spinor superparticle in $D = 10$ [21]. Now, we will show that the cohomology of the operator Q will give us a manifestly super-Poincaré covariant description of D=10 SYM (in the abelian case).

Let us calculate the Q -cohomology at zero momentum. When $P^m = 0$, we are left with $d_\alpha = p_\alpha$. Now, in order to show that the zero momentum Q -cohomology contains the fields, ghosts and their corresponding antifields of D=10 SYM, we will proceed in two steps. First, we will prove that at zero momentum the Q -cohomology is equivalent to the linear \hat{Q} -cohomology²⁰, where

$$\hat{Q} = \hat{\lambda}^\alpha p_\alpha + b_m (\hat{\lambda} \gamma^m \hat{\lambda}) + c^m (\hat{\lambda} \gamma_m f) + (g \gamma^n h) (\hat{\lambda} \gamma_n \hat{\lambda}) - 2g \hat{\lambda} (\hat{\lambda} h) + (j \gamma^n \hat{\lambda}) k_n + r^n (\hat{\lambda} \gamma_n \hat{\lambda}) p \quad (4.2.49)$$

and $\hat{\lambda}^\alpha$ is an unconstrained bosonic spinor. Finally we will calculate the \hat{Q} -cohomology and the corresponding elements in the Q -cohomology. Let us start by explaining the construction of this operator \hat{Q} . If we assume that the 16 components of λ^α are independent (and we write the constraint $\lambda \gamma^m \lambda = 0$ in the action with its corresponding Lagrange multiplier), we obtain the following BRST operator $Q_1 = \hat{\lambda}^\alpha p_\alpha + b_m (\hat{\lambda} \gamma^m \hat{\lambda})$, where b_m is the (fermionic) ghost. But this constraint $\hat{\lambda} \gamma^m \hat{\lambda} = 0$ is reducible, that is there exists a non-trivial function $\tilde{b}_{m\alpha} = (\hat{\lambda} \gamma_m)_\alpha$ satisfying $\tilde{b}_{m\alpha} \hat{\lambda} \gamma^m \hat{\lambda} = 0$ ²¹. Therefore we have a reducible gauge system. It is well known the procedure to treat reducible gauge systems [8]:

1. We have to add a new ghost for each gauge symmetry (with its corresponding antighost or conjugate momentum).
2. We should add to the BRST operator the product of the antighost corresponding to the ghost coming from a gauge transformation, with the corresponding non-trivial function which makes this gauge transformation reducible and the new ghost corresponding to this reducible gauge transformation.
3. In order to have a fermionic BRST operator, the ghosts belonging to an even generation must be fermionic, while those belonging to an odd generation must be bosonic.
4. This procedure will finish when having a linearly independent combination of a gauge transformation which vanishes ($t^m G_m = 0 \rightarrow t^m = 0$, G_m is the generator of a gauge transformation).

¹⁹That is the states in Hilbert space will be independent of λ'_A and S_A , and its respective conjugate momenta w'_A and $S_{\bar{A}}$.

²⁰We mean by linear" that elements in the \hat{Q} -cohomology depend linearly on the variables $(c^m, g_\alpha, j^\alpha, r_m, q)$.

²¹This can be seen as follows: $\tilde{b}_{m\alpha} \hat{\lambda} \gamma^m \hat{\lambda} = (\hat{\lambda} \gamma_m)_\alpha \hat{\lambda} \gamma^m \hat{\lambda} = \hat{\lambda}^\alpha \hat{\lambda}^\beta \hat{\lambda}^\delta (\gamma^m)_{\alpha\beta} (\gamma^m)_{\delta\sigma} = \hat{\lambda}^\alpha \hat{\lambda}^\beta \hat{\lambda}^\delta (\gamma^m)_{(\alpha\beta} (\gamma^m)_{\delta)\sigma} = 0$, because of the identity $(\gamma^m)_{\alpha(\beta} (\gamma^m)_{\delta\sigma)} = 0$.

So the BRST operator should be modified to the following form:

$$Q_2 = \hat{\lambda}^\alpha p_\alpha + b_m(\hat{\lambda}\gamma_m\hat{\lambda}) + c^m(\hat{\lambda}\gamma_m f) \quad (4.2.50)$$

where f^α is a (bosonic) ghost coming from the reducibility function $\tilde{b}_{m\alpha}$ and c^m is the (fermionic) antighost corresponding to b_m . In order to save typing we will just analyze the (reducibility) gauge invariance of the ghosts directly in the expression of the BRST operator (probably with incorrect bosonic or fermionic properties, but this is not a problem because this is just a trick to obtain the reducibility functions in a fast way), and introduce the corresponding new ghosts (and antighosts) directly in the expression of the BRST operator (with its correct bosonic or fermionic properties). So we see that under a transformation on f^α of the form: $\delta f^\alpha = \hat{\lambda}\gamma_n\hat{\lambda}(\gamma^n h)^\alpha - 2\hat{\lambda}^\alpha(\hat{\lambda}h)$, where h_α is an arbitrary (fermionic) spinor, the third term of Q_2 is invariant²² (this is the same to say that we have a reducible gauge invariance. The function $\tilde{f}^{\alpha\beta} = \hat{\lambda}\gamma_n\hat{\lambda}(\gamma^n)^{\alpha\beta} - 2\hat{\lambda}^\alpha\hat{\lambda}^\beta$ satisfies the reducibility condition $(\hat{\lambda}\gamma^m)_\alpha\tilde{f}^{\alpha\beta} = 0$. Then, when constructing the BRST operator this reducibility function will give us the corresponding (fermionic) ghost h_α). We call g_α the conjugate momentum to f^α , so we should redefine our BRST operator:

$$Q_3 = \hat{\lambda}^\alpha p_\alpha + b_m(\hat{\lambda}\gamma^m\hat{\lambda}) + c^m(\hat{\lambda}\gamma_m f) + (g\gamma^n h)(\hat{\lambda}\gamma_n\hat{\lambda}) - 2g\hat{\lambda}(\hat{\lambda}h) \quad (4.2.51)$$

We can see that under the transformation $\delta h_\alpha = (\gamma^n\hat{\lambda})_\alpha k_n$ (where k_n is an arbitrary (bosonic) vector), the sum of the fourth and fifth terms of Q_3 is invariant²³, so we should repeat the same procedure which we have been following above. Let j^α be the conjugate momentum to h_α , so our BRST operator will be now

$$Q_4 = \hat{\lambda}^\alpha p_\alpha + b_m(\hat{\lambda}\gamma^m\hat{\lambda}) + c^m(\hat{\lambda}\gamma_m f) + (g\gamma^n h)(\hat{\lambda}\gamma_n\hat{\lambda}) - 2g\hat{\lambda}(\hat{\lambda}h) + (j\gamma^n\hat{\lambda})k_n \quad (4.2.52)$$

Again, under the transformation $\delta k_n = \hat{\lambda}\gamma_n\hat{\lambda}p$, the sixth term of Q_4 is invariant²⁴, where p is an arbitrary (fermionic) scalar. Let r^n be the conjugate momentum to k_n , so the BRST operator is now:

$$\hat{Q} = \hat{\lambda}^\alpha p_\alpha + b_m(\hat{\lambda}\gamma^m\hat{\lambda}) + c^m(\hat{\lambda}\gamma_m f) + (g\gamma^n h)(\hat{\lambda}\gamma_n\hat{\lambda}) - 2g\hat{\lambda}(\hat{\lambda}h) + (j\gamma^n\hat{\lambda})k_n + r^n(\hat{\lambda}\gamma_n\hat{\lambda})p \quad (4.2.53)$$

We have done, because there is no any gauge transformation which leaves invariant the last term of this BRST operator (which means that we should require $\hat{Q}p = 0$). We have found the following pairs of fields: (b^m, c_m) , (f^α, g_α) , (h_α, j^α) , (k_m, r^m) and (p, q) , where we have added the field q , the conjugate momentum to p . The fields (b^m, c_m) , (h_α, j^α) , (p, q) are fermionic and (f^α, g_α) , (k_m, r^m) are bosonic. We can calculate the ghost number of these fields by defining the ghost number of $\hat{\lambda}^\alpha$ to be one and using the fact that a BRST operator should have ghost number equals to one. So, by following the construction explained above (the gauge transformations

²²Let us show this claim: $\delta(\hat{\lambda}\gamma^m f) = (\hat{\lambda}\gamma^m)_\alpha[(\hat{\lambda}\gamma_n\hat{\lambda})(\gamma^n h)^\alpha - 2\hat{\lambda}^\alpha(\hat{\lambda}h)] = \hat{\lambda}^\beta\hat{\lambda}^\sigma\hat{\lambda}^\rho(\gamma^m)_{\alpha\beta}(\gamma_n)_{\sigma\rho}(\gamma^n)^{\alpha\delta}h_\delta - 2\hat{\lambda}^\alpha\hat{\lambda}^\beta(\gamma^m)_{\alpha\beta}\hat{\lambda}^\delta h_\delta = \hat{\lambda}^\beta\hat{\lambda}^\sigma\hat{\lambda}^\rho[2\eta^{mn}\delta_\beta^\delta(\gamma_n)_{\sigma\rho} - (\gamma^n)_{\alpha\beta}(\gamma_n)_{\sigma\rho}(\gamma^m)^{\alpha\delta}]h_\delta - 2\hat{\lambda}^\alpha\hat{\lambda}^\beta(\gamma^m)_{\alpha\beta}\hat{\lambda}^\delta h_\delta = 2\hat{\lambda}^\beta\hat{\lambda}^\sigma\hat{\lambda}^\rho\gamma_{\sigma\rho}^m h_\beta - 2\hat{\lambda}^\alpha\hat{\lambda}^\beta\hat{\lambda}^\delta\gamma_{\alpha\beta}^m h_\delta - \hat{\lambda}^\beta\hat{\lambda}^\sigma\hat{\lambda}^\rho(\gamma^n)_{\alpha\beta}(\gamma_n)_{\sigma\rho}(\gamma^m)^{\alpha\delta} = -\hat{\lambda}^\beta\hat{\lambda}^\sigma\hat{\lambda}^\rho(\gamma^n)_{\alpha(\beta}(\gamma_n)_{\sigma\rho)}(\gamma^m)^{\alpha\delta} = 0$.

²³This is true because of the following argument: $\delta((g\gamma^n h)(\hat{\lambda}\gamma_n\hat{\lambda}) - 2g\hat{\lambda}(\hat{\lambda}h)) = \hat{\lambda}\gamma_n\hat{\lambda}g_\alpha(\gamma^n)^{\alpha\beta}(\gamma^m)_{\beta\delta}\hat{\lambda}^\delta k_m - 2g\hat{\lambda}(\hat{\lambda}\gamma^m\hat{\lambda})k_m = \hat{\lambda}\gamma_n\hat{\lambda}g_\alpha(2\eta^{mn}\delta_\delta^\alpha - (\gamma^m)^{\alpha\beta}(\gamma^n)_{\beta\delta})\hat{\lambda}^\delta k_m - 2g\hat{\lambda}(\hat{\lambda}\gamma^m\hat{\lambda})k_m = -\hat{\lambda}^\sigma(\gamma_n)_{\sigma\rho}\hat{\lambda}^\rho g_\alpha(\gamma^m)^{\alpha\beta}(\gamma^n)_{\beta\delta}\hat{\lambda}^\delta k_m = \hat{\lambda}^\sigma\hat{\lambda}^\rho\hat{\lambda}^\delta g_\alpha(\gamma^m)^{\alpha\beta}(\gamma^n)_{(\sigma\rho}(\gamma_n)_{\delta)\beta}k_m = 0$.

²⁴This is easy to see: $\delta((j\gamma^n\hat{\lambda})k_n) = j^\alpha(\gamma^n)_{\alpha\beta}\hat{\lambda}^\beta\hat{\lambda}^\sigma(\gamma_n)_{\sigma\rho}\hat{\lambda}^\rho p = j^\alpha\hat{\lambda}^\beta\hat{\lambda}^\sigma\hat{\lambda}^\rho(\gamma^n)_{\alpha(\beta}(\gamma_n)_{\sigma\rho)}p = 0$.

and the final BRST operator) we obtain:

$$(b^m, c_m) \rightarrow \text{gh\#} = (-1, 1) \quad (4.2.54)$$

$$(f^\alpha, g_\alpha) \rightarrow \text{gh\#} = (-1, 1) \quad (4.2.55)$$

$$(h_\alpha, j^\alpha) \rightarrow \text{gh\#} = (-2, 2) \quad (4.2.56)$$

$$(k_m, r^m) \rightarrow \text{gh\#} = (-2, 2) \quad (4.2.57)$$

$$(p, q) \rightarrow \text{gh\#} = (-3, 3) \quad (4.2.58)$$

Now, let us show the equivalence between the cohomologies mentioned above. Let us start with a state $F(\lambda, \theta)$ which satisfies $Q_0 F = (\lambda_0 \gamma^m \lambda_0) \tau_m$ for some τ_m ²⁵. So, we see that²⁶

$$\begin{aligned} \hat{Q}(F - c^m \tau_m) &= \hat{Q}F - (\hat{Q}c^m) \tau_m + c^m \hat{Q}\tau_m \\ &= (\hat{\lambda} \gamma^m \hat{\lambda}) \tau_m - (\hat{\lambda} \gamma^m \hat{\lambda}) \tau_m + c^m \hat{Q}\tau_m \\ \rightarrow \hat{Q}(F - c^m \tau_m) &= c^m \hat{Q}\tau_m \end{aligned}$$

By using the fact that $\hat{Q}^2 = 0$, we get $(\hat{\lambda} \gamma^m \hat{\lambda}) \hat{Q}\tau_m = 0 \rightarrow \hat{Q}\tau_m = \hat{\lambda} \gamma_m \psi$, for some ψ ²⁷. Therefore²⁸

$$\begin{aligned} \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha) &= c^m \hat{Q}\tau_m - (\hat{Q}g_\alpha) \psi^\alpha - g_\alpha \hat{Q}\psi^\alpha \\ &= c^m (\hat{\lambda} \gamma_m \psi) - c^m (\hat{\lambda} \gamma_m \psi) - g_\alpha \hat{Q}\psi^\alpha \\ &= -g_\alpha \hat{Q}\psi^\alpha \\ \rightarrow \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha) &= -g_\alpha \hat{Q}\psi^\alpha \end{aligned}$$

where we used the fact that g_α is bosonic. Once again, using the fact that $\hat{Q}^2 = 0$, we get $\hat{\lambda} \gamma_m \hat{Q}\psi = 0 \rightarrow \hat{Q}\psi^\alpha = (\hat{\lambda} \gamma^n \hat{\lambda})(\gamma_n \rho)^\alpha - 2\hat{\lambda}^\alpha (\rho \hat{\lambda})$, for some ρ_α ²⁹. So³⁰

$$\begin{aligned} \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha + j^\alpha \rho_\alpha) &= \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha) + (\hat{Q}j^\alpha) \rho_\alpha - j^\alpha (\hat{Q}\rho_\alpha) \\ &= -[\hat{\lambda} \gamma^n \hat{\lambda} (g \gamma_n \rho) - 2g \hat{\lambda} (\hat{\lambda} \rho)] + [\hat{\lambda} \gamma^n \hat{\lambda} (g \gamma_n \rho) \\ &\quad - 2g \hat{\lambda} (\hat{\lambda} \rho)] - j^\alpha (\hat{Q}\rho_\alpha) \\ &= -j^\alpha (\hat{Q}\rho_\alpha) \\ \rightarrow \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha + j^\alpha \rho_\alpha) &= -j^\alpha (\hat{Q}\rho_\alpha) \end{aligned}$$

where we used that j^α is fermionic. We can use the nilpotency of \hat{Q} to see that $\hat{Q}\rho_\alpha = (\hat{\lambda} \gamma^m)_\alpha \sigma_m$,

²⁵We are using the same auxiliary notation Q_0, λ_0 as before, but now we impose Q_0 to be nilpotent (because both Q and \hat{Q} are nilpotent). When λ_0^α is replaced by λ^α , $F(\lambda, \theta)$ is annihilated by Q_0 (which in this case it becomes Q) and when λ_0^α is replaced by $\hat{\lambda}^\alpha$, we have the equality: $Q_0 F = \hat{Q}F = (\hat{\lambda} \gamma^m \hat{\lambda}) \tau_m$.

²⁶We used the fact that $\hat{Q}c^m = \hat{\lambda} \gamma^m \hat{\lambda}$, which is easy to see from the form of the operator \hat{Q} .

²⁷This is true because of the γ -matrix identities: $\hat{Q}^2 F = \hat{\lambda} \gamma^m \hat{\lambda} \hat{Q}\tau_m = \hat{\lambda} \gamma^m \hat{\lambda} (\hat{\lambda} \gamma_m \psi) = \hat{\lambda}^\sigma \hat{\lambda}^\delta \hat{\lambda}^\sigma \psi^\rho (\gamma^m)_{\alpha\beta} (\gamma_m)_{\sigma\rho} = \hat{\lambda}^\alpha \hat{\lambda}^\beta \hat{\lambda}^\sigma \psi^\rho (\gamma^m)_{(\alpha\beta)(\gamma_m)_{\sigma\rho}} = 0$.

²⁸We used that $\hat{Q}g_\alpha = c^m (\hat{\lambda} \gamma_m)_\alpha$, as it can be seen easily from the form of \hat{Q} .

²⁹Let us check the truth of this statement: $\hat{Q}^2 \tau_m = (\hat{\lambda} \gamma_m)_\alpha [(\hat{\lambda} \gamma^n \hat{\lambda})(\gamma_n \rho)^\alpha - 2\hat{\lambda}^\alpha (\hat{\lambda} \rho)] = \hat{\lambda}^\sigma \hat{\lambda}^\delta \hat{\lambda}^\beta (\gamma_m)_{\sigma\alpha} (\gamma^n)_{\delta\beta} (\gamma_n)^{\alpha\kappa} \rho_\kappa - 2\hat{\lambda}^\sigma \hat{\lambda}^\alpha \hat{\lambda}^\kappa (\gamma_m)_{\sigma\alpha} \rho_\kappa = \hat{\lambda}^\sigma \hat{\lambda}^\delta \hat{\lambda}^\beta [2\eta_{mn} (\gamma^n)_{\delta\beta} \delta_\sigma^\kappa - (\gamma_n)_{\sigma\alpha} (\gamma^n)_{\delta\beta} (\gamma_m)^{\alpha\kappa}] \rho_\kappa - 2\hat{\lambda}^\sigma \hat{\lambda}^\alpha \hat{\lambda}^\kappa (\gamma_m)_{\sigma\alpha} \rho_\kappa = \hat{\lambda}^\sigma \hat{\lambda}^\delta \hat{\lambda}^\beta (\gamma_n)_{\sigma\alpha} (\gamma^n)_{(\delta\beta)(\gamma_m)_{\sigma\alpha}} \rho_\kappa = 0$.

³⁰We used that $\hat{Q}j^\alpha = (g \gamma^n)^\alpha (\hat{\lambda} \gamma_n \hat{\lambda}) - 2(g \hat{\lambda}) \hat{\lambda}^\alpha$, which is immediately obtained from the expression for \hat{Q} .

for some σ_m ³¹. Hence³²

$$\begin{aligned}\hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha + j^\alpha \rho_\alpha + r_m \sigma^m) &= \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha + j^\alpha \rho_\alpha) + \hat{Q}(r_m) \sigma^m + r_m \hat{Q}(\sigma^m) \\ &= -(\hat{\lambda} \gamma^m j) \sigma_m + (\hat{\lambda} \gamma^m j) \sigma_m + r_m \hat{Q} \sigma^m \\ &= r_m \hat{Q} \sigma^m \\ \rightarrow \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha + j^\alpha \rho_\alpha + r_m \sigma^m) &= r_m \hat{Q} \sigma^m\end{aligned}$$

and we used the bosonic characteristic of r^m . Again, from the nilpotency of \hat{Q} , we get that $\hat{Q} \sigma^m = (\hat{\lambda} \gamma^m \hat{\lambda}) u$, for some u ³³. So, we have

$$\begin{aligned}\hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha + j^\alpha \rho_\alpha + r_m \sigma^m - qu) &= \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha + j^\alpha \rho_\alpha + r_m \sigma^m) - \hat{Q}(qu) \\ &\quad + q \hat{Q}(u) \\ &= r^m (\hat{\lambda} \gamma_m \hat{\lambda}) u - r^n (\hat{\lambda} \gamma_n \hat{\lambda}) u + q \hat{Q} u \\ &= q \hat{Q} u \\ &= 0 \\ \rightarrow \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha + j^\alpha \rho_\alpha + r_m \sigma^m - qu) &= 0\end{aligned}$$

because the nilpotency of \hat{Q} requires that $\hat{Q} u = 0$. We used the fact that q is fermionic³⁴. So we have showed that if we have a state $F(\lambda, \theta)$ which is annihilated by Q , we can find a state

$$\hat{F}(\hat{\lambda}, \theta) = F(\hat{\lambda}, \theta) - c^m \tau_m(\hat{\lambda}, \theta) - g_\alpha \psi^\alpha(\hat{\lambda}, \theta) + j^\alpha \rho_\alpha(\hat{\lambda}, \theta) + r_m \sigma^m(\hat{\lambda}, \theta) - qu(\hat{\lambda}, \theta) \quad (4.2.59)$$

which is linear in the variables $(c^m, g_\alpha, j^\alpha, r_m, q)$, and is annihilated by \hat{Q} .

