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Defects and Bäcklund Transformations for the $\mathcal{N} = 1$ Supersymmetric mKdV Hierarchy

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Resumo

A integrabilidade da hierarquia de Korteweg de-Vries modificada supersimétrica com $\mathcal{N} = 1$ (smKdV) na presença de defeitos é investigada através da construção de sua transformação de Bäcklund supersimétrica. A construção de tal transformação é realizada usando essencialmente dois métodos: a abordagem da matriz de defeito e empregando o operador de recursão. Primeiramente, empregamos a matriz de defeitos associada à hierarquia, que é a mesma para o modelo sinh-Gordon supersimétrico (sshG). O método é geral e válido para todos os fluxos da hierarquia e como exemplo derivamos explicitamente as equações de Bäcklund para os primeiros fluxos, são eles t_1, t_3 e t_5. Em segundo lugar, o operador de recursão relacionando tempos consecutivos é derivado e mostrados que ele relaciona também as transformação de Bäcklund. Finalmente, esta transformação de Bäcklund supersimétrica é empregada para introduzir defeitos do tipo I para a hierarquia supersimétrica mKdV. Outros aspectos de integrabilidade são considerados, através da construção das quantidades conservadas modificadas, derivadas da matriz de defeito.

Palavras Chaves: Hierarquias integráveis; transformações de Bäcklund; defeitos integráveis; smKdV.

Áreas do conhecimento: Física Matemática; Teorias de campos integráveis.

Abstract

The integrability of the $\mathcal{N}=1$ supersymmetric modified Korteweg de-Vries (smKdV) hierarchy in the presence of defects is investigated through the construction of its super Bäcklund transformation. The construction of such transformation is performed by essentially using two methods: the Bäcklund-defect matrix approach and the by employing the recursion operator. Firstly, we employ the defect matrix associated to the hierarchy which turns out to be the same for the supersymmetric sinh-Gordon (sshG) model. The method is general for all flows and as an example we derive explicitly the Bäcklund equations in components for the first few flows of the hierarchy, namely t_1, t_3 and t_5. Secondly, the recursion operator relating consecutive time flows is derived and shown to relate their Bäcklund transformations. Finally, this super Bäcklund transformation is employed to introduce type I defects for the supersymmetric mKdV hierarchy. Further integrability aspects by considering modified conserved quantities are derived from the defect matrix.

Keywords: Integrable hierarchies; Bäcklund transformations; integrable defects; smKdV.

Areas: Mathematical physics; Integrable field theories.

List of publications

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[8] A.R. Aguirre, J.F.Gomes, A.L. Retore, N.I.Spano and A.H.Zimerman, *Recursion Operator and Bäcklund Transformation for Super mKdV Hierarchy*, *QTS10 and LT12 Proceedings*, (2018).

Contents

1 Introduction **1**

2 Systematic construction of integrable and superintegrable hierarchies **5**

2.1 Integrable hierarchies 5

2.2 Supersymmetric integrable hierarchies 8

2.2.1 The supersymmetric mKdV hierarchy 9

2.3 Soliton Solutions 19

3 Construction of the super Bäcklund transformations for the smKdV hierarchy **21**

3.1 Bäcklund transformations 21

3.2 Obtaining the K matrix for the sshG equation 23

3.3 Bäcklund transformations for the smKdV hierarchy 28

3.4 Superspace formalism 31

4 Recursion operator for the smKdV hierarchy **34**

4.1 Rercursion operator for the super integrable equations of motion . . . 34

4.2 Recursion operator for the Bäcklund transformations 39

5 Conserved charges for the smKdV hierarchy **42**

5.1 Conservation laws 42

5.2 Inspecting the conservation 46

6 Integrable defects in the smKdV hierarchy **49**
6.1 Defect contribution for the momentum 51
6.2 Defect contribution for the energy 53
6.3 Bäcklund solutions for super mKdV hierarchy 54

