The Standard Model Effective Field Theory

Integrating UV Models via Functional Methods

FAGNER CINTRA CORREIA

A dissertation submitted to the Institute for Theoretical Physics - Sao Paulo State University in accordance with the requirements for the degree of DOCTOR OF PHILOSOPHY.

Instituto de Física Teórica
UNIVERSIDADE ESTADUAL PAULISTA "JÚLIO DE MESQUITA FILHO"

AUGUST 2017

Número: IFT-T.004/17. Word count: 59394

Dedicado à Camilla.

ABSTRACT

It will be presented the principles behind the use of the Standard Model Effective Field Theory as a consistent method to parametrize New Physics. The concepts of Matching and Power Counting are covered and a Covariant Derivative Expansion introduced to the construction of the operators set coming from the particular integrated UV model. The technique is applied in examples including the SM with a new Scalar Triplet and for different sectors of the 3-3-1 model in the presence of Heavy Leptons. Finally, the Wilson coefficient for a dimension-6 operator generated from the integration of a heavy J-quark is then compared with the measurements of the oblique Y parameter.

Keywords: Covariant Derivative Expansion, Standard Model Effective Field Theory, 3-3-1 Models, Triplet Scalar, Oblique Parameters.

RESUMO

O Modelo Padrão Efetivo é apresentado como um método consistente de parametrizar Física Nova. Os conceitos de Matching e Power Counting são tratados, assim como a Expansão em Derivadas Covariantes introduzida como alternativa à construção do conjunto de operadores efetivos resultante de um modelo UV particular. A técnica de integração funcional é aplicada em casos que incluem o MP com Tripleto de Escalares e diferentes setores do modelo 3-3-1 na presença de Leptons pesados. Finalmente, o coeficiente de Wilson de dimensão-6 gerado a partir da integração de um quark-J pesado é limitado pelos valores recentes do parâmetro obliquo Y.

Palavras-Chave: Expansão em Derivadas Covariantes, Modelo Padrão Efetivo, Modelos 3-3-1, Tripleto Escalar, Parâmetros Oblíquos.

AUTHOR'S DECLARATION

I declare that the work in this dissertation was carried out in accordance with the requirements of the he Institute for Theoretical Physics - UNESP's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

SIGNED: .... DATE: ....

PREFACE

Some time ago I lived the circumstance of having to search for a place on a map. I was with Prof. Pleitez. After a few minutes we were not even close to realize where we were or about the point we were trying to reach. Then he told me something that I will probably never forget - “This is the problem with the maps, you never know where you are going”.

Well, the purpose of a map should be exactly the opposite, of course. It is true, in fact, that some of them may require a small experience as a scout. However, the point about my supervisor’s conclusion was that it remounted me a memory of my first year in college. At that time I was used to think about Physics exactly as a map, but a map made from a previous knowledge on the treasure. The treasure of Higgs phenomenology, of dark matter, neutrino and flavor physics, etc.

After studying the topic of the present work I have realized about our problem that day. The scale of that map was not the approprite one. The place we were looking for was not too close, but not too far. Most of those symbols and lines were indeed not necessary, and we did not need the information about places that we were not able to reach in any case. At the other hand if the map had only our current street and something about the neighbor, it would be still useless. What we needed was a map in an intermediary scale, a simplified one, an effective map.

The present work is about maps and is my ackowledgment to Prof. Vicente Pleitez.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Part I

Introduction

CHAPTER 1

THE METHOD IN HIGH ENERGY PHYSICS

The title of this chapter remounts a complex and diverse area that will not be entirely, or even approximately, covered by the present work. It has been chosen, however, because the set of elements defining a methodology for particle physics will contain (i) a complete quantum field theory (or UV, for short), (ii) how to extract predictions from it and (iii) and how to connect it with the experiment.

Here, these three topics are introduced in a specific form. The starting point consists in a technique for simplifying the computation from the UV model, redefining it like an Effective Field Theory (EFT). It will be seen that the use of an EFT must preserve the overall of quantum field theory paradigms and it can be better considered as a tool, a very consistent one, appropriate for many branches of theoretical physics. In High Energy Physics this technique will play an important role in the context where new degrees of freedom do not emerge asymptotically in the experiments, what is the current status of the measurements coming from the Large Hadron Collider since the discovery of the Higgs boson.

The construction of an EFT may be performed by two approaches of *integration* - functional methods or Feynman diagrams - followed by two views of *matching* - subtraction or integration by regions. The term *integration*, or *to integrate a model*, express the procedure of cutting a heavy sector of the model out of the set of external particles. As mentioned, it follows from the phenomenological fact that new physics have not emerged in this form yet. The term *matching* express the correspondence into the both modes of describing the physical process - At a given scale the EFT implies the same predictions as the UV complete theory.

The content of the Chapter 2 covers one combination of the points above. The purpose is to present a technique of integration that rewrites the UV theory directly into its Effective form by preserving the original symmetries. The method is called *Covariant Derivative Expansion*, a concept revealing its own meaning. The presentation is supported by two recent works of B. Henning et al. ([32] and [33]) and it involves the task of writing a series from the non-local components of the UV theory after the heavy particles have been *integrated out*. The series must preserve the covariant derivative intact.

The functional that must be expanded consists of the first quantum correction to the Effective Action of both theories. These determinants are a more general result than those from a perturbative expansion and they will be called *Log* corrections, in contrast to the term *1-loop* corrections. Since the generators of one-particle-irreducible graphs are the fundamental objects for defining the analytical structure of a theory, the matching is performed by equating the Effective Actions, denoted by Γ, and solving to the so-called Wilson coefficients. It will be seen that the equations are defined order-by-order, what permits the construction of the EFT in a systematic manner. In addition, the matching equations emerge with an important conceptual meaning - the Wilson coefficients, c_i, receives contributions from the difference of Γ_{UV} and Γ_{EFT} Log corrections. Since, by definition, the EFT must agree with the UV in the *soft* region of momentum scales, the c_i is corrected at this level with the information about the heavy line in the *hard* region. By matching through *subtraction* is meant to perform the Log computation in both theories and then to identify the difference between them. At the other, during the match through integration by regions, the many loop integrals are computed already considering $q^2 \sim \Lambda$ inside the integrand, with Λ denoting a heavy scale. The methods are equivalent and must lead to the same results.

In the literature the combination of Feynman diagrams plus subtractions is present, for example, in the work about QCD corrections to weak interactions of Buchalla et. all in [8]. For integration by regions it can be mentioned the textbook of J.Donoghue et. al, [21]. To the functional approach plus integration by regions it follows the recent work of Fuentes-Martin et.al, [28]. As mentioned before, the second article of B.Henning et al. [33] presents the last combination, i.e. functional methods followed by the subtraction, but does not contain a final expression to the case where both light and heavy particles run into the loops. The Chapter 2 propose a complement to this topic.

1.0.1 Renormalizability

The Top-Down approach described above is confirming the statement that the effective theory comprises the principles of a quantum field theory and is in fact addressing any possible controversy about non-renormalizability. The EFT composes an alternative on searching for New Physics through the virtual effects of particles that cannot be produced as free states, and explores how they may enhance the Standard Model parameters. The Chapter 3 develops three fundamental results that may support these assertions - the Operator Product Expansion, the Weinberg Theorem and the Decoupling Theorem.

The presentation about infinities and renormalizability will not intend to saturate or repeat the original fonts, namely the works of Bogolyubov and Shirkov [5], Peskin and Schroeder [42] and more recently Matthew Schwartz [49]. The aim is to explore the total of components behind the EFT technique, thus composing a consistent chain.

The acceptance of the presence of infinities is specially a consequence of a better conceptual comprehension of the theory. If the computation of a divergent loop resulted in a non-analytical piece on the external momentum, this would require an insertion of a non-local term in the Lagrangian, thus violating the locality of the theory. The Weinberg theorem will prove, however, that any divergent piece emerging from the theory must be proportional to polynomials in the external momenta, what is translated into the locality of the counterterms. The presence of divergent terms with a non-analytical structure on the external momenta could be, perhaps, a better definition of non-renormalizability. The fact of having to include terms consistently, although an indefinite number of times, which are suppressed by the completion scale of the theory and that can handle with the divergences is, at the most, a practical issue. If such an infinite, but discrete, set of operators are proposed a priori, this final theory, of no definite form and preserving the symmetries of its complete version, would still be predictive.

It is important to remark that the presence of infinities occurs at the very beginning in quantum field theory. By placing a harmonic oscillator to every point in space-time, for example, at the same time an infinity amount of energy is associated to the universe [49]. The result must not be inconvenient since every measurement in nature is only meaningful once it is performed through a comparison, a difference. Every number should correspond a variation.

At some cases, to perform a variation (or subtraction) can be something trivial, like estimating the position of a body from some reference point. However, this variation in quantum field theory, although very systematic, is not direct. It must occur for every parameter and fields through the *renormalization procedure*. The variables performing the subtraction are the counterterms and rely on the above mentioned feature of the theory - the infinities appears as polynomials in the external momenta, a property connected to the local aspect of the quantum theory. During the renormalization procedure the choice for the subtraction scale must be done, and the final result is independent from it. It is important to emphasize that to state that some piece of the Lagrangian is independent of the renormalization scale is not to say that the couplings does not depend on the physical scale. What is being changed, in truth, is the starting point, the reference where the subtraction was chosen to be performed. That is equivalent to say that the first measurement of the fine-structure constant α could have been done at a larger scale than $\sim m_e$ - This single measurement would be sufficient to provide exactly the same predictions of QED. In other words, the renormalization group equations does not come from a hypothesis of independence of a physical process on the energy, but from the independence of the object (correlation functions) associated with the given process under the change of the scale that defines its components (couplings and operators). A Green's function is invariant under translation as a physical process will also be. Finally, the functional form of a running coupling on the energy scale as well as its Landau pole can contribute to define the properties of a given phenomena.

1.0.2 Top-down approach and Precision Observables

The previous discussion may clarify the question about when the perturbative analysis is important. The answer is - when the scale of the process computed is not exactly the scale where the coupling constant were extracted. If it was, only the tree-level computation would be necessary, although this would not mean that there would not be a subtraction present. The subtraction for tree-level exact results was implicit during the fitting of the constants with the data.

The renormalization group equations will then perform the sum of corrections at once and will transfer them to the couplings, leaving the last task of computing a tree-level matrix element. The point about this discussion, therefore, is that the matching procedure provides the boundary conditions for solving the RGE. The running of the Wilson coefficients will be associated with the many corrections from the light-particles present in the renormalizable sector of the theory, namely the Standard Model. These assertions reveal the advantage of working in a top-down approach. It is clear since the starting point that the aim of making an effective model is nothing more than simplifying the analysis of the original UltraViolet theory. By the correspondence principle one extracts the couplings at the heavy scale that can be run down to the electroweak scale. The final set of operators will then be considered for an arbitrary number of process and through a tree-level analysis.

According to the Decoupling Theorem, the error committed on working exclusively in a Standard Model framework must be proportional to $\frac{1}{m^a}$, being m a heavy-scale and a approximately two. The aim of raising an SMEFT is exactly to reduce this error by including new local interactions suppressed by the same power. What is going to be shown in the Chapter 3, therefore, precedes the matching procedure and can be summarized as follows - (i) Consider the initial UV complete theory as a large set of interactions containing those of the Standard Model as a subset, i.e.

$$\Gamma_{UV} = \Gamma_{SM} + \Gamma_{SM+H} \tag{1.1}$$

where H is meant to be a heavy sector. (ii) To explore the consequences of proposing a low-energy version of the UV model consisting in the SM theory with changed couplings and masses, emerged by cutting out Γ_H from the graphs generated by Γ_{UV}:

$$\Gamma_{UV} \to \Gamma_{\overline{SM}} \tag{1.2}$$

such that the 1LPI graph, i.e. 1PI graphs in the light-fields, generated by $\Gamma^{(n)}_{UV}$ will be

$$\Gamma^{(n)}_{UV} = \Gamma^{(n)}_{\overline{SM}} \times \left[1 + O\left(\frac{1}{m^a}\right)\right] \tag{1.3}$$

The first task is to prove that exists a simplified theory at low energies which is able to provide the same results as the complete theory, at some level of precision. Next, the error is inserted as local operators into the low-energy renormalizable theory, representing the value of the coupling constant in the heavy scale. Since the exchange of light particles involves renormalizable couplings, the coefficients will be run down to the low-energy scale through these light corrections. One comment, the differential equation will also be coupled, which corresponds a dependence of the anomalous dimension for the higher-dimension operators on the SM parameters.

Finally, the Decoupling Theorem confirms that the purpose of constructing an EFT is not of defending the use of non-renormalizable theories as a final description of the nature, but to develop a technique for studying it. The SM is, by hypothesis, a renormalizable low-energy sector of a more complex complete theory.

The present work develops the principles discussed above. The Part II consist of a literature review about the functional methods of matching and fundamentals of Effective Field Theories. The Part III contains the introduction of gauge theories with spontaneously symmetry breaking, where the 3-3-1 models with Heavy leptons have been chosen as the main example. The choice was motivated by the possibility of exploring features that are not present in the Standard Model. Apart from that, it is in general alleged that these theories present a pattern of symmetry breaking following strictly the path $3 \otimes 3 \otimes 1 \to 3 \otimes 2 \otimes 1 \to U(1)$, which includes the Standard Model. Thus, a SMEFT can be extracted from it, leading to a entire connection with the rest of the work. The Chapter 5 apply the results of Part II to a set of models, including some specific sectors of the 3-3-1HL. The defined set of Wilson coefficients must be run down and face some Electroweak Precision Observables in Chapter 6, thus informing about how far the experimental measurements are from being sensible to the theories at the loop level.

Part II

The Covariant Derivative Expansion

Literature Review

CHAPTER 2

THE COVARIANT DERIVATIVE EXPANSION

Consider an UltraViolet complete theory (UV for short) formulated in order to present two sectors separated by different scales. The light sector can be generically denoted by ϕ and the heavy sector by Φ. The generating functional is expressed like [53]

$$Z_{UV}[J_\Phi, J_\phi] = \int D\Phi D\phi e^{i\int_x (\mathscr{L}_{UV}[\Phi,\phi] + J_\phi \phi + J_\Phi \Phi)} \tag{2.1}$$

On what follows the Effective Action for the *Light-UV* theory is given by the following Legendre transform

$$\Gamma_{L,UV}[\phi] = -i \log Z_{UV}[J_\Phi = 0, J_\phi] - \int_x J_\phi \phi \tag{2.2}$$

i.e. the source for heavy fields must be equal to zero. In other words the sector Φ are not allowed to emerge as external particles and the solution for Γ can be obtained through the saddle-point approximation by expanding the integrand around the classical solutions:

$$\frac{\delta S_{UV}[\Phi,\phi]}{\delta \Phi}\Big|_{J_\Phi=0} = 0, \qquad \frac{\delta S_{UV}[\Phi,\phi]}{\delta \phi} = 0 \tag{2.3}$$

such that $\Gamma_{L,UV}[\phi] \equiv \Gamma_{L,UV}[\Phi[\phi],\phi]$ after the replacement of $\Phi[\phi]$ as an implicit functional of the light fields. The UV Effective Action expressed in terms of background fields at the *Log* order is given by

$$\Gamma_{UV}[\phi,\Phi] = S_{UV}[\phi,\Phi] + i\alpha \log\det\left(\frac{\delta^2 S_{UV}}{\delta(\phi,\Phi)^2}\right) \tag{2.4}$$

2.1 Locality

The concept of locality will be present in some of the most important steps of the matching procedure and defines, for example, the correct mode of performing the loop counting in the Effective and in the Light-UV theory.

Consider a theory for two scalar fields defined by the Lagrangian [49]:

$$\mathscr{L} = -\frac{1}{2}\phi\left(\Box + M^2\right)\phi - \frac{1}{2}\pi\left(\Box + m^2\right)\pi + \frac{\lambda}{2}\phi\pi^2 \tag{2.5}$$

The equations of motion for ϕ, when applied back to $\mathscr{L}$, may convert it into a theory for the light fields π. In other words,

$$\frac{\delta S}{\delta \phi} = 0 \quad \rightarrow \quad -\left(\Box + M^2\right)\phi + \frac{\lambda}{2}\pi^2 = 0 \tag{2.6}$$

where the solution may be given in terms of the Green functions to the Klein-Gordon operator

$$\left(\Box + M^2\right)_x G^\phi_{xy} = \delta_{xy} \tag{2.7}$$

and here the continuous index is a shorthand for the variables defining functions and operators or, for example, $\delta_{xy} \equiv \delta(x-y)$. Besides, the upper index denote the dependence on M and will be omitted on what follows. The above expression implies

$$\phi_x = \frac{\lambda}{2}\langle G_{xy}\pi_y^2\rangle \equiv \frac{\lambda}{2}\int_y G_{xy}\pi_y^2 \tag{2.8}$$

where it was used a shorthand notation (by [48]) that will frequently be evoked along this work - Repeated space-time indices inside brackets < ... > are being integrated over the total volume. By plugging Eq.(2.8) back into Eq.(2.5), it follows

$$\begin{aligned}\mathscr{L} &= -\frac{\lambda^2}{8}\langle G_{xz}\pi_z^2\rangle\left(\Box+M^2\right)_x\langle G_{xy}\pi_y^2\rangle - \frac{1}{2}\pi\left(\Box+m^2\right)\pi + \frac{\lambda}{4}\langle G_{xy}\pi_y^2\rangle\pi_x^2 \\ &= \frac{\lambda^2}{8}\pi_x^2\langle G_{xz}\pi_z^2\rangle - \frac{1}{2}\pi\left(\Box+m^2\right)\pi\end{aligned} \tag{2.9}$$

and, thus, the function $\mathscr{L}$ on a particular x is now being simultaneously affected by the total field configuration from the integral inside the brackets. This new quality is what defines the Lagrangian as a *non-local* object. The Fourier transform of G_{xy} is given by

$$G_{xy} = \int dq e^{-iq(x-y)}G_q \qquad \overset{Eq.(2.7)}{\longrightarrow} \qquad G_{xy} = \int dq \frac{e^{-iq(x-y)}}{-q^2+M^2} \tag{2.10}$$

by abbreviating $dq \equiv \frac{dq}{(2\pi)^4}$. In a more symbolic form, Eq.(2.7) can be rewritten as[1]

$$G^{\phi}_{xy} = \left(\Box+M^2\right)_x^{-1}\delta_{xy} \tag{2.11}$$

It is known from the Feynman rules in coordinate space that, at tree-level, the momentum running in Eq.(2.10) must collapse into the momenta p^2 of the external particles. Thus, motivated by the scenario where $\Box \sim p^2 \ll M^2$, the r.h.s. may be expanded [49] into local terms through

$$G^{\phi}_{xy} = \frac{1}{M^2}\left(1 - \frac{\Box}{M^2} + \left(\frac{\Box}{M^2}\right)^2 - \cdots\right)_x \delta_{xy} \tag{2.12}$$

which turns even more spurious when the propagator runs inside loops. As it will repeatedly explored, to force this expansion is in fact what is behind the necessity of a consistent matching procedure.

Finally, the replacement of ϕ by the respective e.o.m. solution encodes its influence to the Green function. The field is *out* of the set of asymptotic particles, but is certainly present through virtual effects.

2.2 On the Effective Action

The Effective Action will always be denoted by the Greek letter Γ. In order to perform the Legendre transformation in the beginning of this chapter it is first needed to solve the set of

[1] Or $\mathscr{O}^{-1}\mathscr{O} \equiv 1$, for a generic operator $\mathscr{O}$. Formally, it is equivalent to $\mathscr{O}_x^{-1}A_x \equiv \int_y G_{xy}A_y$, where $\mathscr{O}_x G_{xy} = \delta_{xy}$.

classical equations in terms of the sources and then replace the solutions in the functional generator formula, commonly obtained in the saddle point approximation. The classical result for the Effective Action (i.e. the first term in the expansion) is composed by the classical action given in terms of the field-sources, here denoted by the sub-index 'c', like in ϕ_c and ψ_c [48]. The aim of this section is to provide both a qualitative and formal explanation about the Effective Action as a generating functional and its role during the construction of Effective Field Theories.

It may be assumed as known the quality of $\Gamma[\phi]$ as the generator of amputated one-particle irreducible (1PI) diagrams. In a theory composed by a single self-interacting field, this means that the corrections coming from the Log in Eq.(2.4), diagrammatically, are represented by loops without external lines and such that it cannot be decomposed into independent graphs through a single cut of propagator.

The saddle-point approximation [48], commonly adopted in order to solve a closed form to the functional generators, consists in a Taylor expansion on the Action made by assuming that the leading contribution to integrals like $\int_x e^{-a(x)}$ is given by the region around the minimum of $a(x)$. Therefore, the procedure is an independent method compared with a perturbation expansion for a quantum field theory. In other words, the Log may enclose simultaneously the quantum correction in different orders of the small parameter of the theory, i.e. it can associate different n-point functions. This assertion will turn clearer when the Universal Formula for the determinant computation is presented (see Section 2.3).

It is still important to distinguish about two concepts already present at this point. Again, from the Effective Action definition it can be seen that it corresponds to an object that can be written in an arbitrarily higher dimension on the classical fields. The term 'effective' here is rather clear - its components are in fact what is connected with physical processes, whose effect is weighted by the coefficient of the correspondent n-point function. In summary, the set of higher-dimension operators defines a set of physical processes. A given operator is accompanied by a coefficient informing the importance (or suppression) of the correspondent scattering, for example. In this sense, the Effective Action is an object able to provide physical intuition about the considered phenomena. Along this work there will be rarely found mention to Feynman diagrams, but it might be important to include some comments. The Effective Action generate n-point functions containing exclusively internal lines or, equivalently, amputated Green's function. Thus, the diagrams are composed by dots and propagators running in loops. By *integrate out* a field, i.e. by replacing it into the theory through the solution of its classical equation of motion, the field will then be forbidden to represent asymptotic degrees of freedom. This new theory must be called *UV-Light* and its respective Γ_{LUV} will generate *1-Light Particle Irreducible* diagrams (or 1-LPI, for short) - Irreducible graphs only on the light internal lines. Therefore, in this framework Γ_{LUV} may generate diagrams containing heavy internal lines. This may be seen, for example, from the first term of Eq.(2.9). One *Effective Field Theory* consists on bringing the 1LUV theory into a local description of the interactions. Here, the respective action for the EFT will again generate dots

FIGURE 2.1. In a $\lambda\phi^3$ theory, for example, the effective action can generate the 4-point function (a) but not (b), where $\bullet$ and $\circ$ denote an insertion of one and two external lines, respectively.

FIGURE 2.2. In a $\lambda\phi^4$ theory, the determinant can generate a set of 4-point functions containing graphs of different order in perturbation theory. Above, the $\bullet$ and $\circ$ denote an insertion of one and two external lines, respectively.

FIGURE 2.3. The Effective Action for the 1LUV of Eq.(2.9) may generate the 4-point functions above, where the dashed and full lines represent the propagation of the light π and the heavy ϕ, respectively.

FIGURE 2.4. One effective vertex (b) from the local expansion of the LUV theory (a).

and loops, although the former is commonly considered sufficient for a well accurate analysis.

In an Effective Theory for π's of the Eq.(2.9) only the classical term, or the linear contribution on the Wilson coefficients, are present. The n-point function in this approximation will be represented only by c_n, in reference to the operator $\mathcal{O}_n$, and a box $\square$ (or $\otimes$).

FIGURE 2.5. In a theory containing self- or mixed interactions of light-fields, the 1LUV (a) and the EFT (b) may present the above graphs. The □ represents an effective vertex;

2.2.1 On the sign of determinant for real, complex scalars and fermions

Along the development of an Universal Formula to the first quantum correction of the Effective Action, the authors in [32] first sought for a general expression[2] related to graphs which include solely heavy lines. The first step consisted in defining a generic expression for the Log piece:

$$\Gamma^{(1)}_{UV} = ic_s \mathrm{Tr}\, \log(-P^2 + m^2 + U_x) \tag{2.13}$$

with $P^\mu \equiv iD^\mu = i\partial_\mu + \mathbf{A}_\mu$ the covariant derivative and $U(x)$ a function representing constant configurations of light fields. One important point observation, therefore, relies on the extraction of the constant c_s, holding the information about the specie of particles being integrated. The following lines are intended to treat this topic and concern fermions, real and complex scalars.

The Eq.(2.13) is derived from the first correction to the generating functional on the saddle point approximation. For a theory containing real and spinor fields [48], it implies the expression

$$e^{iS_{\text{eff}}[\Phi,\Psi^\dagger,\Psi]}|_c = \int \mathscr{D}\Phi\mathscr{D}\Psi^\dagger\mathscr{D}\Psi e^{iS[\Phi,\Psi^\dagger,\Psi]} \tag{2.14}$$

In fact this expression is already assuming the final result for the Effective Action which will be fully demonstrated in the coming text. The action in the r.h.s. must be rotated to the Euclidean space and then expanded through

$$\begin{aligned} S \;=\; & S\Big|_c + \Big\langle \eta^\dagger \frac{\delta S}{\delta \Psi^\dagger}\Big|_c \Big\rangle + \Big\langle \frac{\delta S}{\delta \Psi}\Big|_c \eta \Big\rangle + \Big\langle \frac{\delta S}{\delta \Phi}\Big|_c \rho \Big\rangle \\ & + \frac{1}{2}\Big\{ \Big\langle \rho_x \frac{\delta S}{\delta\Phi_x \delta\Phi_y}\Big|_c \rho_y \Big\rangle + 2\Big\langle \eta^\dagger_x \frac{\delta S}{\delta\Psi^\dagger_x \delta\Psi_y}\Big|_c \eta_y \Big\rangle \\ & + 2\Big\langle \rho_x \frac{\delta S}{\delta\Phi_x \delta\Psi_y}\Big|_c \eta_y \Big\rangle + 2\Big\langle \eta^\dagger_x \frac{\delta S}{\delta\Psi^\dagger_x \delta\Phi_y}\Big|_c \rho_y \Big\rangle \Big\} + \cdots \end{aligned} \tag{2.15}$$

where $\rho \equiv \Phi - \Phi_c$ and $\eta \equiv \Psi - \Psi_c$. The sub-index 'c' stands for the fields on the solution of the classical equations of motion, which eliminate the first derivatives. Moreover, the factor of two inside the brackets accounts for the second derivative of mixed terms. As mentioned before, repeated space-time indices inside $< \cdots >$ are being integrated over.

[2]The notation will be consistently preserved.

It was left implicit in Eq.(2.15) that the action is defined in Euclidean space. At the end of a functional integration, the fields must be rotated back to Minkowski variables, what in summary will imply an overall multiplication by the factor $-i$. The extraction of c_s concerns only the second line and for real scalars it should come from

$$e^{iS_{\text{eff}}} = \int \mathscr{D}\Phi \, \exp\left[-i\int_x \frac{1}{2}\Phi\left(-P^2+M^2+U_x\right)\Phi\right] \tag{2.16}$$

When the expansion (2.15) is performed the corrections are in fact being integrated over Euclidean (or 'bar') variables, such that

$$S_E = \int_{\overline{x}} \frac{1}{2}\Phi\left(-\overline{P}^2+M^2+U_x\right)\Phi \tag{2.17}$$

Note that the minus sign is preserved in the definition just to follow $-P^2 \supset -(i\partial_\mu)^2 = \partial^2 = -\overline{\partial}^2$. By replacing the expansion $S_E = S_E|_c + \frac{1}{2} < \rho_x \frac{\delta^2 S_E}{\delta\Phi_x\delta\Phi_y}\rho_y > + \cdots$ back into Eq.(2.16):

$$e^{-S_E} = e^{-S_E|_c} \times \int \mathscr{D}\Phi \, \exp\left[-\frac{1}{2}\langle\rho_x \frac{\delta^2 S_E}{\delta\Phi_x\delta\Phi_y}\rho_y\rangle\right] \tag{2.18}$$

Since $\int\mathscr{D}\Phi \, e^{-\frac{1}{2}\langle\Phi A\Phi\rangle} = (\det A)^{-\frac{1}{2}} = \exp[-\frac{1}{2}\text{Tr}\log \text{A}]$, after rotating the expressions back to Minkowski space the Effective Action at loop order must be given by

$$e^{iS_{\text{eff}}} = e^{iS|_c} \times \, \exp\left[-\frac{1}{2}\text{Tr}\,\log(-P^2+M^2+U_x)\right] \tag{2.19}$$

or

$$S_{\text{eff}} = S\big|_c + \frac{i}{2}\text{Tr}\,\log(-P^2+M^2+U_x) \tag{2.20}$$

which leads to $c_s = \frac{1}{2}$ (real scalars). It turns out that this half factor, just in front of the trace, originates from the Taylor expansion instead of the Lagrangian. Thus, although the procedure for complex scalars must follow in the same way, now the Taylor coefficient acquires an additional factor of two from mixed terms, as already recorded in Eq.(2.15). From $\int\mathscr{D}\Phi^\dagger\mathscr{D}\Phi \, e^{-\langle\Phi^\dagger A\Phi\rangle} = (\det A)^{-1}$, it follows that $c_s = 1$ (complex scalars).

Finally, the constant for fermions must consider the inverse relation for a Gaussian integration of Grassmann fields, i.e. $\int\mathscr{D}\Psi^\dagger\mathscr{D}\Psi \, e^{-\langle\Psi^\dagger A\Psi\rangle} = (\det A)$. Moreover, this result would still lead to

$$S_{\text{eff}} = S\big|_c - i\text{Tr}\,\log(-\not{P}+M+F_x) \tag{2.21}$$

where F_x is denoting constant light fields. In order to convert the above result into the desired quadratic form of Eq.(2.20) one may resort to the invariance of this trace under flipping the signs of gamma matrices:

$$\begin{aligned} S_{\text{eff}}^{(1)} &= -\frac{i}{2}\text{Tr}[\log(-i\not{D}+M+F_x)+\log(-i\not{D}+M+F_x)] \\ &= -\frac{i}{2}\text{Tr}[\log(-i\not{D}+M+F_x)+\log(i\not{D}+M+F_x)] \\ &= -\frac{i}{2}\text{Tr}\left[\log\left(\not{D}^2+(M+F_x)^2-i[\not{D},F_x]\right)\right] \end{aligned} \tag{2.22}$$

Proceeding according to the notation in [33], from the identity $\not{D}^2 = D^2 - \frac{i}{2}\sigma^{\mu\nu}G'_{\mu\nu}$ and $G'_{\mu\nu} \equiv [D_\mu, D_\nu]$, it follows that

$$S^{(1)}_{\text{eff}} = -\frac{i}{2}\text{Tr}\left[\log\left(D^2 + M^2 + U_{\text{ferm}}\right)\right] \tag{2.23}$$

where

$$U_{\text{ferm}} \equiv -\frac{i}{2}\sigma^{\mu\nu}G'_{\mu\nu} - i[\not{D}, F_x] + 2MF_x + F_x^2 \tag{2.24}$$

and $c_s = -\frac{1}{2}$ (fermions).

2.3 Evaluating the Functional Determinant - On the Universal Formula

The task of determining the first quantum corrections for the Effective Action involves a solid comprehension to the meaning of a functional trace. One complete treatment on this topic was developed by the authors HLM[3] in [32] and, for completeness, it will be summarized in this subsection.

In a general case, the loop correction consists of

$$\text{Tr}\ f(\hat{x}, \hat{q}) \tag{2.25}$$

where the hats refer to the operators form. The representation for the arguments will recline on the choice of a basis, either in momentum or position space. To the former case, the trace can be defined as

$$\text{Tr}\ f(\hat{x}, \hat{q}) = \int dq\ \text{tr}\ \langle q|f(\hat{x}, \hat{q})|q\rangle \tag{2.26}$$

with the small 'tr' now accounting exclusively for internal indices. The representation $\hat{q}$ in the position basis is given by $\hat{q} = i\partial_x$, thus corresponding the commutation relation

$$[\hat{x}, \hat{q}] = -i \tag{2.27}$$

Moreover, the plane wave convention follows from the product $\langle x|q\rangle = e^{-iq\cdot x}$. By proceeding with the notation of HLM, the differentials must always hide the $2\pi^4$ factor like $dq \equiv \frac{d^4q}{2\pi^4}$ and $dx \equiv d^4x$. Finally, the representation for $\hat{x}$ and $\hat{q}$ on the aforementioned basis is such that

$$\langle x|f(\hat{x}, \hat{q}) = f(x, i\partial_x), \qquad \langle q|f(\hat{x}, \hat{q}) \overset{(2.27)}{=} f(-i\partial_q, q) \tag{2.28}$$

Next, a complete set of position eigenstates[4], $1 = \int_x |x\rangle\langle x|$, is inserted in Eq.(2.26), what implies

$$\begin{aligned} \text{Tr}\ f(\hat{x}, \hat{q}) &= \int dx dq\ \text{tr}\ \langle q|x\rangle\langle x|f(\hat{x}, \hat{q})|q\rangle \\ &= \int dx dq\ \text{tr}\ e^{iq\cdot x}\ f(x, i\partial_x)\ e^{-iq\cdot x} \end{aligned} \tag{2.29}$$

[3] Abbreviation for B. Henning, X. Lu and H. Murayama;

[4] Remark that, analogously, the identity for the discrete case arises like $\mathbb{I} = \sum_i |\mathbf{e}_i\rangle\langle\mathbf{e}_i|$, from the set of generators $\{\mathbf{e}_i\}$ defining an orthonormal basis of a n-dimensional vectorial space.

From the Baker-Campbell-Hausdorff formula, the momentum operator is in fact being translated into[5] $i\partial_x \to i\partial_x + q$. By flipping the momentum sign, the final result may be written as

$$\mathrm{Tr}\, f(\hat{x},\hat{q}) = \int dxdq\ \mathrm{tr}\, f(x, i\partial_x - q) \tag{2.30}$$

The Effective Field Theory, under development, is marked by the matching procedure, one equation that must in general reveal a piece like $\frac{\delta^2 S_{\mathrm{UV}}}{\delta\Phi^2}$, noted in HLM as the term defining the set of Wilson coefficients coming from the exclusive case of heavy fields running in the loops. This portion can be expressed like

$$S^{(1)}_{\mathrm{eff}} \supset\ ic_s \mathrm{Tr}\log(-P^2 + m^2 + U_x) \tag{2.31}$$

whose components were already presented in Section (2.2.1). From Eq.(2.29) it is translated into

$$\begin{aligned} S^{(1)}_{\mathrm{eff}} &\supset\ ic_s \int dxdq\ \mathrm{tr}\ e^{iq\cdot x}\ \log(-P^2 + m^2 + U_x)\ e^{-iq\cdot x} \\ &=\ ic_s \int dxdq\ \mathrm{tr}\ \log\left[-(P_\mu - q_\mu)^2 + m^2 + U_x\right] \end{aligned} \tag{2.32}$$

One interesting approach for treating the above expression was proposed by Mary Gaillard in [29] and reproduced in [32] with the intent of reaching a closed formula. The author in [29] proposed a general matrix-valued function g, dependent on the derivatives (∂_p, ∂_x) and under the initial condition

$$g(0,0) = 1 \tag{2.33}$$

such that its expansion around $(0,0)$ would be given by

$$g = 1 + g^{(1)}|_0 + g^{(2)}|_0 + \cdots \tag{2.34}$$

Once g^{-1} acts in the right, it is in fact acting on the identity. Moreover, since the $g^{(i)}$'s are at least of first order in derivatives, it follows that it should be equal to the unit. On acting in the left, the situation would not be different. The expansion of g leaves total derivatives of the operator, what should vanish by the boundary conditions of the integrand. Thus, only $g(0,0)$ acts, and the insertion of g in both sides of Eq.(2.32) is trivial. At the end, the function is assumed with the form

$$g \equiv e^{P\cdot\partial_q} \tag{2.35}$$

[5]This follows from the analyticity of $f(\hat{x},\hat{q})$, resulting that the BCH formula may in fact be applied on the identity vicinity, being sufficient to verify only $e^{iq\cdot\hat{x}}\ \hat{q}\ e^{-iq\cdot\hat{x}}$;

or

$$\begin{aligned}\int e^{P\cdot\partial_q}\mathbb{A}e^{-P\cdot\partial_q} &= \int e^{P\cdot\partial_q}\mathbb{A}(e^{-P\cdot\partial_q}\mathbb{I}) \\ &= \int e^{P\cdot\partial_q}\mathbb{A} \\ &= \int\left(1+\frac{(P\cdot\partial_q)^n}{n!}\right)\mathbb{A}, \qquad n\geq 1 \\ &= \int \mathbb{A}+\mathbb{F}\big|_{-\infty}^{\infty} \\ &= \int \mathbb{A} \end{aligned} \tag{2.36}$$

By applying the Gaillard procedure to Eq.(2.32), the task converts into simplifying the expression

$$S_{\text{eff}}^{(1)} \supset ic_s\int dxdq \text{ tr } e^{P\cdot\partial_q}\log\left[-(P_\mu-q_\mu)^2+m^2+U_x\right]e^{-P\cdot\partial_q} \tag{2.37}$$

by requesting the BCH formula once more:

$$e^{B}Ae^{-B}=\sum_{n=0}^{\infty}\frac{1}{n!}L_B^nA, \qquad \text{where} \qquad L_BA\equiv[B,A] \tag{2.38}$$

The first component, $(P_\mu-q_\mu)$, may be translated after one brief observation:

$$\begin{aligned}\left[P\cdot\partial_q,q_\mu\right] &= P\cdot\partial_q\lfloor q_\mu-q_\mu P\cdot\partial_q \\ &= P_\nu\delta^\nu_\mu+q_\mu P\cdot\partial_q-q_\mu P\cdot\partial_q \\ &= P_\mu \end{aligned} \tag{2.39}$$

and, thus

$$\begin{aligned} e^{P\cdot\partial_q}(P_\mu-q_\mu)e^{-P\cdot\partial_q} &= \sum_{n=0}^{\infty}\frac{1}{n!}(L_{P\partial_q})^nP_\mu-\sum_{n=0}^{\infty}\frac{1}{n!}(L_{P\partial_q})^nq_\mu \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(L_{P\partial_q})^nP_\mu - \left(q_\mu+\sum_{n=1}^{\infty}\frac{1}{n!}(L_{P\partial_q})^nq_\mu\right) \end{aligned} \tag{2.40}$$

where the last sum can be rewritten as

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n!}(L_{P\partial_q})^nq_\mu &\overset{(2.39)}{=} P_\mu+\sum_{n=2}^{\infty}\frac{1}{n!}(L_{P\partial_q})^{n-1}P_\mu \\ &= P_\mu+\sum_{n=1}^{\infty}\frac{1}{(n+1)!}(L_{P\partial_q})^nP_\mu \end{aligned} \tag{2.41}$$

and, in Eq.(2.40),

$$\begin{aligned} e^{P\cdot\partial_q}(P_\mu-q_\mu)e^{-P\cdot\partial_q} &= -q_\mu+\sum_{n=1}^{\infty}\frac{1}{n!}(L_{P\partial_q})^nP_\mu-\sum_{n=1}^{\infty}\frac{1}{(n+1)!}(L_{P\partial_q})^nP_\mu \\ &= -q_\mu+\sum_{n=1}^{\infty}\frac{n}{(n+1)!}(L_{P\partial_q})^nP_\mu \end{aligned} \tag{2.42}$$

The sum can still be simplified via

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n}{(n+1)!}(L_{P\partial_q})^n P_\mu &= \sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}(L_{P\partial_q})^{n+1}P_\mu \\ &= \sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}(L_{P\partial_q})^{n}(L_{P\partial_q}P_\mu) \\ &= -\sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}\left\{L^n_{P\partial_q}\left[D_\nu,D_\mu\right]\right\}\partial_{q_\nu} \end{aligned} \tag{2.43}$$

where it has been considered

$$\begin{aligned} L_{P\partial_q}P_\mu &= P_\nu P_\mu \partial_{q_\nu} + P_\nu(\partial_{q_\nu}P_\mu) - P_\mu P_\nu \partial_{q_\nu} \\ &= \left[P_\nu, P_\mu\right]\partial_{q_\nu} \end{aligned} \tag{2.44}$$

since P_μ is momentum-independent. Finally, by strictly following the HLM notation, all the momentum derivatives inside $L_{P\partial_q}$ can be placed to the right, what implies

$$\begin{aligned} e^{P\cdot\partial_q}(P_\mu - q_\mu)e^{-P\cdot\partial_q} &= -q_\mu - \sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}\left\{L^n_{P_\alpha}\left[D_\nu,D_\mu\right]\right\}\partial^n_{q_\alpha}\partial_{q_\nu} \\ &= -q_\mu - \sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}\left[P_{\alpha_1},\left[\cdots\left[P_{\alpha_n},[D_\nu,D_\mu]\right]\right]\right]\frac{\partial^n}{\partial q_{\alpha_1}\cdots\partial q_{\alpha_n}}\partial_{q_\nu} \\ &\equiv -\left(q_\mu + \tilde{G}_{\nu\mu}\partial_{q_\nu}\right) \end{aligned} \tag{2.45}$$

where $\tilde{G}_{\nu\mu}$ is thus referring a general form to the field-strength.

The same analysis can proceed for the simpler case of the U component inside the determinant:

$$e^{P\cdot\partial_q}Ue^{-P\cdot\partial_q} = \sum_{n=0}^{\infty}\frac{1}{n!}\left[P_{\alpha_1},\left[\cdots\left[P_{\alpha_n},U\right]\right]\right]\frac{\partial^n}{\partial q_{\alpha_1}\cdots\partial q_{\alpha_n}} \equiv \tilde{U} \tag{2.46}$$

These previous expressions permit to represent $S^{(1)}_{\text{eff}}$ in a simplified form

$$\begin{aligned} S^{(1)}_{\text{eff}} &= ic_s\int dxdq \ \text{tr}\ \log\left[-\left(q_\mu + \tilde{G}_{\nu\mu}\partial_{q_\nu}\right)^2 + m^2 + \tilde{U}\right] \\ &\equiv \int dx \mathscr{L}^{(1)}_{\text{eff}} \end{aligned} \tag{2.47}$$

where the second line remark that the correction can be expressed equivalently in terms of a Lagrangian.

For moving to the Log calculation, the prescription first consists in convert it into an integral over m^2, i.e.

$$\mathscr{L}^{(1)}_{\text{eff}} = ic_s\int dqdm^2\ \text{tr}\ \left[-\left(q_\mu + \tilde{G}_{\nu\mu}\partial_{q_\nu}\right)^2 + m^2 + \tilde{U}\right]^{-1} \tag{2.48}$$

and then to expand the squared terms:

$$\begin{aligned} \mathscr{L}^{(1)}_{\text{eff}} &= -ic_s\int dqdm^2\ \text{tr}\ \left[\Delta^{-1} + \{q,\tilde{G}\partial_q\} + (\tilde{G}\partial_q)^2 - \tilde{U}\right]^{-1} \\ &= -ic_s\int dqdm^2\ \text{tr}\ \left\{\Delta^{-1}\left[1 + \Delta\left[\ -(\tilde{G}\partial_q)^2 - \{q,\tilde{G}\partial_q\} + \tilde{U}\ \right]\right]\right\}^{-1} \end{aligned} \tag{2.49}$$

where the omitted Lorentz indices are contracted according to Eq.(2.48). Moreover, $\Delta \equiv (q^2-m^2)^{-1}$ and may be considered as the variable for a matrix expansion like

$$\begin{aligned}\left(A^{-1}(1-AB)\right)^{-1} &= (1-AB)^{-1}A \\ &= \sum_{n=0}^{\infty}(AB)^n A \end{aligned} \tag{2.50}$$

which converts the quantum correction to the effective Lagrangian into a sum of integrals, or

$$\mathscr{L}^{(1)}_{\text{eff}} = -ic_s \sum_{n=0}^{\infty} \mathfrak{I}_n \tag{2.51}$$

where[6]

$$\mathfrak{I}_n \equiv \int dq dm^2 \text{ tr } \left[\Delta\left[\, -(\tilde{G}\partial_q)^2 - \{q,\tilde{G}\}\,\partial_q + \tilde{U}\,\right]\right]^n \Delta \tag{2.54}$$

Above it was also considered the identity[7]

$$\{q_\mu, \tilde{G}_{\nu\mu}\partial_\nu\} = \{q_\mu, \tilde{G}_{\nu\mu}\}\,\partial_\nu + \tilde{G}_{\nu\mu}\left[q_\mu, \partial_\nu\right] \tag{2.55}$$

and, from $\left[q_\mu, \partial_\nu\right] = -\delta_{\mu\nu}$ and the antisymmetry of $\tilde{G}_{\mu\nu}$, it follows that $\{q_\mu, \tilde{G}_{\nu\mu}\partial_\nu\} = \{q_\mu, \tilde{G}_{\nu\mu}\}\,\partial_\nu$.

Thus, the effective Lagrangian at Log level is represented by the expression of Eq.(2.51). There, the $\mathfrak{I}_n$ consists basically in operations on Δ. As it will be recurrently emphasized, a series expansion is always present during the **matching** procedure and must be truncated up to the desired order in the fields dimension after a **power counting** at the level of $\tilde{G}_{\nu\mu}$ and $\tilde{U}$, both containing *higher-dimension operators* (or HDO's for short). In other words, the $\mathscr{L}^{(1)}_{\text{eff}}$ is a series of $\mathfrak{I}_n$, which turns out to be a series in HDO's. By interrupting the sum at some order in the fields, it is established a number n for Eq.(2.51), implying a set that in fact composes the theory. One example - It is known a priori that $\tilde{G}_{\nu\mu}$ and $\tilde{U}$ are at least linear in the light fields. Then, if the expansion is chosen to cease at dim-6 operators, it follows that no $\mathfrak{I}_n$ will contribute for $n > 6$.

2.3.1 Evaluation of Integrals

The formula Eq.(2.51) implies the task of calculating seven integrals. In this section the main tools for achieving this are completely developed. The final result will be, as presented by HLM,

[6]There is one additional information behind the integral after the power series conversion of the log. Being F_x the primitive of f_x, i.e. $f_x = \partial_x F_x$, the transformation is in fact given by

$$F(m^2) - F(m_0^2) = \int_{m_0^2}^{m^2} d\overline{m}^2 f(\overline{m}^2) \tag{2.52}$$

and m_0 can be chosen such that $F(m_0^2) = 0$. In the context, $F_x = \log(a+x)$, the primitive of $\frac{1}{a+x}$. Since the final result must given in terms of a truncated series, the integrand becomes $f_x \simeq g_x$, or

$$\int_{m_0^2}^{m^2} d\overline{m}^2 g(\overline{m}^2) = G(m^2) - G(m_0^2) \tag{2.53}$$

such that $F_x \simeq G_x$. In other words, the primitive for the truncation must still be zero in the inferior limit of the integral;

[7]The lower-upper convention for Lorentz indices will not be followed henceforth;

one *Universal Formula* for the matching of quantum corrections from process involving only heavy internal lines. As demonstrated in last section, this formula must not entail the total set of operators behind a loop calculation, since it does not include process with light internal lines. However, from its simplicity and clarity it might be sufficient in some contexts. The next part of this work will turn this argument clear through some examples of application.

The reason of only seven integrals has been mentioned before and is justified by the truncation of the series up to dimension-six operators. Since the functions inside $\mathfrak{I}_n$ are at least linear in the fields, it follows that it must be interrupted at $n = 6$.

Although apparently $n = 6$ is a small number, the final amount of integrals may be large and arduous to be computed. In HLM all the fundamental elements to for this task were presented and the result for $\mathfrak{I}_1$ completely described. Here this result will be complemented with three new integrals, the most complex ones related with $\mathfrak{I}_2$.

The choice for what integrals to consider is motivated by the presence of different techniques during the computation, including, for instance, counting of divergences, subtraction scheme, Wick rotation, Gamma functions and master integrals, etc. In summary, it is important to identify the complete set of conceptual and technical manipulations which compose the method for treating these objects before the task of writing a final expression.

As mentioned, the first criteria corresponds the counting of divergences and the integration on m^2 can be very informative on this part. For $n \geq 1$ it automatically arises like

$$\int dm^2 \Delta^p = \frac{\Delta^{p-1}}{p-1}, \qquad p \geq 2 \quad \text{and} \quad \Delta = (q^2 - m^2)^{-1} \tag{2.56}$$

where p is at least of order two, since the effect of a q derivation is to increase the Δ degree:

$$\partial_{q_\mu} \Delta^k = -2k \times q^\mu \Delta^{k+1} \tag{2.57}$$

Next, the counting is performed for the integral in q which, due to momentum derivatives, must arise like

$$\int d^4q \; \Delta^k q^{2a} \tag{2.58}$$

where $a \in \mathbb{N}$ and, in Euclidean space

$$\begin{aligned}
\int d^4q \Delta^k q^{2a} \quad &\propto \quad i(-1)^{k+a} \int_0^\infty d\overline{q} \frac{\overline{q}^{3+2a}}{(\overline{q}^2 + m^2)^k}, \qquad \text{where} \quad \overline{q} = \overline{q}_0^2 + \overline{\mathbf{q}}^2 \\
&= \quad i\frac{(-1)^{k+a}}{2} \int_{m^2}^\infty du \frac{\overline{q}^{2(a+1)}}{u^k} \\
&= \quad i\frac{(-1)^{k+a}}{2} \int_{m^2}^\infty du \frac{(u - m^2)^{(a+1)}}{u^k} \\
&= \quad \frac{i}{2} \sum_{j=0}^{a+1} \binom{a+1}{j} (-1)^{k+a+j} (m^2)^j \int_{m^2}^\infty du \frac{u^{(a+1-j)}}{u^k}
\end{aligned} \tag{2.59}$$

which is convergent whenever $k + j - (a+1) > 1$ and, since its minimum occurs for $j = 0$, it follows

$$k > 2 + a \tag{2.60}$$

The above expression is a central criteria and will be constantly required. By looking at the definition of $\mathfrak{I}_n$ of Eq.(2.54), the object inside the brackets is operating on the outside Δ with an arbitrary number of q-derivatives. The counting can be directly made by first excluding one propagator from the integral on m^2 effect, as Eq.(2.56). If at some point the integrand for q is in the form of Δ^k such that $k > 2$, from Eq.(2.57), by applying $(2a)$ additional derivatives over Δ implies the final power $\tilde{k} \equiv k + 2a > 2 + 2a > 2 + a$, and the integrand remains convergent. This last conclusion is therefore sufficient to conclude that no divergence should be found in $\mathfrak{I}_n$ for $n > 2$. Moreover, even for $\mathfrak{I}_2$ all the pieces with derivatives must imply k with a minimum $k = 4$ with $a = 1$, such that only the coefficient for $\mathrm{tr}U^2$ must be regulated. This and other examples are completely worked in Appendix (A.3).

The integral $\mathfrak{I}_1$ was completed calculated in HLM and here only one of its components were chosen to be registered. Nevertheless, the rest of divergent terms will be calculated in detail, namely $\mathfrak{I}_0$ and $\mathfrak{I}_2$, thus supplementing the evaluation section of [32]. The finite part will consist, in general, of systematic application of the master integrals and here the equivalent pieces present in $\mathfrak{I}_2$ will saturate the examples at this topic.

2.3.1.1 Evaluating $\mathfrak{I}_0$

From the formula Eq.(2.54) it is clear that $\mathfrak{I}_0$ is a constant factor and does not play any role in the theory although it can be one important object of illustration. It follows that

$$\mathfrak{I}_0 = \int dq dm^2 \Delta \tag{2.61}$$

For this sort of function it is adequate to first perform the integral on the variable q, calculated in Appendix A.1 and given by Eq.(A.21). Thus[8]

$$\begin{aligned} \mathfrak{I}_0 &= \int dm^2 \left[\frac{i}{(4\pi)^2} m^2 \left(\log\left(\frac{m^2}{\mu^2} \right) - 1 \right) \right] \\ &= \frac{i}{(4\pi)^2} \left\{ \frac{m^4}{2} \log m^2 - \frac{m^4}{2} (\log \mu^2 + 1) - \frac{m^4}{4} \right\} \\ &= \frac{i}{(4\pi)^2} \left\{ -\frac{3}{4} m^4 + \frac{m^4}{2} \log \frac{m^2}{\mu^2} \right\} \end{aligned} \tag{2.63}$$

2.3.1.2 Evaluating $\mathfrak{I}_1$

To recapitulate, the $\mathfrak{I}_1$ is given by[9]

$$\mathfrak{I}_1 \equiv \int dq dm^2 \ \mathrm{tr}\ \Delta \left[\ -(\tilde{G}\partial_q)^2 - \{q, \tilde{G}\} \partial_q + \tilde{U} \ \right] \Delta \tag{2.64}$$

[8] From

$$\int_a^b x \log x = \frac{x^2}{2} \log x \Big|_a^b - \frac{x^2}{4} \tag{2.62}$$

[9] Along the rest of the work the notation $\mathbb{A}$ has been chosen for denoting any matrix under internal indices. However, for simplicity this must not be followed in this chapter;

where

$$\tilde{G}_{\nu\mu} \equiv \sum_{n=0}^{\infty} \frac{n+1}{(n+2)!}\left[P_{\alpha_1},\left[\cdots\left[P_{\alpha_n},[D_\nu,D_\mu]\right]\right]\right]\frac{\partial^n}{\partial q_{\alpha_1}\cdots\partial q_{\alpha_n}} \tag{2.65a}$$

$$\tilde{U} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left[P_{\alpha_1},\left[\cdots\left[P_{\alpha_n},U\right]\right]\right]\frac{\partial^n}{\partial q_{\alpha_1}\cdots\partial q_{\alpha_n}} \tag{2.65b}$$

and $P_\mu = iD_\mu$.