Now we will show that if this F is non Q -exact, the corresponding \hat{F} will also be non \hat{Q} -exact. Let us will do it by contradiction. So, let us suppose that $\hat{F} = \hat{Q} \hat{\Omega}$, for some $\hat{\Omega} = \Omega(\hat{\lambda}, \theta) + c^m \xi_m(\hat{\lambda}, \theta) + \dots$, where ellipsis means terms involving the other remaining fields $(g_\alpha, j^\alpha, r_m, u)$. Hence

$$\hat{F}(\hat{\lambda}, \theta) = \hat{Q}(\Omega(\hat{\lambda}, \theta)) + (\hat{\lambda} \gamma^m \hat{\lambda}) \xi_m(\hat{\lambda}, \theta) + \dots \quad (4.2.60)$$

where ellipsis means terms involving at least one of the new variables $(c^m, g_\alpha, j^\alpha, r_m, q)$ ³⁵. Now, $\hat{F}(\hat{\lambda}, \theta)$ has a term which is independent of the new variables, namely $F(\hat{\lambda}, \theta)$, therefore:

$$F(\hat{\lambda}, \theta) = \hat{Q}(\Omega(\hat{\lambda}, \theta)) + (\hat{\lambda} \gamma^m \hat{\lambda}) \xi_m(\hat{\lambda}, \theta) \quad (4.2.61)$$

So if we replace $\hat{\lambda}^\alpha = \lambda^\alpha$, we obtain $F(\lambda, \theta) = Q\Omega(\lambda, \theta)$, which is clearly a contradiction.

It is also true that if we start with a state $\hat{F}(\hat{\lambda}, \theta)$ in the cohomology of \hat{Q} (which is at most linear in $(c^m, g_\alpha, j^\alpha, r_m, q)$), we obtain a state $F(\lambda, \theta)$ in the cohomology of Q . So, we have

$$\hat{Q} \hat{F} = \hat{Q}(F - c^m \tau_m - g_\alpha \psi^\alpha + j^\alpha \rho_\alpha + r_m \sigma^m - qu) = 0 \quad (4.2.62)$$

$$\rightarrow \hat{Q} F(\hat{\lambda}, \theta) - (\hat{\lambda} \gamma^m \hat{\lambda}) \tau_m + (\text{terms involving the new fields}) = 0 \quad (4.2.63)$$

$$\rightarrow \hat{Q} F(\hat{\lambda}, \theta) - (\hat{\lambda} \gamma^m \hat{\lambda}) \tau_m(\hat{\lambda}, \theta) = 0 \quad (4.2.64)$$

³¹This is so because: $\hat{Q}^2 \psi^\alpha = (\hat{\lambda} \gamma^n \hat{\lambda})(\gamma_n)^{\alpha\beta} (\hat{\lambda} \gamma^m)_\beta \sigma_m - 2\hat{\lambda}^\alpha (\hat{\lambda} \gamma^m \hat{\lambda}) \sigma_m = \hat{\lambda}^\sigma \hat{\lambda}^\delta \hat{\lambda}^\rho [2\eta^{mn} \delta_\rho^\alpha (\gamma_n)_{\sigma\delta} - (\gamma_n)_{\sigma\delta} (\gamma^m)^{\alpha\beta} (\gamma^n)_{\rho\beta}] \sigma_m - 2\hat{\lambda}^\alpha \hat{\lambda}^\sigma \hat{\lambda}^\rho (\gamma^m)_{\sigma\rho} \sigma_m = -\hat{\lambda}^\sigma \hat{\lambda}^\delta \hat{\lambda}^\rho (\gamma_n)_{(\sigma\delta} (\gamma^n)_{\rho)\beta} (\gamma^m)^{\alpha\beta} \sigma_m = 0$.

³²We used the fact that $\hat{Q}(r_m) = (\hat{\lambda} \gamma^m j) \sigma_m$, which is easily obtained from the expression for \hat{Q} .

³³This is immediate: $\hat{Q}^2 \rho_\alpha = \hat{\lambda}^\beta (\gamma^m)_{\beta\alpha} \hat{\lambda}^\delta (\gamma_m)_{\delta\kappa} \hat{\lambda}^\kappa u = \hat{\lambda}^\beta \hat{\lambda}^\delta \hat{\lambda}^\kappa (\gamma^m)_{\alpha(\beta} (\gamma_m)_{\delta\kappa)} u = 0$.

³⁴We also used the result $\hat{Q} q = r^n (\hat{\lambda} \gamma_n \hat{\lambda})$, which is easy to prove, just by looking at the form of \hat{Q} .

³⁵This is so because one can see from the expression for the operator \hat{Q} that this is non-linear in the new variables except in the term corresponding to the field b_m .

so if $\hat{\lambda}^\alpha = \lambda^\alpha \rightarrow QF = 0$, that is F is in the Q -cohomology.

Now, let us show that if \hat{F} is non \hat{Q} -exact, the corresponding F is also non Q -exact. Let us will do it by contradiction too. So let us start with a $F(\lambda, \theta)$ which is Q -exact: $F(\lambda, \theta) = Q\Omega(\lambda, \theta)$. So the corresponding $F(\hat{\lambda}, \theta) = F(\lambda, \theta) + (\hat{\lambda}\gamma^m\hat{\lambda})l_m^{(0)}(\hat{\lambda}, \theta)$ (for some $l_m^{(0)}$) will be

$$F(\hat{\lambda}, \theta) = Q\Omega(\lambda, \theta) + (\hat{\lambda}\gamma^m\hat{\lambda})l_m^{(0)}(\hat{\lambda}, \theta) \quad (4.2.65)$$

$$= \hat{Q}\Omega(\hat{\lambda}, \theta) - (\hat{\lambda}\gamma^m\hat{\lambda})l_m^{(1)}(\hat{\lambda}, \theta) + (\hat{\lambda}\gamma^m\hat{\lambda})l_m^{(0)}(\hat{\lambda}, \theta) \quad (4.2.66)$$

$$= \hat{Q}\Omega(\hat{\lambda}, \theta) + (\hat{\lambda}\gamma^m\hat{\lambda})l_m^{(2)}(\hat{\lambda}, \theta) \quad (4.2.67)$$

where $l_m^{(2)} = l_m^{(0)} - l_m^{(1)}$ and we used the fact that $\hat{Q}\Omega(\hat{\lambda}, \theta) = Q\Omega(\lambda, \theta) + \hat{\lambda}\gamma^m\hat{\lambda}l_m^{(1)}(\hat{\lambda}, \theta)$, for some $l_m^{(1)}$ ³⁶. Hence

$$\hat{F}(\hat{\lambda}, \theta) = F(\hat{\lambda}, \theta) + c^m\tau_m(\hat{\lambda}, \theta) + g_\alpha\psi^\alpha(\hat{\lambda}, \theta) + \dots \quad (4.2.68)$$

$$= \hat{Q}\Omega(\hat{\lambda}, \theta) + (\hat{\lambda}\gamma^m\hat{\lambda})l_m^{(2)}(\hat{\lambda}, \theta) + c^m\tau_m(\hat{\lambda}, \theta) + g_\alpha\psi^\alpha(\hat{\lambda}, \theta) + \dots \quad (4.2.69)$$

$$= \hat{Q}(\Omega(\hat{\lambda}, \theta) + c^ml_m^{(2)}(\hat{\lambda}, \theta)) + c^m(\tau_m + \hat{Q}l_m^{(2)}) + g_\alpha\psi^\alpha(\hat{\lambda}, \theta) + \dots \quad (4.2.70)$$

Because of the nilpotency of \hat{Q} , we obtain $(\hat{\lambda}\gamma^m\hat{\lambda})(\tau_m + \hat{Q}l_m^{(2)}) = 0 \rightarrow \tau_m + \hat{Q}l_m^{(2)} = (\hat{\lambda}\gamma_m\chi)$, for some χ ³⁷. Therefore $\hat{F}(\hat{\lambda}, \theta) = \hat{Q}(\Omega(\hat{\lambda}, \theta) + c^ml_m^{(2)}(\hat{\lambda}, \theta) + g_\alpha\chi^\alpha(\hat{\lambda}, \theta)) + g_\alpha(\psi^\alpha - \hat{Q}\chi^\alpha) + k^\alpha\rho_\alpha(\hat{\lambda}, \theta) + \dots$. Once again, the nilpotency of \hat{Q} requires that $c^m(\hat{\lambda}\gamma_m)(\psi^\alpha - \hat{Q}\chi^\alpha) = 0 \rightarrow (\psi^\alpha - \hat{Q}\chi^\alpha) = [(\hat{\lambda}\gamma^n\hat{\lambda})(\gamma_n\zeta)^\alpha - 2\hat{\lambda}^\alpha(\hat{\lambda}\zeta)]$, for some ζ ³⁸. So, $\hat{F}(\hat{\lambda}, \theta) = \hat{Q}(\Omega(\hat{\lambda}, \theta) + c^ml_m^{(2)}(\hat{\lambda}, \theta) + g_\alpha\chi^\alpha(\hat{\lambda}, \theta) + j^\alpha\zeta_\alpha(\hat{\lambda}, \theta)) + j^\alpha(\rho_\alpha + \hat{Q}\zeta_\alpha) + r_m\sigma^m(\hat{\lambda}, \theta) - qu(\hat{\lambda}, \theta)$. The nilpotency of \hat{Q} implies that $((g\gamma^n)^\alpha(\hat{\lambda}\gamma_n\hat{\lambda}) - 2(\hat{\lambda}g)^\alpha)(\rho_\alpha + \hat{Q}\zeta_\alpha) \rightarrow (\rho_\alpha + \hat{Q}\zeta_\alpha) = (\hat{\lambda}\gamma^n)_{\alpha t_n}$, for some t_m ³⁹. So, we obtain

$$\hat{F}(\hat{\lambda}, \theta) = \hat{Q}(\Omega(\hat{\lambda}, \theta) + c^ml_m^{(2)}(\hat{\lambda}, \theta) + g_\alpha\chi^\alpha(\hat{\lambda}, \theta) + j^\alpha\zeta_\alpha(\hat{\lambda}, \theta) + r^nt_n(\hat{\lambda}, \theta)) + r^n(\sigma_n - \hat{Q}t_n) - qu(\hat{\lambda}, \theta) \quad (4.2.71)$$

Once again, from the nilpotency of \hat{Q} , we obtain $j\gamma^n\hat{\lambda}(\sigma_n - \hat{Q}t_n) = 0 \rightarrow (\sigma_n - \hat{Q}t_n) = (\hat{\lambda}\gamma_n\hat{\lambda})y$, for some y ⁴⁰. Then,

$$\hat{F}(\hat{\lambda}, \theta) = \hat{Q}(\Omega(\hat{\lambda}, \theta) + c^ml_m^{(2)}(\hat{\lambda}, \theta) + g_\alpha\chi^\alpha(\hat{\lambda}, \theta) + j^\alpha\zeta_\alpha(\hat{\lambda}, \theta) + r^nt_n(\hat{\lambda}, \theta) + qy(\hat{\lambda}, \theta)) \quad (4.2.72)$$

because the nilpotency of \hat{Q} requires that $\hat{Q}y = u$. Therefore, we have obtained a contradiction ($\hat{F}(\hat{\lambda}, \theta)$ is non \hat{Q} -exact).

Now, let us calculate the \hat{Q} -cohomology. If we use the quartet argument, we conclude that our states will be represented just by the fields $(1, c^m, g_\alpha, j^\alpha, r_m, q)$ which have the ghost numbers $(0, 1, 1, 2, 2, 3)$, respectively.

To construct the physical states, we have to find the states which are gauge-invariant (annihilated by \hat{Q}). For example, the state $c^m + i(\hat{\lambda}\gamma^m\theta)$ satisfies this condition:

$$\hat{Q}(c^m + i(\hat{\lambda}\gamma^m\theta)) = \hat{\lambda}\gamma^m\hat{\lambda} - \hat{\lambda}\gamma^m\hat{\lambda} = 0 \quad (4.2.73)$$

³⁶Actually we have the following equalities: $\hat{F}(\hat{\lambda}, \theta) = G(\hat{\lambda}, \theta) + (\hat{\lambda}\gamma^m\hat{\lambda})l_m^{(0)}(\hat{\lambda}, \theta)$, where $G(\lambda, \theta) = F(\lambda, \theta)$. Similarly, when \hat{Q} acts on states which depend just on $(\hat{\lambda}, \theta)$ and no on the new variables, we have the relation $\hat{Q}\Omega(\hat{\lambda}, \theta) = K(\hat{\lambda}, \theta) + (\hat{\lambda}\gamma^m\hat{\lambda})l_m^{(1)}$, where $K(\lambda, \theta) = Q\Omega(\lambda, \theta)$. However, because we are interested at the end in the case when $\hat{\lambda}^\alpha = \lambda^\alpha$, we have made some abuses of language (like $F(\hat{\lambda}, \theta) = F(\lambda, \theta) + (\hat{\lambda}\gamma^m\hat{\lambda})l_m^{(0)}$) but with this aclaration we hope they do not cause confusion.

³⁷The proof of this result is analog to the previous ones, it is just a consequence of the γ -matrix identities.

³⁸We have already done this computation, therefore we will omit it.

³⁹This computation was already done before. We will omit it.

⁴⁰This computation was already done.

Analogously, we can show that the state: $g_\alpha + i(\theta\gamma_m)_\alpha c^m + \frac{2}{3}(\hat{\lambda}\gamma_m\theta)(\gamma^m\theta)^\alpha$, is also annihilated by \hat{Q} :

$$\begin{aligned}
\hat{Q}(g_\alpha + i(\theta\gamma_m)_\alpha c^m + \frac{2}{3}(\hat{\lambda}\gamma_m\theta)(\gamma^m\theta)_\alpha) &= c^m(\hat{\lambda}\gamma_m)_\alpha - (\hat{\lambda}\gamma_m)_\alpha c^m - i(\theta\gamma_m)_\alpha(\hat{\lambda}\gamma^m\hat{\lambda}) \\
&+ \frac{2i}{3}(\hat{\lambda}\gamma_m\hat{\lambda})(\gamma^m\theta)_\alpha - \frac{2i}{3}(\hat{\lambda}\gamma_m\theta)(\hat{\lambda}\gamma^m)_\alpha \\
&= -\frac{i}{3}[(\theta\gamma_m)_\alpha(\hat{\lambda}\gamma^m\hat{\lambda}) + 2(\hat{\lambda}\gamma_m\theta)(\gamma^m\hat{\lambda})_\alpha] \\
&= -\frac{i}{3}(\theta^\rho\hat{\lambda}^\beta\hat{\lambda}^\sigma)[(\gamma_m)_{\rho\alpha}(\gamma^m)_{\beta\sigma} + 2(\gamma^m)_{\beta\rho}(\gamma_m)_{\sigma\alpha}] \\
&= -\frac{i}{3}(\theta^\rho\hat{\lambda}^\beta\hat{\lambda}^\sigma)[(\gamma_m)_{\rho\alpha}(\gamma^m)_{\beta\sigma} + (\gamma^m)_{\beta\rho}(\gamma_m)_{\sigma\alpha} \\
&+ (\gamma^m)_{\sigma\rho}(\gamma_m)_{\beta\alpha}] \\
\rightarrow \hat{Q}(g_\alpha + i(\theta\gamma_m)_\alpha c^m + \frac{2}{3}(\hat{\lambda}\gamma_m\theta)(\gamma^m\theta)_\alpha) &= 0
\end{aligned}$$

The corresponding states in the Q -cohomology can be obtained just by setting the new fields to zero. Hence, the states associated to c^m and g_α are $(\lambda\gamma^m\theta)$ and $(\lambda\gamma_m\theta)(\gamma^m\theta)_\alpha$, the gluon and gluino respectively. We can also show that the states corresponding to the fields j^α , r_p and q are $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\theta\gamma_{mn})^\alpha$, $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\theta\gamma_{mnp}\theta)$ and $(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$, the gluino antifield, gluon antifield and ghost antifield respectively (the ghost corresponds to a term λ -independent). It must be emphasized that these states will appear in a ghost number one pure spinor superfield belonging to the Q -cohomology with the following ghost numbers:

$$\text{gh.num.}(\text{ghost}) = +1 \quad (4.2.74)$$

$$\text{gh.num.}(\text{gluon, gluino}) = (0, 0) \quad (4.2.75)$$

$$\text{gh.num.}(\text{gluon antifield, gluino antifield}) = (-1, -1) \quad (4.2.76)$$

$$\text{gh.num.}(\text{ghost antifield}) = -2 \quad (4.2.77)$$

This is the spectrum of D=10 SYM in the BV framework.

Now, let us figure out the consequences coming from requiring the physical condition on a general state $\Psi(x, \theta, \lambda)$:

$$\Psi(x, \theta, \lambda) = C(x, \theta) + \lambda^\alpha A_\alpha(x, \theta) + (\lambda\gamma^{mnpqr}\lambda)A_{mnpqr}^*(x, \theta) + \lambda^\alpha\lambda^\beta\lambda^\gamma C_{\alpha\beta\gamma}^*(x, \theta) + \dots \quad (4.2.78)$$

we obtain

$$Q\Psi(x, \theta, \lambda) = -i\lambda^\beta D_\beta C(x, \theta) - i\lambda^\beta\lambda^\alpha D_\beta A_\alpha - i\lambda^\beta\lambda^\sigma\lambda^\alpha(\gamma^{mnpqr})_{\sigma\alpha} D_\beta A_{mnpqr}^* - i\lambda^\beta\lambda^\sigma\lambda^\alpha\lambda^\gamma D_\beta C_{\sigma\alpha\gamma}^* + \dots \quad (4.2.79)$$

where $d_\alpha = p_\alpha + i(\gamma^m)_{\alpha\beta}\theta^\beta P_m \rightarrow d_\alpha = -i(\frac{\partial}{\partial\theta^\alpha} + i(\gamma^m)_{\alpha\beta}\theta^\beta\frac{\partial}{\partial x^m}) = -iD_\alpha$, and D_α is the supersymmetric derivative. From this we can deduce the following equations:

$$\lambda^\beta D_\beta C(x, \theta) = 0 \quad (4.2.80)$$

$$\lambda^\beta\lambda^\alpha D_\beta A_\alpha = 0 \quad (4.2.81)$$

$$\lambda^\beta\lambda^\sigma\lambda^\alpha(\gamma^{mnpqr})_{\sigma\alpha} D_\beta A_{mnpqr}^* = 0 \quad (4.2.82)$$

$$\lambda^\beta\lambda^\sigma\lambda^\alpha\lambda^\gamma D_\beta C_{\sigma\alpha\gamma}^* = 0 \quad (4.2.83)$$

etc. We can also use the gauge invariance condition $\delta\Psi(x, \theta, \lambda) = Q\Omega(x, \theta, \lambda)$, which tells us that

$$\delta\Psi(x, \theta, \lambda) = Q\Omega(x, \theta, \lambda) \quad (4.2.84)$$

where $\Omega(x, \theta, \lambda) = i\Lambda(x, \theta) + \lambda^\alpha w_\alpha + (\lambda\gamma^{mnpqr}\lambda)u_{mnpqr}(x, \theta) \dots$, therefore

$$\delta C(x, \theta) = 0 \quad (4.2.85)$$

$$\lambda^\alpha \delta A_\alpha(x, \theta) = \lambda^\alpha D_\alpha \Lambda(x, \theta) \quad (4.2.86)$$

$$(\lambda\gamma^{mnpqr}\lambda)A_{mnpqr}^*(x, \theta) = -i\lambda^\beta \lambda^\alpha D_\beta w_\alpha(x, \theta) \quad (4.2.87)$$

$$\lambda^\alpha \lambda^\beta \lambda^\gamma \delta C_{\alpha\beta\gamma}^*(x, \theta) = -i\lambda^\alpha \lambda^\beta \lambda^\gamma (\gamma^{mnpqr})_{\beta\gamma} D_\alpha u_{mnpqr}(x, \theta) \quad (4.2.88)$$

We can solve the equation 4.2.80 by expanding $C(x, \theta) = c(x) + \theta^\alpha C_\alpha^{(1)}(x) + \theta^\alpha \theta^\beta C_{\alpha\beta}^{(2)}(x) + \dots$. Let us see:

$$\begin{aligned} \lambda^\beta D_\beta &= \lambda^\beta \left(\frac{\partial}{\partial \theta^\beta} + i\gamma_{\beta\delta}^m \theta^\delta \frac{\partial}{\partial x^m} \right) C(x, \theta) = 0 \\ \lambda^\beta D_\beta &= \lambda^\beta C_\beta^{(1)}(x) + 2\theta^\beta \theta^\delta C_{\beta\delta}^{(2)}(x) + i(\gamma^m)_{\beta\delta} \theta^\delta \partial_m c(x) \\ &+ i(\lambda\gamma^m \theta) \theta^\alpha \partial_m C_\alpha^{(1)} + i(\lambda\gamma^m \theta) \theta^\alpha \theta^\beta C_{\alpha\beta}^{(2)}(x) + 3\lambda^\beta \theta^\delta \theta^\sigma C_{\beta\delta\sigma}^{(3)} + \dots \\ &= 0 \end{aligned}$$

This will imply that $\partial_m c(x) = 0$, and $C_{\alpha_1 \dots \alpha_n}^{(n)} = 0$, for all $n \geq 1$. This reminds us to the equation of motion for the ghost c in the BV framework (equation 2.3.32). And from the equation 4.2.85, we conclude that this field has no gauge transformation.

