UNIVERSIDADE ESTADUAL PAULISTA
“JÚLIO DE MESQUITA FILHO”
Campus de Ilha Solteira

Julio Cezar Uzinski

A state-space parameterization for perfect-reconstruction wavelet FIR filter banks with special orthonormal basis functions

Ilha Solteira
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CERTIFICADO DE APROVAÇÃO

TÍTULO DA TESE: A state-space parameterization for perfect-reconstruction wavelet FIR filter banks with special orthonormal basis functions

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To my parents: Julio and Maria,
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If I have seen further than others, it is by standing upon the shoulders of giants.
Isaac Newton

Resumo

Esta tese apresenta uma parametrização no espaço de estados para a transformada wavelet rápida. Esta parametrização é baseada em funções de base ortonormal e filtros de resposta finita ao impulso simultaneamente, uma vez que, a transformada rápida wavelet é um algoritmo que consiste em decompor sinais no domínio do tempo em sequências de coeficientes baseados numa base ortogonal de funções wavelet. Deste modo, vantagens apresentadas por ambas as propostas são incorporadas. Modelos de resposta finita ao impulso têm propriedades atrativas como vantagens computacionais e analíticas, garantia de estabilidade BIBO e robustez para a mudança de alguns parâmetros, dentre outras. Por outro lado, séries de funções de base ortonormal têm características que as fazem atrativas para a modelagem de sistemas dinâmicos, como ausência de recursão da saída, a não necessidade de se conhecer previamente a estrutura exata do vetor de regressão, possibilidade de aumentar a capacidade de representação do modelo aumentando-se o número de funções ortonormais utilizadas, desacoplamento natural das saídas em modelos multivariáveis; tolerância a dinâmicas não modeladas. Além disso, a realização no espaço de estados é mínima. A contribuição deste trabalho consiste no desenvolvimento de uma realização no espaço de estados para bancos de filtros wavelet, em que há a presença explícita de parâmetros que podem ser livremente ajustados mantendo as propriedades de reconstrução perfeita e ortonormalidade. Para ilustrar o funcionamento e as vantagens da técnica proposta, alguns exemplos de decomposição de sinais no contexto de processamento de sinais mostrando que ela proporciona os mesmos coeficientes wavelet que a transformada wavelet rápida, e uma aplicação em controle através de realimentação dinâmica de estados também são apresentados nesta tese.

Palavras-chave: Filtros de resposta finita ao impulso. Bases de funções ortonormais. Descrição no espaço de estados. Bancos de filtros wavelet. Regulador linear quadrático discreto.

Abstract

This thesis presents a state-space parameterization for the fast wavelet transform. This parameterization is based on orthonormal basis functions and finite impulse response filters at the same time, since the fast wavelet transform is an algorithm which converts a signal in the time domain into a sequence of coefficients based on an orthogonal basis of small finite wavelet functions. Advantages presented by both proposals are incorporated. Finite impulse response systems have attractive properties, for instance, computational and analytical advantages, BIBO stability and robustness guarantee to some parameter changes, and others. On the other hand, orthonormal basis functions have some characteristics that make them attractive for dynamic systems modeling, examples are, output recursion absence, not requiring prior regression vector exact structure knowledge; possibility of increasing the model representation capacity by increasing the number of orthonormal functions employed; natural outputs uncoupling in multivariable models; tolerance to unmodeled dynamics, and others. Furthermore, the state-space realization is minimal. The contribution of this work consists in the development of a state-space realization for a wavelet filter bank, with the explicit presence of the parameters that can be freely adjusted, keeping perfect-reconstruction and orthonormality guarantees. In order to illustrate advantages and how the proposed technique works, some decomposition examples in signal processing context are presented showing that it provides the same wavelet coefficients as the fast wavelet transform, and an application on dynamic state feedback control is also presented in this thesis.

Keywords: Finite impulse response filters. Orthonormal basis functions. State-space description. Wavelet filter banks. Discrete linear quadratic regulator.

List of Figures

List of abbreviations and acronyms

BIBO	Bounded input – bounded output
CWT	Continuous wavelet transform
dB	Decibel
`db1`	Daubechies wavelet 1
`db3`	Daubechies wavelet 3
`db4`	Daubechies wavelet 4
DLQR	Discrete linear quadratic regulator
DLQR-WFB	Discrete linear quadratic regulator employing wavelets filter banks
DWT	Discrete wavelet transform
e.g.	Abbreviation of *exempli gratia*, which usually means "for example"
ECG	Electrocardiogram
FIR	Finite impulse response
`fr`N	Optimal wavelet filter from Paiva and Galvão (2012) with length $2N$
FT	Fourier transform
FWT	Fast wavelet transform
GOBF	Generalized orthonormal basis functions
`Haar`	Haar wavelet
i.e.	Abbreviation of *id est*, which usually means "that is"
ICWT	Inverse continuous wavelet transform
IDWT	Inverse discrete wavelet transform
IFWT	Inverse fast wavelet transform
IIR	Infinite impulse response

LQR	Linear quadratic regulator
LTI	Linear time-invariant (system)
MIMO	Multiple input – Multiple output
MISO	Multiple input – Single output
MRA	Multiresolution analysis
OBF	Orthonormal basis functions
QMF	Quadrature mirror filter
SQP	Sequential quadratic programming
SNR	Signal-to-noise ratio
SIMO	Single input – Multiple output
SISO	Single input – Single output
STFT	Short-time Fourier transform
`sym4`	Symlet wavelet 4
WT	Wavelet transform

List of symbols

$u(t)$	Continuous-time input signal
$y(t)$	Continuous-time output signal
$x(t)$	Continuous-time state variable
$\dot{x}(t)$	Derivative of the continuous-time state variable
$\mathbf{u(t)}$	Continuous-time input vector
$\mathbf{y(t)}$	Continuous-time output vector
$\mathbf{x(t)}$	Continuous-time state vector
$\dot{\mathbf{x}}(t)$	Vector of the derivative of the continuous-time state variable
$(\mathbf{A}_c(t), \mathbf{B}_c(t), \mathbf{C}_c(t), \mathbf{D}_c(t))$	State-space realization for continuous time-varying systems
$(\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c, \mathbf{D}_c)$	State-space realization for continuous time-invariant systems
$\mathbf{I}$	Identity matrix
$u[k]$	Discrete-time input signal
$y[k]$	Discrete-time output signal
$x[k]$	Discrete-time state variable
$\mathbf{u}[\mathbf{k}]$	Discrete-time input vector
$\mathbf{y}[\mathbf{k}]$	Discrete-time output vector
$\mathbf{x}[\mathbf{k}]$	Discrete-time state vector
$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$	State-space realization for discrete time-invariant systems
$\mathbf{X}(z)$	z-transform for $\mathbf{x}[\mathbf{k}]$
$\mathbf{Y}(z)$	z-transform for $\mathbf{y}[\mathbf{k}]$
$\mathbf{U}(z)$	z-transform for $\mathbf{u}[\mathbf{k}]$
$\mathbf{H}(z)$	Transfer function in z-domain

$\mathbf{R}(z)$	Adjugate matrix of $(z\mathbf{I} - \mathbf{A})$
$\det(.)$	Determinant
$\lvert\lambda\rvert$	Absolute values of eigenvalues
$\mathbf{A}^\dagger$	Conjugate transpose of $\mathbf{A}$
$\langle , \rangle$	Inner product
$\lvert\lvert\mathbf{u}\rvert\rvert$	Euclidean norm for a vector $\mathbf{u}$
f^*	Complex conjugate of f
$\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$	Projection of $\mathbf{v}$ onto $\mathbf{u}$
$\mathbb{D}$	The unit disc: $\{z, \lvert z\rvert < 1\}$
$\mathbb{T}$	The unit circumference around the unit disc $\mathbb{D}$: $\{z, \lvert z\rvert = 1\}$
$\mathbb{E}$	Exterior of the unit disc $\mathbb{D}$: $\{z, \lvert z\rvert > 1\}$
$\mathcal{H}_2(\mathbb{E})$	Hardy space of square integrable functions on $\mathbb{T}$, analytic in the region $\mathbb{E}$
$\mathbb{Z}_+$	Set of non-negative integer numbers
$\mathbb{Z}_+^*$	Set of strictly positive integers
$L_2[0, \infty)$	Space of square-summable sequences
z^{-1}	Unity delay operator
$\mathbf{Re}(\beta)$	Real part of a complex number β
$\mathbf{Im}(\beta)$	Imaginary part of a complex number β
ξ_k	Poles of basis functions
$\mathbb{C}$	Set of Complex numbers
$\downarrow 2$	Downsampling operator
$\downarrow M$	Decimator at rate M
$\uparrow 2$	Upsampling operator
$\uparrow M$	Expander at rate M

$\mathbb{R}$	Set of real numbers
$L^1(\mathbb{R})$	Set of integrable functions in $\mathbb{R}$
$L^2(\mathbb{R})$	Set of square-integrable functions in $\mathbb{R}$
$\hat{f}(\omega)$	Fourier transform of $f(t)$
$\phi[k]$	Discrete-time scale function
$\psi[k]$	Discrete-time wavelet function
$\underline{\mathbf{y}}_i[k]$	Output signal vector for *Algorithme à Trous*
$h[k]$ and h_i	Low-pass filter coefficient
$g[k]$ and g_i	High-pass filter coefficient
N	Angular parameters' sequence length
β_i	Angular parameter with $i = 0, 1, \cdots N-1$
α_i	Angular parameter with $i = 1, 2, \cdots N$
θ	Set of angular parameters $\theta = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_N \end{bmatrix}$ and $\theta_1 = \alpha_N, \theta_2 = \alpha_{N-1}, \cdots, \theta_N = \alpha_1$
dim	Dimension of a matrix

Contents

1 Introduction

Modeling, analysis and control of dynamic systems have been of great interest for engineers for a long time (PALM III, 1983). In a few years, the study of these areas has grown tremendously for several reasons, e.g., advances in computer and information sciences. Nowadays, the use of computers has been expanded allowing the study of more detailed models and complex algorithms for system design and analysis (DECARLO, 1989). With the increase in computational power, it can impose stricter performance specifications in engineering designs, leading to the need of more detailed models to the process under study, especially with regards to forecasting their transient behavior (DA ROSA, 2009).

Real system modeling is of great importance in almost all science fields, it is mainly of interest to engineers, and its importance has grown a lot in recent years. Models help in understanding the characteristics of an analyzed process. Especially in engineering, models are needed for new process designs and for analyzing existing ones. In general, advanced techniques of controller design, optimization and supervision are based on process models, and model quality directly influences solution quality to the problem (MACHADO, 2011; PALM III, 1983).

Among the several modeling techniques of dynamic systems available in the literature, it can be mentioned as one of the most important, models obtained by using orthogonal functions that form a complete basis for the Lebesgue space and orthonormal basis functions that includes Laguerre, Kautz and Takenaka-Malquimist functions (MACHADO, 2011; HEUBERGER; VAN DEN HOF; WAHLBERG, 2005).

Modeling with orthonormal basis functions has presented interesting results as shown by many researches (MACHADO, 2011; WAHLBERG, 1991; DUMONT; FU, 1993; WAHLBERG, 1994; VAN DEN HOF; HEUBERGER; BOKOR, 1995; HEUBERGER; VAN DEN HOF; BOSGRA, 1995; BOKOR; SHIPP, 1998; OLIVEIRA et al., 2000). These models' features make them attractive for dynamical systems modeling: output recursion absence, no vector regression exact structure prior knowledge requirement; possibility to increase the model representation capacity by increasing the number of orthonormal functions employed; natural outputs uncoupling in multivariable models; tolerance to unmodeled dynamics, and others (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005).

The finite impulse response (FIR) system is a common way to represent the transfer function for linear time-invariant dynamical systems. These finite expansion representations are very useful dynamical models in many areas of engineering. Applications examples are signal processing, filter design, communication, control design and system identification (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005).

Finite impulse response systems are widely used in discrete signal processing tasks, which include denoising, subband and multirate filtering, prediction, control, etc. FIR structure does not only guarantee both BIBO (Bounded Input - Bounded Output) stability and robustness to some parameter changes, but also improves the filter divergence problem, and they are naturally stable and robust even for higher order designs (TUQAN; VAIDYANATHAN, 2000; VAIDYANATHAN, 1990a; VAIDYANATHAN, 1990b; UZINSKI et al., 2015b).

Another mathematical tool extensively used in signal processing and control systems is the Wavelet Transform (WT). The algorithm to work with the discrete wavelet transform is known as Fast Wavelet Transform (MALLAT, 1998; MALLAT, 1989c). There exist many applications using this technique, which has several advantages except for cases where the state-space approach is more suitable than other classical modeling methods (ELALI, 2012; ELALI, 2008).

Considering Sherlock and Monro (1998) studies on the space of orthonormal wavelets functions and its relation with finite impulse response, a possibility of parameterization of these basis functions with several promising properties is considered.

1.1 Objectives and justification

The present work brings out a study about orthonormal functions and orthonormal basis functions (MACHADO, 2011; DA ROSA, 2009; HEUBERGER; VAN DEN HOF; WAHLBERG, 2005). However, the main objective is to explore the space of orthonormal functions used by Sherlock and Monro (1998) in order to formulate a set of orthonormal filter banks and parameterize it in the state-space, and, from this initial parameterization, to extend this idea to the case of M decomposition levels. In every case, properties that make the state-space realization attractive like observability, reachability and minimality, besides those advantages previously mentioned for orthonormal basis functions and finite impulse response filters may be kept.

The main justification for orthonormal functions parameterization in the state-space is given by the possibility of applications with interesting results (MACHADO, 2011; DA ROSA, 2009; HEUBERGER; VAN DEN HOF; WAHLBERG, 2005). On the other hand, the FIR structure guarantees BIBO stability, robustness to some parameter changes and improves the filter divergence problem (TUQAN; VAIDYANATHAN, 2000; VAIDYANATHAN, 1990a). The objective in this work combines these two ideas and their advantages.

Sherlock and Monro (1998) orthonormal basis functions, in their turn, have been studied in terms of wavelets filter banks (DUARTE; GALVÃO; PAIVA, 2013; MURUGESAN; TAY, 2012; PAIVA; GALVÃO, 2012; PAIVA; GALVÃO; RODRIGUES, 2009; HADJILOUCAS et al., 2014; PAIVA et al., 2008; GALVÃO et al., 2004; FROESE et al., 2006; UZINSKI, 2013; PAIVA et al., 2009; UZINSKI et al., 2013). This formulation's origin is very related to the study of wavelets and multirate systems (VAIDYANATHAN, 1993; VETTERLI; KOVAČEVIĆ, 1995; STRANG; NGUYEN, 1996).

In order to achieve the central work objective, which means the parameterization for the Sherlock and Monro (1998) formulation functions in the state-space and its extension for M decomposition levels with expected features that make it minimal, in addition to the fact that it should be able to be adjusted by the designer and all the advantages deriving from orthonormal basis functions and finite impulse response filters, some secondary objectives are proposed:

- firstly, writing a highly didactic and well structured text, in this way making the redaction simple and objective, but keeping the scientific rigor. In this sense, some more abstract parts were carefully organized, objectives and didactic examples are also presented;
- making an introduction to the systems in study, and at the same time reviewing the involved applied mathematics concepts, facilitating text reading, as well as introducing basic algebra concepts that are fundamental to the study and comprehension of orthonormal basis functions. Presenting the wavelet and filter bank theory that is of great importance in this work, and after this, elaborating a widely detailed text about Sherlock and Monro (1998) formulation;
- presenting applications of the proposed state-space parameterization in order to illustrate how it works and its advantages.

1.2 Outline of the thesis

Almost everything in this text was written in the discrete-time domain, in some sections or paragraphs it can be changed depending on the necessity. What follows is an explanation of how the text is organized, in order to facilitate its reading.

- **Chapter 2**: the redaction is about the main concepts on dynamical discrete systems, more accurately the description in the state-space and its properties, presenting an introductory text, addressing an introduction to the state-space functions parameterization. In the unique section, basic concepts on discrete-time systems are presented;

- **Chapter 3**: a review on the topics related to the orthogonality and orthonormality for continuous functions and vectors that can represent discrete functions is done. This chapter is divided as follows:
  - in Section 3.1, the main concepts and definitions about orthogonality and orthonormality are presented in a brief way,
  - in Section 3.2, it is presented the Gram-Schmidt procedure, which is an orthogonalization vector set method in a space with inner product,
  - a general idea of orthonormal basis functions is presented in Section 3.3,
  - Section 3.4 presents the corresponding orthogonal state-space model for finite impulse response networks,
  - Laguerre functions and their state-space description are presented in Section 3.5,
  - Kautz Functions and their state-space description are presented in Section 3.6,
  - Section 3.7 presents Takenaka-Malmquist Functions, also known as Generalized Orthonormal Basis Functions (GOBF).
- **Chapter 4**: Filter bank theory, wavelets basic theory, wavelets filter banks concepts and Sherlock and Monro (1998) formulation are presented. The main idea is to present the filter bank theory related to the Wavelet Transform, especially the Discrete Wavelet Transform (DWT). All of them are presented focusing on Sherlock and Monro (1998) formulation.
- **Chapter 5**: The parameterization proposed in this work for those functions presented in Section 4.5 is performed in Chapter 5. This chapter proposes two different parameterization possibilities for a particular orthonormal basis functions in the state-space:
  - a state-space single filter parameterization using the specified basis functions is presented in Section 5.1,
  - the main idea in this work is the specified basis functions parameterization to the system implementation in a lattice structure, presented in Section 5.2.

  After presenting the parameterizations in Sections 5.1 and 5.2, they are extended to the multilevel decomposition case in Section 5.3.
- **Chapter 6**: In order to illustrate how the proposed state-space parameterization works and its advantages, some decomposition examples and an application example are presented in Chapter 6.

- Final remarks, conclusions, contributions and future works are presented in Chapter 7. References and an appendix are presented at the end of the thesis.

2 Models for dynamical discrete systems

This chapter discusses the main concepts of dynamical discrete-time systems, state-space description and its properties. For further discussions, proofs and all concepts involved see Elali (2012), and the version to continuous systems can be seen in Elali (2008).

2.1 The discrete-time systems

A system is the combination of components acting together to achieve a determined objective. The system concept is not restricted to the physics, it can be applied to abstract dynamical phenomena, and they are mathematically modelled for study and applications. Dynamic systems can be described by differential equations, which can be obtained through physics laws that govern them (OGATA, 2010).

Linear systems are defined by the superposition principle. Let k be an arbitrary constant and the system $T\{u\}$ with y, y_1 and y_2 being the system outputs for the inputs u, u_1 and u_2, respectively. This system is linear if, and only if,

$$T\{u_1 + u_2\} = T\{u_1\} + T\{u_2\} = y_1 + y_2$$

and

$$T\{ku\} = kT\{u\} = ky.$$

An important class of systems are the Linear Time-invariant (LTI) systems. These systems obey the superposition principle to be linear. In this way, they can be completely characterized by the impulse response. Besides, they satisfy the time invariance propriety, namely, if $h[k]$ is the response to $\delta[k]$, the response to $\delta[k-m]$ is $h[k-m]$. In other words, Linear Time-Invariant systems are composed of linear time-invariant components, for instance, the spring-mass system that depends on mass and spring constant (CHEN, 1999).

2.1.1 State-space

The state-space approach can be superior to many other classic models of modeling. This theory can handle linear and nonlinear systems, besides, it can handle systems with nonzero initial conditions. Moreover, systems with single input and single output (SISO), single input and multiple outputs (SIMO), multiple inputs and single output (MISO) and multiple inputs and/or multiple outputs (MIMO) can be handled by this approach (ELALI, 2012).

This section introduces the continuous time systems, then the discrete-time systems can be presented through the discretization.

The state of a dynamic system is the smallest set of variables such that the knowledge of these variables at $t = t_0$, together with the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$ (OGATA, 2010).

Considering that the state vector has n state variables, the axis formed by the coordinates of these variables form the n-dimensional space, denominated state-space. Any state can be represented by a point in the state-space. Analysis in the state-space involves three types of variables that are present in the dynamical systems modeling: input variables, output variables and state variables (OGATA, 2010).

Considering a continuous time system with r inputs $u_1(t), u_2(t), \cdots, u_r(t)$ and m outputs $y_1(t), y_2(t), \cdots, y_m(t)$ involving n integrator. Defining the n integrators outputs as state variables $x_1(t), x_2(t), \cdots, x_n(t)$, the system can be written as

$$\begin{aligned}
\dot{x}_1(t) &= f_1(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\
\dot{x}_2(t) &= f_2(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\
&\vdots \\
\dot{x}_n(t) &= f_n(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t),
\end{aligned} \tag{1}$$

and the outputs $y_1(t), y_2(t), \cdots, y_m(t)$ can be written as

$$\begin{aligned}
y_1(t) &= g_1(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\
y_2(t) &= g_2(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \\
&\vdots \\
y_m(t) &= g_m(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t).
\end{aligned} \tag{2}$$

Equations (1) and (2) can be rewritten as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \tag{3}$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{u}, t), \tag{4}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T$, $\mathbf{y}(t) = [y_1(t), y_2(t), \cdots, y_m(t)]^T$ and $\mathbf{u}(t) = [u_1(t), u_2(t), \cdots, u_r(t)]^T$. Equation (3) is the state equation and (4) is the output equation.

Equations (3) and (4) when linearised around an operation point are written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c(t)\mathbf{x}(t) + \mathbf{B}_c(t)\mathbf{u}(t), \tag{5}$$

$$\mathbf{y}(t) = \mathbf{C}_c(t)\mathbf{x}(t) + \mathbf{D}_c(t)\mathbf{u}(t), \tag{6}$$

where $\mathbf{A}(t)$ is the state matrix, $\mathbf{B}(t)$ is the input matrix, $\mathbf{C}(t)$ is the output matrix and $\mathbf{D}(t)$ is the feedthrough or feedforward matrix.

If the system is time-invariant, equations (5) and (6) can be simplified to

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c\mathbf{x}(t) + \mathbf{B}_c\mathbf{u}(t), \tag{7}$$

$$\mathbf{y}(t) = \mathbf{C}_c\mathbf{x}(t) + \mathbf{D}_c\mathbf{u}(t). \tag{8}$$

Considering the continuous time system given by (7) and (8), if the set of equations is to be computed on a digital computer, it must be discretized. Since

$$\dot{\mathbf{x}}(t) = \lim_{T\to 0} \frac{\mathbf{x}(t+T) - \mathbf{x}(t)}{T}, \tag{9}$$

(7) can be approximated as

$$\mathbf{x}(t+T) = \mathbf{x}(t) + \mathbf{A}\mathbf{x}(t)T + \mathbf{B}\mathbf{u}(t)T. \tag{10}$$

Computing $\mathbf{x}(t)$ and $\mathbf{y}(t)$ only at $t = kT$ for $k = 0, 1, \cdots$, then (10) and (8) become

$$\begin{aligned} \mathbf{x}((k+1)T) &= (\mathbf{I} + T\mathbf{A})\mathbf{x}(kT) + T\mathbf{B}\mathbf{u}(kT) \\ \mathbf{y}(kT) &= \mathbf{C}\mathbf{x}(kT) + \mathbf{D}\mathbf{u}(kT). \end{aligned}$$

Another discretization could be $\mathbf{u}(t) = \mathbf{u}(kT)$, and defining this input as

$$\mathbf{u}(kT) = \mathbf{u}(t) = \mathbf{u}[k] \qquad \text{for} \qquad kT \leq t < (k+1)T \tag{11}$$

where $k = 0, 1, \cdots$. Values will be changed only at discrete-time instants. For this input, the solution of (7) still is the same for (5), i.e.,

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau, \tag{12}$$

where τ denotes the time at which the impulse input is applied. Computing (12) at $t = kT$ and $t = (k+1)T$ and defining $\mathbf{x}[k] = \mathbf{x}(kT)$, $\mathbf{x}[k+1] = \mathbf{x}((k+1)T)$, leads to

$$\mathbf{x}[k] = \mathbf{x}(kT) = e^{\mathbf{A}kT}\mathbf{x}(0) + \int_0^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{13}$$

and

$$\mathbf{x}[k+1] = \mathbf{x}((k+1)T) = e^{\mathbf{A}(k+1)T}\mathbf{x}(0) + \int_0^{(k+1)T} e^{\mathbf{A}((k+1)T-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau. \tag{14}$$

Equation (14) can be written as

$$\mathbf{x}[k+1] = e^{\mathbf{A}T}\left[e^{\mathbf{A}kT}\mathbf{x}(0) + \int_0^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\right] + \int_{kT}^{(k+1)T} e^{\mathbf{A}(kT+T-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau. \tag{15}$$

Substituting (11) and (13) and defining a new variable $\eta = kT + T - \tau$ (15) becomes

$$\mathbf{x}[k+1] = e^{\mathbf{A}T}\mathbf{x}[k] + \left(\int_0^T e^{\mathbf{A}\eta} d\eta\right) \mathbf{B}\mathbf{u}[k]. \tag{16}$$

Therefore, if an input changes values only at discrete-time instants kT and if it is computed, only the responses at $t = kT$, (7) and (8) become

$$\mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k], \tag{17}$$

$$\mathbf{y}[k] = \mathbf{C}_d\mathbf{x}[k] + \mathbf{D}_d\mathbf{u}[k], \tag{18}$$

where

$$\mathbf{A}_d = e^{\mathbf{A}T},$$

$$\mathbf{B}_d = \left(\int_0^T e^{\mathbf{A}\tau} d\tau\right) \mathbf{B},$$

$$\mathbf{C}_d = \mathbf{C}$$

and

$$\mathbf{D}_d = \mathbf{D},$$

(CHEN, 1999).

2.1.2 State-space descriptions properties

State and output equations will be denoted in this text as (ELALI, 2012)

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k], \tag{19}$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k], \tag{20}$$

respectively.

Input and output system behaviors are completely determined by the state-space description ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$). This affirmation can be reasoned by writing the transfer matrix in the time domain related to inputs and outputs to matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ (VAIDYANATHAN, 1993).

Taking the z-transform of both sides in (19) and (20) one obtains, considering the initial state vector (initial condition) $\mathbf{x}(0) = 0$,

$$z\mathbf{X}(z) = \mathbf{A}\mathbf{X}(z) + \mathbf{B}\mathbf{U}(z), \tag{21}$$

$$\mathbf{Y}(z) = \mathbf{C}\mathbf{X}(z) + \mathbf{D}\mathbf{U}(z). \tag{22}$$

From equations (21) and (22) we have $\mathbf{Y}(z) = \mathbf{H}(z)\mathbf{U}(z)$, where

$$\mathbf{H}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}. \tag{23}$$

In order to express the output $\mathbf{y}[k]$ in terms of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and the input, it is firstly necessary to write $\mathbf{y}[k]$ in terms of the initial state vector (initial condition) $\mathbf{x}[0]$ and the input $\mathbf{u}[m]$, $0 \leq m \leq k$. Again, from (19) it is found

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{A}^m\mathbf{B}\mathbf{u}[k-m-1], \qquad k \geq 1, \tag{24}$$

and from (20)

$$\mathbf{y}[k] = \mathbf{C}\mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{C}\mathbf{A}^m\mathbf{B}\mathbf{u}[k-m-1] + \mathbf{D}\mathbf{u}[k]. \qquad k \geq 1. \tag{25}$$

Particularly, if $\mathbf{x}[0] = 0$, (25) becomes

$$\mathbf{y}[k] = \mathbf{D}\mathbf{u}[k] + \sum_{i=1}^{k} \mathbf{C}\mathbf{A}^{k-1}\mathbf{B}\mathbf{u}[k-i]. \tag{26}$$

Comparing (26) to the next equation

$$\mathbf{y}[k] = \sum_{i=-\infty}^{\infty} \mathbf{h}[i]\mathbf{u}[k-i] = \sum_{i=-\infty}^{\infty} \mathbf{h}[k-i]\mathbf{u}[i], \tag{27}$$

the impulse response can be written as

$$\mathbf{h}[k] = \begin{cases} \mathbf{D}, & k = 0, \\ \mathbf{C}\mathbf{A}^{k-1}\mathbf{B}, & k > 0. \end{cases} \tag{28}$$

Equation (28) implies that all impulse response coefficients can be computed based on matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ (VAIDYANATHAN, 1993).

If z_p is a pole of at least one element $H_{km}(z)$ in the matrix $\mathbf{H}(z)$, z_p is a pole of $\mathbf{H}(z)$. From equation (23)

$$\mathbf{H}(z) = \mathbf{D} + \frac{\mathbf{C}\mathbf{R}(z)\mathbf{B}}{\det(z\mathbf{I} - \mathbf{A})}, \tag{29}$$

where $\mathbf{R}(z)$ is the adjugate matrix of $(z\mathbf{I} - \mathbf{A})$. Then, if z_p is a pole, it is a zero of the determinant that appears in (29). Therefore,

$$\begin{aligned} z_p \text{ is a pole} \quad &\Rightarrow \quad \det(z_p\mathbf{I} - \mathbf{A}) = 0 \\ &\Rightarrow \quad (z_p\mathbf{I} - \mathbf{A})\mathbf{v} = 0 \qquad (\text{for some } \mathbf{v} \neq 0) \qquad (30) \\ &\Rightarrow \quad \mathbf{A}\mathbf{v} = z_p\mathbf{v}, \qquad \mathbf{v} \neq 0, \qquad (31) \end{aligned}$$

which shows that the poles of $\mathbf{H}(z)$ are eigenvalues of $\mathbf{A}$. All eigenvalues of $\mathbf{A}$ are poles of $\mathbf{H}(z)$ when the state-space description is minimal. In this way, system $\mathbf{H}(z)$ is stable if all of its eigenvalues satisfy $|\lambda_i| < 1$ (i.e., all these eigenvalues are inside the unit circle). It is possible that all eigenvalues of $\mathbf{A}$ are zero, this would be the FIR case (VAIDYANATHAN, 1993).

The definition of minimal realization is presented from the following lemmas. Proofs and further discussion can be found in Vaidyanathan (1993).

Lemma 1. (VAIDYANATHAN, 1993) *A structure with n delays (i.e., n state variables) is reachable if, and only if, the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are such that the matrix*

$$\mathcal{R}_{\boldsymbol{A},\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} \tag{32}$$

has full rank n.

Lemma 2. (VAIDYANATHAN, 1993) *A structure with n delays (i.e., n state variables) is observable if, and only if, the matrices $\boldsymbol{C}$ and $\boldsymbol{A}$ are such that the matrix*

$$\mathcal{S}_{\boldsymbol{C},\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \boldsymbol{C}\boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix} \tag{33}$$

has full rank n.

Lemma 3. (VAIDYANATHAN, 1993) *Suppose that a realization ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$) is reachable and observable. Then it is minimal, i.e., it does not exist an equivalent structure for the same transfer function with fewer delays.*

An interesting property of minimal realizations is that all of them for a system $\mathbf{H}(z)$ are related by similarity transformations.

Lemma 4. (VAIDYANATHAN, 1993) *Let ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$) and ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{D}$) be two minimal realizations of a system $\boldsymbol{H}(z)$. Then there exists a unique similarity transformation $\boldsymbol{T}$ which transforms ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$) to ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{D}$).*

From Lemma 4 another important fact can be stated:

Lemma 5. (VAIDYANATHAN, 1993) *Let ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$) be reachable. Then ($\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{D}$) obtained by a similarity transformation $\boldsymbol{T}$ is also reachable. Same holds for observability.*

The stability of $\mathbf{A}$ can be expressed in terms of a simple algebraic equation:

Lemma 6. ***Discrete-time Lyapunov lemma*** (VAIDYANATHAN, 1993)

1. *Let* $\boldsymbol{A}$ *be* $n \times n$*,* $\boldsymbol{C}$ $p \times n$ *and* $\boldsymbol{C}$*,* $\boldsymbol{A}$ *observable. Let the equation*

$$\boldsymbol{A}^{\dagger}\boldsymbol{P}\boldsymbol{A} + \boldsymbol{C}^{\dagger}\boldsymbol{C} = \boldsymbol{P}, \tag{34}$$

be satisfied for some Hermitian positive definite $\boldsymbol{P}$*. Then* $\boldsymbol{A}$ *is stable.*

2. *Let* $\boldsymbol{Q}$ *be some* $n \times n$ *Hermitian positive semidefinite matrix so that it can be written as* $\boldsymbol{Q} = \boldsymbol{C}^{\dagger}\boldsymbol{C}$ *for some* $\boldsymbol{C}$*. Let* $\boldsymbol{A}$ *be* $n \times n$ *stable with* $\boldsymbol{C}$*,* $\boldsymbol{A}$ *observable. Then the algebric equation (34) has an unique solution* $\boldsymbol{P}$*, and this solution is positive definite.*

A Hermitian matrix $\mathbf{Q}$ is a square matrix with complex entries that is equal to its own conjugate transpose, namely, $q_{i,j} = q^*_{j,i}$. Besides, $\mathbf{Q}$ is positive definite if for an arbitrary $n \times 1$ vector $\mathbf{x}$ it has $\mathbf{x} \neq 0 \Rightarrow \mathbf{x}^T\mathbf{Q}\mathbf{x} > 0$, if $\mathbf{x} \neq 0 \Rightarrow \mathbf{x}^T\mathbf{Q}\mathbf{x} \geq 0$ then matrix $\mathbf{Q}$ is positive semidefinite (SLOTINE; LI, 1991).

2.2 Further comments

A continuous time system accepts continuous time inputs and generates continuous time outputs. On the other hand, a discrete-time system starts with discrete-time inputs and it has discrete-time outputs. In both cases, the system can be causal or non-causal, memoryless or non-memoryless, linear or nonlinear, time-invariant or time-variant.

There are several approaches to work on linear systems theory. One of them is the transfer functions concept, which is used to describe the relations among inputs and outputs in linear time-invariant systems. However, the applicability of this concept is limited to this kind of system. The trend in real system modeling increases its complexity to guarantee high precision. Along with many other features, more complex systems can have multiple inputs and outputs, can be time-varying, etc. An alternative to work with multiple inputs and multiple outputs systems or time-variant systems is the state-space realization (OGATA, 2010).

In this thesis, alternatives to transfer functions through state-space realizations are presented. For this reason, the state-space theory for continuous and discrete-time systems was presented in Subsection 2.1.1. Stability, controllability, observability and minimality are important concepts concerning state-space theory. These concepts were presented in Section 2.1.2 and will also be utilized in Chapter 5. Nevertheless, for a complete review of these contents, refer to the following sources: Chen (1999), Slotine and Li (1991), Elali (2012), Elali (2008).

3 Orthonormal basis functions

When approaching the impulse response of a time-invariant linear system by a finite sum of exponentials, the modeling problem and identification are considerably simplified. In most cases, approximations by finite impulse response are used. However, when using the orthonormal filters with infinite impulse response (IIR) as basis functions, the structures of models, which are much more effective, can be obtained (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005).

Some of the main orthonormal basis functions (OBF) applications are in system identification and adaptive signal processing, where models parameterizations in terms of finite expansion coefficients are attractive in spite of the linear structure of the model parameters. The orthonormal series functions study in signal representation is important in automation, control and signal processing, and it is subject of extensive study nowadays (TANGUY et al., 2002).

The first mention of rational orthonormal basis functions seems to have occurred at the end of the 1920s with the independent work developed by Takenaka (1925) and Malmquist (1926). The basis functions that were considered are now called generalized orthonormal basis functions (GOBF).

In the 1930s and 1940s, applications of the Malmquist's study in discrete and continuous-time were presented in various studies by Walsh, especially Walsh (1935), which brings the majority of his studies. At the same time, Wiener started his studies about continuous-time Laguerre basis, although his work just appeared in 1949, (WIENER, 1949), there are mentions of his ideas in Wiener (1933) and in a publication of one of his students, Lee (1933).

In the 1950s, emerged the studies of Grenander and Szegö (1958), Kautz (1952), Kautz (1954), Head (1956). Based on a mathematical point of view, a standard reference for orthogonal polynomials was presented by Sansone (1959). In the following decade, the highlights are the works of Broome (1965), Mendel (1966), Ross (1964) about GOBF, Clowes (1965) specific on Laguerre bases, and the book of Horowitz (1963) with orthonormal parameterization for control theory applications, while the mathematical point of view stands out in the book of Akhiezer (1965).