By assuming that m^2 commutes with the fields present both in G and U, the trace can be completely separated from the integrals, thus resulting in just a factor to the operators. The generalization for non-commuting m^2 was performed by [22] and, for simplicity, will not be reproduced here. The authors in [32] then start with $\tilde{U}$ by noting that

$$\begin{aligned}[\partial_{x_\mu}+A,U] &= (\partial_{x_\mu}\lfloor +A)\,U - U\,(\partial_{x_\mu}+A)\\ &= (\partial_{x_\mu}U)+[A,U]\end{aligned} \tag{2.66}$$

such that, whenever the internal indices are retained in a single commutator, the trace operation results a total derivative and therefore must vanish[10]. Thus, from Eq.(2.65), only $n=0$ must remain in the series such that

$$\begin{aligned}\mathfrak{I}_1 &\supset \int dq\,dm^2\ \mathrm{tr}\ \Delta\tilde{U}\Delta\\ &= \mathrm{tr}\,U\times\int dq\,dm^2\Delta^2\\ &\overset{(2.56)}{=} \mathrm{tr}\,U\times\int dq\Delta\\ &= \mathrm{tr}\,U\times I_1^0\end{aligned} \tag{2.67}$$

and, finally, from Eq.(A.21)[11]:

$$\mathfrak{I}_1 \supset i\frac{m^2}{(4\pi)^2}\left(\log\left(\frac{m^2}{\mu^2}\right)-1\right)\times\mathrm{tr}\,U \tag{2.68}$$

The authors then move to the anti-commutator piece which, for being a linear function with a commutator, must also be zero, leaving the final term to be

$$\mathfrak{I}_1 \supset -\int dq\,dm^2\ \mathrm{tr}\ \Delta\,\tilde{G}_{\mu\sigma}\tilde{G}_{\nu\sigma}\frac{\partial^2}{\partial_{q_\mu}\partial_{q_\nu}}\Delta \tag{2.69}$$

The above formula may provide the first example of *power counting*. The evaluation of $\mathfrak{I}_1$, as well as any other integral, has been truncated to include operators only up to dim-6 in the fields. During the expansion, it is important to emphasize that the covariant derivative must be counted as dim-1 object, a condition for achieving a covariant formula. Thus, $G_{\nu\mu}$ is dim-2 and, from Eq.(2.65a), both $\tilde{G}$ must cease at $n=2$, i.e.

$$\tilde{G}_{\mu\sigma} = \frac{1}{2}G_{\mu\sigma} + \frac{1}{3}\left[P_\alpha,G_{\mu\sigma}\right]\frac{\partial}{\partial q_\alpha} + \frac{1}{8}\left[P_{\alpha_2},[P_{\alpha_1},G_{\mu\sigma}]\right]\frac{\partial^2}{\partial q_{\alpha_1}\partial q_{\alpha_2}} + O(\mathrm{dim}5) \tag{2.70}$$

[10]This will certainly not be the case for the product of traceless terms;

[11]The following result differs in sign with [32], but is correct in their Universal Formula;

such that the product $\tilde{G}^2$, up to dim-6, implies:

$$\begin{aligned}\tilde{G}_{\mu\sigma}\tilde{G}_{\nu\sigma}\partial^2_{\mu\nu} &= \left\{\frac{1}{4}G_{\mu\sigma}G_{\nu\sigma}\partial^2_{\mu\nu} + \frac{1}{16}G_{\mu\sigma}\Big[P_{\alpha_2},[P_{\alpha_1},G_{\nu\sigma}]\Big]\partial^2_{\alpha_1\alpha_2}\partial^2_{\mu\nu} + \frac{1}{9}[P_\alpha,G_{\mu\sigma}][P_\beta,G_{\nu\sigma}]\partial^2_{\alpha\beta}\partial^2_{\mu\nu} + \right.\\ &\left. + \frac{1}{16}\Big[P_{\alpha_2},[P_{\alpha_1},G_{\mu\sigma}]\Big]G_{\nu\sigma}\partial^2_{\alpha_1\alpha_2}\partial^2_{\mu\nu}\right\} \qquad (2.71)\end{aligned}$$

where it was made implicit that ∂_μ stands for 4-momentum indices. Moreover, the $O(P^5)$ vanish since they are odd under the integrated momentum[12].

From the criteria for convergence of Eq.(2.60) all the terms with four derivatives would correspond at most to $k=5$ and $a=2$, being therefore convergent. Thus, only the dim-4 operator above must present a regulated coefficient. The next result is similarly presented in HLM and here reproduced for a matter of fixing the notation for the Appendix. Thus,

$$\begin{aligned}\int dq dm^2 \Delta\partial^2_{\mu\nu}\Delta &\overset{(2.56)}{=} 2g_{\mu\nu}\int dq\left(-\frac{1}{2}\Delta^2+\frac{1}{3}q^2\Delta^3\right)\\ &\overset{(A.15)}{=} 2g_{\mu\nu}\left(-\frac{1}{2}(iA_2)+\frac{1}{3}(iA_2)\right)\\ &= -i\frac{g_{\mu\nu}}{3}A_2 \qquad (2.72)\end{aligned}$$

It must be noted that the replacement $q_\mu q_\nu \to \frac{q^2}{4}g_{\mu\nu}$ must only be performed at the stage of integration. From Eq.(A.15) the A_2 is given by

$$A_2 = \frac{\Gamma\left(\frac{\epsilon}{2}\right)}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} \qquad (2.73)$$

which can be expanded into

$$A_2 = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon}+\psi(2)+\log\left(\frac{4\pi\mu^2}{m^2}\right)\right) \qquad (2.74)$$

Since in the $\overline{MS}$ scheme the pole is subtracted along with $\log(4\pi)$ and the Euler-Mascheroni constant γ present in $\psi(2)=1-\gamma$, it follows, finally, that

$$A_2 \overset{\overline{MS}}{\to} \frac{1}{(4\pi)^2}\left(1-\log\left(\frac{m^2}{\mu^2}\right)\right) \qquad (2.75)$$

or

$$\mathfrak{I}_1 \supset \frac{i}{(4\pi)^2}\frac{1}{12}\left(1-\log\left(\frac{m^2}{\mu^2}\right)\right)\times \mathrm{tr}\,(G_{\mu\nu}G_{\mu\nu}) \qquad (2.76)$$

After the $\mathfrak{I}_1$ analysis, the authors of HLM simplify the above results through systematic application of the covariant derivative properties that are partially reproduced in Appendix A.2.

[12] For Lorentz violating dim-5 operators, see [6].

2.3.1.3 Evaluation of $\mathfrak{I}_2$

This section intents to extract the last divergent part of the formula for dim-6 operators in an effective Lagrangian at quantum level. From Eq.(2.54) the $\mathfrak{I}_2$ can be expressed like,

$$\mathfrak{I}_2 \equiv \int dq dm^2 \ \mathrm{tr} \ \Delta \left[\ -(\tilde{G}\partial_q)^2 - \{q,\tilde{G}\}\,\partial_q + \tilde{U} \ \right] \Delta \left[\ -(\tilde{G}\partial_q)^2 - \{q,\tilde{G}\}\,\partial_q + \tilde{U} \ \right] \Delta \tag{2.77}$$

By power counting under the dim-6 criteria the following items can be selected

$$\begin{aligned}
\mathfrak{I}_2 \quad \supset \quad & \int dq dm^2 \ \mathrm{tr}\Big\{\Delta \ (\tilde{G}\partial_q)^2 \ \Delta \ \{q,\tilde{G}\}\,\partial_q \ \Delta - && \text{(2.78a)}\\
& -\Delta \ (\tilde{G}\partial_q)^2 \Delta \ \tilde{U} \Delta + && \text{(2.78b)}\\
& +\Delta \ \{q,\tilde{G}\}\,\partial_q \lfloor \ \Delta \ (\tilde{G}\partial_q)^2 \ \Delta + && \text{(2.78c)}\\
& +\Delta \ \{q,\tilde{G}\}\,\partial_q \lfloor \ \Delta \ \{q,\tilde{G}\}\,\partial_q \ \Delta - && \text{(2.78d)}\\
& -\Delta \ \{q,\tilde{G}\}\,\partial_q \lfloor \ \Delta \ \tilde{U} \ \Delta - && \text{(2.78e)}\\
& -\Delta \ \tilde{U} \ \Delta \ \{q,\tilde{G}\}\,\partial_q \ \Delta - && \text{(2.78f)}\\
& -\Delta \ \tilde{U} \ \Delta \ (\tilde{G}\partial_q)^2 \ \Delta + && \text{(2.78g)}\\
& +\Delta \ \tilde{U} \ \Delta \ \tilde{U} \ \Delta\Big\} && \text{(2.78h)}
\end{aligned}$$

In summary, only one term must be neglected a priori, namely, $(\tilde{G}\partial_q)^2(\tilde{G}\partial_q)^2$, for producing operators of, at least, dim-8. On the task of extracting divergences it is worthy to argue the above items in separate:

a. The criteria Eq.(2.60) must be applied after the observation of Eq.(2.56), i.e. the mass integration, that is equivalent to exclude one of the Δ's during the counting. For dim-6 implies that only the $n = 0$ in the series of both $\tilde{G}$ is being taken into account. From Eq.(2.57), the final power for the propagators must be $k = 5$, while $a = 2$. Therefore, the term is **convergent**;

b. Here, by power counting, both the $n = 0,1$ in the expansion for $\tilde{U}$ must be taken. For $n = 1$, however, the integral of q has an old integrand and a vanishing result. For $n = 0$, the counting leads to $a = 1$ and $k = 4$ and no divergence will be present here;

c. The same conclusions for the first line are made to this case;

d. This is a potentially divergent term that will require a more particular treatment over the anti-commutator piece. From power counting, the series for both $\tilde{G}$ must run for $m,n \in [0,1]$, simultaneously[13]. For $(m,n) = (0,0)$, the integrand presents $a = 2$ and $k = 4$, thus breaking

[13] A requirement from the parity on q and disallowance of dim-5 operators;

the convergence criteria. Nevertheless, explicitly it is given by

$$\begin{aligned} 2.78d &= \Delta\ \{q,G\}\partial_q \lfloor \Delta\ \left[q^\mu G_{\nu\mu}\partial^\nu\ \Delta\right] \\ &= \Delta\ \{q,G\}\partial_q \lfloor \Delta\ \left[-2G_{\nu\mu}q^\mu q^\nu\ \Delta\right] \\ &= 0 \end{aligned} \tag{2.79}$$

as a consequence of the $G_{\nu\mu}$ antisymmetry. This result might suggest one incorrect conclusion that the anti-commutators would always vanish for the same field-strength property. To elucidate it, the complete case for $(m,n) = (1,1)$ is treated in Appendix A.3 and may provide an useful intuition on the application of the series expansion during the matching. For completeness, here $a = 1$ and $k = 4$, i.e. the integral is finite;

e. This is also a potentially divergent piece. From power counting, the powers to be analyzed where $m,n \in \{0,1\}$ in the $\tilde{G}$ and $\tilde{U}$ series. Here, if $m+n$ is an odd number they should vanish from the parity of q[14]. For $(m,n) = (1,1)$, the convergence criteria reads with $a = 2$, $k = 5$, the coefficient is finite and has been computed in Appendix A.3 . The term for $(m,n) = (0,0)$ vanishes from $G_{\nu\mu}$ anti-symmetry;

f. Equivalent to the previous case;

g. Equivalent to item 2.78b;

h. Finally, this is the term that could render divergences. Again, from q-parity, each term in the series must run simultaneously. The $(m,n) = (0,0)$ is certainly divergent, since $a = 0$ and $k = 2$. For $(m,n) = (1,1)$, it follows $a = 1$ and $k = 4$, thus corresponding a finite coefficient.

Therefore, to the total $\mathfrak{I}_2$, only the Wilson coefficient for $\mathrm{tr}(U^2)$ must be renormalized, what is extracted straightforwardly from the identities of Appendix A.1 and results:

$$\begin{aligned} \mathfrak{I}_2 &\supset \mathrm{tr}(U^2) \times \int dq dm^2\ \Delta^3 \\ &\overset{(2.56)}{=} \frac{\mathrm{tr}(U^2)}{2} \times \int dq \Delta^2 \\ &\overset{(A.14a)}{=} i\frac{\mathrm{tr}(U^2)}{2} \times A_2 \end{aligned} \tag{2.80}$$

Finally[15],

$$\mathfrak{I}_2 \supset \frac{i}{2(4\pi)^2}\left(1 - \log\left(\frac{m^2}{\mu^2}\right)\right) \times \mathrm{tr}(U^2) \tag{2.81}$$

[14]If the number of momentum and derivatives is odd then, by simple counting in the power of q, the total integrand emerges like an odd function;

[15]The result does not agree with the universal formula in HLM, but is corrected in [53];

2.3.2 The Universal Formula

The previous section presented the totality of Wilson coefficients whose divergences were regularized and then subtracted via the $\overline{MS}$ scheme. In order to recapitulate, the work on those integrals involves the search for a final expression to the quantum correction of the Effective Action whenever it can be expressed in the form

$$S_{\text{eff}}^{(1)} \supset ic_s \text{Tr}\log(-P^2+m^2+U_x) \tag{2.82}$$

which in general corresponds to non-mixed terms of heavy particles in the original UV theory. It is worthy to note the above choice for the symbol S_{eff} instead of Γ. Such conversion will be clarified in the next section.

Along the derivation, a power-counting over the fields was consistently performed and the series truncated at dimension-six operators, one condition that will be made clear at the next chapter. From the definition of a *Covariant Derivative Expansion*, $P^\mu = iD^\mu$ was preserved intact along the derivation and counted like an one-dimensional field. At the other hand, the piece U_x, an arbitrary function of SM fields, can assume order one or two, depending on the nature of the correspondent vertex. Furthermore, the mass matrix m^2 has been presumed to commute with both U and the field-strength.

On what follows, the formula is presented including the notation and results registered in the updated version [33]. The finite coefficients are assumed to be correct and have been reproduced in subsequent extensions like [28] and [53]. Finally,

$$\begin{aligned}
\mathscr{L}_{\text{eff}}^{(1)} \supset\ & \frac{c_s}{(4\pi)^2}\text{tr}\Bigg\{\frac{m^4}{2}\left[\frac{3}{2}-\log\left(\frac{m^2}{\mu^2}\right)\right]+m^2\left[\left(1-\log\left(\frac{m^2}{\mu^2}\right)\right)U\right]+\\
&+m^0\left[\frac{1}{2}\left(1-\log\left(\frac{m^2}{\mu^2}\right)\right)U^2-\frac{1}{12}\left(1-\log\left(\frac{m^2}{\mu^2}\right)\right)G_{\mu\nu}G^{\mu\nu}\right]+\\
&+\frac{1}{m^2}\left[\frac{1}{12}\left[D_\mu,U\right]^2-\frac{1}{12}UG_{\mu\nu}G^{\mu\nu}-\frac{1}{6}U^3+\frac{1}{60}\left[D^\mu,G_{\mu\nu}\right]^2-\frac{1}{90}G_\mu^{\nu}G_\nu^{\rho}G_\nu^{\rho}\right]+\\
&+\frac{1}{m^4}\left[\frac{1}{24}U^4-\frac{1}{12}U\left[D_\mu,U\right]^2+\frac{1}{120}\left[D^\mu,\left[D_\mu,U\right]\right]^2+\frac{1}{60}\left[D_\mu,U\right]\left[D_\nu,U\right]G^{\mu\nu}+\right.\\
&\left.+\frac{1}{40}U^2G_{\mu\nu}G^{\mu\nu}+\frac{1}{60}\left[U,G_{\mu\nu}\right]\left[U,G^{\mu\nu}\right]\right]+\\
&+\frac{1}{m^6}\left[\frac{1}{20}U^2\left[D_\mu,U\right]^2-\frac{1}{60}U^5+\frac{1}{30}U\left[D_\mu,U\right]U\left[D^\mu,U\right]\right]+\\
&+\frac{1}{m^8}\left[\frac{1}{120}U^6\right]\Bigg\}
\end{aligned} \tag{2.83}$$

where it is important to remark the minus sign in the definition of Eq.(2.51).

One important conclusion immediately extracted from the above expression is invariance under the symmetry of the original Lagrangian. As a brief review, for a theory based on arbitrary fields Ψ and symmetric under the transformation

$$\Psi \to \Psi' = V\Psi \tag{2.84}$$

the covariant derivative is defined such that

$$(D_\mu \Psi) \to (D_\mu \Psi)' = V(D_\mu \Psi) \quad \text{or} \quad D_\mu \to D'_\mu = V D_\mu V^{-1} \tag{2.85}$$

The commutator of these objects defines de gauge-field strength

$$[D_\mu, D_\nu] \propto F_{\mu\nu} \quad \text{and, from Eq.(2.85),} \quad F_{\mu\nu} \to F'_{\mu\nu} = V F_{\mu\nu} V^{-1} \tag{2.86}$$

At this point, therefore, the presence of a trace over the internal indices is sufficient to demonstrate the invariance of Eq.(2.83) under V:

$$\mathrm{tr}[VAV^{-1}] = \mathrm{tr}A \tag{2.87}$$

Apart from that, the Universal Formula express the real meaning of quantum corrections to the Effective Action. Although the independent powers for the heavy mass exclusively represents a 1-loop computation, it may be noted that the arbitrary function U is present at different levels of the expansion. This function is what in fact contains the small perturbative parameter and, therefore, the formula contain corrections associated with different n-point functions.

As emphasized during the power counting, the covariant derivative is necessarily of dimension one and the function U can be of dim-2, for example, in the scalar sector. In this context, the formula breaks in an even more simplified version. The most complete case, at the other hand, will certainly occur with trilinear vertex of scalars, what forces the presence of couplings with positive dimension and these functions to be of order one.

Despite of its closed and model-independent form, the UF is limited to the case where the internal lines defining the Effective Action contain exclusively the same field. Nevertheless, a priori there is not a decisive argument that could avoid the additional Wilson coefficients coming from mixed heavy and light internal lines. This scenario motivated the authors of HLM to update their original work into a new version including this topic, namely in [33] and improved in [28] and [53]. As it can be seen, therefore, it has been a very recent and interesting topic of discussion.

The next section is centered on the matter of mixed corrections and the attempt of performing a similar matching through a covariant expansion.

2.4 Evaluating the Functional Determinant - On the Mixed Terms

The last section has tried to elucidate some technical aspects behind the computation of a functional determinant, but did not clarify the conceptual meaning behind the term Effective Theory. One example is the constant lack of consistency on calling the Log correction as an action. The present section develops a more formal presentation to the logical path before the construction of an EFT and its correct connection with Effective Actions. To achieve this task it is important to consider the general case where corrections from both light and internal lines are equally present.

It has been shown that the first term defining the *Effective Action* is just composed by the classical action, i.e. by the functional which defines the theory. Thus, the action is already a generator functional. Although the One-Particle Irreducible graphs are fundamental elements for the analytical properties of a given model, they do not comprise the overall set of n-point functions present in the S-matrix - the classical action can be considered for a tree-level analysis, i.e. for representing any type of process exclusively through tree-level diagrams. The Log correction, therefore, will be used as a source of more accurate predictions, then including in the description the independent elements of the ultraviolet structure of the theory. The 1PI are generated by the Effective Action and the Effective Theory will be constructed by taking advantage on the theoretical strength of this functional generator.

The Lagrangian for the EFT is by definition written as a general series over a set of operators. Their coefficients are just weights on that specific process represented by the matrix element. If the series is defined *a priori* this would be a model independent Effective Theory, a *bottom-up* proposal emerged from principles that will follow a previous knowledge or a primary assumption about the particular phenomena. In many contexts this effective proposal is considered sufficient to the specific analysis - All the operators, for example, that could be generated through Log corrections are then assumed to be out of the current precision. However, the quantum correction to the EFT Effective Action can be computed with no restriction. Analogously, this would correspond to improve the *predictions* in higher orders for the Wilson coefficients, turn them more accurate.

Here, however, the treatment follows a different direction. The Effective Theory will be built from a previous hypothesis on the Ultra-Violet (UV for short) complete model. The purpose is still to raise a series of higher-dimension operators, but now from a different perspective - The hypothesis of a UV complete model, more general than the SM, is necessarily accompanied by the phenomenological fact that any new heavy field, with independent properties, has appeared asymptotically in the experiments. Since these UV versions are in general more complicated, it is important to develop a technique to simplify their analysis. By considering them entirely during the computation of a particular process would represent a large step compared with the experiments achievement, apart from the expected complexity likely to be required. The EFT at this level arises as one attempt to reach new physics information in a more gradual manner. This will define a method and the Effective Actions the most important tool.

From the Decoupling Theorem (see Chapter 3) it is known a priori that if a theory intrinsically contains a heavy sector, this can be eliminated of the complete model by just cutting these heavy lines out of any graph, i.e. by considering exclusively its renormalizable low-energy sector. The procedure consists in a redefinition of fields and couplings of the low sector and - the most important feature - is such that the error committed decreases with the heavy scale. Now, assuming one particular UV theory is to assume the Standard Model as this low-energy model. Since the Effective Action is related with the ultraviolet property of the complete theory, it is therefore the adequate object to translate any information about these heavy particles to inside the constant couplings of these first corrections. Thus, to propose dim-6 operators is to propose a representation to the error committed in assuming the Standard Model as a final theory for describing process at intermediary energy scales.

The composition of $\mathscr{L}_{\text{EFT}}$ must change as the *matching* between the Effective Actions Γ_{EFT} and Γ_{LUV} is done in different order from the saddle-point expansion. These previous sentences are then stating a clear separation into an Effective Theory and a Light-UV Theory. The latter, however, will be definitely present during the rise of the former, following a type of recursive construction.

In summary, there are present two important and distinct concepts for the method:

- Γ_{LUV}: The Effective Action for the Light-UV model, i.e. the theory where heavy degrees of freedom are disallowed to appear as external fields, is constructed by following the steps:

  1. Make the Effective Action for the UV theory, including all the fields at the same level and by following, for example, the method of saddle point approximation for extracting corrections, expanding all the fields around their background.
  2. Afterwards, replace the heavy fields by their classical, and non-local, solution. At this context, there will not be an expansion into local terms.
  3. In this sense, the Effective Action at classical level[16] for the theory of Eq.(2.5) will just be given by Eq.(2.9).
  4. The final and unique model is still the UV theory, but the Effective Action may be represented exclusively by a particular set of fields. The first correction for Γ_{LUV} will not be constructed from the Lagrangian in Eq.(2.9) but from the correspondent correction for the effective action of the complete Eq.(2.5). Again, after this resolution, the heavy fields are then replaced by their classical and non-local representation.

- Γ_{EFT}: The matching is performed at the level of Effective Actions and, therefore, the Effective Field Theory must emerge recursively, according to the desired level of accuracy for the Wilson coefficients. In other words, each level for the matching, either at classical or at quantum level, will imply different and *independent* theories.

  1. The first theory, i.e. the starting point, will certainly be present in any of the improved versions and is defined by equating the $\Gamma^{(0)}_{\text{EFT}}[\phi]$ with $\Gamma^{(0)}_{\text{LUV}}[\hat{\Phi}_c[\phi],\phi]$ when the classical fields Φ_c are made local $\Phi_c \to \hat{\Phi}_c$, employing the notation from [53]. Since $\Gamma^{(0)}$ is just the classical action, the theory of $S^{(0)}_{\text{eff}}$ will be the local version of $S_{\text{LUV}} \equiv \Gamma^{(0)}_{\text{LUV}}$. In the example of Section 2.1, the $\mathscr{L}^{(0)}_{\text{eff}}$ is given by Eq.(2.9) after the insertion of the expansion Eq.(2.12). This theory may be represented, for instance, as the SM plus n

[16] i.e. with no determinants;

higher-dimension operators:

$$\mathscr{L}_{\text{eff}}^{(0)} = \mathscr{L}_{\text{SM}} + \sum_{i=1}^{n} c_i^{(0)} \mathscr{O}_i \tag{2.88}$$

Regardless its origin, $\mathscr{L}_{\text{eff}}^{(0)}$ must be phenomenologically constrained and theoretically explored at any order in its coupling constants .

2. The next theory will certainly entail the previous one but now including a new set of operators apart from additional contributions to the zeroth coefficients. Since this will be considered the ultimate effective theory, it can be written without superscript:

$$\mathscr{L}_{\text{eff}} \equiv \mathscr{L}_{\text{eff}}^{(0)} + \mathscr{L}_{\text{eff}}^{(1)} = \mathscr{L}_{\text{SM}} + \sum_{i=1}^{n} \left(c_i^{(0)} + c_i^{(1)} \right) \mathscr{O}_i + \sum_{j=n+1}^{p} c_j^{(1)} \mathscr{O}_j \tag{2.89}$$

As mentioned before, $\mathscr{L}_{\text{eff}}^{(1)}$ is obtained recursively by equating the quantum correction $\Gamma_{\text{LUV}}^{(1)}[\Phi_c[\phi], \phi]$, expanded into local terms after the Trace computation and truncated by power counting, with the equivalent $\Gamma_{\text{EFT}}^{(1)}[\phi]$, which now includes a determinant of $\frac{\delta^2 S_{\text{eff}}^{(0)}}{\delta \phi^2}$, i.e. the *quantum correction to the Effective Action* of the theory $\mathscr{L}_{\text{eff}}^{(0)}$.

The formal clarification, the consequences and points on how the matching is performed will be treated in detail along the next section.

2.4.1 The Matching Procedure

Before moving to the first application of matching by functional methods, one final procedure must also be developed and, if possible, by involving the same transparent steps as those behind the universal formula. The work [33] was dedicated to address this topic and, although it has been later extended by the authors in [28] and [25], here most of their method must be preserved, apart from one unpretentious attempt to leave the formulas with a simpler aspect. It will be seen that the task of performing a Covariant Derivative Expansion, i.e. of matching while maintaining the covariant derivative intact, is achieved through the expansion of the 1LUV Effective Action into a series of local operators after the Trace computation, being the truncation defined according to a power counting fixed a priori.

On what follows, first it will be done a brief but complete presentation of the entire set of Wilson coefficients from the formalism developed in [33]. At this point, the review is general but covers in particular the Chapter 3 of the referred article. Once the coefficients related to the Universal Formula are fully identified, the focus are then directed to the additional terms, there called *mixed* terms, by manifest reasons. At this point it will be argued that the final form of mixed operators remains in a sense obscure in their work, with a general trace that, although correct, is exhibited without a consistent support.

The present section is in fully agreement on the incontestable importance of all the concepts and results presented in [33], apart from their accuracy. Here, however, is aimed to find the same universality as in Eq.(2.83), via one additional formula, preserving at the same time the conceptual richness contained in both HLM articles.

The Matching procedure may be considered as the *Correspondence Principle* for Effective Theories - The Ultra-Violet complete model and its Effective variant must agree at some scale μ.

In general the matching scale is assumed to be determined by the mass of the particles that were integrated out of the UV theory, since it is being considered excessively heavy to emerge as external degrees of freedom. The procedure consists on equating both theories represented by their respective Effective Actions. If the φ is chosen to represent the complete set of Standard Model fields, i.e. those present as asymptotic particles, then Φ will be denoting the counterpart, a set of fields running exclusively through virtual processes. In the context of this *Standard Model Effective Field Theory* the actions will be represented by

$$S_{UV}[\varphi,\Phi] = S_{SM}[\varphi] + S_{\Phi}[\varphi,\Phi] \tag{2.90a}$$

$$S_{EFT}[\varphi] = S_{SM}[\varphi] + \sum_i c_i(\mu)\, O_i[\varphi] \tag{2.90b}$$

At $\mu = M$, the heavy scale, the 'Light' version of the UV must define the EFT, order-by-order, at the level of *Effective Actions*:

$$\Gamma^{(0)}_{EFT}[\varphi] = \Gamma^{(0)}_{L,UV}[\varphi], \quad \text{at} \quad \mu = M \tag{2.91}$$

$$\Gamma^{(1)}_{EFT}[\varphi] = \Gamma^{(1)}_{L,UV}[\varphi], \quad \text{at} \quad \mu = M \tag{2.92}$$

The UV Effective Action, at the *Log* order is given by

$$\Gamma_{UV}[\varphi,\Phi] = S_{UV}[\varphi,\Phi] + i\alpha \log\det\left(\frac{\delta^2 S_{UV}}{\delta(\varphi,\Phi)^2}\right) \tag{2.93}$$

and, by definition, is being expressed in terms of background fields.

The $\Gamma_{L,UV}$ is then constructed after Γ_{UV} and through the replacement of Φ as an implicit functional of the light fields, from the solution of the classical equation of motion

$$\left.\frac{\delta S_{UV}[\varphi,\Phi]}{\delta\Phi}\right|_{\Phi_c[\varphi]} = 0 \tag{2.94}$$

Here it will be chosen a representation for Φ_c motivated by both the notation in [33] and the general aspect of phenomenologically relevant Lagrangians, with a tree-level piece like

$$\mathscr{L}_{UV} \supset \frac{1}{2}\Phi\, \mathscr{O}^{\Phi}\, \Phi \;+\; \frac{1}{2}\varphi\, \mathscr{O}^{\varphi}\, \varphi \;-\; \Phi B_{\varphi} \;+\; Q_{\Phi} \tag{2.95}$$

where Q_{Φ}, a 'quartic' operator may depend on N-power of heavy fields, with $N \in [3,4]$. The operator $\mathscr{O}^{\Phi}$ may also be a function of the light fields, $\mathscr{O}^{\Phi} = \mathscr{O}^{\Phi}(\varphi)$ and, as one example, in the case of a Scalar Fields it would be

$$\mathscr{O}^{\Phi} \equiv [\Delta^{\Phi}]^{-1} - A(\varphi), \qquad \Delta^{\Phi} \equiv [P^2 - M^2]^{-1} \tag{2.96}$$

As before, $P^\mu \equiv iD^\mu$. The equation defining the classical Φ_c, Eq.(2.94), is chosen to be solved linearly, discarding the contribution from Q, which results in

$$\Phi_c[\varphi] = \left[\mathcal{O}^\Phi\right]^{-1} B_\varphi \tag{2.97}$$

The Effective Action for the LUV, the fundamental object behind the construction of a Effective Theory, will be given by

$$\Gamma_{L,UV}[\varphi] = \Gamma_{UV}[\varphi, \Phi_c[\varphi]] \tag{2.98}$$

At the EFT side the Effective Action requires a thorough conceptual analysis. The action in Eq.(2.90b) is the final representation of a theory which is being constructed recursively and, therefore, remains as an symbolic object. At the quantum level , the correspondent Effective Action is given by the familiar formula

$$\Gamma_{EFT}[\varphi] = S_{EFT}[\varphi] + i\alpha \log\det\left(\frac{\delta^2 S_{EFT}[\varphi]}{\delta\varphi^2}\right) \tag{2.99}$$

Since the Wilson coefficients are determined after the local expansion of $\Gamma_{L,UV}$, order-by-order, the inclusion of the *Log* term in the equation for the still undetermined $c_i^{(1)}$'s is only meaningful once it contains only the zeroth-order action, $S_{EFT}^{(0)}$, emerged after the tree-level matching. It is crucial to remark that in terms of the action for the EFT - instead of the *Effective Action* - the upper-index is denoting the order in which the matching and the Wilson coefficients were extracted. That is to say that the theory is being constructed by steps and following $S_{EFT} = S_{EFT}^{(0)} + S_{EFT}^{(1)} + \cdots$. The logical significance of Eq.(2.99) is acquired by

$$\begin{aligned}
\Gamma_{EFT}[\varphi] &= S_{EFT}[\varphi] + i\alpha \log\det\left(\frac{\delta^2 S_{EFT}^{(0)}[\varphi]}{\delta\varphi^2}\right) \\
&\overset{(2.90)}{=} S_{SM}[\varphi] + \sum_{i=1}^{N} c_i(\mu)\, O_i[\varphi] + i\alpha \log\det\left(\frac{\delta^2 S_{EFT}^{(0)}[\varphi]}{\delta\varphi^2}\right) \\
&= S_{SM}[\varphi] + \sum_{m=1}^{p} \left(c_m^{(0)} + c_m^{(1)}\right) O_m[\varphi] + \sum_{j=p+1}^{N} c_j^{(1)}\, O_j[\varphi] \\
&\quad + i\alpha \log\det\left(\frac{\delta^2 S_{EFT}^{(0)}[\varphi]}{\delta\varphi^2}\right)
\end{aligned} \tag{2.100}$$

where the split of the Wilson coefficients is just turning explicit that the matching at quantum level can originate a new set of operators. Finally, at zeroth-order the Effective Action for the EFT is given by

$$\Gamma_{EFT}^{(0)}[\varphi] = S_{EFT}^{(0)} = S_{SM}[\varphi] + \sum_{m=1}^{p} c_m^{(0)}\, O_m[\varphi] \tag{2.101}$$

and, at the *Log* order,

$$\Gamma_{EFT}^{(1)}[\varphi] = \sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] + i\alpha \log\det\left(\frac{\delta^2 \Gamma_{EFT}^{(0)}[\varphi]}{\delta\varphi^2}\right) \tag{2.102}$$

where $N \geq p$. Note, therefore, that the presence of $S^{(1)}_{EFT}$ in the above expression provides the equation for the Log correction of the EFT, such that the matching $\Gamma^{(1)}_{EFT}[\varphi] = \Gamma^{(1)}_{L,UV}[\varphi]$, is now represented by

$$\sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] + i\alpha \log\det\left(\frac{\delta^2\Gamma^{(0)}_{EFT}[\varphi]}{\delta\varphi^2}\right) = i\alpha\log\det\left(\frac{\delta^2 S_{UV}[\varphi,\Phi]}{\delta(\varphi,\Phi)^2}\Bigg|_{\Phi_c[\varphi]}\right) \tag{2.103}$$

which must be solved for $c_i^{(1)}$ once the r.h.s is expanded through a covariant series of local terms. Having found the corrections to $c_i^{(0)}$, and possible new operators, the EFT is then redefined and corresponds a new and independent theory, more accurate than that raised by the tree-level matching.

The Effective Field Theory is, in general, experimentally constrained only at linear order on the final c_i. That is to say that the action defining the final EFT is explored like a classical generator functional. Thus, the n-point function behind the process of interest is given by the correspondent Feynman diagram with a single dot, usually expressed like $\otimes$, i.e[17]

$$\frac{\delta^n S_{EFT}}{(\delta\phi)^n} \equiv\ c_i \otimes \qquad \text{for}\ \ O_i = \varphi_1\cdots\varphi_n \tag{2.106}$$

The reason for not considering higher perturbative corrections on c_i, for a specific process, is comprehensible - given the observable, the matching at quantum level is assumed to be sufficient precise, and the *loop* corrections including light particles can be summarized from the Renormalization Group Equations. The $c_i(M)$ provide the initial condition to the RGE, which then provide their appropriate evolution to the scale where the matrix element $\langle O_i\rangle$ is in fact evaluated.

As clarified in the Section 3.1, the separation into high and low energy regimes for coefficients and operators, respectively, is one of the most important step composing the method for investigate phenomenology via the EFT approach. Here, it corresponds to the fact that the information about the propagation of the heavy particles is all contained in the Wilson coefficients to the local operators. The evaluation of $\langle O_i\rangle$ is usually performed by non-perturbative methods and, even for classical processes[18], like Kaon decays, still retain large uncertainties.

The task of running $c_i(M)$ down to the scale μ is motivated by the following hypothesis - The improvement of non-perturbative techniques in the determination of the matrix element $\langle O_i\rangle$ will not be sufficient to fully explain the particular anomaly being analyzed. The possible

[17]The Feynman rule can be more transparent in the momentum space:

$$\frac{\delta^n S_{EFT}}{\delta\phi_1\cdots\delta\phi_n} \propto\ c_i\prod_{i=2}^{n}\delta(x_i - x_1) \tag{2.104}$$

which converts into

$$G_p^{(n)} \propto c_i \int d\mathbf{x}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\prod_{i=2}^{n}\delta(x_i - x_1) = c_i\delta\left(\sum_{i=1}^{n} p_i\right) \tag{2.105}$$

where $\mathbf{a} \equiv (a_1, \cdots, a_n)$ and the final delta is therefore representing the momentum conservation;

[18]Here, by *classical* it is meant *historically relevant*;

disagreement between observable and theory should be explained by a precise determination of the Wilson coefficients. The persistence of a anomaly might be a sign of New Physics and the current $\Delta I/2$ rule from the QCD corrections for meson decays is one of the most important examples of this philosophy [21].

The main topic of this section consists on solving the Eq.(2.103) to the $c_i^{(1)}$, what involves one additional step related to the fact that now $\frac{\delta^2 S_{UV}[\varphi,\Phi]}{\delta(\varphi,\Phi)^2}$ is an Hessian matrix including field indices, i.e.

$$\mathscr{H} \equiv \frac{\delta^2 S_{UV}[\varphi,\Phi]}{\delta(\varphi,\Phi)^2} = \begin{pmatrix} \frac{\delta^2 S}{\delta\Phi\delta\Phi} & \frac{\delta^2 S}{\delta\Phi\delta\varphi} \\ \frac{\delta^2 S}{\delta\varphi\delta\Phi} & \frac{\delta^2 S}{\delta\varphi\delta\varphi} \end{pmatrix} \tag{2.107}$$

In the Appendix of [33], however, the authors have shown one important identity that permits $\mathscr{H}$ to be written as a sum of two independent logarithms, what finally provides a very logical meaning to the form of Eq.(2.103). It follows:

$$\log\det\left(\frac{\delta^2 S_{UV}[\varphi,\Phi]}{\delta(\varphi,\Phi)^2}\bigg|_{\Phi_c[\varphi]}\right) = \log\det\left(\frac{\delta^2 S_{UV}[\varphi,\Phi]}{\delta\Phi^2}\bigg|_{\Phi_c[\varphi]}\right) + \log\det\left(\frac{\delta^2\Gamma^{(0)}_{L,UV}[\varphi]}{\delta\varphi^2}\right) \tag{2.108}$$

where $\Gamma^{(0)}_{L,UV}[\varphi] = S_{UV}[\varphi,\Phi[\varphi]]$. Thus, the matching equation, at *Log* level, converts into

$$\begin{aligned} \sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] \quad &= \quad i\alpha\log\det\left(\frac{\delta^2 S_{UV}[\varphi,\Phi]}{\delta\Phi^2}\bigg|_{\Phi_c[\varphi]}\right) + \\ &+ i\alpha\left\{\log\det\left(\frac{\delta^2\Gamma^{(0)}_{L,UV}[\varphi]}{\delta\varphi^2}\right) - \log\det\left(\frac{\delta^2\Gamma^{(0)}_{EFT}[\varphi]}{\delta\varphi^2}\right)\right\} \end{aligned} \tag{2.109}$$

The above equation implies a strong simplification on the procedure. The first line can be promptly identified with the matter of the previous section and its relative operators will then be extracted from the Universal Formula of Eq.(2.83). The correspondent coefficients was denoted by the authors of [33] like $c_{i,\text{heavy}}$, since they are connected with diagrams including only heavy internal lines.

The second term, inside the brackets in Eq.(2.109), is a new object and the correspondent Wilson coefficients were called $c_{i,\text{mixed}}$ by HLM. Here, none of these terminology will be adopted, despite of their importance in a conceptual level. Notwithstanding, the main question to be addressed is what is the real difference between these two similar Logs.

As has been stated recurrently, the determination of c_i is done through a local expansion of $\Gamma^{(0)}_{L,UV}$ in Eq.(2.109) up to some predetermined power in the fields. The $\Gamma^{(0)}_{EFT}$ is originated by a power counted truncation performed at the linear level. Notwithstanding, now the expansion, made a posteriori, is in a non-linear context. This justify the fact that the difference inside the brackets does not trivially vanishes - $\Gamma^{(0)}_{L,UV}$ is intact a priori and only expanded as the argument of a Log function. The correspondent series will certainly contain the $\Gamma^{(0)}_{EFT}$ piece, but in such a way that the cancellation still have operators on the desired field dimension. In summary, the power counting at the quantum level is independent of that performed linearly, what in these terms sounds like a reasonable statement.

The precise cancellation of common terms and the counting in this framework can be explicitly achieved by considering a generic representation for the UV Lagrangian as Eq.(2.95). The classical Effective Action for the LUV corresponds simply to the action $S_{UV}[\varphi,\Phi[\varphi]]$, or:

$$\Gamma^{(0)}_{L,UV} = \int_x \left(\frac{1}{2}\Phi_c\, \mathcal{O}^{\Phi}\, \Phi_c + \frac{1}{2}\varphi\, \mathcal{O}^{\varphi}\, \varphi \; - \; \Phi_c B_{\varphi} \; + \; Q_{\Phi_c}\right) \tag{2.110}$$

and, from Eq.(2.97), it follows that

$$\Gamma^{(0)}_{L,UV} = \int_x \left(\frac{1}{2}\varphi\, \mathcal{O}^{\varphi}\, \varphi \; - \; \frac{1}{2}B_{\varphi}\, \left[\mathcal{O}^{\Phi}\right]^{-1}\, B_{\varphi} \; + \; Q_{\Phi_c}\right) \tag{2.111}$$

The series to be truncated during the raising of $\Gamma^{(0)}_{EFT}$ usually comes exclusively from $\left[\mathcal{O}^{\Phi}\right]^{-1}$, since the Q_{Φ_c} is expected to break the adopted power counting up dim-6. Here this truncation will be denoted like:

$$\left[\mathcal{O}^{\Phi}\right]^{-1} = \overline{\mathcal{O}}^{\Phi}_{(0)} + \mathcal{O}^{\Phi}_{(1)} \tag{2.112}$$

where the 'bar' in the r.h.s is referring to local operators. The $\overline{\mathcal{O}}^{\Phi}_{(0)}$ is the part that saturates the power counting during the matching at tree-level and $\mathcal{O}^{\Phi}_{(1)}$ is the reminiscent non-local part that might be present to the power counting during the *Log* matching.

Since the importance of $\Gamma_{L,UV}$ is based entirely on providing one equation for the Effective Theory construction, the inverse operator can be replaced by 'bar' operators without loss of generality. In other words $\left[\mathcal{O}^{\Phi}\right]^{-1}$ can always be expanded exactly into local and non-local operators via a recursive formula. For example, if $A(\varphi) = 0$ in the representation of Eq.(2.96):

$$\begin{aligned}
[\mathcal{O}^{\Phi}]^{-1} = \Delta^{\Phi} \quad &\overset{1}{=} \quad -\left[\frac{1}{M^2} + \frac{P^2}{M^2}\frac{1}{M^2-P^2}\right] \\
&\overset{2}{=} \quad -\left[\frac{1}{M^2} + \frac{P^2}{M^2}\left(\frac{1}{M^2} + \frac{P^2}{M^2}\frac{1}{M^2-P^2}\right)\right] \\
&\vdots \\
&\overset{n}{=} \quad -\left[\frac{1}{M^2} + \frac{P^2}{(M^2)^2} + \cdots + \frac{(P^2)^{n-1}}{(M^2)^n} + \frac{(P^2)^n}{(M^2)^n}\frac{1}{M^2-P^2}\right] \\
&= \quad -\frac{1}{M^2}\left(1 + \frac{P^2}{M^2} + \cdots + \frac{(P^2)^{n-1}}{(M^2)^{n-1}}\right)\Bigg|_{\text{local}} + \frac{(P^2)^n}{(M^2)^n}\frac{1}{P^2-M^2}
\end{aligned} \tag{2.113}$$

where it also has been assumed commutation into D and M components.

In order to simplify the analysis, the quartic self-interaction term Q_{Φ_c} will be assumed to not enter at this level of the matching. Thus, from the previous arguments, the Eq.(2.111) can be rewritten as

$$\Gamma^{(0)}_{L,UV} = \int_x \left(\frac{1}{2}\varphi\, \mathcal{O}^{\varphi}\, \varphi \; - \; \frac{1}{2}B_{\varphi}\, \left[\overline{\mathcal{O}}^{\Phi}_{(0)} + \mathcal{O}^{\Phi}_{(1)}\right]\, B_{\varphi}\right) \tag{2.114}$$

At the other hand, the $\Gamma^{(0)}_{EFT}$, by definition, corresponds to the same form, although containing the 'zeroth-order' piece, i.e.

$$\Gamma^{(0)}_{EFT} = \int_x \left(\frac{1}{2}\varphi\, \mathcal{O}^{\varphi}\, \varphi \; - \; \frac{1}{2}B_{\varphi}\, \overline{\mathcal{O}}^{\Phi}_{(0)}\, B_{\varphi}\right) \tag{2.115}$$

Back to the brackets of Eq.(2.109), it follows that

$$\begin{aligned}\sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] \;&\supset\; i\alpha\left\{\log\det\left(\frac{\delta^2\Gamma^{(0)}_{L,UV}[\varphi]}{\delta\varphi^2}\right) - \log\det\left(\frac{\delta^2\Gamma^{(0)}_{EFT}[\varphi]}{\delta\varphi^2}\right)\right\} \\ &= i\alpha\left\{\mathrm{Tr}\log\left(\mathscr{O}^{\varphi} - \left[\mathscr{H}_{(0)} + \mathscr{H}_{(1)}\right]\right) - \mathrm{Tr}\log\left(\mathscr{O}^{\varphi} - \mathscr{H}_{(0)}\right)\right\} \qquad (2.116)\end{aligned}$$

where $\mathscr{H}_{(i)}$ is now denoting the Hessian matrix of $\frac{1}{2}B_\varphi\, \mathscr{O}^{\Phi}_{(i)}\, B_\varphi$ over light-fields, i.e. $\mathscr{H}[.] = \frac{\delta^2}{\delta\varphi^2}[.]$. Note that minus sign has been factorized out of the bracket. Moreover, the sum of the first term is then exploring the fact that $\mathscr{H}$ is a linear application[19].

Next, the trace of the light operator can be subtracted through the identity

$$\begin{aligned}\sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] \;&\supset\; i\alpha\left\{\mathrm{Tr}\log\left(1 - \left[\mathscr{O}^{\varphi}\right]^{-1}\left[\mathscr{H}_{(0)} + \mathscr{H}_{(1)}\right]\right) - \mathrm{Tr}\log\left(1 - \left[\mathscr{O}^{\varphi}\right]^{-1}\mathscr{H}_{(0)}\right)\right\} \\ &\equiv i\alpha\left\{\mathrm{Tr}\log\left(1 - \left[\mathscr{A}_{(0)} + \mathscr{A}_{(1)}\right]\right) - \mathrm{Tr}\log\left(1 - \mathscr{A}_{(0)}\right)\right\} \qquad (2.117)\end{aligned}$$

and for future references the definition of $\mathscr{A}_{(i)}$ must be registered explicitly:

$$\mathscr{A}_{(i)} \equiv \left[\mathscr{O}^{\varphi}\right]^{-1}\mathscr{H}\left[\frac{1}{2}B_\varphi\, \mathscr{O}^{\Phi}_{(i)}\, B_\varphi\right] \qquad (2.118)$$

The power counting will finally be made transparent through a series for the logarithm:

$$\sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] \;\supset\; i\alpha\mathrm{Tr}\sum_p \frac{1}{p}\left\{\mathscr{A}^p_{(0)} - \left[\mathscr{A}_{(0)} + \mathscr{A}_{(1)}\right]^p\right\} \qquad (2.119)$$

The cancellation of $\mathscr{A}_{(0)}$, remembering that these are the diagrams coming exclusively from the EFT, can then be directly identified after a binomial expansion of the last bracket:

$$\begin{aligned}\sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] \;&\supset\; i\alpha\mathrm{Tr}\sum_p \frac{1}{p}\left\{\mathscr{A}^p_{(0)} - \sum_{k=0}^{p}\binom{p}{k}\mathscr{A}^{p-k}_{(0)}\mathscr{A}^{k}_{(1)}\right\} \\ &= -i\alpha\mathrm{Tr}\sum_p \frac{1}{p}\left\{\sum_{k=1}^{p}\binom{p}{k}\mathscr{A}^{p-k}_{(0)}\mathscr{A}^{k}_{(1)}\right\} \qquad (2.120)\end{aligned}$$

The above formula is the final complement to the matching at the *log*-level and was introduced in [33] in a generic form, without mentioning the dependence of the coefficients. It is important to mention that the binomial may be expressed in its canonical formula once the operators $\mathscr{A}_{(i)}$ commute. In a more specific case, the binomial coefficients are informing the number of combinations to the non-commuting objects, like for example:

$$(A+B)^4 \supset 6A^2B^2 \quad or \quad 6A^2B^2 \to A^2B^2 + ABAB + BA^2B + AB^2A + BABA + B^2A^2 \qquad (2.121)$$

[19] Consider $f(\mathbf{x})$ and $g(\mathbf{x})$ two functions of multi-variables $\mathbf{x}$, then:

$$\mathscr{H}\left[\alpha f(\mathbf{x}) + \beta g(\mathbf{x})\right] = \alpha\mathscr{H}[f(\mathbf{x})] + \beta\mathscr{H}[g(\mathbf{x})]$$

for constants α and β.

Thus, although in a compact form, the result of Eq.(2.120) can be very arduous to compute, even for the simplest dim-6 criteria.

In summary, the final formula to the matching at log-level will be given by

$$\sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] = i\alpha \mathrm{Tr}\, \log\left(\frac{\delta^2 S_{UV}[\varphi,\Phi]}{\delta\Phi^2}\bigg|_{\Phi_c[\varphi]}\right) + i\alpha\mathrm{Tr}\sum_p \frac{1}{p}\left\{\sum_{k=1}^{p}\binom{p}{k}\mathscr{A}_{(0)}^{p-k}\mathscr{A}_{(1)}^{k}\right\} \tag{2.122}$$

The first trace will be computed through the Universal Formula Eq.(2.83) and represents the set of process containing heavy internal lines only. The second piece, from its complexity, requires additional comments:

- The trace is complementing the set of operators from processes containing both light and heavy internal lines inside the loop;
- The only simplification that has been done on the UV theory concerns the omission of quartic self-interactions represented by Q_Φ in Eq.(2.95). In fact these terms will be more suppressed at this level, since the derivatives are being taken, a posteriori, in the Higgs fields. The Wilson coefficients dependent on the Q_Φ parameters are then being left to the first part of the matching, with heavy-lines only. Nevertheless, in order to be complete, the dependence in λ_Φ corresponds to an additional piece in the Hessian $\mathscr{H}_{(1)}$ and does not imply any change in the derivation of Eq.(2.122);
- As it was mentioned for Eq.(2.96), the generic operator $\mathscr{O}^\Phi$ may still contain a function on the light fields, so-called $A(\varphi)$. Remembering that

$$\mathscr{O}^\Phi \equiv [\Delta^\Phi]^{-1} - A(\varphi), \qquad\qquad \Delta^\Phi \equiv [P^2 - M^2]^{-1} \tag{2.123}$$

In the framework where $A(\varphi)\neq 0$, the expansion of $[\mathscr{O}^\Phi]^{-1}$ will require one additional step given by

$$\begin{aligned}[]
[\mathscr{O}^\Phi]^{-1} &= \frac{1}{P^2 - M^2 - A} \\
&\simeq \Delta^\Phi + \Delta^\Phi\, A\, \Delta^\Phi + \cdots
\end{aligned} \tag{2.124}$$

where the Δ operator, which includes the covariant derivative, is opened on the right. The two terms in the above series are usually sufficient and, then the local expansion of Eq.(2.113) can be taken. All the steps for the derivation of Eq.(2.120) follows in the same way, although now the $\overline{\mathscr{O}}^\Phi_{(1)}$ is complemented with terms in the form $\overline{\mathscr{O}}^\Phi_{(1)} \supset \Delta^\Phi_{(0)}\, A\, \Delta^\Phi_{(1)} + \Delta^\Phi_{(1)}\, A\, \Delta^\Phi_{(0)} + \Delta^\Phi_{(1)}\, A\, \Delta^\Phi_{(1)}$.
- Since the main purpose of the present work is to clarify the light-heavy matching as represented in Eq.(2.122), the $A(\varphi)$ parameters will not be included at this level and their contribution to the Wilson coefficients will be left exclusively to the Universal Formula

application. Notwithstanding, the previous item was completely covered by [33], [28] and [25] for the example of the Electroweak Theory with a Triplet Scalar. In order to compare their results with those coming from the application of the formula Eq.(2.122), the model will also be treated in the Chapter 5.

2.4.2 Summary

In their presentation of mixing terms, the authors of HLM chose a subtraction a posteriori, made inside the functional trace, what may render a lack a clarity. Here it was intended to separate the pieces at the beginning, resulting in a series that can then be used in an algorithmic form. In summary,

- It was first supposed a generic Lagrangian to the UV theory, in Eq.(2.95), considering both tre-level pieces as quartic self-interactions. Although the aspect resembles the scalar case, at principle no additional complexity should be found in the fermionic case, already discussed in Appendix of [33];

- The formula for subtraction at the Log-level, Eq.(2.109), was then applied for both the generic UV and the EFT action;

- The LUV theory was split into local and non-local parts according to the definition of the EFT at tree-level. This marks where the series must be recovered;

- The Log is expanded and the linear local piece is extracted. The Eq.(2.120) is the final expression.