Clearly the equation 4.2.81 is the dynamical constraint already studied in the previous chapter for the abelian case ($\lambda^\beta \lambda^\alpha D_\beta A_\alpha = (\gamma^{mnpqr})^{\alpha\beta} D_\alpha A_\beta = 0$). Now, if we also impose the conventional constraint (which can be always chosen), we obtain: $F_{\alpha\beta} = 0$. This condition gave us the equations of motion for the physical fields of D=10 SYM (equations 2.2.86, 2.2.87 in their corresponding abelian versions, namely the equations 2.3.29 and 2.3.30). In addition to this, in the description via gauge connection, we used the equation 4.2.86 and we reproduced the gauge invariances for the physical fields of D=10 SYM.

By using the gauge transformation 4.2.87, it can be shown that the field $A_{mnpqr}^*(x, \theta)$ can be written in the form:

$$A_{mnpqr}^*(x, \theta) = (\theta\gamma_{[mnp}\theta)(\theta\gamma_{qr]})^\alpha \chi_\alpha^*(x) + (\theta\gamma_{[mnp}\theta)(\theta\gamma_{qr]s}\theta) a^{*s}(x) + \dots \quad (4.2.89)$$

with the gauge invariance:

$$\delta a^* = \partial_n (\partial^n s^m - \partial^m s^n) \quad (4.2.90)$$

where s^m is a gauge parameter. And if we use the equation 4.2.82, we get

$$\partial_p a^{*p} = 0 \quad (4.2.91)$$

In the case of the gaugino we find that this field has the gauge invariance:

$$\delta \chi_\alpha^* = \gamma_{\alpha\beta}^m \partial_m \kappa \quad (4.2.92)$$

where κ is a gauge parameter. After using the equation 4.2.82 we realize that there is not any equation of motion for χ_α^* .

We can use the gauge transformation 4.2.88 to put the field $C_{\sigma\alpha\gamma}^*(x, \theta)$ to the form

$$C_{\sigma\alpha\gamma}^*(x, \theta) = (\gamma^m \theta)_\sigma (\gamma^n \theta)_\alpha (\gamma^p \theta)_\gamma (\theta\gamma_{mnp}\theta) c^*(x) \quad (4.2.93)$$

and, if we use the equation 4.2.83, we do not get any equation of motion for $c^*(x)$. We also have a gauge invariance for this field: $\delta c^* = \partial^m t_m$.

Therefore we have reproduced the spectrum of D=10 SYM (abelian case) with the appropriate equations of motion and gauge invariances in the Batalin-Vilkovisky framework.

These results can be obtained by proposing an action principle as the following one:

$$S = \int d^{10}x \langle \Psi Q \Psi \rangle \quad (4.2.94)$$

where the norm is defined such that $\langle \lambda^3 \theta^5 \rangle = 1$ ⁴¹. This definition picks out the top cohomology making this action invariant under the gauge transformation $\delta \Psi = Q \Lambda$ ⁴². After varying this action in Ψ one finds the equations of motion:

$$Q \Psi = 0 \quad (4.2.95)$$

only for components of $Q \Psi$ involving five θ 's. Clearly this is not a manifestly supersymmetric formulation of linearized $D = 10$ SYM, however it can be shown that the equations involving more than five θ 's just change auxiliary fields by gauge fields and so they do not affect the physical content of the theory.

⁴¹One can see in [19] that this convention is similar to that used in $D = 3$ Chern-Simons theory $\langle c^\mu c^\nu c^\rho \rangle \propto \epsilon^{\mu\nu\rho}$. Indeed in that case the top cohomology is in the term $c^\mu c^\nu c^\rho C_{\mu\nu\rho}^*(x)$ where c^μ is the ghost corresponding to the constraint $P^\mu = 0$ and $C^*(x)$ is the superfield containing the (spacetime) ghost antifield C^* (the top cohomology). This normalization reproduces the BV-action for the $D = 3$ Chern-Simons theory.

⁴²This is so because the (scalar) ghost antifield is in the top cohomology, and this one cannot be written as $\Psi = Q \Sigma$ for any Σ .

4.3 N=1 D=11 superparticle

The generalization of the superparticle action for the case D=11 is:

$$S = \int d\tau (P^c \Pi_c + e P^c P_c) \quad (4.3.1)$$

where $\Pi_c = \dot{X}_c - i\dot{\Theta}^\alpha (\Gamma_c)_{\alpha\beta} \Theta^\beta$, and Θ^α is a Majorana spinor. Now, let us mention some words about the conventions which will be used in the case of D=11 superparticle. We will denote $SO(10,1)$ vector indices by a, b, c, \dots , and $SO(10,1)$ spinor indices by α, β, \dots ($a = 0, \dots, 10$ and $\alpha = 1, \dots, 32$). The $D = 11$ gamma matrices Γ^c are 32×32 symmetric matrices which satisfy $\Gamma_{\alpha\beta}^c \Gamma^{d\beta\gamma} + \Gamma_{\alpha\beta}^d \Gamma^{c\beta\gamma} = 2\eta^{cd} \delta_\alpha^\gamma$ and $\eta_{bc} \Gamma_{(\alpha\beta}^b \Gamma_{\gamma\delta)}^{cd} = 0$. As opposite to the D=10 case, now we have an antisymmetric metric tensor $C^{\alpha\beta}$ (and its inverse $C_{\alpha\beta}^{-1}$) which will allow us to raise and lower indices (for instance $\Gamma^{c\alpha\beta} = C^{\alpha\delta} \Gamma_\delta^{c\beta}$, etc). It is also useful to note that any D=11 antisymmetric bispinor can be decomposed into a scalar, three-form, and four form as $f^{[\alpha\beta]} = C^{\alpha\beta} f + (\Gamma_{bcd})^{\alpha\beta} f^{bcd} + (\Gamma_{bcde})^{\alpha\beta} f^{bcde}$, and any D=11 symmetric bispinor can be written in terms of a one-form, two-form and five-form as $g^{(\alpha\beta)} = \Gamma_c^{\alpha\beta} g^c + (\Gamma_{cd})^{\alpha\beta} g^{cd} + (\Gamma_{bcdef})^{\alpha\beta} g^{bcdef}$. Analogously to the D=10 case, we will have SUSY transformations and κ transformations (and obviously also the reparameterization invariance) under which the action 4.3.1 is invariant. These transformations are just generalizations of the D=10 case:

$$\begin{aligned} \text{SUSY transformations} &\rightarrow \delta\Theta^\alpha = \epsilon^\alpha, \quad \delta X^c = i\Theta^\alpha \Gamma_{\alpha\beta}^c \epsilon^\beta, \quad \delta P_c = \delta e = 0 \\ \kappa \text{ (local) transformations} &\rightarrow \delta\Theta^\alpha = P^c \Gamma_c^{\alpha\beta} \kappa_\beta, \quad \delta X^c = -i\Theta^\alpha \Gamma_{\alpha\beta}^c \delta\Theta^\beta, \quad \delta P_m = 0, \quad \delta e = 2i\dot{\Theta}^\beta \kappa_\beta \end{aligned}$$

We can proceed in the same way as we did in the D=10 superparticle case (because the Γ^c matrices obey the same algebra as γ^m matrices do). Therefore, we will just show a few calculations (the remaining ones can be checked easily mimicking the corresponding computations in the D = 10 case). Let us show the SUSY invariance:

$$\begin{aligned} \delta S &= \int d\tau (\delta\Pi^c P_c + \Pi^c \delta P_c) \\ \delta S &= \int d\tau [(\delta\dot{X}^c - i\dot{\Theta}^\alpha (\Gamma^c)_{\alpha\beta} \delta\Theta^\beta) P_c] \\ \delta S &= \int d\tau [i\dot{\Theta}^\alpha (\Gamma^c)_{\alpha\beta} \epsilon^\beta - i\dot{\Theta}^\alpha (\Gamma^c)_{\alpha\beta} \epsilon^\beta] P_c \\ \delta S &= 0 \end{aligned}$$

We will omit the proof of the κ -invariance for the reasons already explained above. Now, we can compute the conjugate momentum to Θ^α :

$$P_\alpha = \frac{\partial L}{\partial \dot{\Theta}^\alpha} = -i\Gamma_{\alpha\beta}^c \Theta^\beta P_c \quad (4.3.2)$$

Once again, we have a constrained system. The constraints are:

$$d_\alpha = P_\alpha + i\Gamma_{\alpha\beta}^c \Theta^\beta P_c \quad (4.3.3)$$

and they satisfy (considering that⁴³ $\{\Theta^\alpha, P_\beta\}_{P.B} = i\delta_\beta^\alpha$):

$$\begin{aligned} \{d_\alpha, d_\beta\} &= \{P_\alpha + i\Gamma_{\alpha\lambda}^c \Theta^\lambda P_c, P_\beta + i\Gamma_{\beta\sigma}^d \Theta^\sigma P_d\} \\ \{d_\alpha, d_\beta\} &= \{P_\alpha, P_\beta\} + i\Gamma_{\alpha\lambda}^c P_c \{\Theta^\lambda, P_\beta\} + i\Gamma_{\beta\sigma}^d P_d \{P_\alpha, \Theta^\sigma\} - \Gamma_{\alpha\lambda}^c \Gamma_{\beta\sigma}^d P_c P_d \{\Theta^\lambda, \Theta^\sigma\} \end{aligned}$$

⁴³We are using the same conventions as before, that is to say, we write i in front of δ_β^α because the reality of the action will require the complex nature of P_α .

therefore

$$\{d_\alpha, d_\beta\} = -2(\Gamma^c)_{\alpha\beta} P_c \quad (4.3.4)$$

where we should understand $\{, \}$ as being a Poisson bracket. As in the D=10 case, if we define $K^\alpha = -iP^c \Gamma_c^{\alpha\beta} d_\beta$, we find that these ones are the first-class constraints generating the κ -symmetry:

$$\{K^\alpha, K^\beta\} = \{-iP^c \Gamma_c^{\alpha\lambda} d_\lambda, -iP^d \Gamma_d^{\beta\sigma} d_\sigma\} \quad (4.3.5)$$

$$\{K^\alpha, K^\beta\} = -\Gamma_c^{\alpha\lambda} \Gamma_d^{\beta\sigma} P^c P^d \{d_\lambda, d_\sigma\} \quad (4.3.6)$$

$$\{K^\alpha, K^\beta\} = 2\Gamma_c^{\alpha\lambda} \Gamma_d^{\beta\sigma} P^c P^d P^e (\Gamma_e)_{\lambda\sigma} \quad (4.3.7)$$

$$\{K^\alpha, K^\beta\} = 2P^c P^d P^e (\Gamma_c \Gamma_e \Gamma_d)^{\alpha\beta} \quad (4.3.8)$$

and it is easy to see that this quantity vanishes⁴⁴. Now, by using these K^α we restore the κ -transformations defined above. Let us check this statement just for Θ^α :

$$\delta\Theta^\alpha = \{K^\lambda, \Theta^\alpha\} \kappa_\lambda \quad (4.3.9)$$

$$\delta\Theta^\alpha = \{-iP^c \Gamma_c^{\lambda\beta} d_\beta, \Theta^\alpha\} \kappa_\lambda \quad (4.3.10)$$

$$\delta\Theta^\alpha = -iP^c \Gamma_c^{\lambda\beta} \{P_\beta, \Theta^\alpha\} \kappa_\lambda \quad (4.3.11)$$

$$\delta\Theta^\alpha = P^c \Gamma_c^{\alpha\lambda} \kappa_\lambda \quad (4.3.12)$$

We can see that the calculation is analog to the D=10 case. We will omit the explicit calculations for the other fields. Going back to the equation 4.3.4, we realize that we have sixteen first-class constraints and 16 second-class constraints⁴⁵. Once again, there is no simple way to covariantly separate out the second-class constraints. However, we can use the semi light-cone gauge ($P^+ \neq 0$) to figure out the physical spectrum of the theory. Analogously to the D=10 case, we define the light-cone coordinates:

$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^9) \quad (4.3.13)$$

$$X^- = \frac{1}{\sqrt{2}}(X^0 - X^9) \quad (4.3.14)$$

$$\Gamma^+ = \frac{1}{\sqrt{2}}(\Gamma^0 + \Gamma^9) \quad (4.3.15)$$

$$\Gamma^- = \frac{1}{\sqrt{2}}(\Gamma^0 - \Gamma^9) \quad (4.3.16)$$

and we choose a frame where $P^c = (P, 0, \dots, P, 0)$. Therefore:

$$\Gamma_{\alpha\beta}^+ \Gamma^{+\beta\delta} = \frac{1}{2}(\Gamma^0 + \Gamma^9)(\Gamma^0 - \Gamma^9) = \frac{1}{2}(-1 + 1) = 0 \quad (4.3.17)$$

$$P^- = \frac{1}{\sqrt{2}}(P - P) = 0 \quad (4.3.18)$$

⁴⁴Analogously to the D=10 case, we just have to use $P^2 = 0$ and the identity $\Gamma_{\alpha\beta}^c \Gamma^{d\beta\sigma} + \Gamma_{\alpha\beta}^d \Gamma^{c\beta\sigma} = 2\eta^{cd}\delta_{\alpha}^{\sigma}$.

⁴⁵In analogy to the D=10 case, we should to choose a frame where $P^c = (P, 0, \dots, P, 0)$, and also to work with the light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^9)$, $\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9)$. It is clear that $P^- = 0$ and therefore

the equation 4.3.4 takes the form: $\{d_\alpha, d_\beta\} = -2(\Gamma^-)_{\alpha\beta} P^+ \propto \begin{pmatrix} \begin{pmatrix} 1_{8 \times 8} & 0_{8 \times 8} \\ 0_{8 \times 8} & 0_{8 \times 8} \end{pmatrix} & 0_{16 \times 16} \\ 0_{16 \times 16} & \begin{pmatrix} 0_{8 \times 8} & 0_{8 \times 8} \\ 0_{8 \times 8} & 1_{8 \times 8} \end{pmatrix} \end{pmatrix}$.

In light-cone coordinates, the κ -transformation for Θ^α becomes:

$$\delta\Theta^\alpha = -P^+\Gamma^{-\alpha\beta}\kappa_\beta - P^-\Gamma^{+\alpha\beta}\kappa_\beta + P^i\Gamma^{i\alpha\beta}\kappa_\beta = -P^+\Gamma^{-\alpha\beta}\kappa_\beta \quad (4.3.19)$$

because $P^- = 0$ and $P^i = 0$ ($i = 1, \dots, 8, 10$). If we define $\kappa_\beta = -\frac{1}{2P^+}\Gamma_{\beta\lambda}^+\theta^\lambda$, we get $\delta\Theta^\alpha = \frac{1}{2}(\Gamma^-\Gamma^+\Theta)^\alpha$. Now, let us decompose a $SO(10, 1)$ Majorana spinor in terms of its $SO(9)$ spinor components:

$$\Theta^\alpha = \begin{pmatrix} \theta^A \\ \tilde{\theta}^A \end{pmatrix} \quad (4.3.20)$$

So, we can write Θ^α as follows⁴⁶:

$$\Theta^\alpha = -\frac{1}{2}(\Gamma^+\Gamma^-\Theta)^\alpha - \frac{1}{2}(\Gamma^-\Gamma^+\Theta)^\alpha \quad (4.3.24)$$

So, we can always use the κ -transformation to get the gauge $(\Gamma^+\Theta)_\alpha = 0$:

$$\Theta'^\alpha = \Theta^\alpha + \delta\Theta^\alpha = -\frac{1}{2}(\Gamma^+\Gamma^-\Theta)^\alpha - \frac{1}{2}(\Gamma^-\Gamma^+\Theta)^\alpha + \frac{1}{2}(\Gamma^-\Gamma^+\Theta)^\alpha = -\frac{1}{2}(\Gamma^+\Gamma^-\Theta)^\alpha \quad (4.3.25)$$

and now by using the equation 4.3.17:

$$\Gamma_{\alpha\beta}^+\Theta'^\beta = -\frac{1}{2}\Gamma_{\alpha\beta}^+\Gamma^{+\beta\sigma}\Gamma_{\sigma\lambda}\Theta^\lambda = 0 \quad (4.3.26)$$

We will just write θ^α instead of θ'^α . The action becomes⁴⁷

$$S = \int d\tau [P^c \dot{X}_c - iP^c \dot{\Theta} \Gamma_c \Theta + eP^c P_c] \quad (4.3.28)$$

$$S = \int d\tau [P^c \dot{X}_c + iP^+ \dot{\Theta} \Gamma^- \Theta + eP^c P_c] \quad (4.3.29)$$

$$S = \int d\tau [P^c \dot{X}_c - \sqrt{2}iP^+(\dot{\theta}^a \theta^a + \dot{\tilde{\theta}}^{\dot{a}} \tilde{\theta}^{\dot{a}}) + eP^c P_c] \quad (4.3.30)$$

$$S = \int d\tau [P^c \dot{X}_c - \frac{i}{2} \dot{S}^A S^A + eP^c P_c] \quad (4.3.31)$$

⁴⁶In the representation adopted in this work, the Γ^\pm have the following explicit form:

$$\Gamma_{\alpha\beta}^+ = \begin{pmatrix} \gamma^{+AB} & 0 \\ 0 & \gamma_{AB}^+ \end{pmatrix}, \quad \Gamma^{+\alpha\beta} = \begin{pmatrix} \gamma_{AB}^+ & 0 \\ 0 & \gamma^{+AB} \end{pmatrix}, \quad (4.3.21)$$

$$\Gamma_{\alpha\beta}^- = \begin{pmatrix} \gamma^{-AB} & 0 \\ 0 & \gamma_{AB}^- \end{pmatrix}, \quad \Gamma^{-\alpha\beta} = \begin{pmatrix} \gamma_{AB}^- & 0 \\ 0 & \gamma^{-AB} \end{pmatrix} \quad (4.3.22)$$

where γ^m are the matrices seen in the D=10 case. Therefore:

$$\Gamma^\pm \Gamma^\mp = \begin{pmatrix} \gamma^\pm \gamma^\mp & 0 \\ 0 & \gamma^\pm \gamma^\mp \end{pmatrix} \quad (4.3.23)$$

⁴⁷It is easy to see that $\dot{\Theta} \Gamma^i \Theta = 0$:

$$\dot{\Theta} \Gamma^i \Theta = -\frac{1}{2} \dot{\Theta} \Gamma^i (\Gamma^+ \Gamma^- + \Gamma^- \Gamma^+) \Theta = 0 \quad (4.3.27)$$

because $(\Gamma^+ \Theta)_\alpha = 0$ and Γ^+ anticommutes with Γ^i .