Published in the seventies, there were the works of Dewilde, Vieira and Kailath (1978) about the use of GOBFs for optimal prediction of stationary processes and Kailath, Vieira and Morf (1978) with further discussion of polynomials orthonormal bases. At that same time

Laguerre basis for identification systems were studied by King and Paraskevopoulos (1977), King and Paraskevopoulos (1979).

From 1980s to 2000s, there was an explosion of interest in the study of orthonormal parameterizations. In signal processing (DAVIDSON; FALCONER, 1991; DEN BRINKER, 1994; NINNESS; GOMEZ, 1998; PEREZ; TSUJII, 1991; WILLIAMSON, 1995), in control theory oriented system identification setting (AKÇAY; NINNESS, 1998; BODIN; WAHLBERG, 1994; CLEMENT, 1982; GUNNARSSON; WAHLBERG, 1991; CLUETT; WANG, 1992; HEUBERGER; VAN DEN HOF; BOSGRA, 1995; NINNESS; GUSTAFSSON, 1997; WAHLBERG, 1991; WAHLBERG, 1994), in model approximation setting (MÄKILÄ, 1990b; MÄKILÄ, 1990a; PARTINGTON, 1991; WAHLBERG; MÄKILÄ, 1996), in adaptive control problem (DUMONT; FU; ELSHAFEI, 1991; ZERVOS; BÉLANGER; DUMONT, 1988; ZERVOS; DUMONT, 1988), among others.

In the last years, many works have been developed in the context of orthonormal basis functions, on approximation properties, new methods for optimal approximations, state-space realizations, fast algorithms for adaptive signal processing, system identifications and other applications in control theory and signal processing. The list of developed works is very extensive, however, the book of Heuberger, van den Hof and Wahlberg (2005) is a synthesis of all the theory in its current context.

This chapter discusses the main concepts of orthogonality and orthonormality for continuous functions and vectors that can represent discrete functions. In addition to other appropriate concepts for the easy reading of the text. In this thesis, this study is one of the most important concepts. Therefore, ideas about it are explored in the following sections, from the basic concepts until more complex ideas.

Section 3.1 presents definitions about orthogonality and orthonormality. The Gram-Schmidt procedure is shown in Section 3.2. The orthonormal basis functions idea is presented in Section 3.3, FIR networks, in Section 3.4, Laguerre Functions are in Section 3.5, Kautz Functions, in Section 3.6 and Takenaka-Malmquist Functions are in Section 3.7.

3.1 Orthogonality and orthonormality

The main concepts and definitions about orthogonality and orthonormality are presented here in a brief way. For further study concepts regarding orthogonal functions, the following reading are recommended: Bultheel (1999), Jackson (2004), Sansone (1959) and Szegö (1959).

Definition 7. (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005) *Let V be a vector* *space over K with inner product $\langle , \rangle$ and $u, v \in V$. It is said that u and v are orthogonal if $\langle u, v \rangle = 0$.*

Definition 8. (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005) *A subset $\mathcal{A}$ of V is named orthogonal if its elements are orthogonal in pairs, and it is said that $\mathcal{A}$ is an orthonormal subset if it is an orthogonal set and if $\|u\| = 1, \forall u \in \mathcal{A}$. Or else, a base $\{v_1, v_2, \cdots, v_n\}$ of V is named orthonormal if $\langle v_i, v_j \rangle = \delta_{i,j}$, where*

$$\delta_{i,j} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

The previous paragraph defines orthogonality and orthonormality concepts for vectors $u, v \in V$, however such concepts can be extended to continuous functions, as follows.

Definition 9. (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005) *Two functions f and g are named orthogonal in an interval $a \leq x \leq b$ if, and only if, their inner product $\langle f, g \rangle$ is equal to zero for $f \neq g$ in the interval, namely*

$$\langle f, g \rangle = \int_a^b f(x) g(x) dx = 0 \qquad \text{if} \qquad f \neq g. \tag{35}$$

Definition 10. (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005) *An orthogonal set $f_m = (f_1, f_2, \cdots, f_n)$, defined in an interval $a \leq x \leq b$, whose functions have unit norm, is denominated orthonormal if*

$$\langle f_m, f_n \rangle = \int_a^b f_m^* f_n dx = \delta_{m,n}, \tag{36}$$

where

$$\delta_{m,n} = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

The concept of orthonormality for discrete functions can be related to the previous definitions for vectors, namely, as following.

Definition 11. (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005) *Let $k = 1, 2, \cdots$ be an increasing sequence of integer numbers. The set of functions $\{p_m[k]\}$ is named orthonormal in the interval I if*

$$\sum_{k \in I} p_m[k] p_n[k] = \delta_{m,n} = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n, \end{cases} \tag{37}$$

An equivalent definition to (37) for functions defined in the complex domain z is

$$\frac{1}{2\pi j} \oint_C P_m(z) P_n(z^{-1}) \frac{dz}{z} = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n, \end{cases} \tag{38}$$

where $P_m(z) = \mathcal{Z}[p_m(k)]$ and the region C is the unitary circle (BROOME, 1965).

The use of orthonormal basis functions (OBF) has advantages in the representation of models, simplifying the associated problems of identification and control, allows the incorporation of prior knowledge about the dynamic system, there is no feedback of forecast errors, the base is complete, among others. The most used orthonormal functions to represent models using OBFs are Laguerre Functions, Kautz Functions and the Generalized Orthonormal Basis Functions (MACHADO, 2011; CAMPELLO; OLIVEIRA; AMARAL, 2007; OLIVEIRA; CAMPELLO; AMARAL, 2007).

3.2 The Gram-Schmidt procedure

The Gram-Schmidt procedure is a method for a vector set orthogonalization in a space with inner product. This process consists of receiving a finite and linearly independent set of vectors $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and return an orthonormal set $S' = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ that spans the same initial subspace S (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005).

The projection operator is defined by

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u},$$

and The Gram-Schmidt procedure works as following

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), \\
&\vdots \\
\mathbf{u}_\ell &= \mathbf{v}_\ell - \sum_{j=1}^{\ell-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_\ell),
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_1 &= \frac{\mathbf{u}_1}{||\mathbf{u}_1||}, \\
\mathbf{e}_2 &= \frac{\mathbf{u}_2}{||\mathbf{u}_2||}, \\
\mathbf{e}_3 &= \frac{\mathbf{u}_3}{||\mathbf{u}_3||}, \\
&\vdots \\
\mathbf{e}_\ell &= \frac{\mathbf{u}_\ell}{||\mathbf{u}_\ell||}.
\end{aligned}$$

The sequence $\mathbf{u}_1, \cdots, \mathbf{u}_\ell$ is the required system of orthogonal vectors, while the sequence $\mathbf{e}_1, \cdots, \mathbf{e}_\ell$ form an orthonormal set.

The vector space of functions with inner product can also be considered, for example, let $\overline{F}_1(z) = \dfrac{1}{z-a}$ and $\overline{F}_2(z) = \dfrac{1}{z-b}$, $-1 < a, b < 1$, $a \neq b$ be two functions. It is easy to check that $\overline{F}_1(z)$ and $\overline{F}_2(z)$ are linearly independent, but they are not orthonormal in the space $\mathcal{H}_2(\mathbb{E})$.

In fact, through the Gram-Schmidt procedure follows

$$
\begin{aligned}
K_1(z) &= \overline{F}_1(z), \\
F_1(z) &= \frac{\frac{1}{z-a}}{\frac{1}{\sqrt{1-a^2}}} = \frac{\sqrt{1-a^2}}{z-a}, \\
K_2(z) &= \overline{F}_2(z) - \langle \overline{F}_2, F_1 \rangle F_1(z) = \frac{1}{z-b} - \frac{\sqrt{1-a^2}}{1-ab}\frac{\sqrt{1-a^2}}{z-a} = \frac{b-a}{1-ab}\frac{1}{z-b}\frac{1-az}{z-a}, \\
F_2(z) &= \frac{\sqrt{1-b^2}}{z-b}\frac{1-az}{z-a}.
\end{aligned}
$$

Functions $F_1(z)$ and $F_2(z)$ are orthonormal constructions from $\overline{F}_1(z)$ and $\overline{F}_2(z)$, respectively, and they span the same subspace.

Function $F_2(z)$ can be rewritten:

$$F_2(z) = \frac{\sqrt{1-b^2}}{z-b} G_b(z),$$

where

$$G_b(z) = \frac{1-az}{z-a},$$

and $G_b(z)$ is named first-order all-pass transfer function, which satisfies $G_b(z)G_b(1/z) = 1$.

Let

$$\overline{F}_3(z) = \frac{1}{z-c}, \qquad -1 < c < 1, c \neq a, b,$$

be a third function. Then, according to Heuberger, van den Hof and Wahlberg (2005), it has

$$
\begin{aligned}
K_3(z) &= \overline{F}_3(z) - \langle \overline{F}_3, F_2 \rangle F_2(z) - \langle \overline{F}_3 F_1 \rangle F_1(z), \\
F_3(z) &= \frac{K_3(z)}{\|K_3\|},
\end{aligned}
$$

where $\langle \overline{F}_3, F_1 \rangle = F_1(1/c)/c$ and $\langle \overline{F}_3, F_2 \rangle = F_2(1/c)/c$. Further computations give

$$F_3(z) = \left(\frac{\sqrt{1-c^2}}{z-c}\right)\left(\frac{1-az}{z-a}\right)\left(\frac{1-bz}{z-b}\right),$$

which is orthogonal to $F_1(z)$ and $F_2(z)$. In this way, if projecting $\overline{F}_3$ onto the orthogonal space to the two first basis functions corresponds to multiplication by an all-pass function $G_b(z)$ containing the same poles as the previous two basis functions. Since

$$G_b(z) = \frac{(1-az)(1-bz)}{(z-a)(z-b)},$$

Heuberger, van den Hof and Wahlberg (2005) affirms that this example can be generalized to a set of n first-order transfer functions with distinct real poles.

3.3 Orthonormal basis functions

The orthonormal basis functions satisfies the completeness criteria in the square-summable functions space in the interval $[0, \infty)$. Furthermore, functions from these bases can be approximated with arbitrary precision, through a linear combination of finite functions from the basis, of any linear function in the L_2 space (HEUBERGER; VAN DEN HOF; BOSGRA, 1995; CAMPELLO; OLIVEIRA; AMARAL, 2007; OLIVEIRA; CAMPELLO; AMARAL, 2007; MACHADO, 2011).

The output of a digital filter is the linear sequence $\{y[k]\}$ formed from the input sequence $\{u[k]\}$. If the filter is of the recursive type, the output sequence is generated by a recursion relation

$$y[k] = \sum_{n=0}^{R} a_n u[k-n] + \sum_{n=1}^{S} b_n y[k-n] \tag{39}$$

that uses the past outputs and the past and actual inputs to generate the actual output. In equation (39), $R \in \mathbb{Z}_+^*$ and $S \in \mathbb{Z}_+$. The recursive method is a numerical scheme especially effective for both generating sequences and filtering data in digital computers (BROOME, 1965).

According to Broome (1965), the scalar product between the sequences $\{y[k]\}$ and $\{z^k\}$ is

$$\sum_{k=0}^{\infty} y[k]z^k = \sum_{n=0}^{R} a_n \sum_{k=0}^{\infty} u[k-n]z^k + \sum_{n=1}^{S} b_n \sum_{k=0}^{\infty} y[k-n]z^k. \tag{40}$$

Using the following z-transform property

$$\sum_{k=0}^{\infty} y[k-n]z^k = z^n Y(z)$$

and

$$\sum_{k=0}^{\infty} u[k-n]z^k = z^n U(z),$$

then (40) can be written as

$$Y(z) = \sum_{n=0}^{R} a_n z^n U(z) + \sum_{n=1}^{S} b_n z^n Y(z) \tag{41}$$

or

$$Y(z) = \frac{U(z) \sum_{n=0}^{R} a_n z^n}{1 - \sum_{n=1}^{S} b_n z^n}. \tag{42}$$

When the input signal is $u[k] = \delta[k]$, where $\delta[k]$ is the unitary impulse response, it implies that $U(z) = \mathcal{Z}(\delta[k]) = 1$. Then the transfer function $H(z) = Y(z)/U(z)$ is given by

$$H(z) = \frac{\sum_{n=0}^{R} a_n z^n}{1 - \sum_{n=1}^{S} b_n z^n}, \tag{43}$$

or

$$H(z) = \frac{a_0 \prod_{n=1}^{R} (1 - \alpha_n z)}{\prod_{n=1}^{S} (1 - \beta_n z)}, \tag{44}$$

which is the factored form, where the coefficients α_n and β_n are derived from a_n and b_n, respectively. In order to restrict h to real values, a_n and b_n must be real and consequently α_n and β_n also must be real or must happen in conjugate pairs in the product (BROOME, 1965).

Following the procedure for constructing sets of orthonormal sequences, Broome (1965) shows that generally when β_n is a complex pole, functions $H(z)$ have the form

$$H(z) = \frac{C(z - a_1) \prod_{n=1}^{j-1} (\beta_n - z)(\beta_n^* - z)}{\prod_{n=1}^{j} (1 - \beta_n z)(1 - \beta_n^* z)}, \tag{45}$$

where C is a constant that depends on α_n and β_n, and β_n^* represents the complex conjugate of β_n.

Functions such as (45) are named generalized orthonormal functions, and they constitute a complete basis in the Lebesgue space. Laguerre and Kautz functions are some particular cases of these functions (SARROUKH; EIJNDHOVEN; BRINKER, 2001).

In this way, for any function $h[k] \in L^2[0,\infty)$ there is $n \in \mathbb{Z}_+^*$, such that for any $\epsilon > 0$ we have

$$\sum_{k=0}^{\infty}\left(h[k] - \sum_{i=1}^{n} c_i\phi[k]\right)^2 < \epsilon, \tag{46}$$

where $\{\phi_1[k], \cdots, \phi_n[k]\}$ is the set of the n first orthonormal functions of the basis, $c_1, \cdots, c_n$ are not scalars. Then

$$h[k] = \sum_{i=1}^{\infty} c_i\phi_i[k], \tag{47}$$

as a consequence, for any integer $j > 0$

$$\begin{aligned}\sum_{k=0}^{\infty} h[k]\phi_i[k] &= \sum_{k=0}^{\infty}\phi_j[k]\sum_{i=1}^{\infty} c_i\phi_i[k] \\ &= \sum_{i=1}^{\infty} c_i\sum_{k=0}^{\infty}\phi_j[k]\phi_i[k]. \end{aligned}\tag{48}$$

From this result and from the orthonormality condition, it is concluded that the series coefficients are given by

$$c_j = \sum_{k=0}^{\infty} h[k]\phi_j[k]. \tag{49}$$

For a causal linear time-invariant dynamic system, consider $u[k]$ the input, $y[k]$ the output and $h[m]$ the system impulse response. The approximate development of the impulse response in n orthonormal functions is accomplished in the following manner:

$$\begin{aligned} y[k] &= \sum_{m=0}^{k}\hat{h}[m]u[k-m] \\ &= \sum_{m=0}^{k}\sum_{i=1}^{n} c_i\phi_i[m]u[k-m] \\ &= \sum_{i=1}^{n} c_i\sum_{m=0}^{k}\phi_i[m]u[k-m] \\ &= \sum_{i=1}^{n} c_i\varphi_i[k], \end{aligned}\tag{50}$$

where $\varphi_i[k]$ is the convolution of the input signal $u[k]$ with the i-th orthonormal function ϕ_i at an instant k (MACHADO, 2011; CAMPELLO; OLIVEIRA; AMARAL, 2007; OLIVEIRA; CAMPELLO; AMARAL, 2007).

3.4 Finite impulse response networks

Consider Figure 1 that represents a FIR network. The corresponding orthogonal state-space model is obtained by taking

$$x[k] = \left[\begin{array}{cccc} u[k-1] & u[k-2] & \cdots & u[k-n] \end{array}\right]^T.$$

Consequently

$$\begin{aligned} x[k+1] &= \mathbf{A}x[k] + \mathbf{B}u[k], \\ y[k] &= \mathbf{C}x[k] + Du[k], \end{aligned}$$

where

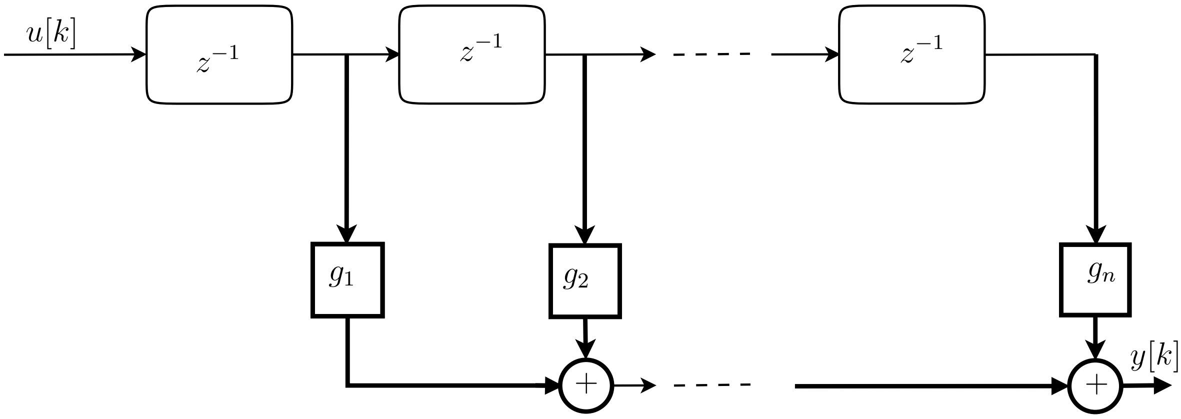
$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} g_1 & g_2 & g_3 & \cdots & g_n \end{bmatrix}$$

and $D = 0$.

Figure 1: FIR network.



Source: Adapted from Heuberger, van den Hof and Wahlberg (2005).

3.5 Laguerre networks

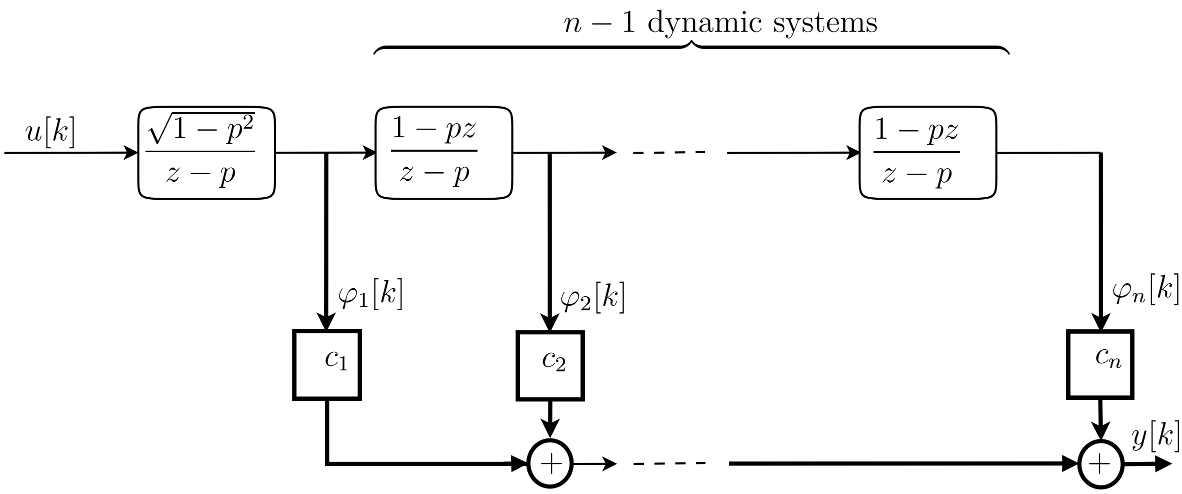
Taking $\beta_n = p$ in (44) where p is a real value, it leads to the special Laguerre case. According to Wahlberg (1994), Laguerre functions are defined in terms of their z-transform, as follows

$$F_i(z) = Z\{\phi_i[k]\} = \frac{\sqrt{1-p^2}}{z-p}\left(\frac{1-pz}{z-p}\right)^{i-1}, \tag{51}$$

where $i = 1, 2, \cdots$ and p is known as Laguerre pole.

According to Silva (1994), Laguerre functions present as an advantage the fact that their transforms are rational functions with very simple repetitive functions. Since each Laguerre function $\phi_i[k] = \mathcal{Z}^{-1}[F_i(z)]$ is null for $k < 0$, it is possible to construct a causal linear time-invariant system with impulse response equal to $\phi_i(k)$, as represented in Figure 2 ($\varphi_i[k]$ is i-th element in the state vector defined as $\varphi[k]$). When $p = 0$ this is known as finite impulse response model representation (SILVA, 1995).

Figure 2: Block representation for Laguerre Functions $\phi_i[k]$.



Source: Adapted from Machado (2011).

3.5.1 State-space description for Laguerre Functions

Campello (2002) affirms that any set of orthonormal functions can be represented in the state-space. In the Laguerre Functions case, the model representation in the state-space for a linear time-invariant system is as follows:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}u[k], \\ y[k] &= \mathbf{C}\mathbf{x}[k] \end{aligned} \tag{52}$$

where $u[k]$ is the input signal and $\mathbf{x}[k] = [\varphi_1[k], \varphi_2[k], \cdots, \varphi_n[k]]^T$ represents the vector of orthonormal states with order n. Matrix $\mathbf{A}$ and vector $\mathbf{B}$ only depend on the orthonormal basis functions, and $\mathbf{C}$ contains the series expansion terms with the orthonormal functions output.

In case of Laguerre functions, in (52) **A**, **B** and **C** are as follow (OLIVEIRA, 1997):

$$\mathbf{A} = \begin{bmatrix} p & 0 & 0 & \cdots & 0 \\ 1-p^2 & p & 0 & \cdots & 0 \\ (-p)(1-p^2) & 1-p^2 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-p)^{n-2}(1-p^2) & (-p)^{n-3}(1-p^2) & (-p)^{n-4}(1-p^2) & \cdots & p \end{bmatrix}, \tag{53}$$

$$\mathbf{B} = \sqrt{1-p^2} \begin{bmatrix} 1 \\ -p \\ (-p)^2 \\ \vdots \\ (-p)^{n-1} \end{bmatrix}, \tag{54}$$

$$\mathbf{C} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}. \tag{55}$$

3.6 Two-parameter Kautz networks

Kautz Functions also are a special case of equation (44), in a similar form to the Laguerre functions, but in this case $\beta_i = \beta$ and $\beta \in \mathbb{C}$. According to Ninness and Gustafsson (1994), in this case, we have the known Kautz Functions that are defined through their z-transforms

$$F_{2i}(z) = \frac{\sqrt{(1-a^2)(1-b^2)}}{z^2 + b(a-1)z - a} \left(\frac{-az^2 + b(a-1)z + 1}{z^2 + b(a-1)z - a} \right)^{i-1}, \tag{56}$$

$$F_{2i-1}(z) = \frac{\sqrt{(1-a^2)}(z-b)}{z^2 + b(a-1)z - a} \left(\frac{-az^2 + b(a-1)z + 1}{z^2 + b(a-1)z - a} \right)^{i-1}, \tag{57}$$

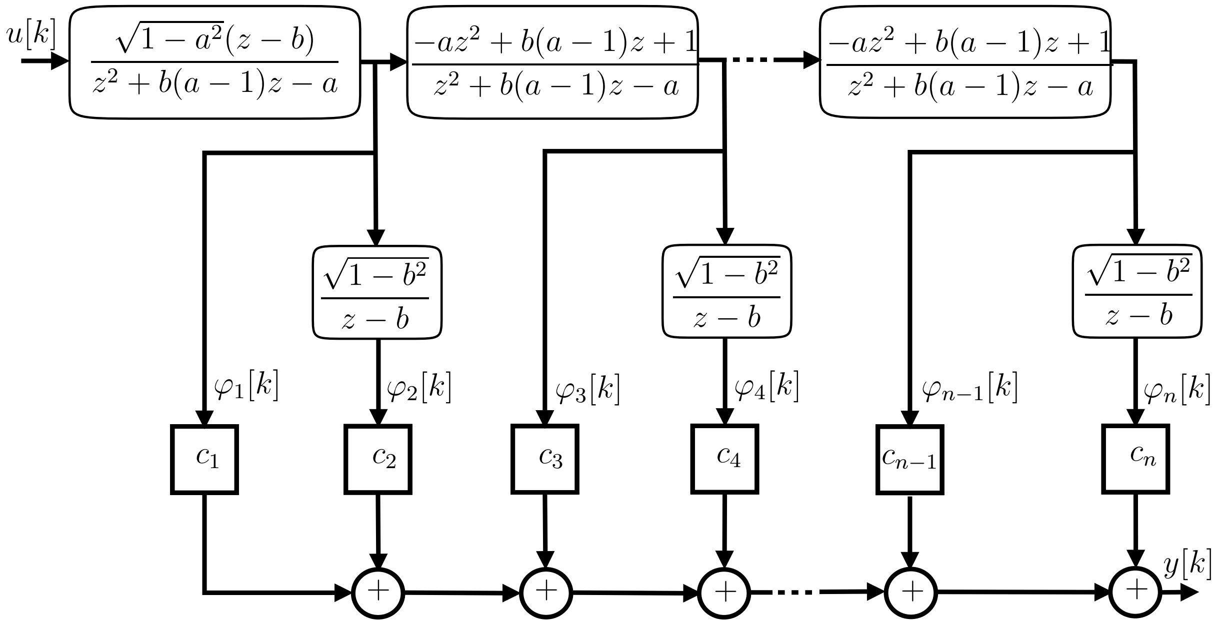
where $i = 1, 2, \cdots$ and the parameters a and b are real constants,

$$b = \frac{\beta + \beta^*}{1 + \beta\beta^*},$$

$$a = -\beta\beta^*,$$

β is named the Kautz pole (NINNESS; GUSTAFSSON, 1994). The Kautz Functions also can be described in a recursive representation and, consequently, in cascade, as represented in Figure 3.

Figure 3: Block representation for Kautz Functions.



Source: Adapted from Machado (2011).

3.6.1 State-space description for Kautz Functions

According to Oliveira (1997), the Kautz Functions defined by equations (56) and (57) also have a representation in the state-space, in the form (52), namely,

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}u[k], \\ y[k] &= \mathbf{C}\mathbf{x}[k] \end{aligned}$$

where:

$$\mathbf{x}[k] = \begin{bmatrix} \varphi_1[k-1] & \varphi_1[k] & \cdots & \varphi_{n-1}[k-1] & \varphi_{n-1}[k] & \varphi_n[k-1] & \varphi_n[k] \end{bmatrix}^T,$$

$$\mathbf{u}[k] = \begin{bmatrix} u[k-1] & u[k] \end{bmatrix}^T,$$

$$\mathbf{A} = \begin{bmatrix} 0 & a_{1,2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{3,4} & \cdots & 0 & 0 & 0 & 0 \\ 0 & a_{4,2} & 0 & a_{4,4} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{2n-3,2n-2} & 0 & 0 \\ a_{2n-2,1} & a_{2n-2,2} & 0 & 0 & \cdots & a_{2n-2,2n-3} & a_{2n-2,2n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{2n-1,2n} \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{2n,2n-2} & 0 & a_{2n,2n} \end{bmatrix}, \tag{58}$$

$$\mathbf{B} = \sqrt{1-a^2} \begin{bmatrix} 0 & 0 \\ b_{2,1} & b_{2,2} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ b_{6,1} & b_{6,2} \\ \vdots & \vdots \\ 0 & 0 \\ b_{2(n-1),1} & b_{2(n-1),2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{59}$$

$$\mathbf{C} = \left[\begin{matrix} 0 & c_1 & 0 & c_2 & \cdots & 0 & c_{n-1} & 0 & c_n \end{matrix} \right. \tag{60}$$

Let i be the line cardinality of matrix $\mathbf{A}$ in (58), and $a_{\ell,m}$ the elements of $\mathbf{A}$, ℓ denotes line while m denotes column:

- when i is an odd number:
  - $a_{i,2i} = 1$,
  - remaining elements are null;
- when $i = 2$:
  - $a_{i,i} = -b(c-1)$,
  - $a_{i,i-1} = c$,
  - remaining elements are null;
- when i is a multiple of 4:

  - $a_{i,i} = b$,
  - $a_{i,i-2} = \sqrt{1-b^2}$,
  - remaining elements are null;

- for other lines with i being an even number:

  - $a_{i,i} = -b(c-1)$,
  - $a_{i,i-1} = c$,
  - $a_{i,i-4j} = (-c)^{j-1}(b(c-1) + (-c)(-b(c-1)))$, $\qquad j = 0, 1, 2, \cdots$ such that $4j < i$,
  - $a_{i,i-(4j+1)} = (-c)^{j-1}((-c)(c) + 1)$, $\qquad j = 0, 1, 2, \cdots$ such that $4j < i$,
  - remaining elements are null.

The elements of matrix **B** in (59) are given by:

- when $i = 2 + 4j$, with $j = 0, 1, 2, \cdots$ such that $j \leq n/2$:

  - $b_{i,1} = (-c)^j$,
  - $b_{i,2} = (-c)^j(-b)$;

- for remaining i's

  - $b_{i,1} = b_{i,2} = 0$.

3.7 The Takenaka-Malmquist Functions

The construction of orthonormal basis using the Gram-Schmidt procedure was explained in the Section 3.2. This construction for the general case with multiple complex poles will possibly lead to the Takenaka-Malmquist Functions (WALSH, 1935; WALSH, 1954; LUKATSKII, 1974; HEUBERGER; VAN DEN HOF; WAHLBERG, 2005):

$$F_k(z) = \frac{\sqrt{1-|\xi_k|^2}}{z-\xi_k} \prod_{i=1}^{k-1} \left(\frac{1-\xi_i^* z}{z-\xi_i} \right), \qquad k = 1, 2, \cdots, \qquad \xi_i \in \mathbb{C}, \ |\xi_i| < 1. \tag{61}$$

This structure can be anticipated from the Cauchy integral formula

$$\langle F_k, F_1 \rangle = F_k \left(\frac{1}{\xi_1^*} \right) \sqrt{1-|\xi_1|^2} = 0, \qquad \forall k > 1,$$

which forces $F_k(z)$ to have a zero at $\dfrac{1}{\xi_1^*}$. Using this argument for $\langle F_k, F_j\rangle$, $j = 2, \cdots, k-1$, we have $F_k\left(\dfrac{1}{\xi_j^*}\right) = 0,\ j < k$.

According to Heuberger, van den Hof and Wahlberg (2005), the projection property of all-pass function

$$G_b(z) = \prod_{i=1}^{k-1}\left(\frac{1-\xi_i^* z}{z-\xi_i}\right),$$

namely,

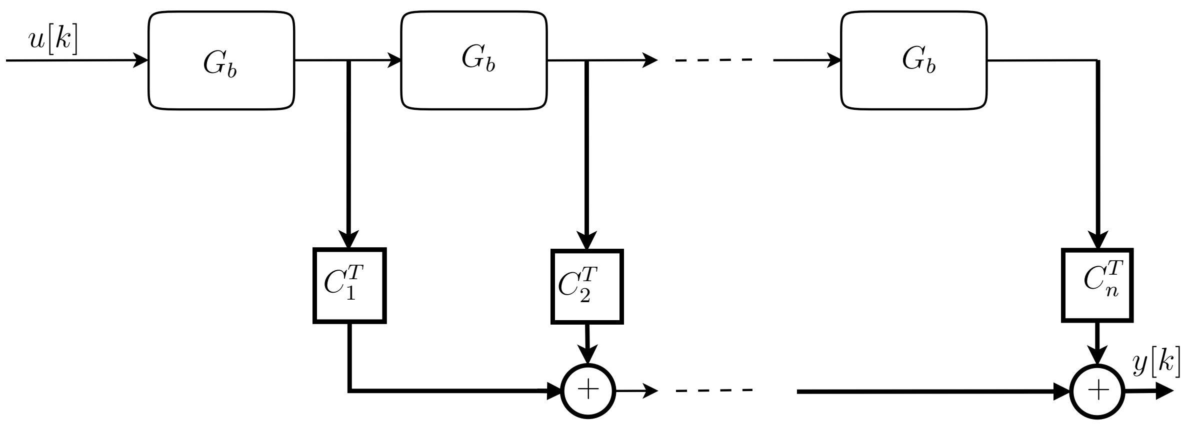
$$\langle F_j, FG_b\rangle = 0,\ j \leq k-1, \qquad \forall F(z) \in \mathcal{H}_2(\mathbb{E}),$$

is known in a somewhat more abstract setting as the Beurling-Lax theorem (REGALIA, 1994; WAHLBERG; MÄKILÄ, 1996).

Summarizing, the generalized orthonormal basis functions (GOBF) class is obtained by connecting n all-pass filters with order n_b, namely, each filter has n_b poles. These poles can be distinct or not, real or complex. A GOBF particular case corresponds to connect different first and second orders all-pass filters, where each all-pass may have distinct poles. These basis functions are also known as Takenaka-Malmquist functions (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005).

A GOBF model block representation is presented in Figure 4. The blocks represented by G_b are orthonormal functions that compose the basis and C_i^T, with $i = 1, \cdots, n$, are the coefficients for the vector that multiplies the series by the functions outputs to form the estimated output $y[k]$.

Figure 4: GOBF model block representation.



Source: Adapted from Machado (2011).

Models with complex conjugate poles can be combined corresponding to similar functions to Kautz functions. Nevertheless, those type of models have some disadvantages when the dynamic characteristics of the system (poles) are unknown and the model must be obtained through identification methods (MACHADO, 2011).

There is another representation for GOBFs which does not present the problems described in Machado (2011). This representation is the known orthonormal basis of functions with generalized internal functions of ladder structure. Laguerre and Kautz models are special cases for GOBFs with internal functions (MACHADO, 2011; HEUBERGER; VAN DEN HOF; WAHLBERG, 2005).

3.7.1 Ladder structures

A rational transfer function $G_b(z)$ is named "internal" if it is stable and satisfies the condition

$$G_b(z)G_b(1/z) = 1, \tag{62}$$

in this manner $G(z)$ is an all-pass filter.

$$G_b(z) = \frac{1 - \xi z}{z - \xi}.$$

In order to write it in the state-space consider $y(z) = G_b(z)u(z)$, that is

$$(z - \xi)y(z) = (1 - \xi z)u(z),$$

which can be rewritten as the difference equation

$$y[k] = \xi y[k-1] - \xi u[k] + u[k-1].$$

Let $x'[k] = \xi y[k-1] + u[k-1]$, then

$$\begin{aligned} x'[k+1] &= \xi y[k] + u[k], \\ y[k] &= x'[k] - \xi u[k], \end{aligned}$$

that can be rewritten as

$$x'[k+1] = \xi x'[k] + u(1 - \xi^2)[k], \tag{63}$$

$$y[k] = x'[k] - \xi u[k]. \tag{64}$$

Let $x[k] = x'[k]/\sqrt{1-\xi^2}$, (63) and (64) become

$$\sqrt{1-\xi^2}x[k+1] = \xi\sqrt{1-\xi^2}x[k] + (1-\xi^2)u[k],$$

$$y[k] \;=\; \sqrt{1-\xi^2}x[k] - \xi u[k],$$

that is

$$\begin{aligned} x[k+1] &= \xi x[k] + \sqrt{1-\xi^2}u[k], \\ y[k] &= \sqrt{1-\xi^2}x[k] - \xi u[k], \end{aligned}$$

or equivalently

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \xi & \sqrt{1-\xi^2} \\ \sqrt{1-\xi^2} & -\xi \end{bmatrix}. \tag{65}$$

This representation with real poles is equivalent to the Laguerre networks representation (MACHADO, 2011; HEUBERGER; VAN DEN HOF; WAHLBERG, 2005).