7 Modified conserved charges for the smKdV hierarchy **56**
7.1 Defect contributions from the defect matrix 56

8 Conclusions and further developments **60**

A Representation of the $\widehat{sl}(2,1)$ affine Lie superalgebra **63**

B $N = 5$ Lax component **64**

C Calculation of the defect matrix **67**

D Type-II defect matrix for the mKdV hierarchy **72**

E Coefficients of the Bäcklund transformations for $N = 5$ member **75**

F Conservation of mKdV momentum defect with respect to t_5 **78**

Bibliography **82**

Chapter 1

Introduction

The study of two-dimensional integrable models is an important and beautiful branch of research in physics. Such systems are described by partial nonlinear differential equations possessing soliton solutions. These solutions are nonlinear localized waves which preserve their profile and velocity during propagation, even after the interaction with another soliton. This is a novel feature having several applications such as describing waves in shallow waters [1] and in blood pressure [2], signal propagation in optical fibres [3], and so on. The stability of these solutions are directly associated with the existence of an infinite number of conserved quantities ensuring the integrability of the model.

Besides, the mathematical structure involved in the formulation and classification of such systems is an interesting topic to explore. In this context an integrable hierarchy is constituted of a set of integrable models. In such a way that equations already known could be studied within the same formalism, this is the case of the sinh-Gordon and mKdV equations, which belongs to the mKdV hierarchy.

The construction of an integrable hierarchy is based on the zero curvature condition in order to guarantee the integrability. This set of time evolution equations (or integrable hierarchy) has in commom the spatial component of the Lax pair, varying only the time component, such property enables the systematic construction of an integrable hyerarchy.

Hence from this commom structure, the equations pertaining to the same hierarchy are related by the so called recursion operator. Basically, applying this operator in some equation gives us the consecutive corresponding time evolution equation.

An attractive topic to study is the introduction of special impurities or defects within two-dimensional integrable models that preserves the integrability properties. Integrable defects, as they are known, were introduced originally in [4, 5], as a set of internal boundary conditions derived from a Lagrangian density located at certain spatial position connecting two types of solutions.

The presence of these special defects has been studied in several models, including sine(h)-Gordon [4], affine Toda field theories [5], the non-linear Schrödinger, and other non-relativistic field theories [6, 7]. These models were studied using the Lagrangian formalism and it was noticed that in order to repair the momentum conservation the defect equations correspond to frozen Bäcklund transformations (BT) [8] and then preserve the momentum of the original bulk theory after including some defect contributions. In the Lax approach, which will be used in this thesis, the total integrability is ensured by the existence of the defect matrix that will generate an infinity set of defect contributions for each conserved charge.

This kind of defect is then named type-I if the fields on either side of it only interact with each other at the defect location. It is called type-II if they interact through additional degrees of freedom associated to the defect itself which only exist at the defect point [9, 11].

Several other interesting issues have been studied for these types of integrable defects, among which the following are worth mentioning: the computation of the higher order modified conserved quantities and their involutivity [12, 13, 14], quantum description [15, 16, 17, 18, 19, 20, 21, 22, 23], fermionic [24, 25, 26] and supersymmetric extensions [27, 28, 29, 30, 31].

The main purpose of this thesis is to propose an extension of the framework of integrable defects for the supersymmetric modified Korteweg-de Vries (smKdV) hierarchy through the construction of the associated super Bäcklund transformation, by using the Lax approach. In refs. [32, 33], authors have shown that the smKdV and the super sinh-Gordon (sshG) equations belong to the same integrable hierarchy based on the $\widehat{sl}(2,1)$ affine super Lie algebra. On the other hand, it was shown in ref. [34] that the spatial part of the bosonic Bäcklund transformation for the mKdV hierarchy is universal within the entire hierarchy. Using this fact, it was proposed in ref. [35] that the associated defect-gauge matrix is also universal and provides the corresponding Bäcklund equations for the entire hierarchy. Therefore, it is quite natural to expect that such property will be preserved for the supersymmetric extension in the sense that the super sinh-Gordon and other models within the hierarchy will share the same defect matrix.

The presence of type I and type II integrable defects in the $\mathcal{N} = 1$ sshG model has been already investigated in [27, 30, 31], through the Lagrangian formalism and the Lax approach, where the associated modified conserved charges was derived directly from the corresponding defect matrices.