where $S_A = \sqrt{\sqrt{2}P^+}(\Gamma^-\Theta)_A$ is a $SO(9)$ Majorana spinor. This looks like very similar to the corresponding light-cone action in the D=10 case (equation 4.1.43). The conjugate momentum to S^A is:

$$p_A = \frac{\partial L}{\partial \dot{S}^A} = -\frac{i}{2}S_A \quad (4.3.32)$$

So, the constraints are:

$$\tilde{d}_A = p_A + \frac{i}{2}S_A \quad (4.3.33)$$

The constraint matrix is calculated as follows:

$$\{\tilde{d}_A, \tilde{d}_B\} = \{p_A + \frac{i}{2}S_A, p_A + \frac{i}{2}S_A\} \quad (4.3.34)$$

$$\{\tilde{d}_A, \tilde{d}_B\} = \{p_A, p_B\} + \{p_A, \frac{i}{2}S_B\} + \{\frac{i}{2}S_A, p_B\} - \frac{1}{4}\{S_A, S_B\} \quad (4.3.35)$$

$$\{\tilde{d}_A, \tilde{d}_B\} = \delta_{AB} \quad (4.3.36)$$

where as usual we considered $\{S_A, p_B\} = -i\delta_{AB}$. So the corresponding constraint matrix is $C_{AB} = \delta_{AB}$ so its inverse is $(C^{-1})^{AB} = \delta^{AB}$. The corresponding Dirack bracket will be

$$\{S_A, S_B\}_D = \{S_A, S_B\}_P - \sum_{E,F} E, F \{S_A, \tilde{d}_E\}_P (C^{-1})^{EF} \{\tilde{d}_F, S_B\}_P \quad (4.3.37)$$

$$\{S_A, S_B\}_D = 0 - \sum_{E,F} (-i\delta_{AE})(\delta^{EF})(-i\delta_{FB}) \quad (4.3.38)$$

$$\{S_A, S_B\}_D = \delta_{AB} \quad (4.3.39)$$

The physical states will be denoted by: $|IJ\rangle$, $|BI\rangle$ and $|LMN\rangle$. They correspond to a $SO(9)$ traceless symmetric tensor, a $SO(9)$ γ -traceless vectorspinor and a $SO(9)$ 3-form⁴⁸, respectively. This is the (on-shell) field content of D=11 SUGRA. These states form a representation of the algebra 4.3.39:

$$S_A|IJ\rangle = \Gamma_{AB}^I|BJ\rangle + \Gamma^J|BI\rangle \quad (4.3.40)$$

$$S_A|BI\rangle = \frac{1}{4}\Gamma_{AB}^J|IJ\rangle + \frac{1}{72}(\Gamma_{AB}^{ILMN} + 6\delta^{IL}\Gamma_{AB}^{MN})|LMN\rangle \quad (4.3.41)$$

$$S_A|LMN\rangle = \Gamma_{AB}^{LM}|BN\rangle + \Gamma_{AB}^{MN}|BL\rangle + \Gamma_{AB}^{NL}|BM\rangle \quad (4.3.42)$$

We can check that indeed these definitions reproduce the desired algebra. Let us show this statement for the state $|IJ\rangle$:

$$\begin{aligned} S_A S^B|IJ\rangle &= \Gamma_{BC}^I S_A|CJ\rangle + \Gamma_{BC}^J S_A|CI\rangle \\ &= \Gamma_{BC}^I \left[\frac{1}{4}\Gamma_{AC}^K|JK\rangle + \frac{1}{72}(\Gamma_{AC}^{JLMN} + 6\delta^{JL}\Gamma_{AC}^{MN})|LMN\rangle \right] \\ &+ \Gamma_{BC}^J \left[\frac{1}{4}\Gamma_{AC}^K|IK\rangle + \frac{1}{72}(\Gamma_{AC}^{ILMN} + 6\delta^{JL}\Gamma_{AC}^{MN})|LMN\rangle \right] \end{aligned}$$

Analogously,

$$\begin{aligned} S_B S_A|IJ\rangle &= \Gamma_{AC}^I S_B|CJ\rangle + \Gamma_{AC}^J S_B|CI\rangle \\ &= \Gamma_{AC}^I \left[\frac{1}{4}\Gamma_{BC}^K|JK\rangle + \frac{1}{72}(\Gamma_{BC}^{JLMN} + 6\delta^{JL}\Gamma_{BC}^{MN})|LMN\rangle \right] \\ &+ \Gamma_{AC}^J \left[\frac{1}{4}\Gamma_{BC}^K|IK\rangle + \frac{1}{72}(\Gamma_{BC}^{ILMN} + 6\delta^{JL}\Gamma_{BC}^{MN})|LMN\rangle \right] \end{aligned}$$

⁴⁸We are denoting $SO(9)$ vector indices by I, J, K, L, \dots , and $SO(9)$ spinor indices by A, B, C, D, \dots

So, the anticommutator is

$$\begin{aligned}
\{S_A, S_B\}|IJ\rangle &= \frac{1}{4}[\Gamma_{BC}^I \Gamma_{AC}^K + \Gamma_{AC}^I \Gamma_{BC}^K]|JK\rangle + \frac{1}{4}[\Gamma_{BC}^J \Gamma_{AC}^K + \Gamma_{AC}^J \Gamma_{BC}^K]|IK\rangle \\
&+ \frac{1}{72}[(\Gamma_{BC}^I \Gamma_{AC}^{JLMN} + \Gamma_{BC}^J \Gamma_{AC}^{ILMN} + \Gamma_{AC}^I \Gamma_{BC}^{JLMN} + \Gamma_{AC}^J \Gamma_{BC}^{ILMN}) \\
&+ 6(\delta^{JL} \Gamma_{BC}^I \Gamma_{AC}^{MN} + \delta^{IL} \Gamma_{BC}^J \Gamma_{AC}^{MN} + \delta^{JL} \Gamma_{AC}^I \Gamma_{BC}^{MN} + \delta^{IL} \Gamma_{AC}^J \Gamma_{BC}^{MN})]|LMN\rangle \\
&= \frac{1}{4}(2\delta^{IK} \delta_{AB}|JK\rangle + 2\delta^{JK} \delta_{AB}|IK\rangle) + \frac{1}{72}[4!(\delta^{I[J} \Gamma_{BA}^{LMN]} + \delta^{J[I} \Gamma_{BA}^{LMN]} \\
&+ \delta^{I[J} \Gamma_{AB}^{LMN]} + \delta^{J[I} \Gamma_{AB}^{LMN]}) + 6(\delta^{JL} \Gamma_{BC}^I \Gamma_{AC}^{MN} + \delta^{IL} \Gamma_{BC}^J \Gamma_{AC}^{MN} \\
&+ \delta^{JL} \Gamma_{AC}^I \Gamma_{BC}^{MN} + \delta^{IL} \Gamma_{AC}^J \Gamma_{BC}^{MN})]|LMN\rangle
\end{aligned}$$

Now, we have to consider the symmetries properties of the Γ -matrices. The 1-form and 4-form are symmetric in its spinor indices, and the 2-form and 3-form are antisymmetric in its spinor indices. Therefore

$$\begin{aligned}
\{S_A, S_B\}|IJ\rangle &= \delta_{AB}|IJ\rangle + \frac{1}{12}(\delta^{JL} \Gamma_{BC}^I \Gamma_{AC}^{MN} + \delta^{IL} \Gamma_{BC}^J \Gamma_{AC}^{MN} \\
&+ \delta^{JL} \Gamma_{AC}^I \Gamma_{BC}^{MN} + \delta^{IL} \Gamma_{AC}^J \Gamma_{BC}^{MN})|LMN\rangle \\
&= \delta_{AB}|IJ\rangle + \frac{1}{12}[\delta^{JL}(\Gamma_{BA}^{IMN} + \delta^{I[M} \Gamma^{N]}) + \Gamma_{AB}^{IMN} + \delta^{I[M} \Gamma_{AB}^{N]}) \\
&+ \delta^{IL}(\Gamma_{BA}^{JMN} + \delta^{J[M} \Gamma^{N]}) + \Gamma_{AB}^{JMN} + \delta^{J[M} \Gamma_{AB}^{N]})]|LMN\rangle \\
&= \delta_{AB}|IJ\rangle + \frac{1}{12}[\delta^{JL} \delta^{IM} \Gamma^N - \delta^{JL} \delta^{IN} \Gamma^M + \delta^{JL} \delta^{IM} \Gamma^N - \delta^{IN} \delta^{JL} \Gamma^M \\
&+ \delta^{IL} \delta^{JM} \Gamma^N - \delta^{IL} \delta^{JN} \Gamma^M + \delta^{IL} \delta^{JM} \Gamma^N - \delta^{IL} \delta^{JN} \Gamma^M]|LMN\rangle \\
&= \delta_{AB}|IJ\rangle + \frac{1}{12}[2\Gamma^N|JIN\rangle - 2\Gamma^M|JMI\rangle + 2\Gamma^N|IJN\rangle - 2\Gamma^M|IMJ\rangle] \\
\rightarrow \{S_A, S_B\}|IJ\rangle &= \delta_{AB}|IJ\rangle
\end{aligned}$$

We can show in the same way which this algebra is satisfied when S_A acts on the other two fields, however we will omit those computations. Therefore the physical spectrum of the D=11 superparticle correspond to D=11 SUGRA. Now, as we did in the D=10 case (SYM), we want a covariant description of D=11 SUGRA. It is precisely the pure spinor version of the D=11 superparticle who do this work.

4.4 N=1 D=11 Semi-pure spinor superparticle

Unlike the D=10 superparticle case, we will not obtain the pure spinor version by adding a new symmetry (and a pair of conjugate fields) in the gauge fixed superparticle action (equation 4.3.31). We will just define (by analogy) the D=11 pure spinor superparticle action by:

$$S = \int d\tau (\dot{x}^c P_c - \frac{1}{2} P^c P_c + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha w_\alpha) \quad (4.4.1)$$

where p_α is an independent variable, λ^α is an $SO(10,1)$ pure spinor ghost variable satisfying⁴⁹:

$$\lambda \Gamma^c \lambda = 0 \text{ for } c = 1 \text{ to } 11 \quad (4.4.2)$$

Let us figure out how many degrees of freedom λ^α has. First, we should decompose λ^α in its $SO(9,1)$ components⁵⁰ λ^μ and $\tilde{\lambda}_\mu$ ⁵¹. Therefore the constraint 4.4.2 becomes

$$\lambda^\alpha \Gamma_{\alpha\beta}^m \lambda^\beta = \lambda^\mu \gamma_{\mu\nu}^m \lambda^\nu + \tilde{\lambda}_\mu \gamma^{m\mu\nu} \tilde{\lambda}_\nu = 0 \text{ for } m = 0 \text{ to } 9 \quad (4.4.3)$$

$$\lambda^\alpha \Gamma_{\alpha\beta}^{10} \lambda^\beta = \lambda^\mu \tilde{\lambda}_\mu = 0 \quad (4.4.4)$$

Now, we are going to show that the first line can imply the second one. We should use the identity $\eta_{de} \Gamma_{(\alpha\beta}^{cd} \Gamma_{\gamma\delta)}^e = 0$:

$$\eta_{de} \Gamma_{(\alpha\beta}^{cd} \Gamma_{\gamma\delta)}^e = \Gamma_{(\alpha\beta}^{ce} (\Gamma_e)_{\gamma\delta)} = 0 \quad (4.4.5)$$

$$\text{multiplying by } \lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta \rightarrow (\lambda \Gamma^{c10} \lambda)(\lambda \Gamma^{10} \lambda) + (\lambda \Gamma^{cm} \lambda)(\lambda \Gamma_m \lambda) = 0 \quad (4.4.6)$$

therefore⁵²

$$(\lambda \Gamma^{c10} \lambda)(\lambda \Gamma^{10} \lambda) = -(\lambda \Gamma^{cm} \lambda)(\lambda \Gamma_m \lambda) \quad (4.4.7)$$

So, if $\lambda \Gamma^m \lambda = 0$, we will require that $\lambda \Gamma^{10} \lambda = 0$, hence the equation 4.4.4 is implied by the equation 4.4.3. Now, we will focus on the equation 4.4.3. Let h^m be an $SO(9,1)$ vector such that $h^m = \lambda^\mu \gamma_{\mu\nu}^m \lambda^\nu$. By using the identity $\gamma_{(\mu\nu)}^m (\gamma_m)_{\kappa\rho} = 0$ we see that $h^m h_m = \lambda^\mu \lambda^\nu \lambda^\kappa \lambda^\rho (\gamma^m)_{\mu\nu} (\gamma_m)_{\kappa\rho} = \lambda^\mu \lambda^\nu \lambda^\kappa \lambda^\rho (\gamma^m)_{(\mu\nu)} (\gamma_m)_{\kappa\rho} = 0$, that is h^m is a $SO(9,1)$ null vector. In addition to this, we see that $\tilde{\lambda}_\mu (\gamma^m)^{\mu\nu} \tilde{\lambda}_\nu = -h^m$, therefore once we fix λ^μ , we get h^m and this gives us 9 constraints on $\tilde{\lambda}_\mu$ (it is not 10 constraints because of the nullity condition). So we have 23 degrees of freedom (16 degrees of freedom in λ^μ and $16 - 9 = 7$ degrees of freedom in $\tilde{\lambda}_\mu$).

As usual, the physical states for the D=11 superparticle are defined as elements in the cohomology of the BRST operator:

$$Q = \lambda^\alpha d_\alpha \quad (4.4.8)$$

which is clearly nilpotent:

$$Q^2 = \frac{1}{2} \{Q, Q\} = \frac{1}{2} \lambda^\alpha \lambda^\beta \{d_\alpha, d_\beta\} = -(\lambda \Gamma^c \lambda) P_c = 0 \quad (4.4.9)$$

⁴⁹We will not consider the condition $\lambda \Gamma^{cd} \lambda = 0$ which is considered by some authors as an extra condition in the definition of a pure spinor in D=12, and this gives the condition mentioned on D=11 pure spinor space. For this reason it would be more appropriate to call these ghost variables *semi-pure spinors*. However, we will continue with the name *pure spinor*, but keeping in mind the kind the object which is being studied here.

⁵⁰We will denote $SO(9,1)$ spinor indices by μ, ν, \dots and $SO(9,1)$ vector indices by m, n, \dots , as before.

⁵¹We are using a hat on λ_μ in order to distinguish spinors of different chiralities (λ^μ is a Weyl spinor and $\tilde{\lambda}_\mu$ is an anti-Weyl spinor).

⁵²When we do not put indices on λ , we should understand that we are treating with a $SO(10,1)$ spinor, namely λ^α .

Now, we will show that the Q -cohomology give us a manifestly super-Poincare covariant description of linearized D=11 supergravity. As we did for the D=10 superparticle, we will start with the Q -cohomology at zero momentum. For that, we also write the BRST operator \hat{Q} coming from writing in the action the constraint $\lambda\Gamma^c\lambda = 0$ and working with unconstrained $\hat{\lambda}^\alpha$:

$$\begin{aligned}
\hat{Q} &= \hat{\lambda}^\alpha p_\alpha + \hat{\lambda}\Gamma^c\hat{\lambda}b_{(-1)c} + c_{(1)}^c(\hat{\lambda}\Gamma_{cd}\hat{\lambda}u_{(-2)}^d + \hat{\lambda}\Gamma^d\hat{\lambda}u_{(-2)[cd]}) + v_{(2)}^c((\hat{\lambda}\Gamma_c)_\alpha b_{(-2)}^\alpha) \\
&+ \hat{\lambda}\Gamma^d\hat{\lambda}b_{(-3)(cd)} + v_{(2)}^{[cd]}\left(\frac{1}{2}(\hat{\lambda}\Gamma_{cd})_\alpha b_{(-2)}^\alpha + \eta_{de}\hat{\lambda}\Gamma^{ef}\hat{\lambda}b_{(-3)(cf)} + \hat{\lambda}\Gamma^e\hat{\lambda}b_{(-3)[cde]}\right) \\
&+ c_{(2)}^\alpha\left(-\hat{\lambda}\Gamma^e\hat{\lambda}u_{(-3)c\alpha} + \frac{1}{2}(\hat{\lambda}\Gamma^{cd})_\alpha(\hat{\lambda}\Gamma_c)^\beta u_{(-3)d\beta}\right) + \frac{1}{2}c_{(3)}^{(de)}(\hat{\lambda}\Gamma_d)^\alpha u_{(-3)e\alpha} \\
&+ \frac{1}{4}c_{(3)}^{[def]}(\hat{\lambda}\Gamma_{ef})^\alpha u_{(-3)d\alpha} + v_{(3)}^{c\alpha}b_{(-4)}^{d\beta}M_{c\alpha d\beta\gamma\delta}\hat{\lambda}^\gamma\hat{\lambda}^\delta + \frac{1}{4}u_{(-4)}^{[def]}(\hat{\lambda}\Gamma_{ef})^\alpha c_{(4)d\alpha} \\
&+ \frac{1}{2}u_{(-4)}^{(de)}(\hat{\lambda}\Gamma_d)^\alpha c_{(4)e\alpha} + u_{(-5)}^\alpha\left(-\hat{\lambda}\Gamma^c\hat{\lambda}c_{(4)c\alpha} + \frac{1}{2}(\hat{\lambda}\Gamma^{cd})_\alpha(\hat{\lambda}\Gamma_c)^\beta c_{(4)d\beta}\right) \\
&+ b_{(-5)}^{[cd]}\left(\frac{1}{2}(\hat{\lambda}\Gamma_{cd})_\alpha v_{(5)}^\alpha + \eta_{de}\hat{\lambda}\Gamma^{ef}\hat{\lambda}v_{(4)(cf)} + \hat{\lambda}\Gamma^e\hat{\lambda}v_{(4)[cde]}\right) \\
&+ b_{(-5)}^c\left((\hat{\lambda}\Gamma_c)_\alpha v_{(5)}^\alpha + \hat{\lambda}\Gamma^d\hat{\lambda}v_{(4)(cd)}\right) + u_{(-6)}^c\left(\hat{\lambda}\Gamma_{cd}\hat{\lambda}c_{(5)}^d + \hat{\lambda}\Gamma^d\hat{\lambda}c_{(5)[cd]}\right) \\
&+ b_{(-7)}\hat{\lambda}\Gamma^c\hat{\lambda}v_{(6)c}
\end{aligned}$$

The way of constructing the first terms of this operator is identical to that developed for $D = 10$ SYM, so we will not spend more time repeating the arguments mentioned there⁵³. However, in this case we have more complicated reducibility functions so an alternative method would help us to avoid the long and direct computation. In [23] one can see that the construction of \hat{Q} is based on two decent proposals. One of them says that the linear terms in $\tilde{\lambda}$ will be the linearized version of the well-known supersymmetry transformations of D=11 supergravity fields, ghosts, antifields, ghost antifields. And the other one says that the quadratic terms $\tilde{\lambda}^2$ will be related with the anticommutator of two supersymmetry transformations (whose algebra is closed up to equations of motion). So the quantity $M_{c\alpha d\beta\gamma\delta}$ is defined by using the anticommutator of two supersymmetry transformations acting on the gravitino as follows:

$$\{\delta_\gamma, \delta_\delta\}\chi_{c\alpha} = \Gamma_{\gamma\delta}^d\partial_d\chi_{c\alpha} + M_{c\alpha d\beta\gamma\delta}\frac{\delta S}{\delta\chi_{d\beta}} \quad (4.4.10)$$

⁵³For example, the term $\hat{\lambda}\Gamma^c\hat{\lambda}b_{(-1)c}$ takes into account the pure spinor constraint ($b_{(-1)c}$ is a fermionic variable). This gauge symmetry is reducible. The corresponding ghost ($b_{(-1)c}$ fermionic) has the gauge symmetry: $\delta b_{(-1)c} = \hat{\lambda}\Gamma_{cd}\hat{\lambda}g^d + \hat{\lambda}\Gamma^d\hat{\lambda}f_{[cd]}$. This can be seen as follows:

$$\begin{aligned}
(\hat{\lambda}\Gamma^c\hat{\lambda})\delta b_{(-1)c} &= \hat{\lambda}\Gamma^c\hat{\lambda}(\hat{\lambda}\Gamma_{cd}\hat{\lambda}g^d + \hat{\lambda}\Gamma^d\hat{\lambda}f_{[cd]}) \\
(\hat{\lambda}\Gamma^c\hat{\lambda})\delta b_{(-1)c} &= \hat{\lambda}\Gamma^c\hat{\lambda}(\hat{\lambda}\Gamma_{cd}\hat{\lambda})g^d + \hat{\lambda}\Gamma^c\hat{\lambda}\hat{\lambda}\Gamma^d\hat{\lambda}f_{[cd]} \\
(\hat{\lambda}\Gamma^c\hat{\lambda})\delta b_{(-1)c} &= \hat{\lambda}^\alpha\hat{\lambda}^\beta\hat{\lambda}^\sigma\hat{\lambda}^\delta\Gamma_{(\alpha\beta}^{[c}\Gamma_{\sigma\delta]}^d]f_{[cd]} \\
(\hat{\lambda}\Gamma^c\hat{\lambda})\delta b_{(-1)c} &= 0
\end{aligned}$$

where the third equality follows from the identity $\eta_{de}\Gamma_{(\alpha\beta}^{cd}\Gamma_{\delta\sigma)}^e = 0$ and the last equality is a simple consequence of the antisymmetrization in c, d and symmetrization in $\alpha, \beta, \sigma, \delta$. Now, the fields $g^d, f_{[cd]}$ (which will become (bosonic) ghost fields in the expression for the BRST operator) will have also gauge symmetries and so these ones will contribute to the BRST operator \hat{Q} . As we know, it must be taken into account each reducible gauge symmetry which appears in the theory in order to have a correct expression for \hat{Q} . As we said, we will omit the full proof of this construction.

where S is the standard $D = 11$ supergravity action. This proposal turned out to be the correct one, as it was proved later by direct computation.