Considering dynamical system models with a pair of complex conjugate poles, ξ and ξ^*, the internal functions are

$$G_1(z) = \frac{1-\xi^* z}{z-\xi}$$

and

$$G_2(z) = \frac{1-\xi z}{z-\xi^*}.$$

Connecting in cascade these two functions and rewriting, it gives

$$G(z) = \frac{1-(\xi+\xi^*)z + \xi\xi^* z^2}{z^2-(\xi+\xi^*)z+\xi\xi^*}. \tag{66}$$

Taking

$$b = \frac{\xi+\xi^*}{1-\xi\xi^*}$$

and

$$c = -\xi\xi^*,$$

equation (66) becomes

$$G(z) = \frac{-cz^2 + b(c-1)z + 1}{z^2 + b(c-1)z - c}. \tag{67}$$

This representation is equivalent to the Kautz functions. Analogous to (65), in the state-space, it leads to

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \left[\begin{array}{cc:c} b & c\sqrt{1-b^2} & \sqrt{1-b^2}\sqrt{1-c^2} \\ \sqrt{1-b^2} & -bc & -b\sqrt{1-c^2} \\ \hdashline 0 & \sqrt{1-c^2} & -c \end{array}\right], \tag{68}$$

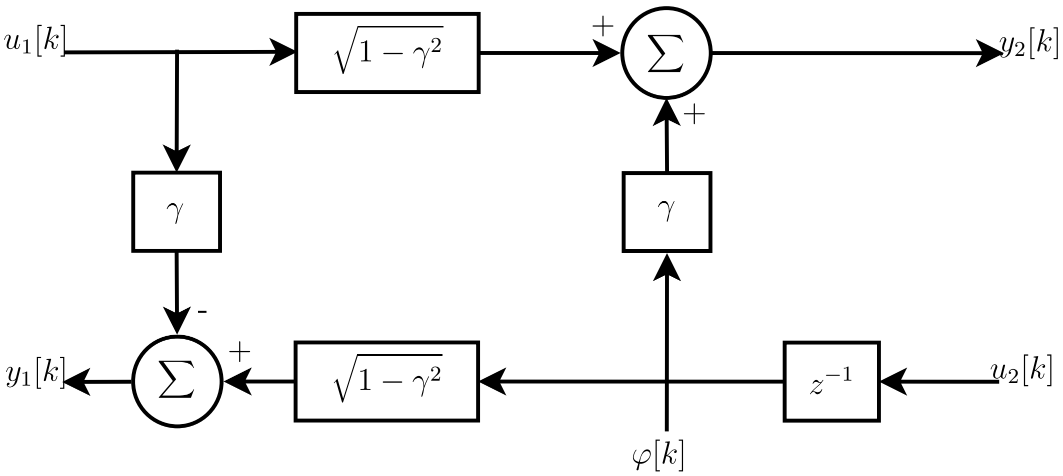
(MACHADO, 2011).

There is another representation to GOBF models with only one pole, containing two inputs and two outputs (GRAY A.; MARKEL, 1975). The model in the state-space is known as ladder structure:

$$\begin{bmatrix} \varphi[k+1] \\ y_1[k] \\ y_2[k] \end{bmatrix} = \left[\begin{array}{c:cc} 0 & 0 & 1 \\ \hdashline \sqrt{1-\gamma^2} & -\gamma & 0 \\ \gamma & \sqrt{1-\gamma^2} & 0 \end{array}\right] \begin{bmatrix} \varphi[k] \\ u_1[k] \\ u_2[k] \end{bmatrix}, \tag{69}$$

where $-1 < \gamma < 1$. Model (69) representation is shown in Figure 5. Further discussions can be found in Machado (2011), Heuberger, van den Hof and Wahlberg (2005).

Figure 5: Block diagram for the normalized ladder structure (two-input two-output all-pass transfer function).



Source: (HEUBERGER; VAN DEN HOF; WAHLBERG, 2005).

4 Wavelet transform and filter banks

The main idea in this chapter is to present the formulation of Sherlock and Monro (1998). In order to do it, filter bank theory related to the Wavelet Transform, especially the Discrete Wavelet Transform, is introduced initially.

4.1 Filter banks

An input signal may arrive in a system in many ways: continuous-time, discrete-time, finite time, and it can be processed in several manners. In this thesis, the greatest interest is in the discrete-time signals that are processed by linear time-invariant operators. These operators are called digital filters. Further details about signals can be found in Oppenheim and Schafer (1975), Oppenheim and Schafer (1989).

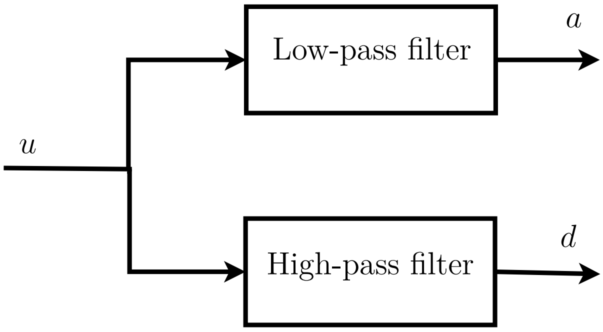
An important point is the relation between filters and frequency. The frequency response can be classified into a number of different bandforms describing which frequency bands the filter passes (the passband) and which it rejects (the stopband). The wavelet filter bank is composed by low-pass and high-pass filters. In a low-pass filter low frequencies are passed and high frequencies are attenuated, while in a high-pass filter high frequencies are passed and low frequencies are attenuated. There are other filters types: band-pass filter, only frequencies in a frequency band are passed, all-pass filter, all frequencies are passed, and others (STRANG; NGUYEN, 1996).

Let u be a signal passing through two filters separately, as shown in Figure 6. Low-pass and high-pass filters split the signal into two parts: a and d, which are approximation and detail signals, respectively.

If this operation is performed on a real digital signal, outputs a and d together have twice as many samples as the initial signal, while the information has not doubled. To overcome this problem we may keep only half of the samples from signals a and d. This is the notion of Downsampling, $(\downarrow 2)$. For M-channel filter banks the sampling is done at the rate $(\downarrow M)$.

The decimation operation is not invertible, nevertheless, in the reconstruction of the initial signal u, there is an operation with opposite role: the Upsampling, $(\uparrow 2)$. Upsampling is the process of lengthening a signal component by inserting zeros between samples, namely, it places zeros into the odd-numbered samples. For M-channel filter banks the sampling is done at the rate $(\uparrow M)$.

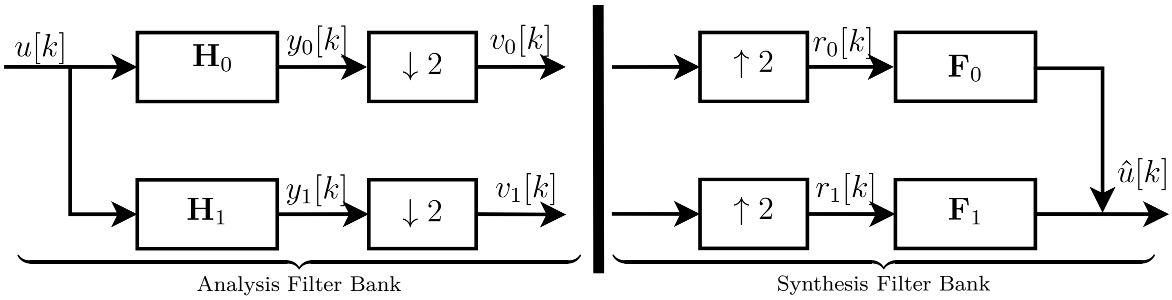
Figure 6: Filtering process by low-pass and high-pass filters.



Source: Adapted from Strang and Nguyen (1996).

A filter bank is a set of filters linked by sampling operators, such as upsampling (decimators) and downsampling (expanders), and sometimes by delays. If the filter bank has two-channel, normally the analysis filters are low-pass and high-pass, as it can be seen in Figure 7 for a two-channel filter bank scheme.

Figure 7: Two-channel filter bank scheme: Analysis and Synthesis Filter Banks.



Source: Adapted from Strang and Nguyen (1996).

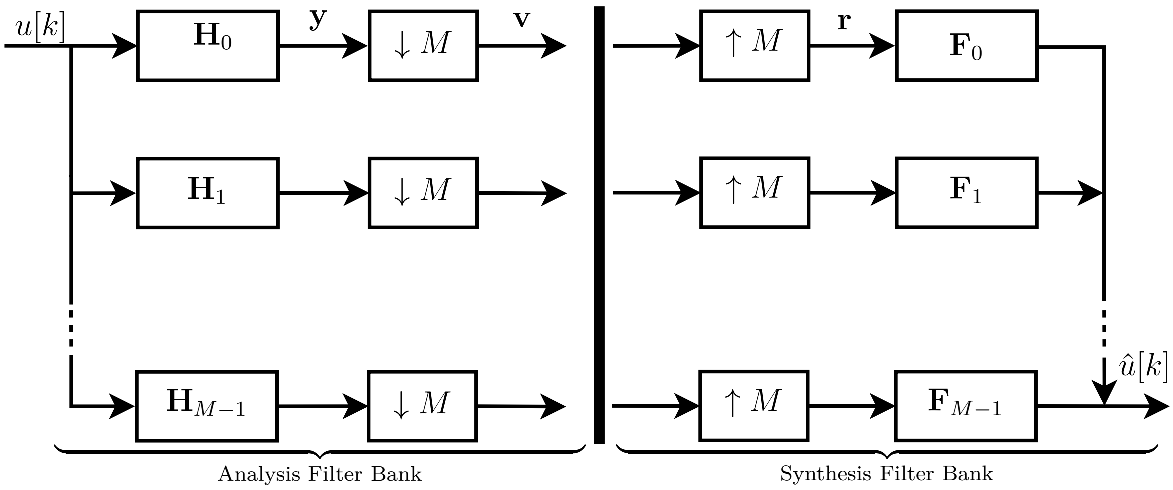
In Figure 7, $u[k]$ is the input signal, $\mathbf{H}_0$ and $\mathbf{H}_1$ are the analysis filters (low-pass and high-pass, respectively), $\downarrow 2$ are the decimators, $\uparrow 2$ are the expanders, $\mathbf{F}_0$ and $\mathbf{F}_1$ are the synthesis filters (low-pass and high-pass, respectively) and $\hat{u}[k]$ is the output signal. This filter bank is biorthogonal, but if the filters for decomposition and reconstruction are the same in reverse order, the filter bank is orthogonal (NIEVERGELT, 1999).

It is worth noting that $\downarrow 2$ is the downsampling operator that removes alternate samples of the filtered signal to avoid the increasing of samples after the filtering process. While, $\uparrow 2$ is the process of lengthening a signal component by inserting zeros between samples. The left part in Figure 7 represents the signal decomposition, whose inverse process is the signal reconstruction, which is represented on the right side.

According to Strang and Nguyen (1996), a filter bank can have M channels, although

$M = 2$ is standard in many applications, it can often be seen $M > 2$. Considering M analysis filters $H_0, H_1, \cdots, H_{M-1}$, the sampling is done at the critical rate ($\downarrow M$) and ($\uparrow M$). For the synthesis the idea is analogous, as we can see in Figure 8 (decomposition is shown in the left side, while reconstruction is shown in the right side of the figure).

Figure 8: M-channel filter bank scheme: Analysis and Synthesis Filter Banks.



Source: Adapted from Strang and Nguyen (1996).

A filter bank is represented by its polyphase matrix, if the filters are known, $H_p(z)$ also is. Otherwise, a suitable matrix can be chosen as a product of simple matrices. In this last case, the filter bank is a "lattice" of simple filters (STRANG; NGUYEN, 1996).

4.2 Wavelet Transform

The most known books about wavelet, such as Daubechies (1992) and Mallat (1998), introduce the wavelet theory from the Fourier transform (FT). Signal analysis using Fourier transform is older, but efficient and there are many works that use it. The name of the FT is a tribute to Jean-Baptiste Joseph Fourier (France, 1768-1830), and it can be seen as a z-transform particular case. Both of them are widely used techniques in signal processing, besides being powerful tools, already known and used for a long time, they continue to play an important role nowadays (OPPENHEIM; SCHAFER, 1975; OPPENHEIM; SCHAFER, 1989).

The FT for a function $f(t)$, according to Daubechies (1992), is defined as

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{70}$$

Signal analysis with Fourier transform is very limited to stationary signals (stationary signals are constant in their statistical parameters over time). This fact can be understood observing equation (70), namely, the signal is analyzed inside the entire interval $-\infty < t < \infty$. In precise words, it means that, although it is efficient to determine the frequency content of a signal, it is not efficient to determine the location on the time when it occurs (MALLAT, 1998).

A known way to overcome this inconvenience is through the Short-time Fourier transform (STFT), which analyses the signal through a short time range within which the signal stays approximately stationary,

$$(T^{WIN}f)(\omega, t) = \int_{-\infty}^{\infty} f(s)\ g(s-t)e^{-i\omega s}ds, \tag{71}$$

where g is the window function (OPPENHEIM; SCHAFER, 1975). The formula in equation (71) is the Short-time Fourier transform, a standard technique for time-frequency localization. For a discrete-time version, t and ω have discrete values: $t = nt_0$, $\omega = m\omega_0$, where $m,\ n\ \in\ \mathbb{Z}$, and $t_0, \omega_0\ >\ 0$ are fixed. Thus, (71) becomes

$$T_{m,n}^{WIN}(f) = \int_{-\infty}^{\infty} f(s)\ g(s-nt_0)e^{-im\omega_0 s}ds, \tag{72}$$

(DAUBECHIES, 1992).

The STFT uses a fixed scale that, once determined, cannot be changed. In this way, once a particular size has been selected for the time window, it will be used for all frequencies. Many signals need a more flexible approach, making possible to vary the window size in order to accurately determine events time or frequency (MALLAT, 1998).

The wavelet theory presents a technique that uses variable windows along the analysis of regions in a signal: the Wavelet Transform. This technique enables the use of long time intervals, where one wants a lower frequency information, and shorter intervals where high frequency information is required (DAUBECHIES, 1990; VELHO; CARVALHO, 2000).

Definition 12. (DAUBECHIES, 1992) *A function $\psi(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such as that the family of functions*

$$\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t - k),$$

where j and k are arbitrary integers numbers, being an orthonormal basis for $L^2(\mathbb{R})$ is called Wavelet Function.

If $\psi(t)$ is a wavelet $\psi_{j,k}(t)$ also is, for any $j, k \in \mathbb{Z}$. The fact that $\psi(t) \in L^2(\mathbb{R})$ means that

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty. \tag{73}$$

If the function $\psi(t) \in L^2(\mathbb{R})$ is a wavelet, its Fourier transform, denoted by $\hat{\psi}(\omega)$, satisfies

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty. \tag{74}$$

Equation (74) is known as admissibility condition (DAUBECHIES, 1992), implying that $\lim_{\omega \to 0} \hat{\psi}(\omega) = 0$. Therefore, being

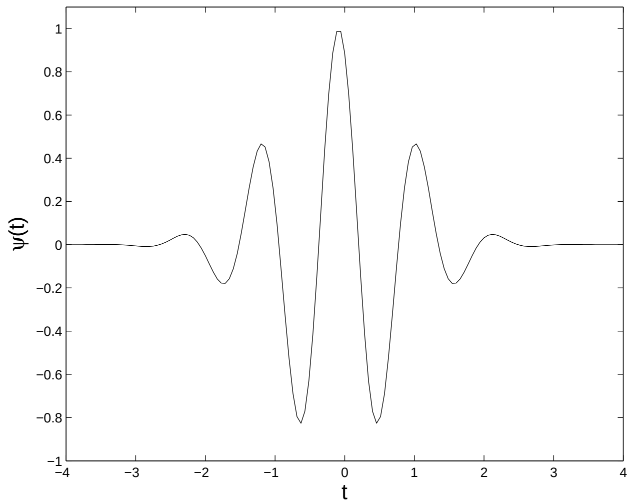
$$\hat{\psi}(\omega) = \int_{-\infty}^{\infty} \psi(t) e^{-j\omega t} dt$$

continuous, then it leads to

$$\int_{-\infty}^{\infty} \psi(t) dt = 0. \tag{75}$$

Equation (75) means that $\psi(t)$ is a wave that oscillates in such a way that the sum of positive and negative areas limited by the curve and the horizontal axis is zero, namely, the graphics of $\psi(t)$ has a short duration waveform as shown in Figure 9.

Figure 9: A wavelet graph (Morlet Wavelet).



Source: (UZINSKI, 2013).

Definition 13. (DAUBECHIES, 1992) *The Continuous Wavelet transform (CWT) is defined as*

$$(T^{WAV} f)(a, b) = |a|^{-1/2} \int_{-\infty}^{\infty} f(t)\ \psi\left(\frac{t-b}{a}\right) dt, \qquad a \neq 0. \tag{76}$$

In equation (76), restricting a and b values to the discrete values $a = a_0^m$ and $b = nb_0 a_0^m$, where $m, n \in \mathbb{Z}$ and $a_0 > 1,\ b_0 > 0$ are fixed, it becomes

$$T_{m,n}^{WAV}(f) = a_0^{-m/2} \int_{-\infty}^{\infty} f(t)\ \psi(a_0^{-m} t - nb_0) dt, \tag{77}$$

which is the Discrete Wavelet Transform.

The parameter a is known as scale parameter and the term $|a|^{-1/2}$ is a normalization factor that ensures that the energy of $\psi_{a,b}(t)$ does not depend on a and b. The parameter b is the translation in the axis t.

Let $\psi_{a,b}(t)$ be the following equation

$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right). \tag{78}$$

The wavelet transform can be written as an inner product between $f(t)$ and $\psi_{a,b}(t)$:

$$(T^{WAV}f)(a,b) = \langle f(t), \psi_{a,b}(t)\rangle = \int_{-\infty}^{\infty} f(t)\psi_{a,b}(t)dt. \tag{79}$$

When $\psi(t)$ is equivalent to $\psi_{1,0}(t)$ it is called "Mother wavelet".

If the continuous signal $f(t)$ was decomposed using a wavelet $\psi(t)$ that satisfies the admissibility condition (74), it is possible to reconstruct it through the inverse continuous wavelet transform (ICWT):

$$f(t) = C_{\psi}^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{a^2}\langle f, \psi_{a,b}\rangle\,\psi_{a,b}da\;db. \tag{80}$$

In the discrete-time domain, in order to reconstruct a function $f(t)$, it is necessary that $\psi_{a,b}$ constitute a frame.

A frame is a collection of functions $\{\phi_n; n \in \mathbb{Z}\}$ for those there exist constants A and B, $0 \leq A < B < \infty$, such that for every function $f \in L^2(\mathbb{R})$

$$A\|f\|^2 \leq \sum_{n=-\infty}^{\infty}|\langle f, \phi_n\rangle|^2 \leq B\|f\|^2.$$

A and B are named the frame infimum and supremum, respectively.

A function or signal $f(t)$ is reconstructed by

$$f(t) = \frac{2}{A+B}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}T_{m,n}^{WAV}(f)\psi_n(t). \tag{81}$$

If the infimum is equal to the supremum we have a tight frame, in the case that $A = B = 1$, all wavelets form an orthonormal basis for $L^2(\mathbb{R})$, then the reconstruction for any function is perfect (DAUBECHIES, 1992).

4.3 Multiresolution analysis (MRA) and Fast Wavelet Transform

Considering the wavelet theory presented previously, the filter bank construction and signal filtering in the basic level, we can present the relationships between wavelets and filter banks. Let $\{h[k]\}$ be the low-pass filter set of coefficients and $\{g[k]\}$ the high-pass set of coefficients. The coefficients $\{g[k]\}$ are obtained from $\{h[k]\}$ through a method called Alternate Flip:

$$g[k] = (-1)^{k+1} h[n-k], \tag{82}$$

therefore $g[k]$ is orthogonal to $h[k]$ (STRANG; NGUYEN, 1996).

Definition 14. (MALLAT, 1998) *Let $\phi[x]$ and $\psi[x]$ be the scale and the wavelet functions, respectively*

$$\phi[x] = \sqrt{2} \sum_{k=-\infty}^{\infty} h[k]\phi[2x-k], \tag{83}$$

$$\psi[x] = \sqrt{2} \sum_{k=-\infty}^{\infty} g[k]\phi[2x-k], \tag{84}$$

again $\phi[x]$ and $\psi[x]$ are orthogonal.

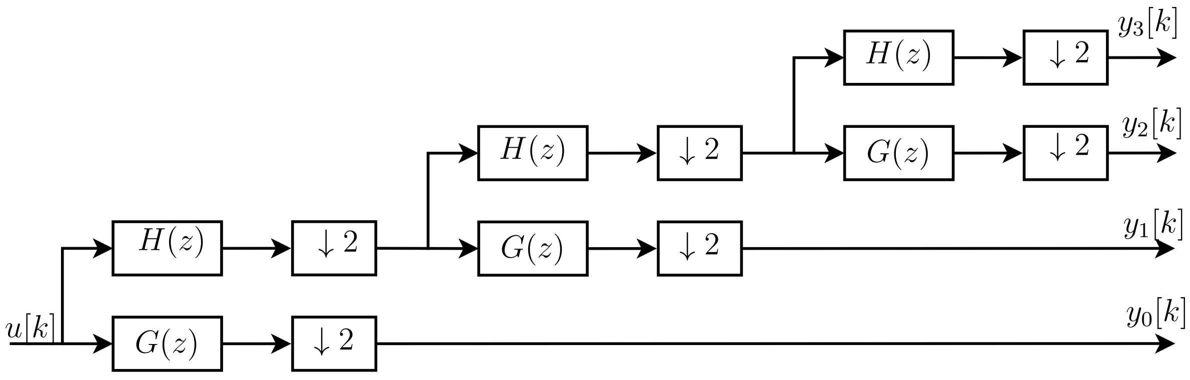
The Fast Wavelet Transform (FWT) is the basic instrument for computational calculus with wavelets. A signal in time domain is converted into a representation in the wavelet basis. On the other hand, the Inverse Fast Wavelet Transform (IFWT) reconstructs the signal from its wavelet representation to the time domain. The wavelet decomposition process of a signal is called analysis and its inverse is called synthesis (VELHO; CARVALHO, 2000).

The algorithm to work with discrete wavelet transform was developed by Mallat (1989c), Mallat (1998). This method uses digital filter banks in a tree structure, as shown in Figure 10 and the equivalent four-channel system in Figure 12. It is worth noting that in these figures just a few decomposition levels are shown, these are examples. The inverse process is shown in Figures 11 and 13.

In this work, notations **G** and **H** were respectively introduced in Figures 10 and 11 for the high-pass and low-pass filters in a two-channel filter bank.

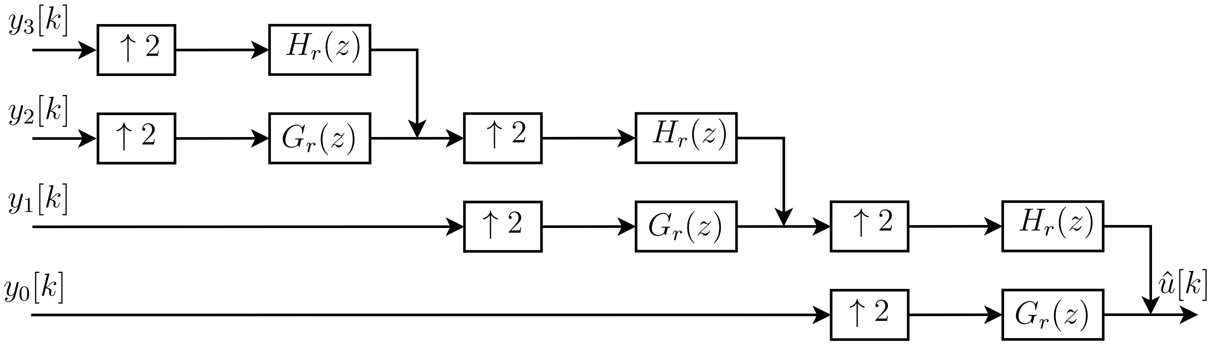
As mentioned in Figures 11 and 13, these filter banks are called Quadrature Mirror Filter Bank (QMF). Works with them were started by Croisier, Esteban and Galand (1976), however the necessary and sufficient conditions to obtain the perfect-reconstruction for the finite impulse response orthogonal filter banks, named quadrature mirror filters, were firstly found by Smith and Barnwell (1984), Mintzer (1985). This theory became complete with the biorthogonal equations in Vetterli (1986) and the paraunitary matrix general theory in Vaidyanathan (1987).

Figure 10: A 3-level binary tree-structured QMF bank.



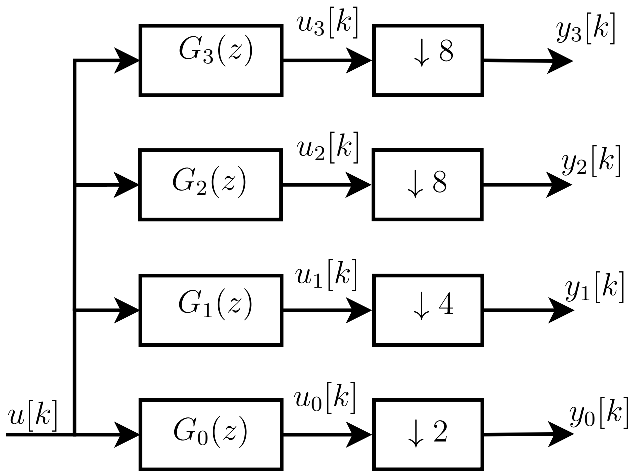
Source: Adapted from Vaidyanathan (1993).

Figure 11: The synthesis bank corresponding to Figure 10.



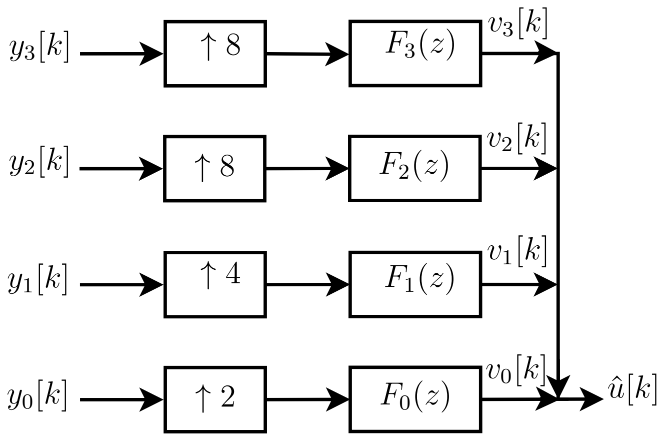
Source: Adapted from Vaidyanathan (1993).

Figure 12: The four channel system equivalent to Figure 10.



Source: Adapted from Vaidyanathan (1993).

Figure 13: The four channel synthesis bank corresponding to Figure 12.

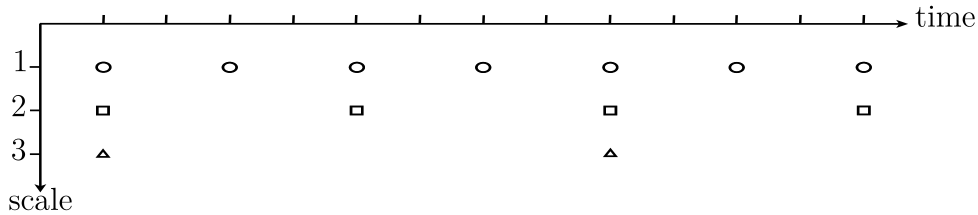


Source: Adapted from Vaidyanathan (1993).

4.4 *Algorithme à Trous*

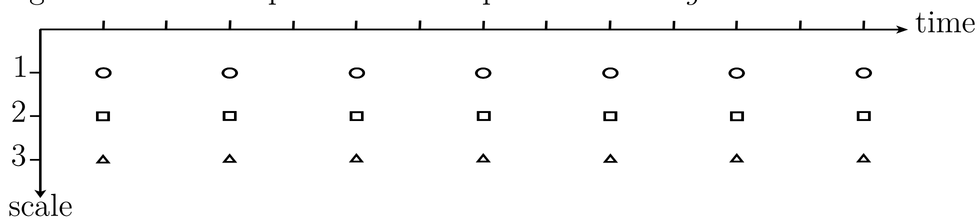
As presented in Section 4.3, the algorithm to work with discrete wavelet transform (Figures 10 and 11) computes the wavelet series expansion on a discrete grid corresponding to scales $a_i = 2^i$ and shifts $b_{i,j} = j2^i$, $i = 0, 1, 2, \cdots$. The samples are shown in Figure 14. According to Vetterli and Kovačević (1995), the associated wavelets form an orthonormal basis, but the transform is not shift-invariant, and it can be a drawback in signal analysis or pattern recognition.

Figure 14: Sampling in the orthogonal discrete-time wavelet series.



Source: Adapted from Vetterli and Kovačević (1995).

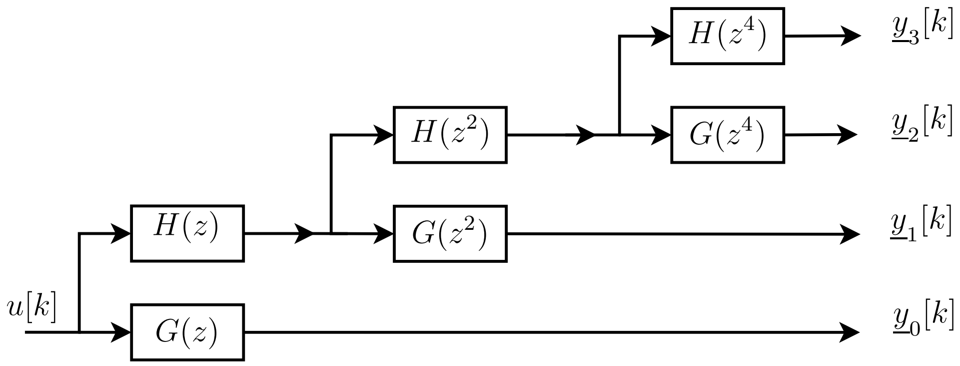
Figure 15: Oversampled time-scale plane in the *Algorithme à Trous*.



Source: Adapted from Vetterli and Kovačević (1995).

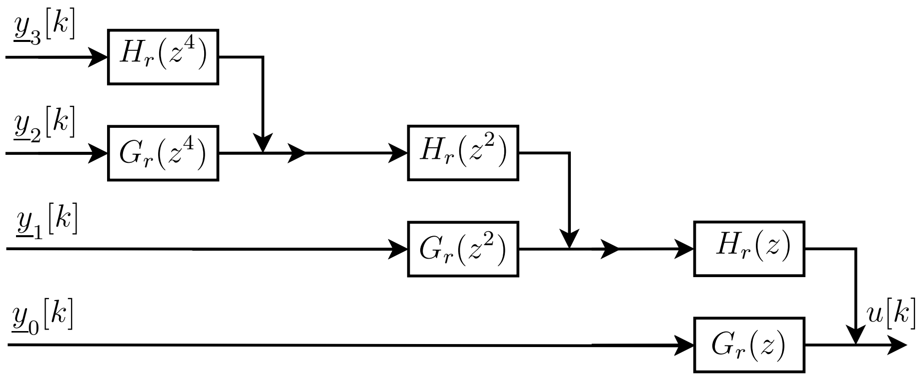
An approach to overcome this problem computes all the shifts, namely, avoiding down-sampling, as presented in Figure 15. In this new algorithm, scales are still restricted to powers of two, while shifts are arbitrary integers. This implementation is called *Algorithme à Trous* (algorithm with holes) (HOLSCHNEIDER et al., 1989), and its correspondent analysis and synthesis filter banks are shown in Figures 16 and 17, respectively.

Figure 16: Equivalent filter bank to the one in Figure 10 using the *Algorithme à Trous*.



Source: Adapted from Vetterli and Kovačević (1995).

Figure 17: Equivalent filter bank to the one in Figure 11 using the *Algorithme à Trous*.



Source: Adapted from Paiva (2005).

As stated by Paiva (2005), the relationship between the coefficients $\mathbf{y}_i[k]$ (FWT) and $\underline{\mathbf{y}}_i[k]$ (*Algorithme à Trous*) is

$$\mathbf{y}_i[k] = (\downarrow 2^i)\underline{\mathbf{y}}_i[k]. \tag{85}$$

4.5 Sherlock and Monro formulation

Sherlock and Monro (1998) developed a formulation to parameterize the orthonormal wavelet space by a set of angular parameters, adapting the work on paraunitary matrices factorization from Vaidyanathan (1993). This formulation is presented in this section, and shall be studied under the scope of orthonormal basis function and as wavelets. It is worth noting that the formulation initially had a weak point, there were no restrictions to ensure a number of vanishing moments greater than one, however, ways to overcome this inconvenience ensuring at least three vanishing moments were obtained by Paiva et al. (2009) and Uzinski (2013), Uzinski et al. (2013), Uzinski et al. (2015a).

It is worth noting that from this paragraph of this text, in some equations the sequences $h[k]$ and $g[k]$ will be denoted by h_i and g_i, respectively, for simplicity and compatibility of notation.

Let the analysis low-pass filter in a two-channel perfect-reconstruction orthonormal filter bank have $2N$ coefficients $\{h_i\}$ and z-transform

$$H(z) = \sum_{i=0}^{2N-1} h_i z^{-i} = H_0(z^2) + z^{-1}H_1(z^2).$$

The terms H_0 and H_1, which denote polyphasic components of $H(z)$, are given by

$$H_0(z) = \sum_{i=0}^{N-1} h_{2i} z^{-1} \tag{86}$$

and

$$H_1(z) = \sum_{i=0}^{N-1} h_{2i+1} z^{-1}, \tag{87}$$

(VETTERLI; KOVAČEVIĆ, 1995).

Vaidyanathan (1993) proposed the polyphase matrix $F_p(z)$ factorization in the following manner

$$\begin{aligned} F_p(z) &= \begin{pmatrix} H_0(z) & H_1(z) \\ G_0(z) & G_1(z) \end{pmatrix} \\ &= \begin{pmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{pmatrix} \prod_{i=1}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} c_i & s_i \\ -s_i & c_i \end{pmatrix}, \end{aligned} \tag{88}$$

where $c_i = \cos(\beta_i)$ and $s_i = \sin(\beta_i)$, and $G_0(z)$ and $G_1(z)$ are the high-pass analysis filter $G(z)$ polyphase components. In this way, this factorization generates all two-channel perfect-reconstruction orthonormal filter banks with an impulse response of length $2N$, namely, any filter bank can be written in terms of N parameters $\beta_i \in [0, 2\pi)$.

According to Sherlock and Monro (1998), in order to the filter bank to correspond to an orthonormal wavelet basis, the first-order regularity constraint

$$\sum_{i=0}^{2N-1} h_i = \sqrt{2}$$

or in an equivalent way

$$\sum_{i=0}^{N-1} \beta_i = \frac{\pi}{4}, \tag{89}$$

must also be satisfied.

Sherlock and Monro formulation which is following shown was developed from (88). Actually, this formulation consists of rewriting (88) in a general recursive form expressing the polyphase matrices corresponding to filters of length $2(N+1)$ in terms of the polyphase matrices for filters of length $2N$

$$F_p^{(N+1)}(z) = F_p^{(N)}(z) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} c_N & s_N \\ -s_N & c_N \end{pmatrix} \tag{90}$$

where $N = 1, 2, \cdots$ and

$$F_p^{(1)} = \begin{pmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{pmatrix}.$$

From (90), recursive formulae expressing the coefficients $\{h_i^{N+1}\}$ for a filter of length $2(N+1)$ in terms of the coefficients $\{h_i^{(N)}\}$ for a filter of length $2N$ are given by

$$H_0^{(N+1)}(z) = H_0^{(N)}(z)c_N - z^{-1}H_1^{(N)}(z)s_N \tag{91}$$

$$H_1^{(N+1)}(z) = H_0^{(N)}(z)s_N + z^{-1}H_1^{(N)}(z)c_N, \tag{92}$$

where $H_0^{(1)}(z) = c_0$ and $H_1^{(1)}(z) = s_0$.