The main goal of this thesis is the extension of the results obtained in [27, 30, 31] for the super sinh-Gordon to the entire smKdV hierachy. This includes the systematic construction of the different time components of the Bäcklund transformations and modified conservation laws. Our results are derived from the invariance under gauge transformations of the algebraic zero curvature representation.

This thesis is organized as follows:

In the chapter 2 we summarise the necessary ingredients to construct an integrable and a superintegrable hierarchies. In particular, we consider the supersymmetric mKdV hierarchy.

In the chapter 3, we will derive the type-I defect matrix for the sshG model then we construct the super Bäcklund transformation for the smKdV hierarchy by assuming the universality of the defect matrix within the hierarchy. The key observation is that the zero curvature representation is invariant under gauge transformation connecting two different field configurations of the same model. This provides a general framework from where the Bäcklund transformations for the various flows can be derived. Explicit examples are worked out for the first three flows, namely t_1, t_3 and t_5.

In the chapter 4 we will propose an alternative construction of these super Bäcklund transformations by employing a recursion operator.

In the chapter 5 we will generate an infinite number of conserved charges. Those are conserved with respect to all flows and in particular, we explicitly verify the conservation of the simplest two charges with respect to the first three flows, namely t_1, t_3 and t_5.

In the chapter 6 we will investigate the introduction of the defect, which requires a modification of the charges in order to ensure its conservation. Again explicit examples are given for t_1, t_3 and t_5.

In the chapter 7 we will extend the results in the chapter 6 to compute higher order of modified conserved charges, this is accomplished systematically assuming the defect matrix to be responsible for the transition from one side to the other of the defect.

The chapter 8 contains some conclusions and comments on future directions to investigate. The explicit representation of the $\widehat{sl}(2,1)$ superalgebra, and some technical computations as well as long expressions for the defect matrix and super Bäcklund transformations are contained in appendices A to F.

Chapter 2

Systematic construction of integrable and superintegrable hierarchies

In this chapter we introduce the main ingredients to systematically construct an integrable hierarchy, as well as the changes that appear when we consider a superintegrable hierarchy. Moreover, we consider the modified Korteweg de Vries (mKdV) hierarchy and its supersymmetric extension (smKdV), as examples.

2.1 Integrable hierarchies

The construction of an integrable hierarchy from the zero curvature condition depends on three important ingredients: a Kac-Moody algebra ($\widehat{\mathcal{G}}$), a grading operator (Q), and a constant element (E).

The algebra $\widehat{\mathcal{G}}$ is obtained from a Lie algebra ($\mathcal{G}$) by introducing the spectral parameter λ. Thus, if T is a generator of $\mathcal{G}$ the generators of $\widehat{\mathcal{G}}$ will be given by

$$T^{(n)} = \lambda^n T, \qquad T \in \mathcal{G}. \tag{2.1}$$

The grading operator Q decomposes the affine Lie algebra (Kac-Moody algebra) in graded subspaces as follows,

$$\widehat{\mathcal{G}} = \sum_{n\in\mathbb{Z}} \widehat{\mathcal{G}}^{(n)}. \tag{2.2}$$

In addition, we assume that

$$\left[Q, \widehat{\mathcal{G}}^{(n)}\right] = n\widehat{\mathcal{G}}^{(n)}, \qquad \left[\widehat{\mathcal{G}}^{(n)}, \widehat{\mathcal{G}}^{(m)}\right] \in \widehat{\mathcal{G}}^{(n+m)}. \tag{2.3}$$

where n is the degree of the subspace $\widehat{\mathcal{G}}^{(n)}$ according to Q. Therefore, the parameter λ is associated with the degree of the generator.

Finally, the constant element E* induces another decomposition into the affine algebra

$$\widehat{\mathcal{G}} = \mathcal{K}(E) \oplus \mathcal{M}(E) \tag{2.4}$$

where $\mathcal{K}(E) = \left\{x \in \widehat{\mathcal{G}}\,/[\,x, E] = 0\right\}$ is the kernel of E and $\mathcal{M}(E) = \left\{x \in \widehat{\mathcal{G}}\,/[\,x, E] \neq 0\right\}$ is the image.