Comments on the last section:

- It is not necessary to carry two calculations, the Eq.(2.120) is already including the subtraction;

- The Hessian matrix is also unique, for both local or non-local parts, what will be illustrated in Chapter 5 and can actually be noted in the definition of Eq.(2.118). Thus, even the Hessian is carried just once;

- When the operators $\mathscr{O}$ contains additional functions of the light fields they may be replaced by the series:

$$[\mathscr{O}^{\Phi}]^{-1} = \frac{1}{P^2 - m^2 - A} = \sum_{n=0}^{\infty}\left[\frac{1}{P^2-m^2}A\right]^n \frac{1}{P^2-m^2} \tag{2.125}$$

2.4.3 On the Meaning of the Subtraction

During these computations some questions must arise in a very natural way: (i) Why is this subtraction needed since the UV theory is known? or in other terms - (ii) Since the real purpose is to integrate out a heavy field, what results in the $\Gamma_{L,UV}$ theory, why it is not sufficient to just plug this non-local object into the equation for the Effective Action, Eq.(2.93), and perform the local expansion a posteriori? Finally, (iii) what is the importance of the matching?

All of these questions are very logical. In fact, one starts with the UV theory and extracts the 1PI generator functional at the Log-level via the Effective Action. Every step can be done once and for all, resulting in a sum of operators informing on the significance for a given physical process. The answer for the above questions relies on a single statement - The Effective Action has the status of a Generating Functional, not of a *Theory*. This consists in an independent and very different perspective.

The Effective Theory will assume the complete set of operators generated by the Effective Action, at leading or Log-level, as the *action* of a new and independent model, valid and very precise at low-energies according to a definite scale. For being a new model, the associate generator functional of 1PI vertex is given by the usual Log correction, of Eq.(2.99). Thus, if this theory was emerged by the LUV only, its Log correction would result in coefficients for the n-point functions which was in reality counted twice. The subtraction is preventing this double counting and making the process consistent.

The previous paragraph can cover the question (i) and leads to a rectification of question (ii) - The procedure of integrating out a field does not resolve the 'real purpose' to the construction of an EFT, but the necessity of calculating complex processes, often involving a large number of particles or events. If it is then feasible to propose a theory with a reduced number of degrees of freedom, it is certain that this would be technically significant.

Finally, the third question is reached. This new theory is by definition an approximate variant of something more universal. It is therefore required a correspondence principle, some coherent argument that guarantees its limit of validity - the principle is the matching procedure.

CHAPTER 3

FUNDAMENTALS OF EFFECTIVE FIELD THEORIES

One of the most important tools during the extraction of physical predictions from a quantum field theory is the set of differential equations composing the renormalization group, and not only for practical purposes, but for its conceptual relevance. The running coupling analysis is a fundamental part of the applicability of Effective Field Theories. The EFT Lagrangian is defined after the replacement of the set composed by non-local product of operators for a sum of local terms whose coefficients will carry all the information about the original small distance of a heavy particle propagation. As mentioned before, the renormalization procedure is based in a fundamental principle which can be stated in many equivalent forms, and in this work it will be chosen as - any measurement in physics consist on variations. In other words, a number may not have any meaning unless it is giving through a comparison and such that the final and relevant physical result do not depend on the reference for the subtraction. This invariance on a 'translation of references' is what gives origin to the renormalization group equations and results in a strong simplification of calculations in perturbation theory.

For illustrating these statements, consider a generic interaction in the form

$$S_{\text{int}} \supset e\chi\chi\sigma \tag{3.1}$$

Here, a mention to the physical nature of the fields χ and σ is not necessary. Assume that the coupling constant e has been measured at some reference scale p_0^2 from the fitting of a particular set of data to the scattering $\chi\chi \to \chi\chi$. The S-matrix element is associated with the time-ordered correlation function $S^{(4)} \sim \langle\Omega|T\{\chi(x_1)\chi(x_2)\chi(x_3)\chi(x_4)\}|\Omega\rangle$, with $|\Omega\rangle$ and χ the vacuum and fields in the interacting scenario, respectively. For perturbative QFT this object is better to be rewritten in terms of correlation functions in the free theory, like

$$S^{(4)} \sim \langle 0|T\{\chi^0(x_1)\chi^0(x_2)\chi^0(x_3)\chi^0(x_4)\; e^{iS_I[\chi^0,\sigma^0]}\}|0\rangle \tag{3.2}$$

where S_I denotes the interacting action and the sub-index zero the free fields operators. The operators on the external points create the free asymptotic particles while, diagrammatically, the fields on S_I represent internal lines connecting these points. In other words, the interaction can be represented by a set of free fields allowed to *virtually* play a role during the short amount of time that the process occurs. By considering the reduced notation $S^{(4)} \sim \langle e^{iS_I}\rangle$ the simplest non-trivial term from the interaction with σ is the tree-level piece

$$S^{(4)} \sim (ie)^2\Big|_{p_0}\langle(\chi\chi\sigma)_x(\chi\chi\sigma)_y\rangle \equiv (ie)^2\Big|_{p_0}\langle - \rangle \tag{3.3}$$

where the small line is just denoting a tree-level computation. The higher-order terms in the expansion of S_I will correspond to Feynman diagrams, for example, from all the products of 1-loop interactions such that, by factorizing e^2, can be represented like:

$$S^{(4)} \sim (ie)^2\Big|_{p_0}\left[\langle-\rangle + (ie)^2\langle\circ\rangle + (ie)^4\langle\circ\times\circ\rangle + \cdots + (ie)^{2n}\langle\circ\times\cdots\times\circ\rangle\right] \equiv (ie)^2\Big|_{p_0}\langle\text{leading log}\rangle\Big|_p^{p_0} \quad (3.4)$$

Along the summation of the first loops, the fields will get rescaled such that the matrix element may contain a dependence in the reference scale p_0. Suppose the process occurs on the physical scale p^2 and that the dependence of the brackets on the subtraction scale can be written like

$$\langle\text{leading log}\rangle\Big|_p^{p_0} \sim \frac{1}{1-\frac{e^2}{12\pi^2}\log\left(\frac{p^2}{p_0^2}\right)} \times \langle-\rangle\Big|_p \quad (3.5)$$

where again $\langle-\rangle\Big|_p$ is just representing a simple matrix element of a tree-level diagram. It is clear that if the first measurement of e had been performed in a different value $\bar{p}_0$, instead of p_0, the correspondent change in $e(\bar{p})$ would be compensated by a change in the log, such that the total $S^{(4)}$, would remain invariant on the particular choice of reference[1].

The *Renormalization Group Equations* emerge as technique for summing logs in a direct manner, by taking advantage of this feature - physical quantities will not depend on the choice of subtraction point. In general, for representing this arbitrariness for the renormalization scale, the symbol μ is preferred instead of p_0 [49]. By replacing Eq.(3.5) in Eq.(3.4), $S^{(4)}$ can be represented like

$$S^{(4)} \sim \left(ie(p^2)\right)^2\langle-\rangle\Big|_p \quad (3.6)$$

with the *effective* coupling constant absorbing the rescaling factor. Thus, the solution of the RGE's will correspond to an efficient way to perform perturbative corrections, turning the predictions more accurate. It is certain, therefore, that if the physical scale of the process was the same of the first measurement, i.e. p_0, there would be no need for calculating corrections. In fact, as it will be treated in more detail, once the renormalization parameters for the fields involved in the desired process are known, there is no necessity for calculating loops at all - It is sufficient to solve the set of differential equations composing the RGE. The final problem will consist on calculating the remaining tree-level matrix element, what is straightforward if the assumption of free fields at asymptotic times holds, but challenging when one uses a theory of quarks to create and annihilate mesons, for example.

Finally, this introductory discussion is clearly supported by the QED and the electric charge, but the same principles and interpretation are valid for other scenarios. The EFT can then be viewed as technique that explores this philosophy. As mentioned, the solution of the differential equation for the coupling constants will require a initial, or boundary, condition. In the EFT approach, for example, the form of extracting this information can define two different

[1]Although this effective coupling is invariant under a change of the subtraction scale it is in fact dependent on the *subtraction scheme* (see [15], [52]);

perspectives - if the extraction is made via an experimental measurement the Effective Theory is being constructed in a bottom-up approach such that the new operators are inserted by following the experimental hints. At the other hand, if a complete theory is assumed a priori, the initial condition is extracted via a *correspondence principle* which states that the both theories must match at a decoupling scale. The procedure in this case will provide the initial condition for the coefficients by following the hints of the complete theory.

+ Power counting only makes sense if the higher-dimension terms in the expansion are in fact gradually suppressed. This is a property that requires a proof. Here, supported by the work of John Collins in [14] it will be treated the elements for demonstrating the validity of this criteria. If it can be shown that the heavy particle can decouple of the theory, whose effects are given by powers of its mass (decoupling theorem), any prediction coming from these vertices will certainly be equally suppressed.

The predictive strength of the EFT must count still with one important aspect of non-renormalizable theories - the locality of the counterterms. This condition can be translated into pattern of the divergences which arises a polynomial in the external momenta. In other words, a given vertex can be renormalized through a derivative interaction, associated with a new divergent vertex and, thus, with a new counterterm etc.. By considering one infinity, but countable, set of derivative interactions the EFT is renormalized, such that any new prediction turns gradually smaller according to the dimension of the respective operator.

+ On what follows, the heavy mass will be generically denoted by "m". Any light mass will receive the correspondent index. It has been present in any step of the previous chapter that the making of an EFT occurs through a local expansion of the operators inside the Effective Action for the Light-UV theory. All the information about the heavy field propagator must be contained in an operator like in Eq.(2.125)

$$[\mathscr{O}^{\Phi}]^{-1} = \frac{1}{P^2 - m^2 - A} = \sum_{n=0}^{\infty}\left[\frac{1}{P^2-m^2}A\right]^n \frac{1}{P^2-m^2} \tag{2.125}$$

The correlation functions generated by Γ_{LUV} will contain products of operators on the light fields at different space-time points, connected through this functional propagator. The matching will then be performed, in covariant form, by an operator product expansion, such that the subtraction with the Γ_{EFT} contributions implies an explicit step of how the information about the presence of a heavy particle is being retained.

Along the construction of the EFT it is fundamental that the global symmetry of the renormalizable sector of the expansion is still preserved by its higher-dimension elements. In general, this feature is present from the integration of gauge-bosons and singlets of the original UV theory, one aspect that produce simplifications, like for the RGE of Wilson coefficients of conserved currents.

3.1 The Operator Product Expansion

The expansion of the effective *action* for the so-called Light-UV theory during the matching procedure condense the information about the heavy fields into the coefficients of local operators. This method summarizes the role of the Operator Product Expansion. This section introduces the topic and follows the presentation by Peskin & Scroeder [42] and Collins [14].

The clearest technical benefit of making an EFT from a Top-Down approach consists in rewriting the a set of more complex interactions into a single vertex of a new action. one additional motivation comes from the experimental fact that new heavy degrees of freedom, once the scale of the process is not sufficient to produce it on-shell, will propagate by virtual lines with a short range, of order $(x-y) \sim \frac{1}{m}$. Suppose, for example, a process consisting in the scattering of k light fields, here represented like $\{\prod_i^k \phi(y_i)\}$ and y_i denoting the asymptotic region. The S-matrix piece is extracted from a series of matrix elements connecting these external fields with the interacting points. This connection may be given by a product of operators O_a, O_b separated by a distance x such that:

$$G_{ab}(x;\{y_k\}) = \langle O_a(x) O_b(0) \prod_{i=1}^{k} \phi_i \rangle \tag{3.7}$$

According to Wilson's hypothesis, this product may be replaced by a sum of local operators defining a basis and preserving the same *global symmetry* of the original product. The coefficients of this linearly independent operators must retain the information about the small distance x. These sentences may be converted into

$$G_{ab}(x;\{y_k\}) = \langle O_a(x) O_b(0) \prod_{i=1}^{k} \phi_i \rangle \quad \rightarrow \quad \sum_n c_n^{ab}(x) \, \langle O_n(0) \prod_{i=1}^{k} \phi_i \rangle \equiv \sum_n c_n^{ab}(x) \, G_n(\{y_k\}) \tag{3.8}$$

Looking in the opposite direction, this new basis of local operators could have been assumed in the original Lagrangian a priori, once the criteria of small distance is valid. Any computation from this new theory should then follow the same basic premises that have led to Eq.(3.4), this time for the coefficients c^{ab}. Both the matrix elements and the c's are defined according to a renormalization scale. The knowledge of the variables at a given subtraction point will provide an initial condition for solving the renormalization group equations.

As mentioned before, the solution for the RGE's provides a direct computation of higher-order terms in perturbation theory, generally expressed as a direct log summation. The solution must give a form for the an effective coupling c_{ab}.

The bare Green's function for the original product can be given by

$$G_{ab}^{(0)}(x;\{y_k\}) = \langle O_a^{(0)}(x) O_b^{(0)}(0) \prod_{i=1}^{k} \phi_i^{(0)} \rangle \tag{3.9}$$

with the index indicating bare fields. The relation with the renormalized G_{ab} can be written like

$$G_{ab}^{(0)} = Z^{\frac{k}{2}} Z_a Z_b G_{ab} \tag{3.10}$$

with $Z's$ the field-strength for the respective field. The invariance of the bare functions on the subtraction scale μ will lead to the differential equation:

$$\mu\frac{dG^{(0)}_{ab}}{d\mu} = 0 \to Z^{\frac{k}{2}} Z_a Z_b \left\{ \frac{k}{2}\frac{\mu}{Z}\frac{dZ}{d\mu} + \frac{\mu}{Z_a}\frac{dZ_a}{d\mu} + \frac{\mu}{Z_b}\frac{dZ_b}{d\mu} + \mu\partial_\mu + \mu\frac{\partial g}{\partial\mu}\partial_g \right\} G_{ab} = 0 \tag{3.11}$$

with Z_i denoting the possible combinations of field-strength coming from the operators. Analogously, the coupling constant g is representing the presence of all the light-fields behind these corrections.

By definition, the anomalous matrix and the β-function are given by

$$\gamma_j \equiv \frac{\mu}{Z_j}\frac{dZ_j}{d\mu}, \qquad \beta(g) = \mu\frac{dg}{d\mu} \tag{3.12}$$

what provides the usual representation for the Callan-Symanzik equation:

$$\left(\mu\partial_\mu + \beta\partial_g + \frac{k}{2}\gamma + \gamma_a + \gamma_b\right) G_{ab}(\mu) = 0 \tag{3.13}$$

The same derivation may be performed for G_n, such that

$$\left(\mu\partial_\mu + \beta\partial_g + \frac{k}{2}\gamma + \gamma_n\right) G_n(\mu) = 0 \tag{3.14}$$

It is common that the operators can mix via log corrections through one anomalous dinesion *matrix* γ_{ij}. In this context, the Callan-Symanzik equations is better represented by a general form for the coefficients. In short notation, consider the OPE written like:

$$G_{ij} = c^n_{ij} G_n \tag{3.15}$$

with a sum in repeated indices. Besides,

$$\left[\mu\partial_\mu + \beta\partial_g\right] G_{ab} = -\frac{k}{2}\gamma G_{ab} - \left[\gamma_{ai}\delta_{jb} + \gamma_{bj}\delta_{ia}\right] G_{ij} \tag{3.16}$$

and

$$\left[\mu\partial_\mu + \beta\partial_g\right] G_n = -\frac{k}{2}\gamma G_n - \gamma_{np} G_p \tag{3.17}$$

From Eq.(3.15) in Eq.(3.16):

$$\left[\mu\partial_\mu + \beta\partial_g\right](c^n_{ab} G_n) = -\frac{k}{2}\gamma(c^n_{ab} G_n) - \left[\gamma_{ai}\delta_{jb} + \gamma_{bj}\delta_{ia}\right](c^n_{ij} G_n) \tag{3.18}$$

or

$$\begin{aligned} G_n\left[\left(\mu\partial_\mu + \beta\partial_g\right) c^n_{ab}\right] &= -c^n_{ab}\left[\left(\mu\partial_\mu + \beta\partial_g\right) G_n\right] - \frac{k}{2}\gamma(c^n_{ab} G_n) - \left[\gamma_{ai}\delta_{jb} + \gamma_{bj}\delta_{ia}\right](c^n_{ij} G_n) \\ &\overset{(3.17)}{=} c^n_{ab}\gamma_{np} G^p - \left[\gamma_{ai}\delta_{jb} + \gamma_{bj}\delta_{ia}\right](c^n_{ij} G_n) \end{aligned} \tag{3.19}$$

and, finally,

$$\left(\mu\partial_\mu + \beta\partial_g\right) c^n_{ab} = c^j_{ab}\gamma_{jn} - c^n_{ib}\gamma_{ai} - c^n_{aj}\gamma_{bj} \tag{3.20}$$

The operators in the product are commonly found to be currents from a global symmetry also preserved by O_n, what implies a zero anomalous matrix. Therefore, the renormalization group equations can be summarized like

$$\left(\mu\partial_\mu + \beta\partial_g\right) c^n_{ab} = c^j_{ab}\gamma_{jn} \tag{3.21}$$

Since the initial product which gave origin to the local set of operators is general, the indices ab can just be hidden. Moreover, the Wilson coefficients will depend explicitly on μ or implicitly via the coupling constants, such that the derivatives inside the brackets can be replaced by a total derivative. By redefining the γ by its transpose, it follows a reduced representation for the RGE:

$$\mu\frac{dc_i(\mu)}{d\mu} = \frac{1}{16\pi^2}\gamma_{ij}c_j \tag{3.22}$$

where the $\frac{1}{16\pi^2}$ factor represents the first-log correction. The anomalous matrix will be determined from the renormalization of every operator inside the basis and will depend on the subtraction scale through the constant couplings. However, as explained in Chapter 6, these matrix are in general assumed to be fixed at the electroweak scale and their couplings do not run. The matrix equation of Eq.(3.22) can be solved in a first approximation by disentangling the correlated coefficients, fixing them at $\mu = \Lambda$, the heavy-scale, and solving for c_i. The procedure results in[2]

$$c_i(m_W) = c_i(\Lambda) - \sum_j \frac{1}{16\pi^2}\gamma_{ij}c_j(\Lambda)\log\left(\frac{\Lambda}{m_W}\right) \tag{3.25}$$

a solution that will be repeatedly used in Chapter 6. The total anomalous dimension matrix for the Standard Model Effective Field Theory, at dim-6, has been computed in [1], [34] and [35].

3.2 The Weinberg Theorem

This section presents a fundamental theoretical element behind the use of Effective Field Theories - The locality of the counterterms and how it serves as strong support for the consideration of non-renormalizable theories as a predictive tool. The proof is given by the Weinberg's theorem and is fully presented in J.Collins [14]. Here the essential aspects of the proof will be presented by

[2]For completeness, the solution to $\frac{dy}{dx} = ay + b$ can be achieved by first assuming $y = \frac{e^{\beta(x)} - b}{a}$, which corresponds to

$$\frac{dy}{dx} = e^{\beta(x)} \quad \text{or} \quad \frac{\dot\beta}{a}e^{\beta(x)} = e^{\beta(x)} \rightarrow \beta(x) - \beta(x_0) = a(x - x_0) \tag{3.23}$$

such that

$$\begin{aligned} y(x) &= \frac{e^{\beta(x_0)+a(x-x_0)} - b}{a} \\ &= y(x_0)\, e^{a(x-x_0)} + \frac{b}{a}\left[e^{a(x-x_0)} - 1\right] \end{aligned} \tag{3.24}$$

In the present case: $x = \log\mu$, $y(x_0) = c_i(\Lambda)$, $a = \frac{1}{16\pi^2}\gamma_{ii}$, $b = \sum_{j\neq i}\frac{\gamma_{ij}}{16\pi^2}c_j(\Lambda)$, and the exponential expanded at first-order in a.

including Collins notation and during the development a classification of quantum field theories renormalizability must be performed.

The kernel for Weinberg's theorem involves the observation that the *degree of divergence* $\delta(\Gamma) \geq 0$ of a 1PI graph Γ, with no subgraphs, can be gradually reduced by one unit through the application of a derivative ∂ in the external momenta, i.e.:

$$\delta(\partial\Gamma) = \delta(\Gamma) - 1 \tag{3.26}$$

which implies that ∂^λ with $\lambda = \delta(\Gamma) + 1$ may leave $\partial^\lambda(\Gamma)$ finite. By defining $R(\Gamma)$ equals to the sum of the 1PI Γ with its counterterms $C(\Gamma)$, it follows that

$$\begin{aligned} R(\Gamma) = \Gamma + C(\Gamma) \rightarrow \partial R(\Gamma) &= \partial\Gamma + \partial C(\Gamma) \\ &= \partial\Gamma + C(\partial\Gamma) \\ &= R(\partial\Gamma) \end{aligned} \tag{3.27}$$

where the second equality is equivalent to assert that since $\partial R(\Gamma)$ corresponds to a finite term, then $\partial C(\Gamma)$ is actually renormalizing $\partial\Gamma$, and therefore can be made equal to $C(\partial\Gamma)$. To ensure the commutation relation $\partial C(\Gamma) = C(\partial\Gamma)$ implies $C(\Gamma)$ as a polynomial of degree $\delta(\Gamma)$ on the external momenta. In fact,

$$R(\partial\Gamma) = \partial\Gamma + \partial C(\Gamma) \rightarrow R(\partial^\lambda\Gamma) = \partial^\lambda\Gamma + \partial^\lambda C(\Gamma) \tag{3.28}$$

Thus, since $\lambda = \delta(\Gamma) + 1$ make $\partial^\lambda\Gamma$ finite, it follows that

$$\partial^\lambda C(\Gamma) = 0 \tag{3.29}$$

and $C(\Gamma)$ is a polynomial of order $\delta(\Gamma)$.

This introduction may sound trivial but the complete argument requires the analysis of Γ when it includes divergent sub-diagrams 1PI, which has been entirely treated in Collins (Chapter 5).

The meaning of Weinberg's theorem will lead to a set of important conclusions for the analysis of a field theory renormalizability. Again, by following J.Collins, if Γ is a one-particle irreducible graph with degree of divergence $\delta(\Gamma)$, it follows that its mass dimension $d(\Gamma)$ plus the dimension of its correspondent couplings, here denoted $\Delta(\Gamma)$, is such that

$$d(\Gamma) = \delta(\Gamma) + \Delta(\Gamma) \tag{3.30}$$

i.e. given the specific vertex, the degree $\delta(\Gamma)$ associated with the integrals is connected with the mass dimension of couplings. The counterterms, being polynomials in the external momenta, can then be written through new couplings and derivatives with a correspondent dependence on the fields according to the particular Γ. In other words, if C is the counterterm to Γ, then

$$\delta(C) + \Delta(C) = d(\Gamma) \tag{3.31}$$

where $\delta(C)$ is the number of derivatives and $\Delta(C)$ the mass dimension of the couplings. Once the order of the polynomial is the degree of divergence of Γ, the maximum number of derivatives is given by $\delta(\Gamma)$, such that

$$\begin{aligned}\Delta(C) &= d(\Gamma)-\delta(C)\\ &\geq d(\Gamma)-\delta(\Gamma)=\Delta(\Gamma)\end{aligned} \tag{3.32}$$

Thus, if the couplings do not have a negative mass dimension, this aspect must be preserved by the couplings of the counterterms. The importance of this statement remounts Eq.(3.30)

$$\delta(\Gamma)=d(\Gamma)-\Delta(\Gamma) \tag{3.33}$$

Since $d(\Gamma)$ must be fixed by the particular process, i.e. n-point function, of interest, the insertion of internal lines in the 1PI graph with couplings of negative dimension will turn the graph increasingly divergent. At the order hand, $d(\Gamma)$ must decrease by recovering the LSZ reduction formula ([49]):

$$\begin{aligned}\delta^4(\sum p)\Gamma \quad\sim\quad & \prod_{i=1}^{b}\int d^4x_i e^{\pm p_i x_i}\Box_i\cdots\prod_{j=1}^{f}\int d^4y_j e^{\pm p_j y_j}\partial\!\!\!/_j\cdots\prod_{k=1}^{s}\int d^4z_k e^{\pm p_k z_k}\Box_k\cdots\times\\ & \times\langle\psi_1\cdots\psi_f B_1\cdots B_b\phi_1\cdots\phi_s\rangle\end{aligned} \tag{3.34}$$

which implies:

$$\begin{aligned}d(\Gamma) &= \frac{3}{2}f+b+s-2b-3f-2s+4\\ &= 4-\left(\frac{3}{2}f+b+s\right)\end{aligned} \tag{3.35}$$

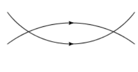
If the couplings have positive mass dimension, from Eq.3.33, the Γ will be finite after some number of couplings insertions (*relevant interactions*). At the other hand, couplings with null dimension may present divergent n-point functions for some values of n (marginal). Finally, as stated before, couplings with negative dimension may turn the graph arbitrarily divergent. For illustrating this, one example from M. Schwartz can be picked [49]:

$$\mathscr{L}=-\frac{1}{2}\phi(\Box+m^2)\phi+\frac{g}{4!}\phi^2\Box\phi^2 \tag{3.36}$$

with $[g]=-2$. The first loop for $\Gamma^{(4)}$ will present, from Eq.(3.35),

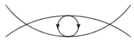
$$d(\Gamma^{(4)})=4-4=0 \tag{3.37}$$

and $\Delta(g^2)=-4$, implying $\delta(\Gamma^{(4)})_{\text{1-loop}}=4$. This could be directly confirmed, once δ comes from the integral:



$$\rightarrow \qquad I=\int d^4k\frac{k^4}{k^4}\rightarrow\delta(\Gamma^{(4)})_{\text{1-loop}}=4 \tag{3.38}$$

By inserting new couplings in $\Gamma(4)$:



$$\rightarrow \qquad \Delta(\Gamma) = (g^{-2})^4 = -8 \text{ or } \int (d^4k)^3 \frac{(k^2)^4}{(k^2)^6} \rightarrow \delta(\Gamma^{(4)})_{\text{3-loop}} = 8 \qquad (3.39)$$

which could have been shown through $d(\Gamma(4)) = 0 \rightarrow \delta(\Gamma^{(4)})_{\text{3-loop}} = 0 + 8$.

The relevant consequence of the above discussion can be read from Eq.(3.32). The counterterms must preserve the renormalizability of the theory, once they are limited by the dimension of the n-point graph. If the original Lagrangian contain couplings up to null mass-dimension, the theory will preserve its functional form, with a definite number of counterterms. In the non-renormalizable scenario, the counterterms must preserve the global and discrete symmetries of the starting theory, thus permitting to count the number of new interactions emerging. From the above example, $\Delta(\Gamma) = g^2 = -4$. In order to preserve the ϕ-parity, the non-renormalizable interactions inserted corresponds to $\Delta(C) \in \{-4, -2, 0\}$, what will certainly lead to new divergences, renormalized systematically by the same procedure.

In summary, the set of counterterms for a non-renormalizable theory, although infinite, is definitely countable and with well-defined local properties. Their insertion a priori in the Lagrangian would compose a theory with a valuable predictive power. Nevertheless, the use of N.R. theories remains under the hypothesis that there must be a finite and closed theory up to some scale. To consider N.R. models it is not a manner to abandon the important field theory paradigms and claims, but, instead, it is a part of a strong reasonable technique to find New physics directions gradually, avoiding the common complexity of the Ultra-Violet theories. According to this principle the next section will show that the N.R. sector inserted into the SM is indeed suppressed by powers of the UV completion scale.

3.3 The Decoupling Theorem

Among the theoretical apparatus previously discussed, perhaps the most related to the perspective present in this work is the Decoupling Theorem [2].

The Operator Product Expansion reveals about the procedure and the validity on transforming a non-local product of operators into a single local operator. The unique assumption - the distance x where the interaction occurs is around to zero compared with the masses involved (free-fields scale). Process involving hard-momentum scattering are examples of application and the Decoupling Theorem must explore the opposite scenario. The distance x of interaction is small because it corresponds to the duration of propagation of a highly off-shell particle, too massive compared with the physical scale. The perspective of the OPE follows equivalently and Wilson coefficients are defined from the extraction of the heavy lines from the totality of 1PI graphs.

Although the theorem will not be fully demonstrated here, a chosen example must certainly contain the necessary components for this. The presentation follows the work of John Collins

([14] - Chapter 8) and is based in the demonstration by T. Appelquist and J. Carazzone [2], who originally stated the theorem like:

> "For any 1PI Feynman graph with external momenta are small relative to m, then apart from coupling-constant and field-strength renormalization the graph will be suppressed by some power of momentum m relative to a graph with the same number of vector mesons but no internal fermions"

The specific nature of the fields involved is irrelevant to the result. According to J.Collins, whenever the physical scale is smaller than the masses from a specific sector of the full theory, these degrees of freedom may simply be eliminated through the *rescaling* of fields and couplings of the renormalizable sub-set of operators, composing the so-called effective Lagrangian. The error is proportional to the power $\frac{1}{m^a}$ with a close to two. In summary the complete theory is simplified into *its renormalizable sub-sector* including only *light fields*.

On what follows, the principles supporting the theorem are developed through the same model chosen by J.Collins, namely, the ϕ^3 theory including a heavy scalar in dimension-six space-time. The main purpose is to illustrate the suppression factor representing the error in considering a low-energy renormalizable version for light-fields, in comparison with the result for the same process computed in the complete theory. The example may also contain two aspects about renormalizability contained in the previous section.

The chosen model contains two scalars, assumed to be one light, φ, and one with a heavy mass, Φ. The procedure involves to rewrite a simplified renormalizable version of the complete model including only one light rescaled field and a new subset of parameters. The theorem will ensure that a particular process involving only light-fields, at low-energies, can be *equivalently* predicted by this low-effective theory, whose accuracy must differ from the prediction of from the full theory only by powers of $\frac{1}{m^a}$, $a > 0$.

The constant couplings in both theories have positive or null mass dimension. Thus, for a Green's function whose $d(\Gamma)$ is negative, the insertion of internal lines cannot change their overall degree of divergence. The possible divergent subdiagrams will be renormalized by the same counterterms defined a priori, during the renormalization of the initial Lagrangian, constant in its form. It is then clear the importance for proving the decoupling: If the final contribution from a graph with $d(\Gamma) < 0$, including a divergent subdiagram with *heavy-lines*, results to be the finite part of the subdiagram times the overall convergent diagram *plus* a finite piece suppressed by $\frac{1}{m^a}$, then the low-energy effective theory would give the same result, unless the suppressed term.

The previous paragraph will be clarified in detail. Despite the fact of consider a specific case, the followed steps will just reproduce the arguments of J.Collins [14] in his complete demonstration of the decoupling theorem. As mentioned before, the model consist in the ϕ^3 theory at dimension six. The chosen graph will be the $\Gamma(4)$ at second order, with a subgraph of *heavy lines*, like in Fig.3.1(a). The process also include, at first order, the graph of Fig.3.1(b). These

(a) (b)

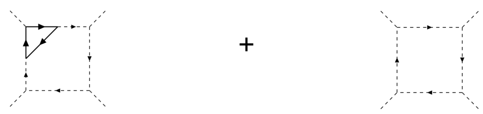
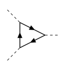


FIGURE 3.1. The Effective Action for the 1LUV of Eq.(2.9) may generate the 4-point function (**figure** 5) where the dashed line represents the propagation of the heavy Φ.

diagrams will be first computed in the context of the full theory:

$$\mathscr{L} = \frac{(\partial\varphi)^2}{2} + \frac{(\partial\Phi)^2}{2} - m_\varphi^2\frac{\varphi^2}{2} - m^2\frac{\Phi^2}{2} - \mu^{3-\frac{d}{2}}\left[g_\varphi\frac{\varphi^3}{3!} + g_\Phi\frac{\varphi\Phi^2}{2}\right] + \text{counterterms} \tag{3.40}$$

The subdiagram in Fig.3.1(a) must be renormalized and leads to

$\Gamma^{(3)} \sim$ 

$$\sim\ I^{(3)} = (ig_\Phi)^3\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}\frac{1}{(k-p)^2-m^2}\frac{1}{(k+p)^2-m^2} \tag{3.41}$$

or,

$$\begin{aligned} I^{(3)} &= -2(g_\Phi)^3\int_0^1\int_0^{1-x}dxdy\int dk\left[x(k^2-m^2)+y((k+p)^2-m^2)+(1-x-y)((k-p)^2-m^2)\right]^{-3} \\ &= -2(g_\Phi)^3\int_0^1\int_0^{1-x}dxdy\int dq\left[q^2-\Delta\right]^{-3} \end{aligned} \tag{3.42}$$

where $\Delta = m^2 - m_\varphi^2[(x+y)-(x-y)^2]$. Finally, from Appendix A.1:

$$\begin{aligned} I^{(3)} &= -2(g_\Phi)^3\int_0^1\int_0^{1-x}dxdy\ I_3^0 \\ &= \frac{i(g_\Phi)^3}{(4\pi)^3}\int_0^1\int_0^{1-x}dxdy\ \Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}} \\ &= \frac{i(g_\Phi)^3}{(4\pi)^3}\int_0^1\int_0^{1-x}dxdy\ \left(\frac{2}{\epsilon}-\log\left(\frac{\Delta}{4\pi\mu^2}\right)-\gamma\right) \end{aligned} \tag{3.43}$$

In the MS subtraction scheme, the divergence would be canceled by the counterterm

$$\delta_\Phi^{(3)} = \frac{g_\Phi^3}{(4\pi)^3}\left(-\frac{1}{\epsilon}\right) \tag{3.44}$$

and the renormalized vertex would be given by

$$\begin{aligned} R(\Gamma^{(3)}) &= -\frac{i(g_\Phi)^3}{(4\pi)^3}\left(\frac{\gamma}{2}+\int_0^1\int_0^{1-x}dxdy\ \log\left(\frac{m^2-m_\varphi^2[(x+y)-(x-y)^2]}{4\pi\mu^2}\right)\right) \\ &\sim -\frac{i(g_\Phi)^3}{(4\pi)^3}\left(\frac{\gamma}{2}+\frac{1}{2}\log\left(\frac{m^2}{4\pi\mu^2}\right)-\frac{m_\varphi^2}{4m^2}\right) \end{aligned} \tag{3.45}$$

The above result is useful both for the definition of the counterterm as for the identification of a finite term that vanishes in the limit $m \to \infty$. The original problem, however, consists in the computation of the 4-point function in Fig.3.1.

Analogously to Eq.(3.35), in six dimensions the mass dimension of a correlation function is given by

$$d(\Gamma) = 6 - \left(\frac{3}{2}f + b + s\right) \tag{3.46}$$

Since the coupling constants are dimensionless, the graphs of Fig.3.1 have overall divergence degree equals to $\delta(\Gamma(4)) = -2$. Thus, the Fig.3.1(b) must be UV finite. Moreover, no divergences must arise from the subdiagram in 3.1(a) when the diagram for the counterterm of Eq.(3.44) is also considered. For simplicity, the graph of Fig.3.1(b) will be just denoted like:

$$\Gamma[3.1(b)] = g_\varphi^4 \times \int_k I_4(k;p) \tag{3.47}$$

such that the graph including the counterterm will be given by

$$\Gamma[\delta_\Phi] = g_\varphi^3 \frac{g_\Phi^3}{(4\pi)^3}\left(-\frac{1}{\epsilon}\right) \times \int_l I_4(l;p) \tag{3.48}$$

Next, the complete Fig.3.1(a) must be computed and the integral on k follows like in the previous way, resulting into

$$\Gamma[3.1(a)] = g_\varphi^3 \frac{g_\Phi^3}{(4\pi)^3} \times \int_l I_4(l;p) \times \int dx\, x \left[\Gamma\left(\frac{\epsilon}{2}\right)\cdot\left(\frac{4\pi\mu^2}{\Delta}\right)^{\frac{\epsilon}{2}}\right] \tag{3.49}$$

where $m \to \infty$ has been considered for neglecting the momentum p inside the heavy loop. In the above expression, $\Delta \equiv [l^2(x^2 - x) + m^2]$, with l in Minkowski space. Finally, it can be written like

$$\Gamma[3.1(a)] = g_\varphi^3 \frac{g_\Phi^3}{(4\pi)^3} \times \int_l I_4(l;p) \times \left[\frac{1}{\epsilon} - \int dx\, x\left(\gamma + \log\left(\frac{\Delta}{4\pi\mu^2}\right)\right)\right] \tag{3.50}$$

The divergent term exactly cancels with the contribution from the counterterm, as expected. Apart from that, one additional step for the analysis about the renormalizability of a theory consists on proving that the insertion of divergent subdiagrams in an generic 1PI convergent graph cannot change its degree of divergence. This present example will serve to illustrate this situation. However, the objective here is to verify that that the finite piece will in fact be suppressed by powers of $\frac{1}{m^a}$ in the limit $m \to \infty$, the hypothesis of the decoupling theorem. In fact, after the subtraction from the counterterm, the renormalized expression can be summarized like:

$$\begin{aligned} R(3.1(a)) \quad = \quad & -g_\varphi^3 \frac{g_\Phi^3}{(4\pi)^3} \times \left\{ \left[\frac{\gamma}{2} + \frac{1}{2}\log\left(\frac{m}{4\pi\mu^2}\right)\right] \int_l I_4(l;p) \; - \right. \\ & \left. - \int dx\, x \int_l I_4(l;p) \times \log\left[\frac{l^2}{m^2}(x^2 - x) + 1\right]\right\} \end{aligned} \tag{3.51}$$

At this point the first observation is that the finite term of Eq.(3.45) is entirely present as a coefficient in the first line. Thus, the task is to identify the factor of suppression in the second line. After one integration by parts, it can be put in the form

$$\int dx\, x\log\left[\frac{l^2}{m^2}(x^2-x)+1\right] \;\rightarrow\; \frac{1}{2}\int_0^1 dx\, \frac{x^2(2x-1)\times l^2}{l^2(x-x^2)-m^2} \tag{3.52}$$

such that

$$\begin{aligned} R(3.1(a)) \;&=\; -g_\varphi^3\frac{g_\Phi^3}{(4\pi)^3}\times\left\{\left[\frac{\gamma}{2}+\frac{1}{2}\log\left(\frac{m}{4\pi\mu^2}\right)\right]\int_l I_4(l;p)\;-\right.\\ &\left.-\frac{1}{2}\int dx\, x^2(2x-1)\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{[l^2f_x-m^2][l^2-m_\varphi^2]^4}\right. \end{aligned} \tag{3.53}$$

with $f_x \equiv x-x^2$. As usual, the integral on momentum can be simplified via the introduction of Feynman parameters, like:

$$\frac{1}{AB^n} = \int_0^1 \frac{ny^{n-1}}{[(1-y)A+yB]^{n+1}} \tag{3.54}$$

which implies

$$\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{[l^2f_x-m^2][l^2-m_\varphi^2]^4} \rightarrow 4\int_0^1 dyy^3\int_l\frac{l^2}{l^2g_{xy}-\Delta_y} \tag{3.55}$$

where now $\Delta_y \equiv m_\varphi^2 y + m^2(1-y)$ and $g_{xy}\equiv y+f_x(1-y)$. After a change of variables[3], the second line of Eq.(3.53) follows

$$R(3.1(a)) \supset ig_\varphi^3\frac{g_\Phi^3}{4(4\pi)^6}\times\left(\frac{1}{m^2}\right)\int_0^1 dx\int_0^1 dy\frac{x^2(2x-1)y^3}{[y+(x-x^2)(1-y)]^4[y(z_\varphi-1)+1]} \tag{3.57}$$

where $z_\varphi \equiv \frac{m_\varphi^2}{m^2}$. For $z_\varphi \sim 0$ the integrals result in

$$\int_0^1 dx\int_0^1 dy\frac{x^2(2x-1)y^3}{[y+(x-x^2)(1-y)]^4[y(z_\varphi-1)+1]} \overset{z_\varphi\to 0}{\sim} \frac{5}{36}-\frac{1}{6}\log(z_\varphi) \tag{3.58}$$

In summary, the factor of suppression is smaller than two by the log term.

$$\begin{aligned} R(3.1(a)) \;&=\; -g_\varphi^3\frac{g_\Phi^3}{(4\pi)^3}\times\left\{\left[\frac{\gamma}{2}+\frac{1}{2}\log\left(\frac{m}{4\pi\mu^2}\right)\right]\int_l I_4(l;p)\;-\right.\\ &\left.-\frac{i}{4(4\pi)^3}\left(\frac{1}{m^2}\right)\left(\frac{5}{36}+\frac{1}{6}\log\left(\frac{m^2}{m_\varphi^2}\right)\right)\right) \end{aligned} \tag{3.59}$$

Thus, from the results of Eq.(3.45) and the first line of Eq.(3.59), it can be seen that a theory containing only the light-fields and a redefinition of the self-interaction coupling g_φ like

$$g_\varphi \rightarrow g_\varphi^* = g_\varphi - \frac{(g_\Phi)^3}{(4\pi)^3}\left(\frac{\gamma}{2}+\frac{1}{2}\log\left(\frac{m^2}{4\pi\mu^2}\right)\right) \tag{3.60}$$

[3]The change of variables leads to

$$4\int_0^1 dy\, y^3\, g_{xy}^{-(\frac{d}{2}+1)}\int_u\frac{u^2}{(u^2-\Delta_y)^5} \tag{3.56}$$

would imply the same results as the original one, with an error suppressed by the factors of a heavy-mass. The complete demonstration for the Decoupling Theorem is evidently more general and complex than this single example here, but it is based on the same principles. Every parameter of the original Lagrangian may be redefined - including those like the mass parameter related with divergent 1PI graphs with divergent subdiagrams - and such that the error committed will vanish in the limit $m \to \infty$. The suppression factor is slightly smaller than two, compensated by a large log including the infrared regulator, i.e. the light mass. By proving that the infrared structure of the original theory cannot overcome its mass suppression, the heavy particle must in fact decouple. When the experiments are performed in a regime which does not reach the large scale, all the process can be equivalently described by the simplified model:

$$\mathscr{L} = \frac{(\partial\varphi^*)^2}{2} - m_\varphi^{*2}\frac{\varphi^{*2}}{2} - \mu^{3-\frac{d}{2}}\left[g_\varphi^*\frac{\varphi^{*3}}{3!}\right] + \text{counterterms} \tag{3.61}$$

Part III

The Standard Model Effective Field Theory

CHAPTER 4

THE 3-3-1 MODEL WITH HEAVY LEPTONS

4.1 Introduction

The new gauge-structure defining the electroweak sector of 3-3-1 models [44], namely $SU(3)_L \otimes U(1)_X$, can render important phenomenological consequences and has been object of attention along the last years. Among some recent analysis, it can be mentioned those based on the context of collider [13] and low-energy physics [10–12, 17, 36, 50], WIMPs [39, 47] and different possible extensions [16].

The model has a total of six versions, two of which filling the lepton content with a conjugate of standard particles, thus defining the minimal 3-3-1 (see [27] and references therein). The present work is a brief review of the variant including new Heavy-Leptons [19, 40, 45, 46], here denoted like 3-3-1HL. It will be shown that a different particle content may be defined according to a discrete variable β limited to a set of four possible values [13, 20]. The total Lagrangian, divided into boson and fermion sectors, is then explored by first introducing the totality of β-independent terms. At this point it is expected to conclude about the presence of stable charged particles, as stated in [13]. These features arise whenever only the universal potential is considered, i.e. when the scalar self-interactions are assumed to be defined exclusively from some generic terms present in all versions of the model. The omission of β-specific interactions will retain the mixing from the potential following the same pattern as that present in the gauge-fixing Lagrangian. Next, the complete case is developed and a method that explore our previous knowledge on the gauge-dependent sector is introduced in order to simplify the search for the mass eigenstates.

The review composes the first part of a work in progress that intends to apply a general integration method for these sort of models, resulting in an Effective Theory that might be directly tested through precision observables. Thus, the first task involves the development of a consistent notation followed by a systematic classification of the interactions among new and standard terms. Those pieces that can generate 6-dimension operators at tree and loop-level must be ultimately selected. Apart from that, one exclusive assumption is consistently considered along this analysis - The first breaking scale must be much larger than the second one, with the notation translated into $u \gg v_\rho, v_\eta$.

Finally, the same steps of the authors in [7] along their presentation of the Standard Model are freely followed. Like any gauge theory with spontaneous symmetry breaking, the total Lagrangian is composed by the gauge-kinetic interactions of scalars and fermions, self-interactions of bosons and a Yukawa, and the chapter has been organized according to this structure.

4.2 Gauge Structure and Scalars in the 3-3-1HL

The root of electroweak interactions is expressed by the structure of the covariant derivative. In the context of a $SU(3)_L \otimes U(1)_X$ gauge group it can be represented like:

$$D_\mu = \partial_\mu + ig\mathbf{W}_\mu \cdot \mathbf{I} + ig_X X W^0_\mu \tag{4.1}$$

where the bold letter express a simple vector $\mathbf{A} \equiv \left(A^1, A^2, \cdots, A^8\right)$ and $I^a = \frac{\lambda^a}{2}$ are the generators of $SU(3)$. By expanding the gauge piece of D_μ explicitly it can be divided into complex (or non-diagonal) and real interactions

$$D^{(CC)}_\mu = i\frac{g}{2}\begin{pmatrix} 0 & W^1_\mu - iW^2_\mu & W^4_\mu - iW^5_\mu \\ W^1_\mu + iW^2_\mu & 0 & W^6_\mu - iW^7_\mu \\ W^4_\mu + iW^5_\mu & W^6_\mu + iW^7_\mu & 0 \end{pmatrix} \tag{4.2a}$$

and

$$D^{(NC)}_\mu = i\begin{pmatrix} \frac{g}{2}\left(W^3_\mu + \frac{W^8_\mu}{\sqrt{3}}\right) + g_X X W^0_\mu & 0 & 0 \\ 0 & \frac{g}{2}\left(-W^3_\mu + \frac{W^8_\mu}{\sqrt{3}}\right) + g_X X W^0_\mu & 0 \\ 0 & 0 & -\frac{g}{\sqrt{3}}W^8_\mu + g_X X W^0_\mu \end{pmatrix} \tag{4.2b}$$

The assertion that the fields presented in Eq.(4.2a) will be associated with charged currents would be a first sign on how to construct both the particle content of the theory and the pattern of the symmetry breaking. Nevertheless, as it is shown in Section 4.2.1, there are in effect two different variants of the model where Neutral Currents may also reside in non-diagonal vertices.

The following representation to the fermionic fields is going to be considered:

- **Leptons** ψ_α: $\left((\nu_\alpha\ l_\alpha) \quad E_\alpha\right)^{\intercal}_L$, $\quad (\mathbf{1},\mathbf{3},X_\psi)$
- **Quarks** Q_i: $\left((d_i\ u_i) \quad J_i\right)^{\intercal}_L$, $\quad (\mathbf{3},\overline{\mathbf{3}},X_Q)$
- **Quarks** Q_3: $\left((u_3\ d_3) \quad J_3\right)^{\intercal}_L$, $\quad (\mathbf{3},\mathbf{3},X_3)$
- u^R_a: $(\mathbf{3},\mathbf{1},\frac{2}{3})$
- d^R_a: $(\mathbf{3},\mathbf{1},-\frac{1}{3})$
- l^R_α: $(\mathbf{1},\mathbf{1},-1)$
- J^R_i: $(\mathbf{3},\mathbf{1},X_J)$
- J^R_3: $(\mathbf{3},\mathbf{1},X_{J_3})$
- E^R_α: $(\mathbf{1},\mathbf{1},X_E)$

where $\alpha = [e, \mu, \tau]$, $a = [1,2,3]$ and $i = [1,2]$. The brackets inside the triplets are denoting that these are in reality doublets of $SU(2)$ or, implicitly, that the gauge group will follow a hierarchy under the symmetry breaking such that, in one of its steps, these objects compose a new symmetric Lagrangian comprising the Standard Model. Apart from that, some of the hypercharges were omitted and their derivation left to the following subsection.

The above mentioned hierarchy is endowing the model with a new scale represented by the letter u and present for the breaking of $SU(3)_L \otimes U(1)_X$ into $SU(2)_L \otimes U(1)_Y$. The scale is introduced via the vacuum expectation value of a new neutral scalar χ^0 singlet of $SU(2)$, composing the triplet

$$\chi = \begin{pmatrix} (\chi^V \ \chi^U) & \chi^0 \end{pmatrix}^{\intercal}$$

The next sections will discuss that the 3-3-1HL must also include two additional scalar triplets which may appear in an intermediate scale via Higgs interactions. Nevertheless, for a matter of clarity and driven by a current phenomenology that disallow new degrees of freedom to show up at low energies, the task of separating the model into standard and new exotic sectors will be performed by considering only the first breaking. In other words, the classical electroweak symmetry breaking can be placed aside, enclosed in a $\mathscr{L}_{SM}$ function. Thus, here the primary focus is about an universe ruled by nine interactions, where five of them are mediated by massive and four by massless vector particles. The former acquire their masses when χ^0 presents a v.e.v. $\langle\chi^0\rangle = u$, or $\langle\chi\rangle \propto (0\ 0\ u)^{\intercal}$, leading the theory to $\mathscr{L}_{331} \to \mathscr{L}_{SM} + \text{NP}$.

Inside the set of nine $SU(3)_L \otimes U(1)_X$ generators, the following four leave the vacuum unbroken and may define a basis for the group $SU(2)_L \otimes U(1)_Y$ [12]

$$\mathbb{T}_1\langle\chi\rangle = \mathbb{T}_2\langle\chi\rangle = \mathbb{T}_3\langle\chi\rangle = (\beta\mathbb{T}_8 + X\mathbb{I})\langle\chi\rangle = 0 \tag{4.3}$$

From our knowledge on the SM symmetry breaking it can be extracted a connection between electromagnetism and weak interactions represented by the so called Gell-Mann-Nishima relation

$$\mathbb{Q} = \mathbb{T}_3 + \frac{\mathbb{Y}}{2} \tag{4.4a}$$

where, in the present case,

$$\frac{\mathbb{Y}}{2} = \beta\mathbb{T}_8 + X\mathbb{I} \tag{4.4b}$$

i.e. a diagonal generator for the SM symmetry group defining the particles hypercharge and the respective conserved currents.

The Eq.(4.4b) introduces a new and decisive variable. The parameter β may completely alter the model phenomenology and is an element of a set limited to four numbers, namely $\{\pm\sqrt{3}, \pm\frac{1}{\sqrt{3}}\}$. As presented in [13], this set is generated by a series of factors which includes the integer nature of the electric charge for asymptotic states and the positiveness for variables of mass. For illustration, this section is concluded with the demonstration of some results and identities in the context of $\beta = -\sqrt{3}$ (or $X_\chi = -1$). This version contains a single charged heavy lepton and the previous relations may provide us, for example, the charges of exotic quarks inside the triplets.

Since the quarks J are singlet of $SU(2)$, it follows

$$Q_3 = \begin{pmatrix} u_3 \\ d_3 \\ J_3 \end{pmatrix} \qquad \to \qquad \mathbb{Q}_{Q_3} = \begin{pmatrix} \frac{2}{3} & & \\ & -\frac{1}{3} & \\ & & q_{J_3} \end{pmatrix} \tag{4.5}$$

$$\begin{aligned} \therefore \quad \frac{2}{3} &= (\mathbb{T}_3)_{11} - \sqrt{3}(\mathbb{T}_8)_{11} + X_3 \\ &= X_3 \end{aligned} \tag{4.6}$$

The J_3 electric charge will then be given by

$$q_{J_3} = -\sqrt{3}(\mathbb{T}_8)_{33} + \frac{2}{3} = \frac{5}{3} \tag{4.7}$$

The gauge boson $(W^4 - iW^5)$, for instance, will couple with the current

$$j^{\mu}_{(uJ)} \equiv \overline{u}_{3L}\gamma^{\mu} J_{3L}$$

thus corresponding to a particle charged by (-1). Similarly,

$$j^{\mu}_{(dJ)} \equiv \overline{d}_{3L}\gamma^{\mu} J_{3L}$$

coupled to the so-denoted U-boson and whose charge, in this case, must be equal to (-2).

By considering the three $SU(2)$ subalgebras of $SU(3)$ defined by the raising and lowering operators:

$$\mathbb{I}_{\pm} = \frac{\mathbb{T}_1 \pm i\mathbb{T}_2}{\sqrt{2}}, \qquad \mathbb{I}_3 = \mathbb{T}_3 \tag{4.8}$$

$$\mathbb{J}_{\pm} = \frac{\mathbb{T}_4 \pm i\mathbb{T}_5}{\sqrt{2}}, \qquad \mathbb{J}_3 = \frac{\sqrt{3}}{2}\mathbb{T}_8 - \frac{\mathbb{T}_3}{2} \tag{4.9}$$

$$\mathbb{L}_{\pm} = \frac{\mathbb{T}_6 \pm i\mathbb{T}_7}{\sqrt{2}}, \qquad \mathbb{L}_3 = \frac{\sqrt{3}}{2}\mathbb{T}_8 + \frac{\mathbb{T}_3}{2} \tag{4.10}$$

and also the gauge bosons (note the changes on the signs for V and U compared with W):

$$W^{\pm} = \frac{W^1 \mp iW^2}{\sqrt{2}}, \quad V^{\pm} = \frac{W^4 \pm iW^5}{\sqrt{2}}, \quad U^{\pm\pm} = \frac{W^6 \pm iW^7}{\sqrt{2}} \tag{4.11}$$

the charged sector of D_μ can be expressed in a short notation as

$$D^{(CC)}_{\mu} = ig(W^+\mathbb{I}_+ + W^-\mathbb{I}_- + V^-\mathbb{J}_+ + V^+\mathbb{J}_- + U^{--}\mathbb{L}_+ + U^{++}\mathbb{L}_-) \tag{4.12}$$

In order to obtain the interactions in the conjugate representation the raising and lowering operators are just changed in sign:

$$D^{*(CC)}_{\mu} = -ig(W^+\mathbb{I}_- + W^-\mathbb{I}_+ + V^-\mathbb{J}_- + V^+\mathbb{J}_+ + U^{--}\mathbb{L}_- + U^{++}\mathbb{L}_+) \tag{4.13}$$

It is common to present the quark representations with a minus sign in the definition of Q_i, in general, aiming to recover the exact aspect of the SM Lagrangian. Nevertheless, such phase insertion does not have any phenomenological implications.