Expanding (91), one obtains

$$\begin{aligned} H_0^{(N+1)}(z) &\equiv \sum_{i=0}^{N} h_{2i}^{(N+1)} z^{-i} \\ &= \left[\sum_{i=0}^{N-1} h_{2i}^{(N)} z^{-i}\right] c_N - z^{-1} \left[\sum_{i=0}^{N-1} h_{2i+1}^{(N)} z^{-i}\right] s_N \\ &= c_N h_0^{(N)} + \sum_{i=1}^{N-1} \left(c_N h_{2i}^{(N)} - s_N h_{2i-1}^{(N)}\right) z^{-i} - s_N h_{2N-1}^{(N)} z^{-N}, \end{aligned}$$

where $h_0^{(1)} = c_0$ and $h_1^{(1)} = s_0$. Equation (92) can be expanded in an analogous manner as (91), and after all, the analysis low-pass filter coefficients is given by

$$\begin{aligned} h_0^{(N+1)} &= \cos(\beta_N)h_0^{(N)} \\ h_{2i}^{(N+1)} &= \cos(\beta_N)h_{2i}^{(N)} - \sin(\beta_N)h_{2i-1}^{(N)} \\ h_{2N}^{(N+1)} &= -\sin(\beta_N)h_{2N-1}^{(N)} \\ \\ h_1^{(N+1)} &= \sin(\beta_N)h_0^{(N)} \\ h_{2i+1}^{(N+1)} &= \sin(\beta_N)h_{2i}^{(N)} + \cos(\beta_N)h_{2i-1}^{(N)} \\ h_{2N+1}^{(N+1)} &= \cos(\beta_N)h_{2N-1}^{(N)} \end{aligned} \tag{93}$$

where $h_0^{(1)} = \cos(\beta_0)$, $h_1^{(1)} = \sin(\beta_0)$ and $i = 0, 1, \cdots, N-1$.

The parameterization based on (93) satisfies the orthonormality conditions for filter banks presented by Strang and Nguyen (1996),

$$\sum_{i=0}^{2N-1} \left[h_i^{(N)}\right]^2 = 1, \;\; N \geq 1, \tag{94}$$

$$\sum_{i=0}^{2N-1-2m} h_i^{(N)} h_{i+2m}^{(N)} = 0, \;\; m = 1, 2, \cdots, N-1, \;\; N \geq 2. \tag{95}$$

For further discussions and proof see Appendix A.

Equations (93) express the $2N$ coefficients $\{h_i\}$ in terms of N angular parameters $\{\beta_i | i = 0, 1, 2, \cdots, N-1\}$ with values in the interval $[0, 2\pi)$. For any choice of $\{\beta_i\}$ with $\sum_{i=0}^{N-1} \beta_i = \frac{\pi}{4}$, it leads to an orthonormal FIR filter bank, and any such system can be expressed for the some choice of $\{\beta_i\}$.

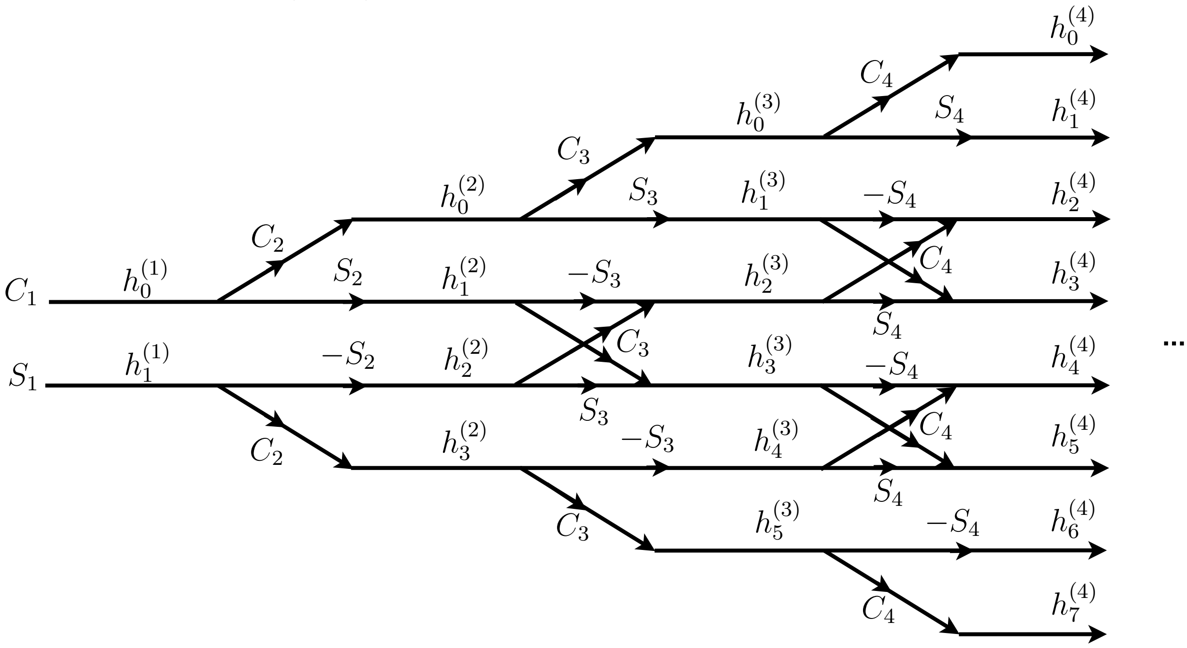
From this section of the text, the angular parameters set of the formulation proposed by Sherlock and Monro (1998) and described as $\{\beta_i | i = 0, 1, 2, \cdots, N-1\}$ will be written as $\{\alpha_i | i = 1, 2, \cdots, N\}$ in accordance with the subsequent works that proposed the extensions (UZINSKI, 2013; PAIVA et al., 2009; UZINSKI et al., 2013; UZINSKI et al., 2015a; UZINSKI et al., 2015b), and to facilitate the notation. For these reasons, equations (93) are written as follows:

$$\begin{aligned}
h_0^{(1)} &= \cos(\alpha_1)\\
h_1^{(1)} &= \sin(\alpha_1)\\[2ex]
h_0^{(N+1)} &= \cos(\alpha_{N+1}) h_0^{(N)}\\
h_{2i}^{(N+1)} &= \cos(\alpha_{N+1}) h_{2i}^{(N)} - \sin(\alpha_{N+1}) h_{2i-1}^{(N)} \qquad\qquad i = 0, 1, \cdots, N-1,\\
h_{2N}^{(N+1)} &= -\sin(\alpha_{N+1}) h_{2N-1}^{(N)}\\[2ex]
h_1^{(N+1)} &= \sin(\alpha_{N+1}) h_0^{(N)}\\
h_{2i+1}^{(N+1)} &= \sin(\alpha_{N+1}) h_{2i}^{(N)} + \cos(\alpha_{N+1}) h_{2i-1}^{(N)}\\
h_{2N+1}^{(N+1)} &= \cos(\alpha_{N+1}) h_{2N-1}^{(N)}
\end{aligned} \tag{96}$$

and the construction of the analysis low-pass filter coefficients in the recursive form previously described is shown in the Figure 18.

As previously mentioned, the constructions of the analysis low-pass filter coefficients are given by the equations expressed in (96) that form an orthonormal basis. These constructions, as already shown, can also be seen in the flow diagram of Figure 18. In what follows, examples are shown.

Figure 18: Flow diagram illustrating the recurrence for generating analysis low-pass filter coefficients (for α).



Source: Adapted from Sherlock and Monro (1998).

Example 1. *For an angular set of parameters* $\{\alpha_i\}$ *with* $N = 1$ *elements, the length of the vector containing the filter coefficients is* $2N = 2$*. Such as the set of angular parameters must satisfy* $\sum_{i=1}^{N} \alpha_i = \frac{\pi}{4}$*, then, necessarily,* $\alpha = \{\frac{\pi}{4}\}$*. Thus, the coefficients are*

$$h_0^{(1)} = \cos(\alpha_1) \textit{ and } h_1^{(1)} = \sin(\alpha_1),$$

that lead to

$$h_0^{(1)} = \frac{\sqrt{2}}{2} \textit{ and } h_1^{(1)} = \frac{\sqrt{2}}{2},$$

whose sum is $\sqrt{2}$*.*

It is worth noting that for $N = 1$ *it has the same coefficients as the* `db1` *wavelet, also known as* `Haar` *wavelet.*

Example 2. *If the set of angular parameters has length* $N = 2$ *the filter has length* $2N = 4$*. It is easy to note that since* $\sum_{i=1}^{N} \alpha_i = \frac{\pi}{4}$*, then the sum of* $\{h_i^{(2)}\}$ *is* $\sqrt{2}$*, but unlike Example 1 there are many ways to write the set* $\{\alpha\}$ *and, consequently, many ways for the coefficients vector. However, from equations (96) it has*

$$\begin{cases} h_0^{(2)} = \cos(\alpha_1)\cos(\alpha_2) \\ h_1^{(2)} = \cos(\alpha_1)\sin(\alpha_2) \\ h_2^{(2)} = -\sin(\alpha_1)\sin(\alpha_2) \\ h_3^{(2)} = \sin(\alpha_1)\cos(\alpha_2). \end{cases}$$

The reasoning presented in Examples 1 and 2 can be applied for any value of N. Besides, for both examples, it can be easily checked that in fact they are orthogonal. This fact is true for any N.

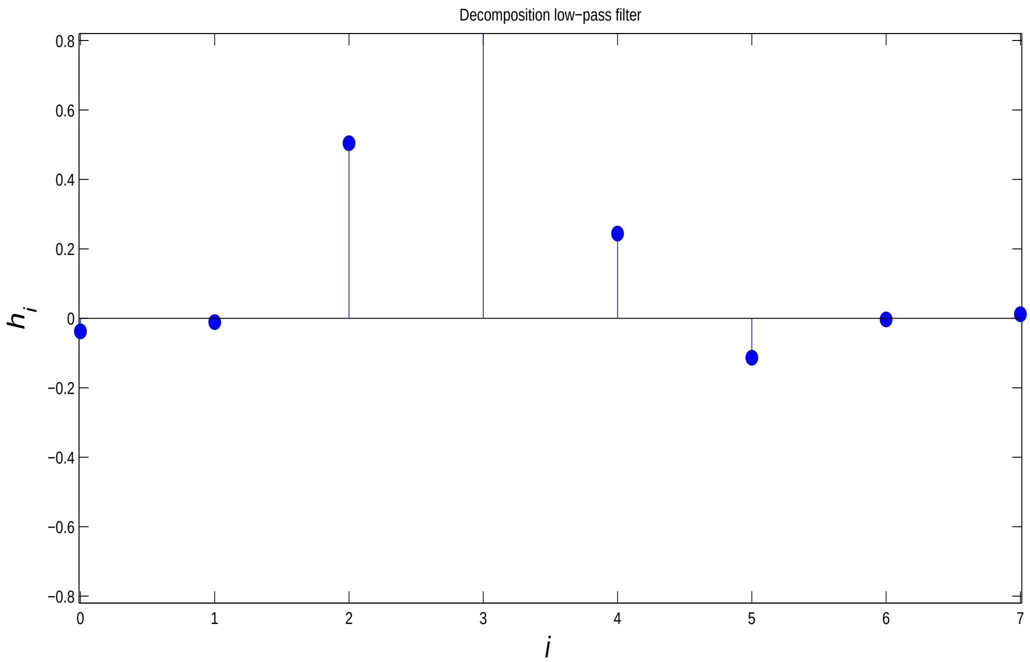
Example 3. *In this example the intention is to show the generated coefficients for a filter bank with length $2N = 8$ by a set with $N = 4$ angular parameters. There are numerous sets of angular parameters with this length and that satisfy the condition to be a wavelet filter bank. However, the expressions that define the coefficients for these filter banks in terms of the angular parameters are given as follows, but an example for $N = 4$ is also presented to illustrate the situation.*

$$\begin{cases} h_0^{(4)} = \cos(\alpha_1)\cos(\alpha_2)\cos(\alpha_3)\cos(\alpha_4) \\ h_1^{(4)} = \cos(\alpha_1)\cos(\alpha_2)\cos(\alpha_3)\sin(\alpha_4) \\ h_2^{(4)} = -\cos(\alpha_4)(\cos(\alpha_1)\sin(\alpha_2)\sin(\alpha_3) + \cos(\alpha_3)\sin(\alpha_1)\sin(\alpha_2)) - \\ \qquad\qquad - \cos(\alpha_1)\cos(\alpha_2)\sin(\alpha_3)\sin(\alpha_4) \\ h_3^{(4)} = \cos(\alpha_1)\cos(\alpha_2)\cos(\alpha_4)\sin(\alpha_3) - \sin(\alpha_4)(\cos(\alpha_1)\sin(\alpha_2)\sin(\alpha_3) + \\ \qquad\qquad + \cos(\alpha_3)\sin(\alpha_1)\sin(\alpha_2)) \\ h_4^{(4)} = -\sin(\alpha_4)(\cos(\alpha_1)\cos(\alpha_3)\sin(\alpha_2) - \sin(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)) - \\ \qquad\qquad - \cos(\alpha_2)\cos(\alpha_4)\sin(\alpha_1)\sin(\alpha_3) \\ h_5^{(4)} = \cos(\alpha_4)(\cos(\alpha_1)\cos(\alpha_3)\sin(\alpha_2) - \sin(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)) - \\ \qquad\qquad - \cos(\alpha_2)\sin(\alpha_1)\sin(\alpha_3)\sin(\alpha_4) \\ h_6^{(4)} = -\cos(\alpha_2)\cos(\alpha_3)\sin(\alpha_1)\sin(\alpha_4) \\ h_7^{(4)} = \cos(\alpha_2)\cos(\alpha_3)\cos(\alpha_4)\sin(\alpha_1). \end{cases}$$

Let $\alpha = \{-0.3033,\ -0.8243,\ 1.6308,\ 0.2822\}$ be the set of angular parameters, whose parameters sum obeys the equality $\sum_{i=1}^{N} \alpha_i = \frac{\pi}{4}$, see Uzinski (2013). Therefore, in accordance to the previously presented expressions for the analysis low-pass filter, the following vector is *obtained, whose sum is* $\sqrt{2}$*, with a graphic representation given in Figure 19,*

$$\begin{cases} h_0^{(4)} = -0.0374 \\ h_1^{(4)} = -0.0108 \\ h_2^{(4)} = 0.5042 \\ h_3^{(4)} = 0.8197 \\ h_4^{(4)} = 0.2438 \\ h_5^{(4)} = -0.1134 \\ h_6^{(4)} = -0.0034 \\ h_7^{(4)} = 0.0117. \end{cases}$$

Figure 19: Analysis low-pass filter coefficients for $\alpha = \{-0.3033,\ -0.8243,\ 1.6308,\ 0.2822\}$.



Source: Elaborated by the author.

A set of angular parameters α whose filter is a vector with eight coefficients was presented in the Example 3. In the following it is shown, in another example, the signal filtering using the filter obtained in Example 3.

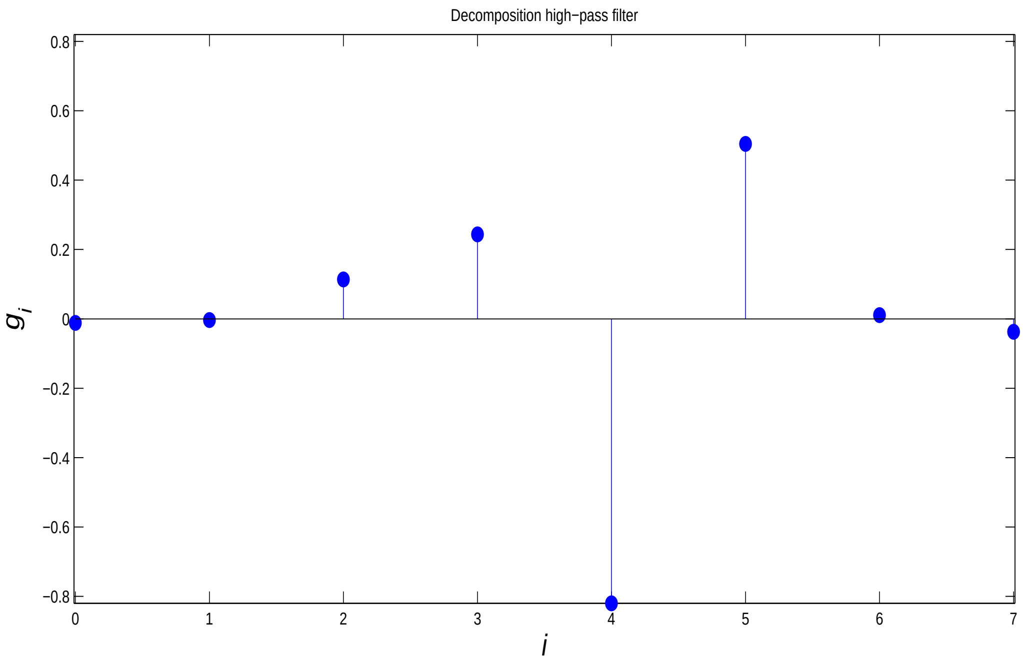
Example 4. *The filter obtained in Example 3 is used to discuss wavelet function, scale function (given by (83) and (84), respectively) and signal filtering in general. In that example the filter coefficients* $h[k]$ *were shown, and now they are following shown along with the high-pass filter coefficients* $g[k]$*, obtained by (82),*

$$\begin{cases} h_0^{(4)} = -0.0374 \\ h_1^{(4)} = -0.0108 \\ h_2^{(4)} = 0.5042 \\ h_3^{(4)} = 0.8197 \\ h_4^{(4)} = 0.2438 \\ h_5^{(4)} = -0.1134 \\ h_6^{(4)} = -0.0034 \\ h_7^{(4)} = 0.0117, \end{cases} \qquad \begin{cases} g_0^{(4)} = -0.0117 \\ g_1^{(4)} = -0.0034 \\ g_2^{(4)} = 0.1134 \\ g_3^{(4)} = 0.2438 \\ g_4^{(4)} = -0.8197 \\ g_5^{(4)} = 0.5042 \\ g_6^{(4)} = 0.0108 \\ g_7^{(4)} = -0.0374. \end{cases}$$

A simple, but relevant, observation is that as expected by the orthogonality and orthonormality, $h[k]g[k]^T = g[k]h[k]^T = 0$ and $g[k]g[k]^T = h[k]h[k]^T = 1$.

A figure for $h[k]$ has already been shown in Example 3, in Figure 19. In an analogous manner, in Figure 20 $g[k]$, is presented.

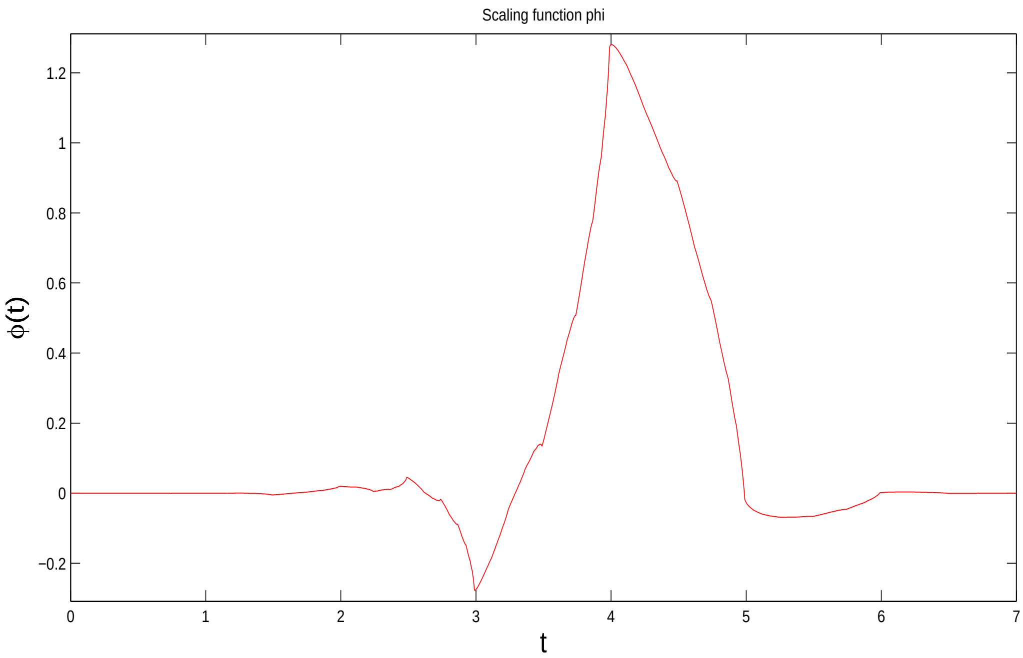
Figure 20: Analysis high-pass filter coefficients for $\alpha = \{-0.3033,\ -0.8243,\ 1.6308,\ 0.2822\}$.



Source: Elaborated by the author.

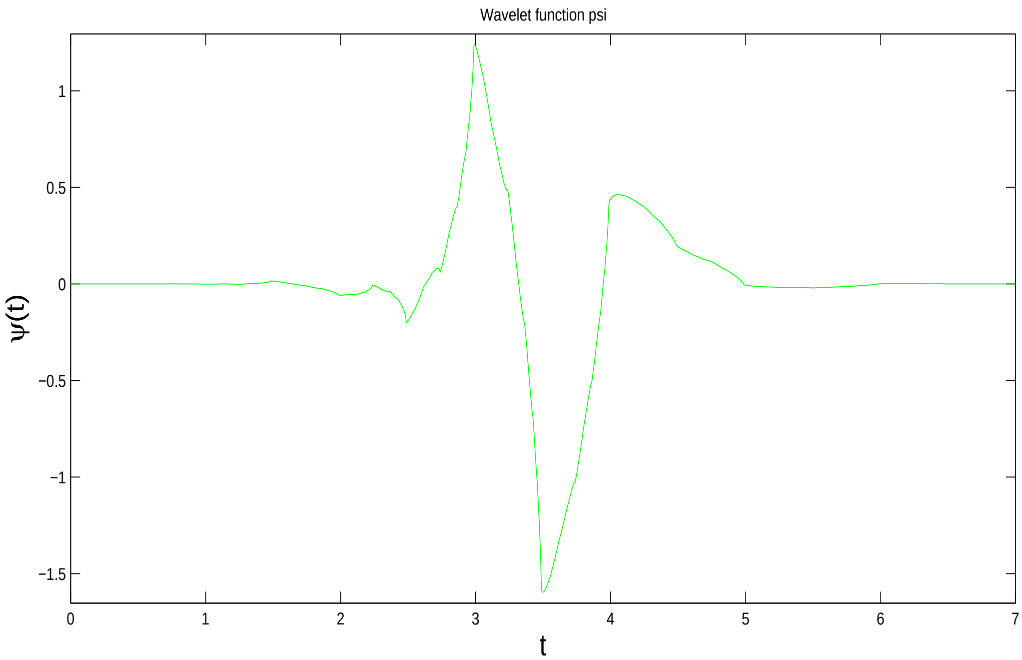
Wavelet and Scale functions are obtained by taking $h[k]$ in (83) and $g[k]$ in (84), respectively. Thus, the Wavelet and Scale functions are shown in Figures 21 and 22, respectively.

Figure 21: Scale function for $\alpha = \{-0.3033,\ -0.8243,\ 1.6308,\ 0.2822\}$.



Source: Elaborated by the author.

Figure 22: Wavelet function for $\alpha = \{-0.3033,\ -0.8243,\ 1.6308,\ 0.2822\}$.



Source: Elaborated by the author.

4.6 Further comments

Discrete-time bases better suited for signal processing achieve good frequency resolution while also keeping good time locality. In order to compute them, the set of basis functions has to be structured. This leads to discrete-time filter banks, described in Section 4.1. The multiresolution analysis is central framework to the wavelet decomposition and establishes the link between wavelets and filter banks (MALLAT, 1989a; MALLAT, 1989b; MALLAT, 1989c; DAUBECHIES, 1992; DAUBECHIES, 1988).

Connection between wavelets and filter banks leads to the construction of wavelets, presented in Section 4.2, from filter banks, as described in Section 4.3 (RIOUL, 1993; VETTERLI; HERLEY, 1992). However, in this thesis it is important to introduce the *Algorithme à Trous*, in Section 4.4, which is a way to work with wavelets, avoiding downsampling (VETTERLI; KOVAČEVIĆ, 1995).

An interesting and simple framework to parameterize the space of orthonormal wavelets by a set of angular parameters was put forth by Sherlock and Monro (1998), as in Section 4.5. This was made in view of schemes for parameterization of the filter coefficients in a way to express the existing degrees of freedom in terms of variables more easily tunable by the designer (PAIVA et al., 2009). Such an approach has been explored in a number of applications (DAAMOUCHE et al., 2012; FROESE et al., 2006; COELHO et al., 2003; PAIVA et al., 2008; GALVÃO et al., 2003; HADJILOUCAS et al., 2004; DUARTE; GALVÃO; PAIVA, 2013; NAZARAHARI et al., 2015). A methodology to get the angular parameters from the filter coefficients is given by Galvão et al. (2004).

The framework proposed by Sherlock and Monro (1998) does not ensure that the parameterized wavelet has a predetermined number of vanishing moments. It is a weak point since various useful properties of wavelets are related to vanishing moments, e.g., sparsity of signal representation in the wavelet domain and characterization of singularities based on the decay of the coefficients across scale (PAIVA et al., 2009).

There are two restrictions that can be applied to the Sherlock and Monro formulation. The first was developed by Paiva et al. (2009) and it is a way to ensure at least two vanishing moments to the original formulation, while the second was developed by Uzinski et al. (2013), Uzinski (2013), Uzinski et al. (2015a) and ensures at least three vanishing moments.

5 Orthogonal filter banks parameterization in the state space

This chapter introduces the main idea of this work, which is the proposition of two different possibilities for parameterization of particular orthonormal basis functions in the state-space. These orthonormal basis functions were presented in the Chapter 4, and consist of wavelet basis functions obtained through the formulation of Sherlock and Monro (1998).

The first state-space realization developed in this chapter is from FIR filters. To use the Sherlock and Monro's orthonormal basis functions in this approach, it is necessary that just after obtaining the state-space matrices, the filter coefficients must be replaced by the results from (96) and (82).

The main idea in this thesis is the parameterization of orthogonal filter banks in the state-space from system implementations in lattice structure. This work achieves a minimal discrete-time state-space description for the fast wavelet transform with multiple decomposition levels. Firstly, the parameterization for one-single-level i of the FWT algorithm is made in Section 5.2. In Section 5.3, the extension for multiple levels FWT starting from the idea presented for one-single-level i will be shown.

5.1 Parameterization for a single FIR filter in state-space

In mathematics, a discrete-time system is defined as a transformation that maps an input sequence $u[k]$ into an output sequence $y[k]$. This transformation can be denoted as

$$y[k] = T\{u[k]\}.$$

The system is linear when $y_1[k]$ and $y_2[k]$ are outputs of the system with inputs $u_1[k]$ and $u_2[k]$ and considering arbitrary constants a and b, it is observed that

$$T\{au_1[k] + bu_2[k]\} = aT\{u_1[k]\} + bT\{u_2[k]\}.$$

If for all value of k_0 the input sequence $u_1[k] = u[k - k_0]$ produces an output sequence with values $y_1[k] = y[k - k_0]$, the system is called a Time-Invariant System.

Considering the unit sample sequence $\delta[k]$ it is true that any sequence $u[k]$ can be written as

$$u[k] = \sum_{m=-\infty}^{\infty} u[m]\delta[k - m]. \tag{97}$$

An important class of systems are the linear time-invariant systems (LTI) that combines linearity and time invariance properties. This kind of system can be completely characterized by the impulse response (OPPENHEIM; SCHAFER, 1975; OPPENHEIM; SCHAFER, 1989).

Let $h_m[k]$ be the system impulse response to $\delta[k-m]$, an impulse occurring in $k=m$, then

$$y[k] = T\left\{\sum_{m=-\infty}^{\infty} u[m]\delta[k-m]\right\}. \tag{98}$$

From the linearity condition and according to time invariance property (98) becomes

$$y[k] = \sum_{m=-\infty}^{\infty} u[m]h_m[k]. \tag{99}$$

Consider a two-channel orthonormal filter bank with length-$2N$ filters. Let $H^{(N)}(z)$ and $G^{(N)}(z)$ be the transfer functions of the low-pass and high-pass filters, respectively, such that

$$H^{(N)}(z) = \sum_{i=0}^{2N-1} h_i^{(N)} z^{-i}, \tag{100}$$

$$G^{(N)}(z) = \sum_{i=0}^{2N-1} g_i^{(N)} z^{-i}. \tag{101}$$

Recalling that the coefficients $h_i^{(N)}$ to equation (100) are given by

$$\begin{aligned} h_0^{(1)} &= \cos(\alpha_1) \\ h_1^{(1)} &= \sin(\alpha_1) \end{aligned}$$

$$\begin{aligned} h_0^{(N+1)} &= \cos(\alpha_{N+1})h_0^{(N)} \\ h_{2i}^{(N+1)} &= \cos(\alpha_{N+1})h_{2i}^{(N)} - \sin(\alpha_{N+1})h_{2i-1}^{(N)} \qquad\qquad i = 0, 1, \cdots, N-1, \\ h_{2N}^{(N+1)} &= -\sin(\alpha_{N+1})h_{2N-1}^{(N)} \\ \\ h_1^{(N+1)} &= \sin(\alpha_{N+1})h_0^{(N)} \\ h_{2i+1}^{(N+1)} &= \sin(\alpha_{N+1})h_{2i}^{(N)} + \cos(\alpha_{N+1})h_{2i-1}^{(N)} \\ h_{2N+1}^{(N+1)} &= \cos(\alpha_{N+1})h_{2N-1}^{(N)} \end{aligned} \tag{102}$$

while the coefficients $g_i^{(N)}$ to equation (101) are given by

$$g_i^{(N)} = (-1)^{i+1} h_{2N-1-i}^{(N)}, \qquad i = 0, 1, \cdots, 2N-1. \tag{103}$$

Based on equations (99) and (100), the following paragraph presents a parameterization to the filters, low-pass and high-pass. In the first case involving the results from equations (102) and in second case, equation (103).

From (99) and (100) it follows that signal $y_1[k]$ created by the approximation coefficients is given by

$$\begin{aligned} y_1[k] &= \sum_{i=0}^{2N-1} u[k]h[k-i], \\ &= h_0u[k] + h_1u[k-1] + \cdots + h_{2N-1}u[k-2N+1]. \end{aligned} \tag{104}$$

Defining

$$\begin{aligned} x_1[k] &= u[k-2N+1], \\ x_2[k] &= u[k-2N+2], \\ &\vdots \\ x_{2N-2}[k] &= u[k-2], \\ x_{2N-1}[k] &= u[k-1], \end{aligned} \tag{105}$$

it follows that

$$\begin{aligned} x_1[k+1] &= u[k-2N+2] = x_2[k], \\ x_2[k+1] &= u[k-2N+3] = x_3[k], \\ &\vdots \\ x_{2N-2}[k+1] &= u[k-1] = x_{2N-1}[k], \\ x_{2N-1}[k+1] &= u[k]. \end{aligned} \tag{106}$$

And (104) can be rewritten as

$$y_1[k] = h_{2N-1}x_1[k] + h_{2N-2}x_2[k] + \cdots + h_1x_{2N-1}[k] + h_0u[k]. \tag{107}$$

Based on equations (106) and (107), in equations

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \tag{108}$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k], \tag{109}$$

we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{110}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{111}$$

$$\mathbf{C} = \left[\begin{array}{cccc} h_{2N-1} & h_{2N-2} & \cdots & h_1 \end{array} \right], \tag{112}$$

and

$$D = h_0, \tag{113}$$

where $\mathbf{A}$ is a $(2N-1) \times (2N-1)$ matrix, $\mathbf{B}$ has order $(2N-1) \times 1$ and $\mathbf{C}$ is $1 \times (2N-1)$.

In high-pass filters case, equation (104) becomes

$$y_2[k] = g_0 u[k] + g_1 u[k-1] + \cdots + g_{2N-1} u[k-2N+1]. \tag{114}$$

Considering (106), matrices $\mathbf{A}$ and $\mathbf{B}$ are the same as for low-pass filters, while in an analog way to (107),

$$y_2[k] = g_{2N-1} x_1[k] + g_{2N-2} x_2[k] + \cdots + g_1 x_{2N-1}[k] + g_0 u[k]. \tag{115}$$

From equation (115), $\mathbf{C}$ and $\mathbf{D}$ for high-pass filters are

$$\mathbf{C} = \left[\begin{array}{cccc} g_{2N-1} & g_{2N-2} & \cdots & g_1 \end{array} \right], \tag{116}$$

and

$$D = g_0. \tag{117}$$

Example 5. *Considering the* `db4` *wavelet, the set of low-pass filter coefficients is*

$$h = [\begin{array}{cccccccc} -0.0106 & 0.0329 & 0.0308 & -0.1870 & -0.0280 & 0.6309 & 0.7148 & 0.2304 \end{array}]$$

and the high-pass set is

$$g = [\begin{array}{cccccccc} -0.2304 & 0.7148 & -0.6309 & -0.0280 & 0.1870 & 0.0308 & -0.0329 & -0.0106 \end{array}].$$

Matrices $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are*

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{118}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{119}$$

For the output signal y_1 filtered by $H^{(N)}(z)$, $\boldsymbol{C}$ and D are

$$\boldsymbol{C} = \begin{bmatrix} 0.2304 & 0.7148 & 0.6309 & -0.0280 & -0.1870 & 0.0308 & 0.0329 \end{bmatrix}, \tag{120}$$

$$D = -0.0106. \tag{121}$$

For the output signal y_2 filtered by $G^{(N)}(z)$, $\boldsymbol{C}$ and D are

$$\boldsymbol{C} = \begin{bmatrix} -0.0106 & -0.0329 & 0.0308 & 0.1870 & -0.0280 & -0.6309 & 0.7148 \end{bmatrix}, \tag{122}$$

$$D = -0.2304. \tag{123}$$

An important theorem about observability and reachability properties in the proposed formulation is now presented.

Theorem 15. *A realization ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$) with those matrices given by (110), (111), (112) and (113) or (110), (111), (116) and (117), respectively, with nonzero h_{2N-1} and g_{2N-1}, is minimal because it is reachable and observable.*

Proof. For convenience of notation, matrix $\mathbf{C}$ from (112) and (116) is written in a special manner, denoting its elements by c_k, as described in (124), and for matrices $\mathbf{A}$ and $\mathbf{B}$ we will remember equations (110) and (111),

$$\mathbf{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{2N-1} \end{bmatrix}. \tag{124}$$

The controllability matrix is denoted by $\mathcal{C}_{\mathbf{A},\mathbf{B}}$ and given by

$$\mathcal{C}_{\mathbf{A},\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{2N-2}\mathbf{B} \end{bmatrix}, \tag{125}$$

resulting in a $(2N-1) \times (2N-1)$ matrix, as follows

$$\mathcal{C}_{\mathbf{A},\mathbf{B}} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}. \tag{126}$$

Clearly, the determinant of $\mathcal{C}_{\mathbf{A},\mathbf{B}}$ is nonzero. This matrix has all rows and columns linearly independent, in other words, this matrix has full rank. In this way, as the controllability matrix has full rank $2N-1$ then the presented realization is reachable.

On the other hand, the observability matrix is denoted by $\mathcal{O}_{\mathbf{C},\mathbf{A}}$ and defined as

$$\mathcal{O}_{\mathbf{C},\mathbf{A}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{2N-2} \end{bmatrix}, \tag{127}$$

and, in this case, this parameterization results in a $(2N-1) \times (2N-1)$ matrix, as follows:

$$\mathcal{O}_{\mathbf{C},\mathbf{A}} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{2N-1} \\ 0 & c_1 & \cdots & c_{2N-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_1 \end{bmatrix}. \tag{128}$$

Matrix $\mathcal{O}_{\mathbf{C},\mathbf{A}}$ is not singular, since it has all rows and columns linearly independent, namely, this matrix has full rank $2N-1$. In this way the presented realization is observable.