Therefore the states in the linear \hat{Q} -cohomology are represented by (after using the quartet argument)

$$[1, c_{(1)}^c, v_{(2)}^c, v_{(2)}^{[cd]}, c_{(2)}^\alpha, c_{(3)}^{[cde]}, v_{(3)}^{c\alpha}, c_{(4)}^{[c\alpha]}, v_{(4)}^{[cde]}, v_{(4)}^{(cd)}, v_{(5)}^\alpha, c_{(5)}^{cd}, c_{(5)}^e, v_{(6)}^c, c_{(7)}] \quad (4.4.11)$$

The field content is the following: The states of ghost number 3 corresponding to $\{c_{(3)}^{(cd)}, c_{(3)}^{[cde]}, v_{(3)}^{c\alpha}\}$ are the physical fields of linearized $D = 11$ SUGRA (graviton, 3-form potential and gravitino, respectively) and the states of ghost number 4 corresponding to $\{v_{(4)}^{(cd)}, v_{(4)}^{[cde]}, c_{(4)}^{c\alpha}\}$ are their respective antifields. The states of ghost number 2 corresponding to $\{v_{(2)}^c, c_{(2)}^\alpha, v_{(2)}^{[cd]}\}$ are the ghosts coming from supersymmetry, reparameterization invariance and the 3-form gauge invariance ($\delta b_{cde} = \partial_{[c}\Lambda_{de]}$) and the states of ghost number 5 corresponding to $\{c_{(5)}^c, v_{(5)}^\alpha, c_{(5)}^{[cd]}\}$ are their respective antifields. The state of ghost number 1 corresponding to $c_{(1)}^c$ is the ghost-for-ghost coming from the gauge invariance of the 2-form gauge parameter ($\delta\Lambda_{de} = \partial_{[d}\Lambda'_{e]}$) and the state of ghost number 6 corresponding to $v_{(6)}^c$ is its respective antifield. The state of ghost number 0 is the ghost-for-ghost-for-ghost coming from the gauge invariance of the 1-form gauge parameter ($\delta\Lambda'_e = \partial_e\Lambda''$) and the state of ghost number 7 corresponding to $c_{(7)}$ is its respective antifield.

Once again, we should find the BRST invariant expressions and then eliminate the new fields, the result will be the state associated in the Q -cohomology. So, for example, the expression $c_{(1)}^c - i\hat{\lambda}\Gamma^c\theta$ is BRST closed⁵⁴, this means that the corresponding state in the Q -cohomology is in $\lambda\Gamma^c\theta$. Using the same line of reasoning one can show that the BRST closed expression for $v_{(2)}^{[bc]}$ is $v_{(2)}^{[bc]} - c_{(1)}^{[b} \hat{\lambda}\Gamma^{c]}\theta + \frac{1}{2}(\hat{\lambda}\Gamma^b\theta)(\hat{\lambda}\Gamma^c\theta)$ and the corresponding state in the Q -cohomology is $\frac{1}{2}(\lambda\Gamma^b\theta)(\lambda\Gamma^c\theta)$, etc.

It must be emphasized that these states will appear in a ghost number three pure spinor superfield belonging to the Q -cohomology with the following ghost numbers:

$$\text{gh.num.}(\text{ghost-for-ghost-for-ghost}) = +3 \quad (4.4.12)$$

$$\text{gh.num.}(\text{ghost-for-ghost}) = +2 \quad (4.4.13)$$

$$\text{gh.num.}(\text{ghost, SUSY ghost, repa. inv. ghost}) = (+1, +1, +1) \quad (4.4.14)$$

$$\text{gh.num.}(\text{graviton, gravitino, 3-form}) = (0, 0, 0) \quad (4.4.15)$$

$$\text{gh.num.}(\text{gluon antifield, gluino antifield, 3-form antifield}) = (-1, -1, -1) \quad (4.4.16)$$

$$\text{gh.num.}(\text{ghost anti., SUSY ghost anti., rep. inv. ghost anti.}) = (-2, -2, -2) \quad (4.4.17)$$

$$\text{gh.num.}(\text{ghost-for-ghost antifield}) = -3 \quad (4.4.18)$$

$$\text{gh.num.}(\text{ghost-for-ghost-for-ghost antifield}) = -4 \quad (4.4.19)$$

This is the field content obtained via BV-quantization of $D = 11$ SUGRA.

As it was done for $D = 10$ SYM, we can find a state $\hat{\Psi}(\tilde{\lambda}, \theta)$ in the (at zero momentum) \hat{Q} -cohomology corresponding to a state $\Psi(\lambda, \theta)$ in the (at zero momentum) Q -cohomology.

⁵⁴This is so because:

$$\hat{Q}(c_{(1)}^c - i\hat{\lambda}\Gamma^c\theta) = \hat{\lambda}\Gamma^c\hat{\lambda} - \hat{\lambda}\Gamma^c\hat{\lambda} = 0$$

where we used the fact that $\{\theta^\alpha, p_\beta\} = -i\delta_\beta^\alpha$.

After replacing the pure spinor condition (that is λ^α will be constrained in this expression), we obtain:

$$\begin{aligned}
\Psi(\lambda, \theta) = & w_{(3)} + (\lambda\theta)^c w_{(2)c} + (\lambda^2\theta^2)^{[cd]} w_{(1)[cd]} + (\lambda^2\theta^2)^c \rho_{(1)c} + (\lambda^2\theta^3)^\alpha \xi_{(1)\alpha} \\
& + (\lambda^3\theta^3)^{[cde]} b_{(0)[cde]} + (\lambda^3\theta^3)^{(cd)} g_{(0)(cd)} + (\lambda^3\theta^4)^{c\alpha} \chi_{(0)c\alpha} \\
& + (\lambda^4\theta^5)^{c\alpha} \chi_{(-1)c\alpha}^* + (\lambda^4\theta^6)^{(cd)} g_{(-1)(cd)}^* + (\lambda^4\theta^6)^{[cde]} b_{(-1)[cde]}^* \\
& + (\lambda^5\theta^6)^\alpha \xi_{(-2)\alpha}^* + (\lambda^5\theta^7)^c \rho_{(-2)c}^* + (\lambda^5\theta^7)^{[cd]} w_{(-2)[cd]}^* \\
& + (\lambda^6\theta^8)^c w_{(-3)c}^* + (\lambda^7\theta^9) w_{(-4)}^*
\end{aligned}$$

where the powers in λ and θ just denote the order of the term in these variables, so for example $(\lambda\theta)^c w_{(2)c}$ should be understood as $(\lambda\Gamma^c\theta)w_{(2)c}$, etc. We can see here in a clearer way the field content at zero momentum of the pure spinor superfield, which coincides with the description mentioned above.

Now we will explain what occurs when we require $\Psi(x, \theta, \lambda)$ to be an element of the Q -cohomology. Let us expand the pure spinor superfield $\Psi(x, \theta, \lambda)$ in powers of λ :

$$\Psi(x, \theta, \lambda) = C_{(3)}(x, \theta) + \lambda^\alpha C_{(2)\alpha}(x, \theta) + \lambda^\alpha \lambda^\beta C_{(1)\alpha\beta}(x, \theta) + \lambda^\alpha \lambda^\beta \lambda^\delta C_{(0)\alpha\beta\delta} + \dots \quad (4.4.20)$$

Let us analyze the ghost number 3 term: $\lambda^\alpha \lambda^\beta \lambda^\delta C_{(0)\alpha\beta\delta}$. This term is fully symmetric in their indices and the condition of physical state on this can be read:

$$Q(\lambda^\alpha \lambda^\beta \lambda^\delta C_{(0)\alpha\beta\delta})(x, \theta) = \lambda^\gamma \lambda^\alpha \lambda^\beta \lambda^\delta D_\gamma C_{(0)\alpha\beta\delta}(x, \theta) = 0 \quad (4.4.21)$$

and

$$\delta C_{(0)\alpha\beta\delta}(x, \theta) = D_{(\alpha} \Lambda_{\beta\delta)}(x, \theta) \quad (4.4.22)$$

where $\Lambda^{\beta\delta}$ is the term at ghost number 2 in the λ -expansion of the superfield $\Lambda(x, \theta, \lambda)$:

$$\Lambda(x, \theta, \lambda) = \dots + \lambda^\alpha \lambda^\beta \Lambda_{\alpha\beta}(x, \theta) + \dots \quad (4.4.23)$$

After doing the θ -expansion of the superfield $C_{\alpha\beta\delta}(x, \theta)$ we find that

$$\begin{aligned}
\lambda^\alpha \lambda^\beta \lambda^\delta C_{\alpha\beta\delta}(x, \theta) = & (\lambda\Gamma^a\theta)(\lambda\Gamma^b\theta)(\lambda\Gamma^c\theta) b_{abc}(x) + (\lambda\Gamma^{(a}\theta)(\lambda\Gamma^{b)c}\theta)(\lambda\Gamma_c\theta) g_{ab}(x) \\
& + (\lambda\Gamma^b\theta)[(\lambda\Gamma^c\theta)(\lambda\Gamma^d\theta)(\Gamma_{cd}\theta)_\alpha - (\lambda\Gamma^{cd}\theta)(\lambda\Gamma_c\theta)(\theta\Gamma_d)_\alpha] \chi_b^\alpha(x) + \dots
\end{aligned}$$

where b_{abc} is a 3-form field, g_{ab} is a symmetric 2-tensor and χ_b^α is a vectorspinor, and \dots contains terms involving more than four θ 's. The corresponding equations of motion and gauge invariances satisfied by these field can be obtained from the equations 4.4.21 and 4.4.22:

$$\partial^d \partial_{[a} b_{bcd]} = 0 \quad , \quad \delta b_{abc} = \partial_{[a} w_{bc]} \quad (4.4.24)$$

$$\partial^a (\partial_a g_{bc} - 2\partial_{(b} g_{c)a}) + \partial_b \partial_c (\eta^{de} g_{de}) = 0 \quad , \quad \delta g_{ab} = \partial_{(a} \rho_{b)} \quad (4.4.25)$$

$$\gamma_{\alpha\beta}^{abc} \partial_b \chi_c^\beta = 0 \quad , \quad \delta \chi_a^\beta = \partial_a \chi^\beta \quad (4.4.26)$$

These are the equations of motion and gauge invariances of linearized $D = 11$ SG fields. It can be shown that after imposing the condition of physical state on $C_{\alpha\beta\delta}(x, \theta)$ the only independent fields which will appear in its θ -expansion are the graviton (g_{bc}), the gravitino (χ_c^α) and the 3-form potential (b_{cde}).

We can also see what the implications are after imposing the condition of physical state on

$\Psi(\lambda, \theta)$ (but now with component fields depending on x). So after using $Q\Psi = 0$ and $\delta\Psi = Q\Lambda$ one finds

$$\partial^a b_{abc}^* = 0 \quad , \quad \delta b_{abc}^* = \partial^d \partial_{[a} \rho_{bcd]} \quad (4.4.27)$$

$$\partial^a g_{ab}^* - \frac{1}{2} \partial_b (\eta^{de} g_{de}^*) = 0 \quad , \quad \delta g_{bc}^* = \partial^a (\partial_a w_{bc} - 2\partial_{(b} w_{c)a}) + \partial_b \partial_c (\eta^{de} w_{de}) \quad (4.4.28)$$

$$\partial^a \chi_{a\beta}^* = 0 \quad , \quad \delta \chi_{a\beta}^* = (\gamma_{abc})_{\alpha\beta} \partial^b \xi^{c\alpha} \quad (4.4.29)$$

and all ghosts and antighosts have trivial cohomology. The result is clear, we have the equations of motion and gauge invariances of linearized $D = 11$ SG in the BV framework.

As we did for $D = 10$ SYM we can propose an action principle from the results found:

$$S = \int d^{11}x \langle \Psi Q\Psi \rangle \quad (4.4.30)$$

where the norm is defined such that $\langle \lambda^7 \theta^9 \rangle = 1$. Once again, this definition picks up the top cohomology making this action invariant under the gauge transformation $\delta\Psi = Q\Lambda$ ⁵⁵. After varying this action in Ψ one finds the equations of motion:

$$Q\Psi = 0 \quad (4.4.31)$$

only for components of $Q\Psi$ involving nine θ 's. Clearly this is not a manifestly supersymmetric formulation of linearized $D = 11$ SG, however it can be shown that the equations involving more than nine θ 's just change auxiliary fields by gauge fields and so they do not affect the physical content of the theory.

⁵⁵This is so because the (scalar) ghost-for-ghost-for-ghost antifield is in the top cohomology, and this one cannot be written as $\Psi = Q\Sigma$ for any Σ .

Chapter 5

Pure Spinor Superfield Formalism

So far we have only studied the linearized versions of $D = 10$ SYM and $D = 11$ SG. We realized that we could obtain the equations of motion and gauge invariances of the corresponding theories from the following action principle:

$$S = \int [dx] \langle \Psi Q \Psi \rangle \quad (5.1)$$

where $\langle \rangle$ was defined adequately for each theory¹, Ψ is a pure spinor superfield² and Q is the usual pure spinor BRST operator. Even though this prescription was studied in this work only for the linearized version of $D = 10$ SYM, it was also constructed an action for non abelian $D = 10$ SYM [19]:

$$S = \int d^{10}x \langle \frac{1}{2} \Psi Q \Psi + \frac{1}{3} \Psi \Psi \Psi \rangle \quad (5.2)$$

where Ψ is a pure spinor superfield in the adjoint representation of the gauge group. This was expected to reproduce the non abelian $D = 10$ SYM theory (in the same manner as 5.2 reproduces the abelian $D = 10$ SYM theory).

We saw that this prescription was not manifestly supersymmetric but it described correctly the (linearized) theories studied. However, we want to have a manifestly supersymmetric way to formulate the interacting theories. In the search for a systematic procedure to construct (pure spinor) actions and at the same time preserving explicitly supersymmetry, we are lead to a new framework which will have many similarities with the BV framework because the field content (matter fields, ghosts, antifields, ghost antifields) within a single pure spinor superfield.

We will start by finding a adequate measure which should present manifest supersymmetry and satisfy certain requirements (depending on each model). It is at this point where the non-minimal version of the pure spinor formalism comes into play [24] [41]. We will not introduce this formalism in a constructive way but just describe the way in which this is useful for our purposes. Besides the usual pure spinor variable (λ^α), we will introduce the *non-minimal* variables $\bar{\lambda}_\alpha$ which will be also a pure spinor variable ($\bar{\lambda} \gamma^m \bar{\lambda} = 0$), and r_α which will be a fermionic spinor and pure relative to $\bar{\lambda}_\alpha$ ($\bar{\lambda} \gamma^m r = 0$). The ghost numbers will be -1 and 0 for $\bar{\lambda}_\alpha$ and r_α respectively. The mass dimension will be $\frac{1}{2}$ for both $\bar{\lambda}_\alpha$ and r_α . The BRST operator will be modified to:

$$Q_{n.m} = Q_m + \bar{w}^\alpha r_\alpha \quad (5.3)$$

¹For SYM $\langle \lambda^3 \theta^5 \rangle = 1$ and for SG $\langle \lambda^7 \theta^9 \rangle = 1$ [19] [23]. It is clear that this measure is not manifestly supersymmetric, in the sense that we just integrate over 3 θ 's (9 θ 's) instead of integrating over 16 θ 's (32 θ 's) for SYM (SG).

²For SYM this pure spinor superfield will be defined to have ghost number $+1$, and for SG this will have ghost number $+3$.

which after using the quartet argument it can be seen that it possesses the same cohomology as that of Q_m . A naive candidate for the measure including these new variables would be:

$$\text{For SYM} \rightarrow [dZ] = d^{10}x d^{16}\theta d^{11}\lambda d^{11}\bar{\lambda} d^{11}r \quad (5.4)$$

$$\text{For SG} \rightarrow [dZ] = d^{11}x d^{32}\theta d^{23}\lambda d^{23}\bar{\lambda} d^{23}r \quad (5.5)$$

where we just considered the independent degrees of freedom of each variable. These measures are manifestly spacetime supersymmetric.

However, we have not still mentioned the requirements that this measure should obey. If we want to construct an action of the type showed in the equation 5.2, we must have³

$$\text{For SYM} \rightarrow \text{gh.num.}([dZ]) = -3 \quad (5.6)$$

$$\text{For SG} \rightarrow \text{gh.num.}([dZ]) = -7 \quad (5.7)$$

and

$$\text{For SYM} \rightarrow \text{mass.dim.}(\frac{1}{g^2} \int d^{10}x d^{16}\theta) = 4 \rightarrow \text{mass.dim.}[dZ^*] = -4 \quad (5.8)$$

$$\text{For SG} \rightarrow \text{gh.num.}(\frac{1}{\kappa^2} \int d^{11}x d^{32}\theta) = 14 \rightarrow \text{mass.dim.}[dZ^*] = -8 \quad (5.9)$$

where $[dZ^*] = [d\lambda][d\bar{\lambda}][dr]$. Clearly the naive measures proposed above does not satisfy these requirements. A way to repair this problem is to propose the following measures

$$\text{For SYM} \rightarrow [dZ] = \lambda^{-3} d^{10}x d^{16}\theta d^{11}\lambda d^{11}\bar{\lambda} d^{11}r \quad (5.10)$$

$$\text{For SG} \rightarrow [dZ] = \lambda^{-7} d^{11}x d^{32}\theta d^{23}\lambda d^{23}\bar{\lambda} d^{23}r \quad (5.11)$$

These are just pictorial expressions. The exact expressions are more complicated [24] [30] [36] but the structure showed above coincide with those showed in the references cited. In addition to this we will need to regularize the integrals. To this end we introduce a BRST invariant regularization factor $e^{-\{Q,\chi\}}$. After choosing $\chi = \theta^\alpha \bar{\lambda}_\alpha$, we obtain the regularization factor: $e^{-(\lambda\bar{\lambda}-\theta r)}$. We see that 11 (23) r 's will saturate the fermionic integral ($\int d^{11}r$) and at the same time 11 (23) θ 's will be integrated out. We will be just left with 5 (9) integrals in θ which will pick out the top cohomology as before.

We will define the *antibracket* looking at the field-antifield symmetry of the BRST cohomology, this suggests us to think Ψ as a self-conjugate field with respect to this antibracket. Therefore a natural definition is

$$(A, B) = \int A \overleftarrow{\frac{\delta}{\delta\Psi(Z)}} [dZ] \overrightarrow{\frac{\delta}{\delta\Psi(Z)}} B \quad (5.12)$$

Our guiding principle to construct actions in this framework will be the *master equation*:

$$(\mathcal{S}, \mathcal{S}) = 0 \quad (5.13)$$

where \mathcal{S} will be called the (*pure spinor*) *master action*.