Then, considering the zero curvature condition,

$$[\partial_x + A_x, \partial_{t_N} + A_{t_N}] = 0, \tag{2.5}$$

where, A_x and A_{t_N} are the Lax pair that belong to a Kac-Moody algebra $(\widehat{\mathcal{G}})$ and t_N represents the time flow of an integrable equation.

Thus, as the Lax pairs belong to the Kac-Moody algebra they can also be decomposed into kernel and image so we can define the Lax pair as

$$A_x = E + A_0 \tag{2.6}$$
$$A_{t_N} = D^{(N)} + D^{(N-1)} + ... + D^{(1)} + D^{(0)} \tag{2.7}$$

where $A_0 \in \widehat{\mathcal{G}}^{(0)} \in \mathcal{M}(E)$ and it depends on the fields of the theory, and $D^{(m)} \in \widehat{\mathcal{G}}^{(m)}$.

Now, substituting (2.6) and (2.7) in (2.5) we get

$$\left[\partial_x + E + A_0, \partial_{t_N} + D^{(N)} + D^{(N-1)} + ... + D^{(1)} + D^{(0)}\right] = 0. \tag{2.8}$$

*For simplicity we will assume that $E \in \widehat{\mathcal{G}}^{(1)}$.

The decomposition of the above equation into graded subspaces yields to the following system

$$
\begin{aligned}
(N+1): &\quad \left[E, D^{(N)}\right] = 0,\\
(N): &\quad \partial_x D^{(N)} + \left[A_0, D^{(N)}\right] + \left[E, D^{(N-1)}\right] = 0,\\
&\quad \vdots\\
(1): &\quad \partial_x D^{(1)} + \left[A_0, D^{(1)}\right] + \left[E, D^{(0)}\right] = 0,\\
(0): &\quad \partial_x D^{(0)} + \left[A_0, D^{(0)}\right] - \partial_{t_N} A_0 = 0.
\end{aligned}
\tag{2.9}
$$

It is important to note that each $D^{(k)}$ can be decomposed as (2.4), such that

$$D^{(k)} = a_k D^{(k)}_{\mathcal{K}} + b_k D^{(k)}_{\mathcal{M}}. \tag{2.10}$$

Then, we have a system with $N+2$ equations, each one in its respective subspace. This set of equations (2.9) can be recursively solved, starting with the highest grade equation until the lowest one, which corresponds to the equation of motion for the field in A_0.

In particular, the construction of the mKdV hierarchy is based on the $\widehat{\mathcal{G}} = \widehat{sl}(2)$ affine Lie algebra, the grading operator is $Q = 2d + \frac{1}{2}h_1^{(0)}$ and the constant element is $E = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)}$.

The generators are $\{h^{(n)}, E_{\alpha}^{(n)}, E_{-\alpha}^{(n)}\}$ and from (2.3) we get,

$$[Q, E_{\pm\alpha}^{(n)}] = (2n \pm 1)E_{\pm\alpha}^{(n)}, \qquad [Q, h^{(n)}] = 2nh^{(n)}. \tag{2.11}$$

Then the algebra can be splitted in even and odd subspaces, respectively,

$$
\begin{aligned}
&\widehat{\mathcal{G}}^{(2n)} : \{h^{(n)}\},\\
&\widehat{\mathcal{G}}^{(2n+1)} : \{E_{\alpha}^{(n)}, E_{-\alpha}^{(n+1)}\}.
\end{aligned}
\tag{2.12}
$$

Besides, from the commutation relations we can verify that

$$\mathcal{K}(E) = \{E_{\alpha}^{(n)} + E_{-\alpha}^{(n+1)}\}, \quad \mathcal{M}(E) = \{h^{(n)}, E_{\alpha}^{(n)} - E_{-\alpha}^{(n+1)}\} \tag{2.13}$$

Since $A_0 \in \widehat{\mathcal{G}}^{(0)} \in \mathcal{M}(E)$ we get that

$$A_0 = u(x, t_N)h^{(0)} \tag{2.14}$$

where $u(x, t_N)$ is the field of the theory.