- **Note**: To find the Gell-Mann-Nishima relation to the fields in conjugate representation it must be recovered that the conjugation of the generators $\mathbb{T}_3$ and $\mathbb{T}_8$ is defined as

$$\mathbb{T}_3^* = -\mathbb{T}_3, \qquad \mathbb{T}_8^* = -\mathbb{T}_8 \tag{4.14}$$

and, therefore,

$$\mathbb{Q} = -\mathbb{T}_3 + \sqrt{3}\,\mathbb{T}_8 + X\mathbb{I} \tag{4.15}$$

Consider, for example, X_Q and q_{J_i}:

$$\begin{aligned} \mathbb{Q}_{Q_i} = \begin{pmatrix} -\frac{1}{3} & & \\ & \frac{2}{3} & \\ & & q_{J_i} \end{pmatrix} \quad \to \quad \frac{2}{3} &= (-\mathbb{T}_3)_{22} + \sqrt{3}(\mathbb{T}_8)_{22} + X_Q \\ &= \frac{1}{2} + \frac{1}{2} + X_Q \end{aligned}$$

Thus,

$$\therefore \qquad X_Q = -\frac{1}{3} \qquad \to \qquad q_{J_i} = \sqrt{3}(\mathbb{T}_8)_{33} - \frac{1}{3} = -\frac{4}{3} \tag{4.16}$$

Now the diagonal terms of the covariant derivative, those corresponding to the neutral current interactions, are

$$D_\mu^{(NC)} = ig(\mathbb{T}_3 W^3 + \mathbb{T}_8 W^8) + ig_X X\mathbb{I} W^0 \tag{4.17}$$

After the first breaking by $\langle\chi\rangle$, W^0 will mix with W^8 and can be rotated into the mass eigenstates by

$$\begin{pmatrix} W^0 \\ W^8 \end{pmatrix} = \begin{pmatrix} c_x & -s_x \\ s_x & c_x \end{pmatrix} \begin{pmatrix} B \\ Z' \end{pmatrix} \tag{4.18}$$

which implies

$$\begin{aligned} D_\mu^{(NC)} &= ig\mathbb{T}_3 W^3 + ig\mathbb{T}_8(Bs_x + Z'c_x) + ig_X X\mathbb{I}(Bc_x - Z's_x) \\ &= i(\mathbf{g}_{W^3} W^3 + \mathbf{g}_B B + \mathbf{g}_{Z'} Z') \end{aligned} \tag{4.19}$$

where it has been defined

$$\mathbf{g}_{W^3} \equiv g\mathbb{T}_3, \qquad \mathbf{g}_B \equiv g\mathbb{T}_8 s_x + g_X X\mathbb{I} c_x, \qquad \mathbf{g}_{Z'} \equiv \mathbb{T}_8 c_x - g_X X\mathbb{I} s_x \tag{4.20}$$

The mixing between B and W^3 after the second symmetry breaking can also be anticipated without loss of generality

$$\begin{pmatrix} B \\ W^3 \end{pmatrix} = \begin{pmatrix} c_w & -s_w \\ s_w & c_w \end{pmatrix} \begin{pmatrix} A \\ Z \end{pmatrix} \tag{4.21}$$

or, finally,

$$D_\mu^{(NC)} = i(\mathbf{g}_A A + \mathbf{g}_Z Z + \mathbf{g}_{Z'} Z') \tag{4.22}$$

where,

$$\mathbf{g}_A \equiv s_w \mathbf{g}_{W^3} + c_w \mathbf{g}_B, \qquad \mathbf{g}_Z \equiv c_w \mathbf{g}_{W^3} - s_w \mathbf{g}_B \tag{4.23}$$

In order to require that electromagnetic interactions be reproduced, it follows

$$\mathbf{g}_A = e\mathbb{Q} \tag{4.24}$$

or

$$e\mathbb{Q} = g\mathbb{T}_3 s_w + g\mathbb{T}_8 s_x c_w + g_X X\mathbb{I} c_x c_w \tag{4.25}$$

The Eq.(4.25) when applied for different quarks and leptons may result in some important relations to the coupling constants. For instance, to the quark-d_1 ($q_d = -\frac{1}{3}, X_Q = -\frac{1}{3}$):

$$-\frac{e}{3} = -\left(\frac{g}{2}s_w + \frac{g}{2\sqrt{3}}s_x c_w\right) - \frac{g_X}{3}c_x c_w \tag{4.26}$$

From the following Eq.(4.28) and by assuming $e = g s_w$:

$$-\frac{e}{3} = -\left(\frac{e}{2} - \frac{e}{2}\right) - \frac{g_X}{3}c_x c_w \quad \rightarrow \quad g_X c_x c_w = e$$

or, finally, if $g' = \frac{e}{c_w}$,

$$g_X c_x = g' \tag{4.27}$$

which connects the hypercharge sectors of the Standard Model and the 3-3-1HL. Besides, from the results of the next section, derived for a generic version of the model, it can be shown that the Eq.(4.27) is in fact independent of β. It must be observed, however, that the same relations raised in the context of the Standard Model were considered, what is to assert both angles and couplings as equivalent.

By repeating the use of Eq. (4.25) to the case of neutrinos ($q_\nu = 0, X_\psi = 0$) it follows that

$$\frac{g}{2}s_w + \frac{g}{2\sqrt{3}}s_x c_w = 0 \qquad \rightarrow \qquad s_x = -\sqrt{3}\tan\theta_w \qquad (\text{or } s_x = \beta\tan\theta_w) \tag{4.28}$$

which provides a connection between the Weinberg angle and the mixing on the first breaking. From the identities proved in the next section it is straightforward to obtain the general formula inside the brackets.

Here the rotation of (B, W^3) into (A, Z) is in reality neglecting the mixing with Z' emerged after the second breaking. This assumption is also motivated by the large difference between the scales, $u \gg v$.

The Eqs. (4.28) and (4.27) will also induce a set of simplified expressions for the Z and Z' couplings. For instance,

$$\begin{aligned}
\mathbf{g}_Z &= g\mathbb{T}_3 c_w - (g\mathbb{T}_8 s_x + g_X X\mathbb{I} c_x) s_w \\
&\overset{(4.27)}{=} g\mathbb{T}_3 c_w - (g\mathbb{T}_8 s_x + g' X\mathbb{I}) s_w \\
&\overset{(4.28)}{=} g c_w \mathbb{T}_3 + \sqrt{3} g \tan\theta_w s_w \mathbb{T}_8 - g' s_w X\mathbb{I} \\
&= g c_w \mathbb{T}_3 + \sqrt{3} g \frac{s_w^2}{c_w} \mathbb{T}_8 - g \frac{s_w^2}{c_w} X\mathbb{I} \\
&= g c_w \mathbb{T}_3 + g \frac{s_w^2}{c_w} (\sqrt{3}\mathbb{T}_8 - X\mathbb{I}) \\
&= g c_w \mathbb{T}_3 + g \frac{s_w^2}{c_w} (\mathbb{T}_3 - \mathbb{Q}) \\
&= \frac{g}{c_w} (\mathbb{T}_3 - \mathbb{Q} s_w^2)
\end{aligned} \tag{4.29}$$

i.e. the well-known coupling of Z with the SM particles. The fourth and sixth equality sign consider $g' = g\tan\theta_w$ and Eq.(4.15), respectively, and the identity is β-independent. A similar expression can also be achieved for Z' through

$$\begin{aligned}
\mathbf{g}_{Z'} &= g\mathbb{T}_8 c_x - g_X X\mathbb{I} s_x \\
&\overset{(4.27)}{=} g\mathbb{T}_8 c_x - \frac{g'}{c_x} X\mathbb{I} s_x \\
&= g(\mathbb{T}_8 c_x - \tan\theta_w \tan\theta_x X\mathbb{I}) \\
&\overset{(4.28)}{=} \frac{g}{c_x} (\mathbb{T}_8 c_x^2 + \sqrt{3}\tan^2\theta_w X\mathbb{I})
\end{aligned} \tag{4.30}$$

The relation (4.29) also exhibits the massless nature of Z after the first breaking

$$\frac{g}{c_w}((\mathbb{T}_3)_{33} - 0 * s_w^2) = 0$$

i.e. there is no coupling with the neutral scalar in χ.

Thus, the section may be concluded with the total covariant derivative expressed under a simplified notation,

$$\begin{aligned}
D_\mu &= \mathbb{I}\partial_\mu + i[g(W^+\mathbb{I}_+ + W^-\mathbb{I}_-) + \mathbf{g}_Z Z + e\mathbb{Q}A] + \\
&\quad + i[g(V^-\mathbb{J}_+ + V^+\mathbb{J}_- + U^{--}\mathbb{L}_+ + U^{++}\mathbb{L}_-) + \mathbf{g}_{Z'} Z']
\end{aligned} \tag{4.31}$$

The first line reproduces exactly the Standard Model contribution and will make simpler the task of dividing the 3-3-1HL into SM and New Physics elements.

As mentioned previously, this work discuss how the different sectors of the Lagrangian will connect the new degrees of freedom with the standard fields at tree- and loop-level. In order to be consistent and provide an independent review, some of the main aspects of the model can be registered, being however substantially supported by previous works like [13], [44] and [12]. When the context allows, the first breaking $SU(3)_L \otimes U(1)_X \to SU(2)_L \otimes U(1)_Y$ is considered with priority. In the next section some identities for a general 3-3-1HL version are extracted.

4.2.1 Particle Content in Different Versions

First, recall the electric charge operator:

$$\mathbb{Q} = \mathbb{T}_3 + \beta\mathbb{T}_8 + X\mathbb{I} \tag{4.32}$$

First consider to find the β-dependence of the electric charges. From the first line of $D_\mu\chi$, for example, it follows $q_{\chi_1} = 1 + q_{\chi_2} = q_V$, which can also justify why the first entry of χ has been defined as χ_V. Besides, the second line would give $q_U = q_{\chi_2}$, and is denoted like $\chi_2 \equiv \chi_U$.

The neutral third component of χ implies $X_\chi = \frac{\beta}{\sqrt{3}}$, i.e.

$$\begin{aligned} q_{\chi_V} = q_V &= \frac{1}{2} + \frac{\beta}{2\sqrt{3}} + \frac{\beta}{\sqrt{3}} \\ &= \frac{\sqrt{3}}{2}\beta + \frac{1}{2} \qquad \rightarrow \qquad q_U = \frac{\sqrt{3}}{2}\beta - \frac{1}{2} \end{aligned} \tag{4.33}$$

By repeating this simple procedure to the quarks and leptons representations one obtain their respective hypercharges. For example, the value of X_Q, defined in conjugate representation, will be given by

$$X_Q = \frac{1}{6} + \beta\frac{1}{2\sqrt{3}} \quad \rightarrow \quad q_{j_i} = \frac{1}{6} + \frac{\sqrt{3}}{2}\beta \tag{4.34}$$

or, for the triplet in fundamental representation,

$$X_3 = \frac{1}{6} - \frac{\beta}{2\sqrt{3}} \quad \rightarrow \quad q_{j_3} = \frac{1}{6} - \frac{\sqrt{3}}{2}\beta \tag{4.35}$$

As mentioned before, the electric charges of the new gauge bosons, as well as the Z' mass, prevent β of assuming any possible value [13], restricting it to the set

$$\beta \in \left[\pm\frac{1}{\sqrt{3}}, \pm\sqrt{3}\right] \tag{4.36}$$

From Eq.(4.34) and Eq.(4.35) it is clear the effect of a $\pm$ sign for quarks is merely of inverting the electric charges into different representations. For $|\beta| = \sqrt{3}$ the possible values are $q_J \in \left[-\frac{4}{3}, \frac{5}{3}\right]$ and for $|\beta| = \frac{1}{\sqrt{3}}$, $q_J \in \left[-\frac{1}{3}, \frac{2}{3}\right]$.

Similarly, for the leptons:

$$\psi_\alpha : \begin{pmatrix} \nu_\alpha \\ l_\alpha \\ E_\alpha \end{pmatrix}_L \qquad \rightarrow \qquad X_\psi = -\left(\frac{1}{2} + \frac{\beta}{2\sqrt{3}}\right) \tag{4.37}$$

or

$$q_E = -\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\beta\right) \tag{4.38}$$

Thus, depending on the selected β among the four possible values, the model may include from neutral to doubled charged heavy leptons:

$$q_E = [-2,-1,0,+1] \quad \text{for} \quad \beta = \left[\sqrt{3}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\sqrt{3}\right] \tag{4.39}$$

From phenomenological reasons the heavy neutral particles must also be analyzed in detail. The Section 4.2.2 resolve, in the context of Scalars, that the most general potential can be formulated regardless the sign of β, i.e. it is sufficient to consider each of the two possible modulus. Thus, in the remaining of this work those models with $\beta = \left[-\sqrt{3}, -\frac{1}{\sqrt{3}}\right]$ are taken with priority.

Finally, the general hypercharge of triplets and anti-triplets can be summarized like:

- **Leptons** ψ_α: $\left(\mathbf{1},\mathbf{3},-\left(\frac{1}{2}+\frac{\beta}{2\sqrt{3}}\right)\right)$
- **Quarks** Q_i: $\left(\mathbf{3},\overline{\mathbf{3}},\frac{1}{6}+\frac{\beta}{2\sqrt{3}}\right)$
- **Quarks** Q_3: $\left(\mathbf{3},\mathbf{3},\frac{1}{6}-\frac{\beta}{2\sqrt{3}}\right)$
- u_a^R: $\left(\mathbf{3},\mathbf{1},\frac{2}{3}\right)$
- d_a^R: $\left(\mathbf{3},\mathbf{1},-\frac{1}{3}\right)$
- J_i^R: $\left(\mathbf{3},\mathbf{1},\frac{1}{6}+\beta\frac{\sqrt{3}}{2}\right)$
- J_3^R: $\left(\mathbf{3},\mathbf{1},\frac{1}{6}-\beta\frac{\sqrt{3}}{2}\right)$
- l_α^R: $(\mathbf{1},\mathbf{1},-1)$
- E_α^R: $\left(\mathbf{1},\mathbf{1},-\left(\frac{1}{2}+\beta\frac{\sqrt{3}}{2}\right)\right)$

where $\alpha = [e,\mu,\tau]$, $a = [1,2,3]$ and $i = [1,2]$.

As a brief comment on the Scalar triplets, the gauge-fixing Lagrangian, discussed in Section 4.2.4, in general is defined from the kinetic sector of the Scalars and contain terms proportional to their product with a correspondent gauge boson. For example, the first line of the product $D^\mu\chi$ contains the sum $\partial^\mu\chi_V + uV$ such that its squared will produce the bilinear $uV\partial\chi_V$. Therefore, the following notation intends to leave explicit how the gauge-structure of the model will connect the Scalars with Vector Bosons:

$$\chi = \begin{pmatrix} \chi^V \\ \chi^U \\ \frac{u+\overline{H}+i\chi_g}{\sqrt{2}} \end{pmatrix}, \qquad \left(\mathbf{1},\mathbf{3},\frac{\beta}{\sqrt{3}}\right) \tag{4.40a}$$

$$\rho = \begin{pmatrix} \rho^W \\ \frac{v_\rho+H_\rho+i\rho_g}{\sqrt{2}} \\ \rho^{-U} \end{pmatrix}, \qquad \left(\mathbf{1},\mathbf{3},\frac{1}{2}-\frac{\beta}{2\sqrt{3}}\right) \tag{4.40b}$$

$$\eta = \begin{pmatrix} \frac{v_\eta+H_\eta+i\eta_g}{\sqrt{2}} \\ \eta^{-W} \\ \eta^{-V} \end{pmatrix}, \qquad \left(\mathbf{1},\mathbf{3},-\left(\frac{1}{2}+\frac{\beta}{2\sqrt{3}}\right)\right) \tag{4.40c}$$

where the minus sign indicates that their charges are in opposite sign as those defined in Eq.(4.33). As will be repeatedly explored, when these connections are not broken by β-dependent terms, they may control the pattern of scalar mixing and correspond to residual symmetries that avoid the heavy leptons to decay. These features, along with their phenomenological consequences, have been addressed by the authors in [13], and here it must be complemented by exposing the vertices allowed by the gauge symmetry that could create decaying channels into the light asymptotic fields. Therefore, the complete set of elements for the potential are considered and a Z_2 symmetry is not assumed a priori. It can be demonstrated that the pattern of scalar mixing present in the incomplete and universal[1] potential eliminates the totality of tree-level interactions, leaving the heavy-lepton stable.

4.2.2 Self-Interactions of Scalars

Here is examined the algebraic path followed by the symmetry breaking of a generic potential which may conduce the gauge theories to their correct spectra. It starts by presenting some aspects of this sector in the Standard Model, where:

$$V(\Phi) = \mu\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 \tag{4.41}$$

and

$$\Phi = \begin{pmatrix} \varphi_g^+ & \frac{\varphi^0+i\varphi_g^0}{\sqrt{2}} \end{pmatrix}^\mathsf{T} \tag{4.42}$$

The subindex ‘g’ on some of the above fields is referring to the Goldstone bosons. In fact, an intermediate step along the development of the Goldstone theorem is supported by the condition that the vacuum of the theory is located on the point where one of the neutral fields, namely φ^0, assume a value different from zero. This Vacuum Stability Condition (VSC) will correspond to one or a set of equations that, when applied back to Eq.(4.41), cancel the mass terms for these ‘g’ degrees of freedom. The contribution to the mass of this particles will be gauge-dependent

[1]The term ‘universal’ refers to the components present in all versions of the model.

and comes from the insertion of a gauge-fixing Lagrangian. That is what was called the path for identifying these massless particles, effectively traced by the theorem.

The vacuum expectation value (v.e.v) is usually denoted by v, such that the vacuum condition can be read like

$$\frac{\partial V(\Phi)}{\partial \varphi^0}\Big|_{\langle\varphi^0\rangle=v} = 0 \tag{4.43}$$

The minimum point represented as $\langle\varphi^0\rangle = v$ denotes that the derivative is being taken on the fields vacuum expectation value. The φ^0 is then expanded around its v.e.v, $\varphi^0 = v + H$, and the above condition rewritten as:

$$\frac{\partial H}{\partial \phi^0}\frac{\partial V}{\partial H}\Big|_{\langle H\rangle=0} = 0, \qquad \text{or, shortly,} \qquad \partial_H V\Big|_{H=0} = 0 \tag{4.44}$$

The potential $V(H)$ is a polynomial and the Eq.(4.44) implies, therefore, that its linear term on H must vanish, or

$$\mu + \lambda v^2 = 0 \tag{4.45}$$

which enable to rewrite Eq.(4.41) like

$$V(\Phi) = \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2 \tag{4.46}$$

Thus, the zeroth order term vanishes inside the brackets, leaving only the Higgs with a mass term in the function.

The VSC defines a set of equations which will lead to the correct mass matrix coming from the potential. This is seen, for example, in the simplest one dimensional case of the SM. The rotation to the mass eigenstates must also include the contribution coming from quadratic mixed terms present in the gauge-kinetic scalar Lagrangian. These terms are strictly correlated to the gauge-fixing piece, whose treatment leads to the correct mass of the Goldstone bosons. Finally, the Goldstone theorem is essentially based on the vacuum stability and reveal, as a corollary, that these mass matrices, i.e. those coming from the potential and the gauge-fixing, must reside in orthogonal subspaces. On what follows, this result is treated in detail.

By focusing on the potential, it might be important to consider the whole spectra of scalar fields. In order to the standard quarks and leptons acquire their masses, two additional triplets are introduced with the following components:

$$\beta = -\sqrt{3}: \ \rho = \begin{pmatrix}\Phi_\rho \\ \rho_U^{++}\end{pmatrix} \quad (\mathbf{1},\mathbf{3},1), \quad \eta = \begin{pmatrix}\tilde{\Phi}_\eta \\ \eta_V^{+}\end{pmatrix} \quad (\mathbf{1},\mathbf{3},0) \tag{4.47}$$

where Φ_ρ and $\tilde{\Phi}_\eta$ are just denoting that these fields have a similar structure to the doublet in Eq.(4.42) and its conjugate. The expansion of χ^0 around its v.e.v, $\chi^0 = u + \overline{H}$, introduces the new heavy Higgs, $\overline{H}$. As a matter of counting, it will be properly identified those eight scalar fields which are connected, as additional degrees of freedom, to the vectors $V^\pm, U^{\pm\pm}, W^\pm, Z'$ and Z .

It has been emphasized in (4.2.4) that the choice of a specific gauge can be made independently for each gauge interaction. Through their couplings with the SM gauge bosons, the fields in Φ_ρ and $\tilde{\Phi}_\eta$ settle a pattern of mixing that will be followed by the composition of the potential presented below. Thus, it may be more elucidative to keep developing the model under an arbitrary gauge. For $\beta = -\sqrt{3}$,

$$\rho = \begin{pmatrix} \rho_W^+ \\ \frac{(v_\rho + H_\rho) + i\rho_g}{\sqrt{2}} \\ \rho_U^{++} \end{pmatrix}, \qquad \eta = \begin{pmatrix} \frac{(v_\eta + H_\eta) + i\eta_g}{\sqrt{2}} \\ \eta_W^- \\ \eta_V^+ \end{pmatrix} \tag{4.48}$$

and finally introduce the most general β-independent potential [12]

$$\begin{aligned} V(\chi,\rho,\eta) \;=\;& \mu_\chi \chi^\dagger \chi + \lambda_\chi (\chi^\dagger \chi)^2 + \mu_\eta \eta^\dagger \eta + \lambda_\eta (\eta^\dagger \eta)^2 + \mu_\rho \rho^\dagger \rho + \lambda_\rho (\rho^\dagger \rho)^2 + \\ & + \lambda_{\rho\eta} (\rho^\dagger \rho)(\eta^\dagger \eta) + \lambda_{\rho\chi} (\rho^\dagger \rho)(\chi^\dagger \chi) + \lambda_{\eta\chi} (\eta^\dagger \eta)(\chi^\dagger \chi) + \\ & + \overline{\lambda}_{\rho\eta} (\rho^\dagger \eta)(\eta^\dagger \rho) + \overline{\lambda}_{\rho\chi} (\rho^\dagger \chi)(\chi^\dagger \rho) + \overline{\lambda}_{\eta\chi} (\eta^\dagger \chi)(\chi^\dagger \eta) + \\ & + \sqrt{2}\zeta(\epsilon_{ijk} \rho^i \eta^j \chi^k + \text{h.c.}) \end{aligned} \tag{4.49}$$

The first two entries of each triplet define a $SU(2)$ doublet, such that any of the above terms are $SU(3), SU(2), U(1)_X, U(1)_Y$ invariant. In addition, from the definitions of Eq.(4.40), there can also exist a set of new terms added to Eq.(4.49), in the context of specific values for β. In other words, depending on the specific model being regarded, new elements might emerge to the potential.

4.2.3 Vacuum Stability Condition

In this framework, the vacuum stability condition is given by

$$\frac{\partial V(\chi^0,\rho^0,\eta^0)}{\partial \chi^0}\Big|_{\langle \chi^0 \rangle = u} = 0, \quad \frac{\partial V(\chi^0,\rho^0,\eta^0)}{\partial \rho^0}\Big|_{\langle \rho^0 \rangle = v_\rho} = 0, \quad \frac{\partial V(\chi^0,\rho^0,\eta^0)}{\partial \eta^0}\Big|_{\langle \eta^0 \rangle = v_\eta} = 0 \tag{4.50}$$

where ρ^0 and η^0 denote the neutral component of the respective triplet. The relations above are translated into

$$\partial_{\overline{H}} V(\overline{H}, H_\rho, H_\eta)\Big|_{\overline{H}=0} = 0, \quad \partial_{H_\rho} V(\overline{H}, H_\rho, H_\eta)\Big|_{H_\rho=0} = 0, \quad \partial_{H_\eta} V(\overline{H}, H_\rho, H_\eta)\Big|_{H_\eta=0} = 0 \tag{4.51}$$

i.e. the coefficient of linear terms must vanish. For illustration, a geometrical picture can be drawn to the potential vacuum as follows. First, rewrite Eq.(4.49) as

$$\begin{aligned} V(\chi,\rho,\eta) \;=\;& V|_{\rho,\eta=0} + V|_{\chi,\eta=0} + V|_{\chi,\rho=0} + \\ & + \lambda_{\rho\eta} (\rho^\dagger \rho)(\eta^\dagger \eta) + \lambda_{\rho\chi} (\rho^\dagger \rho)(\chi^\dagger \chi) + \lambda_{\eta\chi} (\eta^\dagger \eta)(\chi^\dagger \chi) + \\ & + \overline{\lambda}_{\rho\eta} (\rho^\dagger \eta)(\eta^\dagger \rho) + \overline{\lambda}_{\rho\chi} (\rho^\dagger \chi)(\chi^\dagger \rho) + \overline{\lambda}_{\eta\chi} (\eta^\dagger \chi)(\chi^\dagger \eta) + \\ & + \sqrt{2}\zeta(\epsilon_{ijk} \rho^i \eta^j \chi^k + \text{h.c.}) \end{aligned} \tag{4.52}$$

The first line of Eq.(4.52) is composed by the three functions defining the scenario where the symmetry breaking occur independently, i.e. when just one of the triplets are present in the theory.

The $V|_{\rho,\eta=0}$, for instance, comprises the piece regarding solely the breaking $SU(3)_L \otimes U(1)_X \to SU(2)_L \otimes U(1)_Y$. The vacuum condition to it provides

$$\partial_{\overline{H}}\left(V|_{\rho,\eta=0}\right)\Big|_{\overline{H}=0} = 0 \quad \to \quad \mu_\chi + \lambda_\chi u^2 = 0 \tag{4.53}$$

Now, when the VSC is also assumed for the second and third terms of the first line, it implies

$$\partial_{H_\rho}\left(V|_{\chi,\eta=0}\right)\Big|_{H_\rho=0} = 0 \quad \to \quad \mu_\rho + \lambda_\rho v_\rho^2 = 0 \tag{4.54a}$$

$$\partial_{H_\eta}\left(V|_{\chi,\rho=0}\right)\Big|_{H_\eta=0} = 0 \quad \to \quad \mu_\eta + \lambda_\eta v_\eta^2 = 0 \tag{4.54b}$$

When both Eq.(4.53) and Eq.(4.54) are taken, it is in fact being assumed that the points $(\chi^0, \rho^0, \varphi^0) = \left[(u,0,0),(0,v_\rho,0),(0,0,v_\eta)\right]$ are either local or global minimum of $V(\chi,\rho,\eta)$, along with the point (u, v_ρ, v_η). Thus, the vacuum could be described by the following ellipsoid

$$x^2 + y^2 + z^2 + xy + xz + yz - \frac{(e^2+u^2)}{u}x - \frac{(e^2+v_\rho^2)}{v_\rho}y - \frac{(e^2+v_\eta^2)}{v_\eta}z + e^2 = 0 \tag{4.55}$$

where $e^2 = \frac{uv_\rho + v_\rho v_\eta + uv_\eta}{2}$ and $(x,y,z) \in \left[(u,0,0),(0,v_\rho,0),(0,0,v_\eta),(u,v_\rho,v_\eta)\right]$ defines a subset of possible solutions.

Finally, by applying Eq.(4.51) simultaneously to the total potential it follows the final equations

$$\partial_{\overline{H}} V\Big|_{\overline{H}=0} = 0 \quad \to \quad \mu_\chi + \lambda_\chi u^2 + \frac{v_\eta^2}{2}\lambda_{\eta\chi} + \frac{v_\rho^2}{2}\lambda_{\rho\chi} - \frac{v_\rho v_\eta}{u}\zeta = 0 \tag{4.56a}$$

$$\partial_{H_\rho} V\Big|_{H_\rho=0} = 0 \quad \to \quad \mu_\rho + \lambda_\rho v_\rho^2 + \frac{u^2}{2}\lambda_{\rho\chi} + \frac{v_\eta^2}{2}\lambda_{\rho\eta} - \frac{uv_\eta}{v_\rho}\zeta = 0 \tag{4.56b}$$

and

$$\partial_{H_\eta} V\Big|_{H_\eta=0} = 0 \quad \to \quad \mu_\eta + \lambda_\eta v_\eta^2 + \frac{u^2}{2}\lambda_{\eta\chi} + \frac{v_\rho^2}{2}\lambda_{\rho\eta} - \frac{uv_\rho}{v_\eta}\zeta = 0 \tag{4.56c}$$

As will be detailed, these three conditions are sufficient to leave the model with a well defined mass spectrum to the scalars.

4.2.4 Gauge-Fixing Lagrangian

The gauge interactions of the scalars come from the invariant kinetic term

$$\mathscr{L}_s \supset (D_\mu \chi)^\dagger (D^\mu \chi) \tag{4.57}$$

For $\beta = -\sqrt{3}$, the scalar can be defined as

$$\chi = \begin{pmatrix} (\chi_g^- \ \chi_g^{--}) & \frac{\chi_u^0 + i\chi_g}{\sqrt{2}} \end{pmatrix}^\intercal \tag{4.58}$$

and the g index emphasizes that those states are exactly the Goldstone bosons in the context of a single breaking scale. The Gell-Mann-Nishima relation of Eq.(4.4a) also promptly implies $X_\chi = -1$.

The covariant derivative (4.31), in matrix form, is given by

$$D_\mu = \mathbb{I}\partial_\mu + i\frac{g}{\sqrt{2}}\begin{pmatrix} 0 & W^+ & V^- \\ W^- & 0 & U^{--} \\ V^+ & U^{++} & 0 \end{pmatrix} + i\ \mathrm{diag}\left[\begin{pmatrix} -e & \frac{g(\frac{1}{2}+s_w^2)}{c_w} & g\left(\frac{c_x}{2\sqrt{3}} - \sqrt{3}\mathrm{tg}_w\mathrm{tg}_x\right) \\ -2e & \frac{g(-\frac{1}{2}+2s_w^2)}{c_w} & g\left(\frac{c_x}{2\sqrt{3}} - \sqrt{3}\mathrm{tg}_w\mathrm{tg}_x\right) \\ 0 & 0 & g\left(-\frac{c_x}{\sqrt{3}} - \sqrt{3}\mathrm{tg}_w\mathrm{tg}_x\right) \end{pmatrix}\begin{pmatrix} A \\ Z \\ Z' \end{pmatrix}\right] \tag{4.59}$$

where 'diag' requires the vector inside the brackets to define a diagonal 3×3 matrix. The Eq. (4.31) provides a direct representation whence can be seen how the gauge interactions play with the isospin components of the triplets, aside from the manifest separation into the SM and New Physics terms, one of the main purposes of this work. Nevertheless, in this section the complete vertices concerning scalars and vectors are searched, and the expansion in Eq. (4.59) can make their identification simpler.

From the complete result of Eq.(4.57) only the terms that will be indeed related in the coming analysis must be selected. First, they include those canceled out by the gauge-fixing Lagrangian. Since each factor in $D_\mu\chi$ is at least linear in the fields, it can be set up the quadratic parts plus all the χ_u^0 dependent ones, which yield mass to the gauge bosons and introduce the new physical Higgs $\overline{H}$. This corresponds to

$$\begin{aligned}(D_\mu\chi)^\dagger(D^\mu\chi) \quad = \quad & \frac{1}{2}(\partial\chi_u^0 - i\partial\chi_g^0 - in_{33}\chi_u^0 Z')(\partial\chi_u^0 + i\partial\chi_g^0 + in_{33}\chi_u^0 Z') + \\ & +(\partial\chi_g^- + i\frac{g}{2}\chi_u^0 V^-)(\partial\chi_g^+ - i\frac{g}{2}\chi_u^0 V^+) + \\ & +(\partial\chi_g^{--} + i\frac{g}{2}\chi_u^0 U^{--})(\partial\chi_g^{++} - i\frac{g}{2}\chi_u^0 U^{++}) + \\ & +\mathcal{O}(\text{non-quadratic terms}) \qquad (\beta = -\sqrt{3})\end{aligned} \tag{4.60}$$

where $n_{33} \equiv g\left(-\frac{c_x}{\sqrt{3}} - \sqrt{3}\mathrm{tg}_w\mathrm{tg}_x\right)$. The final vertices will be given by:

$$\begin{aligned}(D_\mu\chi)^\dagger(D^\mu\chi) \quad = \quad & (\partial\chi_u^0)(\partial\chi_u^0) + (\partial\chi_g^0)(\partial\chi_g^0) + (\partial\chi_g^-)(\partial\chi_g^+) + (\partial\chi_g^{--})(\partial\chi_g^{++}) + \\ & +\frac{g^2}{4}\chi_u^0\chi_u^0 V^- V^+ + \frac{g^2}{4}\chi_u^0\chi_u^0 U^{--}U^{++} + \frac{n_{33}^2}{2}\chi_u^0\chi_u^0 Z'Z' + n_{33}(\partial\chi_g^0)\chi_u^0 Z' + \\ & +i\frac{g}{2}(\partial\chi_g^+)\chi_u^0 V^- - i\frac{g}{2}(\partial\chi_g^-)\chi_u^0 V^+ + i\frac{g}{2}(\partial\chi_g^{++})\chi_u^0 U^{--} - i\frac{g}{2}(\partial\chi_g^{--})\chi_u^0 U^{++} + \\ & +\mathcal{O}(\text{non-quadratic terms}) \qquad (\beta = -\sqrt{3})\end{aligned} \tag{4.61}$$

The gauge-fixing factor ξ_{GB} may be defined independently for each gauge-boson and the entire analysis will be considered in the arbitrary t'Hooft gauge. The total Lagrangian to the scalars is given by

$$\mathcal{L}_s = (D_\mu\chi)^\dagger(D^\mu\chi) + (D_\mu\eta)^\dagger(D^\mu\eta) + (D_\mu\rho)^\dagger(D^\mu\rho) \tag{4.62}$$

and accounts for the gauge-bosons acquisition of mass. At this point it appears one of the possible arguments to explain the absence of connection with the SM through non-diagonal interactions. First, designate the pairs $12, 13, 23$ of the triplet entries in Eq.(4.40) as the first, second and third doublets, respectively. Now, since the generators $\mathbb{I}_\pm$ will act only in the first doublet, the above Lagrangian always present the vertices with the 3-component in pairs of exotic particles, forbidding all the tree-level contributions.

From Eq.(4.61) one quadratic mixed term is extracted as

$$\mathscr{L}_s \supset u\frac{g}{2}(\partial\chi_V^+)V^- - u\frac{g}{2}(\partial\chi_V^-)V^+ \tag{4.63}$$

These interactions, in general, are related to a transversal propagator for the gauge-bosons and can be equivalently replaced via the insertion of a gauge-fixing Lagrangian. Apart from that, the scalars involved in the second breaking will also contribute to the mass of V and U, including a similar momentum-dependent vertex

$$\mathscr{L}_s \supset v_\eta\frac{g}{2}(\partial\eta_V^-)V^+ - v_\eta\frac{g}{2}(\partial\eta_V^+)V^- \tag{4.64}$$

The Goldstones will be identified after the rotation of the correlated degrees of freedom inside χ and η. To the above case, it can be achieved via the orthogonal matrix

$$R^V_{\eta\chi} = \frac{1}{\overline{u}_\eta}\begin{pmatrix} u & -v_\eta \\ v_\eta & u \end{pmatrix}, \qquad \overline{u}_\eta = \sqrt{u^2+v_\eta^2} \tag{4.65}$$

or

$$\begin{pmatrix} u, & -v_\eta \end{pmatrix}\partial\begin{pmatrix} \chi_V \\ \eta_V \end{pmatrix}V^- \to \begin{pmatrix} \overline{u}_\eta, & 0 \end{pmatrix}\partial\begin{pmatrix} \overline{\chi}_V \\ \overline{\eta}_V \end{pmatrix}V^- \tag{4.66}$$

where

$$\begin{pmatrix} \overline{\chi}_V \\ \overline{\eta}_V \end{pmatrix} = R^V_{\eta\chi}\begin{pmatrix} \chi_V \\ \eta_V \end{pmatrix} \tag{4.67}$$

From Eq.(4.66) the $\overline{\chi}_V$ can be identified as the Goldstone of the theory. The procedure thus described follows from the group structure and is independent of β. In Section 4.2.7 it will be seen that the rotation Eq.(4.67) defines a larger block diagonal matrix that plays an important role on the diagonalization of the total mass matrix. After the insertion of additional self-interactions, the potential acquires a new structure of mixing that differs from the gauge-fixing mass matrix. However, one may consistently make use of a general consequence of the Goldstone theorem which constrains these two components to lie on orthogonal subspaces, and then verify that the previous knowledge about the rotation of $\mathbb{M}_\xi$ can simplify the procedure of finding the total mass matrix, $\mathbb{M}_s$, diagonal.

In the context of Eq.(4.52), the new matrix on the basis $\begin{pmatrix}\chi_V & \eta_V\end{pmatrix}$ will appear like

$$\begin{pmatrix} \mu_\chi + \lambda_\chi u^2 + \frac{v_\eta^2}{2}\overline{\lambda}_{\eta\chi} + \frac{v_\eta^2}{2}\lambda_{\eta\chi} + \frac{v_\rho^2}{2}\lambda_{\rho\chi} & \frac{uv_\eta}{2}\overline{\lambda}_{\eta\chi} + v_\rho\zeta \\ \frac{uv_\eta}{2}\overline{\lambda}_{\eta\chi} + v_\rho\zeta & \mu_\eta + \lambda_\eta v_\eta^2 + \frac{u^2}{2}\overline{\lambda}_{\eta\chi} + \frac{u^2}{2}\lambda_{\eta\chi} + \frac{v_\rho^2}{2}\lambda_{\rho\eta} \end{pmatrix} \tag{4.68}$$

which can be simplified by applying the VSC of Eq.(4.56) to

$$\begin{aligned} V(\chi,\rho,\eta) \supset \quad & \begin{pmatrix}\chi_V^* & \eta_V^*\end{pmatrix}\begin{pmatrix} \frac{v_\eta^2}{2}\overline{\lambda}_{\eta\chi} + \frac{v_\rho v_\eta}{u}\zeta & \frac{uv_\eta}{2}\overline{\lambda}_{\eta\chi} + v_\rho\zeta \\ \frac{uv_\eta}{2}\overline{\lambda}_{\eta\chi} + v_\rho\zeta & \frac{u^2}{2}\overline{\lambda}_{\eta\chi} + \frac{uv_\rho}{v_\eta}\zeta \end{pmatrix}\begin{pmatrix}\chi_V \\ \eta_V\end{pmatrix} \\ &= \frac{\lambda_V}{2}\begin{pmatrix}\chi_V^* & \eta_V^*\end{pmatrix}\begin{pmatrix} v_\eta^2 & uv_\eta \\ uv_\eta & u^2 \end{pmatrix}\begin{pmatrix}\chi_V \\ \eta_V\end{pmatrix} \end{aligned} \tag{4.69}$$

where

$$\frac{\lambda_V}{2} = \frac{\overline{\lambda}_{\eta\chi}}{2} + \frac{v_\rho \zeta}{u v_\eta} \tag{4.70}$$

The matrix in Eq.(4.69) can also be directly diagonalized via $R^V_{\eta\chi}$, leading to

$$\frac{\lambda_V}{2}\begin{pmatrix}\overline{\chi}^*_V & \overline{\eta}^*_V\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & \overline{u}^2_\eta\end{pmatrix}\begin{pmatrix}\overline{\chi}_V\\ \overline{\eta}_V\end{pmatrix} \quad\rightarrow\quad m^2_\eta = \lambda_V \overline{u}^2_\eta \tag{4.71}$$

The Goldstone theorem is then being manifest. The quadratic mass coming from the gauge-fixing Lagrangian is

$$\mathbb{M}^2_\xi = \xi_V \begin{pmatrix} u^2 & -u v_\eta \\ -u v_\eta & v^2_\eta \end{pmatrix} \quad\rightarrow\quad \mathbb{M}^2_\xi \cdot \begin{pmatrix} v^2_\eta & u v_\eta \\ u v_\eta & u^2 \end{pmatrix} = \mathbb{O} \tag{4.72}$$

and its orthogonality to the matrix in Eq.(4.69) has been registered.

4.2.5 Gauge-Boson Masses

The gauge-dependent mass matrix is sufficient to exhibit the gauge-boson masses which, for completeness, are presented here as extracted from the Scalar Lagrangian:

$$m^2_W = \frac{g^2}{4}\overline{v}^2, \qquad m^2_V = \frac{g^2}{4}\overline{u}^2_\rho, \qquad m^2_U = \frac{g^2}{4}\overline{u}^2_\eta, \qquad m^2_Z = \frac{g^2}{4c^2_w}\overline{v}^2 \tag{4.73}$$

In order to be general one distinguish the U and V masses above. However, on the assumption $u \gg v_\rho, v_\eta$, their values would be equal and different of the Z' via a β dependent factor. For $\beta = -\sqrt{3}$ it follows

$$m^2_{Z'} = \frac{g^2}{3}u^2 c^2_x \qquad (\beta = -\sqrt{3}) \tag{4.74}$$

The bilinears for neutral vectors are given by

$$\mathscr{L}_s \supset -g\frac{c_x}{\sqrt{3}}\partial\chi_g Z' - g\frac{v_\rho}{2c_w}\partial\rho_g Z + g\frac{v_\eta}{2c_w}\partial\eta_g Z \tag{4.75}$$

corresponding, after the $\mathscr{L}_{g.f.}$ insertion, to the correct ξ-dependent Z, Z' propagators and a new mass matrix to the scalars:

$$\mathbb{M}^2_\xi\Big|_{Z,Z'} = \begin{pmatrix} \frac{\xi_Z}{4c^2_w}\begin{pmatrix} v^2_\eta & -v_\eta v_\rho \\ -v_\eta v_\rho & v^2_\rho \end{pmatrix} & \mathbf{0} \\ \mathbf{0}^\intercal & \xi_{Z'}\frac{u^2}{3}c^2_x \end{pmatrix} \tag{4.76}$$

on the basis (η_g, ρ_g, χ_g). It can be verified that the above result is in fact orthogonal to the $\mathbb{P}$ matrix presented in the next section.

4.2.6 Scalar Masses

This section is introduced by approaching the masses for the remaining scalars. The expansion of Eq.(4.52) reveals the pattern of mixing between the charged particles and a result similar to Eq.(4.71) must arise. For the U-type scalars on the basis $\begin{pmatrix}\chi_U & \rho_U\end{pmatrix}$ it follows

$$\begin{pmatrix} \mu_\chi + \lambda_\chi u^2 + \frac{v_\rho^2}{2}\overline{\lambda}_{\rho\chi} + \frac{v_\eta^2}{2}\lambda_{\eta\chi} + \frac{v_\rho^2}{2}\lambda_{\rho\chi} & \frac{uv_\eta}{2}\overline{\lambda}_{\rho\chi} + v_\eta\zeta \\ \frac{uv_\eta}{2}\overline{\lambda}_{\rho\chi} + v_\eta\zeta & \mu_\rho + \lambda_\rho v^2 + \frac{u^2}{2}\overline{\lambda}_{\rho\chi} + \frac{u^2}{2}\lambda_{\rho\chi} + \frac{v_\eta^2}{2}\lambda_{\rho\eta} \end{pmatrix} \tag{4.77}$$

such that, with the help of Eq.(4.56), it can be simplified to

$$\frac{\lambda_U}{2}\begin{pmatrix}\chi_U^* & \rho_U^*\end{pmatrix}\begin{pmatrix} v_\rho^2 & uv_\rho \\ uv_\rho & u^2\end{pmatrix}\begin{pmatrix}\chi_U \\ \rho_U\end{pmatrix} \quad \to \quad \frac{\lambda_U}{2} = \frac{\overline{\lambda}_{\rho\chi}}{2} + \frac{v_\eta\zeta}{uv_\rho} \tag{4.78}$$

or

$$\frac{\lambda_U}{2}\begin{pmatrix}\overline{\chi}_U^* & \overline{\rho}_U^*\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \overline{u}_\rho^2\end{pmatrix}\begin{pmatrix}\overline{\chi}_U \\ \overline{\rho}_U\end{pmatrix} \quad \to \quad m_\rho^2 = \lambda_U \overline{u}_\rho^2 \tag{4.79}$$

where the rotation matrix is given by

$$R_{\rho\chi}^U = \frac{1}{\overline{u}_\rho}\begin{pmatrix} u & -v_\rho \\ v_\rho & u\end{pmatrix} \quad \to \quad \overline{u}_\rho = \sqrt{u^2 + v_\rho^2} \tag{4.80}$$

To conclude the non-diagonal sector, there are those scalars coupled with the SM gauge bosons:

$$\frac{\lambda_W}{2}\begin{pmatrix}\eta_W^* & \rho_W^*\end{pmatrix}\begin{pmatrix} v_\rho^2 & v_\rho v_\eta \\ v_\rho v_\eta & v_\eta^2\end{pmatrix}\begin{pmatrix}\eta_W \\ \rho_W\end{pmatrix} \quad \to \quad \frac{\lambda_W}{2} = \frac{\overline{\lambda}_{\rho\eta}}{2} + \frac{u\zeta}{v_\rho v_\eta} \tag{4.81}$$

or

$$\frac{\lambda_W}{2}\begin{pmatrix}\overline{\eta}_W^* & \overline{\rho}_W^*\end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & \overline{v}^2\end{pmatrix}\begin{pmatrix}\overline{\eta}_W \\ \overline{\rho}_W\end{pmatrix} \quad \to \quad m_{\rho_W}^2 = \lambda_W \overline{v}^2 \tag{4.82}$$

Equivalently,

$$R_{\rho\eta}^W = \frac{1}{\overline{v}}\begin{pmatrix} v_\eta & -v_\rho \\ v_\rho & v_\eta\end{pmatrix} \quad \to \quad \overline{v} = \sqrt{v_\rho^2 + v_\eta^2} \tag{4.83}$$

where

$$\begin{pmatrix}\overline{\eta}_W \\ \overline{\rho}_W\end{pmatrix} = R_{\rho\eta}^W \begin{pmatrix}\eta_W \\ \rho_W\end{pmatrix} \tag{4.84}$$

From the beginning of this review, the Z' has been assumed to not mix with Z and Photon, a claim justified from the hierarchy between the breaking scales. one may expect, therefore, the same pattern to be also present in the neutral scalar sector. On what follows this feature is identified in detail.

First consider the mass matrix of pseudo-scalars on the following basis

$$\begin{pmatrix}\eta_g & \rho_g & \chi_g\end{pmatrix}\mathbb{P}\begin{pmatrix}\eta_g \\ \rho_g \\ \chi_g\end{pmatrix} \tag{4.85}$$

such that the entries of $\mathbb{P}$ are given by

$$\begin{pmatrix} \frac{1}{2}(\mu_\eta + \lambda_\eta v_\eta^2 + \frac{u^2}{2}\lambda_{\eta\chi} + \frac{v_\rho^2}{2}\lambda_{\rho\eta}) & \frac{u\zeta}{2} & \frac{v_\rho\zeta}{2} \\ \frac{u\zeta}{2} & \frac{1}{2}(\mu_\rho + \lambda_\rho v_\rho^2 + \frac{u^2}{2}\lambda_{\rho\chi} + \frac{v_\eta^2}{2}\lambda_{\rho\eta}) & \frac{v_\eta\zeta}{2} \\ \frac{v_\rho\zeta}{2} & \frac{v_\eta\zeta}{2} & \frac{1}{2}(\mu_\chi + \lambda_\chi u^2 + \frac{v_\eta^2}{2}\lambda_{\eta\chi} + \frac{v_\rho^2}{2}\lambda_{\rho\chi}) \end{pmatrix} \tag{4.86}$$

or, simply,

$$\mathbb{P} = \frac{u\zeta}{2}\begin{pmatrix} \frac{v_\rho}{v_\eta} & 1 & \frac{v_\rho}{u} \\ 1 & \frac{v_\eta}{v_\rho} & \frac{v_\eta}{u} \\ \frac{v_\rho}{u} & \frac{v_\eta}{u} & \frac{v_\rho v_\eta}{u^2} \end{pmatrix} \approx \frac{u\zeta}{2}\begin{pmatrix} \frac{v_\rho}{v_\eta} & 1 & 0 \\ 1 & \frac{v_\eta}{v_\rho} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\lambda_P}{2}\begin{pmatrix} v_\rho^2 & v_\rho v_\eta & 0 \\ v_\rho v_\eta & v_\eta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.87}$$

The second signal is representing the absence of contribution from the low-energy breaking to the Z' mass and must not be understood as a formal limit. The constant is given by

$$\lambda_P = \frac{u\zeta}{v_\rho v_\eta} \tag{4.88}$$

Thus the pseudo-scalars are diagonalized to their mass eigenstates by the same matrix $R^W_{\rho\eta}$ that the standard charged scalars. Finally, the potential introduces a single pseudo-scalar, here denoted ρ_P, such that

$$\rho_P : \qquad m_P^2 = \lambda_P \bar{v}^2 \tag{4.89}$$

The presence of this particle is β-independent.

The real scalars follow a similar analysis. In the basis

$$\begin{pmatrix} H_\eta & H_\rho & \overline{H} \end{pmatrix} \mathbb{S} \begin{pmatrix} H_\eta \\ H_\rho \\ \overline{H} \end{pmatrix} \tag{4.90}$$

and after the Eq.(4.56) insertion, the $\mathbb{S}$ entries are

$$\mathbb{S} = \begin{pmatrix} \lambda_\eta v_\eta^2 + \zeta\frac{u v_\rho}{2v_\eta} & \lambda_{\rho\eta}\frac{v_\eta v_\rho}{2} - \frac{u\zeta}{2} & 0 \\ \lambda_{\rho\eta}\frac{v_\eta v_\rho}{2} - \frac{u\zeta}{2} & \lambda_\rho v_\rho^2 + \zeta\frac{u v_\eta}{2v_\rho} & 0 \\ 0 & 0 & \lambda_\chi u^2 + \zeta\frac{v_\eta v_\rho}{2u} \end{pmatrix} \tag{4.91}$$

The above result includes that, in the context of $u \gg v_\rho v_\eta$, the third non-diagonal elements are small compared with the remaining and thus give a neglected contribution to the eigenvalues. Their original values are:

$$\mathbb{S}_{13} \quad : \quad \zeta\frac{v_\rho}{2} - \frac{v_\eta}{u}(\mu_\eta + \lambda_\eta v_\eta^2) - \lambda_{\rho\eta}\frac{v_\rho^2 v_\eta}{2u} \tag{4.92}$$

$$\mathbb{S}_{23} \quad : \quad \zeta\frac{v_\eta}{2} - \frac{v_\rho}{u}(\mu_\rho + \lambda_\rho v_\rho^2) - \lambda_{\rho\eta}\frac{v_\eta^2 v_\rho}{2u} \tag{4.93}$$

The matrix $\mathbb{S}$ can be diagonalized by

$$R_S = \begin{pmatrix} c_s & -s_s & 0 \\ s_s & c_s & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.94}$$

and their simplified eigenvalues are:

$$\begin{aligned} m^2_{H_\eta} &= \frac{1}{v_\eta v_\rho}(\lambda_\eta v_\eta^3 v_\rho + \lambda_\rho v_\rho^3 v_\eta) \\ &= \lambda_\eta v_\eta^2 + \lambda_\rho v_\rho^2 \end{aligned} \tag{4.95}$$

$$\begin{aligned} m^2_{H_\rho} &= \frac{1}{v_\eta v_\rho}(\lambda_\eta v_\eta^3 v_\rho + \lambda_\rho v_\rho^3 v_\eta + u\zeta(v_\rho^2 + v_\eta^2)) \\ &= \lambda_\eta v_\eta^2 + \lambda_\rho v_\rho^2 + u\zeta\frac{(v_\eta^2 + v_\rho^2)}{v_\eta v_\rho} \end{aligned} \tag{4.96}$$

$$m^2_{\overline{H}} = 2\lambda_\chi u^2 + \zeta\frac{v_\eta v_\rho}{u} \tag{4.97}$$

Note, therefore, an approximate relation between the scalar and pseudo-scalar masses:

$$m^2_{H_\rho} = m^2_{H_\eta} + m^2_P \tag{4.98}$$

Thus, as in the Standard Model, the Vacuum Stability Condition prevents the mass terms of some charged scalars to arise in the potential, leaving their contribution exclusively in the gauge-fixing Lagrangian. In the previous discussion one identified this feature occurring for the version-independent 3-3-1HL potential and the VSC leading the model to a correct scalar spectrum. Nevertheless, it has not been mentioned yet about the allowed β-dependent pieces, like for example

$$V(\chi,\rho,\eta)\Big|_{\beta=-\sqrt{3}} \supset V^\chi_{\rho\eta} \equiv \lambda^\chi_{\rho\eta}(\chi^\dagger\eta)(\rho^\dagger\eta) + h.c. \tag{4.99}$$

which could connect the scalars (η_W,ρ_W) with (χ_V,η_V). Along the rest of this work it will be verified that the omission of a term like Eq.(4.99) may imply a complete dissociation of the charged exotic sector with standard particles at tree-level, leaving the Yukawa interactions as the ultimate chance to connect them. In Section (4.2.8) this possibility is disclosed and one conclude that assuming Eq.(4.99) equal to zero is equivalent to assume a discrete symmetry for this variant of the 3-3-1HL.