In this way, we can construct manifestly supersymmetric actions for the maximally supersymmetric theories studied in this work. For SYM we obtain:

$$\mathcal{S} = \int [dZ] \left(\frac{1}{2} \Psi Q \Psi + \frac{1}{3} \Psi^3 \right) \quad (5.14)$$

³We must keep in mind that the ghost number and mass dimension of the pure spinor superfield is 1 (3) and 0 (-3) for SYM (SG), respectively.

which leads to equations of motion

$$Q\Psi + \Psi^2 = 0 \quad (5.15)$$

which at ghost number zero level yields $\lambda^\alpha \lambda^\beta F_{\alpha\beta} = 0^4$.

For SG the analysis is more complicated [30], [31], [32], [33], [36]. Basically the idea is to apply the master equation and to get all of the terms allowed by this equation. However we have two ways to describe linearized D=11 SG by using pure spinor superfields. One of them is by using a vector pure spinor superfield $\Phi^a(x, \theta, \lambda)^5$ and the other one is by using a scalar pure spinor superfield $\Psi(x, \theta, \lambda)$. The cohomologies at zero momentum of each one of these fields can be seen in [35]. Basically they have the following properties:

1. Φ^a has to satisfy the additional condition $\Phi^a \approx \Phi^a + (\lambda\gamma^a\rho)$ (shift symmetry [30] [34]), where ρ is any spinor, to reproduce the right spectrum. Its cohomology at zero momentum contains the superdiffeomorphism ghosts. All of the supergravity fields appear in ghost number zero. However, the (bosonic) 3-form C enters this description just through its (super) field strength 4-form H . There is no symmetry in this description between fields and antifields.
2. Ψ presents a cohomology at zero momentum which also contains all the ghosts and higher order ghosts relevant for the tensor gauge symmetries and superdiffeomorphisms. All of the supergravity fields appear in ghost number zero. Here, the (bosonic) 3-form C enters this description with a 3-form superfield C . There exists symmetry in this description between fields and antifields.

For these reasons and also the fact that we want to construct a Chern-Simons-like term (inspired by the (bosonic) analogous term in the component action), the field Ψ will be considered as a fundamental field. So the field Φ^a should be obtained from Ψ . It is precisely the operator R^a , constructed in [30] which makes this work:

$$\Phi^a = R^a\Psi \quad (5.16)$$

which must have ghost number -2 and mass dimension 2 .

For the 3-point coupling (Chern-Simons term) we propose⁶

$$(\lambda\gamma_{ab}\lambda)\Psi\Phi^a\Phi^b \quad (5.17)$$

which has the correct properties: ghost number 7 , mass dimension -6 and shift symmetry (because in $D = 11$: $(\lambda\gamma_{ab}\lambda)(\gamma^a\lambda) = 0$). So we have a good candidate (indeed this will be the correct one) for the 3-point coupling, and so our action would be:

$$\mathcal{S} = \int [dZ] \left(\frac{1}{2} \Psi Q \Psi + \frac{1}{6} (\lambda\gamma_{ab}\lambda) \Psi \Phi^a \Phi^b \right) \quad (5.18)$$

Now we must use the master equation and check if this action is correct (that is if it is a master action or not). When this is done, one finds that a new term has to be introduced: The 4-point

⁴This is the dynamical constraint which is used on superspace formulation to obtain the known SYM theory.

⁵This superfield has ghost number 1 and mass dimension -1 .

⁶A naive guess would be $\Psi\Phi^2$. However this quantity does not satisfy the properties (ghost number and mass dimension) studied above by analyzing the kinetic term $(\Psi Q \Psi)$. This expression has ghost number 5 and mass dimension -5 .

coupling. This term will involve a new nilpotent operator T whose properties will ensure that the master action is satisfied to all orders. Therefore, our full (master) action is:

$$S = \int [dZ] \left[\frac{1}{2} \Psi Q \Psi + \frac{1}{6} (\lambda \gamma_{ab} \lambda) \left(1 - \frac{3}{2} T \Psi \right) \Psi R^a \Psi R^b \Psi \right] \quad (5.19)$$

For a detailed discussion of the 4-point coupling we refer to ref [31].

Chapter 6

Future work

We have arrived at a framework which allows us to formulate action principles with manifest supersymmetry for maximally supersymmetric gauge theories ($D = 10$ SYM and $D = 11$ SG). One of the many questions that can arise at this point is how this formalism can be used to compute scattering amplitudes. In order to answer this one should gauge fix the master action as one usually does in the BV-formalism¹. However in the pure spinor superfield formalism we cannot use the same gauge fixing procedure as in the BV formalism because our pure spinor superfield is self-conjugate². So we need another mechanism to gauge fix the master action. Inspired by string theory we usually use the Siegel gauge:

$$b\Psi = 0 \tag{6.1}$$

where b is the famous b -ghost, which must obey the condition:

$$\{Q, b\} = \square \tag{6.2}$$

This b -ghost usually comes up after gauge fixing reparameterization invariance of a theory (it is in this way how this b -ghost with the properties 6.1 and 6.2 arises in bosonic string theory). However in our model we do not have such a symmetry which after gauge fixing provide us a b -ghost. Nevertheless a composite operator³ satisfying the b -ghost properties can be constructed. For $D = 10$ SYM this b -ghost was constructed in [24] and for $D = 11$ SG this b -ghost was constructed in [36]. Once we have gauge fixed the theory we are ready to compute scattering amplitudes. Therefore our future work will be addressed to the following issues:

1. We have formulated the pure spinor superfield framework for maximally supersymmetric gauge theories. So the natural question is: How about theories with less supersymmetry? This is an interesting question and it could help us to understand better how this new framework works.
2. The action proposed for $D = 11$ SG 5.19 is consistent with all of the requisites of our new framework. However it has not been proved yet if this action provide us the well-known equations of motion for $D = 11$ SG (even though it has already been showed that this action reproduces the Chern-Simons term of $D = 11$ SG, as it can be seen in [31]).

¹In our short review of the BV formalism we saw that the gauge fixing is given by $\Phi_I = \frac{\delta_L S}{\delta \Phi^I}$.

²That is the pure spinor superfield contains both field and antifields, and they cannot be gauge fixed unless the pure spinor superfield is separated into components.

³This means an operator which depends on the variables of the theory.

3. Usually the (negative ghost number) operators constructed in this formalism are expressions very complicated (for instance the b -ghost in $D = 11$ SG [36]). This makes the computation of scattering amplitudes be a mess. If we achieved to simplify those expressions it would be a nice thing for the calculation of scattering amplitudes by using this framework.
4. We can try to use this formalism to calculate the 4-point amplitude in 7 loops⁴ and try to figure out how is the ultraviolet behaviour of $D = 11$ SG. It is usually assumed that in 7 loops $D = 11$ SG present divergences.

⁴The 4-point amplitude in 6 loops was already done in [36].

Chapter 7

Conclusions

We have studied in detail the maximally supersymmetric models (abelian) $D = 10$ SYM and $D = 11$ SG in their component and superspace formulations. We saw that after imposing some adequate constraints on some superfields, these formulations turned out to be equivalent. The study of the standard $D = 10$ ($D = 11$) superparticle gave us the spectrum of $D = 10$ SYM ($D = 11$ SG). We introduced the pure spinor version for the $D = 10$ superparticle by introducing a new gauge symmetry and a new couple of fermionic fields in the gauge fixed version of the Brink-Schwarz superparticle. After gauge fixing and proving equivalences among cohomologies we were able to get the $D = 10$ pure spinor superparticle together with its BRST operator. The quantization of this model gave us the same spectrum as that obtained for linearized $D = 10$ SYM by using the BV formalism. Inspired by the $D = 10$ case, we introduced the $D = 11$ semi pure spinor superparticle and its corresponding BRST operator. The quantization of this theory gave us the same spectrum as that obtained for linearized $D = 11$ SG by using the BV formalism. A natural action principle arose from these studies and together with the search for manifest supersymmetry we were led to the pure spinor superfield formalism which uses non-minimal variables and pure spinor superfields. In this way it could be constructed manifestly supersymmetric actions for these maximally supersymmetric theories. The existence of an operator b -ghost-like plays an essential role in the search for a gauge fixing condition and so for the computation of scattering amplitudes.

Appendix A

Spinors in $D=(9,1)$ and $D=(10,1)$

We will start studying spinors in $D = (9, 1)$ and follow the conventions discussed in [25] and [40]. We construct our real gamma matrices Γ^m in $D = (9, 1)$ as follows: $\Gamma^m = (I \otimes (i\tau_2), \sigma^i \otimes \tau_1, \chi \otimes \tau_1)$, where I is the 8×8 identity matrix, τ_1, τ_2, τ_3 are the well-known Pauli matrices, σ^i are eight real symmetric 16×16 off-diagonal Dirac matrices for $D = (8, 0)$ and χ is the corresponding real 16×16 diagonal chirality matrix in $D = (8, 0)$ ¹. More explicitly:

$$\Gamma^m = \begin{pmatrix} 0 & (\sigma^m)^{\alpha\dot{\beta}} \\ (\sigma^m)_{\dot{\beta}\gamma} & 0 \end{pmatrix} \quad (\text{A.1})$$

where $\sigma^m = \{I, \sigma^i, \chi\}$ and $\tilde{\sigma}^m = \{-I, \sigma^i, \chi\}$. Note that we are using two kinds of indices: dotted ($\dot{\alpha}$) and undotted (α) indices. However, later we will show that it just suffices to use one kind of index and the only important thing is its position (up or down). Now, we can find the chirality matrix for $D = (9, 1)$:

$$\begin{aligned} \Gamma &= \Gamma^0 \dots \Gamma^9 \\ \Gamma &= (I \otimes (i\tau_2)) \cdot (\sigma^1 \otimes \tau_1) \dots (\sigma^8 \otimes \tau_1) \cdot (\chi \otimes \tau_1) \\ \Gamma &= \chi^2 \otimes (i\tau_2\tau_1) \\ \Gamma &= I \otimes \tau_3 \end{aligned}$$

where we used that $\tau_i\tau_j = \delta_{ij} + i\epsilon_{ijk}\tau_k$ and $(\tau_1)^2 = I$. Explicitly,

$$\Gamma = \begin{pmatrix} I_{16 \times 16} & 0 \\ 0 & -I_{16 \times 16} \end{pmatrix} \quad (\text{A.2})$$

We can also construct the charge conjugation matrix (C). We know that by definition this matrix must satisfy $C\Gamma^m = -\Gamma^{m,T}C$. It is not difficult to see that $C = \Gamma^0$ is the desired matrix. So,

$$C = \begin{pmatrix} 0 & \delta_{\alpha}^{\dot{\beta}} \\ -\delta_{\dot{\beta}}^{\gamma} & 0 \end{pmatrix} \quad (\text{A.3})$$

At this point, we can see why we cannot raise or lower indices in $D = (9, 1)$ as follows: If we write an arbitrary spinor $\Psi = (\lambda^\alpha, \xi_{\dot{\beta}})$, we can see that we have Weyl and anti-Weyl spinors, λ^α and $\xi_{\dot{\beta}}$ respectively. Let us show that these representations of $SO(9, 1)$ are inequivalent. The Lorentz generators are:

$$L^{mn} = \frac{1}{2}(\Gamma^m\Gamma^n - \Gamma^n\Gamma^m) = \begin{pmatrix} \frac{1}{2}\sigma^{m,\alpha\dot{\beta}}\tilde{\sigma}_{\dot{\beta}\gamma}^n - m \leftrightarrow n & 0 \\ 0 & \frac{1}{2}\tilde{\sigma}_{\dot{\alpha}\beta}^m\sigma^{n,\beta\dot{\gamma}} - m \leftrightarrow n \end{pmatrix} \quad (\text{A.4})$$

¹Explicitly $\chi = \sigma^1 \dots \sigma^8$, and satisfies the properties $\chi^T = \chi$, $\chi^2 = 1$.

We should remember that $\sigma^m = \{I, \sigma^i, \chi\}$ and $\tilde{\sigma}^m = \{-I, \sigma^i, \chi\}$. Therefore we get

$$L_{chiral}^{oi} = \sigma^i \quad , \quad L_{chiral}^{09} = \chi \quad (A.5)$$

$$L_{antichiral}^{oi} = -\sigma^i \quad , \quad L_{antichiral}^{09} = -\chi \quad (A.6)$$

$$(A.7)$$

So, if we assume that there exists a matrix S which satisfies $SL_{chiral}^{0m} = L_{antichiral}^{0m}S$ then $S\sigma^i = -\sigma^iS$ and $S\chi = -\chi S$. But $\chi = \sigma^1 \dots \sigma^8 \rightarrow S\chi = +\chi S$, therefore we obtain a contradiction. Hence, these representations are inequivalent. They are going to be denoted by 16 and 16' (λ^α and $\xi_{\dot{\beta}}$, respectively). As usual², one defines the conjugate spinor as $\Psi^C = C\Psi$, and forms the Lorentz-invariant quantity $\Psi^T\Psi^C$, which in this case is identically zero³. For this reason, this prescription does not work in $D = (9, 1)$, and so we cannot raise or lower indices.

Usually we will need to work with $C\Gamma^m$:

$$C\Gamma^m = \begin{pmatrix} (\tilde{\sigma}^m)_{\alpha\beta} & 0 \\ 0 & -(\sigma^m)^{\dot{\beta}\dot{\alpha}} \end{pmatrix} \equiv \begin{pmatrix} (\gamma^m)_{\alpha\beta} & 0 \\ 0 & (\gamma^m)^{\dot{\beta}\dot{\alpha}} \end{pmatrix} \quad (A.8)$$

These are the real 16×16 symmetric matrices which have been used throughout all of this work. Now, we are going to explain why we can use just one kind of index, that is to say there is no difference between dotted and undotted indices (we just have to worry in their corresponding positions). As we explained above, we want to construct Lorentz-invariant quantities (this is the motivation to construct "a metric" $\begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\beta}\dot{\alpha}} \end{pmatrix}$ in D=4, which allows us to raise and lower indices). When we used the conjugation matrix, we saw that it does not work. So, now let us start by saying that k_α and $\eta^{\dot{\alpha}}$ are spinors which transform under Lorentz transformations such that $k_\alpha\lambda^\alpha$ and $\eta^{\dot{\alpha}}\xi_{\dot{\alpha}}$ are Lorentz-invariant objects. If we denote the generators which act on λ^α by (γ^{kl}, γ^k) ($k = 1, \dots, 8$), then the generators acting on $\xi_{\dot{\beta}}$ are $(\gamma^{kl}, -\gamma^k)$. Now we will have the Lorentz-invariant objects mentioned above only if k_α transforms with the generators $(\gamma^{kl}, -\gamma^k)$ and $\eta^{\dot{\alpha}}$ with the generators (γ^{kl}, γ^k) : Let us check this:

$$\begin{aligned} k_\alpha\lambda^\alpha \rightarrow k'_\alpha\lambda'^\alpha &= k_\lambda(\delta_\delta^\lambda + (\gamma^{kl,T})^\lambda_\delta w_{kl} - \theta_k(\gamma^{k,T})^\lambda_\delta)(\delta_\alpha^\delta + (\gamma^{pq})^\delta_\alpha w_{pq} + \theta_p(\gamma^p)^\delta_\alpha)\lambda^\alpha \\ &= k_\alpha\lambda^\alpha - k_\lambda(\gamma^{kl})^\lambda_\delta w_{kl}\lambda^\delta - k_\lambda\theta_k(\gamma^k)^\lambda_\delta\lambda^\delta + k_\lambda(\gamma^{pq})^\lambda_\alpha w_{pq}\lambda^\alpha + k_\lambda\theta_p(\gamma^p)^\lambda_\alpha\lambda^\alpha \\ &= k_\alpha\lambda^\alpha \end{aligned}$$

at first order in the parameters of the transformation. The same procedure for the dotted indices:

$$\begin{aligned} \eta^{\dot{\alpha}}\xi_{\dot{\alpha}} \rightarrow \eta'^{\dot{\alpha}}\xi'_{\dot{\alpha}} &= \eta^\lambda(\delta_\lambda^{\dot{\alpha}} + (\gamma^{kl,T})_\lambda^{\dot{\alpha}} w_{kl} + \theta_k(\gamma^{k,T})_\lambda^{\dot{\alpha}})(\delta_{\dot{\alpha}}^{\dot{\beta}} + (\gamma^{pq})_{\dot{\alpha}}^{\dot{\beta}} w_{pq} - \theta_p(\gamma^p)_{\dot{\alpha}}^{\dot{\beta}})\xi_{\dot{\beta}} \\ &= \eta^{\dot{\alpha}}\xi_{\dot{\alpha}} - \eta^\lambda(\gamma^{kl})_\lambda^{\dot{\alpha}}\xi_{\dot{\alpha}}w_{kl} + \eta^\lambda\theta_k(\gamma^k)_\lambda^{\dot{\alpha}}\xi_{\dot{\alpha}} + \eta^{\dot{\alpha}}(\gamma^{pq})_{\dot{\alpha}}^{\dot{\beta}}\xi_{\dot{\beta}}w_{pq} - \eta^\lambda\theta_p(\gamma^p)_\lambda^{\dot{\beta}}\xi_{\dot{\beta}} \\ &= \eta^{\dot{\alpha}}\xi_{\dot{\alpha}} \end{aligned}$$

at first order in the parameters of the transformation.

Summarizing:

$$\begin{aligned} \lambda^\alpha \text{ and } \eta^{\dot{\alpha}} \text{ transform in the same way under Lorentz transformations} &: (\gamma^{kl}, \gamma^k) \\ \lambda_{\dot{\alpha}} \text{ and } \eta_\alpha \text{ transform in the same way under Lorentz transformations} &: (\gamma^{kl}, -\gamma^k) \end{aligned}$$

²This is done in $D = 4$ as it can be seen in [4].

³The fact that this procedure works in D=4, is because the chirality matrix is diagonal.

Therefore we will just write undotted indices but taking into account the position of them (upper and lower).

Now let us move on to the $D = (10, 1)$ case. We will denote $SO(10, 1)$ vector indices by a, b, \dots and $SO(9, 1)$ vector indices by m, n, \dots . Also, we will denote $SO(10, 1)$ spinor indices by α, β, \dots and $SO(9, 1)$ spinor indices by μ, ν, \dots ⁴. As usual, we should add a new matrix (Γ^{10}) to the previous set of Γ -matrices ($\{\Gamma^m\}$). This new Γ -matrix will be numerically equal to the chirality matrix in $D = (9, 1)$ (which will be denoted by $\Gamma^{(9,1)}$):

$$\Gamma^{10} = \Gamma^{(9,1)} = \begin{pmatrix} I_{16 \times 16} & 0 \\ 0 & -I_{16 \times 16} \end{pmatrix} \quad (\text{A.9})$$

We have already seen that this matrix satisfies the properties $\{\Gamma^m, \Gamma^{10}\} = 0$ for $m = 0, \dots, 9$, and $(\Gamma^{10})^2 = 1$. To calculate the chirality matrix (in $D = (10, 1)$), which will be denoted just by Γ , we should compute:

$$\Gamma = \Gamma^0 \Gamma^1 \dots \Gamma^9 \Gamma^{10} = \Gamma^{(9,1)} \Gamma^{10} = (\Gamma^{10})^2 = 1 \quad (\text{A.10})$$

so we do not have Weyl (Anti-Weyl) spinors in this case. However, it is clear that we have Majorana spinors (because of the reality of the Γ -matrices).

Now, by using the definition of the charge conjugation matrix $C\Gamma^a = -(\Gamma^a)^T C$, we find that $C = \Gamma^0$ satisfies this requirement⁵.

Usually one uses the following property which is satisfied by Majorana spinors: $\bar{\Theta} \Gamma^a \Psi = \Theta^T C \Gamma^a \Psi$, which can be viewed in terms of its $SO(9, 1)$ components:

$$\Theta^T C \Gamma^m \Psi = (\Theta^\mu \quad \Theta_\mu) \begin{pmatrix} \gamma_{\mu\nu}^m & 0 \\ 0 & (\gamma^m)^{\mu\nu} \end{pmatrix} (\Psi^\nu \quad \Psi_\nu) \quad (\text{A.11})$$

where $\gamma_{\mu\nu}^m$ and $(\gamma^m)^{\mu\nu}$ are the $SO(9, 1)$ γ -matrices, and also

$$\Theta^T C \Gamma^{10} \Psi = (\Theta^\mu \quad \Theta_\mu) \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} (\Psi^\nu \quad \Psi_\nu) \quad (\text{A.12})$$

It is useful to mention that the index structure of the charge conjugation matrix is: $C_{\alpha\beta}$. So the Γ -matrices have the index structure $(\Gamma^m)^\alpha_\beta$ and when are multiplied by the charge conjugation matrix (or its inverse) we obtain the corresponding matrices $(\Gamma^m)_{\alpha\beta}$ and $(\Gamma^m)^{\alpha\beta}$.