Since it is simultaneously observable and controllable, the description in the state-space proposed by the matrices **A**, **B**, **C** and **D** in equations (110), (111), (112) and (113) or (110), (111), (116) and (117) is minimal. □

In this section two descriptions were proposed, that together form one parameterization in the state-space using Sherlock and Monro (1998) orthonormal basis functions. In the first case, the system has single output and matrices **A**, **B**, **C** and **D** are given by (110), (111), (112) and (113), respectively. While in the second case, matrices **A** and **B** remain the same, but **C** and **D**, are now given by (116) and (117), respectively.

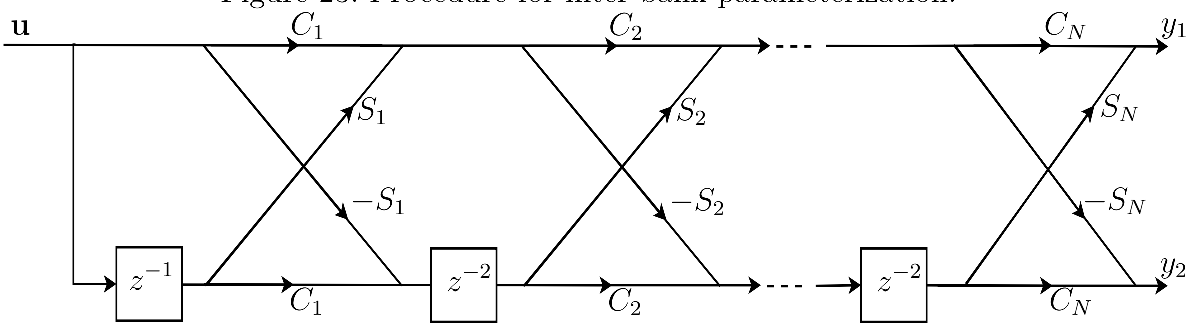
Theorem 15 is an important result, showing that in both cases the descriptions in the state-space are reachable and observable. These features allow the design of controllers and observers, in addition to the allocation of poles and studies of stability through the minimal property.

5.2 System implementation in lattice structure

Lattice filters have particular properties that make them interesting for various applications, for instance, modularity, low sensitivity to the effects of quantization parameters, and a simple criterion for ensuring filter stability (HAYES, 1999).

In particular the proposed structure in Galvão et al. (2003), Hadjiloucas et al. (2004), Froese et al. (2006) is used in the present parameterization. This structure is shown in Figure 23.

Figure 23: Procedure for filter bank parameterization.



Source: Adapted from Akansu and Haddad (2001).

This equationing follows the procedure described in Figure 23, where the first delay operator is z^{-1} and the remainders are z^{-2}. C_i and S_i denote $\sin(\theta_i)$ and $\cos(\theta_i)$, respectively, $\theta = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_N \end{bmatrix}$ and $\theta_1 = \alpha_N, \theta_2 = \alpha_{N-1}, \cdots, \theta_N = \alpha_1$. In Figure 23, y_1 is the output signal corresponding to the input signal $\mathbf{u}$ filtered by the low-pass filter and y_2 corresponds to the input signal $\mathbf{u}$ filtered by the high-pass filter.

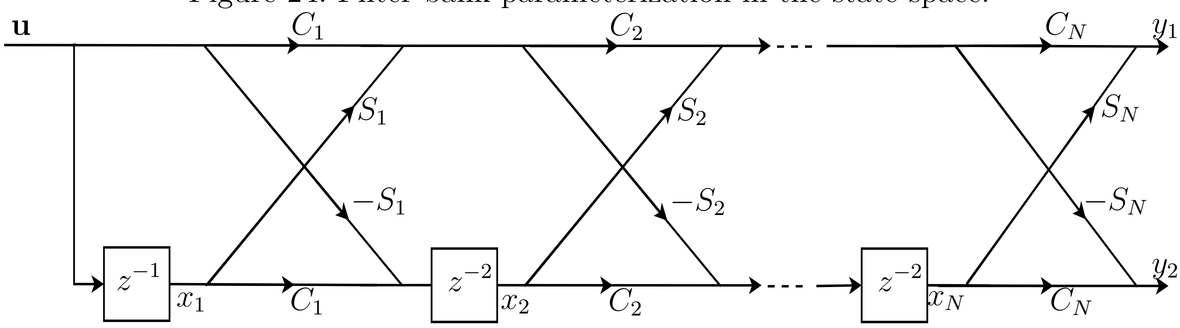
Considering that the z-transform of the output is equivalent to the product of z-transforms of the input signal and the transfer function, which means filtering process, it follows that

$$Y(z) = H(z)U(z). \tag{129}$$

And denoting the transformed signal in the level i by x_i, as shown in Figure 24, the state-space model with state equations and output equation is as follows:

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k], & (130)\\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]. & (131) \end{aligned}$$

Figure 24: Filter bank parameterization in the state-space.



Source: Elaborated by the author.

From Figure 24, it follows:

$$\begin{cases} x_1[k+1] = u[k], \\ x_2[k+2] = C_1x_1[k] - S_1u[k], \\ x_3[k+2] = C_2x_2[k] - S_2S_1x_1[k] - S_2C_1u[k], \\ x_4[k+2] = C_3x_3[k] - S_3S_2x_2[k] - S_3C_2S_1x_1[k] - S_3C_2C_1u[k], \\ \quad\vdots \\ x_{N-1}[k+2] = C_{N-2}x_{N-2}[k] - S_{N-2}S_{N-3}x_{N-3}[k] - \\ \qquad \displaystyle\sum_{j=2}^{N-3}\left[S_{N-2}S_{j-1}x_{j-1}[k]\prod_{i=j}^{N-3}C_i\right] - S_{N-2}\prod_{i=1}^{N-3}C_iu[k], \\ x_N[k+2] = C_{N-1}x_{N-1}[k] - S_{N-1}S_{N-2}x_{N-2}[k] - \\ \qquad \displaystyle\sum_{j=2}^{N-2}\left[S_{N-1}S_{j-1}x_{j-1}[k]\prod_{i=j}^{N-2}C_i\right] - S_{N-1}\prod_{i=1}^{N-2}C_iu[k]. \end{cases} \tag{132}$$

Defining

$$\begin{cases} x_{N+1}[k] = x_2[k+1], \\ x_{N+2}[k] = x_3[k+1], \\ x_{N+3}[k] = x_4[k+1], \\ \quad\vdots \\ x_{2N-1}[k] = x_N[k+1], \end{cases} \tag{133}$$

implying

$$\begin{cases} x_{N+1}[k+1] = x_2[k+2], \\ x_{N+2}[k+1] = x_3[k+2], \\ x_{N+3}[k+1] = x_4[k+2], \\ \quad\vdots \\ x_{2N-1}[k+1] = x_N[k+2], \end{cases} \tag{134}$$

it follows from (132), (133) and (134) that

$$
\begin{cases}
x_1[k+1] = u[k], \\
x_2[k+1] = x_{N+1}[k], \\
x_3[k+1] = x_{N+2}[k], \\
\quad \vdots \\
x_N[k+1] = x_{2N-1}[k], \\
x_{N+1}[k+1] = C_1 x_1[k] - S_1 u[k], \\
x_{N+2}[k+1] = C_2 x_2[k] - S_2 S_1 x_1[k] - S_2 C_1 u[k], \\
x_{N+3}[k+1] = C_3 x_3[k] - S_3 S_2 x_2[k] - S_3 C_2 S_1 x_1[k] - S_3 C_2 C_1 u[k], \\
\quad \vdots \\
x_{2N-2}[k+1] = C_{N-2} x_{N-2}[k] - S_{N-2} S_{N-3} x_{N-3}[k] - \\
\qquad \sum_{j=2}^{N-3} \left[S_{N-2} S_{j-1} x_{j-1}[k] \prod_{i=j}^{N-3} C_i \right] - S_{N-2} \prod_{i=1}^{N-3} C_i u[k], \\
x_{2N-1}[k+1] = C_{N-1} x_{N-1}[k] - S_{N-1} S_{N-2} x_{N-2}[k] - \\
\qquad \sum_{j=2}^{N-2} \left[S_{N-1} S_{j-1} x_{j-1}[k] \prod_{i=j}^{N-2} C_i \right] - S_{N-1} \prod_{i=1}^{N-2} C_i u[k].
\end{cases}
\tag{135}
$$

The state matrix (or system matrix) $\mathbf{A}$ presented in (130) is given by

$$
\mathbf{A} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\
C_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-S_2 S_1 & C_2 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-S_3 C_2 S_1 & -S_3 S_2 & C_3 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
S_{N-1} S_1 \prod_{i=2}^{N-2} C_i & S_{N-1} S_2 \prod_{i=3}^{N-2} C_i & S_{N-1} S_3 \prod_{i=4}^{N-2} C_i & \cdots & C_{N-1} & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix},
\tag{136}
$$

and the input matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -S_1 \\ -S_2C_1 \\ \vdots \\ -S_{N-1}\prod_{i=1}^{N-2} C_i \end{bmatrix}. \tag{137}$$

The output matrix $\mathbf{C}$ and the feedthrough (or feedforward) matrix $\mathbf{D}$ presented in (131) are given by

$$\mathbf{C} = \begin{bmatrix} S_1\prod_{i=2}^{N} C_i & S_2\prod_{i=3}^{N} C_i & \cdots & S_{N-2}\prod_{i=N-1}^{N} C_i & S_{N-1}C_N & S_N & 0 & \cdots & 0 \\ -S_NS_1\prod_{i=2}^{N-1} C_i & -S_NS_2\prod_{i=3}^{N-1} C_i & \cdots & -S_NS_{N-2}\prod_{i=N-1}^{N-1} C_i & -S_NS_{N-1} & C_N & 0 & \cdots & 0 \end{bmatrix}, \tag{138}$$

$$\mathbf{D} = \begin{bmatrix} \prod_{i=1}^{N} C_i \\ -S_N\prod_{i=1}^{N-1} C_i \end{bmatrix}. \tag{139}$$

In order to facilitate the understanding of this proposal, an example is given using the Daubechies `db4` wavelet. The procedure proposed in Galvão et al. (2004) was employed to define the set of angular parameters representing filter coefficients.

Example 6. *The set of angular parameters* $\alpha = \{-1.5248,\ -0.2537,\ 0.6814,\ 1.8826\}$ *is obtained from the* `db4` *wavelet low-pass filter coefficients. Therefore, from (136) - (139),* $\boldsymbol{A}$*,* $\boldsymbol{B}$*,* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *are*

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -0.3067 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5995 & 0.7767 & 0 & 0 & 0 & 0 & 0 \\ 0.1856 & 0.1581 & 0.9680 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -0.9518 \\ 0.1932 \\ -0.0598 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 0.0329 & 0.0280 & -0.0115 & -0.9989 & 0 & 0 & 0 \\ 0.7148 & 0.6091 & -0.2507 & 0.0460 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\boldsymbol{D} = \begin{bmatrix} -0.0106 \\ -0.2304 \end{bmatrix}.$$

It is worth noting that the sizes of the matrices vary according to the number of angular parameters θ.

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be the minimal realization for $H(z)$, then it can be verified that λ is a pole of the $H(z)$ if, and only if, it is an eigenvalue for $\mathbf{A}$. Then the stability condition is equivalent to the condition that $|\lambda_i| < 1$ (VAIDYANATHAN, 1993). When $H(z)$ is a FIR filter all $\mathbf{A}$ eigenvalues are zero (i.e., all these eigenvalues are inside the unit circle). The definition of minimal realization presented in the Lemmas 1, 2 and 3 is now remembered for the purpose of verifying if this realization is minimal.

The state-space realization proposed is really minimal, according to the next result.

Theorem 16. *A realization ($\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$) given by (136), (137), (138) and (139), respectively, with all angles different from 0, $\pi/2$, π and $3\pi/2$, is minimal because it is reachable and observable.*

Proof. From (32), (136) and (137) a $(2N-1) \times (2N-1)$ matrix is obtained, where, for simplicity of notation, $\varphi_{i,j}$ represents nonzero elements:

$$\mathcal{R}_{\mathbf{A},\mathbf{B}} = \begin{bmatrix} \varphi_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varphi_{2,2} & \varphi_{2,3} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varphi_{3,2} & \varphi_{3,3} & \varphi_{3,4} & \varphi_{3,5} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varphi_{4,2} & \varphi_{4,3} & \varphi_{4,4} & \varphi_{4,5} & \varphi_{4,6} & \varphi_{4,7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \varphi_{N-1,2} & \varphi_{N-1,3} & \varphi_{N-1,4} & \varphi_{N-1,5} & \varphi_{N-1,6} & \varphi_{N-1,7} & \cdots & 0 & 0 \\ 0 & \varphi_{N,2} & \varphi_{N,3} & \varphi_{N,4} & \varphi_{N,5} & \varphi_{N,6} & \varphi_{N,7} & \cdots & \varphi_{N,2N-2} & \varphi_{N,2N-1} \\ \varphi_{N+1,1} & \varphi_{N+1,2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \varphi_{N+2,1} & \varphi_{N+2,2} & \varphi_{N+2,3} & \varphi_{N+2,4} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \varphi_{N+3,1} & \varphi_{N+3,2} & \varphi_{N+3,3} & \varphi_{N+3,4} & \varphi_{N+3,5} & \varphi_{N+3,6} & 0 & \cdots & 0 & 0 \\ \varphi_{N+4,1} & \varphi_{N+4,2} & \varphi_{N+4,3} & \varphi_{N+4,4} & \varphi_{N+4,5} & \varphi_{N+4,6} & \varphi_{N+4,7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{2N-2,1} & \varphi_{2N-2,2} & \varphi_{2N-2,3} & \varphi_{2N-2,4} & \varphi_{2N-2,5} & \varphi_{2N-2,6} & \varphi_{2N-2,7} & \cdots & 0 & 0 \\ \varphi_{2N-1,1} & \varphi_{2N-1,2} & \varphi_{2N-1,3} & \varphi_{2N-1,4} & \varphi_{2N-1,5} & \varphi_{2N-1,6} & \varphi_{2N-1,7} & \cdots & \varphi_{2N-1,2N-2} & 0 \end{bmatrix}.$$

In order to facilitate the perception that all rows and columns in $\mathcal{R}_{\mathbf{A},\mathbf{B}}$ are linearly independent, a row equivalent matrix is presented, which is obtained just by swapping rows of the original matrix:

$$\mathcal{R}'_{\mathbf{A},\mathbf{B}} = \begin{bmatrix} \varphi_{1,1} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \varphi_{N+1,1} & \varphi_{N+1,2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varphi_{2,2} & \varphi_{2,3} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \varphi_{N+2,1} & \varphi_{N+2,2} & \varphi_{N+2,3} & \varphi_{N+2,4} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varphi_{3,2} & \varphi_{3,3} & \varphi_{3,4} & \varphi_{3,5} & 0 & 0 & \cdots & 0 & 0 \\ \varphi_{N+3,1} & \varphi_{N+3,2} & \varphi_{N+3,3} & \varphi_{N+3,4} & \varphi_{N+3,5} & \varphi_{N+3,6} & 0 & \cdots & 0 & 0 \\ 0 & \varphi_{4,2} & \varphi_{4,3} & \varphi_{4,4} & \varphi_{4,5} & \varphi_{4,6} & \varphi_{4,7} & \cdots & 0 & 0 \\ \varphi_{N+4,1} & \varphi_{N+4,2} & \varphi_{N+4,3} & \varphi_{N+4,4} & \varphi_{N+4,5} & \varphi_{N+4,6} & \varphi_{N+4,7} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varphi_{2N-1,1} & \varphi_{2N-1,2} & \varphi_{2N-1,3} & \varphi_{2N-1,4} & \varphi_{2N-1,5} & \varphi_{2N-1,6} & \varphi_{2N-1,7} & \cdots & \varphi_{2N-1,2N-2} & 0 \\ 0 & \varphi_{N,2} & \varphi_{N,3} & \varphi_{N,4} & \varphi_{N,5} & \varphi_{N,6} & \varphi_{N,7} & \cdots & \varphi_{N,2N-2} & \varphi_{N,2N-1} \end{bmatrix}.$$

All rows and columns in $\mathcal{R}_{\mathbf{A},\mathbf{B}}$ are linearly independent. Then the determinant is nonzero (i.e., $\mathcal{R}_{\mathbf{A},\mathbf{B}}$ is nonsingular) implying that this matrix has full rank. Therefore Lemma 1 is checked (the realization is reachable).

From (33), (136) and (137) a $(4N-2) \times (2N-1)$ matrix is obtained, again $\varphi_{i,j}$ represent nonzero elements:

$$
\mathcal{S}_{\mathbf{C},\mathbf{A}}=\begin{bmatrix}
\varphi_{1,1} & \varphi_{1,2} & \varphi_{1,3} & \cdots & \varphi_{1,N} & 0 & 0 & 0 & \cdots & 0\\
\varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3} & \cdots & \varphi_{2,N} & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & \varphi_{3,N+1} & \varphi_{3,N+2} & \varphi_{3,N+3} & \cdots & \varphi_{3,2N-1}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\varphi_{4N-15,1} & \varphi_{4N-15,2} & \varphi_{4N-15,3} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
\varphi_{4N-14,1} & \varphi_{4N-14,2} & \varphi_{4N-14,3} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & \varphi_{4N-13,N+1} & \varphi_{4N-13,N+2} & \varphi_{4N-13,N+3} & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & \varphi_{4N-12,N+1} & \varphi_{4N-12,N+2} & \varphi_{4N-12,N+3} & \cdots & 0\\
\varphi_{4N-11,1} & \varphi_{4N-11,2} & \varphi_{4N-11,3} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
\varphi_{4N-10,1} & \varphi_{4N-10,2} & \varphi_{4N-10,3} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & \varphi_{4N-9,N+1} & \varphi_{4N-9,N+2} & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & \varphi_{4N-8,N+1} & \varphi_{4N-8,N+2} & 0 & \cdots & 0\\
\varphi_{4N-7,1} & \varphi_{4N-7,2} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
\varphi_{4N-6,1} & \varphi_{4N-6,2} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & \varphi_{4N-5,N+1} & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \cdots & 0 & \varphi_{4N-4,N+1} & 0 & 0 & \cdots & 0\\
\varphi_{4N-3,1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0\\
\varphi_{4N-2,1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}.
$$

Here, in an analogous manner to the $\mathcal{R}_{\mathbf{A},\mathbf{B}}$ matrix, a row equivalent matrix, $\mathcal{S}'_{\mathbf{C},\mathbf{A}}$, can be obtained by swapping rows of the original matrix, $\mathcal{S}_{\mathbf{C},\mathbf{A}}$. The procedure shall be done in order to obtain a new matrix such that is easy to realize that all $\mathcal{S}_{\mathbf{C},\mathbf{A}}$ columns are linearly independent.

Summarizing, all $\mathcal{S}_{\mathbf{C},\mathbf{A}}$ columns are linearly independent. Then this matrix has $2N-1$ rank. Therefore lemma 2 is checked (the realization is observable). According to Lemma 3 and the previous paragraphs, the realization is minimal. □

As set out by Theorem 16 the proposed realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is minimal (reachable and observable) and, as previously mentioned, the characteristic polynomial of $\mathbf{A}$, which has size $(2N-1)\times(2N-1)$ is S^{2N-1}. This fact implies that when $H(z)$ is a FIR filter, all of $\mathbf{A}$ eigenvalues are zero. In other words, all these eigenvalues are inside the unit circle, fulfilling the stability condition.

5.3 The state-space parameterization for multiple decomposition levels

The ideas presented in the previous sections can be extended for the multiple decomposition levels case, which is the Fast Wavelet Transform.

Denoting the state vector associated with the lattice in the i-th decomposition level as

$$\mathbf{x}_i[k] = \begin{bmatrix} x_{i,1}[k] \\ x_{i,2}[k] \\ x_{i,3}[k] \\ \vdots \\ x_{i,2N-1}[k] \end{bmatrix}, \tag{140}$$

the output in the i-th decomposition level with $\underline{y}_{i,1}[k]$ and $\underline{y}_{i,2}[k]$ associated with the low-pass and high-pass channels, respectively, by

$$\underline{\mathbf{y}}_i[k] = \begin{bmatrix} \underline{y}_{i,1}[k] \\ \underline{y}_{i,2}[k] \end{bmatrix}, \tag{141}$$

besides, matrices $\mathbf{C}$ and $\mathbf{D}$ are conveniently rewritten as

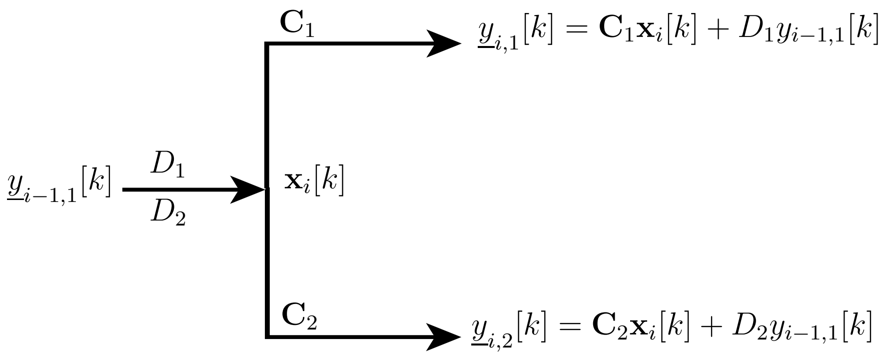
$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}$$

and

$$\mathbf{D} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$

The state-space description for one single decomposition level i of a wavelet filter bank in the *Algorithme à Trous* as previously presented and its denotations are shown in Figure 25. In Figure 25, the input $\underline{y}_{i-1,1}[k]$ at level i is the low-pass output at the level $i-1$. The elements $\mathbf{x}_i[k]$, $\underline{y}_{i,1}[k]$ and $\underline{y}_{i,2}[k]$ are the state variable and two outputs in the i-th decomposition level (algorithm with "holes"), respectively.

Figure 25: State-space description representation for one single decomposition level i in a wavelet filter bank.



Source: Elaborated by the author.

Assuming that the lattice model in the i-th decomposition level (using *Algorithme à Trous*) has the form

$$\mathbf{x}_1[k+1] = \mathbf{A}\mathbf{x}_1[k] + \mathbf{B}u[k], \tag{142}$$

$$\underline{\mathbf{y}}_1[k] = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \mathbf{x}_1[k] + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} u[k], \tag{143}$$

while for $i > 1$ it is

$$\mathbf{x}_i[k+2^{i-1}] = \mathbf{A}\mathbf{x}_i[k] + \mathbf{B}\underline{y}_{i-1,1}[k], \tag{144}$$

$$\underline{\mathbf{y}}_i[k] = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix} \mathbf{x}_i[k] + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \underline{y}_{i-1,1}[k]. \tag{145}$$

The *Algorithme à Trous* (Figure 11) employs filters of the form $H(z^2)$, $H(z^4)$, $H(z^8)$ and so on. It should be noted that $H(z^2)$ is twice as long as $H(z)$. This difference in the filter length is taken into account in (144). To understand this point see the Example 7.

Example 7. *Let $i = 2$ be in (144). In this case*

$$\boldsymbol{x}_2[k+2] = \boldsymbol{A}\boldsymbol{x}_2[k] + \boldsymbol{B}\underline{y}_{1,1}[k] \tag{146}$$

Considering an additional state vector $\boldsymbol{w}[k] = \boldsymbol{x}_2[k+1]$, (146) could be rewritten as

$$\boldsymbol{w}[k+1] = \boldsymbol{A}\boldsymbol{x}_2[k] + \boldsymbol{B}\underline{y}_{1,1}[k] \tag{147}$$

$$\boldsymbol{x}_2[k+1] = \boldsymbol{w}[k], \tag{148}$$

that is:

$$\left[\begin{array}{c} \boldsymbol{w} \\ \hdashline \boldsymbol{x}_2 \end{array}\right][k+1] = \left[\begin{array}{c;{2pt/2pt}c} 0 & \boldsymbol{A} \\ \hdashline \boldsymbol{I} & 0 \end{array}\right] \left[\begin{array}{c} \boldsymbol{w} \\ \hdashline \boldsymbol{x}_2 \end{array}\right][k] + \left[\begin{array}{c} \boldsymbol{B} \\ \hdashline 0 \end{array}\right] \underline{y}_{1,1}[k], \tag{149}$$

where $\boldsymbol{I}$ is the identity matrix and 0 is the null matrix. Therefore, using $\boldsymbol{x}_2[k+2]$ in (144) is equivalent to use more states to describe a longer filter. For $i = 2$, $\boldsymbol{x}_2[k+2]$, twice as many states are necessary. For $i = 3$, $\boldsymbol{x}_3[k+4]$ means four times as many states, and so on.

By taking the entire analysis filter bank with M decomposition levels, the following state vector can be defined as the filter bank, as presented in Figure 26,

$$\mathbf{x}[k] = \begin{bmatrix} \mathbf{x}_1[k] \\ \mathbf{x}_2[k] \\ \mathbf{x}_3[k] \\ \vdots \\ \mathbf{x}_M[k] \end{bmatrix}. \tag{150}$$

Considering the "holes", the equation for the filter bank can be written as

$$\begin{bmatrix} \mathbf{x}_1[k+1] \\ \mathbf{x}_2[k+2] \\ \vdots \\ \mathbf{x}_M[k+2^{M-1}] \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathbf{x}_1[k] + \mathbf{B}u[k] \\ \mathbf{A}\mathbf{x}_2[k] + \mathbf{B}\underline{y}_{1,1}[k] \\ \vdots \\ \mathbf{A}\mathbf{x}_M[k] + \mathbf{B}\underline{y}_{M-1,1}[k] \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}\mathbf{x}_1[k] + \mathbf{B}u[k] \\ \mathbf{A}\mathbf{x}_2[k] + \mathbf{B}(\mathbf{C}_1\mathbf{x}_1[k] + D_1 u[k]) \\ \vdots \\ \mathbf{A}\mathbf{x}_M[k] + \mathbf{B}(\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_1\underline{y}_{M-2,1}[k]) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}\mathbf{x}_1[k] + \mathbf{B}u[k] \\ \mathbf{A}\mathbf{x}_2[k] + \mathbf{B}\mathbf{C}_1\mathbf{x}_1[k] + \mathbf{B}D_1 u[k] \\ \vdots \\ \mathbf{A}\mathbf{x}_M[k] + \mathbf{B}\mathbf{C}_1\mathbf{x}_{M-1}[k] + \mathbf{B}D_1\underline{y}_{M-2,1}[k] \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}\mathbf{x}_1[k] + \mathbf{B}u[k] \\ \mathbf{A}\mathbf{x}_2[k] + \mathbf{B}\mathbf{C}_1\mathbf{x}_1[k] + \mathbf{B}D_1 u[k] \\ \vdots \\ \mathbf{A}\mathbf{x}_M[k] + \mathbf{B}\mathbf{C}_1\mathbf{x}_{M-1}[k] + \mathbf{B}D_1(\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_1\underline{y}_{M-3,1}[k]) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}\mathbf{x}_1[k] + \mathbf{B}u[k] \\ \mathbf{A}\mathbf{x}_2[k] + \mathbf{B}\mathbf{C}_1\mathbf{x}_1[k] + \mathbf{B}D_1 u[k] \\ \vdots \\ \mathbf{A}\mathbf{x}_M[k] + \mathbf{B}\mathbf{C}_1\mathbf{x}_{M-1}[k] + \mathbf{B}D_1\mathbf{C}_1\mathbf{x}_{M-2}[k] + \mathbf{B}D_1^2\underline{y}_{M-3,1}[k] \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{A}\mathbf{x}_1[k] + \mathbf{B}u[k] \\ \mathbf{A}\mathbf{x}_2[k] + \mathbf{B}\mathbf{C}_1\mathbf{x}_1[k] + \mathbf{B}D_1 u[k] \\ \vdots \\ \left(\begin{array}{r} \mathbf{A}\mathbf{x}_M[k] + \mathbf{B}\mathbf{C}_1\mathbf{x}_{M-1}[k] + \mathbf{B}D_1\mathbf{C}_1\mathbf{x}_{M-2}[k] + \mathbf{B}D_1^2\mathbf{C}_1\mathbf{x}_{M-3}[k] + \cdots + \\ \mathbf{B}D_1^{M-2}\mathbf{C}_1\mathbf{x}_1[k] + \mathbf{B}D_1^{M-1}u[k] \end{array} \right) \end{bmatrix}.$$

It can be appropriately written as

$$\begin{bmatrix} \mathbf{x}_1[k+1] \\ \mathbf{x}_2[k+2] \\ \vdots \\ \mathbf{x}_M[k+2^{M-1}] \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 \\ \mathbf{B}\mathbf{C}_1 & \mathbf{A} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}D_1^{M-2}\mathbf{C}_1 & \cdots & \mathbf{B}\mathbf{C}_1 & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1[k] \\ \mathbf{x}_2[k] \\ \vdots \\ \mathbf{x}_M[k] \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{B}D_1 \\ \vdots \\ \mathbf{B}D_1^{M-1} \end{bmatrix} u[k]. \quad (151)$$

Let $\underline{\mathbf{y}}[k]$ be the output vector, comprising the approximation in the last level and details in all levels, defined as

$$\underline{\mathbf{y}}[k] = \begin{bmatrix} \underline{y}_{M,1}[k] \\ \underline{y}_{M,2}[k] \\ \underline{y}_{M-1,2}[k] \\ \vdots \\ \underline{y}_{1,2}[k] \end{bmatrix}, \tag{152}$$

the output equation can be finally written for the filter bank as follows:

$$\begin{aligned}
\underline{\mathbf{y}}[k] &= \begin{bmatrix} \mathbf{C}_1\mathbf{x}_M[k] + D_1\underline{y}_{M-1,1}[k] \\ \mathbf{C}_2\mathbf{x}_M[k] + D_2\underline{y}_{M-1,1}[k] \\ \mathbf{C}_2\mathbf{x}_{M-1}[k] + D_2\underline{y}_{M-2,1}[k] \\ \vdots \\ \mathbf{C}_2\mathbf{x}_1[k] + D_2u[k] \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{C}_1\mathbf{x}_M[k] + D_1(\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_1\underline{y}_{M-2,1}[k]) \\ \mathbf{C}_2\mathbf{x}_M[k] + D_2(\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_1\underline{y}_{M-2,1}[k]) \\ \mathbf{C}_2\mathbf{x}_{M-1}[k] + D_2(\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_1\underline{y}_{M-3,1}[k]) \\ \vdots \\ \mathbf{C}_2\mathbf{x}_1 + D_2u[k] \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{C}_1\mathbf{x}_M[k] + D_1\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_1^2\underline{y}_{M-2,1}[k] \\ \mathbf{C}_2\mathbf{x}_M[k] + D_2\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_2D_1\underline{y}_{M-2,1}[k] \\ \mathbf{C}_2\mathbf{x}_{M-1}[k] + D_2\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_2D_1\underline{y}_{M-3,1}[k] \\ \vdots \\ \mathbf{C}_2\mathbf{x}_1 + D_2u[k] \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{C}_1\mathbf{x}_M[k] + D_1\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_1^2(\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_1\underline{y}_{M-3,1}[k]) \\ \mathbf{C}_2\mathbf{x}_M[k] + D_2\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_2D_1(\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_1\underline{y}_{M-3,1}[k]) \\ \mathbf{C}_2\mathbf{x}_{M-1}[k] + D_2\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_2D_1(\mathbf{C}_1\mathbf{x}_{M-3}[k] + D_1\underline{y}_{M-4,1}[k]) \\ \vdots \\ \mathbf{C}_2\mathbf{x}_1 + D_2u[k] \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{C}_1\mathbf{x}_M[k] + D_1\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_1^2\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_1^3\underline{y}_{M-3,1}[k] \\ \mathbf{C}_2\mathbf{x}_M[k] + D_2\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_2D_1\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_2D_1^2\underline{y}_{M-3,1}[k] \\ \mathbf{C}_2\mathbf{x}_{M-1}[k] + D_2\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_2D_1\mathbf{C}_1\mathbf{x}_{M-3}[k] + D_2D_1^2\underline{y}_{M-4,1}[k] \\ \vdots \\ \mathbf{C}_2\mathbf{x}_1 + D_2u[k] \end{bmatrix}
\end{aligned}$$

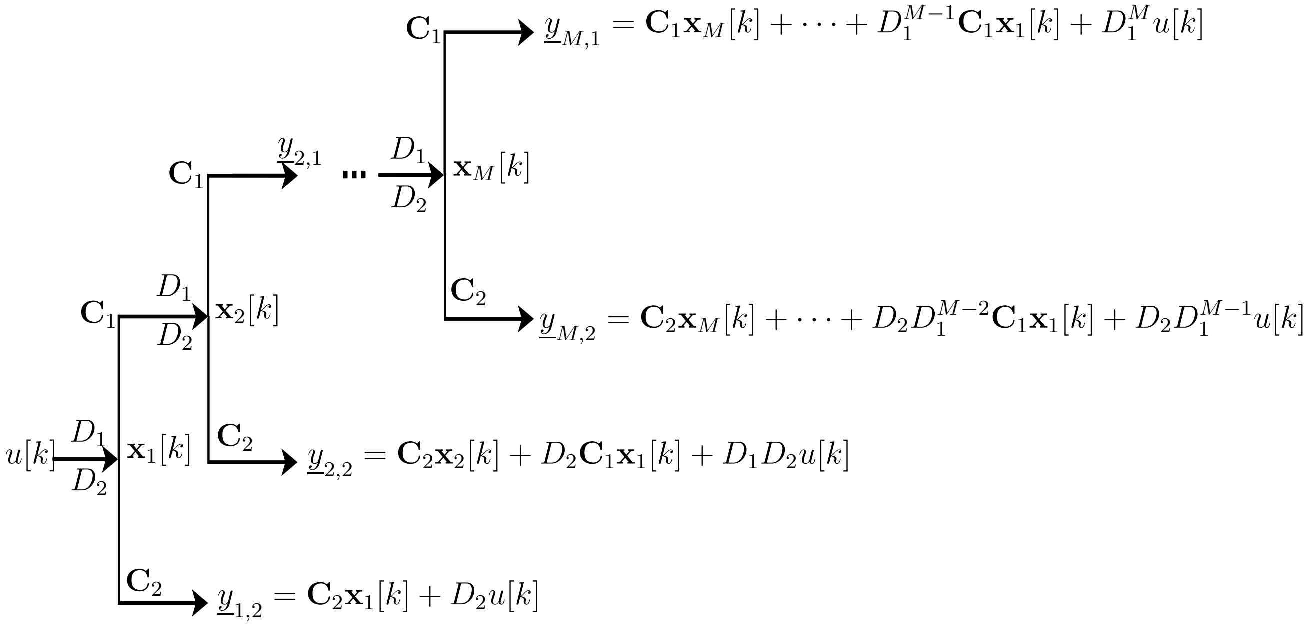
$$= \begin{bmatrix} \mathbf{C}_1\mathbf{x}_M[k] + D_1\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_1^2\mathbf{C}_1\mathbf{x}_{M-2}[k] + \cdots + D_1^{M-1}\mathbf{C}_1\mathbf{x}_1[k] + D_1^M u[k] \\ \mathbf{C}_2\mathbf{x}_M[k] + D_2\mathbf{C}_1\mathbf{x}_{M-1}[k] + D_2D_1\mathbf{C}_1\mathbf{x}_{M-2}[k] + \cdots + D_2D_1^{M-2}\mathbf{C}_1\mathbf{x}_1[k] + D_2D_1^{M-1}u[k] \\ \mathbf{C}_2\mathbf{x}_{M-1}[k] + D_2\mathbf{C}_1\mathbf{x}_{M-2}[k] + D_2D_1\mathbf{C}_1\mathbf{x}_{M-3}[k] + \cdots + D_2D_1^{M-3}\mathbf{C}_1\mathbf{x}_1[k] + D_2D_1^{M-2}u[k] \\ \vdots \\ \mathbf{C}_2\mathbf{x}_1 + D_2u[k] \end{bmatrix},$$

and, writing it more compactly,

$$\begin{bmatrix} \underline{y}_{M,1}[k] \\ \underline{y}_{M,2}[k] \\ \underline{y}_{M-1,2}[k] \\ \vdots \\ \underline{y}_{1,2}[k] \end{bmatrix} = \begin{bmatrix} D_1^{M-1}\mathbf{C}_1 & \cdots & D_1^2\mathbf{C}_1 & D_1\mathbf{C}_1 & \mathbf{C}_1 \\ D_2D_1^{M-2}\mathbf{C}_1 & \cdots & D_2D_1\mathbf{C}_1 & D_2\mathbf{C}_1 & \mathbf{C}_2 \\ D_2D_1^{M-3}\mathbf{C}_1 & \cdots & D_2\mathbf{C}_1 & \mathbf{C}_2 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{C}_2 & \cdots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1[k] \\ \mathbf{x}_2[k] \\ \mathbf{x}_3[k] \\ \vdots \\ \mathbf{x}_M[k] \end{bmatrix} + \begin{bmatrix} D_1^M \\ D_2D_1^{M-1} \\ D_2D_1^{M-2} \\ \vdots \\ D_2 \end{bmatrix} u[k]. \tag{153}$$

From (151) and (153), the analysis wavelet filter bank with M decomposition levels can be seen as the representation given in Figure 26.