4.2.7 The Potential for particular models

As mentioned previously, the term $\lambda^\chi_{\rho\eta}(\chi^\dagger\eta)(\rho^\dagger\eta)$ could a priori be included in the potential[2] for $\beta = -\sqrt{3}$, creating a leading order connection with the SM from one additional mixing between the four charged scalars.

[2]For $\beta = +\sqrt{3}$ the new term is similar to Eq.(4.99), with $\rho \leftrightarrow \eta$.

First, the mass matrix will emerge in the basis like $\begin{pmatrix}\eta_W & \rho_W & \eta_V & \chi_V\end{pmatrix}$ after this insertion:

$$\mathbb{M}_V^2 = \begin{pmatrix} \frac{\lambda_W}{2}\begin{pmatrix} v_\rho^2 & v_\rho v_\eta \\ v_\rho v_\eta & v_\eta^2 \end{pmatrix} & \lambda_{\rho\eta}^{\chi}\begin{pmatrix} u v_\rho & v_\eta v_\rho \\ u v_\eta & v_\eta^2 \end{pmatrix} \\ \lambda_{\rho\eta}^{\chi}\begin{pmatrix} u v_\rho & u v_\eta \\ v_\eta v_\rho & v_\eta^2 \end{pmatrix} & \frac{\lambda_V}{2}\begin{pmatrix} u^2 & u v_\eta \\ u v_\eta & v_\eta^2 \end{pmatrix} \end{pmatrix} \tag{4.100}$$

The determinant of $\mathbb{M}_V^2$ above is in fact zero, a necessary condition to leave the theory with a well-defined spectrum.

The gauge-fixing Lagrangian also contributes with a gauge-dependent squared mass-matrix with:

$$\mathbb{M}_\xi^2 = \begin{pmatrix} \xi_W\begin{pmatrix} v_\eta^2 & -v_\rho v_\eta \\ -v_\rho v_\eta & v_\rho^2 \end{pmatrix} & \mathbb{O} \\ \mathbb{O} & \xi_V\begin{pmatrix} v_\eta^2 & -u v_\eta \\ -u v_\eta & u^2 \end{pmatrix} \end{pmatrix} \tag{4.101}$$

As mentioned before, one important consequence of Goldstone theorem requires that these two matrices must lie in orthogonal subspaces, such that

$$\mathbb{M}_V^2 \cdot \mathbb{M}_\xi^2 = 0 \qquad (\mathbb{M}_\xi^2 \cdot \mathbb{M}_V^2 = 0) \tag{4.102}$$

This can be checked directly by requesting a simple identity for matrix products. Consider a general even $n \times n$ matrix:

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_1 & \mathbb{M}_2 \\ \mathbb{M}_3 & \mathbb{M}_4 \end{pmatrix} \tag{4.103}$$

where $\mathbb{M}_k$ are $\frac{n}{2} \times \frac{n}{2}$. Then, it follows that

$$\mathbb{A} \cdot \mathbb{B} = \begin{pmatrix} (\mathbb{A}_1 \cdot \mathbb{B}_1 + \mathbb{A}_2 \cdot \mathbb{B}_3) & (\mathbb{A}_1 \cdot \mathbb{B}_2 + \mathbb{A}_2 \cdot \mathbb{B}_4) \\ (\mathbb{A}_3 \cdot \mathbb{B}_1 + \mathbb{A}_4 \cdot \mathbb{B}_3) & (\mathbb{A}_3 \cdot \mathbb{B}_2 + \mathbb{A}_4 \cdot \mathbb{B}_4) \end{pmatrix} \tag{4.104}$$

which can be proved by considering a single element

$$\mathbb{M}_{ij} \in \begin{cases} \mathbb{M}_1, & i \in [1, \dots \frac{n}{2}], j \in [1, \dots \frac{n}{2}] \\ \mathbb{M}_2, & i \in [1, \dots \frac{n}{2}], j \in [\frac{n}{2}+1, \dots n] \\ \mathbb{M}_3, & i \in [\frac{n}{2}+1, \dots n], j \in [1, \dots \frac{n}{2}] \\ \mathbb{M}_4, & i \in [\frac{n}{2}+1, \dots n], j \in [\frac{n}{2}+1, \dots n] \end{cases} \tag{4.105}$$

and the general rule for matrix products, $(\mathbb{A} \cdot \mathbb{B})_{ij} = \mathbb{A}_{ik}\mathbb{B}_{kj} + \mathbb{A}_{il}\mathbb{B}_{lj}$, with $k \in [1, \dots \frac{n}{2}]$ and $l \in [\frac{n}{2}+1, \dots n]$. For instance, if $(ij) \in \mathbb{M}_2$ then $(\mathbb{A} \cdot \mathbb{B})_{ij} = (\mathbb{A}_1 \cdot \mathbb{B}_2 + \mathbb{A}_2 \cdot \mathbb{B}_4)_{i(j-\frac{n}{2})}$.

Finally, the Eq.(4.104), for $\mathbb{B}_2 = \mathbb{B}_3 = 0$, implies

$$\mathbb{A} \cdot \mathbb{B} = \begin{pmatrix} (\mathbb{A}_1 \cdot \mathbb{B}_1) & (\mathbb{A}_2 \cdot \mathbb{B}_4) \\ (\mathbb{A}_3 \cdot \mathbb{B}_1) & (\mathbb{A}_4 \cdot \mathbb{B}_4) \end{pmatrix} \tag{4.106}$$

and the demonstration of Eq.(4.102) follows straightforwardly.

- The result of Eq.(4.104) can be also extended for odd matrices, and to the 3×3 case we register one useful identity. First, consider the generic matrix

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_1 & \mathbf{m}_2 \\ \mathbf{m}_3^{\mathsf{T}} & m_4 \end{pmatrix} \tag{4.107}$$

where $\mathbf{m}$ are two-dimensional vectors and m_4 is a c-number. Thus,

$$\mathbb{A}\cdot\mathbb{B} = \begin{pmatrix} \left(\mathbb{A}_1\cdot\mathbb{B}_1 + \mathbf{a}_2\cdot\mathbf{b}_3^{\mathsf{T}}\right) & \left(\mathbb{A}_1\cdot\mathbf{b}_2 + \mathbf{a}_2 b_4\right) \\ \left(\mathbf{a}_3^{\mathsf{T}}\cdot\mathbb{B}_1 + a_4\mathbf{b}_3^{\mathsf{T}}\right) & \mathbf{a}_3^{\mathsf{T}}\cdot\mathbf{b}_2 + a_4 b_4 \end{pmatrix} \tag{4.108}$$

Now, define the total squared mass

$$\mathbb{M}_s^2 \equiv \mathbb{M}_V^2 + \mathbb{M}_\xi^2 \tag{4.109}$$

From Eq.(4.102) it follows that

$$\mathbb{M}_s^2\cdot\mathbb{M}_\xi^2 = \mathbb{M}_\xi^2\cdot\mathbb{M}_s^2 = (\mathbb{M}_\xi^2)^2 \tag{4.110}$$

i.e. these matrices must commute:

$$\left[\mathbb{M}_s^2, \mathbb{M}_\xi^2\right] = 0 \tag{4.111}$$

and, therefore, there must be a matrix $\mathbb{U}$ able to diagonalize them simultaneously. Apart from that, as already seen the matrix

$$\mathbb{D} = \begin{pmatrix} \frac{1}{\bar{v}}\begin{pmatrix} v_\eta & -v_\rho \\ v_\rho & v_\eta \end{pmatrix} & \mathbb{O} \\ \mathbb{O} & \frac{1}{\bar{u}_\eta}\begin{pmatrix} u & v_\eta \\ -v_\eta & u \end{pmatrix} \end{pmatrix} \tag{4.112}$$

is such that

$$\mathbb{D}\mathbb{M}_\xi^2\mathbb{D}^{\mathsf{T}} = \mathbb{X}_\xi^2 \tag{4.113}$$

with $\mathbb{X}_\xi^2$ diagonal. Thus, from Eq.(4.111), it follows that $\mathbb{D}\left[\mathbb{M}_s^2, \mathbb{M}_\xi^2\right]\mathbb{D}^{\mathsf{T}} = 0$, or

$$\left[\overline{\mathbb{M}}_s^2, \mathbb{X}_\xi^2\right] = 0, \quad \text{where} \quad \overline{\mathbb{M}}_s^2 \equiv \mathbb{D}\mathbb{M}_s^2\mathbb{D}^{\mathsf{T}} \tag{4.114}$$

i.e. the rotation of the total mass matrix by $\mathbb{D}$ will commute with the diagonal mass matrix of the Goldstone bosons. In terms of components the above result can be written as

$$\sum_k (\overline{\mathbb{M}}_s^2)_{ik}(\mathbb{X}_\xi^2)_{kj} = \sum_k (\mathbb{X}_\xi^2)_{ik}(\overline{\mathbb{M}}_s^2)_{kj} \tag{4.115}$$

or

$$\sum_k \left[(\overline{\mathbb{M}}_s^2)_{ik}\delta_{kj}(\mathbb{X}_\xi^2)_{jj} - \delta_{ik}(\overline{\mathbb{M}}_s^2)_{kj}(\mathbb{X}_\xi^2)_{ii}\right] = 0 \tag{4.116}$$

and, finally,

$$(\overline{\mathbb{M}}_s^2)_{ij}\left\{(\mathbb{X}_\xi^2)_{ii} - (\mathbb{X}_\xi^2)_{jj}\right\} = 0 \tag{4.117}$$

Thus, the $\overline{\mathbb{M}}_s^2$ must emerge into a block diagonal form, whose dimension will agree with the number of eigenvalues inside $\mathbb{X}_\xi^2$ with multiplicity one.

The second step consists on finding a matrix $\overline{\mathbb{D}}$ to diagonalize $\overline{\mathbb{M}}_s^2$ and, in general, simple to define. Again, from Eq.(4.114), the matrix $\mathbb{X}_\xi^2$ would remain block-diagonal after a rotation by $\overline{\mathbb{D}}$, such that

$$(\overline{\mathbb{D}}\mathbb{D})\mathbb{M}_\xi^2(\overline{\mathbb{D}}\mathbb{D})^\intercal = \mathbb{X}_\xi^2 \quad \text{and} \quad (\overline{\mathbb{D}}\mathbb{D})\mathbb{M}_s^2(\overline{\mathbb{D}}\mathbb{D})^\intercal = \mathbb{X}_s^2 \tag{4.118}$$

with $\mathbb{X}_s^2$ diagonal.

In other words, the matrix that can simultaneously diagonalize the total and the gauge-fixing mass matrices is being constructed by parts, after claiming a general Goldstone theorem corollary. It is certainly necessary to look only at the total $\mathbb{M}_s^2$. However, the direct identification of the Goldstone bosons becomes clear and definitive through these steps since one requires the gauge-dependent sector to play a role during the process.

The conclusions can be applied in this particular case, from Eq.(4.100) and Eq.(4.101). The $\mathbb{X}_\xi^2$ was previously obtained and is given by

$$\mathbb{X}_\xi^2 = \begin{pmatrix} \xi_W \begin{pmatrix} \overline{v}^2 & \\ & 0 \end{pmatrix} & \mathbb{O} \\ \mathbb{O} & \xi_V \begin{pmatrix} 0 & \\ & \overline{u}_\eta^2 \end{pmatrix} \end{pmatrix} \tag{4.119}$$

and, from Eq.(4.117), one must expect $(\mathbb{D})\mathbb{M}_s^2(\mathbb{D})^\intercal$ to contain a 2×2 block. In fact,

$$(\mathbb{D})\mathbb{M}_s^2(\mathbb{D})^\intercal = \begin{pmatrix} \xi_W \overline{v}^2 & & \\ & \begin{pmatrix} \frac{\lambda_W}{2}\overline{v}^2 & \lambda_{\rho\eta}^\chi\, \overline{v}\, \overline{u}_\eta \\ \lambda_{\rho\eta}^\chi\, \overline{v}\, \overline{u}_\eta & \frac{\lambda_V}{2}\overline{u}_\eta^2 \end{pmatrix} & \\ & & \xi_V \overline{u}_\eta^2 \end{pmatrix} \tag{4.120}$$

The central matrix can be diagonalized after the insertion of a new $\overline{\mathbb{D}}$, such that

$$\overline{\mathbb{D}} = \begin{pmatrix} 1 & & \\ & \frac{1}{\overline{y}} \begin{pmatrix} a_1 & \lambda_{\rho\eta}^\chi\, \overline{v}\, \overline{u}_\eta \\ \lambda_{\rho\eta}^\chi\, \overline{v}\, \overline{u}_\eta & a_4 \end{pmatrix} & \\ & & 1 \end{pmatrix} \tag{4.121}$$

$$a_1 \equiv \frac{1}{2}\left(\frac{\lambda_W}{2}\overline{v}^2 - \frac{\lambda_V}{2}\overline{u}_\eta^2 - \sqrt{\left(\frac{\lambda_W}{2}\overline{v}^2 - \frac{\lambda_V}{2}\overline{u}_\eta^2\right)^2 + (2\lambda_{\rho\eta}^\chi\, \overline{v}\, \overline{u}_\eta)^2} \right) \tag{4.122}$$

$$a_4 \equiv \frac{1}{2}\left(\frac{\lambda_V}{2}\overline{u}_\eta^2 - \frac{\lambda_W}{2}\overline{v}^2 + \sqrt{\left(\frac{\lambda_W}{2}\overline{v}^2 - \frac{\lambda_V}{2}\overline{u}_\eta^2\right)^2 + (2\lambda_{\rho\eta}^\chi\, \overline{v}\, \overline{u}_\eta)^2} \right) \tag{4.123}$$

$$\overline{y} \equiv \sqrt{(\lambda_{\rho\eta}^\chi\, \overline{v}\, \overline{u}_\eta)^2 + a_1^2} \tag{4.124}$$

and, finally,

$$\mathbb{X}_s^2 \equiv (\mathbb{D}\overline{\mathbb{D}})\mathbb{M}_s^2(\mathbb{D}\overline{\mathbb{D}})^\intercal = \begin{pmatrix} \xi_W \overline{v}^2 & & \\ & \begin{pmatrix} m^2_{\rho_W} & \\ & m^2_\eta \end{pmatrix} & \\ & & \xi_V \overline{u}^2_\eta \end{pmatrix} \tag{4.125}$$

where

$$\begin{aligned} m^2_{\rho_W} &\equiv \frac{1}{2}\left(\frac{\lambda_W}{2}\overline{v}^2 + \frac{\lambda_V}{2}\overline{u}^2_\eta - \sqrt{\left(\frac{\lambda_W}{2}\overline{v}^2 - \frac{\lambda_V}{2}\overline{u}^2_\eta\right)^2 + (2\lambda^\chi_{\rho\eta}\,\overline{v}\,\overline{u}_\eta)^2}\right) \\ m^2_\eta &\equiv \frac{1}{2}\left(\frac{\lambda_W}{2}\overline{v}^2 + \frac{\lambda_V}{2}\overline{u}^2_\eta + \sqrt{\left(\frac{\lambda_W}{2}\overline{v}^2 - \frac{\lambda_V}{2}\overline{u}^2_\eta\right)^2 + (2\lambda^\chi_{\rho\eta}\,\overline{v}\,\overline{u}_\eta)^2}\right) \end{aligned}$$

In summary, the β-independent potential follows a mixing pattern in precise accordance with the $SU(2)$ subalgebras, i.e. by pairs of the scalars entailed by the same gauge-interactions. In the notation of Eq.(4.40) it corresponds to $\left(\chi^V\ \eta^V\right), \left(\chi^U\ \eta^U\right)$ and $\left(\rho^W\ \eta^W\right)$. When this pattern is broken by the insertion of specific allowed terms, the physical fields inside these pairs can be connected through an additional rotation, leaving the gauge-sector unchanged.

The previous analysis on the additional term in the potential holds for $|\beta| = \sqrt{3}$. In addition, from Eq.(4.40), for $|\beta| = \frac{1}{\sqrt{3}}$ one of the triplets ρ or η has the same hypercharge as χ. The authors in [51] introduce a new non-Z_2 piece in the context of neutral heavy leptons, i.e for $\beta = -\frac{1}{\sqrt{3}}$, where χ and η share the same X. These new contributions are given by

$$V(\chi,\rho,\eta)\Big|_{\beta=-\frac{1}{\sqrt{3}}} \supset V_{\chi\eta} \equiv \mu_{\chi\eta}\chi^\dagger\eta + \lambda^\eta_{\chi\eta}(\chi^\dagger\eta)(\eta^\dagger\eta) + \lambda^\rho_{\chi\eta}(\chi^\dagger\eta)(\rho^\dagger\rho) + \lambda^\chi_{\chi\eta}(\chi^\dagger\eta)(\chi^\dagger\chi) + h.c. \tag{4.126}$$

In this version of the model, the gauge-boson V has electric charge equals to zero. The above expression presents a linear term in the complex neutral field χ^0_V, whose coefficient must be zero in order to leave $V(\chi,\rho,\eta)$ with a lower limit at $\langle\chi^0_V\rangle = 0$. This condition converts into the equation:

$$\mu_{\chi\eta} + \lambda^\eta_{\chi\eta}\frac{v^2_\eta}{2} + \lambda^\rho_{\chi\eta}\frac{v^2_\rho}{2} + \lambda^\chi_{\chi\eta}\frac{u^2}{2} = 0 \tag{4.127}$$

Nevertheless, the bilinear $(\chi^\dagger\eta)$ in Eq. (4.126) turns factorized by the r.h.s of the above expression, implying that the $\mu_{\chi\eta}$ portion along with the mixing terms for the charged scalars $\left(\chi_U\ \eta_W\right)$ will vanish. The Eq.(4.126) is rewritten like

$$V_{\chi\eta} = \lambda^\eta_{\chi\eta}(\chi^\dagger\eta)(\eta^\dagger\eta)\big|_{v^2_\eta=0} + \lambda^\rho_{\chi\eta}(\chi^\dagger\eta)(\rho^\dagger\rho)\big|_{v^2_\rho=0} + \lambda^\chi_{\chi\eta}(\chi^\dagger\eta)(\chi^\dagger\chi)\big|_{u^2=0} + h.c. \tag{4.128}$$

and emphasize that the notation excludes only the quadratic v.e.v's, preserving the trilinear interactions.

Thus far the exotic sector is still connected to the SM particles through tree-level interactions with real Higgs bosons. There will still be remaining bilinears involving the real scalars and the neutral (χ_V, η_V):

$$\begin{aligned} V_{\chi\eta} \quad &\supset \quad \frac{1}{\sqrt{2}}\left(\lambda^{\eta}_{\chi\eta} u v_\eta \; \eta_V H_\eta + \lambda^{\rho}_{\chi\eta} u v_\rho \; \eta_V H_\rho + \lambda^{\chi}_{\chi\eta} u^2 \; \eta_V \overline{H} + h.c.\right) + \\ &+ \quad \frac{1}{\sqrt{2}}\left(\lambda^{\eta}_{\chi\eta} v_\eta^2 \; \chi_V H_\eta + \lambda^{\rho}_{\chi\eta} v_\eta v_\rho \; \chi_V H_\rho + \lambda^{\chi}_{\chi\eta} u v_\eta \; \chi_V \overline{H} + h.c.\right) \end{aligned} \tag{4.129}$$

For real $\lambda' s$, these couplings mix the Higgs sector with the real part of the V-scalars, here denoted as (χ^r_V, η^r_V).

The gauge-fixing Lagrangian to the complex neutral boson V will arise similarly to Eq.(4.63) and Eq.(4.64), now with complex conjugates. Thus, on the basis $\left(\chi^r_V \; \eta^r_V \; H_\eta \; H_\rho \; \overline{H}\right)$, only the first block of $\mathbb{M}_\xi$ differs from zero and is equal to the result of Eq.(4.72):

$$\mathbb{M}_{\xi_V} = \begin{pmatrix} \xi_V \begin{pmatrix} u^2 & -u v_\eta \\ -u v_\eta & v_\eta^2 \end{pmatrix} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} \tag{4.130}$$

The contribution from $V(\chi, \rho, \eta)$ is given by

$$\mathbb{M}_V\Big|_{\beta=-\frac{1}{\sqrt{3}}} = \begin{pmatrix} \mathbb{M}_{\chi\eta} & \mathbb{M}_2 \\ \mathbb{M}_2^\dagger & \mathbb{S} \end{pmatrix} \quad \text{where} \quad \mathbb{M}_2 \equiv \sqrt{2}\begin{pmatrix} \lambda^{\eta}_{\chi\eta} u v_\eta & \lambda^{\rho}_{\chi\eta} u v_\rho & \lambda^{\chi}_{\chi\eta} u^2 \\ \lambda^{\eta}_{\chi\eta} u v_\eta & \lambda^{\rho}_{\chi\eta} u v_\rho & \lambda^{\chi}_{\chi\eta} u^2 \end{pmatrix} \tag{4.131}$$

and $\mathbb{M}_{\chi\eta}$ is defined from Eq.(4.69). In order to build an invertible matrix $\mathbb{D}$ for the diagonalization of $\mathbb{M}_\xi$, one can fill it with the rotation matrices as presented in Eq.(4.65) and Eq.(4.94). Finally, from the previous discussion, by applying a rotation for the total mass matrix $\mathbb{M} = \mathbb{M}_{\xi_V} + \mathbb{M}_V$ through $\mathbb{D}$ results a block-diagonal matrix, here represented by 4×4, that can be fully diagonalized after the insertion of an additional $\overline{\mathbb{D}}$.

4.2.8 Self-Interactions of Gauge Bosons

The self-interactions of the gauge-bosons are mediated by

$$\mathscr{L}_{g.b.} = -\frac{1}{4}\mathbf{W}_{\mu\nu}\cdot\mathbf{W}^{\mu\nu} - \frac{1}{4}W^0_{\mu\nu}W^{\mu\nu}_0 \tag{4.132}$$

where $\mathbf{W}_{\mu\nu} = \left(W^1_{\mu\nu}, W^2_{\mu\nu}, \cdots, W^8_{\mu\nu}\right)$, and

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{aki} W^k_\mu W^i_\nu \tag{4.133}$$

or, in a simplified notation,

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - g\mathbf{Tr}\left[\mathbb{F}^a \cdot \mathbb{W}_{\mu\nu}\right] \tag{4.134}$$

with $(\mathbb{F}^a)_{ij} = f^{aij}$ the structure constants, and $(\mathbb{W}^{\mu\nu})_{ij} \equiv W^\mu_i W^\nu_j$.

Before perform the sum in Eq.(4.132), a set of transformations to the non-diagonal fields may be applied:

$$\begin{pmatrix} W^4 \; (W^6) \\ W^5 \; (W^7) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} V^{+q_V} \; (U^{+q_U}) \\ V^{-q_V} \; (U^{-q_U}) \end{pmatrix} \tag{4.135}$$

$$\begin{pmatrix} W^1 \\ W^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} W^+ \\ W^- \end{pmatrix} \tag{4.136}$$

and, finally,

$$\begin{pmatrix} W^0 \\ W^3 \\ W^8 \end{pmatrix} = \begin{pmatrix} -s_x & c_x c_w & -c_x s_w \\ 0 & s_w & c_w \\ c_x & s_x c_w & -s_x s_w \end{pmatrix} \begin{pmatrix} Z' \\ A \\ Z \end{pmatrix} \tag{4.137}$$

One important feature of $\mathscr{L}_{g.b.}$ is the absence of Z' interactions with standard vectors at tree- and loop-level. On the other hand, the charged bosons will couple with SM at loop-level and, for illustration, the vertices for VW and the triple VZA are:

$$VW:\ \mathscr{L}_{g.b.} \supset -\frac{g^2}{2}\left(V_\nu^- V^{+\nu} W_\mu^- W^{+\mu} + V_\mu^- W^{-\mu} V_\nu^+ W^{+\nu} - 2V_\mu^+ W^{-\mu} V_\nu^- W^{+\nu}\right) \tag{4.138}$$

$$\begin{aligned} VZA:\ \mathscr{L}_{g.b.} \supset \quad & \tfrac{g^2}{4}\left(\sqrt{3}s_x(1-2s_w^2) + c_w s_w(1-3s_x^2)\right) \times \\ & \times \left(V_\nu^- A^\nu V_\mu^+ Z^\mu + V_\nu^+ A^\nu V_\mu^- Z^\mu - 2A^\nu Z^\nu V_\mu^+ V^{-\mu}\right) \end{aligned} \tag{4.139}$$

Additional Feynman rules can be extracted from [13].

4.3 Fermions in the 3-3-1HL

This section is a brief description of the fermions in the 3-3-1HL. Among four possible versions, those models with $|\beta| = \frac{1}{\sqrt{3}}$ might require a special attention. This because they introduce heavy quarks and leptons with the same electric charges as the standard particles (see Eq.(4.34) - Eq.(4.38)), corresponding a new pattern of mixing among different generations and then to a large set of original vertices at leading order [45]. Their fermion content can be summarized like:

- $\beta = -\frac{1}{\sqrt{3}}$:
  - Three additional neutral heavy leptons (or right-handed neutrinos);
  - Two additional flavors for D quarks;
  - One new flavor for U quarks.
- $\beta = \frac{1}{\sqrt{3}}$:
  - Three additional heavy leptons with the electron charge;
  - Two additional flavors for U quarks;
  - One new flavor for D quarks.

4.3.1 Gauge interactions of the Fermions

The gauge interactions for fermions may be better represented in a short notation if their left-handed triplets are written like

$$\psi_\alpha^L = \begin{pmatrix} \mathbf{L}^\alpha & E^\alpha \end{pmatrix}_L; \qquad Q_i^L = \begin{pmatrix} \mathbf{q}_i & J_i \end{pmatrix}_L; \qquad Q_3^L = \begin{pmatrix} \mathbf{q}_3 & J_3 \end{pmatrix}_L \tag{4.140}$$

where the index $\alpha = [e,\mu,\tau]$, $i = [1,2]$, and the boldface is just indicating the separation between SM doublets and exotic singlets. The right-handed fields follow the usual notation for the SM, i.e. q^R_α, l^R_α for quarks and leptons, J^R_α, E^R_α for the new particles. Thus, the total gauge-kinetic Lagrangian can be written as

$$\begin{aligned}\mathscr{L}_{\rm kin} \;=\;& i\Big\{\sum_{\alpha=e,\mu,\tau}\overline{\psi}^L_\alpha \not{D}\psi^L_\alpha + \sum_{i=1,2}\overline{Q}^L_i \not{D} Q^L_i + \overline{Q}^L_3 \not{D} Q^L_3 + \\ & \sum_{\alpha=1,2,3}\overline{d}^R_\alpha \not{D} d^R_\alpha + \sum_{\alpha=1,2,3}\overline{u}^R_\alpha \not{D} u^R_\alpha + \sum_{i=1,2}\overline{J}^R_i \not{D} J^R_i + \overline{J}^R_3 \not{D} J^R_3 \\ & \sum_{\alpha=e,\mu,\tau}\overline{l}^R_\alpha \not{D} l^R_\alpha + \sum_{\alpha=e,\mu,\tau}\overline{E}^R_\alpha \not{D} E^R_\alpha\Big\}\end{aligned} \tag{4.141}$$

and each field associates its correspondent covariant derivative.

The following identity might be useful during the task of dividing the 3-3-1 interactions into SM and New Physics::

$$\begin{pmatrix}\mathbf{a}^\dagger & b^*\end{pmatrix}\begin{pmatrix}\mathbb{A} & \mathbf{x}\\ \mathbf{x}^\dagger & z\end{pmatrix}\begin{pmatrix}\mathbf{a}\\ b\end{pmatrix} = \mathbf{a}^\dagger\mathbb{A}\mathbf{a} + \mathbf{a}^\dagger\mathbf{x}b + b^*\mathbf{x}^\dagger\mathbf{a} + b^* z b \tag{4.142}$$

such that the 2×2 matrix $\mathbb{A}$ for the covariant derivative will be given by

$$\mathbb{A} = D^{SM} + i\mathbf{g}^{Z'}_{2\times 2} \tag{4.143}$$

Apart from that, it can be defined define a vector of gauge bosons

$$\mathbf{x}_\mu \propto \frac{g}{\sqrt{2}}\begin{pmatrix}V_\mu & U_\mu\end{pmatrix}^\intercal$$

where the proportionality corresponds to the complex number i, for both $\mathbf{x}$ and $\mathbf{x}^\dagger$. The b component represents the new heavy degrees of freedom, like in the field definition of Eq.(4.140). By applying Eq.(4.142) in Eq.(4.141) it follows

$$\begin{aligned}\mathscr{L}_{\rm kin} \;=\;& i\Big\{\overline{\mathbf{L}}^L_\alpha \not{D}\mathbf{L}^L_\alpha + \overline{\mathbf{q}}^L_\alpha \not{D}\mathbf{q}^L_\alpha + \overline{l}^R_\alpha \not{D} l^R_\alpha + \overline{d}^R_\alpha \not{D} d^R_\alpha + \overline{u}^R_\alpha \not{D} u^R_\alpha\Big\} + \\ & +i\Big\{\overline{\mathbf{L}}^L_\alpha i\mathbf{g}_{Z'}\not{Z}'\mathbf{L}^L_\alpha + \overline{\mathbf{L}}^L_\alpha \not{\mathbf{x}} E^L_\alpha + \overline{E}^L_\alpha \not{\mathbf{x}}^\dagger \mathbf{L}^L_\alpha + \overline{E}^L_\alpha(\not{\partial} + ig_{Z'}\not{Z}' + ig_Z\not{Z} + ieQ\not{A})E^L_\alpha\Big\} + \\ & +i\Big\{\overline{\mathbf{q}}^L_i i\mathbf{g}^*_{Z'}\not{Z}'\mathbf{q}^L_i + \overline{\mathbf{q}}^L_i \not{\mathbf{x}}^* J^L_i + \overline{J}^L_i \not{\mathbf{x}}^\intercal \mathbf{q}^L_i + \overline{J}^L_i(\not{\partial} + ig^*_{Z'}\not{Z}' + ig^*_Z\not{Z} + ieQ\not{A})J^L_i\Big\} + \\ & +i\Big\{\overline{\mathbf{q}}^L_3 i\mathbf{g}_{Z'}\not{Z}'\mathbf{q}^L_3 + \overline{\mathbf{q}}^L_3 \not{\mathbf{x}} J^L_3 + \overline{J}^L_3 \not{\mathbf{x}}^\dagger \mathbf{q}^L_3 + \overline{J}^L_3(\not{\partial} + ig_{Z'}\not{Z}' + ig_Z\not{Z} + ieQ\not{A})J^L_3\Big\} + \\ & +i\Big\{\overline{E}^R_\alpha \not{D} E^R_\alpha + \overline{J}^R_i \not{D} J^R_i + \overline{J}^R_3 \not{D} J^R_3\Big\}\end{aligned} \tag{4.144}$$

and note:

- The first bracket corresponds to the SM gauge kinetic piece and the meaningless difference in this piece due to the conjugate representation has been ignored. Thus, all the indices are running through the total three generations;

- The second contains the interactions for the heavy leptons, in addition to a tree-level term with Z';
- The third contains new exotic quarks and the Z' at tree-level. After the rotation of SM fields to their mass eigenstates, the matrix components of $\mathbf{g}_{Z'}$ give rise to flavor changing currents due to a distinct coupling with the third generation. The couplings must take into account the properties of the Gell-Mann matrices under conjugation and indices cover the first two generations;
- The fourth bracket, in the fundamental representation, is similar to the previous line;
- The last bracket contains the right-handed exotic fields. The leptonic indices cover the three generations while the following term contains the first two heavy quarks;
- The breaking of $SU(3)_L \otimes U(1)_X$ does not lead exactly to a theory invariant under $SU(2)_L \otimes U(1)_Y$, unless the mass of the vectors inside the doublet $\mathbf{x}$ are equal, valid only in the limit $u \gg v_\eta, v_\rho$.

Moreover, the next section discuss the presence of β-dependent Yukawas that mix standard and exotic fermions, thus expanding the first line into additional tree-level contributions. The Eq.(4.144) can be symbolically written as

$$\mathscr{L}_{\text{kin}} = \mathscr{L}_{\text{kin}}^{SM} + \mathscr{L}_{\text{kin}}^{E_L} + \mathscr{L}_{\text{kin}}^{J_{aL}} + \mathscr{L}_{\text{kin}}^{J_{3L}} + \mathscr{L}_{\text{kin}}^{R} \tag{4.145}$$

A brief digression on the described vertices - The present chapter composes the first part of a work that intends to cover the complete integration of these new heavy fields. Thus, the method will result into an effective version of the Standard Model raised through the 3-3-1 gauge structure. As one example, the only gauge-interactions generated at leading order are those with Z'. In the next chapter, it must also be explored the contributions of mixed terms involving exotic fields, loop-suppressed in the expansion of dimension-six operators. Thus, the complete set of terms involving $\mathbf{x}$ only contribute at this level, concealing the interactions with the non-diagonal gauge bosons. The one-loop sector of this Effective Theory will contain only the electromagnetic covariant derivative, i.e. the $U(1)$ invariant piece, apart from the interactions with Z. The terms with three exotic particles must be discarded. Finally, as mentioned in the previous paragraph, for specific β-dependent interactions, the model suppression is reduced from the appearance of linear vertices on the heavy quarks.

4.3.2 Yukawa Lagrangian

The last section discussed the division of Yukawa interactions into those pieces independent of the 3-3-1HL version and those only valid for a particular value of β. In the general case, the interactions are extracted from

$$\begin{aligned}\mathscr{L}_Y \;=\; & \Big(\lambda^d_{i,a}\,\overline{Q}^i_L\eta^* d^a_R + \lambda^d_{3,a}\,\overline{Q}^3_L\rho d^a_R + \lambda^u_{i,a}\,\overline{Q}^i_L\rho^* u^a_R + \\ & + \lambda^u_{3,a}\,\overline{Q}^3_L\eta u^a_R + \lambda^j_{i,k}\,\overline{Q}^i_L\chi^* J^k_R + \lambda^j_{3,3}\,\overline{Q}^3_L\chi J^3_R + \\ & + \lambda^{d*}_{i,a}\,\overline{d}^a_R\eta^\intercal Q^i_L + \lambda^{d*}_{3,a}\,\overline{d}^a_R\rho^\dagger Q^3_L + \lambda^{u*}_{i,a}\,\overline{u}^a_R\rho^\intercal Q^i_L + \\ & + \lambda^{u*}_{3,a}\,\overline{u}^a_R\eta^\dagger Q^3_L + \lambda^{J*}_{i,k}\,\overline{J}^k_R\chi^\intercal Q^i_L + \lambda^{J*}_{3,3}\,\overline{J}^3_R\chi^\dagger Q^3_L\Big) + \\ & \Big(\lambda^l_{a,b}\,\overline{\psi}^a_L\rho l^b_R + \lambda^E_{a,b}\,\overline{\psi}^a_L\chi E^b_R + \lambda^{l*}_{a,b}\,\overline{l}^a_R\rho^\dagger\psi^a_L + \lambda^{E*}_{a,b}\,\overline{E}^b_R\chi^\dagger\psi^a_L\Big) \end{aligned} \tag{4.146}$$

where it was tried to unify the notation presented in [44], [12] and [27]. The indices run as $a,b = 1,2,3$ and $i,k = 1,2$. On what follows it is clarified the reason for changing the lepton notation.

There are a few important comments about the above β-independent Yukawa. Note that both the heavy leptons and the new exotic quarks couples only with the first breaking triplet χ. If the Scalar, Vector and Kinetic Lagrangian cannot connect the high sector with the SM at leading order, the forced absence of mixing between the χ components and standard scalars may compel the new physics to emerge always by pairs of exotic fields, leaving the heavy leptons stable.

One additional form to contour this feature is by considering the complete set of allowed Yukawa terms in the framework of specific versions. For the leptons:

- $\overline{\psi}_L\rho E_R$ and $\overline{\psi}_L\chi l_R$:

  The total hypercharge for these terms is $X = \frac{1}{2} - \beta\frac{\sqrt{3}}{2}$ and would be invariant for $\beta = \frac{1}{\sqrt{3}}$, i.e. in the version with a neutral U gauge boson and where the heavy leptons have the same charge as the electron. The (Ee) mixing creates decay channels $E_\alpha \to SM$ via Eq.(4.144).

- $\overline{\psi}_L\eta E_R$:

  Since $X_\psi = X_\eta$, the total hypercharge is equal to the H.L. electric charge, or $X = q_E$, which is neutral for $\beta = -\frac{1}{\sqrt{3}}$. Apart from that, V is the complex neutral gauge-boson and the SM portal comes from the mixing with neutrino, $(E\nu)$.

Since both triplets are in the fundamental representation, terms with conjugated scalars are forbidden for the gauge symmetry.

These additional lepton interactions appeared in the context of $|\beta| = \frac{1}{\sqrt{3}}$, where the new quarks have the same electric charges as the standard fermions. For $|\beta| = \sqrt{3}$, however, the quarks J have exotic charges and cannot mix with the U or D type. Thus, whenever the β-dependent interactions are omitted from the potential, one may not expect new decay channels for the leptons in these type of 3-3-1 models. Since the Yukawa is the last component of the total Lagrangian, from Eq.(4.39) these stable particles would be electrically charged with -2 or $+1$.

Similarly, new vertices can also be extracted for the quarks. The remaining $\mathscr{L}_Y$ components are finally classified in terms of the β sign:

- $\beta = +\frac{1}{\sqrt{3}}$:

$$\begin{aligned}\mathscr{L}_Y \supset \quad & \lambda^{\chi u}_{i,a}\,\overline{Q}^i_L \chi^* u^a_R + \lambda^{\chi d}_{3,a}\,\overline{Q}^3_L \chi d^a_R + \lambda^{\rho J}_{i,k}\,\overline{Q}^i_L \rho^* J^k_R + \\ & \lambda^{\eta J}_{i,3}\,\overline{Q}^i_L \eta^* J^3_R + \lambda^{\rho J}_{3,3}\,\overline{Q}^3_L \rho J^3_R + \lambda^{\eta J}_{3,i}\,\overline{Q}^3_L \eta J^i_R + h.c.\end{aligned} \tag{4.147}$$

- $\beta = -\frac{1}{\sqrt{3}}$:

$$\begin{aligned}\mathscr{L}_Y \supset \quad & \lambda^{\chi d}_{i,a}\,\overline{Q}^i_L \chi^* d^a_R + \lambda^{\chi u}_{3,a}\,\overline{Q}^3_L \chi u^a_R + \lambda^{\rho J}_{i,3}\,\overline{Q}^i_L \rho^* J^3_R + \\ & +\lambda^{\eta J}_{i,k}\,\overline{Q}^i_L \eta^* J^k_R + \lambda^{\rho J}_{3,i}\,\overline{Q}^3_L \rho J^i_R + \lambda^{\eta J}_{3,3}\,\overline{Q}^3_L \eta J^3_R + h.c.\end{aligned} \tag{4.148}$$

As before, the indices run as $a = 1,2,3$ and $i,k = 1,2$.

There is one important remark on the Yukawa components above - By returning to the high-energy scenario where the symmetry breaking is given exclusively by χ, there will still be one generation of standard leptons and quarks that acquire mass from their mixing with the new heavy fields. In other words, the two first terms of Eq.(4.148), for example, is breaking the $SU(3)_L \otimes U(1)_X$ directly into $U(1)_q$ leaving the bottom and the top-quark massive along with J_i and J_3. On the basis $(u\ c\ t\ J_3)$ this feature can be illustrated as:

$$\begin{pmatrix} & & & a_{14} \\ & \mathbb{O} & & a_{24} \\ & & & a_{34} \\ a^*_{14} & a^*_{24} & a^*_{34} & a_{44} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & m_t & \\ & & & m_J \end{pmatrix}$$

Thus, the theory suggests that a mass hierarchy might be originated from the presence of at least two distinct breaking scales.

The splitting of $\mathscr{L}_Y$ into SM and new terms is almost trivial, in the sense that only the mass Lagrangian of U and D quarks will contribute to the standard part. Notwithstanding, the diagonalization matrices V^U_L and V^U_R, in order to consent with the recent Higgs phenomenology, might strongly constrain the parameters of these new Higgs sector. In the universal 3-3-1HL, i.e. in the context of only β-independent processes, the exotic quarks J_i, for $i = [1,2]$, must mix through a similar pattern as the standard quarks.

4.4 Conclusions

In this brief presentation of the 3-3-1HL components it was intended to arrange a detailed separation between Standard terms and New Physics in order to prepare a model integration. In other words, one focused on the task of select the totality of pieces that must compose a set of effective operators generated at tree- and loop-level. It has been stressed the importance of the variable β to define a particle content and paid a special attention on an universal context defined by the β-independent vertices. Since, in principle, there is not a strong reason for a specific 3-3-1HL choice, it has been claimed that the first step to test its original gauge-structure is by considering those processes present in all possible variants of the model. In this scenario the Yukawa sector, for example, is loop-supressed due to the exclusive presence of exotic mixed terms[3].

The work intended to be complementary to the review section of [13] and it was considered the most general scalar self-interactions by including new β-specific allowed terms, as present in [51]. Finally, one concluded that the omission of these particular contributions is equivalent to assume a discrete symmetry for the variant where $|\beta| = \sqrt{3}$, leaving the theory in the presence of stable particles.

[3]Namely, by vertices in pairs of new fields.

CHAPTER 5

MATCHING IN THE FUNCTIONAL FORMALISM

Once developed the concepts and results supporting the Covariant Derivative Expansion, this chapter intends to provide the first examples of application and comparison with different approaches adopted before.

The previous part of the present work has brought a good amount of new concepts and techniques that must be accompanied by some specific applications. The present chapter will mark a progressive transition, from a review to a minor set of original results. By 'minor' it refers to a small improvement on the examples already present in the main adopted references.

The method behind the construction of an EFT was proposed by defining the matching procedure at tree- and quantum-level. This was performed in two independent scenarios. In the simplest one, the quantum corrected Wilson coefficients were extracted by an Universal Formula, thus called for having been raised in a model independent form and without focusing on any particular statistics. The formula applies whenever the covariant propagator assumes a standard form but is still limited to the case where only the same heavy fields are present in the internal lines. The generalization for heavy and light fields follows directly, although it requires the power counting to be performed depending on the particular context. This condition comes from the various combinations of operators found during the expansion of the Ligth-UV theory into a local Lagrangian. A Covariant Derivative Expansion consists essentially in preserving the covariant derivative untouched during this expansion.

As mentioned before, the case of mixed Wilson coefficients was recently a theme of intense debate in the literature, like in [28] and [53]. The necessity of these terms is present, for example, during the classical matching for the QCD corrections to meson decays, completely developed in [8] and that persists as one important example for the strength of the Effective Field Theory approach to the theoretical analysis of precision observables.

The EFT methodology is equally encompassed by perturbative corrections to the Wilson coefficients coming from the light fields and mediated by the Renormalization Group Equations. The running analysis requires the choice of an operator basis, what occurs according to different criteria. In general, only in the framework of a theory generating multiple coefficients at tree-level that the choice for the complete Warsaw basis of the SMEFT at dim-6 [31] is required. The examples present in this chapter are most related with Higgs phenomenology and Electroweak Precision Observables, whose analysis may be performed for the smaller subset of Bosonic operators of Table 5.1, assuming the remaining Wilson coefficients equal to zero at the electroweak scale.

Table 5.1: The set of dimension-six Bosonic operators. The basis must be complemented by CP-odd operators treated in the Appendix A.4. (see [32])

$O_H = \frac{1}{2}(\partial_\mu \lvert H\rvert^2)^2$	$O_{WB} = 2gg' H^\dagger \tau^a H W^a_{\mu\nu} B^{\mu\nu}$
$O_T = \frac{1}{2}(H^\dagger \overleftrightarrow{D}^\mu H)^2$	$O_W = ig(H^\dagger \tau^a \overleftrightarrow{D}^\mu H) D^\nu W^a_{\mu\nu}$
$O_R = \lvert H\rvert^2 \lvert D_\mu H\rvert^2$	$O_B = ig' Y_H (H^\dagger \overleftrightarrow{D}^\mu H)\partial^\nu B_{\mu\nu}$
$O_D = \lvert D^2 H\rvert^2$	$O_{3G} = \frac{1}{3!} g_s f^{abc} G^{a\mu}_\rho G^{b\nu}_\mu G^{c\rho}_\nu$
$O_6 = \lvert H\rvert^6$	$O_{3W} = \frac{1}{3!} g\epsilon^{abc} W^{a\mu}_\rho W^{b\nu}_\mu W^{c\rho}_\nu$
$O_{GG} = g_s^2 \lvert H\rvert^2 G^a_{\mu\nu} G^{a,\mu\nu}$	$O_{2G} = -\frac{1}{2}(D^\mu G^a_{\mu\nu})^2$
$O_{WW} = g^2 \lvert H\rvert^2 W^a_{\mu\nu} W^{a,\mu\nu}$	$O_{2W} = -\frac{1}{2}(D^\mu W^a_{\mu\nu})^2$
$O_{BB} = g'^2 \lvert H\rvert^2 B_{\mu\nu} B^{\mu\nu}$	$O_{2B} = -\frac{1}{2}(\partial^\mu B^a_{\mu\nu})^2$

5.1 Triplet Scalar

One of the most interesting models picked by HLM as an application source of the CDE technique consists in the Electroweak Theory supplemented with a triplet scalar. This model was chosen in the subsequent works [28] and [25]. In this section it will be considered with the aim of exploring the total formula for the Wilson coefficients and to compare the use of Eq.(2.122) with the results in HLM.

From a phenomenological point of view the model also offers a set of contributions to precision observables, perhaps the most important those related with the T oblique parameter. A careful analysis is left to the Chapter 5.

In the context of an Electroweak Triplet Scalar, the UV Lagrangian is given by

$$\mathscr{L}[\Phi, H] = \mathrm{tr}\left[(D_\mu\Phi)^\dagger(D_\mu\Phi)\right] - m^2\mathrm{tr}\left[\Phi^\dagger\Phi\right] + 2\kappa H^\dagger\Phi H - 2\eta\lvert H\rvert^2\mathrm{tr}\left[\Phi^\dagger\Phi\right] - \frac{1}{2}\lambda_\Phi \mathrm{tr}^2\left[\Phi^\dagger\Phi\right] \quad (5.1)$$

where $\Phi = \phi^a\tau^a$ is a real triplet with $Y_\Phi = 0$, $\tau^a = \frac{\sigma^2}{2}$ and σ^a the Pauli matrices. The Covariant Derivative is defined as $D_\mu\Phi = [D_\mu, \Phi] = (D_\mu\phi^a)\tau^a$, since Φ transforms in the adjoint representation[1]. From $\mathrm{tr}(\tau_a\tau_b) = \dfrac{\delta_{ab}}{2}$, the Eq.(5.1) can be rewritten in a more direct form

$$\mathscr{L}[\Phi, H] = \frac{1}{2}(D_\mu\phi^a)^2 - \frac{1}{2}m^2\phi^a\phi_a + 2\kappa H^\dagger\tau^a H\phi_a - \eta\lvert H\rvert^2\phi^a\phi_a - \frac{1}{4}\lambda_\Phi\left[\phi^a\phi_a\right]^2 \quad (5.4)$$

[1] In fact,

$$\begin{aligned} \Phi \to \Phi' &= U\Phi U^{-1} \\ &= \left(e^{i\alpha_j\tau^j}\tau_a e^{-i\alpha_j\tau^j}\right)\phi^a \quad \to \quad \left(\tau^a + i\alpha_j\epsilon^{jac}\tau_c\right)\phi_a \end{aligned} \quad (5.2)$$

such that

$$\tau_c\phi^c \to \tau_c\left(\phi^c + i\alpha_j(T^j)^{ca}\phi_a\right) \quad or \quad \vec{\phi} \to (\mathbb{I} + i\alpha_j T^j)\vec{\phi} \quad (5.3)$$

where $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$ and T^i the $SU(2)$ generators in the adjoint representation.

For comparison with the results in [33], it is important to connect their notation with that introduced in the Chapter 2, which is achieved through

$$\mathscr{L}_{UV} = \frac{1}{2}\vec{\phi}^{\intercal}(P^2 - m^2 - A)\vec{\phi} \;+\; \vec{\phi}\cdot\vec{B} \;-\; \frac{1}{4}\lambda_{\Phi}(\vec{\phi}\cdot\vec{\phi})^2 \tag{5.5}$$

with

$$A = 2\eta|H|^2 \qquad \text{and} \qquad \vec{B} = 2\kappa H^{\dagger}\vec{\tau}H \tag{5.6}$$

The classical equation of motion defining $\vec{\phi}_c$ is then providing

$$\left.\frac{\delta S_{UV}}{\delta\vec{\phi}}\right|_c = 0 \qquad \rightarrow \qquad \vec{\phi}_c = -[\mathscr{O}^{\phi}]^{-1}\vec{B} \tag{5.7}$$

such that, by neglecting quartic terms, at this moment, the Effective Action at leading order can be expressed like[2]

$$\begin{aligned}\Gamma^{(0)}_{L,UV}[H] \;&\supset\; \frac{1}{2}\vec{B}^{\intercal}[\mathscr{O}^{\phi}]^{-1}\vec{B} - \vec{B}^{\intercal}[\mathscr{O}^{\phi}]^{-1}\vec{B} \\ &=\; -\frac{1}{2}\vec{B}^{\intercal}[\mathscr{O}^{\phi}]^{-1}\vec{B}\end{aligned} \tag{5.8}$$

The EFT Effective Action at tree-level is constructed by the dim-6 local expansion of $\Gamma^{(0)}_{L,UV}$. Since $\vec{B}$ is of dim-2 in the Higgs field, the series must also break at dim-2 terms, i.e.

$$[\mathscr{O}^{\phi}]^{-1} = \frac{1}{P^2 - m^2 - A} \overset{(2.124)}{\supset} -\frac{1}{m^2}\left[1 + \frac{P^2}{m^2}\right] + \frac{A}{m^4} \tag{5.9}$$

and, by replacing it in Eq.(5.8):

$$\Gamma^{(0)}_{EFT}[H] = \frac{1}{2m^2}\vec{B}\cdot\vec{B} + \frac{1}{2m^4}\vec{B}\cdot(P^2 - A)\vec{B} + \text{dim-8} \tag{5.10}$$

From the property

$$\tau^a_{ij}\otimes\tau^a_{kl} = \frac{1}{2}\left(\delta_{il}\otimes\delta_{jk} - \frac{1}{2}\delta_{ij}\otimes\delta_{lk}\right) \tag{5.11}$$

the $\vec{B}\cdot\vec{B}$ may be converted into

$$\begin{aligned}B^aB^a \;&=\; 4\kappa^2(H^{\dagger}\tau^aH)(H^{\dagger}\tau^aH) \\ &=\; \kappa^2|H|^4\end{aligned} \tag{5.12}$$

and, analogously, $B^aAB^a = 2\eta\kappa^2|H|^6$. By recovering the notation $P^{\mu} = iD^{\mu}$, it follows that $\vec{B}\cdot P^2\vec{B} = \vec{B}\cdot(-D^2)\vec{B} = (D_{\mu}\vec{B})^2$. Moreover, from the properties of covariant derivatives in A.2:

$$\begin{aligned}D_{\mu}B^a \;&=\; 2\kappa D_{\mu}(H^{\dagger}\tau^aH) \\ &=\; 2\kappa\left[(D_{\mu}H^{\dagger})\tau^aH + H^{\dagger}\tau^a(D_{\mu}H)\right]\end{aligned} \tag{5.13}$$

[2]The space-time integral is implicit;

which, from Eq.(5.11), will lead to[3]

$$(D_\mu B^a)(D_\mu B^a) = 2\kappa^2 \left\{ 2|D_\mu H|^2|H|^2 + \frac{1}{2}[(D_\mu H^\dagger)H - H^\dagger(D_\mu H)]^2 \right\} \tag{5.15}$$

By defining $H^\dagger \overleftrightarrow{D} H \equiv [(D_\mu H^\dagger)H - H^\dagger(D_\mu H)]$ and the operators $O_T = \frac{1}{2}(H^\dagger \overleftrightarrow{D} H)^2$, $O_R = |D_\mu H|^2|H|^2$ and $O_6 = |H|^6$, the tree-level EFT will be given by

$$S^{(0)}_{EFT} = \frac{\kappa^2}{2m^2}|H|^4 + \frac{\kappa^2}{m^4}(O_T + 2O_R) - \frac{\eta\kappa^2}{m^4}O_6 \tag{5.16}$$

5.1.1 Matching at the Log-Level

This section proposes to apply the Universal Formula for the Triplet Scalar action. As developed in the last chapter, the formula was derived for the *log* in the matching equation strictly connected with heavy fields inside the loop, which in this context is given by

$$\sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] \supset i\alpha \mathrm{Tr}\ \log\left(\left.\frac{\delta^2 S_{UV}[\varphi,\Phi]}{\delta\Phi^2}\right|_{\Phi_c[\varphi]} \right) = \frac{i}{2}\mathrm{Tr}\log[-P^2 + m^2 + U] \tag{5.17}$$

where now

$$U = A + \lambda_\Phi[\vec{\phi}_c^\intercal \vec{\phi}_c + 2\vec{\phi}_c\vec{\phi}_c^\intercal] \tag{5.18}$$

and the notation for Eq.(2.83) is recovered, with $c_s = \frac{1}{2}$. Next, since U is at least order two in the fields, the formula provides terms up to the order $\frac{1}{m^2}$. Moreover, ϕ^c is of dim-2, such that the brackets in Eq.(5.18) will enter only in the m^2 and m^0 lines.