To finish this appendix we will show explicitly the form of some relevant Γ -matrices [40].

⁴Be careful with this notation, this does nothing to do with the *Einstein* and *Lorentz* indices in a curved space, explained in the Appendix C. This notation is adopted just to distinguish objects in different dimensions ($D = (9, 1)$ and $D = (10, 1)$).

⁵We already saw that in the case $D = (9, 1)$, $C^{(9,1)} = \Gamma^0$ is the charge conjugation matrix, so what remains to show is that $C = \Gamma^0$ obeys $C\Gamma^{10} = -(\Gamma^{10})^T C$, which is trivial from the fact that Γ^{10} is symmetric and $\{\Gamma^{10}, \Gamma^0\} = 0$.

For $D = (9, 1)$, we have:

$$\begin{aligned}
(\gamma^0)^{\alpha\beta} &= \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & 1_{8 \times 8} \end{pmatrix} & (\gamma^0)_{\alpha\beta} &= \begin{pmatrix} -1_{8 \times 8} & 0 \\ 0 & -1_{8 \times 8} \end{pmatrix} \\
(\gamma^9)^{\alpha\beta} &= \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & -1_{8 \times 8} \end{pmatrix} & (\gamma^9)_{\alpha\beta} &= \begin{pmatrix} 1_{8 \times 8} & 0 \\ 0 & -1_{8 \times 8} \end{pmatrix} \\
(\gamma^i)^{\alpha\beta} &= \begin{pmatrix} 0 & \sigma_{aa}^i \\ \sigma_{bb}^i & 0 \end{pmatrix} & (\gamma^i)_{\alpha\beta} &= \begin{pmatrix} 0 & \sigma_{aa}^i \\ \sigma_{bb}^i & 0 \end{pmatrix} \\
(\gamma^+)^{\alpha\beta} &= \begin{pmatrix} \sqrt{2}_{8 \times 8} & 0 \\ 0 & 0 \end{pmatrix} & (\gamma^+)_{\alpha\beta} &= \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{2}_{8 \times 8} \end{pmatrix} \\
(\gamma^-)^{\alpha\beta} &= \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}_{8 \times 8} \end{pmatrix} & (\gamma^-)_{\alpha\beta} &= \begin{pmatrix} -\sqrt{2}_{8 \times 8} & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

where γ^\pm are defined by

$$\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9)$$

and the σ^i matrices are defined by

$$\begin{aligned}
\sigma_{aa}^1 &= \epsilon \otimes \epsilon \otimes \epsilon & \sigma_{aa}^5 &= \tau^3 \otimes \epsilon \otimes 1 \\
\sigma_{aa}^2 &= 1 \otimes \tau^1 \otimes \epsilon & \sigma_{aa}^6 &= \epsilon \otimes 1 \otimes \tau^1 \\
\sigma_{aa}^3 &= 1 \otimes \tau^3 \otimes \epsilon & \sigma_{aa}^7 &= \epsilon \otimes 1 \otimes \tau^3 \\
\sigma_{aa}^4 &= \tau^1 \otimes \epsilon \otimes 1 & \sigma_{aa}^8 &= 1 \otimes 1 \otimes 1
\end{aligned}$$

where $\epsilon = i\tau^2$ and τ^1, τ^2, τ^3 are the usual Pauli matrices. They are symmetric ($\sigma_{aa}^i = (\sigma_{aa}^i)^T$) and satisfy the following relations:

$$\begin{aligned}
\sigma_{aa}^i \sigma_{ab}^j + \sigma_{aa}^j \sigma_{ab}^i &= 2\delta^{ij} \delta_{ab} \\
\sigma_{aa}^i \sigma_{ab}^j + \sigma_{aa}^j \sigma_{ab}^i &= 2\delta^{ij} \delta_{ab} \\
\sigma_{ab}^i \sigma_{ac}^i + \sigma_{aa}^i \sigma_{bc}^i &= 2\delta_{ab} \delta_{ac}
\end{aligned}$$

and for $D = (10, 1)$, we have:

$$\begin{aligned}
(\Gamma^+)^{\alpha\beta} &= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{2} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \\
(\Gamma^-)^{\alpha\beta} &= \begin{pmatrix} \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{pmatrix} \end{pmatrix} \\
(\Gamma^+)_{\alpha\beta} &= \begin{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \end{pmatrix} \\
(\Gamma^-)_{\alpha\beta} &= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}
\end{aligned}$$

Appendix B

Solution of the pure spinor constraint

We have to solve the equation:

$$\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0 \quad (\text{B.1})$$

where λ^α is a complex bosonic $D = (9, 1)$ Weyl spinor. Let us define the following matrices:

$$a_1 = \frac{1}{2}(\Gamma^1 + i\Gamma^2) , \quad a_2 = \frac{1}{2}(\Gamma^3 + i\Gamma^4) , \quad \dots , \quad a_5 = \frac{1}{2}(\Gamma^9 - \Gamma^0) \quad (\text{B.2})$$

$$a^1 = \frac{1}{2}(\Gamma^1 - i\Gamma^2) , \quad a^2 = \frac{1}{2}(\Gamma^3 - i\Gamma^4) , \quad \dots , \quad a^5 = \frac{1}{2}(\Gamma^9 + \Gamma^0) \quad (\text{B.3})$$

So we see that $a^j = a_j^\dagger$. It is easy to show that $\{a_i, a^j\} = \delta_i^j$. If we define the vacuum by $|0\rangle$ with $a_i|0\rangle = 0$ for $i = 1, \dots, 5$, then we can construct the following states: $|0\rangle, a^i|0\rangle, a^i a^j|0\rangle, a^i a^j a^k|0\rangle, a^i a^j a^k a^l|0\rangle, a^i a^j a^k a^l a^m|0\rangle$, which give us a total of $1+5+10+10+5+1 = 32$ independent states². In order to adopt a convention of normalization, we make the following definition:

$$\text{If } |A\rangle = a^{i_1} \dots a^{i_k} |0\rangle \rightarrow \langle A| = \langle 0| a_{i_k} \dots a_{i_1} \quad (\text{B.4})$$

In this way, $\langle A|B\rangle = \delta_B^A$. This convention allows us to write the completeness property in the form $\sum_C |C\rangle \langle C| = I$. So the matrix elements $\langle B|a^j|A\rangle = (a^j)_A^B$ form a representation of the Clifford algebra³. Now, let us express the chirality matrix in terms of these new variables a_i and a^i :

$$\Gamma = \Gamma^0 \Gamma^1 \dots \Gamma^9 \quad (\text{B.5})$$

$$\Gamma = -\Gamma^1 \dots \Gamma^9 \Gamma^0 \quad (\text{B.6})$$

$$\Gamma = -(a_1 + a^1)(a_1 - a^1) \dots (a_5 + a^5)(a_5 - a^5) \quad (\text{B.7})$$

but $(a_1 + a^1)(a_1 - a^1) = a^1 a^1 + a^1(-a_1) + a_1 a^1 - a_1 a_1 = 2a_1 a^1 - 1$. Therefore,

$$\Gamma = -(2a_1 a^1 - 1)(2a_2 a^2 - 1)(2a_3 a^3 - 1)(2a_4 a^4 - 1)(2a_5 a^5 - 1) \quad (\text{B.8})$$

¹We just have to use the Clifford algebra $\{\Gamma^m, \Gamma^n\} = 2\eta^{mn}$.

²We can do the same for a bra vacuum $\langle 0|$, with $\langle 0|a^i = 0$. We create states with the operators $a_i: \langle 0|, \dots, \langle 0|a_i a_j a_k a_l a_m$, and we also obtain a total of 32 independent states.

³In the sense that $\{(a_i), (a^j)\}_A^B = (\delta_i^j)_A^B$.

Just for consistency, let us check that this matrix obeys the usual property of a chirality matrix, that is $\Gamma^2 = I^4$:

$$\Gamma^2 = (2a_1a^1 - 1) \dots (2a_5a^5 - 1)(2a_1a^1 - 1) \dots (2a_5a^5 - 1) \quad (\text{B.9})$$

$$\Gamma^2 = (2a_1a^1 - 1)^2 \dots (2a_5a^5 - 1)^2 \quad (\text{B.10})$$

$$\Gamma^2 = 1.1 \dots 1 \quad (\text{B.11})$$

$$\Gamma^2 = 1 \quad (\text{B.12})$$

where we used $(2a_1a^1 - 1)^2 = (2a_1a^1 - 1)(2a_1a^1 - 1) = 4a_1a^1a_1a^1 - 2a_1a^1 - 2a_1a^1 + 1 = 4a_1a^1 - 4a_1a^1 + 1 = 1$, etc. In addition to this, we can show that $\{(2a_1a^1 - 1), a^1\}0(2a_1a^1 - 1)a^1 + a^1(2a_1a^1 - 1) = 2a_1(a^1)^2 - a^1 + 2a^1a_1a^1 - a^1 = -2a^1 + 2a^1 = 0$ and the same for $\{(2a_1a^1 - 1), a_1\} = 0$. This allows us to show the following result:

$$\{\Gamma, a^1\} = (2a_1a^1 - 1) \dots (2a_5a^5 - 1)a^1 + a^1(2a_1a^1 - 1) \dots (2a_5a^5 - 1) \quad (\text{B.13})$$

$$\{\Gamma, a^1\} = \{(2a_1a^1 - 1), a^1\}(2a_2a^2 - 1) \dots (2a_5a^5 - 1) \quad (\text{B.14})$$

$$\{\Gamma, a^1\} = 0 \quad (\text{B.15})$$

and the same is valid for a_1 and the other operators a_2, a^2 , etc. Now, let us see how Γ acts on the vacuum $|0\rangle$:

$$\Gamma|0\rangle = (2a_1a^1 - 1) \dots (2a_5a^5 - 1)|0\rangle \quad (\text{B.16})$$

$$= (2a_1a^1 - 1) \dots (2 - 2a^5a_5 - 1)|0\rangle \quad (\text{B.17})$$

$$= (2a_1a^1 - 1) \dots (2a_4a^4 - 1)|0\rangle \quad (\text{B.18})$$

$$\rightarrow \Gamma|0\rangle = |0\rangle \quad (\text{B.19})$$

Therefore the vacuum $|0\rangle$ has positive chirality, and so (with our conventions) is a Weyl spinor. Also, it can be seen that the matrix representation of $\Gamma^1, \Gamma^3, \Gamma^5, \Gamma^7, \Gamma^9$ is real and symmetric, of $\Gamma^2, \Gamma^4, \Gamma^6, \Gamma^8$ is purely imaginary and antisymmetric and of Γ^0 is real and antisymmetric. Let us see this:

$$\langle A|a^j \pm a_j|B\rangle = \langle B|\pm a^j + a_j|A\rangle \quad (\text{B.20})$$

$$\rightarrow (\Gamma^1)_B^A = \langle A|a^1 + a_1|B\rangle = \langle B|a^1 + a_1|A\rangle = (\Gamma^1)_A^B \quad (\text{B.21})$$

$$\rightarrow (\Gamma^2)_B^A = -i \langle A|a^1 - a_1|B\rangle = -i \langle B|-a^1 + a_1|A\rangle = -(\Gamma^2)_A^B \quad (\text{B.22})$$

⁴This can be seen immediately from the definition of Γ :

$$\begin{aligned} \Gamma^2 &= \Gamma^1\Gamma^2 \dots \Gamma^9\Gamma^0\Gamma^1\Gamma^2 \dots \Gamma^9\Gamma^0 \\ \Gamma^2 &= (-1)^9(\Gamma^1)^2(-1)^8(\Gamma^2)^2 \dots (-1)^1(\Gamma^0)^2 \\ \Gamma^2 &= (-1)^{\frac{9 \cdot 10}{2}}(-1) \\ \Gamma^2 &= 1 \end{aligned}$$

We can also see that $\Gamma^\dagger = \Gamma$ as follows:

$$\begin{aligned} \Gamma^\dagger &= (\Gamma^0\Gamma^1 \dots \Gamma^9)^\dagger \\ \Gamma^\dagger &= \Gamma^9\Gamma^8 \dots \Gamma^1(-\Gamma^0) \\ \Gamma^\dagger &= (-1) \cdot (-1)^{10} \cdot \Gamma^8 \dots \Gamma^1\Gamma^0\Gamma^9 \\ \Gamma^\dagger &= (-1) \cdot (-1)^{10} \cdot (-1)^9 \cdot \Gamma^7 \dots \Gamma^1\Gamma^0\Gamma^8\Gamma^9 \\ \Gamma^\dagger &= (-1) \cdot (-1)^{\frac{10 \cdot 9}{2}} \Gamma^0\Gamma^1 \dots \Gamma^9 \\ \Gamma^\dagger &= \Gamma^0\Gamma^1 \dots \Gamma^9 \\ \Gamma^\dagger &= \Gamma \end{aligned}$$

where we used the fact that $(\Gamma^0)^\dagger = -\Gamma^0$ and $(\Gamma^i)^\dagger = \Gamma^i$.

and so on.

In this representation matrix of Γ matrices, the charge conjugation matrix is given by $C = -\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0 = (a_1 - a^1)(a_2 - a^2) \dots (a_5 - a^5)$, where the sign in front of the product of gamma matrices is just a convention matter. Let us check that indeed this C satisfies $C\Gamma^m = -\Gamma^{m,T}C$. Let us start with the odd matrices $\Gamma^1, \Gamma^3, \Gamma^5, \Gamma^7, \Gamma^9$:

$$\begin{aligned}\Gamma^1 C &= \Gamma^1(-\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0) = -(-\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0)\Gamma^1 = -C\Gamma^1 \\ &\vdots \\ \Gamma^9 C &= \Gamma^9(-\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0) = -(-\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0)\Gamma^9 = -C\Gamma^9\end{aligned}$$

In addition, we already showed that these odd matrices are symmetric. Therefore:

$$C\Gamma^m = -\Gamma^{m,T}C, \quad \text{for } m = 1, 3, 5, 7, 9 \quad (\text{B.23})$$

Now for the even matrices $\Gamma^2, \Gamma^4, \Gamma^6, \Gamma^8$:

$$\begin{aligned}\Gamma^2 C &= \Gamma^2(-\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0) = -\Gamma^4\Gamma^6\Gamma^8\Gamma^0\Gamma^2\Gamma^2 = (-\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0)\Gamma^2 = C\Gamma^2 \\ &\vdots \\ \Gamma^8 C &= \Gamma^8(-\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0) = +\Gamma^2\Gamma^4\Gamma^6\Gamma^0\Gamma^8\Gamma^8 = (-\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0)\Gamma^8 = C\Gamma^8\end{aligned}$$

In addition, we already showed that these even matrices are antisymmetric. Therefore:

$$C\Gamma^m = -\Gamma^{m,T}C, \quad \text{for } m = 2, 4, 6, 8 \quad (\text{B.24})$$

and finally with Γ^0 , we see that:

$$\begin{aligned}\Gamma^0 C &= \Gamma^0(-\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0) \\ &= (-1)\Gamma^2\Gamma^4\Gamma^6\Gamma^8(-1) \\ &= -\Gamma^2\Gamma^4\Gamma^6\Gamma^8\Gamma^0\Gamma^0 \\ &= C\Gamma^0\end{aligned}$$

and we know that Γ^0 is antisymmetric, so

$$C\Gamma^0 = -\Gamma^{0,T}C \quad (\text{B.25})$$

Hence, we have showed that this matrix C is indeed a charge conjugation matrix.

Now we will show that any chiral (Weyl) spinor λ can be expanded as follows:

$$|\lambda \rangle = \lambda_+ |0 \rangle + \frac{1}{2!} \lambda_{ij} a^j a^i |0 \rangle + \frac{1}{4!} \lambda^i \epsilon_{ijklm} a^j a^k a^l a^m |0 \rangle \quad (\text{B.26})$$

where the coefficients are $\lambda_+ = \langle 0 | \lambda \rangle$, $\lambda_{ij} = \langle 0 | a_i a_j | \lambda \rangle$ and $\lambda^j = \frac{1}{4!} \epsilon^{pqrsj} \langle 0 | a_p a_q a_r a_s | \lambda \rangle$. Let us prove this statement. We know that $\Gamma |0 \rangle = |0 \rangle$ and $\{\Gamma, a^k\} = 0$, so every even combination of operators a^k will commute with Γ , this implies that $\Gamma |\lambda \rangle = |\lambda \rangle$, and so $|\lambda \rangle$ is a Weyl spinor. Now we can compute the respective coefficients in the expansion above. Let us multiply by $\langle 0 |$ on each side of the equality:

$$\langle 0 | \lambda \rangle = \lambda_+ \quad (\text{B.27})$$

where we used that $\langle 0|a^i = 0$ for all $i = 1, 2, 3, 4, 5$, and $\langle 0|0 \rangle = 1$. We can also multiply by $\langle 0|a_l a_m$ and get:

$$\begin{aligned}\langle 0|a_l a_m|\lambda \rangle &= \frac{1}{2!}\lambda_{ij} \langle 0|a_l a_m a^j a^i|0 \rangle \\ \langle 0|a_l a_m|\lambda \rangle &= \frac{1}{2}\lambda_{ij}[\delta_m^j \langle 0|a_l a^i|0 \rangle - \delta_m^i \langle 0|a_l a^j|0 \rangle] \\ \langle 0|a_l a_m|\lambda \rangle &= \frac{1}{2}\lambda_{ij}(\delta_m^j \delta_l^i - \delta_m^i \delta_l^j) \\ \langle 0|a_l a_m|\lambda \rangle &= \frac{1}{2}(\lambda_{lm} - \lambda_{ml})\end{aligned}$$

therefore

$$\langle 0|a_l a_m|\lambda \rangle = \lambda_{lm} \quad (\text{B.28})$$

We can also multiply by $\langle 0|a_p a_q a_r a_s$ and get:

$$\begin{aligned}\langle 0|a_p a_q a_r a_s|\lambda \rangle &= \frac{1}{4!}\lambda^i \epsilon_{ijklm} \langle 0|a_p a_q a_r a_s a^j a^k a^l a^m|0 \rangle \\ \langle 0|a_p a_q a_r a_s|\lambda \rangle &= \frac{1}{4!}\lambda^i \epsilon_{ijklm} [\langle 0|\delta_s^j a_p a_q a_r a^k a^l a^m|0 \rangle - \langle 0|\delta_s^k a_p a_q a_r a^j a^l a^m|0 \rangle \\ &\quad + \langle 0|a_p a_q a_r a^j a^k a_s a^l a^m|0 \rangle] \\ \langle 0|a_p a_q a_r a_s|\lambda \rangle &= \frac{1}{4!}\lambda^i \epsilon_{ijklm} [\delta_s^j \langle 0|a_p a_q a_r a^k a^l a^m|0 \rangle - \delta_s^k \langle 0|a_p a_q a_r a^j a^l a^m|0 \rangle \\ &\quad + \delta_s^l \langle 0|a_p a_q a_r a^j a^k a^m|0 \rangle - \delta_s^m \langle 0|a_p a_q a_r a^j a^k a^l|0 \rangle]\end{aligned}$$