Figure 26: Wavelet filter bank with M decomposition levels seen as state-space description.



Source: Elaborated by the author.

The final state-space realization is not given by (151) and (153). It can be noted in (151) that states are missing. Some changes in this equation are required, and it will imply other changes in (153).

Firstly, it is necessary to define new state variables

$$
\begin{aligned}
&\left\{ x_{M+1}[k] = x_2[k+1], \right. \\
&\begin{cases} x_{M+2}[k] = x_3[k+1], \\ x_{M+3}[k] = x_{M+2}[k+1], \\ x_{M+4}[k] = x_{M+3}[k+1], \end{cases} \\
&\begin{cases} x_{M+5}[k] = x_4[k+1], \\ x_{M+6}[k] = x_{M+5}[k+1], \\ \quad\vdots \\ x_{M+11}[k] = x_{M+10}[k+1], \end{cases} \\
&\quad\vdots \\
&\begin{cases} x_{M+\varepsilon+1}[k] = x_M[k+1], \\ x_{M+\varepsilon+2}[k] = x_{M+\varepsilon+1}[k+1], \\ \quad\vdots \\ x_{M+\varepsilon+2^{M-1}-1}[k] = x_{M+\varepsilon+2^{M-1}-2}[k+1], \end{cases}
\end{aligned}
\tag{154}
$$

where $\varepsilon = \sum_{i=2}^{M-1} \left[2^{i-1} - 1\right]$, hence

$$
\begin{cases} x_{M+1}[k+1] = x_2[k+2], \\ x_{M+4}[k+1] = x_3[k+4], \\ x_{M+11}[k+1] = x_4[k+8], \\ \quad\vdots \\ x_{M+\varepsilon+2^{M-1}-1}[k+1] = x_M[k+2^{M-1}]. \end{cases}
\tag{155}
$$

Thus, vector $\mathbf{x}[k]$ becomes

$$
\mathbf{x}[k] = \begin{bmatrix} \mathbf{x}_1[k] \\ \mathbf{x}_2[k] \\ \vdots \\ \mathbf{x}_M[k] \\ \mathbf{x}_{M+1}[k] \\ \mathbf{x}_{M+2}[k] \\ \vdots \\ \mathbf{x}_\eta[k] \end{bmatrix},
\tag{156}
$$

with $\eta = M + \sum_{i=2}^{M-1} \left[2^{i-1} - 1\right] + 2^{M-1} - 1$, namely, $\eta = M + \varepsilon + 2^{M-1} - 1$.

The final state-space realization can be shown now, but it is necessary to denote the new matrices for the new state-space description to avoid confusion with matrices for the realization for one single decomposition level. For M decomposition levels, matrices are be denoted as $\mathbf{A}_M$, $\mathbf{B}_M$, $\mathbf{C}_M$ and $\mathbf{D}_M$.

From (151), (153), (154) and (155), matrix $\mathbf{A}_M$ takes the form

$$\mathbf{A}_M = \begin{bmatrix} \mathbf{A} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \mathbf{I} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}\mathbf{C}_1 & \mathbf{A} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \mathbf{I} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \mathbf{I} \\ \mathbf{B}D_1^{M-2}\mathbf{C}_1 & \mathbf{B}D_1^{M-3}\mathbf{C}_1 & \cdots & \mathbf{A} & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{157}$$

where $\mathbf{I}$ is the $(2N-1) \times (2N-1)$ identity matrix. In the same way, each element denoted by 0 is a $(2N-1) \times (2N-1)$ null matrix. While $\mathbf{B}_M$ is defined as

$$\mathbf{B}_M = \begin{bmatrix} \mathbf{B} \\ 0 \\ \vdots \\ \mathbf{B}D_1 \\ 0 \\ \vdots \\ 0 \\ \mathbf{B}D_1^{M-1} \end{bmatrix}, \tag{158}$$

with 0 being a $(2N-1) \times 1$ null vector. Matrix $\mathbf{C}_M$ is given by

$$\mathbf{C}_M = \begin{bmatrix} D_1^{M-1}\mathbf{C}_1 & \cdots & D_1^2\mathbf{C}_1 & D_1\mathbf{C}_1 & \mathbf{C}_1 & 0 & \cdots & 0 \\ D_2D_1^{M-2}\mathbf{C}_1 & \cdots & D_2D_1\mathbf{C}_1 & D_2\mathbf{C}_1 & \mathbf{C}_2 & 0 & \cdots & 0 \\ D_2D_1^{M-3}\mathbf{C}_1 & \cdots & D_2\mathbf{C}_1 & \mathbf{C}_2 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ \mathbf{C}_2 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{159}$$

with 0 being an $1 \times (2N-1)$ null vector. Vector $\mathbf{D}_M$ is equal to

$$\mathbf{D}_M = \begin{bmatrix} D_1^M \\ D_2 D_1^{M-1} \\ D_2 D_1^{M-2} \\ \vdots \\ D_2 \end{bmatrix}. \tag{160}$$

The dimension of matrices (157), (158), (159) and (160) are:

- $\dim(\mathbf{A}_M) = \eta(2N-1) \times \eta(2N-1)$;
- $\dim(\mathbf{B}_M) = \eta(2N-1) \times 1$;
- $\dim(\mathbf{C}_M) = (M+1) \times \eta(2N-1)$;
- $\dim(\mathbf{D}_M) = (M+1) \times 1$.

Finally, matrices (157), (158), (159) and (160) are the state-space description for analysis wavelet FIR filter banks with multiple decomposition levels. This proposal is based on the *Algorithme à Trous*, therefore it is necessary to remember that the relationship among the coefficients $\mathbf{y}_i[k]$ (FWT) and $\underline{\mathbf{y}}_i[k]$ (*Algorithme à Trous*) is given by (85), in other words, the convenient decimation must be done after state-space description.

An important result about the state-space description is the minimality condition, which is the combination of controllability and observability conditions. For $M = 1$, the proposed state-space description is minimal, as shown in Theorem 16. For an arbitrary M, the minimality condition is exploited in Theorem 18. The proof of this theorem is made by using mathematical induction. For this reason, Lemma 17 states that the proposed state-space description is minimal for $M = 2$ (for $M = 1$ there are no delay operators included in the parameterization).

Let $\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}$ and $\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}$ be the controllability and observability matrices, then

- $\dim(\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}) = \eta(2N-1) \times \eta(2N-1)$;
- $\dim(\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}) = (M+1)\eta(2N-1) \times \eta(2N-1)$.

Lemma 17. *A realization ($\boldsymbol{A}_M$, $\boldsymbol{B}_M$, $\boldsymbol{C}_M$, $\boldsymbol{D}_M$) for $M = 2$ given by (157), (158), (159) and (160), respectively, with all angles different from 0, $\pi/2$, π and $3\pi/2$, is minimal.*

Proof. For $M = 2$, matrices $\mathbf{A}_M$, $\mathbf{B}_M$ and $\mathbf{C}_M$ become

$$\mathbf{A}_M = \begin{bmatrix} \mathbf{A} & 0 & 0 \\ 0 & 0 & \mathbf{I} \\ \mathbf{B}\mathbf{C}_1 & \mathbf{A} & 0 \end{bmatrix}, \tag{161}$$

$$\mathbf{B}_M = \begin{bmatrix} \mathbf{B} \\ 0 \\ \mathbf{B}D_1 \end{bmatrix} \tag{162}$$

and

$$\mathbf{C}_M = \begin{bmatrix} D_1\mathbf{C}_1 & \mathbf{C}_1 & 0 \\ \mathbf{C}_2\mathbf{C}_1 & \mathbf{C}_2 & 0 \\ \mathbf{C}_2 & 0 & 0 \end{bmatrix}. \tag{163}$$

By the mathematical induction principle it is necessary to verify that the lemma is true for $N = 1$ and if it is true for a certain N, this implies that it will be true for $N + 1$. It is made as follows.

- For $N = 1$: $\mathbf{A} = 0$, $\mathbf{B} = 1$, $\mathbf{C} = [\ S_1 \quad C_1\ ]^T$ and $\mathbf{D} = [\ C_1 \quad S_1\ ]^T$. By replacing these values in (161), (162) and (163):

$$\mathbf{A}_M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ S_1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B}_M = \begin{bmatrix} 1 \\ 0 \\ C_1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C}_M = \begin{bmatrix} C_1S_1 & S_1 & 0 \\ C_1C_1 & C_1 & 0 \\ C_1 & 0 & 0 \end{bmatrix},$$

  and after that, computing the controllability and observability matrices, could be seen that both of them have three linearly independent rows and columns.

- Let N be an arbitrary positive integer number. For $N + 1$, matrices (161), (162) and (163) hold the same form for N, only the dimensions of these matrices will be changed. Consequently, the same fact is valid for matrices $\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}$ and $\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}$. Therefore, if matrices $\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}$ and $\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}$ have $\eta(2N - 1)$ linearly independent rows and columns for N, they have $\eta(2(N + 1) - 1)$ linearly independent rows and columns for $N + 1$ (in η, $M = 2$).

By the mathematical induction principle it is demonstrated that matrices $\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}$ and $\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}$ have $\eta(2N - 1)$ linearly independent rows and columns. Thus, any realization ($\mathbf{A}_M$, $\mathbf{B}_M$, $\mathbf{C}_M$, $\mathbf{D}_M$) (with all angles different from 0, $\pi/2$, π and $3\pi/2$) for $M = 2$ is minimal. □

Theorem 18. *A realization ($\boldsymbol{A}_M$, $\boldsymbol{B}_M$, $\boldsymbol{C}_M$, $\boldsymbol{D}_M$) given by (157), (158), (159) and (160), respectively, with all angles different from 0, $\pi/2$, π and $3\pi/2$, is minimal because it is reachable and observable.*

Proof. By the mathematical induction principle and the following conclusions, the proof of the theorem is achieved as follows.

- For $M = 1$ it is demonstrated in Theorem 16 that $\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}$ and $\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}$ have $\eta(2N-1)$ linearly independent rows and columns (in η, $M = 1$).
- For $M = 2$ it is demonstrated by Lemma 17 that $\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}$ and $\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}$ have $\eta(2N-1)$ linearly independent rows and columns (in η, $M = 2$).
- For any $M \geq 2$ the forms of $\mathbf{A}_M$, $\mathbf{B}_M$ and $\mathbf{C}_M$ are the same as stated by (157), (158), (159) and (160), only the dimensions change according to M. The same fact is valid for matrices $\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}$ and $\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}$. Therefore, if $\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}$ and $\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}$ have $\eta(2N-1)$ linearly independent rows and columns, it implies that $\mathcal{R}_{\mathbf{A}_{M+1},\mathbf{B}_{M+1}}$ and $\mathcal{S}_{\mathbf{C}_{M+1},\mathbf{A}_{M+1}}$ have $\eta(2N-1)$ (in η, the value of M is $M+1$) linearly independent rows and columns.

By the mathematical induction principle it is demonstrated that matrices $\mathcal{R}_{\mathbf{A}_M,\mathbf{B}_M}$ and $\mathcal{S}_{\mathbf{C}_M,\mathbf{A}_M}$ have $\eta(2N-1)$ linearly independent rows and columns. Thus, any realization ($\mathbf{A}_M$, $\mathbf{B}_M$, $\mathbf{C}_M$, $\mathbf{D}_M$) (with all angles different from 0, $\pi/2$, π and $3\pi/2$) is minimal. □

As set out by Theorem 18 the proposed realization ($\mathbf{A}_M$, $\mathbf{B}_M$, $\mathbf{C}_M$, $\mathbf{D}_M$) is minimal (reachable and observable) and, as previously mentioned, the characteristic polynomial of $\mathbf{A}_M$ is $S^{\eta(2N-1)}$. This fact implies that, when $H(z)$ is a FIR filter, all of $\mathbf{A}_M$ eigenvalues are zero. In other words, all these eigenvalues are inside the unit circle, fulfilling the stability condition.

5.4 Further discussions

As previously mentioned in the introductory chapter of this thesis, the central objective is to explore the orthornormal basis functions developed by Sherlock and Monro (1998), parameterize it in the state-space, and develop the extension of this idea for M decomposition levels of the filter bank therewithal. In these proposals, it is a main concern to keep the properties that make the state-space realization attractive like observability, reachability and minimality. Moreover, the existing advantages for orthonormal basis functions and finite impulse response filters should be automatically held.

In Section 5.1, an initial idea to parameterize a finite impulse response filter in the state-space was proposed. That approach is interesting and is a novelty when using the filter coefficients from Sherlock and Monro formulation. Also, it is interesting because the state-space realization achieved is minimal. Nevertheless, this thesis is focused in the system implementation in lattice structure.

Lattice filter structures have a range of interesting and important properties that make them very common in applications. Among these properties, one can cite modularity, low sensitivity to parameter quantization effects and a simple stability test (KAPLAN; ERER; KENT, 2007). A FIR filter implemented in lattice structure was used to obtain the Sherlock and Monro orthonormal basis functions, and the same approach was used in Section 5.2 to obtain the state-space realization from these functions. The obtained state-space realization is minimal.

For one single FIR filter and for the lattice structure, the minimality condition is satisfied, proofs are in Theorems 15 and 16. The single filter parameterization is simpler while the lattice structure carries some advantages related to this kind of structure. It is worth noting that after the achievement of this results, both ideas will be equally easy for implementation.

The state-space realization obtained for one-single decomposition level was extended to the case of M decomposition levels in Section 5.3. This is equivalent to the fast wavelet transform with M decomposition levels. The difficulties associated with the downsampling operators in the filter bank were overcome by exploiting the concept of the *Algorithme à Trous*.

The final state-space realization, for M decomposition levels, holds the properties verified for one-single decomposition level state-space realization, e.g., observability, reachability and minimality, as shown in Theorem 18.

6 Application and decomposition examples

The state-space description for the fast wavelet transform and its properties are developed in Chapter 5. In order to illustrate the proposed technique, this chapter presents some decomposition examples in Section 6.1, and one application example in Section 6.2.

First, the state-space realization used in all cases is presented. In Section 6.1, comparisons among results from FWT and the equivalent state-space description are presented for some applications: Electrocardiogram (ECG) signal in Subsection 6.1.1, Random white Gaussian noise signal in Subsection 6.1.2, Pseudo-random binary signal in Subsection 6.1.3, speech signal in Subsection 6.1.4, and a simulation model of the lateral dynamics of a Boeing 747 aircraft in landing configuration in Subsection 6.1.5.

Finally, in Section 6.2 an application on dynamic state feedback control is presented. A dynamic state feedback approach employing the state-space description for the FWT is presented in order to illustrate the advantages of the theory developed in Chapter 5.

6.1 Comparisons between FWT and the proposed state-space description

For standardization purposes, in all applications, the same wavelet filter bank is used, `db3` wavelet and four decomposition levels were chosen. In this way, filters have considerable length and number of decomposition levels. Thus, **A**, **B**, **C** and **D**, given respectively by (136), (137), (138) and (139), are:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.3812 & 0 & 0 & 0 & 0 \\ 0.4431 & 0.8777 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0.9245 \\ -0.1827 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} -0.0854 & 0.0505 & 0.9944 & 0 & 0 \\ 0.8069 & -0.4766 & 0.1053 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} 0.0352 \\ -0.3327 \end{bmatrix},$$

and $\mathbf{A}_M$, $\mathbf{B}_M$, $\mathbf{C}_M$ and $\mathbf{D}_M$, given respectively by (157), (158), (159) and (160), are:

$$
\mathbf{A}_M = \begin{bmatrix}
\mathbf{A} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 0 \\
0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 & \cdots & 0 \\
\mathbf{B}\mathbf{C}_1 & \mathbf{A} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{B}D_1\mathbf{C}_1 & \mathbf{B}\mathbf{C}_1 & \mathbf{A} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \mathbf{I} \\
\mathbf{B}D_1^2\mathbf{C}_1 & \mathbf{B}D_1\mathbf{C}_1 & \mathbf{B}\mathbf{C}_1 & \mathbf{A} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \qquad
\mathbf{B}_M = \begin{bmatrix}
\mathbf{B} \\ 0 \\ 0 \\ 0 \\ \mathbf{B}D_1 \\ 0 \\ 0 \\ \mathbf{B}D_1^2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \mathbf{B}D_1^3
\end{bmatrix},
$$

$$
\mathbf{C}_M = \begin{bmatrix}
D_1^3\mathbf{C}_1 & D_1^2\mathbf{C}_1 & D_1\mathbf{C}_1 & \mathbf{C}_1 & 0 & \cdots & 0 \\
D_2D_1^2\mathbf{C}_1 & D_2D_1\mathbf{C}_1 & D_2\mathbf{C}_1 & \mathbf{C}_2 & 0 & \cdots & 0 \\
D_2D_1\mathbf{C}_1 & D_2\mathbf{C}_1 & \mathbf{C}_2 & 0 & 0 & \cdots & 0 \\
D_2\mathbf{C}_1 & \mathbf{C}_2 & 0 & 0 & 0 & \cdots & 0 \\
\mathbf{C}_2 & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix} \quad \text{and} \quad
\mathbf{D}_M = \begin{bmatrix}
D_1^4 \\ D_2D_1^3 \\ D_2D_1^2 \\ D_2D_1 \\ D_2
\end{bmatrix}.
$$

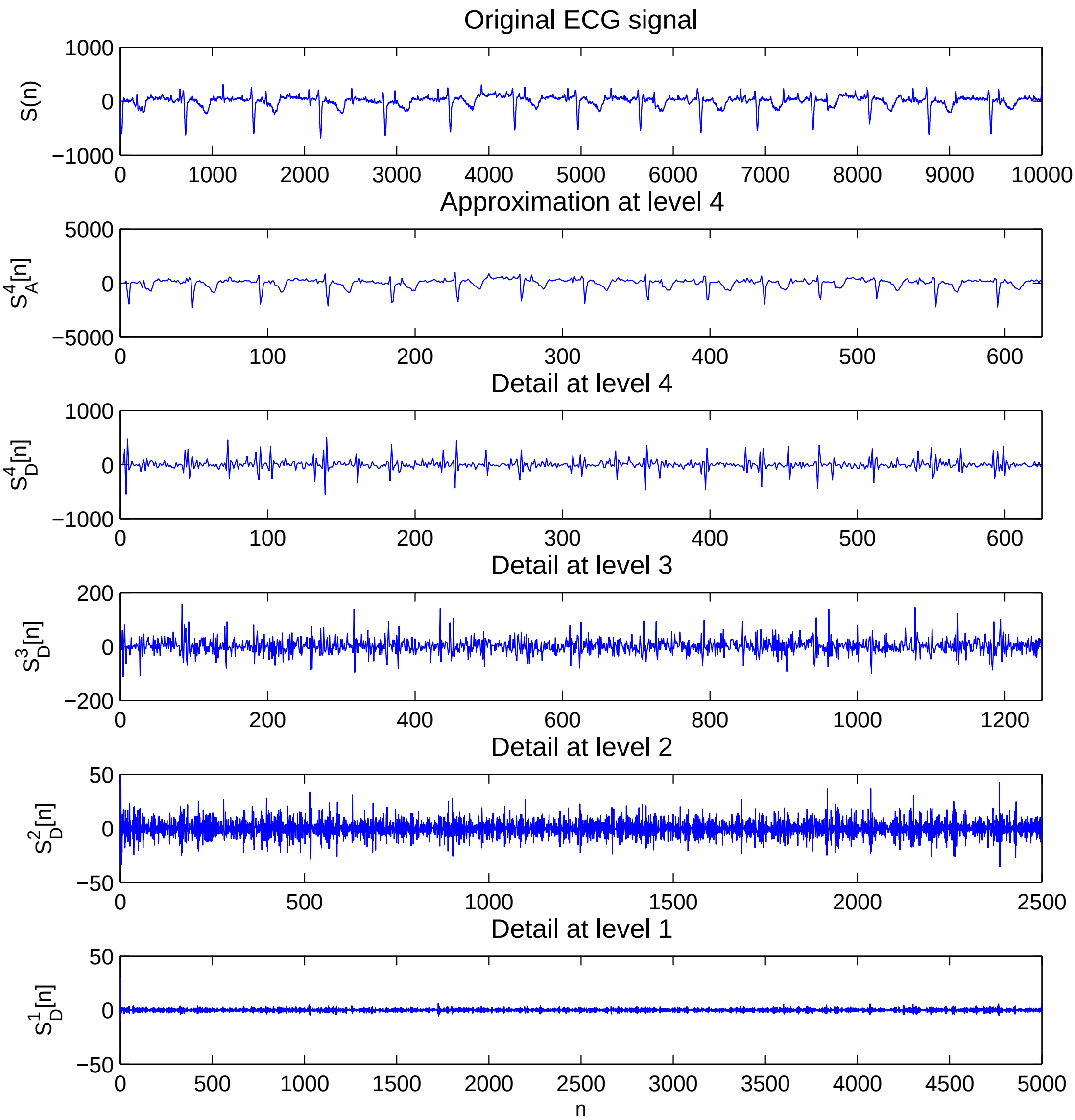
Fast wavelet transform is widely used in signal processing and control theory. Five examples of broadly studied signals are processed by FWT and the proposed state-space description. These examples are ECG, Random white Gaussian noise, Pseudo-random binary and signals from a Boeing 747 aircraft model in landing configuration. Subsequently, is shown each signal and its approximation and details in all levels as result of the processing by the state-space proposal after convenient decimation. There are no differences when these signals are decomposed by FWT, i.e., state-space description and FWT obtain the same curves when used to decompose a signal.

6.1.1 Example 1: Electrocardiogram signal

An abdominal electrocardiogram (ECG) signal was taken from MIT-BIH Arrhythmia Database (GOLDBERGER et al., 2000; MOODY; MARK, 2001), it has 10000 samples recorded in ten seconds. Figure 27 shows the ECG signal and its approximation and details in all levels.

In this section, for all decomposition examples the original signal is denoted by $\mathsf{S}(\mathsf{n})$, samples or time is n, approximation at level four signal is $\mathsf{S}_\mathsf{A}^4[\mathsf{n}]$, details at level i is $\mathsf{S}_\mathsf{D}^\mathsf{i}[\mathsf{n}]$.

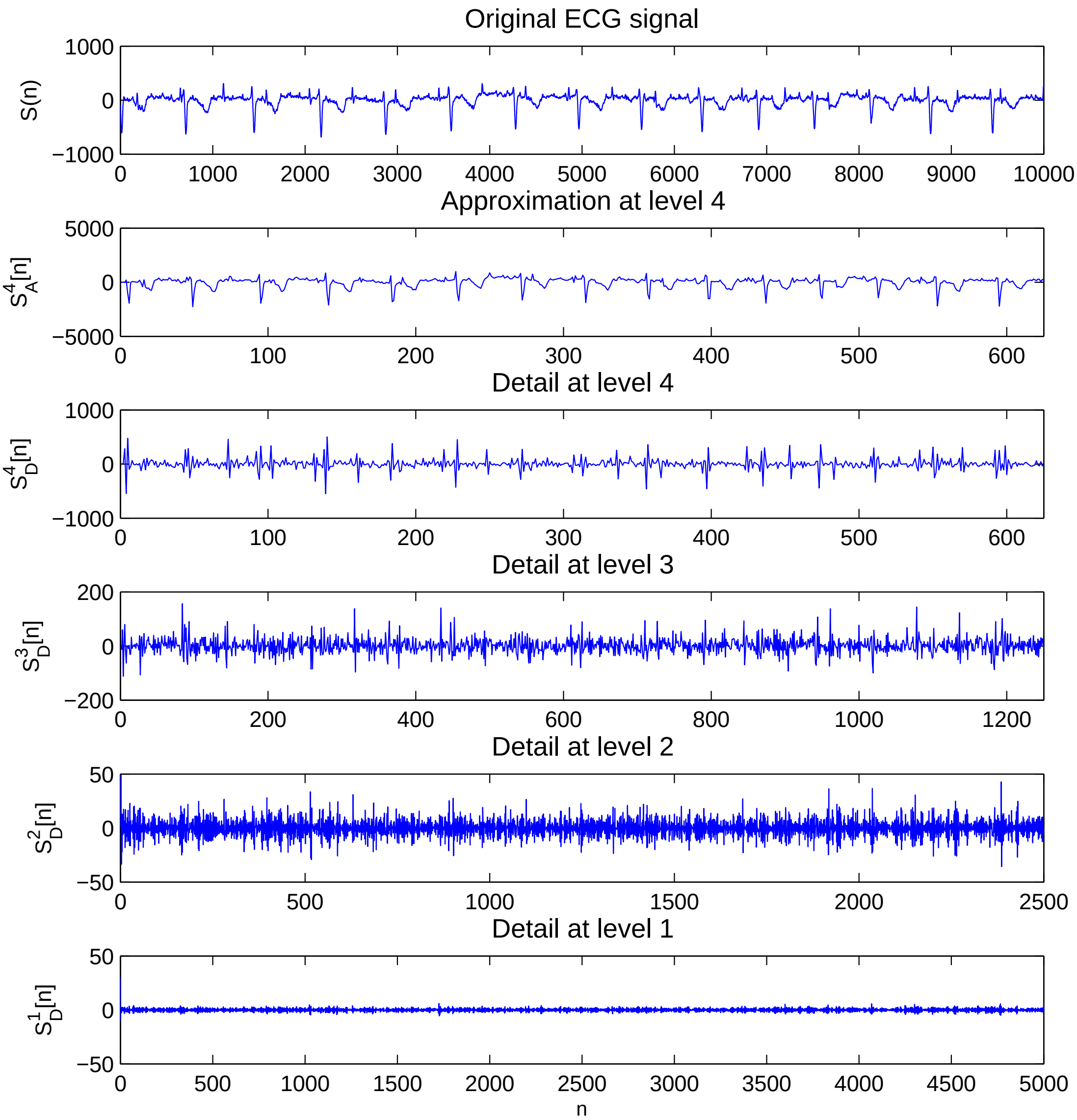
Figure 27: ECG signal, its approximation at level 4 and details in 4 levels processed using the state-space description.



Source: Elaborated by the author.

Figure 28 shows approximation and detail ECG decomposition using FWT. As it can be seen, there are no differences between Figures 27 and 28, i.e., the proposal wavelet state-space description and FWT provide the same decompositions for the ECG signal in all decomposition levels, which was verified for the other signals tested in the remainder of this section. For this reason, for the other signals presented here, only decompositions in the state-space description will be shown, always remembering that FWT provides the same decompositions.

Figure 28: ECG signal, its approximation at level 4 and details in 4 levels decomposed by FWT.

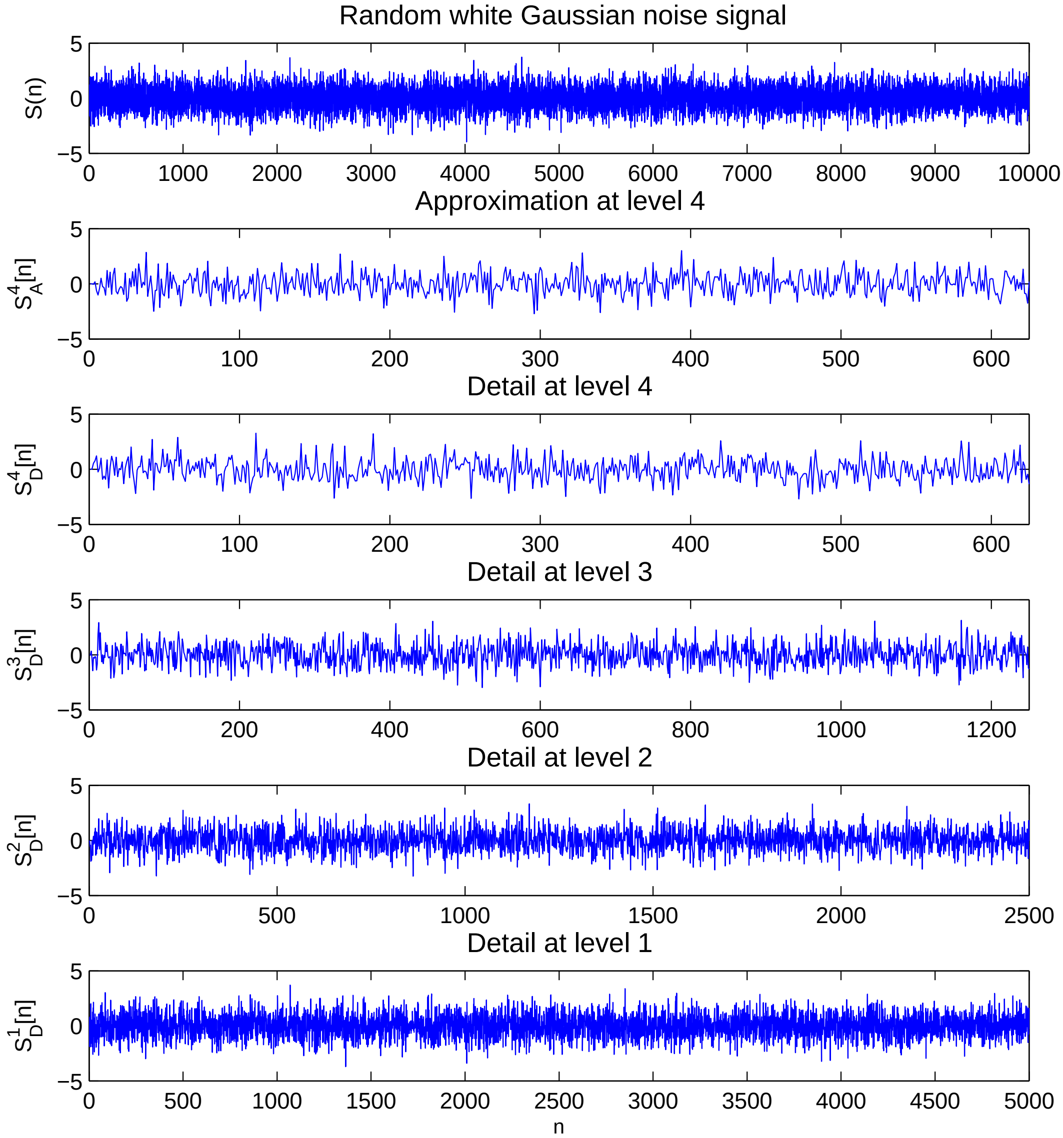


Source: Elaborated by the author.

6.1.2 Example 2: Random white Gaussian noise signal

Figure 29 shows a random white Gaussian noise signal and its approximation and details in all levels.

Figure 29: Random white Gaussian noise signal, its approximation at level 4 and details in 4 levels processed using the state-space description. FWT provides similar results.

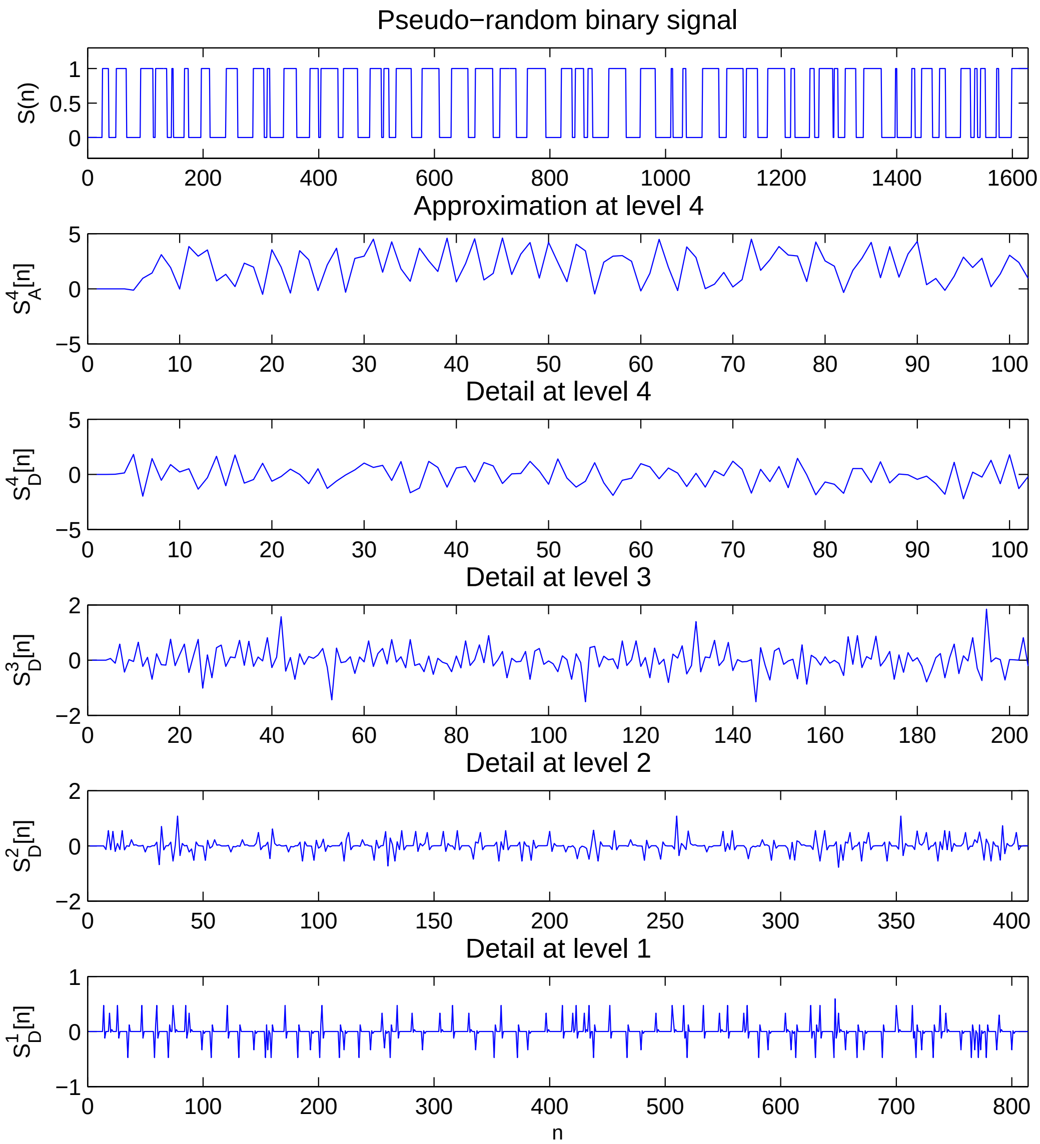


Source: Elaborated by the author.

6.1.3 Example 3: Pseudo-random binary signal

Figure 30 shows a pseudo-random binary signal and its approximation and details in all levels.

Figure 30: Pseudo-random binary signal, its approximation at level 4 and details in 4 levels processed by the state-space description. Results when this signal is decomposed by FWT are the same.