- The first piece is for tr U, or

$$\mathscr{L}^{(1)}_{EFT} \supset m^2\left(1 - \log\left(\frac{m^2}{\mu}\right)\right)\mathrm{tr}U \tag{5.19}$$

where

$$\mathrm{tr}U = \mathrm{tr}A + \lambda_\Phi \mathrm{tr}[\vec{\phi}_c^\intercal \vec{\phi}_c + 2\vec{\phi}_c\vec{\phi}_c^\intercal] \tag{5.20}$$

As a comment on notation, all the capital letters without an arrow are 3×3 matrices, in contrast with the vectors. Thus, $\vec{B}\vec{B}^\intercal$ is a normal matrix product such that $\mathrm{tr}[\vec{B}\vec{B}^\intercal] = B^a B^a$. To the above case $\vec{\phi}_c = (\phi^1, \phi^2, \phi^3)^\intercal$ and the trace over the brackets can be rewritten like

$$\mathrm{tr}[\vec{\phi}_c^\intercal \vec{\phi}_c + 2\vec{\phi}_c\vec{\phi}_c^\intercal] = 5\vec{\phi}_c^\intercal \vec{\phi}_c \tag{5.21}$$

or, finally,

$$\mathrm{tr}U = \mathrm{tr}A + 5\lambda_\Phi \mathrm{tr}[[\mathscr{O}^\Phi]^{-1}\vec{B}]\cdot[[\mathscr{O}^\Phi]^{-1}\vec{B}] \tag{5.22}$$

[3]For example, the first term of the expansion corresponds to

$$(D_\mu B^a)(D_\mu B^a) \supset [(D_\mu H^\dagger)\tau^a_{ij}H][H^\dagger\tau^a_{kl}(D_\mu H)] = (D_\mu H^\dagger)(D_\mu H)|H|^2 - \frac{1}{2}[(D_\mu H^\dagger)H][H^\dagger(D_\mu H)] \tag{5.14}$$

and the expansion of the second term is settled to stop at dim-6 operator, i.e.

$$[[\mathcal{O}^{\Phi}]^{-1}\vec{B}]\cdot[[\mathcal{O}^{\Phi}]^{-1}\vec{B}] = \frac{1}{m^4}B^aB^a + \frac{2}{m^6}[B^a(P^2B^a) - AB^aB^a] \tag{5.23}$$

From Eq.(5.14),

$$\mathrm{tr}[B^a(P^2B^a)] = 2\kappa^2(O_T + 2O_R), \quad \mathrm{tr}[B^aB^a] = \kappa^2|H|^4, \quad \mathrm{tr}[AB^aB^a] = 2\eta\kappa^2 O_6 \tag{5.24}$$

Since $\mathrm{tr}A = 6\eta|H|^2$ it follows that

$$\mathrm{tr}U = 6\eta|H|^2 + 5\frac{\lambda_\Phi}{m^4}\left[\kappa^2|H|^4 - \frac{4\eta\kappa^2}{m^2}O_6 + \frac{4\kappa^2}{m^2}(O_T + 2O_R)\right] \tag{5.25}$$

- The next piece is $\mathrm{tr}U^2$. From power counting only the terms below must contribute:

$$\begin{aligned}
\mathrm{tr}U^2 &\supset \mathrm{tr}[A^2 + 2A\lambda_\Phi[\vec{\phi}_c^\intercal\vec{\phi}_c + 2\vec{\phi}_c\vec{\phi}_c^\intercal]] \\
&= \mathrm{tr}A^2 + 10A\phi_c^a\phi_c^a \\
&= \mathrm{tr}A^2 + \frac{10}{m^4}AB^aB^a + O(\text{dim-8}) \\
&= 12\eta^2|H|^4 + \frac{20}{m^4}\lambda_\Phi\eta\kappa^2 O_6
\end{aligned} \tag{5.26}$$

- Next, the first trace involving field-strength. For remind, $G_{\mu\nu} = [D_\mu, D_\nu]$, where $D_\mu = \partial_\mu - igW_\mu^aT^a$, being T^a the generators on the respective representation of the fields under application, i.e. the adjoint in the actual context. In other words, $[D_\mu, D_\nu] = -igW_{\mu\nu}^aT^a$ where

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\epsilon^{abc}W_\mu^bW_\nu^c \tag{5.27}$$

Finally,

$$\begin{aligned}
\mathrm{tr}G_{\mu\nu}G^{\mu\nu} &= -g^2W_{\mu\nu}^aW^{b,\mu\nu}\mathrm{tr}(T^aT^b) \\
&\quad -2g^2(W_{\mu\nu}^a)^2
\end{aligned} \tag{5.28}$$

since $(T^aT^b) = 2\delta_{ab}$.

- $\mathrm{tr}[D_\mu, U]^2$: Again, from power counting, only the A inside the definition of U must contribute. Besides, since A is a singlet, $[D_\mu, A] \to \partial_\mu A$, or

$$\begin{aligned}
\mathrm{tr}[D_\mu, U]^2 &= 12\eta(\partial_\mu|H|^2)^2 \\
&= 24\eta O_H
\end{aligned} \tag{5.29}$$

- $\mathrm{tr}(UG_{\mu\nu}G^{\mu\nu}) \supset -g^2AW_{\mu\nu}^aW^{b,\mu\nu}\mathrm{tr}(T^aT^b)$, or

$$\mathrm{tr}(UG_{\mu\nu}G^{\mu\nu}) \supset -4\eta g^2(W_{\mu\nu}^a)^2|H|^2 \equiv O_{WW} \tag{5.30}$$

- $\mathrm{tr}U^3$:

$$\mathrm{tr}U^3 \supset \mathrm{tr}A^3 = 24\eta^3 O_6 \tag{5.31}$$

- $\mathrm{tr}[D^\mu, G_{\mu\nu}]^2$:

$$\begin{aligned} \mathrm{tr}[D^\mu, G_{\mu\nu}]^2 &= \mathrm{tr}([D^\mu, G_{\mu\nu}][D^\alpha, G_{\alpha\nu}]) \\ &= -g^2[D^\mu, W^a_{\mu\nu}][D^\alpha, W^{b,\alpha\nu}]\,\mathrm{tr}[T^a T^b] \\ &= -2g^2[D^\mu, W^a_{\mu\nu}]^2 \equiv 4g^2 O_{2W} \end{aligned} \tag{5.32}$$

- Finally, $\mathrm{tr}(G^\nu_\mu G^\rho_\nu G^\mu_\rho)$:

$$\begin{aligned} \mathrm{tr}(G^\nu_\mu G^\rho_\nu G^\mu_\rho) &= ig^3 W^{a,\nu}_\mu W^{b,\rho}_\nu W^{c,\mu}_\rho\,\mathrm{tr}(T^a T^b T^c) \\ &= -g^3\epsilon_{abc} W^{a,\nu}_\mu W^{b,\rho}_\nu W^{c,\mu}_\rho \equiv -6g^2 O_{3W} \end{aligned} \tag{5.33}$$

The above results can then be collected in a final expression through Eq.(2.83):

$$\begin{aligned} \mathscr{L}^{(1)}_{EFT} &= \frac{c_s}{(4\pi)^2}\Big\{6\eta m^2|H|^2 + 5\lambda_\Phi\frac{\kappa^2}{m^2}|H|^4 - 10\eta\lambda_\Phi\frac{\kappa^2}{m^4}O_6 + 20\lambda_\Phi\frac{\kappa^2}{m^4}(O_T + 2O_R) + 6\eta^2|H|^4 + \\ &\quad + \frac{g^2}{6}(W^a_{\mu\nu})^2 + 2\frac{\eta}{m^2}O_H + g^2\frac{\eta}{3m^2}O_{WW} - 4\frac{\eta^3}{m^2}O_6 + \frac{g^2}{15m^2}O_{2W} - \frac{g^2}{10m^2}O_{3W}\Big\} \end{aligned} \tag{5.34}$$

with $c_s = \frac{1}{2}$ and at $\mu = m$.

5.1.2 Wilson Coefficients from Mixed Loops

The derivation of Eq.(2.120) was supported by a very general definition of 'light-fields', denoted by φ, comprising the entire set of Standard Model particles. The Hessian matrix present in $\mathscr{A}_i$ in reality is counting with contributions of all interactions present in the SM. This is coherent, since those results are present in the scenario with both heavy and light fields running in the loops. Notwithstanding, the computation of this large matrix can be a very arduous task and the choice of which sector to include in the analysis is something commonly done in practice. The main argument behind this selection might be related with the search of operators which were already produced at tree-level. As it must be discussed in detail on the following sections, the Wilson coefficients generated at leading order will be significant at the level of running them down to low-energy scales and will be presented inside a pre-selected subset of the SMEFT operator basis. Therefore, on what follows the analysis of mixed terms will focus specifically on Higgs operators and thus consider only the H sector as the light-fields. For simplicity, the quartic and self-interactions of heavy fields will be left out of the computation, being their influence represented by the previous results.

The UV theory will usually contain tri- or quartic interactions, such the operator $\mathscr{O}^\varphi$ inside Eq.(2.116) may be accompanied by a new Hessian over the SM action. Here this additional matrix will be denoted like

$$\frac{\delta^2_{SM}}{\delta\varphi^2} \to \mathscr{O}^\varphi + \mathscr{H}_\lambda \tag{5.35}$$

with a clear reference of the Higgs self-couplings. The inverse $[\mathscr{O}^{\varphi}]^{-1}$ in the definition of $\mathscr{A}_{(i)}$ in Eq.(2.118) must then be shifted by

$$[\mathscr{O}^{\varphi}]^{-1} \to [\mathscr{O}^{\varphi} + \mathscr{H}_{\lambda}]^{-1} \tag{5.36}$$

which, according to Eq.(2.125), will expand like

$$[\mathscr{O}^{\varphi} + \mathscr{H}_{\lambda}]^{-1} = \Delta^{\varphi} - \Delta^{\varphi}\mathscr{H}_{\lambda}\Delta^{\varphi} + \cdots \tag{5.37}$$

and for the coming computations these first two parts must saturate the dim-6 criteria.

The main purpose of this section concerns the application of Eq.(2.120), registered below, what goes in a algorithmically mode.

$$\sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] \quad \supset \quad i\alpha \mathrm{Tr} \sum_{p} \frac{1}{p} \left\{ \sum_{k=1}^{p} \binom{p}{k} \mathscr{A}_{(0)}^{p-k} \mathscr{A}_{(1)}^{k} \right\} \tag{5.38}$$

As mentioned before the Eq.(2.116) is a exact representation of the subtraction and $\mathscr{H}_{(1)}$ can actually differ in form from its local counterpart $\mathscr{H}_{(0)}$ when quartic self-interactions of heavy-fields are included. Nevertheless, these terms are more suppressed at this part of the matching than in the previous one, with exclusive heavy fields, and can be assumed to not enter on the computation. This implies that the both matrices are equivalent in their dependence with light-fields, i.e. with

$$\mathscr{A}_{(i)} \equiv \left[\mathscr{O}^{\varphi}\right]^{-1} \mathscr{H}\left[\frac{1}{2} B_{\varphi}\, \mathscr{O}^{\Phi}_{(i)}\, B_{\varphi}\right] \tag{5.39}$$

and the derivations can then be taken once, independent if it is for local or non-local operators. The steps to determine Eq.(5.38) can be summarized like:

- Computation of $\mathscr{H}_{(i)} \equiv \mathscr{H}\left[\frac{1}{2} B_{\varphi}\, \mathscr{O}^{\Phi}_{(i)}\, B_{\varphi}\right]$:

  A Hessian matrix for complex-valued functions $g(z_a, z_b)$ is given by[4]

$$\begin{pmatrix} \partial^2_{z_a \bar{z}_a} & \partial^2_{z_a \bar{z}_b} \\ \partial^2_{z_b \bar{z}_a} & \partial^2_{z_b \bar{z}_b} \end{pmatrix} \quad \to \quad \begin{pmatrix} \delta^2_{HH^\dagger} & \delta^2_{HH} \\ \delta^2_{H^\dagger H^\dagger} & \delta^2_{H^\dagger H} \end{pmatrix} \tag{5.40}$$

  where the second one is the functional representation for the present case.

  The B_{φ}, as defined in Eq.(5.6), is given by $\vec{B}_{\varphi} = 2\kappa H^\dagger \vec{\tau} H$, thus converting $\mathscr{H}_{(i)}$ into[5]

$$\mathscr{H}_{(i)} = 4\kappa^2 \begin{pmatrix} \tau^a \mathscr{O}_{(i)}(H^\dagger \tau_a H) + (\tau^a H)\mathscr{O}_{(i)}(H^\dagger \tau_a) & (H^\dagger \tau_a)^\intercal \mathscr{O}_{(i)}(H^\dagger \tau_a) \\ (\tau^a H)\mathscr{O}_{(i)}(\tau^a H)^\intercal & (\tau^a)^\intercal \mathscr{O}_{(i)}(H^\dagger \tau_a H) + (H^\dagger \tau_a)^\intercal \mathscr{O}_{(i)}(\tau_a H)^\intercal \end{pmatrix} \tag{5.44}$$

[4] In order to convert it into the real-components of $z = x + iy$, the Wirting derivatives $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ must be applied;

[5] For clarity, the notation considers the usual product with line-vectors on the right and column-vectors on the left, implying a final 2×2 matrix. Apart from that, the operator will commute with the Higgs fields through two integration by parts. In fact, the derivatives are performed a priori, at the level of action. For instance,

$$\text{if } S[\phi] = \int_x \phi P^2 \phi, \quad \text{thus} \quad \frac{\partial S}{\partial \phi_y} = \int_x \delta_{xy} P^2 \phi + \phi P^2 \delta_{xy} = 2\int_x \delta_{xy} P^2 \phi \tag{5.41}$$

- Definition of $\mathscr{O}_{(i)}$:

  From Eq.2.112 the definition of $\overline{\mathscr{O}}^{\Phi}_{(0)}$ remounts the local-expansion of $[\mathscr{O}^{\Phi}]^{-1}$ up to dim-6 operators at tree-level. For the purpose of this section, the series is given by Eq.2.113

$$[\mathscr{O}^{\Phi}]^{-1} = -\frac{1}{M^2}\left(1+\frac{P^2}{M^2}+\cdots+\frac{(P^2)^{n-1}}{(M^2)^{n-1}}\right)\Bigg|_{\text{local}} + \frac{(P^2)^n}{(M^2)^n}\frac{1}{P^2-M^2} \tag{5.45}$$

  and ran up to $n=2$, which implies:

$$\overline{\mathscr{O}}^{\Phi}_{(0)} \;=\; -\frac{1}{m^2}\left(1+\frac{P^2}{m^2}\right), \qquad \mathscr{O}^{\Phi}_{(1)} \;=\; \frac{(P^2)^2}{(m^2)^2}\frac{1}{P^2-m^2} \tag{5.46}$$

- Computation of $[\mathscr{O}^{\varphi}]^{-1}$:

  As asserted before, for comparison with HLM[6], the SM Higgs self-interactions was the sector of light fields chosen to be explored here, i.e.

$$\mathscr{L}_{SM} \supset H^{\dagger}(P^2-m_H^2)H - \frac{\lambda}{4}(H^{\dagger}H)^2 \tag{5.47}$$

  where m_H has been taken as a matter of convenience with the notation and can be set to zero at the end of the calculation.

  The presence of λ will shift $\mathscr{O}^{\varphi}$ by a Hessian denoted as $\mathscr{H}_{\lambda}$ and given by

$$\mathscr{H}_{\lambda} = 2\begin{pmatrix} (H^{\dagger}H)\mathbb{I}+HH^{\dagger} & H^{*}H^{\intercal} \\ HH^{\intercal} & (H^{\dagger}H)\mathbb{I}+H^{*}H^{\intercal} \end{pmatrix} \tag{5.48}$$

  such that $\mathscr{O}^{\varphi} \to \mathscr{O}^{\varphi}+\mathscr{H}_{\lambda}$ in the formula for $\mathscr{A}_{(i)}$. By recovering Eq.(2.125)

$$[\mathscr{O}]^{-1} \;=\; \frac{1}{P^2-m^2-A} \;=\; \sum_{n=0}^{\infty}\left[\frac{1}{P^2-m^2}A\right]^n \frac{1}{P^2-m^2} \tag{2.125}$$

  at first order in λ it implies

$$[\mathscr{O}^{\varphi}]^{-1} \to \frac{1}{P^2-m_H^2} + \frac{\lambda}{4}\,\frac{1}{P^2-m_H^2}\mathscr{H}_{\lambda}\,\frac{1}{P^2-m_H^2} \tag{5.49}$$

As one example for $\mathscr{H}_{(i)}$, the last entry is determined like:

$$(H^{\dagger}\vec{\tau}H)\mathscr{O}^{\Phi}_{(i)}(H^{\dagger}\vec{\tau}H) = (H_k^{*}\tau^{c}_{kj}H_j)\mathscr{O}^{\Phi}_{(i)}(H_l^{*}\tau^{c}_{lp}H_p) \tag{5.42}$$

such that

$$\begin{aligned} \delta_{H_aH_b^*} &= 2\delta_{H_a}(\tau^{c}_{bj}H_j)\mathscr{O}^{\Phi}_{(i)}(H_l^{*}\tau^{c}_{lp}H_p) \\ &= 2\left[\tau^{c}_{ba}\mathscr{O}^{\Phi}_{(i)}(H_l^{*}\tau^{c}_{lp}H_p)+(\tau^{c}_{bj}H_j)\mathscr{O}^{\Phi}_{(i)}(H_l^{*}\tau^{c}_{la})\right] \\ \delta_{H^{\dagger}H} &= 2\left[(\tau^{c})^{\intercal}\mathscr{O}^{\Phi}_{(i)}(H^{\dagger}\tau^{c}H)+(H^{\dagger}\tau^{c})^{\intercal}\mathscr{O}^{\Phi}_{(i)}(\tau^{c}H)^{\intercal}\right] \end{aligned} \tag{5.43}$$

[6]The abbreviation ‘HLM’ may refer either to [32] or to [33], since both compose a work on the general topic of SMEFT;

- Computation of $\mathscr{A}_{(i)}$:

  Finally, from the above expression it follows that

$$\mathscr{A}_{(i)} = \Delta^H \ \mathscr{H}_{(i)} + \frac{\lambda}{4} \ \Delta^H \mathscr{H}_\lambda \ \Delta^H \ \mathscr{H}_{(i)} \tag{5.50}$$

In order to estimate in which point the series Eq.(5.38) should be interrupted, the power criteria cannot be used yet. The reason is that, once the Covariant Derivative is still present inside a functional trace, from the method presented in Section 2.3, D^μ must acquire a shift $D \to D + q$ and thus does not correspond to a dim-1 object. Nevertheless, it is still possible to conclude that both $\mathscr{A}_i$ will be at least of dim-2, such that:

$$\begin{aligned} i\alpha \mathrm{Tr} \sum_p \frac{1}{p} \left\{ \sum_{k=1}^{p} \binom{p}{k} \mathscr{A}_{(0)}^{p-k} \mathscr{A}_{(1)}^{k} \right\} \quad &= \quad i\alpha \mathrm{Tr} \Big\{ \mathscr{A}_{(1)} + \frac{1}{2} \left[2\mathscr{A}_{(0)}\mathscr{A}_{(1)} + \mathscr{A}_{(1)}^2 \right] + \\ &+ \frac{1}{3} \left[3 \left(\mathscr{A}_{(0)}^2 \mathscr{A}_{(1)} + \mathscr{A}_{(0)} \mathscr{A}_{(1)}^2 \right) + \mathscr{A}_{(1)}^3 \right] + \text{dim-8} \Big\} \end{aligned} \tag{5.51}$$

Before continuing with the examples, some comments could prevent any apparent peculiarity over all these steps. At principle, it may seem that this entire algorithm are comparable in complexity with the Feynman diagrams approach. It may be emphasized, however, that these corrections are being performed in a covariant mode and can cover a large set of process from the single criteria of dimension-6 operators.

For comparison and illustration, here it must be traced the first and second elements of the series Eq.(5.51), with the results present in [33] properly identified:

- Tr $[\mathscr{A}_{(1)}]$:

  From Eq.(5.50) it follows that

$$\mathscr{A}_{(1)} = \Delta^H \ \mathscr{H}_{(1)} + \frac{\lambda}{4} \ \Delta^H \mathscr{H}_\lambda \ \Delta^H \ \mathscr{H}_{(1)} \tag{5.52}$$

  where $\mathscr{H}_1$ is given Eq.(5.44) and

$$\mathscr{O}_{(1)}^{\Phi} \ = \ \frac{(P^2)^2}{(m^2)^2} \frac{1}{P^2 - m^2} \tag{5.46}$$

  The first contribution implies

$$\begin{aligned} \mathrm{Tr} \left[\Delta^H \mathscr{H}_{(1)} \right] \quad &= \quad \mathrm{Tr} \left[\Delta^H \mathscr{H}_{(1)}^{H^\dagger H} \right] + \mathrm{Tr} \left[\Delta^H \mathscr{H}_{(1)}^{HH^\dagger} \right] \\ &\supset \quad \frac{4\kappa^2}{m^4} \mathrm{Tr} \left[\Delta^H \tau^a \left\{ \frac{(P^2)^2}{P^2 - m^2} (H^\dagger \tau^a H) \right\} + \Delta^H (\tau^a H) \frac{(P^2)^2}{P^2 - m^2} (H^\dagger \tau^a) \right] \\ &= \quad \frac{4\kappa^2}{m^4} \mathrm{Tr} \left[\Delta^H (\tau^a H) \frac{(P^2)^2}{P^2 - m^2} (H^\dagger \tau^a) \right] \end{aligned} \tag{5.53}$$

  where the first piece vanish by the traceless Pauli matrices. The above result is precisely what [33] defined as S_K. The next step consist in the functional trace and will be performed in the next section.

The second contribution is given by

$$\begin{aligned}\frac{\lambda}{4}\text{Tr}\left[\Delta^H \mathscr{H}_\lambda\, \Delta^H\, \mathscr{H}_{(1)}\right] &= \frac{\lambda}{4}\text{Tr}\Big[\Delta^H \mathscr{H}_\lambda^{H^\dagger H}\, \Delta^H\, \mathscr{H}_{(1)}^{H^\dagger H} + \Delta^H \mathscr{H}_\lambda^{HH}\, \Delta^H\, \mathscr{H}_{(1)}^{H^\dagger H^\dagger} + \\ &\quad + \Delta^H \mathscr{H}_\lambda^{H^\dagger H^\dagger}\, \Delta^H\, \mathscr{H}_{(1)}^{HH} + \Delta^H \mathscr{H}_\lambda^{HH^\dagger}\, \Delta^H\, \mathscr{H}_{(1)}^{HH^\dagger}\Big] \qquad (5.54)\end{aligned}$$

where the $\mathscr{H}_\lambda$ components are written in Eq.(5.48). Considering one term

$$\begin{aligned}\frac{\lambda}{4}\text{Tr}\left[\Delta^H \mathscr{H}_\lambda\, \Delta^H\, \mathscr{H}_{(1)}\right] &\supset \frac{\lambda}{4}\text{Tr}\left[\Delta^H \mathscr{H}_\lambda^{H^\dagger H}\, \Delta^H\, \mathscr{H}_{(1)}^{H^\dagger H}\right] \\ &= 2\lambda\kappa^2\text{Tr}\Big[\Delta^H\, [(H^\dagger H)\mathbb{1} + H\, H^\dagger] \times \\ &\quad \times \Delta^H[\tau^a \mathscr{O}_{(1)}(H^\dagger \tau_a H) + (\tau^a H)\, \mathscr{O}_{(1)}\, (H^\dagger \tau_a)]\Big] \qquad (5.55)\end{aligned}$$

or[7]

$$\frac{\lambda}{4}\text{Tr}\left[\Delta^H \mathscr{H}_\lambda\, \Delta^H\, \mathscr{H}_{(1)}\right] \supset \frac{2\lambda\kappa^2}{m^4}\text{Tr}\Big[\Delta^H\, [(H^\dagger H)\mathbb{1} + H\, H^\dagger]\Delta^H[(\tau^a H)\, \frac{(P^2)^2}{P^2 - m^2}\, (H^\dagger \tau_a)]\Big] \qquad (5.56)$$

thus corresponding to the piece denoted as S_1 by [33] in their Eq.(D7)[8]. All the results must certainly be equivalent. The only difference between the techniques explored here and that of HLM concerns the stage where the subtraction is chosen to be done - In [33] it has been performed at the level of the Trace computation and here at the level of the Log expansion, finally summarized through a series (namely Eq.(2.120)).

For concluding the examples, the first crossed piece of Eq.(5.51) may also be drawn:

- Tr $[\mathscr{A}_{(0)}\mathscr{A}_{(1)}]$:

$$\begin{aligned}\text{Tr}[\mathscr{A}_{(0)}\mathscr{A}_{(1)}] &\supset \text{Tr}\left[\Delta^H \mathscr{H}_{(0)}\, \Delta^H\, \mathscr{H}_{(1)}\right] \\ &= \text{Tr}\left[\Delta^H \mathscr{H}_{(0)}^{H^\dagger H}\, \Delta^H\, \mathscr{H}_{(1)}^{H^\dagger H}\right] + \text{Tr}\left[\Delta^H \mathscr{H}_{(0)}^{HH}\, \Delta^H\, \mathscr{H}_{(1)}^{H^\dagger H^\dagger}\right] + \\ &\quad + \text{Tr}\left[\Delta^H \mathscr{H}_{(0)}^{H^\dagger H^\dagger}\, \Delta^H\, \mathscr{H}_{(1)}^{HH}\right] + \text{Tr}\left[\Delta^H \mathscr{H}_{(0)}^{HH^\dagger}\, \Delta^H\, \mathscr{H}_{(1)}^{HH^\dagger}\right] \qquad (5.57)\end{aligned}$$

Before to proceed, it may important to remark that Δ^H is a open operator, acting to the right, while the $\mathscr{H}$ is closed. Next, from the first piece,

$$\begin{aligned}\text{Tr}[\mathscr{A}_{(0)}\mathscr{A}_{(1)}] &\supset 16\kappa^4\, \text{Tr}\Big[\Delta^H\{\tau^a \mathscr{O}_{(0)}(H^\dagger \tau_a H) + (\tau^a H)\, \mathscr{O}_{(0)}\, (H^\dagger \tau_a)\} \times \\ &\quad \times \Delta^H\, \{\tau^a \mathscr{O}_{(1)}(H^\dagger \tau_a H) + (\tau^a H)\, \mathscr{O}_{(1)}\, (H^\dagger \tau_a)\}\Big] \\ &\supset \frac{16\kappa^4}{m^6}\text{Tr}\Big[\Delta^H\{(\tau^a H)\left(1 + \frac{P^2}{m^2}\right)(H^\dagger \tau_a)\} \times \\ &\quad \times \Delta^H\{(\tau^a H)\left(\frac{(P^2)^2}{P^2 - m^2}\right)(H^\dagger \tau_a)\}\Big] \qquad (5.58)\end{aligned}$$

As a last explanation, the above expression presents a higher suppression compared with the expected $\frac{1}{m^2}$ due to the presence of the dim-1 constant κ. Finally, it corresponds the term S_2 in [33].

[7] Unlike the previous result, the Pauli matrices will not imply a vanishing term in this expression, due the product with $HH^\dagger$;

[8] With the difference of the factor c_s, still not included in the expression;

5.1.2.1 Functional Traces

In order to complete one example of computation for the Wilson coefficients in mixed diagrams, the functional trace will be performed on the result of Eq.(5.56):

$$\begin{aligned}\mathrm{Tr}\,\left[\mathscr{A}_{(1)}\right] &\supset \frac{2\lambda\kappa^2}{m^4}\mathrm{Tr}\Big[\Delta^H\,[(H^\dagger H)\mathbb{I}+H\,H^\dagger]\Delta^H[(\tau^a H)\,\frac{(P^2)^2}{P^2-m^2}\,(H^\dagger\tau_a)]\Big]\\ &\supset \frac{2\lambda\kappa^2}{m^4}\mathrm{Tr}\Big[\Delta^H\,|H|^2\Delta^H[(\tau^a H)\,\frac{(P^2)^2}{P^2-m^2}\,(H^\dagger\tau_a)]\Big]\end{aligned} \tag{5.59}$$

In Section 2.3 it has been developed the formal technique of computation of a functional trace, resulting into an integral over momenta and space-time coordinates in the form

$$\mathrm{Tr}\,f(\hat{x},\hat{q}) = \int dxdq\;\mathrm{tr}\,f(x,i\partial_x - q) \tag{5.60}$$

where at the r.h.s the trace runs over internal indices and $dq \equiv \frac{dq^4}{(2\pi)^4}$. For the common cases in this section, it converts into

$$\mathrm{Tr}\Big[\frac{1}{P^2-m^2}\,B\Big] \quad\to\quad \int dxdq\;\mathrm{tr}\,\Big[\frac{1}{(P-q)^2-m^2}\,B\Big] \tag{5.61}$$

At this stage, the covariant expansion is defined from the series on local terms of the denominator, like

$$\Big[\frac{1}{(P-q)^2-m^2}\,B\Big] \to \left[P^2-2q\cdot P+q^2-m^2\right]^{-1}\,B = \bar{\Delta}\left[1+\bar{\Delta}(P^2-2q\cdot P)\right]^{-1}\,B \tag{5.62}$$

with $\bar{\Delta}\equiv(q^2-m^2)^{-1}$. Since the operator inside the brackets is not acting on $\bar{\Delta}$, it follows that

$$\Big[\frac{1}{(P-q)^2-m^2}\,B\Big] \to \bar{\Delta}\sum_{n=0}^{\infty}\left[\bar{\Delta}(2q\cdot P-P^2)\right]^n\,B \tag{5.63}$$

which will then be used for power counting. By replacing it in Eq.(5.59):

$$\begin{aligned}\mathrm{Tr}\,\left[\mathscr{A}_{(1)}\right] \supset\; &\frac{2\lambda\kappa^2}{m^4}\int dxdq\;\mathrm{tr}\Big[\Big((\bar{\Delta}^H)^2\bar{\Delta}^\Phi\Big)\times\sum_{n=0}^{\infty}\Big[\bar{\Delta}^H(2q\cdot P-P^2)\Big]^n_{\lfloor}\;|H|^2\times\\ &\times\sum_{m=0}^{\infty}\Big[\bar{\Delta}^H(2q\cdot P-P^2)\Big]^m_{\lfloor}\times\\ &\times\left[(\tau^a H)\,\sum_{p=0}^{\infty}\left[\bar{\Delta}^\Phi(2q\cdot P-P^2)\right]^p\,(P^\mu-q^\mu)^4(H^\dagger\tau_a)\right]\Big]\end{aligned} \tag{5.64}$$

where the symbol $\lfloor$ is just remarking that the operators associate with the light Higgs act on the right. By picking the zeroth order components of the series and, from the dim-6 criteria, the

$2q^2P^2$ of $(P-q)^4$, one of the generated operators is[9]

$$\begin{aligned}
\mathrm{Tr}\,\left[\mathscr{A}_{(1)}\right] &\supset \frac{4\lambda\kappa^2}{m^4}\int dxdq\left(q^2(\bar{\Delta}^H)^2\bar{\Delta}^\Phi\right)\times \mathrm{tr}\left[|H|^2(\tau^a H)\,P^2(H^\dagger\tau_a)\right]+c.c. \\
&= \frac{4\lambda\kappa^2}{m^4}\int dx\, I_\Phi^{(2)}\times\left[|H|^2(P^2H^\dagger\tau_a)\cdot(\tau^a H)\;+\;|H|^2(H^\dagger\tau_a)\cdot(\tau^a P^2 H)\right] \\
&= -\frac{3\lambda\kappa^2}{m^4}\int dx\, I_\Phi^{(2)}\times |H|^2\left[(D^2H^\dagger)\cdot H\;+\;H^\dagger\cdot(D^2H)\right] \qquad (5.66)
\end{aligned}$$

where $P^\mu \to iD^\mu$ and the last piece of Eq.(5.54) was included in the second line. The $I_\Phi^{(2)}$ is one element of a set of master integrals registered for a large number of cases in [53]. In the $\overline{MS}$ scheme it is given by

$$I_\Phi^{(2)} \equiv \int dq\left(\frac{q^2}{(q^2-m_H^2)^2(q^2-m_\Phi^2)}\right)=\frac{1}{4}\left(\frac{3}{2}-\log\frac{m_\Phi^2}{\mu^2}\right) \qquad (5.67)$$

where μ is the renormalization scale and and the Higgs mass was already set to zero. To conclude, the operator part of the result can be rewritten in terms of the generators for the chosen CP conserving basis of bosonic operators:

$$|H|^2\left[(D^2H^\dagger)\cdot H\;+\;H^\dagger\cdot(D^2H)\right]\quad\to\quad -2(O_H+O_R) \qquad (5.68)$$

as demonstrated in Appendix (A.2). Finally,

$$\mathrm{Tr}\,\left[\mathscr{A}_{(1)}\right]=\frac{3\lambda\kappa^2}{2m^4}\left(\frac{3}{2}-\log\frac{m_\Phi^2}{\mu^2}\right)\int dx\,(O_H+O_R) \qquad (5.69)$$

5.2 Comments on the Triplet Case

The treatment about the constant c_s was not discussed up to this point, since it requires an additional comment. As mentioned before, in the case where mixed fields are coupled, a rotation is performed at the level of the Gaussian integration, and then the c_s is identified. This allow the above results to receive $c_s = 1$, as in the case of complex scalars demonstrated in Section 2.2.1, and which disagree with the choice in HLM. A rotation must also be performed in order to disentangle vertex including mixed statistics. This feature is present in Yukawa interactions, for example, and has been discussed in detail in [33].

[9] For illustration, the trace on internal indices in this case follows

$$\begin{aligned}
\mathrm{tr}[(\tau_a H)(H^\dagger\tau_a)] &= \sum_{i=1,2}(\tau_a H)_i(H^\dagger\tau_a)_i \\
&= (H^\dagger\tau_a)\cdot(\tau_a H) \\
&= \frac{3}{4}|H|^2 \qquad (5.65)
\end{aligned}$$

since $\sum\tau^a\tau^a=\frac{3}{4}$;

5.3 Theories with Kinetic Mixing of Gauge-Bosons

Influenced by HLM, this section starts a new set of applications of the results in the Chapter 2. The theory presents a kinetic-mixing term with $U(1)_Y$ from a new heavy $U(1)$ boson, denoted by K_μ and mass m:

$$\mathscr{L} \supset -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}K_{\mu\nu}K^{\mu\nu} + \frac{1}{2}m^2K^\mu K_\mu - \frac{\lambda}{2}B_{\mu\nu}K^{\mu\nu} \tag{5.70}$$

From $A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ some identities can be directly recovered, like:

$$\begin{aligned} K_{\mu\nu}B^{\mu\nu} &= (\partial_\mu K_\nu - \partial_\nu K_\mu)B^{\mu\nu} \\ &= \partial_\mu(K_\nu B^{\mu\nu}) - K_\nu\partial_\mu B^{\mu\nu} + K_\mu\partial_\nu B^{\mu\nu} - \partial_\nu(K_\mu B^{\mu\nu}) \end{aligned} \tag{5.71}$$

and, by eliminating total derivatives:

$$K_{\mu\nu}B^{\mu\nu} = 2K_\mu(\partial_\nu B^{\mu\nu}) \tag{5.72}$$

Now, to the squared terms it is useful to continuing from Eq.(5.71):

$$\begin{aligned} K_{\mu\nu}K^{\mu\nu} &= 2K_\mu(\partial_\nu K^{\mu\nu}) \\ &= 2K_\mu(\partial_\nu\partial^\mu K^\nu - \partial_\nu\partial^\nu K^\mu) \end{aligned} \tag{5.73}$$

such that

$$\mathscr{L} \supset \frac{1}{2}K_\mu(g^{\mu\nu}(\partial^2 + m^2) - \partial^\mu\partial^\nu)K_\nu + \frac{1}{2}B_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)B_\nu - \lambda K_\mu(\partial_\nu B^{\mu\nu}) \tag{5.74}$$

thus reproducing the generic representation of Eq.(2.118):

$$\mathscr{L}_{UV} \supset \frac{1}{2}\Phi\,\mathscr{O}^\Phi\,\Phi \;+\; \frac{1}{2}\varphi\,\mathscr{O}^\varphi\,\varphi \;-\; \Phi B_\varphi \;+\; Q_\Phi \tag{2.118}$$

with B_φ a function of light fields, here given by $B_\varphi \to \lambda(\partial_\nu B^{\mu\nu})$. It is also possible to conclude that the computation of the determinant will lead to $c_s = \frac{1}{2}$, similarly to the case of real scalars.

The procedure of integrating out K^μ follows the usual prescription, namely, to replace in Eq.(5.70) the solution of the classical equation of motion

$$\frac{\delta S_{UV}[B,K]}{\delta K}\bigg|_{K_c[B]} = 0 \tag{5.75}$$

or

$$(g^{\mu\nu}(\partial^2 + m^2) - \partial^\mu\partial^\nu)K_\nu = \lambda(\partial_\nu B^{\mu\nu}) \tag{5.76}$$

i.e. the Proca equation with a conserved source $j^\mu = \lambda(\partial_\nu B^{\mu\nu})$. By taking one derivative

$$\partial_\mu K^\mu = \frac{\lambda}{m^2}\partial_\mu j^\mu = 0 \tag{5.77}$$

providing a second equation for K_c^μ as:

$$(\partial^2 + m^2)K_c^\mu = j^\mu \tag{5.78}$$

The UV theory at the classical solution can then be rewritten like

$$\begin{aligned}\mathscr{L}_{LUV} \quad &\supset \quad \frac{1}{2}j^\mu(\partial^2+m^2)^{-1}j_\mu + \frac{1}{2}B_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)B_\nu - j^\mu(\partial^2+m^2)^{-1}j_\mu \\ &= \quad \frac{1}{2}B_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)B_\nu - \frac{1}{2}j^\mu(\partial^2+m^2)^{-1}j_\mu \end{aligned}\tag{5.79}$$

and reproducing the generic form of Eq.(2.111). Here, at dim-6, the tree-level piece for the Effective Lagrangian emerges from the first term in the expansion of the non-local operator, and results in:

$$\mathscr{L}_{EFT} = \frac{1}{2}B_\mu(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)B_\nu + \frac{\lambda^2}{m^2}O_{2B} \tag{5.80}$$

with $O_{2B} \equiv -(\partial_\nu B^{\mu\nu})^2$.

The absence of loop contributions may be seen directly from the non-invariance of triple-gauge couplings. Technically, this may be seen from $U = 0$ and $G_{\mu\nu} = [\partial_\mu, \partial_\nu] = 0$ in the Universal Formula of Eq.(2.83).

5.4 Integrating the Z' of the 3-3-1HL

The subject of gauge-bosons can be continued now by considering the new neutral weak-boson of the 3-3-1HL. Motivated by the notation of [31], the subset of interactions considered here can be given by:

$$\mathscr{L}_{331} \supset \frac{1}{2}Z'_\mu\left(g^{\mu\nu}\partial^2 + m^2 - \partial^\mu\partial^\nu\right)Z'_\nu + \overline{q}_a \not{P} q_a - \sum_{a=1}^{3} g^a_{Z'}\left[\overline{q}_a\gamma^\mu q_a\right]Z'_\mu \tag{5.81}$$

here, for simplicity, only the left-handed sector will be consider and the index L was made implicit. Moreover, the matrix notation for the coupling $g_{Z'}$ was altered to a simple number since the first two entries of $\mathbb{T}^8$ is proportional to the identity. The index a is then for assigning the fields in the fundamental or conjugated representation. Indeed, it originates from this distinction the presence of Flavor Changing Neutral Currents in these sort of 3-3-1 models. Since $g^*_{Z'} \neq g_{Z'}$, the first two generations will acquire a coefficient distinct from the third one, such that, after the second symmetry breaking, the rotation to the mass-eigenstates results will not retain the interactions diagonal.

The couplings can also be estimated in terms of the weak-coupling g and according to the 3-3-1HL variant, i.e. to the particular choice of β. In the Chapter 4 it was demonstrated that (see Eq.(4.30))

$$g^*_{Z'} = -g\left(\frac{c_x}{2\sqrt{3}} + \tan\theta_w\tan\theta_x\left(\frac{1}{6}+\frac{\beta}{2\sqrt{3}}\right)\right); \quad g_{Z'} = g\left(\frac{c_x}{2\sqrt{3}} - \tan\theta_w\tan\theta_x\left(\frac{1}{6}-\frac{\beta}{2\sqrt{3}}\right)\right) \tag{4.30}$$

Moreover, the angle θ_x is related to the weak-angle through Eq.(4.28), i.e.

$$s_x = \beta \tan\theta_w, \quad \text{and} \quad \beta \in \left[\pm\frac{1}{\sqrt{3}}, \pm\sqrt{3}\right] \tag{4.28}$$

Thus, by replacing it into the Z' coupling, it follows that

$$g^*_{Z'} = -g[0.64,\ 0.27,\ 0.33,\ 1.19]; \qquad \text{for} \quad \beta = \left[-\sqrt{3}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \sqrt{3}\right], \text{respectively.} \tag{5.82}$$

with $s_w^2 = 0.231$ at $\mu = m_Z$ ($\overline{MS}$, see [41]). To the fields in the fundamental representation, the values are reversed along the sign of β.

5.4.1 At tree-level

The tree-level integration follows similarly to the case of Section 5.3. The classical equation

$$\frac{\delta S_{331}}{\delta Z'}\Big|_{Z'_c[q,\overline{q}]} = 0 \tag{5.83}$$

corresponds to

$$Z'_\mu = [\partial^2 + m^2]^{-1} \sum_a j^a_\mu, \quad \rightarrow \quad j^a_\mu = g^a_{Z'}[\overline{q}_a \gamma_\mu q_a] \tag{5.84}$$

After replacing it back into Eq.(5.81):

$$\mathscr{L}_{L,331} \supset \overline{q}_a \not{P} q_a - \frac{1}{2}\sum_{a,b}\left[j^a_\mu [\partial^2 + m^2]^{-1} j^{\mu,b}\right] \tag{5.85}$$

The field-dimension of j_μ is three. Thus, at dim-6 the operator must be truncated at the first order. The EFT at tree-level results in:

$$\mathscr{L}^{(0)}_{EFT} \supset \overline{q}_a \not{P} q_a - \frac{1}{2}\frac{1}{m^2}\sum_{a,b} c^{(0)}_{ab} Q_{ab}; \tag{5.86}$$

where, from Grzadkowski at al.[31] notation, $Q_{ab} = [\overline{q}_a \gamma_\mu q_a][\overline{q}_b \gamma^\mu q_b]$ and $c^{(0)}_{ab} = g^a_{Z'} g^b_{Z'}$.

5.4.2 At Loop-Level: Mixed Terms

The Z' may also be taken as a source of application for the matching at the log-level involving mixed heavy-light particles. The formula was computed in Section 2.4 and is based in objects like

$$\mathscr{A}_{(i)} \equiv \left[\mathscr{O}^\varphi\right]^{-1} \mathscr{H}\left[\frac{1}{2} B_\varphi\, \mathscr{O}^\Phi_{(i)}\, B_\varphi\right] \tag{2.118}$$

where φ and Φ denotes light and heavy fields, respectively, and $[\mathscr{O}^\varphi]^{-1}$ is, for the overall Lagrangian representations, a light covariant propagator. Apart from that, the indices (i) indicates (0) local or (1) non-local operators, such that $\overline{\mathscr{O}}_{(0)}$ is the truncated part during the tree-level matching of the heavy propagator expansion, while the $\mathscr{O}_{(1)}$ is the remaining non-local term. The power-counting is performed through

$$\sum_{i=1}^{N} c_i^{(1)}\, O_i[\varphi] \supset -i\alpha \mathrm{Tr} \sum_p \frac{1}{p} \left\{ \sum_{k=1}^{p} \binom{p}{k} \mathscr{A}_{(0)}^{p-k} \mathscr{A}_{(1)}^{k} \right\} \tag{5.87}$$

Finally, the light operators B_φ are given by the current

$$B_\varphi \to j_\mu^a = g_{Z'}^a [\overline{q}^a \gamma_\mu q^a] \tag{5.88}$$

where q are quark doublets and $g_{Z'}$ the coupling constant. For simplifying the notation, here only one generation will be considered, what implies

$$\mathscr{A}_{(i)} \equiv [\not{P}]^{-1} \mathscr{H} \left[\frac{1}{2} j_\mu\, \mathscr{O}_{(i)}^{Z'}\, j^\mu \right] \tag{5.89}$$

From the tree-level integration it is also clear that the truncated $\overline{\mathscr{O}}_{(0)}$ is defined by

$$[\mathscr{O}^{Z'}]^{-1} \quad \to \quad \overline{\mathscr{O}}_{(0)}^{Z'} = \frac{1}{m^2} \tag{5.90}$$

which implies from Eq.(2.113)

$$[\mathscr{O}^{Z'}]^{-1} \quad \to \quad \mathscr{O}_{(1)}^{Z'} = \frac{\partial^2}{m^2} \frac{1}{(\partial^2 - m^2)} \tag{5.91}$$

The Hessian matrix follows similarly to the Scalar Triplet case, despite the fact that the current is already a gauge-covariant object. From the linearity of the trace, for illustration only the operators coming from the first entry of $\mathscr{H}$ will be considered:

$$\mathscr{H}_{(i)} \supset \frac{1}{2} \delta^2_{\overline{q}q} = g_{Z'}^2 \left[\gamma_\mu \mathscr{O}_{(i)} (\overline{q}\gamma_\mu q) + (\gamma^\mu q) \mathscr{O}_{(i)} (\overline{q}\gamma_\mu) \right] \tag{5.92}$$

The first term in the series Eq.(5.87) consists of $\mathscr{A}_{(1)}$, i.e.

$$\mathscr{A}_{(1)} \supset \frac{g_{Z'}^2}{m^2} \left\{ \frac{\not{P}}{P^2} \left[\gamma_\mu \frac{p^2}{(p^2 - m^2)} (\overline{q}\gamma^\mu q) + (\gamma_\mu q) \frac{p^2}{(p^2 - m^2)} (\overline{q}\gamma^\mu) \right] \right\} \tag{5.93}$$

where now the small $p \equiv i\partial$. To the trace,

$$\mathrm{Tr}[\mathscr{A}_{(1)}] \supset 4 \frac{g_{Z'}^2}{m^2} \mathrm{Tr} \left\{ \frac{P^\mu}{P^2} \left[\frac{p^2}{(p^2 - m^2)} (\overline{q}\gamma_\mu q) \right] \right\} \tag{5.94}$$

where $tr\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$ has been used. The next step involves the functional trace and the prescription of Eq.(5.61)

$$\begin{aligned} \mathrm{Tr}[\mathscr{A}_{(1)}] \quad \supset \quad & 4 \frac{g_{Z'}^2}{m^2} \int dx dk \Big\{ \overline{\Delta} \sum_{n=0}^{\infty} [\overline{\Delta}(2k \cdot P - P^2)]^n (P^\mu - k^\mu) \times \\ & \times \overline{\Delta}^{Z'} \sum_{m=0}^{\infty} [\overline{\Delta}^{Z'} (2k \cdot p - p^2)]^m (p^\mu - k^\mu)^2 (\overline{q}\gamma_\mu q) \Big\} \end{aligned} \tag{5.95}$$

and, as before $\overline{\Delta}^{Z'} = [k^2 - m_{Z'}^2]^{-1}$ and $\overline{\Delta} = [k^2]^{-1}$. At the zeroth-order in the series expansion it can be seen that the total covariant acting on an invariant term will leave a total derivative and must vanish. The only even integral in the integrated momentum is

$$\mathrm{Tr}[\mathscr{A}_{(1)}] \supset 4\frac{g_{Z'}^2}{m^2}\int dxdk\left\{\overline{\Delta}\overline{\Delta}^{Z'}k^\mu k^\nu[2p_\nu(\overline{q}\gamma_\mu q)]\right\} \tag{5.96}$$

and, again, since after integration $k^\mu k^\nu \to g^{\mu\nu}$, the final result converts into a total derivative of the current.

The previous result may be interpreted as a zero correction to the quark masses, as well as those from the remaining terms[10]. The derivative over the quark fields inside $\mathscr{O}_{(i)}$ reveals that every correction should be proportional to the fermion mass, which remains zero at the end.

The next term for the series in Eq.(5.87) will be partially performed as one attempt to verify the type of integrals present during the matching and the information in fact translated to the Wilson coefficients during the subtraction. The product is given by

$$c_{ab}^{(1)}Q_{ab} \supset \mathrm{Tr}[\mathscr{A}_{(0)}\mathscr{A}_{(1)}] \tag{5.98}$$

The $\mathscr{A}_{(0)}$ is can be constructed as in Eq.(5.93), via the replacement $\mathscr{O}_{(1)} \to \mathscr{O}_{(0)}$. By taking the first product in $\mathscr{A}_{(0)}\mathscr{A}_{(1)}$, it follows that

$$\mathrm{Tr}[\mathscr{A}_{(0)}\mathscr{A}_{(1)}] \supset \frac{g_{Z'}^4}{m^4}\mathrm{Tr}\left\{\frac{\not{P}}{P^2}\gamma_\mu(\overline{q}\gamma^\mu q)\frac{\not{P}}{P^2}\gamma_\nu\frac{p^2}{(p^2-m^2)}(\overline{q}\gamma^\nu q)\right\} \tag{5.99}$$

To perform the functional trace, again only the zeroth-order element of each propagator expansion will be taken:

$$\mathrm{Tr}[\mathscr{A}_{(0)}\mathscr{A}_{(1)}] \supset \frac{g_{Z'}^4}{m^4}\int dxdk\,\mathrm{tr}\left\{\overline{\Delta}^2\overline{\Delta}_{Z'}(\not{P}-\not{k})\gamma^\mu(\overline{q}\gamma_\mu q)(\not{P}-\not{k})\gamma_\nu(p-k)^2(\overline{q}\gamma^\nu q)\right\} \tag{5.100}$$

Now the above expression contains different degrees of divergence. This degree can be count from the number of momentum in the numerator like

$$\#k \in \{0,2,4\} \quad \text{and thus} \quad \delta(\Gamma^{(4)}) = \#k + 4 - 6 \quad \text{or} \quad \delta(\Gamma^{(4)}) \in \{-2,0,2\} \tag{5.101}$$

The result reproduces the degree of divergence of the diagrams in Fig. In summary, the subtraction discussed in Chapter 2 involving the quantum corrections to the Effective Theory and to the Light-UV theory indeed presents different regions for the integrated momenta, from the low-, where the Z' lines can just be cut-off, to the hard- region where the heavy lines must be activated.

[10]For completeness, the second trace for $\mathscr{A}_{(i)}$ can also be performed:

$$\begin{aligned}\mathrm{Tr}[\mathscr{A}_{(1)}] \supset \mathrm{Tr}\frac{g_{Z'}^2}{m^2}\left\{\frac{\not{P}}{P^2}[(\gamma_\mu q)\mathscr{O}_{(1)}(\overline{q}\gamma^\mu)]\right\} \to \mathrm{Tr}\{P_\nu\gamma^\nu(\gamma_\mu q)[\mathscr{O}_{(1)}(\overline{q}\gamma^\mu)]\} &= P_\nu(\gamma^\nu)_{ab}(\gamma_\mu q)_b[\mathscr{O}_{(1)}(\overline{q}\gamma^\mu)]_a \\ &= P_\nu(\gamma^\nu)_{ab}(\gamma_\mu)_{bi}q_i[\mathscr{O}_{(1)}\overline{q}]_j(\gamma^\mu)_{ja} \\ &= P_\nu(\gamma_\mu\gamma^\nu\gamma^\mu)_{ji}q_i[\mathscr{O}_{(1)}\overline{q}]_j\end{aligned}$$

or

$$\mathrm{Tr}\{P_\nu\gamma^\nu[(\gamma_\mu q)\mathscr{O}_{(1)}(\overline{q}\gamma^\mu)]\} = -2P_\nu[\overline{q}\gamma^\nu\mathscr{O}_{(1)}q] \tag{5.97}$$

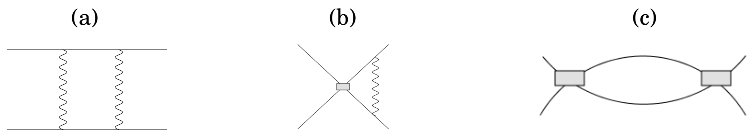


FIGURE 5.1. The degree of divergence of the above graphs are in fact present in the mixed heavy-light matching formula.