It is necessary to calculate $\langle 0|a_p a_q a_r a^k a^l a^m|0 \rangle$, so let us do it:

$$\begin{aligned}\langle 0|a_p a_q a_r a^k a^l a^m|0 \rangle &= \delta_r^k \langle 0|a_p a_q a^l a^m|0 \rangle - \langle 0|a_p a_q a^k a_r a^l a^m|0 \rangle \\ \langle 0|a_p a_q a_r a^k a^l a^m|0 \rangle &= \delta_r^k \langle 0|a_p a_q a^l a^m|0 \rangle - \delta_r^l \langle 0|a_p a_q a^k a^m|0 \rangle \\ &\quad + \delta_r^m \langle 0|a_p a_q a^k a^l|0 \rangle \\ \langle 0|a_p a_q a_r a^k a^l a^m|0 \rangle &= \delta_r^k (\delta_q^l \langle 0|a_p a^m|0 \rangle - \langle 0|a_p a^l a_q a^m|0 \rangle) + \dots \\ \langle 0|a_p a_q a_r a^k a^l a^m|0 \rangle &= \delta_r^k (\delta_q^l \delta_p^m - \delta_p^l \delta_q^m) + \dots \\ \langle 0|a_p a_q a_r a^k a^l a^m|0 \rangle &= \delta_r^k \delta_q^l \delta_p^m - \delta_r^k \delta_p^q \delta_q^m - \delta_r^l \delta_q^k \delta_p^m \\ &\quad + \delta_r^l \delta_q^m \delta_p^k + \delta_r^m \delta_q^k \delta_p^l - \delta_r^m \delta_p^k \delta_q^l\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\langle 0|a_p a_q a_r a_s|\lambda \rangle &= \frac{1}{4!}\lambda^i \epsilon_{ijklm} \{ + \delta_s^j [\delta_r^k \delta_q^l \delta_p^m - \delta_r^k \delta_p^l \delta_q^m - \delta_r^l \delta_q^k \delta_p^m + \delta_r^l \delta_p^m \delta_q^k + \delta_r^m \delta_q^k \delta_p^l - \delta_r^m \delta_p^k \delta_q^l] \\ &\quad - \delta_s^k [\delta_r^j \delta_q^l \delta_p^m - \delta_r^j \delta_p^l \delta_q^m - \delta_r^l \delta_q^j \delta_p^m + \delta_r^l \delta_p^j \delta_q^m + \delta_r^m \delta_p^l \delta_q^j - \delta_r^m \delta_q^l \delta_p^j] \\ &\quad + \delta_s^l [\delta_r^j \delta_q^k \delta_p^m - \delta_r^j \delta_p^k \delta_q^m - \delta_r^k \delta_q^j \delta_p^m + \delta_r^k \delta_p^j \delta_q^m + \delta_r^m \delta_q^j \delta_p^k - \delta_r^m \delta_q^k \delta_p^j] \\ &\quad - \delta_s^m [\delta_r^j \delta_q^k \delta_p^l - \delta_r^j \delta_p^k \delta_q^l - \delta_r^k \delta_q^j \delta_p^l + \delta_r^k \delta_p^j \delta_q^l + \delta_r^l \delta_p^k \delta_q^j - \delta_r^l \delta_q^k \delta_p^j] \} \\ \rightarrow \langle 0|a_p a_q a_r a_s|\lambda \rangle &= \frac{1}{4!}\lambda^i \epsilon_{isrqp} \cdot 4! \\ \rightarrow \langle 0|a_p a_q a_r a_s|\lambda \rangle &= \lambda^i \epsilon_{ipqrs}\end{aligned}$$

Now let us multiply by ϵ^{jpqrs} :

$$\epsilon^{jpqrs} \langle 0|a_p a_q a_r a_s|\lambda \rangle = \lambda^i \epsilon_{ipqrs} \epsilon_{jpqrs} = 4! \lambda^j \quad (\text{B.29})$$

$$\rightarrow \lambda^j = \frac{1}{4!} \epsilon^{jpqrs} \langle 0|a_p a_q a_r a_s|\lambda \rangle \quad (\text{B.30})$$

Now we are going to write the pure spinor constraint $\lambda\gamma^m\lambda = 0$ in a more convenient way: $\lambda^T C \Gamma^m \lambda = 0$, and this form allows us to write the pure spinor constraint in terms of the a_j and a^{j5}

$$\lambda^T C a^j \lambda = 0 \quad (\text{B.31})$$

$$\lambda^T C a_j \lambda = 0 \quad (\text{B.32})$$

which can be written as follows

$$\langle \lambda | C a^j | \lambda \rangle = 0 \quad (\text{B.33})$$

$$\langle \lambda | C a_j | \lambda \rangle = 0 \quad (\text{B.34})$$

It is useful to note that $a^i C = -C a_i$. We will show it explicitly just for two cases:

$$\begin{aligned} a^1 C &= a^1 (a_1 - a^1) (a_2 - a^2) \dots (a_5 - a^5) = (a^1 a_1 - 0) (a_2 - a^2) \dots (a_5 - a^5) \\ &= (a^1 a_1 - a_1 a_1) (a_2 - a^2) \dots (a_5 - a^5) \\ &= (a^1 - a_1) (a_2 - a^2) \dots (a_5 - a^5) a_1 \\ &= -(a_1 - a^1) (a_2 - a^2) \dots (a_5 - a^5) a_1 \\ &\rightarrow a^1 C = -C a^1 \end{aligned}$$

Analogously,

$$\begin{aligned} a^3 C &= a^3 (a_1 - a^1) (a_2 - a^2) \dots (a_5 - a^5) = (a_1 - a^1) \dots (a^3 a_3 - 0) \dots (a_5 - a^5) \\ &= (a_1 - a^1) \dots (a^3 a_3 - a_3 a_3) \dots (a_5 - a^5) \\ &= (a_1 - a^1) \dots (a^3 - a_3) \dots (a_5 - a^5) a_3 \\ &= -(a_1 - a^1) \dots (a_3 - a^3) \dots (a_5 - a^5) a_3 \\ &\rightarrow a^3 C = -C a_3 \end{aligned}$$

Using the same ideas we can also prove that $a_i C = -C a^i$ for all $i = 1, 2, 3, 4, 5$. On the other hand, we can calculate the action of C on the vacuum $|0\rangle$:

$$C|0\rangle = (a_1 - a^1)(a_2 - a^2) \dots (a_5 - a^5)|0\rangle \quad (\text{B.35})$$

$$C|0\rangle = (-a^1)(-a^2)(-a^3)(-a^4)(-a^5)|0\rangle \quad (\text{B.36})$$

$$C|0\rangle = -a^1 a^2 a^3 a^4 a^5 |0\rangle \quad (\text{B.37})$$

and also

$$\langle 0|C = \langle 0|(a_1 - a^1)(a_2 - a^2) \dots (a_5 - a^5) \quad (\text{B.38})$$

$$\langle 0|C = \langle 0|a^1 a^2 a^3 a^4 a^5 \quad (\text{B.39})$$

One more property which will be useful to solve the pure spinor constraint is the fact that $\langle A|C|B\rangle \neq 0$ if and only if $A^\dagger B \propto a^1 a^2 a^3 a^4 a^5$. This can be seen as follows: We put all a_j in $\langle A|$ to the right of C and we obtain (up to an overall sign) $\langle A|C|B\rangle \propto \langle 0|C A^\dagger B|0\rangle$, and this latter will not vanish only when $A^\dagger B$ matches the a_k 's in $\langle 0|C$. Therefore $A^\dagger B|0\rangle \propto a^1 a^2 a^3 a^4 a^5 |0\rangle$.

⁵We just have to use the definition of a_j and a^j .

Now we are ready to solve the pure spinor constraint. Let us start with the first constraint:

$$\begin{aligned}
& \langle \lambda | C a^{i_0} | \lambda \rangle \propto \langle 0 | C (\lambda_+ + \frac{1}{2!} \lambda_{ij} a^i a^j + \frac{1}{4!} \lambda^i a^j a^k a^l a^m \epsilon_{ijklm}) a^{i_0} | \lambda \rangle = 0 \\
& \langle 0 | C (\lambda_+ + \frac{1}{2!} \lambda_{ij} a^i a^j + \frac{1}{4!} \lambda^i a^j a^k a^l a^m \epsilon_{ijklm}) a^{i_0} | (\lambda_+ + \frac{1}{2!} \lambda_{pq} a^q a^p + \frac{1}{4!} \lambda^b a^c a^d a^e a^f \epsilon_{bcdef}) | 0 \rangle = 0 \\
& \frac{1}{4!} \langle 0 | C \lambda^i a^j a^k a^l a^m \epsilon_{ijklm} a^{i_0} \lambda_+ | 0 \rangle + \frac{1}{4} \langle 0 | C \lambda_{ij} a^i a^j a^{i_0} \lambda_{pq} a^p a^q | 0 \rangle \\
& \quad + \frac{1}{4!} \langle 0 | C \lambda_+ a^{i_0} \lambda^b a^c a^d a^e a^f \epsilon_{bcdef} | 0 \rangle = 0 \\
& \frac{1}{4!} \lambda^i \epsilon^{ijklm i_0} \epsilon_{ijklm} \lambda_+ + \frac{1}{4} \lambda_{ij} \lambda_{pq} \epsilon^{ij i_0 pq} + \frac{1}{4!} \lambda_+ \epsilon^{i_0 cdef} \epsilon_{bcdef} \lambda^b = 0 \\
& \quad 2 \lambda^{i_0} \lambda_+ + \frac{1}{4} \epsilon^{i_0 i j p q} \lambda_{ij} \lambda_{pq} = 0
\end{aligned}$$

therefore

$$\lambda^{i_0} = -\frac{1}{8 \lambda_+} \epsilon^{i_0 i j p q} \lambda_{ij} \lambda_{pq} \quad (\text{B.40})$$

Now it remains to solve the second constraint:

$$\langle \lambda | C a_{i_0} | \lambda \rangle = 0 \quad (\text{B.41})$$

However, it turns out that this one is implied by the first constraint. Let us show how this works:

$$\begin{aligned}
& \langle \lambda | C a_{i_0} | \lambda \rangle \propto \langle 0 | C (\lambda_+ + \frac{1}{2!} \lambda_{ij} a^i a^j + \frac{1}{4!} \lambda^i a^j a^k a^l a^m \epsilon_{ijklm}) a_{i_0} | \lambda \rangle = 0 \\
& \langle 0 | C (\lambda_+ + \frac{1}{2!} \lambda_{ij} a^i a^j + \frac{1}{4!} \lambda^i a^j a^k a^l a^m \epsilon_{ijklm}) a_{i_0} | (\lambda_+ + \frac{1}{2!} \lambda_{pq} a^q a^p + \frac{1}{4!} \lambda^b a^c a^d a^e a^f \epsilon_{bcdef}) | 0 \rangle = 0 \\
& \langle 0 | C (\frac{1}{2} \lambda_{ij} a^i a^j a_{i_0} \frac{1}{4!} \lambda^a a^b a^c a^d a^e \epsilon_{abcde}) | 0 \rangle + \langle 0 | C (\frac{1}{4!} \lambda^i a^j a^k a^l a^m \epsilon_{ijklm} a_{i_0} \frac{1}{2} \lambda_{ip} a^p a^i) | 0 \rangle = 0 \\
& \frac{1}{2} \lambda_{i i_0} \frac{1}{4!} \epsilon^{i b c d e} \epsilon_{a b c d e} \lambda^a - \frac{1}{2} \lambda_{i_0 j} \frac{1}{4!} \epsilon^{j b c d e} \epsilon_{a b c d e} \lambda^a + \frac{1}{2} \lambda^i \epsilon^{j k l m r} \frac{1}{4!} \epsilon_{ijklm} \lambda_{r i_0} - \frac{1}{2} \lambda^i \epsilon^{j k l m r} \epsilon_{ijklm} \frac{1}{4!} \lambda_{i_0 r} = 0 \\
& \quad \frac{1}{2} \lambda_{j i_0} \lambda^j - \frac{1}{2} \lambda_{i_0 j} \lambda^j + \frac{1}{2} \lambda^j \lambda_{j i_0} - \frac{1}{2} \lambda^j \lambda_{i_0 j} = 0 \\
& \quad \rightarrow \langle \lambda | C a_{i_0} | \lambda \rangle \propto -2 \lambda_{i_0 j} \lambda^j = 0
\end{aligned}$$

which by using the first constraint is equivalent to

$$\frac{1}{\lambda_+} \lambda_{ij} \lambda_{ab} \lambda_{cd} \epsilon^{iabcd} = 0 \quad (\text{B.42})$$

Now, we will show that this is an identity. We know that $T_{[jklmin]} = 0$ (where $i, j, k, l, m, n = 1, \dots, 5$), so let us define $T_{jklmin} = \lambda_{jk} \lambda_{lm} \lambda_{in}$ and $T_n = \epsilon^{jklmi} \lambda_{jk} \lambda_{lm} \lambda_{in} = 0$, therefore:

$$T_{[jklmin]} = 0 \quad (\text{B.43})$$

$$T_{[jklmi]n} - T_{[jklm|n|i]} + T_{[jkl|n|m]i} - T_{[jk|n|lm]i} + T_{[j|n|klm]i} - T_{n[jklmi]} = 0 \quad (\text{B.44})$$

$$\epsilon^{jklmi} (T_{[jklmi]n} - T_{[jklm|n|i]} + T_{[jkl|n|m]i} - T_{[jk|n|lm]i} + T_{[j|n|klm]i} - T_{n[jklmi]}) = 0 \quad (\text{B.45})$$

$$T_n - (-T_n) - T_n - T_n + T_n - (-T_n) = 0 \quad (\text{B.46})$$

$$\rightarrow T_n = 0 \quad (\text{B.47})$$

This implies that by using the first constraint, the second constraint

$$\langle \lambda | C a_{i_0} | \lambda \rangle \propto -2 \lambda_{i_0 j} \lambda^j = -\frac{2}{\lambda_+} \lambda_{ij} \epsilon^{jabcd} \lambda_{ab} \lambda_{cd} = 0 \quad (\text{B.48})$$

becomes an identity.

Appendix C

On-shell and off-shell degrees of freedom

What will be explained here is based on the reference [28]. It is important to distinguish two kind indices: m, n, \dots (μ, ν, \dots) which will be called *curved or Einstein indices*, and a, b, \dots (α, β, \dots) which will be called *flat or Lorentz indices*. Let us start by studying the off-shell case:

1. A real scalar field has one degree of freedom.
2. A Majorana spinor field ψ^α (assuming that the Majorana condition can be imposed) has $2^{\frac{D}{2}}$ degrees of freedom.
3. A gauge field A_m has D independent components. However, the theory is unchanged under gauge transformations $\delta A_m = D_m \Lambda$, where Λ is the gauge parameter. So, if we use this transformation we can choose one particular Λ to fix one particular component of A^μ . Therefore we just have $D - 1$ degrees of freedom.
4. A spinorvector ψ_m^α has $D \times 2^{\frac{D}{2}}$ independent components. But, we can also use the gauge invariance (local supersymmetry) to fix some components of this field: $\delta \psi_m^\alpha = D_m \epsilon^\alpha$, where ϵ^α is the local supersymmetry parameter. Therefore we are left with $2^{\frac{D}{2}}(D - 1)$ degrees of freedom.
5. A spin-2 field (graviton) g_{mn} has $\frac{D(D+1)}{2}$ independent components. However, the components of this field can be set to fixed values by using general coordinate transformations with gauge parameter ξ^m , therefore we have $\frac{D(D-1)}{2}$ degrees of freedom¹.
6. An antisymmetric tensor $A_{m_1 \dots m_n}$ whose field strength is $F_{m_1 \dots m_{n+1}} = \partial_{[m_1} A_{m_2 \dots m_{n+1}]}$ has $\binom{D}{r} = \frac{D!}{(D-r)!r!}$ independent components. By using the gauge invariance $\delta A_{m_1 \dots m_n} =$

¹We can always see this in terms of the vielben field e_m^a . This field has D^2 independent components which can be fixed by using the general coordinate transformations (with parameter ξ^m) and local Lorentz transformations (with parameter λ^{ab}). Therefore we are left with $D^2 - D - \frac{D(D-1)}{2} = \frac{D(D-1)}{2}$ degrees of freedom.

$\partial_{m_1} \Lambda_{m_2 \dots m_n}$, we can fix $\binom{D-1}{r-1} = \frac{(D-1)!}{(D-r)!(r-1)!}$ components of $A_{m_1 \dots m_n}$. Therefore have

$$\begin{aligned} \binom{D}{r} - \binom{D-1}{r-1} &= \frac{D \cdot (D-1) \dots 2 \cdot 1}{(r \cdot (r-1) \dots 2 \cdot 1)(D-r) \cdot (D-r+1) \dots 2 \cdot 1} \\ &\quad - \frac{(D-1) \cdot (D-2) \dots 2 \cdot 1}{((r-1) \cdot (r-2) \dots 2 \cdot 1)(D-r) \cdot (D-r+1) \dots 2 \cdot 1} \\ &= \frac{(D-r)(D-1)(D-2) \dots 2 \cdot 1}{(r \cdot (r-1) \dots 2 \cdot 1)(D-r) \cdot (D-r+1) \dots 2 \cdot 1} \\ &= \frac{(D-1)(D-2) \dots 2 \cdot 1}{(r \cdot (r-1) \dots 2 \cdot 1)(D-r+1) \dots 2 \cdot 1} \\ \rightarrow \binom{D}{r} - \binom{D-1}{r-1} &= \binom{D-1}{r} \end{aligned}$$

degrees of freedom.

Now, we move on to the on-shell case:

1. The Klein-Gordon equation just restricts the functional form of the scalar field (in momentum space, $p^2 = 0$). So we still have one degree of freedom for the scalar field².
2. The Dirac equation $(\not{p} - m)\psi(p) = 0$ relates the half of components of $\psi(p)$ with the other half. Therefore we have $2^{\frac{D}{2}-1}$ degrees of freedom.
3. The gauge field A_m satisfies the equation of motion $\partial_m F^{mn} = \partial_m(\partial^m A^n - \partial^n A^m) = 0$. In order to see how this restricts the degrees of freedom in A_m , we can choose a particular gauge, for example the Lorenz gauge $\partial^m A_m = 0$. This simplifies the equation of motion to $\partial_m \partial^m A^n = 0$, which is a KG equation and we already saw that this equation does not affect the number of degrees of freedom. However, the gauge choice implies the following condition $p_m \epsilon^m = 0$, where ϵ^m is the polarization vector and p_m is the momentum. So we are left with $(D-1) - 1 = D-2$ degrees of freedom (the transverse components).
4. The graviton g_{mn} has to satisfy the Einstein equation. However at linearized order we can just work with the Pauli-Fierz lagrangian:

$$\mathcal{L} = \frac{1}{2} h_{mn,p}^2 + \frac{1}{2} h_m^2 - h^m h_{,m} + \frac{1}{2} h_{,m}^2 \quad (\text{C.1})$$

where $h_{mn} = g_{mn} - \eta_{mn}$, $h_n = \partial^m h_{mn}$ and $h = h_m^m$. Again, we will choose a particular gauge in order to solve the corresponding equation of motion deduced from the Pauli-Fierz lagrangian. The Donder gauge condition is

$$\partial^m \bar{h}_{mn} = 0 \quad (\text{C.2})$$

where $\bar{h}_{mn} = h_{mn} - \frac{1}{2} \eta_{mn} h$. This gauge condition reduces the equation of motion to $\partial^p \partial_p \bar{h}_{mn} = 0$, which is a KG equation. Therefore we will just have D constraints coming from the orthogonality condition $p^m \epsilon_{mn}$, where ϵ_{mn} is the polarization tensor. So we are left with $\frac{D(D-1)}{2} - D = \frac{D(D-3)}{2}$ degrees of freedom.

²It is necessary to be more clear at this point. We can have two types of scalar fields: *Auxiliary* and *propagating* scalar fields. In the off-shell case, both have one degree of freedom. In the on-shell case the auxiliary field has no degrees of freedom whereas that the propagating field has one degree of freedom.

5. The gravitino ψ_m^α (spin- $\frac{3}{2}$ field) has $2^{\frac{D}{2}-1}(D-2) - 2^{\frac{D}{2}-1} = 2^{\frac{D}{2}-1}(D-3)$ degrees of freedom. This is so because the full representation can be decomposed in irreducible representations as: $\frac{1}{2} \otimes 1 = \frac{1}{2} \oplus \frac{3}{2}$. So we should subtract the spin- $\frac{1}{2}$ part. This constraint can be written as $\gamma_{\alpha\beta}^m \psi_m^\beta = 0$, hence we have $2^{\frac{D}{2}-1}$ constraints on ψ_m^α . This explains the counting of degrees of freedom obtained above.

6. An antisymmetric tensor $A_{m_1 \dots m_r}$ with field strength $F_{m_1 \dots m_{r+1}} = \partial_{[m_1} A_{m_2 \dots m_{r+1}]}$ and gauge invariance $\delta A_{m_1 \dots m_r} = \partial_{[m_1} \Lambda_{m_2 \dots m_r]}$, satisfies a certain equation of motion which becomes into a KG equation $\partial^p \partial_p A_{m_1 \dots m_p} = 0$ after imposing the gauge condition $\partial^{m_1} A_{m_1 \dots m_p} = 0$ (which in momentum space can be read off as $p^{m_1} \epsilon_{m_1 \dots m_r} = 0$, a transversality condition). This restricts the degrees of freedom to $\binom{D-2}{r}$.

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