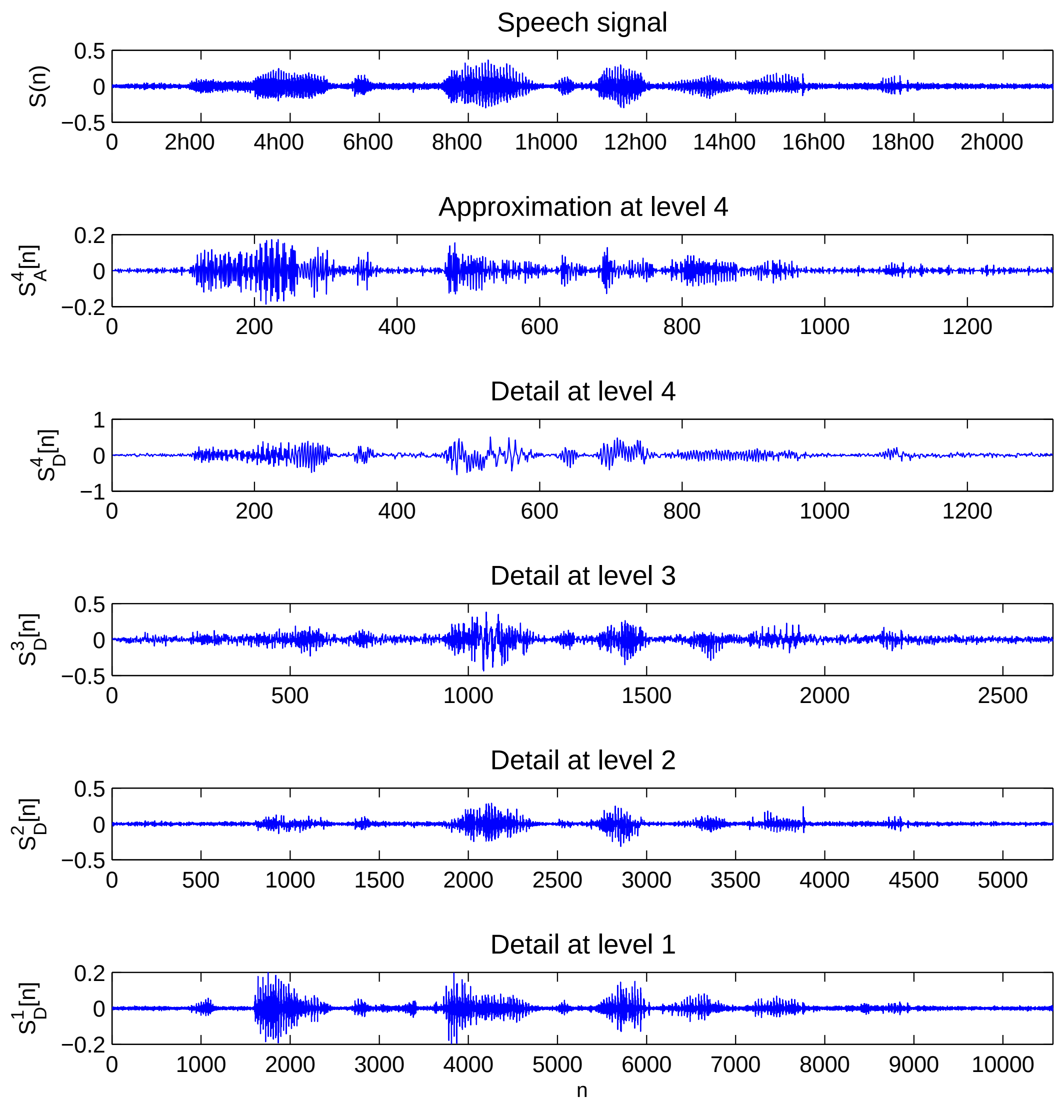


Source: Elaborated by the author.

6.1.4 Example 4: Speech signal

A speech signal was obtained from NOIZEUS database (HU; LOIZOU, 2007). An airport noise signal was added to the speech signal at SNR of 10dB. The resulting signal and its approximation at level 4 and details in all levels are shown in Figure 31.

Figure 31: Speech signal, its approximation at level 4 and details in 4 levels processed by the state-space description. Results when this signal is decomposed by FWT are the same.



Source: Elaborated by the author.

6.1.5 Example 5: Boeing 747 aircraft in landing configuration

This example consists of a simulation model of the lateral dynamics of a Boeing 747 aircraft in landing configuration (BRYSON, 1994). Signals from Boeing were specially considered here since they are obtained from a state-space model.

The lateral dynamics model of a Boeing 747 aircraft in landing configuration (BRYSON,

1994) is

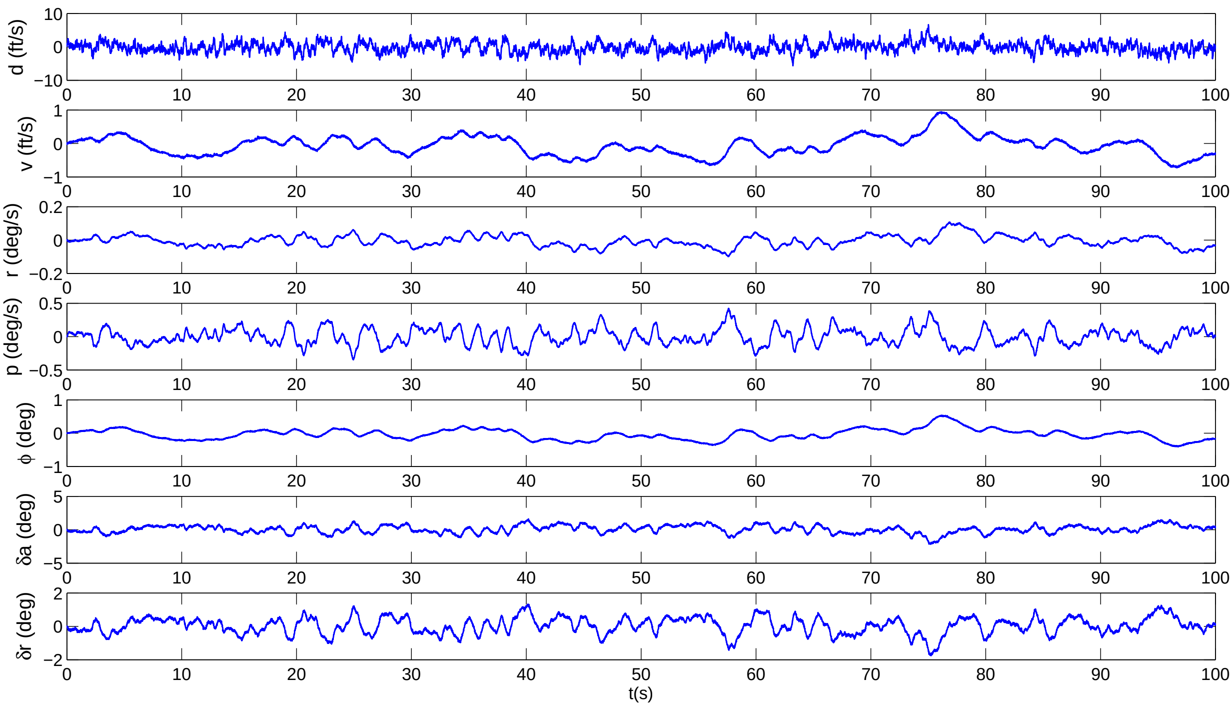
$$\begin{bmatrix} \dot{v} \\ \dot{r} \\ \dot{p} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -0.089 & -2.19 & 0.328 & 0.319 \\ 0.076 & -0.217 & -0.166 & 0 \\ -0.602 & 0.327 & -0.975 & 0 \\ 0 & 0.150 & 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ r \\ p \\ \phi \end{bmatrix} + \begin{bmatrix} 0 & 0.0327 \\ 0.0264 & -0.151 \\ 0.227 & 0.0636 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} + \begin{bmatrix} 0.089 \\ -0.076 \\ 0.602 \\ 0 \end{bmatrix} d, \tag{164}$$

where

$$\begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} = - \begin{bmatrix} -4.15 & 7.6 & 5.36 & 5.57 \\ 3.43 & -14.24 & 0.62 & -0.24 \end{bmatrix} \begin{bmatrix} v \\ r \\ p \\ \phi \end{bmatrix}, \tag{165}$$

v is the sideslip velocity, r is the yaw rate, p is the roll rate, ϕ is the roll angle, δ_a is the aileron angle, δ_r is the rudder angle, and d is an exogenous disturbance. The adopted units are feet, seconds, and crad (0.01 rad) (DUARTE; GALVÃO; PAIVA, 2013). These signals are shown in Figure 32.

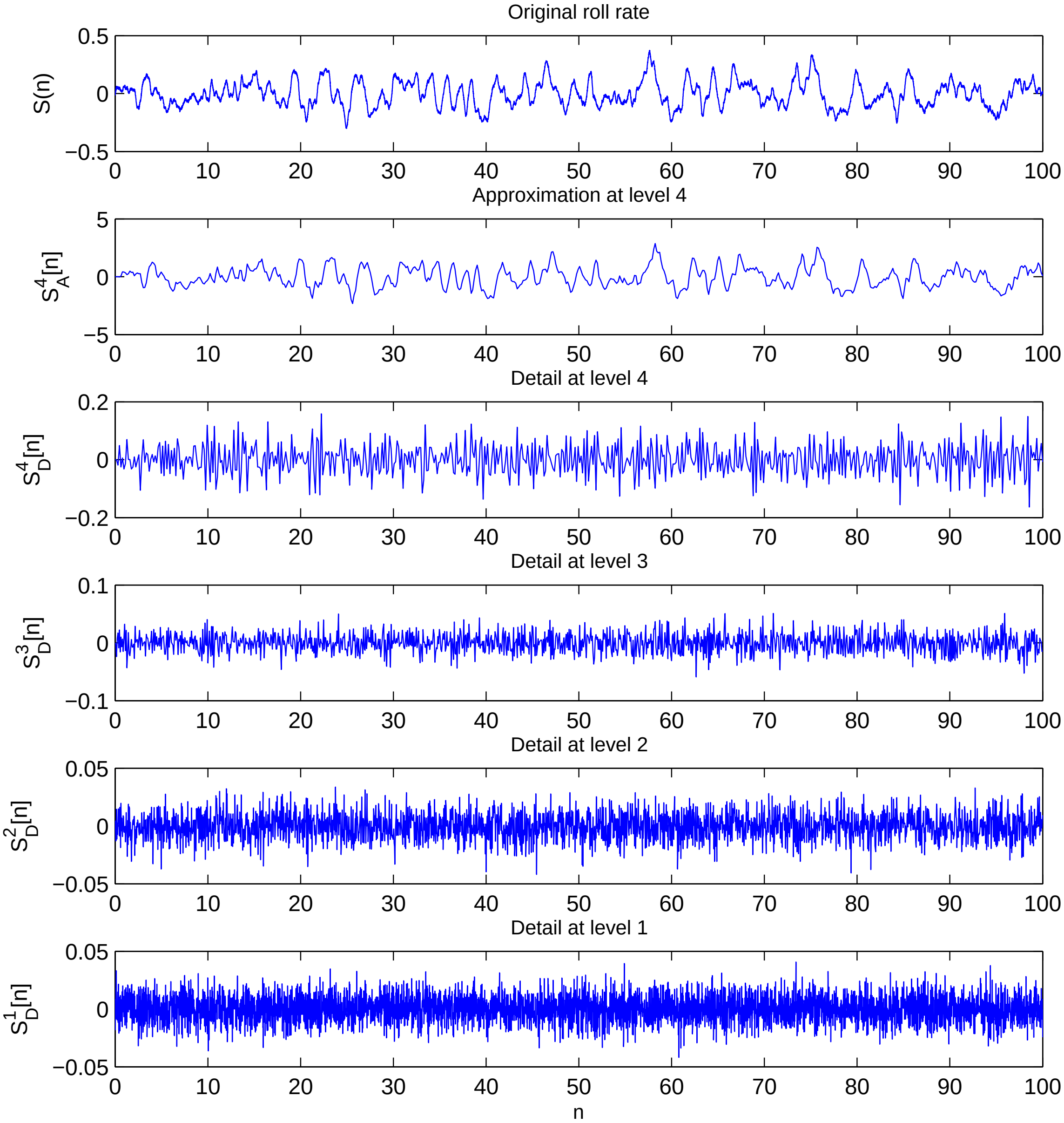
Figure 32: Simulation of the 747 aircraft model (angles in degrees).



Source: Elaborated by the author.

Following, in Figure 33, p signal (roll rate), its approximation and details in all levels as a processing result by the proposed state-space and after convenient decimation are shown.

Figure 33: Roll rate signal p, its approximation at level 4 and details in 4 levels processed using the state-space description. Decomposition results by FWT are the same.



Source: Elaborated by the author.

As expected, in every case (electrocardiogram signal, random white Gaussian noise signal, pseudo-random binary signal, speech signal and a Boeing roll rate signal) the proposed state-space description leads to the same results as FWT algorithm. Therefore, the researcher could choose the technique best suited to the problem to be considered.

6.2 A dynamic state feedback approach employing the state-space description for FWT

Considering that the examples presented in the previous section illustrate the equivalency between FWT and the proposed state-space description, an application example beyond simple decomposition becomes necessary. In order to achieve it, a dynamic state feedback approach employing the FWT state-space description is presented.

Feedback gains are obtained through a discrete linear quadratic regulator (DLQR) formulation, with cost weights adjusted according to suitable performance metrics. Linear quadratic regulator (LQR) is an automated way of finding an appropriate state feedback controller and it provides a key role in many control design methods. Besides being a powerful design method, it is, in many aspects, the principle of several current systematic control design procedures (LEVINE, 1996).

6.2.1 Preliminaries: Discrete linear quadratic regulator (DLQR)

Consider a plant described by the discrete-time model

$$\mathbf{x}_p[k+1] \quad = \quad \mathbf{A}_p\mathbf{x}_p[k] + \mathbf{B}_p\mathbf{u}_p[k], \tag{166}$$

$$\tag{167}$$

where $\mathbf{x}_p[k] \in \mathbb{R}^n$, $\mathbf{u}_p[k] \in \mathbb{R}^p$ are the state and control vectors, and $\mathbf{A}_p$, $\mathbf{B}_p$, are matrices with compatible dimensions. It is assumed that the plant is controllable and the state $\mathbf{x}_p[k]$ is available for feedback. The quadratic cost function for the states and the control signals is

$$J = \sum_{k=0}^{\infty} \left(\mathbf{x}_p^T[k]\mathbf{Q}_p\mathbf{x}_p[k] + \mathbf{u}_p^T[k]\mathbf{R}_p\mathbf{u}_p[k] \right), \tag{168}$$

with positive-definite weight matrices $\mathbf{Q}_p, \mathbf{R}_p$. The feedback is obtained with the control law:

$$\mathbf{u}_p[k] = -\mathbf{K}_p\mathbf{x}_p[k], \tag{169}$$

wherein the solution $\mathbf{u}[k]$ for the optimal quadratic control problem can be expressed as

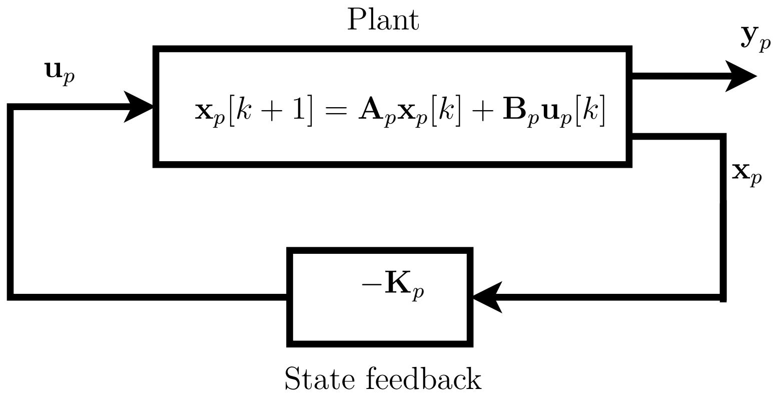
$$\mathbf{K}_p = (\mathbf{B}_p^T\mathbf{P}\mathbf{B}_p + \mathbf{R}_p)^{-1}\mathbf{B}_p^T\mathbf{P}\mathbf{A}_p, \tag{170}$$

where $\mathbf{P}$ is the positive-definite solution of the following Riccati equation, (LEWIS, 1986),

$$\mathbf{P} = \mathbf{A}_p^T\mathbf{P}\mathbf{A}_p - \mathbf{A}_p^T\mathbf{P}\mathbf{B}_p(\mathbf{R}_p + \mathbf{B}_p^T\mathbf{P}\mathbf{B}_p)^{-1}\mathbf{B}_p^T\mathbf{P}\mathbf{A}_p + \mathbf{Q}_p. \tag{171}$$

Figure 34 shows the block diagram for the resulting control loop.

Figure 34: Discrete linear quadratic regulator.



Source: Adapted from Franklin, Powell and Workman (1998).

6.2.2 Discrete linear quadratic regulator employing a wavelet filter bank (DLQR-WFB)

In the DLQR-WFB approach, the plant is coupled to a filter bank, namely, a filter bank is included in the feedback path, described in Figure 35. Therefore, each plant state is decomposed by the Fast Wavelet Transform, in order to obtain an augmented state vector $\mathbf{x}_{pw} = [\mathbf{x}_p^T \ \ \mathbf{x}_w^T]^T$, where $\mathbf{x}_w$ comprises the filter bank state variables. Since the fast wavelet transform is applied to each of the n components of the plant state $\mathbf{x}_p$, the filter bank state $\mathbf{x}_w$ is formed as

$$\mathbf{x}_w = \begin{bmatrix} \mathbf{x}_{FB_1} \\ \mathbf{x}_{FB_2} \\ \vdots \\ \mathbf{x}_{FB_n} \end{bmatrix}, \tag{172}$$

where $\mathbf{x}_{FB_i}$ is a vector with the filter bank states involved in the decomposition of the i-th plant state.

The dynamics of the plant coupled with the filter bank can then be described by a state equation of the form

$$\mathbf{x}_{pw}[k+1] = \mathbf{A}_{pw}\mathbf{x}_{pw}[k] + \mathbf{B}_{pw}\mathbf{u}_p[k], \tag{173}$$

where

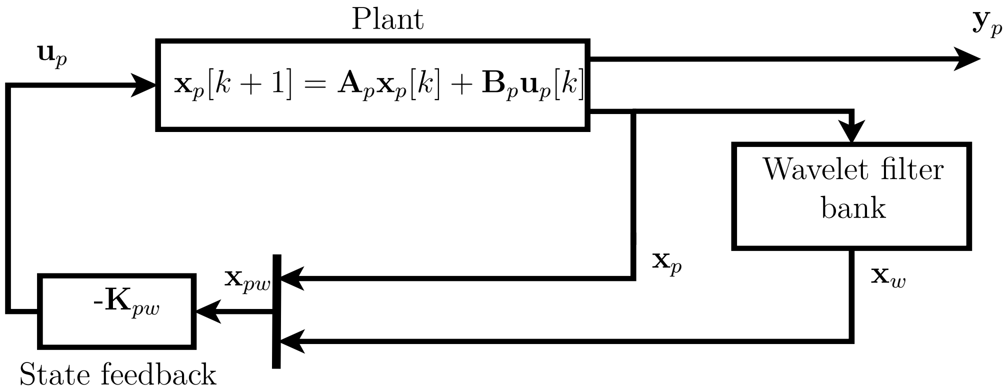
$$\mathbf{x}_{pw}[k] = \begin{bmatrix} \mathbf{x}_p[k] \\ \mathbf{x}_w[k] \end{bmatrix}, \ \ \mathbf{A}_{pw} = \begin{bmatrix} \mathbf{A}_p & 0 \\ \mathbf{B}_w & \mathbf{A}_w \end{bmatrix}, \ \ \mathbf{B}_{pw} = \begin{bmatrix} \mathbf{B}_p \\ 0 \end{bmatrix}, \tag{174}$$

and $\mathbf{A}_w$ and $\mathbf{B}_w$ are obtained from the filter bank equations as

$$\mathbf{A}_w = \begin{bmatrix} \mathbf{A}_M & 0 & \cdots & 0 \\ 0 & \mathbf{A}_M & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_M \end{bmatrix}, \quad \mathbf{B}_w = \begin{bmatrix} \mathbf{B}_M & 0 & \cdots & 0 \\ 0 & \mathbf{B}_M & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}_M \end{bmatrix}. \tag{175}$$

Therefore, a DLQR with weight matrices $\mathbf{Q}_{pw}$, $\mathbf{R}_{pw}$ can be designed for the augmented system (173) in order to obtain a feedback gain $\mathbf{K}_{pw}$, as described in Figure 35.

Figure 35: Discrete linear quadratic regulator employing a wavelet filter bank.



Source: Elaborated by the author.

An important remark is the fact that, when the dynamical system has an unsatisfactorily large number of states, an order reduction method can be applied to the states $\mathbf{x}_M$ of the description (157), (158), (159) and (160).

6.2.3 Numerical comparisons between DLQR and DLQR-WFB approaches

In what follows, the effectiveness of the proposed approach has been interpreted in terms of the sensitivity of the controller's performances to changes in the control inputs. Both approaches, employing the filter bank or not, are evaluated and compared in terms of the sensitivity of the systems to changes in the control inputs.

6.2.3.1 Sensitivity metrics

Let $S(z)$ be a sensitivity function defined for the DLQR case as

$$S(z) = \frac{1}{1 + \mathbf{K}_p(z\mathbf{I} - \mathbf{A}_p)^{-1}\mathbf{B}_p}. \tag{176}$$

Consider a similar definition of the sensitivity function in the DLQR-WFB case (replacing $\mathbf{K}_p$, $\mathbf{A}_p$, $\mathbf{B}_p$ by $\mathbf{K}_{pw}$, $\mathbf{A}_{pw}$, $\mathbf{B}_{pw}$, respectively).

This sensitivity function can be used as a robustness measure, since the value $|S(e^{j\omega})|$ is the reciprocal for the distance between the Nyquist curve and the critical point -1, considering the loop broken at the plant input (FRANKLIN; POWELL; WORKMAN, 1998). In this sense, a possible design goal may consist of obtaining small values of $|S(e^{j\omega})|$ at a given set of frequencies ω. This can be achieved by using a numerical optimization method to adjust the control and state weights in the DLQR or DLQR-WFB formulations.

In these examples, diagonal weight matrices are adopted for simplicity. The control weight is set to a fixed scalar value and the state weights are optimized by using the Sequential Quadratic Programming (SQP) method (NOCEDAL; WRIGHT, 2006). The index to be minimized is defined as $|S(e^{j\omega_1})| + \rho|S(e^{j\omega_2})|$, where $\omega_1 = \pi/2$, $\omega_2 = \pi/4$ and ρ is a positive scalar that can be adjusted to place larger emphasis on the minimization of $|S(e^{j\omega_1})|$ or $|S(e^{j\omega_2})|$.

Let

$$\mathbf{A}_p = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}_p = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}. \tag{177}$$

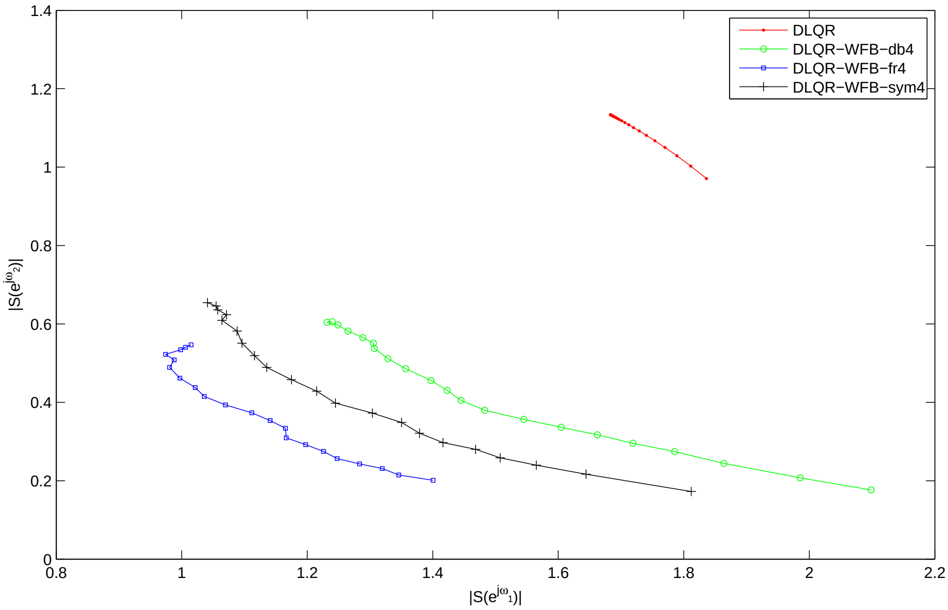
be a double integrator plant of the form (166). Model (177) can be regarded as a simplified representation of translational or rotational movements with one degree of freedom, and it is often used for illustration in control studies (RAO; BERNSTEIN, 2001). Different wavelets filters banks are considered for the DLQR-WFB approach. By using balanced realizations and removing the states corresponding to small Hankel singular values (LAUB et al., 1987), the model order for all filter banks is reduced.

Firstly, for the DLQR-WFB are considered the `db4`, `sym4`, and an optimized orthonormal wavelet filter from Paiva and Galvão (2012) (with length $2N = 8$ and denoted here as `fr`N, namely, `fr`4) decomposed in four levels. Now, the intention is to compare DLQR and DLQR-WFB (`db4`, `sym4`, `fr`4) results. Such the filters have the same length ($2N = 8$), but from different wavelet families.

DLQR and DLQR-WFB (`db4`, `sym4`, `fr`4) results obtained by varying the value of ρ are presented in Figure 36. As can be seen, smaller values of $|S(e^{j\omega_1})|$ and $|S(e^{j\omega_2})|$ can be achieved by using the proposed DLQR-WFB formulation, mainly, when using `fr`4.

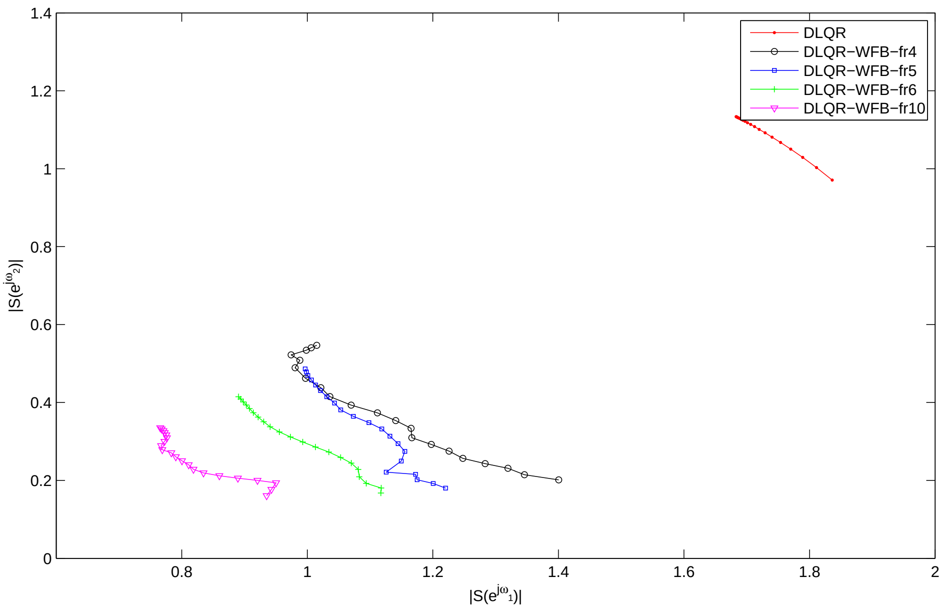
Considering the results obtained with DLQR-WFB, in the following are presented results for DLQR and DLQR-WFB using optimized wavelet filters from Paiva and Galvão (2012), but with different lengths ($2N = 8$, $2N = 10$, $2N = 12$ and $2N = 20$). In all cases, $M = 4$. Results are displayed in Figure 37. It can be noted that smaller values of $|S(e^{j\omega_1})|$ and $|S(e^{j\omega_2})|$ are achieved when using DLQR-WFB formulation with longer filters. It is worth noting that longer filters lead to more states variables.

Figure 36: Sensitivity values at $\omega_1 = \pi/2$ and $\omega_2 = \pi/4$ obtained by optimizing the state weights in the DLQR and DLQR-WFB formulations.



Source: Elaborated by the author.

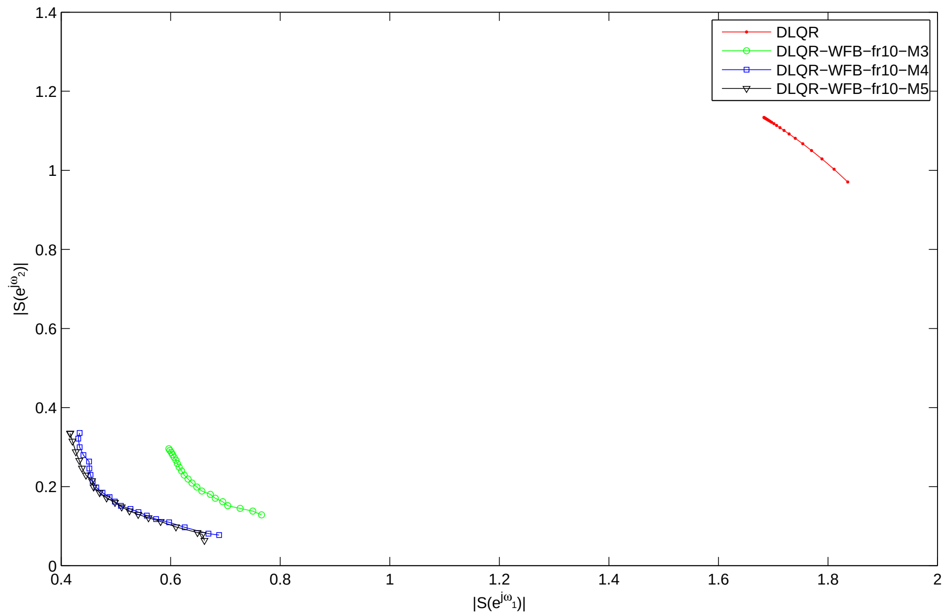
Figure 37: Sensitivity values at $\omega_1 = \pi/2$ and $\omega_2 = \pi/4$ obtained by optimizing the state weights in the DLQR and DLQR-WFB (using optimized wavelet filters) formulations.



Source: Elaborated by the author.

Optimized orthonormal wavelet filters lead to better results, mainly for longer filters, e.g. $2N = 20$, as seen in Figures 36 and 37. In Figure 38, results comparing DLQR and DLQR-WFB (with the longer filter presented by Paiva and Galvão (2012), $2N = 20$, but with the wavelet filter bank decomposed in three, four and five levels) are presented.

Figure 38: Sensitivity values at $\omega_1 = \pi/2$ and $\omega_2 = \pi/4$ obtained by optimizing the state weights in the DLQR and DLQR-WFB (using optimized wavelet filters for $M = 3$, $M = 4$ and $M = 5$) formulations.



Source: Elaborated by the author.

From Figures 36, 37 and 38: smaller values of $|S(e^{j\omega_1})|$ and $|S(e^{j\omega_2})|$ can be achieved by using the DLQR-WFB formulation. Those results can be improved by using optimized wavelet filters from Paiva and Galvão (2012), mainly by using longer filters with fewer decomposition levels.

6.2.3.2 External disturbances and measurement noise effects

In the following, both state feedback systems (DLQR and DLQR-WFB) are exposed to external disturbances and measurement noise in order to compare the effects over them.

Consider the plant model (177) with a scalar output y_p defined as

$$y_p[k] = \mathbf{C}_p \mathbf{x}_p[k], \tag{178}$$

where $\mathbf{C}_p$ is a row vector of compatible dimension. Here, the output is defined as the first state variable, i.e. $\mathbf{C}_p = [1 \;\; 0]$.

Assume that a disturbance d is applied at the plant input. In the DLQR case, the disturbance effect on the plant output y_p can be evaluated by replacing (169) with

$$u_p[k] = -\mathbf{K}_p\mathbf{x}_p[k] + d[k]. \tag{179}$$

From (166), (178) and (179), a transfer function $H_{dy}(z)$ can be obtained as

$$H_{dy}(z) = \frac{Y_p(z)}{D(z)} = \mathbf{C}_p(z\mathbf{I} - \mathbf{A}_p + \mathbf{B}_p\mathbf{K}_p)^{-1}\mathbf{B}_p. \tag{180}$$

Disturbance effect on the plant output can then be evaluated in terms of the H_2 norm of $H_{dy_p}(z)$, which is defined as, (BUNSE-GERSTNER et al., 2010),

$$\|H_{dy}\|_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} |H_{dy}(e^{j\omega})|^2 d\omega}. \tag{181}$$

A similar transfer function can be obtained in the DLQR-WFB case, replacing $\mathbf{K}_p$, $\mathbf{A}_p$, $\mathbf{B}_p$ and $\mathbf{C}_p$ by $\mathbf{K}_{pw}$, $\mathbf{A}_{pw}$, $\mathbf{B}_{pw}$ and $[\mathbf{C}_p \;\; 0]$, respectively.

On the other hand, instead of considering an input disturbance, assume that the state values employed in the feedback control law are corrupted by a measurement noise term n such that

$$u_p[k] = -\mathbf{K}_p(\mathbf{x}_p[k] + \mathbf{E}_p n[k]) \tag{182}$$

in the DLQR case, where $\mathbf{E}_p$ is a column vector of compatible dimension. Here, noise is included in the state measurement, i.e. $\mathbf{E}_p = [1 \;\; 0]^T$. From (166), (178) and (182), a transfer function $H_{ny}(z)$ can be obtained as

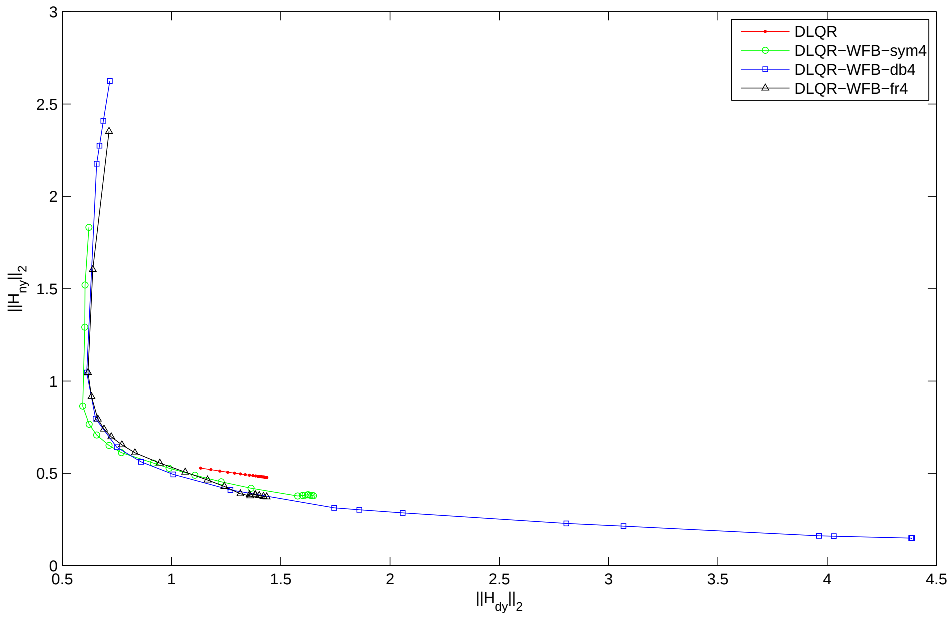
$$H_{ny}(z) = \frac{Y_p(z)}{N(z)} = -\mathbf{C}_p(z\mathbf{I} - \mathbf{A}_p + \mathbf{B}_p\mathbf{K}_p)^{-1}\mathbf{B}_p\mathbf{K}_p\mathbf{E}_p. \tag{183}$$

Thus, H_2 norm of $H_{ny}(z)$ can be used to evaluate the measurement noise effect on the plant output. A similar transfer function can be obtained in the DLQR-WFB case, replacing $\mathbf{K}_p$, $\mathbf{A}_p$, $\mathbf{B}_p$, $\mathbf{C}_p$ and $\mathbf{E}_p$ by $\mathbf{K}_{pw}$, $\mathbf{A}_{pw}$, $\mathbf{B}_{pw}$, $[\mathbf{C}_p \;\; 0]$ and $[\mathbf{E}_p^T \;\; 0]^T$, respectively.