5.5 The J-quarks in the 3-3-1HL

The Chapter 4 tried to elucidate the circumstances when the 3-3-1HL model may comprise the Standard Model. After the triplet χ acquire a vacuum expectation value, the $SU(3)_L \otimes U(1)_X$ symmetry breaks into $SU(2)_L \otimes U(1)_Y$ due to a there called β-independent scalar interactions, i.e. a potential whose form is present in all possible variants of the model. It has also been shown that, in this single breaking scenario, the non-standard particles, apart from the previous neutral Z', completely decouples with the SM at tree-level, a consequence of the pattern of gauge interactions settled by the structure of $SU(3)$. Under these assumptions, the criteria of separation

$$\mathscr{L}_{331} = \mathscr{L}_{SM} + \mathscr{L}_{SM+NP} \tag{5.102}$$

is then satisfied and the Universal Formula developed in Chapter 2 appropriate for the 3-3-1HL integration.

It has also been argued that the β-independent scenario, the same of considering a Z_2 parity, is a reasonable assumption once one is primarily interested on testing this new $SU(3)$ gauge-structure.

The Section 3.3 has developed one of the basic principles for defining the EFT technique, valid whenever the complete theory intrinsically presents very separated energy scales. At a first level the integration of the 3-3-1HL would imply a redefinition of the renormalizable SM like

$$\mathscr{L}_{331} = \mathscr{L}_{\overline{SM}} + O\left(\frac{1}{u^a}\right) \tag{5.103}$$

with rescaled fields and couplings. Here u denotes de heavy scale.

As soon the experiments reach larger scales, the suppressed part must start to be filled with higher-dimension operators. The aim of this section is to accomplish part of this task, now through the integration of the heavy J-quarks in the *conjugated representation*, independently of the specific model, i.e. regardless the value of β.

As mentioned before, the quarks J_i are singlets of $SU(2)_L$ and charged under hypercharge according to the coupling g_B^* with the field B^μ. The Covariant Derivative is given by

$$\not{P} = i\not{D} = i(\not{\partial} + ig_B^*\not{B}) \tag{5.104}$$

where, in terms of the original couplings

$$g_B^* = \frac{g}{\sqrt{3}} s_x + g_X \left(\frac{1}{6} + \frac{\beta}{2\sqrt{3}}\right) c_x = g' \left(\frac{1}{6} + \frac{\beta\sqrt{3}}{2}\right) \tag{5.105}$$

and θ_x the 3-3-1HL mixing angle. To the application of the Universal Formula, Eq.(2.83), the vertex

$$\overline{J}_i(\not{P} - m)J^i \tag{5.106}$$

is better to be rewritten after the trick of Section 2.2.1, namely

$$-i\mathrm{Tr}\log(-\not{P} + m) \quad \rightarrow \quad -\frac{i}{2}\mathrm{Tr}\log[D^2 + m^2 + U_J] \tag{5.107}$$

where now $U_J \equiv -\frac{i}{2}\sigma^{\mu\nu}[D_\mu, D_\nu]$. For the Abelian case $[D_\mu, D_\nu] = ig_B^* B_{\mu\nu}$. Thus, the total components for the application of the Eq.(2.83) are done. Since U_J is of dim-2 in the fields, the sum must break at $\frac{1}{m^2}$. From $tr[\sigma_{\mu\nu}] = 0$, the m^2 piece vanishes. In fact, any piece with a single U_J must vanish. In the m^0 term, the U_J^2 term also vanishes by the antisymmetry of $B_{\mu\nu}$[11]. The remaining operator is just proportional to the kinetic term:

$$\mathscr{L}_{EFT}^{(1)} \supset \frac{c_s}{(4\pi)^2} \frac{g_B^{*2}}{12} \left(1 - \log\left(\frac{m^2}{\mu^2}\right)\right) B_{\mu\nu} B^{\mu\nu} \tag{5.108}$$

At dim-6 operators, the triple interaction vanishes by the antisymmetry of $B_{\mu\nu}$[12], as well as the pieces containing U_J. The single contribution comes from

$$\mathscr{L}_{EFT}^{(1)} \supset \frac{c_s}{(4\pi)^2} \frac{g_B^{*2}}{m^2} \frac{O_{2B}}{30} \tag{5.109}$$

where $O_{2B} \equiv -\frac{1}{2}(\partial_\mu B^{\mu\nu})^2$.

[11] Here, $trU_J^2 \propto tr[\sigma^{\alpha\mu}\sigma^{\beta\nu}]B_{\alpha\mu}B_{\beta\nu} = 4[g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta}]B_{\alpha\mu}B_{\beta\nu} = 0$;

[12] Unlike in the non-Abelian case, where $tr[W_{\mu\nu}W_{\nu\alpha}W_{\alpha\mu}] = W^a_{\mu\nu}W^b_{\nu\alpha}W^c_{\alpha\mu} \times tr[T^aT^bT^c]$. Here, $B_{\mu\nu}B_{\nu\alpha}B_{\alpha\mu} = -B_{\nu\mu}B_{\mu\alpha}B_{\alpha\nu} = 0$

CHAPTER 6

PRECISION OBSERVABLES

The last part of the work is dedicated to apply not only the formulas developed in the previous chapters, but mainly the conceptual ideas contributing to the strength of Effective Field Theories. The source taken is the Lagrangian raised after the integration of the heavy quarks in the 3-3-1HL. It will be seen that, although a single Wilson coefficient has been generated, its effects can be translated into many distinct observables through the mixing implied by the anomalous dimension matrix. After running down the operators to the electroweak scale, a linear combination of the coefficients may define an *observable basis*, following the Elias-Miró et al. terminology [24]. In other words, a linear combination will affect the same observable such that, as the experimental measurements acquires more precision, more narrow become the correlations among the many parameters of the original UV model. This methodology will be verified through the new class of oblique parameters presented by Barbieri et al. [4] and currently measured at per-mille level [3].

6.1 Operator Mixing and Anomalous Matrix

This Section provides a brief review about the origin of the anomalous matrix and the consequent operator mixing, and is supported on the historical example of the QCD corrections to the Fermi operator O_1 (see [8], [49]-Chap. 31):

$$O_1 = [\bar{c}_L \gamma^\mu b_L]_x [\bar{d}_L \gamma^\mu u_L]_x \tag{6.1}$$

where 'x' indicates a local product and the brackets a color contraction. This operator is then complementing a low-energy Lagrangian given by:

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}(G_{\mu\nu}^a)^2 + \sum_q \bar{q}(i\not{D} - m_q)q - \sum_n C_n O_n \tag{6.2}$$

The introduction of the operator in Eq.(6.1) is clearly motivated by the phenomenology of hadronic process like $B \to D$ decays, supposed to be well represented by the channel $b \to c\bar{u}d$. As mentioned before, a theory which creates and annihilates quarks applied for hadronic process will in general imply large uncertainties from the matrix elements evaluation. Therefore, any attempt to make the perturbative computation more precise is proper to be considered.

At tree-level the Low-energy Lagrangian is given by:

$$\mathscr{L} = \mathscr{L}_{kin} - C_1 O_1 \tag{6.3}$$

with $C_1 = 4\frac{G_F}{\sqrt{2}}V_{cb}V^*_{ud}$ in terms of CKM matrix elements. The Chapter 3 has shown that the matching procedure constitute a form of adding information from a heavy particle (or, of the complete theory) to the Wilson coefficients. Thus, the matching at 1-loop will complement these informations through higher-order process involving the integrated particles. This task is, therefore, independent of the running analysis for the effective operators. as clarified before, the running will sum up perturbative corrections that, in the case of the EFT, originates from the many exchange of light-particles. The matching, i.e. the equivalent of a Correspondence Principle, will provide the initial condition to the set of renormalization group equations. Therefore, a clearly benefit of the matching procedure is that, since the renormalization of the operators is already performed in this context, it may directly provide the anomalous matrix to be used during the running analysis.

Back to the QCD corrections, the first step, that of the matching, is fully performed in many textbooks and articles, like in Buchalla et al. [8], Peskin and Schroeder [42] and J. Donoghue et al. [21]. In the last case the authors have also provided an example of matching by integration by regions. In the case of M.Schwartz [49], the procedure is performed like detailed in Chapter 2, namely, by equating the full computation, including a W-line, with the EFT calculation including, as the effective vertex, only the Wilson coefficient at tree-level.

The problem of QCD corrections to heavy-quark channels is the first to illustrate an operator mixing. The many gluons exchange will generate $O_2 = [\overline{c}_L\gamma^\mu u_L][\overline{d}_L\gamma^\mu b_L]$ such that the Lagrangian after the 1-loop matching will turn into

$$\mathscr{L} = \mathscr{L}_{kin} - C_1 Z_1 O_1 - C_2 Z_2 O_2 \tag{6.4}$$

with the operator strength $Z_i = 1 + \delta_i$. It is important to emphasize that the renormalized operators O_i are already depending on the renormalized fields. Again, the boundary conditions to the renormalization group equations of the C_i's are extracted from matching at the subtraction scale $\tilde{\mu} = m_W$ and are given by:

$$C_1 = G\left[1 - \frac{\alpha_s}{2\pi}\left(\frac{1}{2}\log\frac{\tilde{\mu}^2}{m_W^2} + \frac{3}{4}\right)\right] \tag{6.5a}$$

$$C_2 = G\left[\frac{3\alpha_s}{2\pi}\left(\frac{1}{2}\log\frac{\tilde{\mu}^2}{m_W^2} + \frac{3}{4}\right)\right] \tag{6.5b}$$

with G just denoting the tree-level coefficient. At the matching scale, the log function above must vanish. In general, these expressions are written preserving these functions in order to represent the summation performed by the RGE. However, the above results are *not* functions of μ, but numbers defined before the correspondence principle - the EFT and the UV must agree at $\tilde{\mu} = m_W$[1].

Now, the normalization of the effective operators were already given by the computation of QCD corrections *in the context of the EFT*. Here the highlighted sentence is just to clarifying that

[1]All the couplings α, G are indeed dependent on the renormalization scale;

the running analysis is in fact independent of the matching procedure, although this last one will already imply the extraction of the anomalous matrix. From [49], the operator-strength resulted in

$$Z_1 = 1 + \frac{\alpha_s}{2\pi\epsilon}\left(3\frac{C_2}{C_1} - \frac{11}{3}\right), \qquad Z_1 = 1 + \frac{\alpha_s}{2\pi\epsilon}\left(3\frac{C_2}{C_1} - \frac{11}{3}\right) \tag{6.6}$$

Since the quark-fields defining O_i were already the renormalized fields, in terms of bare fields, the vertex can be given by:

$$C_i Z_i O_i \quad \rightarrow \quad \frac{C_i Z_i}{Z_q^2}[q^{(0)}\gamma^\mu q^{(0)}][q^{(0)}\gamma_\mu q^{(0)}] \tag{6.7}$$

Being the bare fields independent on the renormalization scale, the group equations can be summarized like

$$\mu\frac{d}{d\mu}\left(C_i\frac{Z_i}{Z_q^2}\right) = 0 \tag{6.8}$$

To solve it in detail, first consider $Z_i = \frac{1}{C_i}\left(C_i + \frac{\alpha_s}{2\pi\epsilon}\left(3C_j - \frac{11}{3}C_i\right)\right)$. Thus

$$\mu\frac{d}{d\mu}\left(\frac{C_i}{Z_q^2} + \frac{\alpha_s}{2\pi\epsilon}\left(3C_j - \frac{11}{3}C_i\right)\frac{1}{Z_q^2}\right) = 0 \tag{6.9}$$

$$\therefore \frac{\mu}{Z_q^2}\frac{dC_i}{d\mu} = -C_i\mu\frac{d}{d\mu}\frac{1}{Z_q^2} - \mu\frac{d}{d\mu}\left[\frac{\alpha_s}{2\pi\epsilon}\left(3C_j - \frac{11}{3}C_i\right)\frac{1}{Z_q^2}\right] \tag{6.10}$$

the first term in the r.h.s, at order α_s corresponds to

$$\begin{aligned} -C_i\mu\frac{d}{d\mu}\frac{1}{Z_q^2} &= -\frac{4C_i}{Z_q^3}\frac{\beta(\alpha_s)}{3\pi\epsilon} \\ &= \frac{4C_i\alpha_s}{3\pi Z_q^3} + O(\alpha_s^2) \end{aligned} \tag{6.11}$$

where it has been used $\beta(\alpha_s) \sim -\epsilon\alpha_s + O(\alpha_s^2)$. Finally,

$$\begin{aligned} \mu\frac{dC_i}{d\mu} &= \frac{4\alpha_s}{3\pi Z_q}C_i - \frac{\beta}{2\pi\epsilon}\left(3C_j - \frac{11}{3}C_i\right) \\ &= \frac{4\alpha_s}{3\pi Z_q}C_i + \frac{\alpha_s}{2\pi}\left(3C_j - \frac{11}{3}C_i\right) \\ &= \frac{\alpha_s}{2\pi}\left(-C_i + 3C_j\right) \end{aligned} \tag{6.12}$$

and in the last equal sign Z_q at tree-level has been taken. The above equation will then represent the mixture between C_1 and C_2 via:

$$\mu\frac{dC_i}{d\mu} = \gamma_{ij}C_j \quad \rightarrow \quad \gamma_{ij} = \frac{\alpha_s}{2\pi}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} \tag{6.13}$$

The previous discussion has tried to elucidate the methodology behind the extraction of the anomalous matrix. In fact, γ_{ij} will appear from any light particle exchange which is able to

Table 6.1: The set of dimension-six Bosonic operators. The basis must be complemented by CP-odd operators treated in the Appendix A.4. (see [32])

$O_H = \frac{1}{2}(\partial_\mu \lvert H\rvert^2)^2$	$O_{WB} = 2gg' H^\dagger \tau^a H W^a_{\mu\nu} B^{\mu\nu}$
$O_T = \frac{1}{2}(H^\dagger \overleftrightarrow{D}^\mu H)^2$	$O_W = ig(H^\dagger \tau^a \overleftrightarrow{D}^\mu H) D^\nu W^a_{\mu\nu}$
$O_R = \lvert H\rvert^2 \lvert D_\mu H\rvert^2$	$O_B = ig' Y_H (H^\dagger \overleftrightarrow{D}^\mu H)\partial^\nu B_{\mu\nu}$
$O_D = \lvert D^2 H\rvert^2$	$O_{3G} = \frac{1}{3!} g_s f^{abc} G^{a\mu}_\rho G^{b\nu}_\mu G^{c\rho}_\nu$
$O_6 = \lvert H\rvert^6$	$O_{3W} = \frac{1}{3!} g \epsilon^{abc} W^{a\mu}_\rho W^{b\nu}_\mu W^{c\rho}_\nu$
$O_{GG} = g_s^2 \lvert H\rvert^2 G^a_{\mu\nu} G^{a,\mu\nu}$	$O_{2G} = -\frac{1}{2}(D^\mu G^a_{\mu\nu})^2$
$O_{WW} = g^2 \lvert H\rvert^2 W^a_{\mu\nu} W^{a,\mu\nu}$	$O_{2W} = -\frac{1}{2}(D^\mu W^a_{\mu\nu})^2$
$O_{BB} = g'^2 \lvert H\rvert^2 B_{\mu\nu} B^{\mu\nu}$	$O_{2B} = -\frac{1}{2}(\partial^\mu B^a_{\mu\nu})^2$

correct a given operator. The total matrix for the entire basis of the Standard Model Effective Field Theory, at dimension-six has been presented in the literature as, for example, in the Jenkins et al. work [34], [35] and [1], or for the subset of bosonic-operators in Elias-Miró et al. [24].

In the Chapter 3, the running equation for the Wilson Coefficients, at leading-order, was obtained like

$$c_i(m_W) = c_i(m) - \frac{1}{16\pi^2}\sum_j \gamma_{ij} c_j(m) \log\left(\frac{m}{m_W}\right) \tag{6.14}$$

and represents the translation of the coupling at the heavy scale m to the electroweak scale m_W. Since the total basis of the SMEFT is too large to turn the above relation simple to be considered, some plausible assumptions can be adopted [32]. First, a coefficient that was generated from a loop with a heavy particle is likely to give a negligible correction for a formula representing loop corrections from light particles (i.e. a two-loop size correction). In other words, the index "j" can be closed to the coefficients generated at tree-level only. Second, if on the other hand c_i was generated at tree-level a loop correction is likely to be negligible and as a first approximation $c_i(m) \sim c_i(m_W)$. The use of Eq.(6.14) is then motivated by the attempt to improve loop-level results from tree-level terms[2], a perspective that reduces γ_{ij} into a smaller sub-basis.

For models whose phenomenology is centered in Higgs physics, it is common to adopt the set of Bosonic operators of Table 6.1. This set can also be complemented by a few of CP-odd operators discussed in Appendix A.4.

The Standard Model with a Triplet Scalar introduced in Chapter 5 is indeed closed in this subset and can be taken for illustrating the use of Eq.(6.14). Following the aforementioned criteria, from Eq.(5.16), the operators generated at tree-level were

$$S^{(0)}_{EFT} = \frac{\kappa^2}{2m^2}\lvert H\rvert^4 + \frac{\kappa^2}{m^4}(O_T + 2O_R) - \frac{\eta\kappa^2}{m^4} O_6 \quad \to O_{tree} \in \{O_T, O_R, O_6\} \tag{5.16}$$

which reduce the index 'j' for three values. At one-loop level it was generated

$$O_{1-\text{loop}} \in \{O_T, O_R, O_6, O_H, O_{2W}, O_{3W}\} \tag{6.15}$$

[2]It is also motivated whenever the observable has been measured with high precision;

In the work of J.Elias-Miró et al. [23] the authors have registered the anomalous matrix for the operators of Table 6.1, from which one can conclude that only c_H must be corrected, from c_H and c_T (see Table 7 in [24]):

$$c_H(m_W) = 2\eta - \frac{1}{16\pi^2}\left[\frac{3}{2}(2g^2 + g'^2 - 4\lambda)\frac{2\kappa^2}{m^2} + (8\lambda - 6g^2 - \frac{3}{2}g'^2)\frac{2\kappa^2}{m^2}\right]\log\left(\frac{m}{m_W}\right) \tag{6.16}$$

since $c_H(m) = 2\eta$. Besides, (λ, g', g) are quartic Higgs coupling, the hypercharge and the weak coupling, respectively. For simplification, once the coefficients for mixed heavy-light loops were not fully computed, only the results from Eq.5.34 were considered.

6.2 Running the Wilson Coefficients - Considerations

It has been argued hitherto that the equations for running the Wilson coefficients down to the electroweak scale may be reduced depending on the specific process under consideration. Equivalently, some elements inside the set of 59 dim-6 operators of the SMEFT, listed in [31], can just be assumed a priori to not provide any measurable effect. The operators of Table 6.1, for instance, are privileged in the sense that they can serve to test the electroweak symmetry sector.

In the example for the running of the Triplet Scalar operators only the correlations between coefficients generated at tree-level into those at loop level was accounted. However, this simplification must in fact depend on the precision level for the observable and on what follows it will be relaxed. Moreover, since the running corresponds to the sum many light particles exchange, the anomalous matrix must depend on the Standard Model parameters in a very intricate form, as can be noted in Eq.(6.16). These couplings are still scale dependent. Notwithstanding, the values of the SM couplings at high energies are in general not considered and their running has already not entered in the solution of the renormalization equations.

Finally, the equation considered for the running analysis, namely Eq.6.14, was achieved at the leading log, one valid simplification whenever the difference between the UV and the electroweak scale is not very large. Usually, $m \sim O(TeV)$.

6.3 J-quarks and Electroweak Precision Observables

This Section presents a brief introduction on how to constrain the UV model parameters through Electroweak Precision Observables. In the last chapter the integration of the J-quark resulted in two contributions. The first is a consequence of a single loop inside the gauge-boson propagator, and was given by:

$$\mathscr{L}_{EFT} \supset \frac{c_s}{(4\pi)^2}\frac{{g_B^*}^2}{12}B_{\mu\nu}B^{\mu\nu} \tag{6.17}$$

at the heavy scale $\Lambda = m$ and with

$$g_B^* \equiv \frac{g}{\sqrt{3}}s_x + g_X c_x\left(\frac{1}{6} + \frac{\beta}{2\sqrt{3}}\right) = g'\left(\frac{1}{6} + \frac{\beta\sqrt{3}}{2}\right) \tag{6.18}$$

and θ_x the 3-3-1HL mixing angle. At dimension-six there was also a single operator, namely O_{2B}:

$$\mathscr{L}_{EFT} \supset \frac{c_s}{(4\pi)^2} \frac{g_B^{*\,2}}{m^2} \frac{O_{2B}}{30} \tag{6.19}$$

On what follows the meaning of the kinetic term must be addressed. Next, motivated by the analysis of J.Elias-Miró et.al [24], the single Wilson coefficient c_{2B} must be tested under a new class of oblique parameters.

Despite the fact that just one dimension-six operator was generated during the integration of J-quarks, the coupling evolution, down to the electroweak scale, may spread the information about their presence to different classes of observables. On what follows, it is presented a brief review on the analysis introduced by J.Elias-Miró et al. in [24], whose anomalous matrix for the set of operators below was entirely computed:

$$O_{\text{dim-6}} \in \{O_H, O_T, O_W, O_B, O_{2W}, O_{2B}, O_{WW}, O_{WB}, O_{BB}, O_{3W}\} \tag{6.20}$$

Based on their results, some coefficients like c_B or c_{WB}, zero at $\Lambda \sim m$ may assume a non-zero value at $\mu \sim m_W$. These coefficients are then two-loop suppressed and will produce reasonable bounds as the observables related to them turn more precise.

The analysis will follow a general assumption that the Wilson coefficients must not exceed their respective bounds. The c_{2B}, for example, is directly associated with a new class of oblique parameters, namely Y, associated with higher-order corrections of the B^μ propagator. The bounds on c_Y are currently measured at the per-mille level [4], as well as those from different *Electroweak Precision Observables* (EWPO for short), like the remaining oblique parameters *S, T, U* and Triple Gauge Couplings. Notwithstanding, since the propagation of c_Y into different W.C's occurs only through γ_{ij}, its direct bound may be considered sufficiently strong to constrain the J mass. Thus, here the running is performed as a matter of illustration.

In the work of J.Elias-Miró et al. the authors have defined a new class of constants by considering, instead of a single Wilson coefficient, a linear combination of the overall terms that may contribute to a particular observable. Preserving their notation, the method can be represented like

$$(\text{obs})_i = \kappa_i + w_{ij} c_j \equiv \kappa_i + \hat{c}_i \tag{6.21}$$

The κ_i sum up the Standard Model part. The $\hat{c}_i$ represents any possible disagreement and comprises, linearly, the total of Wilson coefficients enhancing the same observable. The authors have called these 'hat' numbers *observable couplings* composing an *observable basis*. These redefinitions permit to obtain the correct anomalous matrix for the hat coefficients and each operator in this basis can be directly bounded from the 'strength' of its coupling. In other words, the w_{ij} will define a basis transformation from γ_{ij} into a new $\hat{\gamma}_{ij}$. As illustrated in [24], the

Table 6.2: The column c_{2B} of the anomalous dimension matrix for the dim-6 Bosonic operators, according to J.Elias-Miró et al in [24].

$\gamma_{c_T,\cdot}$	$\gamma_{c_B,\cdot}$	$\gamma_{c_W,\cdot}$	$\gamma_{c_{2B},\cdot}$	$\gamma_{c_{2W},\cdot}$	$\gamma_{c_{WB},\cdot}$
$3g'^4+\frac{9}{8}g'^2g^2+3\lambda g'^2$	$\frac{59}{4}g'^2$	$\left(\frac{29}{8}-\frac{53g'^2}{4g^2}\right)g'^2$	$\frac{94}{3}g'^2$	$\left(\frac{53}{12}-\frac{53g'^2}{4g^2}\right)g'^2$	0

observable coupling $\hat{c}_{\gamma Z}$ can parametrize corrections to the process $h\to\gamma Z$ through the sum

$$\hat{c}_{\gamma Z}=\frac{m_W^2}{\Lambda^2}\left(2c^2_{\theta_W}c_{WW}-2s^2_{\theta_W}c_{BB}-(c^2_{\theta_W}-s^2_{\theta_W})c_{WB}\right) \tag{6.22}$$

which composes the matrix line $w_{\gamma Z,j}$. One important observation is that the entries of w is dependent on SM parameters and therefore run with the scale through the evolution of g,g' and v. As mentioned before, the authors state a reasonable argument already followed in the derivation of the approximate evolution equation - the UV values of SM parameters are in general considered constant around their values aat the electroweak scale. In summary, $w_{ij}\to w_{ij}(m_h)$.

Finally, the scale-dependent coupling $\hat{c}_i(\mu)\equiv w_{ij}(m_h)c_j(\mu)$ is correlated with the complete set of observables through:

$$\delta(\text{obs})_i\Big|_{m_h}=\hat{c}_i(m_h)=\hat{c}_i(m)-\frac{1}{16\pi^2}\sum_j\hat{\gamma}_{ij}\hat{c}_j(m)\log\frac{m}{m_h} \tag{6.23}$$

with

$$\hat{\gamma}_{ij}=w_{ik}(m_h)\gamma_{kl}w^{-1}_{lj}(m_h) \tag{6.24}$$

In the Table 6.2 is registered the column $\gamma_{\cdot,c_{2B}}$ computed in the basis of Eq.(6.20), extracted from [24], and revealing the dependence on the set of SM parameters

$$\{g',g,\lambda,y_t\}\quad\to\quad\{U(1)_Y,SU(2)_L,\mathscr{L}_{\text{Higgs}},\mathscr{L}_Y(top)\} \tag{6.25}$$

with y_t the top Yukawa coupling. The Table also clarify how the anomalous matrix indeed depends on the entire configuration of light particles correcting the operator.

After solving the Eq.(6.23), $\hat{c}_i$ can then be tested before the criteria

$$\delta(\text{obs})_i\Big|_{m_h}=\hat{c}_i(m_h)=\sum_j w_{ij}(m_h)c_j(m_h)\quad\in\quad[\epsilon_i^{\text{low}},\epsilon_i^{\text{up}}] \tag{6.26}$$

being ϵ_i the lower and upper bounds from the measurements to the observable i.

Therefore, the analysis remains essentially the same - To translate, in a direct manner, informations from one measured sector to another. As in the Triplet scalar example, a tree-level Wilson coefficient which may not be well limited can propagate its effects into a perhaps more constrained process. However, the method will be useful when the final values have about the same size. On what follows, this possibility for the quark-J is explored, first by clarifying the meaning of its contribution to the kinetic term $B_{\mu\nu}B^{\mu\nu}$. Next, the effective operator O_{2B} must face the class of electroweak parameters $\hat{S},\hat{T}$ and Y, all measured at the per-mille level.

6.3.1 Scaled Fields

In the Chapter 5 it was shown that the J-quark integration resulted in a shift of the kinetic term $B_{\mu\nu}B^{\mu\nu}$. It provides, therefore, the first example of the field redefinition presented during the Decoupling Theorem analysis of Chapter 3. Although the top-down approach chosen for constructing the Effective Field Theory implies the previous knowledge on the Ultra-Violet complete theory, the application of the final EFT is definitely independent. Once it is assumed that the physical universe is ruled by the UV model and that, for experimental reasons, a space of new particles has not been discovered yet, the philosophy is to state that the experiments have seen only the 'bar' Standard Model, or the *Standard Model Effective Field Theory* (SMEFT), whose couplings and fields already include the parameters of the complete theory. Symbolically,

$$\mathscr{L}_{331} = \mathscr{L}_{SM} + \mathscr{L}_{NP} \tag{6.27}$$

where an abstract expression, the $\mathscr{L}_{331}$, has been divided in a renormalizable piece, $\mathscr{L}_{SM}$, plus some undiscovered properties, $\mathscr{L}_{NP}$. The physical process, at low-energies, have known about the New Physics parameters only through virtual lines, such that what is in fact measured is

$$\mathscr{L}_{SMEFT} = \overline{\mathscr{L}}_{SM} + \sum_{\text{dim-6}} c_i O_i \tag{6.28}$$

i.e. when one assumes the existence of a full theory that is to assert all the SM as a set of Effective operators. Being the 3-3-1HL the final theory, what the experiments have seen at low-energies consists in the SM with shifted fields and couplings. In summary, no additional information can be extracted from a heavy-particle integration which results in operators of the renormalizable low-energy theory.

The contribution to the kinetic term has been originated from one piece of the 1-loop result for these exotic particles to the B_μ propagator. The oblique parameter S [43] is a quantity defined as a *zeroth-order natural relation* [42], i.e. from relations involving the Lagrangian parameters that can result in predictions of the theory. As discussed in Chapter 3 the final form of a renormalizable Lagrangian remains intact under renormalization and thus any possible tree-level connection into its parameters. The zeroth-order natural variables are UV finite. In other words, any relation emerged at tree-level and involving observables may render predictions when the number of parameters necessary to define the theory is smaller than its overall set. The UV finiteness of loop corrections to natural relations, although logical is difficult to verify explicitly. One example for the W-mass was fully performed in M. Schwartz [49], chapter 31.

Thus, the S parameter constitute an observable of the Standard Model, identically equals to zero, and relates the contributions to the neutral bosons propagators from New Physics models. Once the 1-loop piece from the J-quark to the SMEFT can be absorbed by rescaling the gauge-boson B_μ, there is no deformation in the SM Lagrangian and S must persist equals to zero. Nevertheless, this conclusion does not imply that the J-quark cannot change the value of S - it does, at higher-loop level. In the next section it will be shown that c_{2B} will run down to m_W and contribute to the Wilson coefficient c_S.

The fact that S will change due to exotic quarks can be interpreted both in the context of the UV theory as in the SMEFT. In the 3-3-1HL the natural relation behind the definition of S is being altered by the presence of new parameters. It seems a logical statement. In the SMEFT scenario, the insertion of O_{2B} deforms the analytical property of the SM Lagrangian, a criteria to the definition of S. The non-renormalizability of the SMEFT avoids the appearance of natural relations[3].

6.3.2 The Y Parameter

The previous section stated that the oblique parameter S may receive corrections from J-quarks at higher-loop level, induced by the operator O_{2B}. The authors of [24] have parametrized the Lagrangian for Electroweak Precision Observables through:

$$\mathscr{L}_{\text{EWPO}} = -\frac{\hat{T}}{2}\frac{m_Z^2}{2}Z_\mu Z^\mu - \frac{\hat{S}}{4m_W^2}\frac{gg'v^2}{2}(W^3_{\mu\nu}B^{\mu\nu}) - \frac{W}{2m_W^2}(\partial^\mu W^3_{\mu\nu})^2 - \frac{Y}{2m_W^2}(\partial^\mu B_{\mu\nu})^2 \tag{6.33}$$

such that

$$\hat{T} = \hat{c}_T(m_W) = \frac{v^2}{m^2}c_T(m_W); \quad \hat{S} = \hat{c}_S(m_W) = \frac{m_W^2}{m^2}[c_W(m_W) + c_B(m_W) + 4c_{WB}(m_W)] \tag{6.34a}$$

$$Y = \hat{c}_Y(m_W) = \frac{m_W^2}{m^2}c_{2B}(m_W); \qquad W = \hat{c}_W(m_W) = \frac{m_W^2}{m^2}c_{2W}(m_W) \tag{6.34b}$$

The J-quark is directly enhancing the parameter Y, measured at per-mille level [26]. Now, from Table 6.2 it can be seen that, although the $c_i(m)$ are zero for the remaining observables, they are in fact different from zero at $\mu \sim m_W$. Recovering the evolving equation:

$$c_i(m_W) = c_i(m) - \frac{1}{16\pi^2}\sum_j \gamma_{ij}c_j(m)\log\frac{m}{m_W} \tag{6.35}$$

---

[3]The argument follows - (i) The parameter S is zero in the SM. (ii) The 1-loop corrections from J-quarks does not alter the SM Lagrangian. (iii) Thus, the 1-loop from J's cannot alter the S parameter. By definition [49] it follows:

$$S \propto \left[\Pi_{ZZ}(m_Z^2) - \Pi_{ZZ}(0) - \frac{c^2 - s^2}{cs}\Pi_{Z\gamma}(m_Z^2) - \Pi_{\gamma\gamma}(m_Z^2)\right] \tag{6.29}$$

$s = s_{\theta_W}$. In order to prove $S = 0$ from J-quarks it is important to preserve the couplings in terms of A^μ and Z_μ:

$$g_Z = -eqtg_W, \qquad g_A = eq \tag{6.30}$$

with q the electric charge. By the singlet isospin structure of J_i ($T_3 = 0$), the many vaccum polarization amplitudes will be given by (N is the number of colors):

$$\Pi_{\gamma\gamma}(p^2) = Nq^2\Pi_{VV}(p^2), \quad \Pi_{\gamma Z}(p^2) = -\frac{Ns}{c}q^2\Pi_{VV}(p^2), \quad \Pi_{ZZ}(p^2) = Nq^2\frac{s^2}{c^2}\Pi_{VV}(p^2) \tag{6.31}$$

with Π_{VV} the sum of the overall chiralities ($\Pi_{VV}(0) = 0$). Replacing it in S:

$$S \propto \left[\frac{s^2}{c^2} - \frac{c^2 - s^2}{cs}\frac{s}{c} - 1\right] = 0 \tag{6.32}$$

Now, higher-order loop corrections from J does not imply the SM - The S parameter *may* receive a correction.

Table 6.3: The 95 % CL measurement to the Precision Observables [24].

$\hat{c}_S(m_t)$ [3]	$\hat{c}_T(m_t)$ [3]	$\hat{c}_Y(m_t)$ [4]
$[-1,2]\times 10^{-3}$	$[-1,2]\times 10^{-3}$	$[-3,3]\times 10^{-3}$

the c_T would be given by

$$c_T(m_W) = -\frac{1}{16\pi^2}[3g'^4 + \frac{9}{8}g'^2 g^2 + 3\lambda g'^2]c_{2B}(m)\log\frac{m}{m_W} \tag{6.36}$$

with[4]

$$c_{2B}(m) = \frac{c_s}{30}\frac{g_B^{*2}}{(4\pi)^2}; \quad g_B^* = g\tan_{\theta_W}\left(\frac{1}{6}+\frac{\beta\sqrt{3}}{2}\right) \tag{6.37}$$

where the relation $s_{\theta_x} = \beta\tan_{\theta_W}$ has been considered. The $c_B(m_W)$, entering in the $\hat{S}$, it will not be so suppressed:

$$c_B(m_W) = -\frac{1}{16\pi^2}\frac{59}{4}g'^2 c_{2B}(m)\log\frac{m}{m_W} \tag{6.38}$$

The Table 6.2 brings the $\gamma_{i,2B}$ column to the remaining Wilson coefficients entering in the EWPOs.

Despite the fact that the renormalization group equations were then able to propagate the c_{2B} into new observables, from Table 6.3, the measurements are equivalent in precision and the analysis can be performed exclusively to $\hat{c}_Y(m_t)$:

$$\hat{c}_Y(m_t) = \frac{m_t^2}{m^2}c_{2B}(m)\left(1-\frac{94}{3}\frac{1}{16\pi^2}g'^2\log\frac{m}{m_t}\right) \tag{6.39}$$

which can be tested in the interval (see [4])

$$\hat{c}_Y(m_t) \in [-3,3]\times 10^{-3} \tag{6.40}$$

For completeness, as mentioned by J.Elias-Miró et al., it must be expected to future measurements not to include the zero in the allowed interval, such that

$$0 < \epsilon_i^{\text{low}} < |\delta(\text{obs})|_i < \epsilon_i^{\text{low}} \tag{6.41}$$

The Eq.(6.39) will then be testing how much tune is necessary to accommodate the heavy mass m inside the interval. In terms of β the final equation must be

$$\hat{c}_Y(m_t) = c_s\frac{m_t^2}{m^2}\frac{g'^2}{30(4\pi)^2}\left(\frac{1}{6}+\frac{\sqrt{3}\beta}{2}\right)^2\left(1-\frac{94}{3}\frac{1}{16\pi^2}g'^2\log\frac{m}{m_t}\right) \tag{6.42}$$

with $g' = g\tan_{\theta_W}$, $g = 0.65$, $\sin^2_{\theta_W}(m_Z) = 0.231$ and $m_t = 173\text{GeV}$ (from Particle Data Group Review [41]). Thus, the correction to c_{2B} is second order in g'^2.

[4]Here c_s is the coefficient coming from the functional determinant and is not related with the Wislon coefficient c_S.

6.3.3 The Role of the c_s

The last three equations are crucial to the final conclusions. The first observation is that, despite the fact that the constant c_s, coming from a Gaussian integral, was already computed in Section 2.2.1, it has been left intact at this point. The reason is that the chosen observable, namely the Y parameter, is sensible to the sign of the Wilson coefficient. This can be interpreted as the potential associated with the many exchange of B-bosons may be either attractive or repulsive. Second, the entire content of the present work is based on the importance of Log corrections to Effective Theories. The results, therefore, may inform what is the level of precision required to the experiments in order to some particular sectors of models beyond the Standard Model can be indeed tested. The case of the J-quarks in the 3-3-1HL serves as one example. These heavy particles are already model suppressed, not coupled with SM fields at tree-level. The first loop suppression in the result of Eq.(6.42), the factorized piece, is already of order $O(10^{-5})$ for $m \sim m_t$, thus corresponding a safe result compared with the level of precision of Eq.(6.40). The c_s, however, is negative for quarks ($c_s = -\frac{1}{2}$), a quality that must be flipped by the Log inside the brackets. The sign for the Y parameter is then testing the type of particles correcting the propagator. If the chosen case was a scalar particle, the situation could be the opposite.

Therefore, the condition of Eq.(6.41) can be very informative to these sort of models. For instance, if only quarks generated the coefficient c_{2B}, a closure in the positive side would be very constraining. In fact, even when c_{2B} is generated from both fermions and bosons there may be no possibility to accommodate both signs in the tune. The 3-3-1HL may also provide one example of this. After the first breaking, the model presents a new heavy Higgs, $\overline{H}$ singlet of $SU(2)_L$ but neutral under hypercharge, $g_B^\chi = 0$. The additional triplets ρ and η could also be applied before the second breaking, since now their singlet components are charged under g_B. By considering the scalar ρ^U of Eq.(4.40), with $g_B^\rho = g'\left(\frac{1}{2} - \frac{\sqrt{3}\beta}{2}\right)$the formula for $\hat{c}_Y(m_t)$ will be corrected to:

$$
\begin{aligned}
\hat{c}_Y(m_t) &= \frac{m_t^2}{m^2}\left[c_{2B}^J + c_{2B}^\rho\right]\left(1 - \frac{94}{3}\frac{1}{16\pi^2}g'^2 \log\frac{m}{m_t}\right) \\
&= \frac{m_t^2}{m^2}\frac{1}{30(4\pi)^2}\left[c_s^\rho g_B^\rho + c_s^J g_B^{*2}\right]\left(1 - \frac{94}{3}\frac{1}{16\pi^2}g'^2 \log\frac{m}{m_t}\right) \\
&= \frac{m_t^2}{m^2}\frac{1}{60(4\pi)^2}\left[g_B^{\rho 2} - g_B^{*2}\right]\left(1 - \frac{94}{3}\frac{1}{16\pi^2}g'^2 \log\frac{m}{m_t}\right)
\end{aligned}
\tag{6.43}
$$

where it has been assumed, for simplicity, both particles with the same mass m. It turns out that, for variants of the model with positive β, the $\left[g_B^{\chi 2} - g_B^{*2}\right]$ is still negative, while the sign changes for β negative. From Chapter 4, the entire phenomenology of the 3-3-1HL depends on the particular choice of the β parameter. Thus, the exclusion of the zero in the allowed region of Y could be informative to these sort of models.

CHAPTER 7

CONCLUSIONS

The work was a brief presentation about the principles behind the Standard Model Effective Field Theory and has been separated in two parts:

Part II

- One *Effective Field Theory* consists on bringing the 1LUV theory into a local description of the interactions. The expansion can be made in a covariant form via the CDE method. An universal formula was registered to the case where loop graphs involves heavy particles only. For mixed heavy-light terms a new expression has been presented.
- The introduction of new terms in order to deal with divergences of a specific Green's function can be performed systematically. The presence of field strength-, mass- and charge renormalization factors is the conceptual manner in QFT to provide predictions consisting of variations.
- The matching procedure will also contain one independent concept of subtraction. The procedure is such that it preserves the "hard" modes of propagation of the heavy particle. This is to say that the Wilson coefficients will carry the information about the integrated degree of freedom along a small distance of propagation (information of non-locality).
- The heavy mass propagate exclusively in the internal lines and during a short amount of time compared with the length of the whole interaction. The product of operators at different points can be replaced by a sum of local operators.
- The use of new non-renormalizable theory is legitimate both by the Weinberg's and the decoupling theorem. If the EFT is computed at higher-level, the possible divergences must always be accompanied by polinomials in the external momenta, thus corresponding to local counterterms that could be inserted a priori [49]. These insertions leads to new interactions from higher-dim operators, also gradually more suppressed by powers of the heavy scale. In summary, close to the decoupling limit, the finite terms will correspond to small corrections. The theory is predictive.
- The Decoupling theorem states that a renormalizable theory including heavy degrees of freedom will provide the same predictions at low energies as an equivalent renormalizable theory obtained by just cutting out the heavy field and redefining its couplings. The error for a graph Γ will be proportional to $\frac{1}{m^a}$.

- The dim-6 truncation is one attempt to gradually insert corrections to a renormalizable low-energy theory, thus reducing the error when one eliminates heavy particles of a complete theory.

- The Decoupling theorem presents a very coherent argument about the validity of transforming a large UV complete theory into a simplified and renormalizable variant - The corrections are suppressed by powers of the heavy sector integrated out. It must be expected, therefore, that as soon as the experiments can achieve larger scales, these corrections can become gradually more important and predictions from the low-effective theory less accurate. It is important to emphasize that both, full and low theories, are renormalizable and differ for a change in the couplings of the effective variant which includes a finite shift after the renormalization of the original vertices. In the present scenario, it is assumed that the experiments can in fact reach these suppressed corrections, but the energies are not sufficient yet to produce the heavy particles asymptotically.

- The operators added to the low theory are not completely arbitrary but must follow the symmetry of the original theory. In the case of the Standard Model, for example, the complete set of dimension-six operators was first classified by W. Buchmüller and D.Wyler in 1986 [9], and recently improved in B. Grzadkowski et al., whose work establish the [31] so-called Warsaw basis for the dim-6 SMEFT operators.

- The claims in favor of Effective Field Theories also justify a choice for the *top-down* approach. The structure of the model, i.e. gauge symmetry, representation, particle-content.., will conduct the emerging set of operators and eventually accentuate a subset of elements in the Warsaw basis. This can give an important hint on where the experiments should concentrate their searches.

- The examples to the Decoupling theorem in Chapter 3 illustrated the procedure for construction of the Effective Theory as a recursive logic. The weight of the new local terms, i.e. their couplings, must correspond to the difference between the complete and the previous EFT, at the heavy scale. Thus, the *matching* is present as a form of introducing a Correspondence Principle into the two theories and to control the presence of large logs. The resulting coefficients will finally compose a set of boundary conditions to the Renormalization Group Equations.

- Effective Field Theories are in fact dotted of all the fundamental concepts behind a QFT and can be seen as a robust technique for simplifying the phenomenological analysis of complex UV complete models.

- The particle content for the 3-3-1 model with heavy-leptons arises according to the so-denoted variable β. For β-independent interactions, i.e. for vertices present in any variant of the model, the $SU(3)\otimes U(1)$ can be broken into the Standard Model via a new scalar triplet.

- In the above scenario, all the heavy sector turns loop-suppressed, apart from a neutral gauge-boson, Z'.

- The integration of heavy-quarks resulted in a contribution to the kinetic-term of the gauge-boson B^{μ}, in addition to the dim-6 operator O_{2B};

- The kinetic term is just rescaling a piece already present in the SM Lagrangian, like in the example for the decoupling theorem, and must not alter any natural relation emerged in the SM framework. This correction comes from 1-loop with J-quarks to the boson propagator. The oblique parameter S is a ultraviolet finite defined as a natural relation and identically zero in the SM. Having the contributions from J-quarks resulted in $B_{\mu\nu}B^{\mu\nu}$ cannot imply a contribution to it.

- In fact, the S parameter may be corrected by the presence of these exotic quarks as higher-loop effect and it can be shown through the mixing of the operator O_{2B} during the coupling evolution. Thus, the changing in tree-level relations will appear in the EFT scenario as a deformation in the analytical structure of the theory, while in the UV complete framework it will be given due to the presence of a new sector for the SM Lagrangian.

- The loop contribution of c_{2B} to the Y parameter is two order of magnitude ($O(10^{-5})$) smaller than the current precision ($O(10^{-3})$). However, the fermion loops implies a negative sign to the observable, a feature that cannot be tuned by running the coupling.

- The inclusion of a Bosonic loop, through a heavy Higgs, may flip the sign of c_Y for negative values of the β-parameter. The verification of a region for the Y parameter which eliminates the zero value would be very informative to the phenomenology of 3-3-1HL models.

APPENDIX A

FORMULAE

A.1 On Gamma Functions and $\overline{\text{MS}}$ scheme

This section presents some of the properties of Gamma functions that originate the constants subtracted during the $\overline{MS}$ scheme.

The Gamma function is defined like

$$\Gamma(z) = \int_0^\infty dt\; t^{z-1} e^{-t} \tag{A.1}$$

such that, by a changing of variables,

$$\Gamma(z) = \frac{t^z}{z} e^{-t}\Big|_0^\infty + \int dt \frac{t^z}{z} e^{-t} \quad \rightarrow \quad z\Gamma(z) = \Gamma(z+1) \tag{A.2}$$

It is worthy to extract some identities for these objects from a series expansion [37]. First,

$$\begin{aligned}\Gamma(2+\epsilon) &= (1+\epsilon)\,\Gamma(1+\epsilon) \\ &= (1+\epsilon)\,\epsilon\,(1-\epsilon)\,\Gamma(\epsilon-1)\end{aligned} \tag{A.3}$$

Apart from that, by Taylor expanding $\Gamma(2+\epsilon)$ around $\epsilon = 0$:

$$\begin{aligned}\Gamma(2+\epsilon) &= \Gamma(2) + \Gamma'(2)\,\epsilon + \frac{1}{2}\Gamma''(2)\,\epsilon^2 + O^3 \\ \frac{\Gamma(2+\epsilon)}{\Gamma(2)} &= 1 + \psi(2)\,\epsilon + \frac{1}{2}\frac{\Gamma''(2)}{\Gamma(2)}\,\epsilon^2 + O^3\end{aligned} \tag{A.4}$$

where

$$\psi(z) \equiv \frac{\Gamma'(z)}{\Gamma(z)} \tag{A.5}$$

and thus, with $\left(d_z \equiv \frac{d}{dz}\right)$,

$$\begin{aligned}\psi(z) &= d_z \log\Gamma(z) \\ &= d_z \log[(z-1)\Gamma(z-1)] \\ &= d_z \log[(z-1)] + d_z \log[\Gamma(z-1)] \\ &= \frac{1}{z-1} + \psi(z-1)\end{aligned} \tag{A.6}$$

or

$$\psi'(z) = -\frac{1}{(z-1)^2} + \psi'(z-1) \tag{A.7}$$

Besides, from the definition of ψ of Eq.(A.5):

$$\begin{aligned}\psi'(z) &= \frac{\Gamma''(z)}{\Gamma(z)} - \left(\frac{\Gamma'(z)}{\Gamma(z)}\right)^2 \\ &\stackrel{(A.5)}{=} \frac{\Gamma''(z)}{\Gamma(z)} - \psi(z)^2 \end{aligned} \tag{A.8}$$

Thus corresponding, through Eq.(A.6), to

$$\frac{\Gamma''(z)}{\Gamma(z)} = \psi'(z-1) - \frac{1}{(z-1)^2} + \psi(z)^2 \tag{A.9}$$

This last formula can be placed back to the power series Eq.(A.4), leading to

$$\begin{aligned}\frac{\Gamma(2+\epsilon)}{\Gamma(2)} &= 1 + \psi(2)\,\epsilon + \frac{1}{2}\left(\psi'(1) - 1 + \psi(2)^2\right)\,\epsilon^2 + O^3 \\ &\stackrel{(A.7)}{=} 1 + \psi(2)\,\epsilon + \frac{1}{2}\left(\psi'(2) + \psi(2)^2\right)\,\epsilon^2 + O^3 \end{aligned} \tag{A.10}$$

From these results it is possible to extract some frequently required values present in loop-integrals calculation. The final expressions for this part of the treatment are, from Eq.(A.3),

$$\Gamma(\epsilon - 1) = \frac{\Gamma(2+\epsilon)}{(1+\epsilon)\,\epsilon\,(1-\epsilon)} \tag{A.11}$$

requiring

$$\psi(2) \stackrel{(A.6)}{=} 1 + \psi(1), \qquad \psi'(2) \stackrel{(A.7)}{=} \psi'(1) - 1 \tag{A.12}$$

The constants $\psi(1)$ and $\psi'(1)$ are known and given by

$$\psi(1) = -\gamma \simeq 0.578, \qquad \psi'(1) = \frac{\pi^2}{6} \tag{A.13}$$

and γ the so-called Euler-Mascheroni constant.

The fundamental formula for the matching procedure at quantum level, namely Eq.(2.51), will generally result in d-dimensional loop integrals in the form

$$I_n^0 \equiv \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left(q^2 - m^2\right)^n} = i \frac{(-1)^n}{\Gamma(n)} A_n \tag{A.14a}$$

$$I_n^2 \equiv \int \frac{d^d q}{(2\pi)^d} \frac{q^2}{\left(q^2 - m^2\right)^n} = i \frac{(-1)^{n-1}}{\Gamma(n)} \frac{d}{2} A_{(n-1)} \tag{A.14b}$$

where the A_n piece is be given by (with $\epsilon = 4 - d$):

$$\begin{aligned} A_n &\equiv \mu^{n\epsilon} \frac{\Gamma(n - d/2)}{(4\pi)^{d/2}} \left(\frac{1}{m^2}\right)^{n-d/2} \\ &= \mu^{\epsilon(n-1)} \frac{\Gamma\left((n-2) + \frac{\epsilon}{2}\right)}{(4\pi)^2} \left(m^2\right)^{(2-n)} \left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} \end{aligned} \tag{A.15}$$

and the mass parameter μ is inserted in order to leave the coupling constants of the theory in their correct dimension. It must be noted, therefore, that the presence of μ^n refers to insertion of n external points for each loop-diagram.

Next, a power series can be performed for ϵ around zero by requesting the result

$$f(x) \equiv a^x = 1 + d_x(a^x)\big|_0 \, x + O(x^2) \tag{A.16}$$

and

$$\text{if } a^x = b \to x \log a = b \quad \to \quad \frac{d_x b}{b} = \log a \tag{A.17}$$

or, in summary,

$$d_x(a^x) = a^x \log a \tag{A.18}$$

such that, when applied back to Eq.(A.16) implies

$$a^x = 1 + x \log a + O(x^2) \tag{A.19}$$

The previous formulas can finally be applied for some examples. The first is chosen to be the simplest I_1^0, which then requires

$$\begin{aligned}
A_1 &= \frac{\Gamma\left(\frac{\epsilon}{2}-1\right)}{(4\pi)^2}\left(m^2\right)\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} \\
&\overset{(A.11)}{=} \frac{m^2}{(4\pi)^2}\frac{2}{\epsilon}\Gamma\left(\frac{\epsilon}{2}+2\right)\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} \\
&\overset{(A.19)}{=} \frac{m^2}{(4\pi)^2}\frac{2}{\epsilon}\Gamma\left(\frac{\epsilon}{2}+2\right)\left(1+\frac{\epsilon}{2}\log\left(\frac{4\pi\mu^2}{m^2}\right)\right) \\
&\overset{(A.10)}{=} \frac{m^2}{(4\pi)^2}\frac{2}{\epsilon}\left(1+\psi(2)\,\frac{\epsilon}{2}\right)\left(1+\frac{\epsilon}{2}\log\left(\frac{4\pi\mu^2}{m^2}\right)\right) \\
&= \frac{m^2}{(4\pi)^2}\left(\frac{2}{\epsilon}+\psi(2)+\log\left(\frac{4\pi\mu^2}{m^2}\right)\right)
\end{aligned} \tag{A.20}$$

In the second line, higher order terms over ϵ were just neglected as well as those remaining in the expansion of $\Gamma(2+\epsilon)$, since they must vanish in the limit $\epsilon \to 0$. Apart from that, in the $\overline{\text{MS}}$ scheme both the pole and the Euler-Mascheroni constant present in $\psi(2) = 1-\gamma$ are subtracted along with $\log 4\pi$. Finally, by replacing A_1 back to the expression for I_1^0, it follows

$$I_1^0 = i\frac{m^2}{(4\pi)^2}\left(\log\left(\frac{m^2}{\mu^2}\right)-1\right) \tag{A.21}$$

Additional results are presented in the main text.