State weights in the DLQR and DLQR-WFB formulations are optimized by using the SQP method, while the index to be minimized is defined as $\|H_{dy}\|_2 + \rho\|H_{ny}\|_2$.

Figure 39 presents DLQR and DLQR-WFB (using `db4`, `sym4` and `fr4` wavelet filters decomposed at level four) results obtained by varying the value of ρ. Note that, in general, DLQR-WFB-`db4` and DLQR-WFB-`sym4` approaches lead to better results, in that smaller values of $\|H_{dy}\|_2$ and $\|H_{ny}\|_2$ can be achieved as compared to the DLQR formulation. It is worth mentioning that ρ varies from 0.1 to 10. Smaller ρ values lead to smaller $\|H_{ny}\|_2$ values, mainly with `db4`, while for ρ values higher than 2.5 `sym4` provides better results mainly for $\|H_{dy}\|_2$.

Figure 39: Comparative evaluation for DLQR and DLQR-WFB formulations in terms of external disturbance and measurement noise effects.



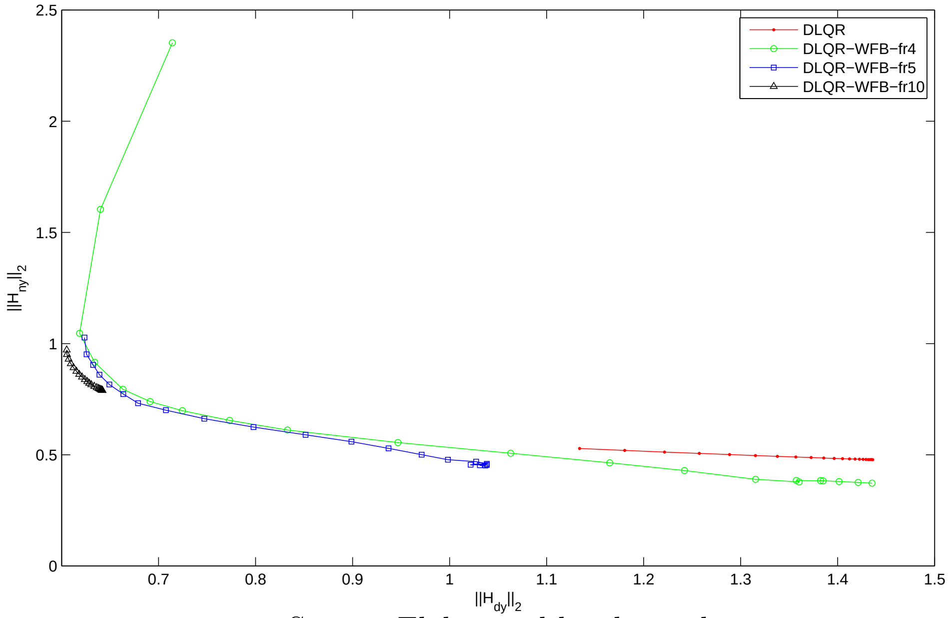
Source: Elaborated by the author.

Now, wavelets filters with different lengths shall be compared. As in the sensitivity case, optimized wavelets filters `fr`N with different lengths ($2N = 8$, $2N = 10$ and $2N = 20$) are employed in the DLQR-WFB and compared with DLQR approach, as shown in Figure 40. It can be noted that a better adjustment between measurement noise and disturbance effects in the system is reached with the system DLQR-WFB-`fr5`. `Fr10` also produces good values, but they are greater than the ones provided by `fr5`. In the other systems, there is not a balance between $\|H_{dy}\|_2$ and $\|H_{ny}\|_2$, i.e., minimizing one will not assure the minimization to the other.

The last comparison between DLQR and DLQR-WFB approaches in order to evaluate external disturbances and measurement noise effects over them is by using different decomposition levels. `Fr10` wavelet filter is evaluated with three, four and five decomposition levels, as shown in Figure 41.

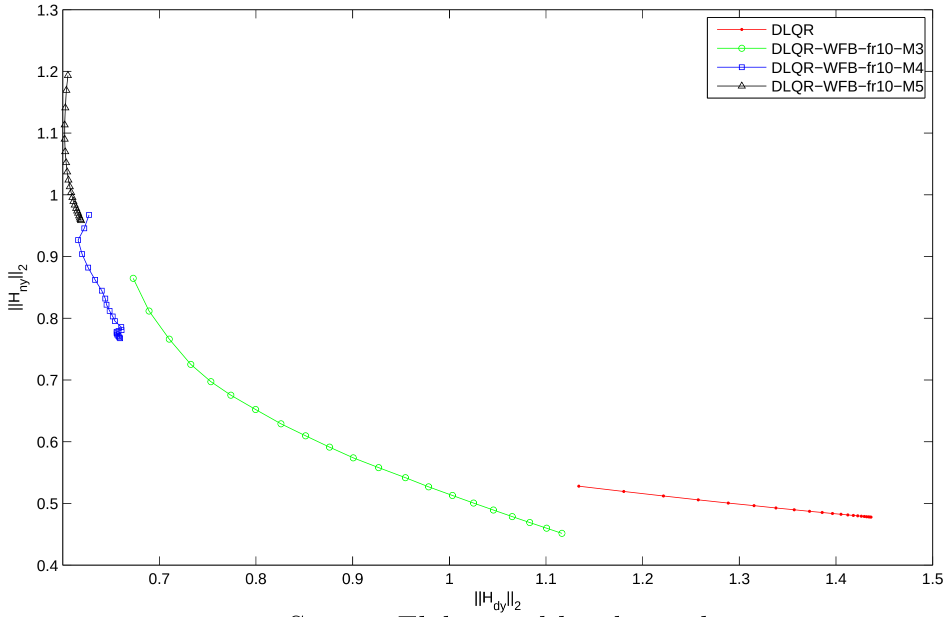
According to the plots shown in Figure 41, there are no advantages in increasing the number of decomposition levels when the system is subject to measurement noise and external disturbance, since there is no balance in the minimization of noise and disturbance

Figure 40: Comparative evaluation of the DLQR and DLQR-WFB (using optimized wavelet filters) formulations subject to external disturbance and measurement noise.



Source: Elaborated by the author.

Figure 41: Comparative evaluation of the DLQR and DLQR-WFB (using optimized wavelet filters for $M = 3$, $M = 4$ and $M = 5$) formulations subject to external disturbance and measurement noise.

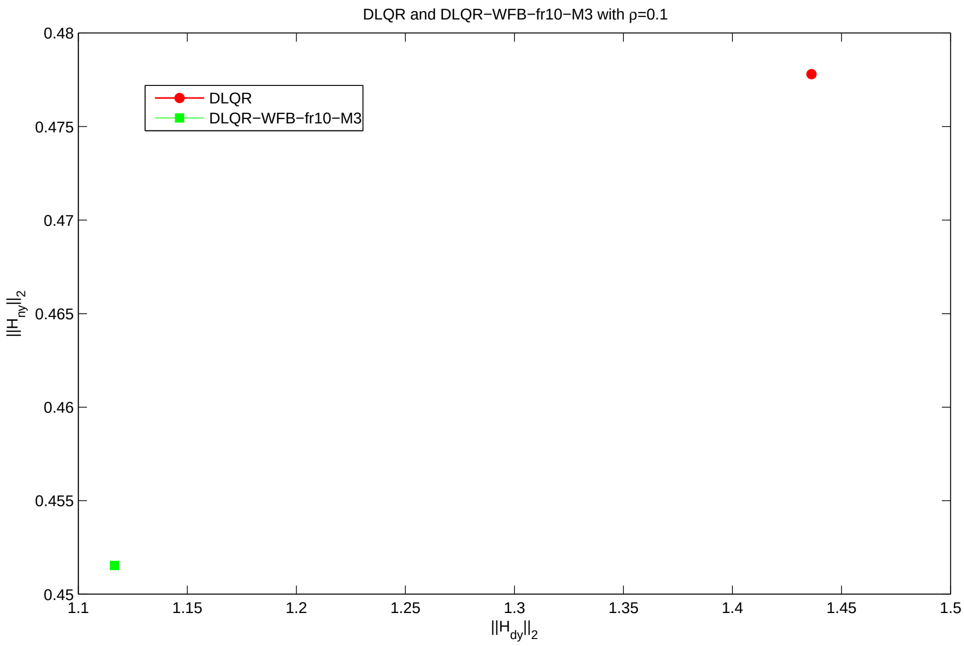


Source: Elaborated by the author.

effects. In general, it is noted in Figure 41 that the higher is the decomposition level the greater will be noise measurements and disturbance effects in the system.

In order to make easy noting the dominance of the system DLQR-WFB-`fr`10-M3 over DLQR, $\rho = 0.1$ is shown separately in Figure 42.

Figure 42: Comparative evaluation of the DLQR and DLQR-WFB-`fr`10-M3 formulations in term of external disturbance and measurement noise effects, when $\rho = 0.1$.



Source: Elaborated by the author.

6.2.4 Multiple wavelet filter banks: multiple inputs and multiple outputs approach

It is worth noting that the state-space description for the FWT has multiple outputs and one input (SIMO system). Here, it is shown that n SIMO systems can be seen as one MIMO system. This extension to MIMO is presented here only for future works purposes, with no application shown.

Consider n systems, $F_1,\ F_2, \cdots F_n$, such that:

$$\left\{\begin{array}{lcl} \mathbf{x}_1[k+1] & = & \mathbf{A}_1\mathbf{x}_1[k] + \mathbf{B}_1 u_1[k], \\ \mathbf{y}_1[k] & = & \mathbf{C}_1\mathbf{x}_1[k] + \mathbf{D}_1 u_1[k], \\ \mathbf{x}_2[k+1] & = & \mathbf{A}_2\mathbf{x}_2[k] + \mathbf{B}_2 u_2[k], \\ \mathbf{y}_2[k] & = & \mathbf{C}_2\mathbf{x}_2[k] + \mathbf{D}_2 u_2[k], \\ \vdots & & \\ \mathbf{x}_n[k+1] & = & \mathbf{A}_n\mathbf{x}_n[k] + \mathbf{B}_n u_n[k], \\ \mathbf{y}_n[k] & = & \mathbf{C}_n\mathbf{x}_n[k] + \mathbf{D}_n u_n[k]. \end{array}\right. \tag{184}$$

Writing the state vector, input and output, respectively, as:

$$\left\{\begin{array}{l} \mathbf{x}[k] = [\mathbf{x}_1[k]^T \ \ \mathbf{x}_2[k]^T \ \ \cdots \ \ \mathbf{x}_n[k]^T]^T, \\ \mathbf{y}[k] = [\mathbf{y}_1[k]^T \ \ \mathbf{y}_2[k]^T \ \ \cdots \ \ \mathbf{y}_n[k]^T]^T, \\ \mathbf{u}[k] = [u_1[k] \ \ u_2[k] \ \ \cdots \ \ u_n[k]]^T. \end{array}\right.$$

Thus, state equation and outputs from (184) become

$$\begin{bmatrix} \mathbf{x}_1[k+1] \\ \mathbf{x}_2[k+1] \\ \vdots \\ \mathbf{x}_n[k+1] \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_1[k] \\ \mathbf{x}_2[k] \\ \vdots \\ \mathbf{x}_n[k] \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{B}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}_n \end{bmatrix} \begin{bmatrix} u_1[k] \\ u_2[k] \\ \cdots \\ u_n[k] \end{bmatrix}, \tag{185}$$

$$\begin{bmatrix} \mathbf{y}_1[k] \\ \mathbf{y}_2[k] \\ \vdots \\ \mathbf{y}_n[k] \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{C}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{C}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_1[k] \\ \mathbf{x}_2[k] \\ \vdots \\ \mathbf{x}_n[k] \end{bmatrix} + \begin{bmatrix} \mathbf{D}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{D}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}_n \end{bmatrix} \begin{bmatrix} u_1[k] \\ u_2[k] \\ \cdots \\ u_n[k] \end{bmatrix}. \tag{186}$$

The FWT can be applied to a plant with multiple inputs when each filter bank is separated (PAIVA; GALVÃO; RODRIGUES, 2009; DUARTE; GALVÃO; PAIVA, 2013). Nevertheless, when using the proposed state-space for the FWT, all filter banks can be considered as one single MIMO system given by (185) and (186).

7 Final remarks and future works

7.1 Discussions

Representing transfer functions for linear dynamical systems through the Finite Impulse Response model is highly common, and these kind of systems are very useful in several engineering areas since they have attractive properties, computational and analytical advantages, the guarantee of BIBO stability and robustness to some parameter changes, and others. On the other hand, representing transfer functions for linear time-invariant dynamic systems through the orthonormal basis functions also is very interesting. This representation has some characteristics that make it attractive for dynamical systems modeling: absence of output recursion, not requiring prior knowledge of the exact structure of the vector of regression; possibility to increase the model capacity of representation by increasing the number of orthonormal functions employed; natural outputs uncoupling in multivariable models; tolerance to unmodeled dynamics, and others.

The possibility of applications with interesting results for representing transfer functions for linear dynamic systems through the orthonormal basis functions and the fact that FIR structure guarantee BIBO stability, robustness to some parameter changing and improving the filter divergence problem justifies the combination of these two representations.

Wavelet transform is also a widely used mathematical tool in engineering to work on signal processing and control. There is a relation between FIR filters and wavelet transform which is discussed in this thesis (Chapter 4). Furthermore, it can be stated that the Fast Wavelet Transform is an algorithm that turns a signal in the time domain into a sequence of coefficients based on an orthogonal basis of small finite wavelet functions.

The three concepts previously mentioned are about transfer functions, and it is possible to represent them in the state-space form. There are cases where the state-space approach is more suitable than other classical methods of modeling, for instance, transfer functions. Some advantages of the state-space form are: it is easier to work with MIMO systems, it is best suited both for the theoretical treatment of control systems and for numerical calculations, and it gives a better insight into the inner system behavior, for instance, the study of system controllability and observability.

This work concerns a study of orthonormal functions, orthonormal basis functions and finite impulse response models, where the main objective was the exploitation of the orthonormal functions space in order to formulate a set of orthonormal filter banks and parameterize it in the state-space. An extension of this idea for M decomposition levels for the filter bank from this initial parameterization, and everything carrying important properties was also proposed.

7.2 Conclusions and contributions

This work consisted of developing a realization in the state-space for a wavelet filter bank with M decomposition levels, with the explicit presence of parameters that can be freely adjusted keeping the guarantees of perfect-reconstruction and orthonormality.

Parameterizations in the state-space using special orthonormal basis functions that forms FIR filter banks were proposed in two different manners: for one single filter and for the lattice structure. In both cases, all properties that make them attractive like observability, reachability and minimality are satisfied. The single filter parameterization is simpler while the lattice structure carries some advantages related to this kind of structure. It is worth noting that after the achievement of these results both ideas will be equally easy to implement.

Lattice structure features mentioned in the previous paragraph, e.g., modularity and low sensitivity to the effects of quantization parameters, are the reasons that make the parameterization in the state-space using this kind of structure the most important one in this research.

An extension for the state-space formulation to multiple levels of decomposition was also developed. Those features that make it attractive, e.g., observability, reachability and minimality, were held for the fast wavelet transform state-space realization with M decomposition levels. This extension can be applied for both kinds of parameterizations considering that this extension was done in a general way.

Summarizing, both state-space realizations and the extension for M decomposition levels provide a combination between two different ways to represent transfer functions for discrete-time linear dynamic systems: systems through Finite Impulse Response model and through orthonormal basis functions. In this way, the advantages that make each idea attractive are combined in one technique. In addition to it, a new approach to work with the fast wavelet transform was achieved.

Some broadly studied signals in signal processing by using wavelets were decomposed by the state-space approach with the intention of showing the equivalence of these mathematical tools. However, in order to show advantages of using the state-space description, an application on dynamic state feedback control was presented, which is a dynamic state feedback approach employing the proposed state-space realization.

7.2.1 Published and submitted papers

- Julio C. Uzinski, Henrique M. Paiva, Marco A.Q. Duarte, Roberto K.H. Galvão, Francisco Villarreal, A state-space description for perfect-reconstruction wavelet FIR filter banks with special orthonormal basis functions, Journal of Computational and Applied Mathematics, Volume 290, 15 December 2015, Pages 290-297. (Published)
- Julio C. Uzinski, Henrique M. Paiva, Roberto K.H. Galvão, Edvaldo Assunção, Marco A.Q. Duarte, Francisco Villarreal, A dynamic state feedback approach employing a new state-space description for the fast wavelet transform with multiple decomposition levels, Journal of Control, Automation and Electrical Systems. (Submitted).

7.3 Future works

There are many applications that could be managed with the state-space realization proposed in this thesis. In Chapter 6, some applications were shown on signal processing and control. However, other engineering areas also use wavelets in some manner, finite impulse response filters or orthonormal basis functions, and they could be considered for further studies.

For some of the fast wavelet transform applications, it is important to use the synthesis filter bank, which means that the development of a state-space realization that would be the inverse of the one developed in this thesis would be interesting.

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APPENDIX A – Orthonormality for the Sherlock and Monro formulation (PAIVA, 2003; UZINSKI, 2013)

Lemma 19. *The formulation given by (93):*

$$\begin{aligned} h_0^{(N+1)} &= \cos(\beta_N) h_0^{(N)} \\ h_{2i}^{(N+1)} &= \cos(\beta_N) h_{2i}^{(N)} - \sin(\beta_N) h_{2i-1}^{(N)} \\ h_{2N}^{(N+1)} &= -\sin(\beta_N) h_{2N-1}^{(N)} \\ \\ h_1^{(N+1)} &= \sin(\beta_N) h_0^{(N)} \\ h_{2i+1}^{(N+1)} &= \sin(\beta_N) h_{2i}^{(N)} + \cos(\beta_N) h_{2i-1}^{(N)} \\ h_{2N+1}^{(N+1)} &= \cos(\beta_N) h_{2N-1}^{(N)} \end{aligned} \tag{187}$$

where $h_0^{(1)} = \cos(\beta_0)$, $h_1^{(1)} = \sin(\beta_0)$ and $i = 0, 1, \cdots, N-1$, satisfies the orthonormality requirements, (STRANG; NGUYEN, 1996),

$$\sum_{i=0}^{2N-1} \left[h_i^{(N)}\right]^2 = 1, \ \ N \geq 1, \tag{188}$$

$$\sum_{i=0}^{2N-1-2m} h_i^{(N)} h_{i+2m}^{(N)} = 0, \ \ m = 1, 2, \cdots, N-1, \ \ N \geq 2. \tag{189}$$

Proof. **Condition (188):**

For $N = 1$:

$$\sum_{i=0}^{1} \left[h_i^{(1)}\right]^2 = \left[h_0^{(1)}\right]^2 + \left[h_1^{(1)}\right]^2 = \cos^2\alpha_1 + \sin^2\alpha_1 = 1$$

Inductive step:

If $\sum_{i=0}^{2N-1} \left[h_i^{(N)}\right]^2 = 1$ check that $\sum_{i=0}^{2N+1} \left[h_i^{(N+1)}\right]^2 = 1$.

From (93) it has

$$\begin{aligned} \sum_{i=0}^{2N+1} \left[h_i^{(N+1)}\right]^2 \ = \ & \left[\cos\alpha_N h_0^{(N)}\right]^2 + \sum_{i=1}^{N-1} \left[\cos\alpha_N h_{2i}^{(N)} - \sin\alpha_N h_{2i-1}^{(N)}\right]^2 + \left[\sin\alpha_N h_{2N-1}^{(N)}\right]^2 + \\ & \left[\sin\alpha_N h_0^{(N)}\right]^2 + \sum_{i=1}^{N-1} \left[\sin\alpha_N h_{2i}^{(N)} + \cos\alpha_N h_{2i-1}^{(N)}\right]^2 + \left[\cos\alpha_N h_{2N-1}^{(N)}\right]^2 \end{aligned}$$

$$
\begin{aligned}
&= (\sin^2\alpha_N + \cos^2\alpha_N)\left[h_0^{(N)}\right]^2 + (\sin^2\alpha_N + \cos^2\alpha_N)\left[h_{2N-1}^{(N)}+\right]^2 + \\
&\quad \sum_{i=1}^{N-1}\left\{\cos^2\alpha_N\left[h_{2i}^{(N)}\right]^2 - 2\sin\alpha_N\cos\alpha_N h_{2i}^{(N)}h_{2i-1}^{(N)} + \sin^2\alpha_N\left[h_{2i-1}^{(N)}\right]^2 + \right.\\
&\quad \left.\sin^2\alpha_N\left[h_{2i}^{(N)}\right]^2 + 2\sin\alpha_N\cos\alpha_N h_{2i}^{(N)}h_{2i-1}^{(N)} + \cos^2\alpha_N\left[h_{2i-1}^{(N)}\right]^2\right\} \\
&= \left[h_0^{(N)}\right]^2 + \left[h_{2N-1}^{(N)}+\right]^2 + \\
&\quad \sum_{i=1}^{N-1}\left\{(\sin^2\alpha_N + \cos^2\alpha_N)\left[h_{2i}^{(N)}\right]^2 + (\sin^2\alpha_N + \cos^2\alpha_N)\left[h_{2i-1}^{(N)}\right]^2\right\} \\
&= \left[h_0^{(N)}\right]^2 + \left[h_{2N-1}^{(N)}+\right]^2 + \sum_{i=1}^{N-1}\left\{\left[h_{2i}^{(N)}\right]^2 + \left[h_{2i-1}^{(N)}\right]^2\right\} \\
&= \left[h_0^{(N)}\right]^2 + \left[h_{2N-1}^{(N)}+\right]^2 + \sum_{i=1}^{2N-2}\left[h_i^{(N)}\right]^2 \\
&= \sum_{i=1}^{2N-1}\left[h_i^{(N)}\right]^2 = 1
\end{aligned}
$$

thus $\sum_{i=0}^{2N+1}\left[h_i^{(N+1)}\right]^2 = 1$.

Condition (189):

For $N = 2$:

$$\sum_{i=0}^{1} h_i^{(2)}h_{i+2}^{(2)} = h_0^{(2)}h_2^{(2)} + h_1^{(2)}h_3^{(2)} = (\cos\alpha_2\cos\alpha_1)(-\sin\alpha_2\sin\alpha_1) + (\sin\alpha_2\cos\alpha_1)(\cos\alpha_2\sin\alpha_1) = 0$$

Inductive step:

If

$$\sum_{i=0}^{2N-1-2m} h_i^{(N)}h_{i+2m}^{(N)} = 0 \tag{190}$$

check that

$$\sum_{i=0}^{2N+1-2m} h_i^{(N+1)}h_{i+2m}^{(N+1)} = 0 \tag{191}$$

for $N \geq 2$ $N \in \mathbb{Z}$.

This proof is made in three stages:

a) $m = N$**:**

Equation (191) can be written as

$$\sum_{i=0}^{1} h_i^{N+1} h_{i+2N}^{N+1} = h_0^{(N+1)} h_{2N}^{(N+1)} + h_1^{(N+1)} h_{2N+1}^{(N+1)}. \tag{192}$$

Replacing (93) in (192) it leads to

$$\begin{aligned}\sum_{i=0}^{1} h_i^{N+1} h_{i+2N}^{N+1} &= (\cos(\alpha_N) h_0^{(N)})(-\sin(\alpha_N) h_{2N-1}^{(N)}) + (\sin(\alpha_N) h_0^{(N)})(\cos(\alpha_N) h_{2N-1}^{(N)}) \\ &= (-\sin(\alpha_N)\cos(\alpha_N) h_0^{(N)} h_{2N-1}^{(N)}) + (\sin(\alpha_N)\cos(\alpha_N) h_0^{(N)} h_{2N-1}^{(N)}) = 0.\end{aligned}$$

b) $m = N - 1$**:**

Now, equation (191) can be written as

$$\sum_{i=0}^{3} h_i^{(N+1)} h_{i+2N-2}^{(N+1)} = h_0^{(N+1)} h_{2N-2}^{(N+1)} + h_1^{(N+1)} h_{2N-1}^{(N+1)} + h_2^{(N+1)} h_{2N}^{(N+1)} + h_3^{(N+1)} h_{2N+1}^{(N+1)} \tag{193}$$

Replacing (93) in (193) one obtains

$$\begin{aligned}\sum_{i=0}^{3} h_i^{(N+1)} h_{i+2N-2}^{(N+1)} &= (\cos(\alpha_N) h_0^{(N)})(\cos(\alpha_N) h_{2N-2}^{(N)} - \sin(\alpha_N) h_{2N-3}^{(N)}) + (\sin(\alpha_N) h_0^{(N)}) \\ &\quad (\sin(\alpha_N) h_{2N-2}^{(N)} + \cos(\alpha_N) h_{2N-3}^{(N)}) + (\cos(\alpha_N) h_2^{(N)} - \sin(\alpha_N) h_1^{(N)}) \\ &\quad (-\sin(\alpha_N) h_{2N-1}^{(N)}) + (\sin(\alpha_N) h_2^{(N)} + \cos(\alpha_N) h_1^{(N)})(\cos(\alpha_N) h_{2N-1}^{(N)}) \\ &= \cos^2(\alpha_N) h_0^{(N)} h_{2N-2}^{(N)} - \sin(\alpha_N)\cos(\alpha_N) h_0^{(N)} h_{2N-3}^{(N)} + \sin^2(\alpha_N) h_0^{(N)} h_{2N-2}^{(N)} + \\ &\quad \sin(\alpha_N)\cos(\alpha_N) h_0^{(N)} h_{2N-3}^{(N)} - \cos(\alpha_N)\sin(\alpha_N) h_2^{(N)} h_{2N-1}^{(N)} + \\ &\quad \sin^2(\alpha_N) h_1^{(N)} h_{2N-1}^{(N)} + \cos(\alpha_N)\sin(\alpha_N) h_2^{(N)} h_{2N-1}^{(N)} + \cos^2 h_1^{(N)} h_{2N-1}^{(N)} \\ &= (\cos^2(\alpha_N) + \sin^2(\alpha_N)) h_0^{(N)} h_{2N-2}^{(N)} + (\sin(\alpha_N)\cos(\alpha_N) - \sin(\alpha_N)\cos(\alpha_N)) \\ &\quad h_0^{(N)} h_{2N-3}^{(N)} + (\cos^2(\alpha_N) + \sin^2(\alpha_N)) h_1^{(N)} h_{2N-1}^{(N)} + \\ &\quad (\sin(\alpha_N)\cos(\alpha_N) - \sin(\alpha_N)\cos(\alpha_N)) h_2^{(N)} h_{2N-1}^{(N)} \\ &= h_0^{(N)} h_{2N-2}^{(N)} + h_1^{(N)} h_{2N-1}^{(N)} = \sum_{i=0}^{1} h_i^{(N)} h_{i+2N-2}^{(N)}\end{aligned} \tag{194}$$

By the inductive step (190), when $m = N - 1$ $\sum_{i=0}^{1} h_i^{(N)} h_{i+2N-2}^{(N)} = 0$, so

$$\sum_{i=0}^{3} h_i^{(N+1)} h_{i+2N-2}^{(N+1)} = \sum_{i=0}^{1} h_i^{(N)} h_{i+2N-2}^{(N)} = 0. \tag{195}$$

c) $0 < m < N-1$**:**

$$
\begin{aligned}
\sum_{i=0}^{2N+1-2m} h_i^{(N+1)} h_{i+2m}^{(N+1)} &= h_0^{(N+1)} h_{2m}^{(N+1)} + h_1^{(N+1)} h_{1+2m}^{(N+1)} + \left[\sum_{i=2}^{2N-1-2m} h_i^{(N+1)} h_{i+2m}^{(N+1)}\right] + \\
& h_{2N-2m}^{(N+1)} h_{2N}^{(N+1)} + h_{2N+1-2m}^{(N+1)} h_{2N+1}^{(N+1)} \\
&= h_0^{(N+1)} h_{2m}^{(N+1)} + h_1^{(N+1)} h_{1+2m}^{(N+1)} + \sum_{i=1}^{N-m-1} \left[h_{2i}^{(N+1)} h_{2i+2m}^{(N+1)} + h_{2i+1}^{(N+1)} h_{2i+1+2m}^{(N+1)}\right] + \\
& h_{2N-2m}^{(N+1)} h_{2N}^{(N+1)} + h_{2N+1-2m}^{(N+1)} h_{2N+1}^{(N+1)} \\
&= h_0^{(N+1)} h_{2m}^{(N+1)} + \left[\sum_{i=1}^{N-m-1} h_{2i}^{(N+1)} h_{2i+2m}^{(N+1)}\right] + h_{2N-2m}^{(N+1)} h_{2N}^{(N+1)} + \\
& h_1^{(N+1)} h_{1+2m}^{(N+1)} + \left[\sum_{i=1}^{N-m-1} h_{2i+1}^{(N+1)} h_{2i+1+2m}^{(N+1)}\right] + h_{2N+1-2m}^{(N+1)} h_{2N+1}^{(N+1)}. \qquad (196)
\end{aligned}
$$

Replacing (93) in (196) it leads to

$$
\begin{aligned}
\sum_{i=0}^{2N+1-2m} h_i^{(N+1)} h_{i+2m}^{(N+1)} &= (\cos(\alpha_N) h_0^{(N)})(\cos(\alpha_N) h_{2m}^{(N)} - \sin(\alpha_N) h_{2m-1}^{(N)}) + \\
& \sum_{i=1}^{N-m-1} \left[(\cos(\alpha_N) h_{2i}^{(N)} - \sin(\alpha_N) h_{2i-1}^{(N)})(\cos(\alpha_N) h_{2i+2m}^{(N)} - \sin(\alpha_N) h_{2i-1+2m}^{(N)})\right] \\
& +(\cos(\alpha_N) h_{2N-2m}^{(N)} - \sin(\alpha_N) h_{2N-2m-1}^{(N)})(-\sin(\alpha_N) h_{2N-1}^{(N)}) + \\
& (\sin(\alpha_N) h_0^{(N)})(\sin(\alpha_N) h_{2m}^{(N)} + \cos(\alpha_N) h_{2m-1}^{(N)} + \\
& \sum_{i=1}^{N-m-1} \left[(\sin(\alpha_N) h_{2i}^{(N)} + \cos(\alpha_N) h_{2i-1}^{(N)})(\sin(\alpha_N) h_{2i+2m}^{(N)} + \cos(\alpha_N) h_{2i-1+2m}^{(N)})\right] \\
& +(\sin(\alpha_N) h_{2N-2m}^{(N)} + \cos(\alpha_N) h_{2N-2m-1}^{(N)})(\cos(\alpha_N) h_{2N-1}^{(N)}) \\
&= \cos^2(\alpha_N) h_0^{(N)} h_{2m}^{(N)} - \sin(\alpha_N)\cos(\alpha_N) h_0^{(N)} h_{2m-1}^{(N)} + \\
& \sum_{i=1}^{N-m-1} \left[\cos^2(\alpha_N) h_{2i}^{(N)} h_{2i+2m}^{(N)} - \sin(\alpha_N)\cos(\alpha_N) h_{2i}^{(N)} h_{2i-1+2m}^{(N)} - \right. \\
& \left. \sin(\alpha_N)\cos(\alpha_N) h_{2i-1}^{(N)} h_{2i+2m}^{(N)} + \sin^2(\alpha_N) h_{2i-1}^{(N)} h_{2i-1+2m}^{(N)}\right] - \\
& \sin(\alpha_N)\cos(\alpha_N) h_{2N-1}^{(N)} h_{2N-2m}^{(N)} + \sin^2(\alpha_N) h_{2N-1}^{(N)} h_{2N-2m-1}^{(N)} + \\
& \sin^2(\alpha_N) h_0^{(N)} h_{2m}^{(N)} + \sin(\alpha_N)\cos(\alpha_N) h_0^{(N)} h_{2m-1}^{(N)} + \\
& \sum_{i=1}^{N-m-1} \left[\sin^2(\alpha_N) h_{2i}^{(N)} h_{2i+2m}^{(N)} + \sin(\alpha_N)\cos(\alpha_N) h_{2i}^{(N)} h_{2i-1+2m}^{(N)} + \right. \\
& \left. \sin(\alpha_N)\cos(\alpha_N) h_{2i-1}^{(N)} h_{2i+2m}^{(N)} + \cos^2(\alpha_N) h_{2i-1}^{(N)} h_{2i-1+2m}^{(N)}\right] - \\
& \sin(\alpha_N)\cos(\alpha_N) h_{2N-1}^{(N)} h_{2N-2m}^{(N)} + \cos^2(\alpha_N) h_{2N-1}^{(N)} h_{2N-2m-1}^{(N)}
\end{aligned}
$$

$$
\begin{aligned}
&= \cos^2(\alpha_N)h_0^{(N)}h_{2m}^{(N)} - \sin(\alpha_N)\cos(\alpha_N)h_0^{(N)}h_{2m-1}^{(N)} - \\
&\quad \sin(\alpha_N)\cos(\alpha_N)h_{2N-1}^{(N)}h_{2N-2m}^{(N)} + \sin^2(\alpha_N)h_{2N-1}^{(N)}h_{2N-2m-1}^{(N)} + \\
&\quad \sin^2(\alpha_N)h_0^{(N)}h_{2m}^{(N)} + \sin(\alpha_N)\cos(\alpha_N)h_0^{(N)}h_{2m-1}^{(N)} + \\
&\quad \sin(\alpha_N)\cos(\alpha_N)h_{2N-1}^{(N)}h_{2N-2m}^{(N)} + \cos^2(\alpha_N)h_{2N-1}^{(N)}h_{2N-2m-1}^{(N)} + \\
&\quad \sum_{i=1}^{N-m-1}\Big[\cos^2(\alpha_N)h_{2i}^{(N)}h_{2i+2m}^{(N)} - \sin(\alpha_N)\cos(\alpha_N)h_{2i}^{(N)}h_{2i-1+2m}^{(N)} - \\
&\quad \sin(\alpha_N)\cos(\alpha_N)h_{2i-1}^{(N)}h_{2i+2m}^{(N)} + \sin^2(\alpha_N)h_{2i-1}^{(N)}h_{2i-1+2m}^{(N)} + \\
&\quad \sin^2(\alpha_N)h_{2i}^{(N)}h_{2i+2m}^{(N)} + \sin(\alpha_N)\cos(\alpha_N)h_{2i}^{(N)}h_{2i+2m-1}^{(N)} + \\
&\quad \sin(\alpha_N)\cos(\alpha_N)h_{2i-1}^{(N)}h_{2i+2m}^{(N)} + \cos^2(\alpha_N)h_{2i-1}^{(N)}h_{2i+2m-1}^{(N)}\Big] \\
&= (\cos^2(\alpha_N) + \sin^2(\alpha_N))h_0^{(N)}h_{2m}^{(N)} + (\cos^2(\alpha_N) + \sin^2(\alpha_N))h_{2N-1}^{(N)}h_{2N-2m-1}^{(N)} \\
&\quad + \sum_{i=1}^{N-m-1}\Big[(\cos^2(\alpha_N) + \sin^2(\alpha_N))h_{2i}^{(N)}h_{2i+2m}^{(N)} + \\
&\quad (\cos^2(\alpha_N) + \sin^2(\alpha_N))h_{2i-1}^{(N)}h_{2i-1+2m}^{(N)}\Big] \\
&= h_0^{(N)}h_{2m}^{(N)} + h_{2N-1}^{(N)}h_{2N-2m-1}^{(N)} + \sum_{i=1}^{N-m-1}\Big[h_{2i}^{(N)}h_{2i+2m}^{(N)} + h_{2i-1}^{(N)}h_{2i-1+2m}^{(N)}\Big] \\
&= h_0^{(N)}h_{2m}^{(N)} + h_{2N-1}^{(N)}h_{2N-2m-1}^{(N)} + \sum_{i=1}^{2N-2m-2} h_i^{(N)}h_{i+2m}^{(N)} \\
&= \sum_{i=1}^{2N-2m-1} h_i^{(N)}h_{i+2m}^{(N)} \qquad (197)
\end{aligned}
$$

From (197) and (190) it leads to

$$
\sum_{i=0}^{2N+1-2m} h_i^{(N+1)}h_{i+2m}^{(N+1)} = 0. \qquad (198)
$$

□