In general, the evaluation of these master integrals requires the integrand to be Wick rotated into Euclidean space as an intermediate step, what corresponds to the redefinition

$$q^0 \to i\overline{q}^0, \qquad q^2 \to -\overline{q}^2 \tag{A.22}$$

such that the inner product for bar variables goes with the Cartesian metric.

For concluding, the integrals of Eq.(A.14) can be complemented with the fourth and sixth order, both present during the evaluation of the various $\mathfrak{I}_n$'s. Here, the former case is presented according to [42] like

$$I_n^4 \equiv \int \frac{d^d q}{(2\pi)^d} \frac{(q^2)^2}{\left(q^2 - m^2\right)^n} = i\frac{(-1)^n}{\Gamma(n)} \frac{d(d+2)}{4} A_{(n-2)} \tag{A.23a}$$

$$(I_n^4)^{\mu\nu\rho\sigma} \equiv \int \frac{d^d q}{(2\pi)^d} \frac{q^\mu q^\nu q^\rho q^\sigma}{\left(q^2 - m^2\right)^n} = i\frac{(-1)^n}{\Gamma(n)} \frac{1}{4}\left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right) A_{(n-2)} \tag{A.23b}$$

For the analysis of higher-order integrals see the Appendix of [32] and [53].

A.2 Properties for Covariant Derivatives

The Covariant Derivative[1] is a conceptual object able to generate translation both in space-time and in the internal space. These infinitesimal gauge transformations are performed by the Wilson line, whose property under a internal rotation U_x is defined like

$$W(x,y) \equiv W_{xy} \to U_x W_{xy} U_y^{-1} \tag{A.24}$$

If the fields ϕ and ψ transform like

$$\phi \to \phi' = U\phi \tag{A.25}$$

the Covariant Derivative on ϕ is then defined like

$$D_\mu \phi \equiv \lim_{y\to x} \frac{W_{xy}\phi_y - \phi_x}{(y-x)} \tag{A.26}$$

with $\phi_x \equiv \phi(x)$. A general product like $\phi\psi$ would transform like $\phi\psi \to \phi'\psi' = (U\phi)(U\psi)$ such that

$$D_\mu(\phi\psi) = \lim_{y\to x} \frac{W_{xy}\phi_y W_{xy}\psi_y - \phi_x\psi_x}{(y-x)} \tag{A.27}$$

which can be factorized like

$$\begin{aligned} D_\mu(\phi\psi) &= \lim_{y\to x} \frac{W_{xy}\phi_y W_{xy}\psi_y + \phi_x W_{xy}\psi_y - \phi_x W_{xy}\psi_y - \phi_x\psi_x}{(y-x)} \\ &= \lim_{y\to x} \frac{(W_{xy}\phi_y - \phi_x)W_{xy}\psi_y + \phi_x(W_{xy}\psi_y - \psi_x)}{(y-x)} \\ &= (D_\mu\phi)\psi + \phi(D_\mu\psi) \end{aligned} \tag{A.28}$$

where it has been considered the property of limit for products and $W_{xx} = 1$. The above result, namely the product rule, is one of the important properties adopted along the main text.

From a general notation

$$D_\mu = \partial_\mu - W_\mu \tag{A.29}$$

[1]Review based in the Chapter 12 - Nikhef notes `https://www.nikhef.nl/~t45/ftip/Ch12.pdf`;

with $W_\mu = t^a W^a_\mu$ and t^a antisymmetric, it has also been defined the field-strength as

$$[D_\mu, D_\nu] = -G_{\mu\nu} \tag{A.30}$$

where $G_{\mu\nu} = (\partial_\mu W_\nu) - (\partial_\nu W_\mu) - [W_\mu, W_\nu]$. In terms of components, it follows

$$G_{\mu\nu} = G^a_{\mu\nu} t_a \quad \text{with} \quad G^a_{\mu\nu} = (\partial_\mu W^a_\nu) - (\partial_\nu W^a_\mu) - f^a_{bc}[W^b_\mu, W^c_\nu] \tag{A.31}$$

Now, the $G_{\mu\nu}$ property of transformation under U, i.e.

$$G_{\mu\nu} \to G'_{\mu\nu} = U G_{\mu\nu} U^{-1} \tag{A.32}$$

will imply a new, but equivalent, definition for the action of D_μ. In fact, for any function with the transformation rule of Eq.(A.32), the Covariant Derivative must be given by

$$D_\mu G^{\mu\nu} = \lim_{\delta x \to 0} \frac{W_{x,x+\delta x} G^{\mu\nu}_{x+\delta x} W_{x+\delta x, x} - G^{\mu\nu}_x}{\delta x^\mu} \tag{A.33}$$

such that, since $W_{xy} \to U_x W_{xy} U_y^{-1}$

$$D_\mu G \to U(D_\mu G)U^{-1} \tag{A.34}$$

If the Wilson line for infinitesimal transformation around $W_{xx} = 1$ is given by

$$\begin{aligned} W(x, x+\delta x) &= 1 - W_\alpha \delta x^\alpha + O(\delta x^2) \\ W(x, x+\delta x) &= 1 + W_\alpha \delta x^\alpha + O(\delta x^2) \end{aligned} \tag{A.35}$$

where W^α are the gauge-fields, in Eq.(A.33) it follows:

$$\begin{aligned} D_\mu G^{\mu\nu} &= \lim_{\delta x \to 0} \frac{(1 - W_\alpha \delta x^\alpha) G^{\mu\nu}_{x+\delta x} (1 + W_\alpha \delta x^\alpha) - G^{\mu\nu}_x}{\delta x^\mu} \\ &= (\partial_\mu G^{\mu\nu}) - [W_\mu, G^{\mu\nu}] \end{aligned} \tag{A.36}$$

which is therefore the Covariant Derivative for matrices. It may be noted that this property can be equivalently written like

$$D_\mu G^{\mu\nu} = [D_\mu, G^{\mu\nu}] \tag{A.37}$$

a choice frequently adopted along the main text.

In practice, however, the gauge fields are rescaled with the presence of a coupling constant and hermitian generators, such that

$$D_\mu = \partial_\mu - igW_\mu \tag{A.38}$$

This justify, for example, the prime index in $G'_{\mu\nu}$, i.e.

$$G'_{\mu\nu} \equiv [D_\mu, D_\nu] = -igG_{\mu\nu} \tag{A.39}$$

and $G_{\mu\nu} = (\partial_\mu W_\nu) - (\partial_\nu W_\mu) - ig[W_\mu, W_\nu]$, with components $G^a_{\mu\nu} = (\partial_\mu W^a_\nu) - (\partial_\nu W^a_\mu) + gf^a_{bc}[W^b_\mu, W^c_\nu]$.

Some examples often encountered are:

$$\begin{aligned} -tr[D_\mu, D_\nu]^2 &= g^2 tr(G_{\mu\nu}G^{\mu\nu}) \\ &= g^2 G^a_{\mu\nu}G^{b,\mu\nu} tr(T^a T^b) \\ &= g^2 G^a_{\mu\nu}G^{a,\mu\nu} T(R) \end{aligned} \tag{A.40}$$

and $T(R)$ the index of the representation, where $T(F) = \frac{1}{2}$ in the fundamental and $T(A) = N$ in the adjoint of $SU(N)$. Another case:

$$\begin{aligned} -tr[D_\mu, [D^\mu, D^\nu]]^2 &= g^2 tr[D_\mu, G^{\mu\nu}]^2 \\ &= g^2 tr(D_\mu G^{\mu\nu})^2 \\ &= g^2 (D_\mu G^{a,\mu\nu})(D^\alpha G^b_{\alpha\nu}) tr(T^a T^b) \\ &= g^2 (D_\mu G^{a,\mu\nu})^2 T(R) \end{aligned} \tag{A.41}$$

where $(D_\mu G^{a,\mu\nu})$ follows from Eq.(A.36) as

$$\begin{aligned} D_\mu G^{\mu\nu} &= (\partial_\mu G^{\mu\nu}) - ig[W_\mu, G^{\mu\nu}] \\ &= (\partial_\mu G^{\mu\nu}) - ig[T^a, T^b] W^a_\mu, G^{b,\mu\nu} \\ &= (\partial_\mu G^{\mu\nu}) - igf^{abc} T^c W^a_\mu, G^{b,\mu\nu} \end{aligned} \tag{A.42}$$

$$\therefore \quad D_\mu G^{a,\mu\nu} = (\partial_\mu G^{a,\mu\nu}) + gf^{bca} W^b_\mu, G^{c,\mu\nu} \tag{A.43}$$

Thus, although the commutators in the Universal Formula have been preserved, they can be replaced by an equivalent notation $[D_\mu, G] \to (D_\mu G)$, with Eq.(A.43) implicit. For completeness, the action of D_μ on the field-strength of an Abelian theory reduces to a partial derivative.

Two additional examples may also be useful along the application of the results in the text. The first involves the covariant derivative of a function A instead of a field-strength, like for example the generic matrix of light fields inside the covariant propagator $P^2 - m^2 + A_x$. Again, the rule $D_\mu G = (\partial_\mu G) - ig[W_\mu, G]$ is general, whenever the function G transforms like $G \to G' = UGU^{-1}$. as illustrated in Chapter 5, there are some cases where the matrix under consideration is composed by the fields in a particular way. Consider, for instance, $A \equiv HH^\dagger$, with H the Higgs doublet. It follows that,

$$\begin{aligned} D_\mu A &= (\partial_\mu A) - ig[W_\mu, A] \\ &= (\partial_\mu A) - ig(W_\mu HH^\dagger - HH^\dagger W_\mu) \\ &= (\partial_\mu H)H^\dagger + H(\partial_\mu H^\dagger) + (-igW_\mu H)H^\dagger + H(-igW_\mu H)^\dagger \\ &= (D_\mu H)H^\dagger + H(D_\mu H)^\dagger \end{aligned} \tag{A.44}$$

now, $D_\mu H$ being just the usual application. The last form of the above relation can make the trace computation clearer.

The last example consists in the case where the CD acts in a product of matrices AB. Thus, $(W_\mu \equiv -igW_\mu)$

$$\begin{aligned} D(AB) &= \partial(AB) + [W_\mu, AB] \\ &= \partial(AB) + [W_\mu, A]B + A[W_\mu, B] \\ &= (DA)B + A(DB) \end{aligned} \tag{A.45}$$

Then, from the product rule

$$\begin{aligned} \int_x (DA)B &= \int_x trD(AB) - \int_x trA(DB) \\ &= \int_x (\partial_\mu(AB) + tr[W_\mu, AB]) - \int_x A(DB) \end{aligned} \tag{A.46}$$

Since the trace of a commutator vanishes

$$\therefore \quad \int_x (DA)B = -\int_x A(DB) \tag{A.47}$$

and follows the chain rule.

In Section 5.1.2.1 it was considered the identity

$$|H|^2\left[(D^2H)^\dagger \cdot H \; + \; H^\dagger \cdot (D^2H)\right] \quad \rightarrow \quad -2(O_H + O_R) \tag{A.48}$$

which follows from the product rule like

$$D^2|H|^4 = 0 \;\rightarrow\; D^\mu|H|^2 D_\mu|H|^2 + |H|^2 D^2|H|^2 = 0 \tag{A.49}$$

or, from $O_H \equiv \frac{1}{2}(D^\mu|H|^2)^2$ and $O_R \equiv |H|^2|D_\mu H|^2$,

$$\begin{aligned} 2O_H &= -|H|^2 D^2|H|^2 \\ &= -|H|^2 D^\mu \left[(D_\mu H)^\dagger \cdot H + H^\dagger \cdot (D_\mu H)\right] \\ &= -|H|^2 \left[(D^2H)^\dagger \cdot H + H^\dagger \cdot (D^2H)\right] - 2|H|^2|D_\mu H|^2 \end{aligned} \tag{A.50}$$

and, finally,

$$|H|^2\left[(D^2H^\dagger) \cdot H \; + \; H^\dagger \cdot (D^2H)\right] = -2(O_H + O_R) \tag{A.51}$$

A.3 Finite Components of $\mathfrak{I}_2$

To work with the finite components of Eq.(2.78d) and (2.78e) is important to restore some conventions on the Lorentz indices and remark that q and ∂_q necessarily commute with the fields present in $\tilde{G}, \tilde{U}$. The terms are expressed like

$$\begin{aligned} (2.78d) &= \Delta\ \{q, \tilde{G}\}\partial_q \lfloor \Delta\ \{q, \tilde{G}\}\partial_q\ \Delta \\ (2.78e) &= \Delta\ \{q, \tilde{G}\}\partial_q \lfloor \Delta\ \tilde{U}\ \Delta \end{aligned}$$

and both tilde fields are being evaluated in the second order of the series expansion, i.e.

$$\tilde{G}_{\nu\mu} = \frac{1}{3}[P_\alpha, G_{\nu\mu}]\partial_{q_\alpha}, \qquad \tilde{U} = [P_\alpha, U]\partial_{q_\alpha} \tag{A.52}$$

For the (2.78d) case, first consider the last commutator, that can be rewritten as

$$\{q, \tilde{G}\}\,\partial_q\, \Delta = \left(2[P_\alpha, G_{\nu\mu}]q^\mu \partial^\alpha + [P^\mu, G_{\nu\mu}]\right)\partial^\nu\, \Delta \tag{A.53}$$

where it was made implicit $\partial_{q_\alpha} \equiv \partial^\alpha$. Moreover, from $\partial^\nu \Delta^p = -2pq^\nu \Delta^{(p+1)}$,

$$\begin{aligned}
\{q, \tilde{G}\}\,\partial_q\, \Delta &= -2\left(2[P_\alpha, G_{\nu\mu}]q^\mu \partial^\alpha + [P^\mu, G_{\nu\mu}]\right) q^\nu \Delta^2 \\
&= -2q^\nu \Delta^2\, [P^\mu, G_{\nu\mu}] - 4q^\mu \partial^\alpha (q^\nu \Delta^2)\, [P_\alpha, G_{\nu\mu}] \\
&= 2q^\nu \Delta^2\, [P^\mu, G_{\nu\mu}] + 16 q^\alpha q^\nu q^\mu \Delta^3\, [P_\alpha, G_{\nu\mu}]
\end{aligned} \tag{A.54}$$

Next, this result must be plugged back at Eq.(2.78d), such that

$$\begin{aligned}
(2.78d) &\overset{(A.53)}{\supset} 2\Delta\, \left(2[P_\beta, G_{\epsilon\lambda}]q^\lambda \partial^\beta + [P^\beta, G_{\epsilon\beta}]\right)\partial^\epsilon \left(q^\nu \Delta^3\right)\, [P^\mu, G_{\nu\mu}] \\
&= 4\Delta\, q^\lambda \partial^\beta \partial^\epsilon \left(q^\nu \Delta^3\right)[P_\beta, G_{\epsilon\lambda}][P^\mu, G_{\nu\mu}] + 2\Delta\, \partial^\epsilon \left(q^\nu \Delta^3\right)[P^\beta, G_{\epsilon\beta}][P^\mu, G_{\nu\mu}]
\end{aligned}$$

both convergent, with $(a,k) = (2,5)$ and $(a,k) = (1,4)$ for Eq.(2.60). By considering

$$\begin{aligned}
\partial^\beta \partial^\epsilon \left(q^\nu \Delta^3\right) &= \partial^\beta \left(g^{\epsilon\nu}\Delta^3 - 6\Delta^4 q^\epsilon q^\nu\right) \\
&= -6q^\beta g^{\epsilon\nu}\Delta^4 + 48\Delta^5 q^\beta q^\epsilon q^\nu
\end{aligned} \tag{A.55}$$

it follows

$$\begin{aligned}
(2.78d) \supset\; & 4\Delta\, \left(48\Delta^5 q^\beta q^\epsilon q^\lambda q^\nu - 6q^\beta q^\lambda g^{\epsilon\nu}\Delta^4\right) \times [P_\beta, G_{\epsilon\lambda}][P^\mu, G_{\nu\mu}] + \\
& + 2\Delta\, \left(g^{\epsilon\nu}\Delta^3 - 6\Delta^4 q^\epsilon q^\nu\right)[P^\beta, G_{\epsilon\beta}][P^\mu, G_{\nu\mu}]
\end{aligned}$$

Finally, from the second operator in Eq.(A.54), it follows

$$\begin{aligned}
(2.78d) &\supset 16\Delta\, \left(2[P_\beta, G_{\epsilon\lambda}]q^\lambda \partial^\beta + [P^\beta, G_{\epsilon\beta}]\right)\partial^\epsilon \left(q^\alpha q^\nu q^\mu \Delta^4\right)\, [P_\alpha, G_{\nu\mu}] \\
&= 32\Delta\, q^\lambda \partial^\beta \partial^\epsilon \left(q^\alpha q^\nu q^\mu \Delta^4\right)[P_\beta, G_{\epsilon\lambda}][P_\alpha, G_{\nu\mu}] + 16\Delta\, \partial^\epsilon \left(q^\alpha q^\nu q^\mu \Delta^4\right)[P^\beta, G_{\epsilon\beta}][P_\alpha, G_{\nu\mu}]
\end{aligned}$$

which is finite and requires an integral over sixth power to the momentum. Once the convergence is stated it must be mentioned that, for a carefully reading, a set of useful identities for large dervatives were already presented in [32].

In the case of (2.78e) it follows that

$$\begin{aligned}
(2.78e) &= \Delta\, \{q, \tilde{G}\}\,\partial_q \llcorner\, \Delta\, [P_\beta, U]\partial^\beta \Delta \\
&\overset{(A.53)}{=} -\frac{2}{3}\Delta\, \left(2[P_\alpha, G_{\nu\mu}]q^\mu \partial^\alpha + [P^\mu, G_{\nu\mu}]\right)\partial^\nu \left(q^\beta \Delta^3\right)[P_\beta, U] \\
&= -\frac{2}{3}\Big(\Delta\, \partial^\nu \left(q^\beta \Delta^3\right)[P^\mu, G_{\nu\mu}][P_\beta, U] + \\
&\quad + 2\Delta\, q^\mu \partial^\alpha \partial^\nu \left(q^\beta \Delta^3\right)[P_\alpha, G_{\nu\mu}][P_\beta, U]\Big)
\end{aligned} \tag{A.56}$$

Again, from Eq.(A.55), the first line becomes

$$(2.78e) \supset -\frac{2}{3}\left(g^{\nu\beta}\Delta^4 - 6q^\nu q^\beta \Delta^5\right)[P^\mu, G_{\nu\mu}][P_\beta, U] \tag{A.57}$$

and, finally,

$$(2.78e) \supset -\frac{2}{3}\left(48q^\mu q^\alpha q^\nu q^\beta \Delta^6 - 6g^{\nu\beta}q^\mu q^\alpha \Delta^5\right)[P_\alpha, G_{\nu\mu}][P_\beta, U] \tag{A.58}$$

A.4 On CP-odd Operators

This section introduces the remaining CP-odd components for the set of Bosonic operators. In order to remark how to extract the properties of these operators under CP transformations, it is sufficient to consider that the vector-fields are such that [7]:

$$PG^a_\mu P^\dagger = G^{a\mu}, \qquad CG^a_\mu C^\dagger = -s^a G^a_\mu \tag{A.59}$$

where $s_a = +1$ for $a \in [1,3,4,6,8]$ and -1 for the remaining. These rules follow from the properties of the bilinears under C and P. Apart from that, it also includes our previous knowledge about the symmetries of the strong interactions. From the field-strength $G^a_{\mu\nu}$ and $s_b s_c = -s_a$ for $f_{abc} \neq 0$:

$$PG^a_{\mu\nu} P^\dagger = G^{a\mu\nu}, \qquad CG^a_{\mu\nu} C^\dagger = -s^a G^a_{\mu\nu} \tag{A.60}$$

Above, the change on the spacial coordinates does not contain additional information once one integral is taken over all the space-time. The dual of $G^a_{\mu\nu}$, i.e. $\tilde{G}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta,a}$ defines a new gauge-invariant operator of the type

$$G^a_{\mu\nu}\tilde{G}^{\mu\nu,a} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}G^a_{\mu\nu}G^a_{\alpha\beta} \tag{A.61}$$

Since the total antisymmetric tensor satisfies $\epsilon^{\mu\nu\alpha\beta} = -\epsilon_{\mu\nu\alpha\beta}$, it follows that

$$\begin{aligned} PG^a_{\mu\nu}\tilde{G}^{\mu\nu,a}P^\dagger &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}PG^a_{\mu\nu}G^a_{\alpha\beta}P^\dagger \\ &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}PG^{\mu\nu,a}G^{\alpha\beta,a}P^\dagger \\ &= -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}PG^{\mu\nu,a}G^{\alpha\beta,a}P^\dagger \\ &= -G\cdot\tilde{G} \end{aligned} \tag{A.62}$$

i.e. the product is odd under parity. Since C is preserved, the operator is also CP-odd. A similar analysis can be performed to prove that $O_{H\tilde{B}} \equiv (D^\mu H)\dagger(D^\nu H)\tilde{B}_{\mu\nu}$ is also such that $(CP)O_{H\tilde{B}}(CP)^\dagger = -O_{H\tilde{B}}$.

APPENDIX B

EXAMPLE FROM A DISCRETE PICTURE

This section aims to cover the concepts behind the construction of an Effective Action from a discrete framework. The components parts of this analysis, however, will not reach the reasonableness expected for a physical theory and, thus, the following text and the overall expressions must be seen merely as an alternative source of intuition.

The first object of analysis must be called a 'discrete' energy functional, denoted as $E[J]$ on the source J, and related with processes extracted from a theory of two fields, namely Φ and φ. It must be reinforced the importance on not associating these symbols with any specific physical object. Here they just represent two independent quantities.

On this idealized discrete scenario, the universe is arranged into a lattice form, and here it is considered in the simplest case of a two points for the space-time, denoted as x_a and x_b. The fields on these points are the *independent variables* denoted in the set $\{\Phi_a, \Phi_b, \varphi_a, \varphi_b\}$.

The discrete functional $E[J]$ is defined from the expression (where $J_i \equiv J(x_i)$):

$$Z[J] = e^{-E[J]} = \int \left[\prod_{i=a,b} d\varphi_i \right] \left[\prod_{i=a,b} d\Phi_i \right] \times \exp \left[S[\varphi, \Phi] + \sum_n J_n \varphi_n \right] \tag{B.1}$$

where the absence of index corresponds to a vector over a, b, like $\varphi \to (\varphi_a, \varphi_b)$. Note, moreover, that there are not sources for Φ.

For an illustrative model, the functional S will be given by

$$S[\varphi, \Phi] = \sum_{n=a,b} \frac{1}{2} \Phi_n \mathcal{O}_n^{\Phi} \Phi_n + \frac{1}{2} \varphi_n \mathcal{O}_n^{\varphi} \varphi_n - \frac{\mu}{3!} \Phi \varphi^3 \tag{B.2}$$

Here the $\mathcal{O}_n$'s are numbers that, in a more reliable theory, would contain information on the vicinity of x_n. It is also worthy noting that all the products involve the same space-time point , representing then a local function to the discrete case.

Now, through a Legendre transformation on $E[J]$ it may be defined a new functional denoted by Γ and so-referred *Effective Action*, such that

$$\Gamma[\varphi^c] = -E[J] - \sum_{n=a,b} J_n \varphi_n^c \tag{B.3}$$

and the *classical field*

$$\frac{\delta E[J]}{\delta J_n} \equiv -\varphi_n^c \tag{B.4}$$

implying then a functional dependence of $J[\varphi^c]$.

The previous expression will certainly need a clarification or an appropriate definition for the operation $\frac{\delta}{\delta J}$. Since the exponent in the r.h.s. of Eq.(B.1) is in fact composed by a sum, it is natural to require

$$\frac{\delta J_a}{\delta J_i} = \delta^a_i \tag{B.5}$$

with δ^a_i a Kronecker delta, resulting straightforwardly to the same standard rules for functional derivatives. Therefore,

$$\begin{aligned} \frac{\delta E[J]}{\delta J_n} &= -\frac{\delta \ln Z}{\delta J_n} \\ &= -\frac{1}{Z}\int \left[\prod_{i=a,b} d\varphi_i\right]\left[\prod_{i=a,b} d\Phi_i\right] \times \varphi_n \exp\left[S[\varphi,\Phi] + \sum_i J_i \varphi_i\right] \end{aligned} \tag{B.6}$$

or, the vacuum expectation value for φ_n. Apart from that, the Effective Action is in fact independent on the source J:

$$\begin{aligned} \frac{\delta \Gamma}{\delta J_n} &= -\frac{\delta E}{\delta J_n} - \varphi^c_n \\ &\stackrel{(B.4)}{=} 0 \end{aligned} \tag{B.7}$$

and satisfies

$$\begin{aligned} \frac{\delta \Gamma}{\delta \varphi^c_n} &= -\frac{\delta E}{\delta \varphi^c_n} - \sum_i \frac{\delta J_i}{\delta \varphi^c_n}\varphi^c_i - J_n \\ &= -\sum_i \frac{\delta J_i}{\varphi^c_n}\frac{\delta E}{\delta J_i} - \sum_i \frac{\delta J_i}{\delta \varphi^c_n}\varphi^c_i - J_n \\ &\stackrel{(B.4)}{=} -J_n \end{aligned} \tag{B.8}$$

The computation of $\Gamma[\varphi^c]$, following [42], may proceed for instance from the saddle point approximation over the exponent in Eq.(B.1). The expansion is performed around φ^c and Φ^c, the latter raised from the solution of

$$\frac{\delta S[\varphi,\Phi]}{\delta \Phi}\Big|_{\Phi=\Phi^c} = 0 \tag{B.9}$$

Finally, by defining $\overline{S}[\varphi,\Phi,J] \equiv S[\varphi,\Phi] + J\cdot\varphi$, it follows

$$\begin{aligned} \overline{S}[\varphi,\Phi,J] \simeq\ & \overline{S}[\varphi^c,\Phi^c,J] + \sum_{n=a,b}(\Phi_n - \Phi^c_n)\frac{\delta \overline{S}}{\delta \Phi_n}\Big|_c + \sum_{n=a,b}(\varphi_n - \varphi^c_n)\frac{\delta \overline{S}}{\delta \varphi_n}\Big|_c + \\ & + \frac{1}{2}\sum_{m,n=a,b}\Bigg[(\Phi_m - \Phi^c_m)\frac{\delta^2 \overline{S}}{\delta \Phi_m \delta \Phi_n}\Big|_c (\Phi_n - \Phi^c_n) + (\varphi_m - \varphi^c_m)\frac{\delta^2 \overline{S}}{\delta \varphi_m \delta \varphi_n}\Big|_c (\varphi_n - \varphi^c_n) + \\ & + (\Phi_m - \Phi^c_m)\frac{\delta^2 \overline{S}}{\delta \Phi_m \delta \varphi_n}\Big|_c (\varphi_n - \varphi^c_n) + (\varphi_m - \varphi^c_m)\frac{\delta^2 \overline{S}}{\delta \varphi_m \delta \Phi_n}\Big|_c (\Phi_n - \Phi^c_n)\Bigg] + \cdots \end{aligned} \tag{B.10}$$

where the $|_c$ is indicating derivatives at the classical fields. Moreover, as it was detailed in the main text, the φ_c implies

$$\frac{\delta \overline{S}}{\delta \varphi}\Big|_c = 0 \qquad \text{or} \qquad \frac{\delta S}{\delta \varphi}\Big|_c + J = 0 \tag{B.11}$$

to hold exactly. Thus, the linear terms Eq.(B.10) vanish and the integral over $\left[\prod d\varphi_i\right][\prod d\Phi_i]$ converts the approximation for $Z[J]$ into a Gaussian integral. At this order the correction is then given by

$$Z[J] = e^{\overline{S}[\varphi^c,\Phi^c,J]} \times \det[\mathscr{H}]^{-\frac{1}{2}} \tag{B.12}$$

where the *Hessian* $\mathscr{H}$ matrix here is composed by 2×2 blocks like

$$\mathscr{H} = \left.\begin{pmatrix} \frac{\delta^2\overline{S}}{\delta\varphi\delta\varphi} & \frac{\delta^2\overline{S}}{\delta\Phi\delta\varphi} \\ \frac{\delta^2\overline{S}}{\delta\varphi\delta\Phi} & \frac{\delta^2\overline{S}}{\delta\Phi\delta\Phi} \end{pmatrix}\right|_c \tag{B.13}$$

Besides, the blocks on the diagonal of $\mathscr{H}$ are, by the local structure of S, already in diagonal form.

The *connected* functional generator $E[J]$ may be written as

$$-E[J] \simeq \overline{S}[\varphi^c,\Phi^c,J] - \frac{1}{2}\log\det[\mathscr{H}] \tag{B.14}$$

corresponding to a first correction to the Effective Action like

$$\begin{aligned} \Gamma[\varphi^c,\Phi^c] \quad &\simeq \quad \overline{S}[\varphi^c,\Phi^c,J] - \frac{1}{2}\log\det[\mathscr{H}] - J\cdot\varphi \\ &\overset{(B.1)}{=} \quad S[\varphi^c,\Phi^c] - \frac{1}{2}\log\det[\mathscr{H}] \end{aligned} \tag{B.15}$$

This is the final formula for a field theory including quantum corrections and, on what follows, it will be the basis for constructing two distinct theories, namely the Light-UV and the correspondent EFT.

Every step leading to Γ could similarly have been taken from the realistic continuous case, by replacing $\prod d \to \mathscr{D}$ and $\sum_n \to \int_x$. The choice for a discrete scenario might provide a simple algebraic form to the calculation of determinants and, as commented before, was chosen with an illustrative intent. Rigorously, the expression of Eq.(B.2) is dimensionally incorrect, since it does not include a factor correspondent to the size of the space-time lattice. In other words, the expression for the vacuum amplitude, including for simplicity only φ (see [30]), should be given by

$$Z[0] = \lim_{N\to\infty}\int \left[\prod d\varphi_i\right]\left[\prod d\pi_i\right] \times \exp\left[\epsilon\sum_n \pi_n \frac{\varphi_{n+1}-\varphi_n}{\epsilon} + H\left(\pi_n, \frac{\varphi_{n+1}+\varphi_n}{2}\right)\right] \tag{B.16}$$

where $n+1$ refers to the grid point in time for $t_{n+1} = t_n + \epsilon$. Furthermore, the canonical momenta is defined as $\pi_n \equiv \frac{\varphi(t_n+\epsilon)-\varphi(t_n)}{\epsilon} = \frac{\varphi_{n+1}-\varphi_n}{\epsilon}$. Thus far, only the time has been fragmented into a discrete set of points. In general, a last step for the functional generator definition consists in writing the Lagrangian function on t as a weighted average of a new set of fields, $\hat{\varphi}$, on the space lattice:

$$\varphi_n = \sum_j a_j(t_n)\,\frac{\hat{\varphi}_j}{N} \tag{B.17}$$

where the index j runs over the points in space. The expression draws a picture of the space filled with a constant, but not necessarily uniform, field whose points may be pondered with a time-dependent sequence of weights $\{a_j\}$ in order to define a new field on t. In fact, since the functional form of these entities does not a play an important role in quantum field theory, the expression can be rearranged in a simpler form like

$$\varphi_n \to \xi \sum_j \varphi_j(t_n) \tag{B.18}$$

where ξ is a constant for the grid size in space. For a more details see [30].

The above discussion was made for fields but must be directly extended to the Lagrangians, what leads to the concept of density Lagrangian. In the example of Eq.(B.2) the constant of volume dimension has been absorbed by the functions, an option that implies a change on their dimension. Thus, for consistency, the numbers $\mathcal{O}$ have dimension of mass squared and assume the representation

$$\mathcal{O}_n^{\Phi} = a_n - b^{\Phi}, \qquad \mathcal{O}_n^{\varphi} = a_n - b^{\varphi}, \quad \text{where} \quad [\mathcal{O}] = 2 \tag{B.19}$$

and b are constants associated to the respective fields. Finally, in order to leave the action dimensionless, it follows that

$$[\varphi] \ = \ [\Phi] \ = \ -1, \qquad [\mu] = 4 \tag{B.20}$$

Moving forward on the conceptual treatment of this section, the theories Γ_{LUV} and Γ_{EFT} can finally be constructed. As treated along the main text, at leading order the $\Gamma^{(0)}_{\text{LUV}} \equiv S_{\text{UV}}[\varphi, \Phi_c[\varphi]]$ where $\Phi^c[\varphi]$ is given as an implicit functional on φ after solving Eq.(B.9). For the present model

$$\Phi_n^c = \mu(\mathcal{O}_n^{\Phi})^{-1}\varphi_n^3 \tag{B.21}$$

such that, after replaced in Eq.(B.2):

$$\Gamma^{(0)}_{\text{LUV}} = \sum_n \frac{1}{2}\varphi_n \mathcal{O}_n^{\varphi} \varphi_n - \frac{\mu^2}{72}\varphi_n^3(\mathcal{O}_n^{\Phi})^{-1}\varphi_n^3 \tag{B.22}$$

thus defining the *final* and *exact* LUV theory. As mentioned before, the presence of $(\mathcal{O}_n^{\Phi})^{-1}$ in a realistic physical context represents a non-local aspect of the theory and in practice is associated with internal propagation of the *integrated-out* degrees of freedom.

The Effective Field Theory, on the other hand, is raised through the so-called procedure of **matching**, performed order-by-order on the effective action and after a local expansion of $\Gamma^{(i)}_{\text{LUV}}$. The local series must run up to the preferred order on the fields dimension, what in general corresponds to an equivalent order on the suppression parameter b, for operators in the form of Eq.(B.19).

In fact, by considering ordinary numbers for simplicity, the series can always be written as

$$
\begin{aligned}
(\mathcal{O}_n^{\Phi})^{-1} &\overset{1}{=} -\left[\frac{1}{b}+\frac{a}{b}\frac{1}{b-a}\right] \\
&\overset{2}{=} -\left[\frac{1}{b}+\frac{a}{b}\left(\frac{1}{b}+\frac{a}{b}\frac{1}{b-a}\right)\right] \\
&\vdots \\
&\overset{n}{=} -\left[\frac{1}{b}+\frac{a}{b^2}+\cdots+\frac{a^{n-1}}{b^n}+\frac{a^n}{b^n}\frac{1}{b-a}\right] \\
&= -\frac{1}{b}\left(1+\frac{a}{b}+\cdots+\frac{a^{n-1}}{b^{n-1}}\right)\Bigg|_{\text{local}}+\frac{a^n}{b^n}\frac{1}{a-b}
\end{aligned}
\tag{B.23}
$$

It follows that the local part of the series will converge to the exact result in the low-energy region where $a \ll b$. On the matter of *matching*, the EFT will emerge from the truncation of Γ_{LUV} resolved by a **power counting**, eventually taking into account the field dimension of the factor a.

The log contribution for Γ_{LUV} is given by Eq.(B.15) with the replacement $\Phi \to \Phi_c[\varphi]$ made a posteriori. It might be clearer, therefore, if the Hessian could be separated into disjoint second derivatives of heavy and light fields. This task was performed in [33] and will be fully reproduced here, resulting in the expression

$$
\log\det\left[\frac{\delta^2 S_{UV}[\Phi,\varphi]}{\delta(\Phi,\varphi)^2}\Big|_{\Phi_c}\right] = \log\det\left[\frac{\delta^2 S_{UV}[\Phi,\varphi]}{(\delta\Phi)^2}\Big|_{\Phi_c}\right] + \log\det\left[\frac{\delta^2 S_{UV}[\Phi_c[\varphi],\varphi]}{(\delta\varphi)^2}\right]
\tag{B.24}
$$

where in the second term in the r.h.s. the $\Phi_c[\varphi]$ must be replaced a priori, and the derivative on the light fields finally taken. In order to derive the above identity, the first observation involves the properties of determinants in a block form matrix:

$$
\frac{\delta^2 S_{UV}[\Phi,\varphi]}{\delta(\Phi,\varphi)^2}\Big|_{\Phi_c} = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{pmatrix} \;\to\; \det\left(\frac{\delta^2 S_{UV}[\Phi,\varphi]}{\delta(\Phi,\varphi)^2}\right) = \det(\mathbb{D})\det(\mathbb{A}-\mathbb{B}\mathbb{D}^{-1}\mathbb{C})
\tag{B.25}
$$

From the equation defining the classical $\frac{\delta S_{UV}[\Phi,\varphi]}{\delta\Phi}\Big|_{\Phi_c} = 0$ it follows that

$$
\begin{aligned}
0 &= \frac{\delta}{\delta\varphi}\left(\frac{\delta S_{UV}[\Phi,\varphi]}{\delta\Phi}\Big|_{\Phi_c}\right) \\
&= \frac{\delta^2 S_{UV}[\Phi,\varphi]}{\delta\varphi\delta\Phi}\Big|_{\Phi_c} + \frac{\delta\Phi_c}{\delta\varphi}\frac{\delta^2 S_{UV}[\Phi,\varphi]}{\delta\Phi^2}\Big|_{\Phi_c}
\end{aligned}
\tag{B.26}
$$

where it has been considered the chain rule[1]

$$
\frac{\delta}{\delta\varphi} = \frac{\delta}{\delta\varphi}\Big|_{\Phi_c} + \frac{\delta\Phi_c}{\delta\varphi}\frac{\delta}{\delta\Phi_c}
\tag{B.28}
$$

[1]Here the $|_{\Phi_c}$ denotes a partial functional derivative. Analogously, for functions:

$$
f(x(y),y) \;\to\; \frac{d}{dy}f = \partial_y f + \frac{dx}{dy}\partial_x f
\tag{B.27}
$$

Thus, back to Eq.(B.25) it follows

$$
\begin{aligned}
\mathbb{A}-\mathbb{B}\mathbb{D}^{-1}\mathbb{C} &= \frac{\delta^2 S_{UV}}{\delta\varphi^2}\bigg|_{\Phi_c} - \frac{\delta^2 S_{UV}}{\delta\varphi\delta\Phi}\bigg|_{\Phi_c}\left(\frac{\delta^2 S_{UV}}{\delta\Phi^2}\bigg|_{\Phi_c}\right)^{-1}\frac{\delta^2 S_{UV}}{\delta\Phi\delta\varphi}\bigg|_{\Phi_c} \\
&\overset{(B.26)}{=} \frac{\delta^2 S_{UV}}{\delta\varphi^2}\bigg|_{\Phi_c} + \frac{\delta\Phi_c}{\delta\varphi}\frac{\delta^2 S_{UV}}{\delta\Phi\delta\varphi}\bigg|_{\Phi_c} \\
&\overset{(B.28)}{=} \frac{\delta^2 S_{UV}[\varphi,\Phi_c[\varphi]]}{\delta\varphi^2}
\end{aligned} \tag{B.29}
$$

which, by replacing in Eq.(B.25), proves the Eq.(B.24). Since, by definition, $S_{UV}[\varphi,\Phi_c[\varphi]] = \Gamma^{(0)}_{L,UV}[\varphi]$, the expression for $\Gamma^{(1)}_{L,UV}$ is clearer expressed like

$$
\Gamma^{(1)}_{L,UV} = -\frac{1}{2}\left\{\log\det\left[\frac{\delta^2 S_{UV}[\Phi,\varphi]}{(\delta\Phi)^2}\bigg|_{\Phi_c}\right] + \log\det\left[\frac{\delta^2\Gamma^{(0)}_{L,UV}[\varphi]}{(\delta\varphi)^2}\right]\right\} \tag{B.30}
$$

As mentioned in the Chapter 2, the construction of the EFT will follow from the matching procedure at the level of Effective Actions through

$$
\Gamma^{(1)}_{EFT} \equiv \sum_i c_i^{(1)} O_i - \frac{1}{2}\log\det\frac{\delta^2\Gamma^{(0)}_{EFT}[\varphi]}{(\delta\varphi)^2} = \Gamma^{(1)}_{LUV}, \qquad \text{at} \qquad \mu = M \tag{B.31}
$$

where here the $\mu = M$ is just remarking the scale where the equality holds. Finally, the Wilson coefficients can extracted from

$$
\sum_i c_i^{(1)} O_i = \frac{1}{2}\left\{\log\det\left[\frac{\delta^2\Gamma^{(0)}_{EFT}[\varphi]}{(\delta\varphi)^2}\right] - \log\det\left[\frac{\delta^2\Gamma^{(0)}_{L,UV}[\varphi]}{(\delta\varphi)^2}\right]\right\} - \frac{1}{2}\log\det\left[\frac{\delta^2 S_{UV}[\Phi,\varphi]}{(\delta\Phi)^2}\bigg|_{\Phi_c}\right] \tag{B.32}
$$

In HLM the coefficients extracted from the first bracket were called *mixed* terms, since they are related with loops containing both light and heavy particles. The above expression also clarify the correct mode of performing the power counting - During the subtraction of equivalent terms present in the log corrections for the *EFT* and the *LUV* theories.

After this point, since the trace computation from a discrete to a continuous point of view may substantially differ, the rest of the computation must be left to the main text.

BIBLIOGRAPHY

[1] R. Alonso, E. E. Jenkins, A. V. Manohar, and M. Trott, *Renormalization Group Evolution of the Standard Model Dimension Six Operators III: Gauge Coupling Dependence and Phenomenology*, JHEP, 04 (2014), p. 159.

[2] T. Appelquist and J. Carazzone, *Infrared Singularities and Massive Fields*, Phys. Rev., D11 (1975), p. 2856.

[3] M. Baak, M. Goebel, J. Haller, A. Hoecker, D. Kennedy, R. Kogler, K. Moenig, M. Schott, and J. Stelzer, *The Electroweak Fit of the Standard Model after the Discovery of a New Boson at the LHC*, Eur. Phys. J., C72 (2012), p. 2205.

[4] R. Barbieri, A. Pomarol, R. Rattazzi, and A. Strumia, *Electroweak symmetry breaking after LEP-1 and LEP-2*, Nucl. Phys., B703 (2004), pp. 127–146.

[5] N. N. Bogolyubov and D. V. Shirkov, *QUANTUM FIELDS*, 1983.

[6] P. A. Bolokhov and M. Pospelov, *Classification of dimension 5 Lorentz violating interactions in the standard model*, Phys. Rev., D77 (2008), p. 025022.

[7] G. Branco, L. Lavoura, and J. Silva, *CP Violation*, Clarendon Press, 1999.

[8] G. Buchalla, A. J. Buras, and M. E. Lautenbacher, *Weak decays beyond leading logarithms*, Rev. Mod. Phys., 68 (1996), pp. 1125–1144.

[9] W. Buchmuller and D. Wyler, *Effective Lagrangian Analysis of New Interactions and Flavor Conservation*, Nucl. Phys., B268 (1986), pp. 621–653.

[10] A. J. Buras and F. De Fazio, *331 Models Facing the Tensions in $\Delta F = 2$ Processes with the Impact on ε'/ε, $B_s \to \mu^+\mu^-$ and $B \to K^*\mu^+\mu^-$*, JHEP, 08 (2016), p. 115.

[11] ——, *ε'/ε in 331 Models*, JHEP, 03 (2016), p. 010.

[12] A. J. Buras, F. De Fazio, J. Girrbach, and M. V. Carlucci, *The Anatomy of Quark Flavour Observables in 331 Models in the Flavour Precision Era*, JHEP, 02 (2013), p. 023.

[13] Q.-H. Cao and D.-M. Zhang, *Collider Phenomenology of the 3-3-1 Model*, (2016).

[14] J. C. Collins, *Renormalization*, vol. 26 of Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1986.

[15] M. CZAKON, *The Four-loop QCD beta-function and anomalous dimensions*, Nucl. Phys., B710 (2005), pp. 485–498.

[16] G. DE CONTO, A. C. B. MACHADO, AND V. PLEITEZ, *Minimal 3-3-1 model with a spectator sextet*, Phys. Rev., D92 (2015), p. 075031.

[17] G. DE CONTO AND V. PLEITEZ, *Neutron electric dipole moment in the minimal 3-3-1 model*, (2016).

[18] C. DEGRANDE, N. GREINER, W. KILIAN, O. MATTELAER, H. MEBANE, T. STELZER, S. WILLENBROCK, AND C. ZHANG, *Effective field theory: A modern approach to anomalous couplings*, Annals of Physics, 335 (2013), pp. 21 – 32.

[19] R. A. DIAZ, R. MARTINEZ, AND F. OCHOA, *The Scalar sector of the SU(3)(c) x SU(3)(L) x U(1)(X) model*, Phys. Rev., D69 (2004), p. 095009.

[20] ——, *SU(3)(c) x SU(3)(L) x U(1)(X) models for beta arbitrary and families with mirror fermions*, Phys. Rev., D72 (2005), p. 035018.

[21] J. F. DONOGHUE, E. GOLOWICH, AND B. R. HOLSTEIN, *Dynamics of the standard model*, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol., 2 (1992), pp. 1–540.
[Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.35(2014)].

[22] A. DROZD, J. ELLIS, J. QUEVILLON, AND T. YOU, *The Universal One-Loop Effective Action*, JHEP, 03 (2016), p. 180.

[23] J. ELIAS-MIRO, J. R. ESPINOSA, E. MASSO, AND A. POMAROL, *Higgs windows to new physics through d=6 operators: constraints and one-loop anomalous dimensions*, JHEP, 11 (2013), p. 066.

[24] J. ELIAS-MIRÓ, C. GROJEAN, R. S. GUPTA, AND D. MARZOCCA, *Scaling and tuning of EW and Higgs observables*, JHEP, 05 (2014), p. 019.

[25] S. A. R. ELLIS, J. QUEVILLON, T. YOU, AND Z. ZHANG, *Mixed heavy–light matching in the Universal One-Loop Effective Action*, Phys. Lett., B762 (2016), pp. 166–176.

[26] M. FARINA, G. PANICO, D. PAPPADOPULO, J. T. RUDERMAN, R. TORRE, AND A. WULZER, *Energy helps accuracy: electroweak precision tests at hadron colliders*, (2016).

[27] R. M. FONSECA AND M. HIRSCH, *Lepton number violation in 331 models*, Phys. Rev., D94 (2016), p. 115003.

[28] J. FUENTES-MARTIN, J. PORTOLES, AND P. RUIZ-FEMENIA, *Integrating out heavy particles with functional methods: a simplified framework*, JHEP, 09 (2016), p. 156.

[29] M. K. GAILLARD, *The Effective One Loop Lagrangian With Derivative Couplings*, Nucl. Phys., B268 (1986), pp. 669–692.

[30] W. GREINER AND J. REINHARDT, *Field quantization*, 1996.

[31] B. GRZADKOWSKI, M. ISKRZYNSKI, M. MISIAK, AND J. ROSIEK, *Dimension-Six Terms in the Standard Model Lagrangian*, JHEP, 10 (2010), p. 085.

[32] B. HENNING, X. LU, AND H. MURAYAMA, *How to use the Standard Model effective field theory*, JHEP, 01 (2016), p. 023.

[33] ——, *One-loop Matching and Running with Covariant Derivative Expansion*, (2016).

[34] E. E. JENKINS, A. V. MANOHAR, AND M. TROTT, *Renormalization Group Evolution of the Standard Model Dimension Six Operators I: Formalism and lambda Dependence*, JHEP, 10 (2013), p. 087.

[35] ——, *Renormalization Group Evolution of the Standard Model Dimension Six Operators II: Yukawa Dependence*, JHEP, 01 (2014), p. 035.

[36] C. KELSO, H. N. LONG, R. MARTINEZ, AND F. S. QUEIROZ, *Connection of $g-2_\mu$, electroweak, dark matter, and collider constraints on 331 models*, Phys. Rev., D90 (2014), p. 113011.

[37] H. KLEINERT AND V. SCHULTE-FROHLINDE, *Critical properties of phi**4-theories*, 2001.

[38] A. V. MANOHAR, *Effective field theories*, Lect. Notes Phys., 479 (1997), pp. 311–362.

[39] J. K. MIZUKOSHI, C. A. DE S. PIRES, F. S. QUEIROZ, AND P. S. RODRIGUES DA SILVA, *WIMPs in a 3-3-1 model with heavy Sterile neutrinos*, Phys. Rev., D83 (2011), p. 065024.

[40] M. OZER, *SU(3)-L x U(1)-x model of the electroweak interactions without exotic quarks*, Phys. Rev., D54 (1996), pp. 1143–1149.

[41] C. PATRIGNANI ET AL., *Review of Particle Physics*, Chin. Phys., C40 (2016), p. 100001.

[42] M. E. PESKIN AND D. V. SCHROEDER, *An Introduction to quantum field theory*, 1995.

[43] M. E. PESKIN AND T. TAKEUCHI, *Estimation of oblique electroweak corrections*, Phys. Rev., D46 (1992), pp. 381–409.

[44] F. PISANO AND V. PLEITEZ, *An SU(3) x U(1) model for electroweak interactions*, Phys. Rev., D46 (1992), pp. 410–417.

[45] V. PLEITEZ, *New fermions and a vector - like third generation in SU(3) (C) x SU(3) (L) x U(1) (N) models*, Phys. Rev., D53 (1996), pp. 514–526.

[46] V. PLEITEZ AND M. D. TONASSE, *Heavy charged leptons in an SU(3)-L x U(1)-N model*, Phys. Rev., D48 (1993), pp. 2353–2355.

[47] S. PROFUMO AND F. S. QUEIROZ, *Constraining the Z' mass in 331 models using direct dark matter detection*, Eur. Phys. J., C74 (2014), p. 2960.

[48] P. RAMOND, *FIELD THEORY. A MODERN PRIMER*, Front. Phys., 51 (1981), pp. 1–397. [Front. Phys.74,1(1989)].

[49] M. D. SCHWARTZ, *Quantum Field Theory and the Standard Model*, Cambridge University Press, 2014.

[50] M. SINGER, J. W. F. VALLE, AND J. SCHECHTER, *Canonical Neutral Current Predictions From the Weak Electromagnetic Gauge Group SU(3) X u(1)*, Phys. Rev., D22 (1980), p. 738.

[51] B. L. SÁNCHEZ-VEGA, E. R. SCHMITZ, AND J. C. MONTERO, *New constraints on the 3-3-1 model with right-handed neutrinos*, (2016).

[52] J. A. M. VERMASEREN, S. A. LARIN, AND T. VAN RITBERGEN, *The four loop quark mass anomalous dimension and the invariant quark mass*, Phys. Lett., B405 (1997), pp. 327–333.

[53] Z. ZHANG, *Covariant diagrams for one-loop matching*, (2016).

ACKNOWLEDGEMENTS

Professor Vicente Pleitez, muito obrigado por sua paciência, atenção e ética.

Professores Juan Montero, Ricardo Matheus, Eduardo Ponton, Bruto Pimentel - obrigado pelos inúmeros bons exemplos que um dia espero ter a capacidade de manifestar como professor.

Rosane, Luzinete, Edna, Meire, Jô, Ana de Paula, Hellen, Neila, Rodrigo, Rodrigo Marques, Heitor - o privilégio da atenção de vocês me forneceu sete anos de sorrisos de gratidão e encantamento.

Professor Marcelo Yamashita, me lembrarei sempre daquela partida de xadrez que o senhor perdeu pra mim... e esquecerei das outras oitenta e sete que o senhor ganhou..

Elaine Fortes, obrigado por insistir em me dizer que o mundo é grande..

Carlisson, Thiago Peixoto, George, Ana e Javier - quanto alívio e alegria nossas conversas me deram, e que boas lembranças sempre me darão..

Meus companheiros de Quarteto Fantástico - Ernany, Renato e Guilherme - que escalaram comigo até a lua..

Nathália Queiroz - Bob Crusoé - que me deixou ser o Sexta-Feira..

Meus companheiros três mosqueteiros - Marcelo, Eduardo e Fernando - que me deixaram ser o d'Artagnan..

Eloy Sinclair camarada, Kaloni, Minoti, Brunna, Jaqueline, Fernanda, Daniel e Nanci - meu maior entusiasmo será sempre o da circunstância de poder revê-los.

Mãe, que é eu inteiro.. Pai, que brincou comigo quando eu era criança.. Camilla, meu sopro e meu alumbramento.. Vó, meu sorriso, Bruno, meu orgulho..

Tia Andreia, Gustavo, Tia Nara, Isa, Luccas e Júlio - que tornaram leve minha visão de mundo e possível minha vida em São Paulo.

Aos professores Svjetlana Fajfer e Antonio Pich, meu eterno obrigado pela honra de me receberem como seu aluno.

Dario, Ursa, Allen, Victor, Luis e a todos companheiros do Instituto Jozef Stefan; Javier Fuentes, Pablo, Antonio, Laura, companheiros de IFIC - Obrigado pela amizade e paciência de todos. Javier Castellano, ainda jogaremos basquete juntos, amigo - gracias siempre.

Amparo, Javier, Eva - minha bonita família Valenciana..

Vinicius, que me disse pra seguir os bons, Marcelo que verá pronta nossa caixa-de-gelo solar, e Vera com quem pude ver que afeto, cuidado e simplicidade são coisas reais..

Bruna, moça bonita que olha pra mim de um balão já muito alto..

A vontade de realizar esse trabalho me permitiu estar com vocês - essa foi a minha grande sorte e será sempre a minha maior força.

Sinceros agradecimentos à Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), ao Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) e à Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) pelo apoio financeiro ao longo de minha formação.

Agradecimentos a University of Bristol pelo template LaTex.