For $N = 3$ we get

$$\left[\partial_x + E + A_0, \partial_{t_3} + D^{(3)} + D^{(2)} + D^{(1)} + D^{(0)}\right] = 0. \tag{2.15}$$

From the fisrt equation in the system (2.9) we obtain that $D^{(3)} \in \mathcal{K}(E)$ and each $D^{(k)}$ is decomposed as follows

$$\begin{aligned} &D^{(3)} = a_3(E_\alpha^{(1)} + E_{-\alpha}^{(2)}), \quad D^{(2)} = b_2 h^{(1)}, \quad D^{(0)} = b_0 h^{(0)}, \\ &D^{(1)} = a_1(E_\alpha^{(0)} + E_{-\alpha}^{(1)}) + c_1(E_\alpha^{(0)} - E_{-\alpha}^{(1)}). \end{aligned} \tag{2.16}$$

By solving the system (2.15) we get the so called mKdV equation,

$$4\partial_{t_3} u = \partial_x^3 u - 6u^2 \partial_x u. \tag{2.17}$$

2.2 Supersymmetric integrable hierarchies

In this section we extend the construction of the integrable hierarchy to the supersymmetric case. Whereas in the last section we dealt with a Lie algebra, now we consider a super Lie algebra $\widehat{\mathcal{G}}$, which has bosonic and fermionic generators. Therefore, the Lax operators change by including semi-integer grades related to the fermionic generators, as follows

$$A_x = E + A_0 + A_{1/2}, \tag{2.18}$$

$$A_{t_N} = D^{(N)} + D^{(N-1/2)} + ... + D^{(1/2)} + D^{(0)} \tag{2.19}$$

where $A_0 \in \widehat{\mathcal{G}}^{(0)} \in \mathcal{M}(E)$, $A_{1/2} \in \widehat{\mathcal{G}}^{(1/2)} \in \mathcal{M}(E)$ and their respective components are the bosonic and fermionic fields of the theory and $D^{(m)} \in \widehat{\mathcal{G}}^{(m)}$. In addition, the zero curvature equation becomes,

$$\left[\partial_x + E + A_0 + A_{1/2}, \partial_{t_N} + D^{(N)} + D^{(N-1/2)} + ... + D^{(1/2)} + D^{(0)}\right] = 0. \tag{2.20}$$

The decomposition of the equation (2.20) into graded subspaces yields the following system

$$
\begin{aligned}
&(N+1): && \left[E, D_N^{(N)}\right] = 0,\\
&(N+1/2): && \left[E, D^{(N-1/2)}\right] + \left[A_{1/2}, D^{(N)}\right] = 0,\\
&(N): && \partial_x D^{(N)} + \left[A_0, D^{(N)}\right] + \left[E, D^{(N-1)}\right] + \left[A_{1/2}, D^{(N-1/2)}\right] = 0,\\
& && \vdots\\
&(1): && \partial_x D^{(1)} + \left[A_0, D^{(1)}\right] + \left[E, D^{(0)}\right] + \left[A_{1/2}, D^{(1/2)}\right] = 0,\\
&(1/2): && \partial_x D^{(1/2)} + \left[A_0, D^{(1/2)}\right] + \left[A_{1/2}, D^{(0)}\right] - \partial_{t_N} A_{1/2} = 0,\\
&(0): && \partial_x D^{(0)} + \left[A_0, D^{(0)}\right] - \partial_{t_N} A_0 = 0.
\end{aligned}
\tag{2.21}
$$

Now, we can recursively solve these equations and by substituting in the equation for (1/2) grade we get ,

$$\mathcal{K}: \quad \partial_x D_{\mathcal{K}}^{(1/2)} = \left[D_{\mathcal{M}}^{(1/2)}, A_0\right] + \left[D_{\mathcal{M}}^{(0)}, A_{1/2}\right] \tag{2.22}$$

$$\mathcal{M}: \quad \partial_{t_N} A_{1/2} = \partial_x D_{\mathcal{M}}^{(0)} \tag{2.23}$$

and (0)

$$\mathcal{K}: \quad \partial_{t_N} A_0 = \partial_x D_{\mathcal{M}}^{(0)}. \tag{2.24}$$

The equations (2.23) and (2.24) are the equations of motion for the bosonic fields in A_0 and the fermionic in $A_{1/2}$. The next section is dedicated to the supersymmetric mKdV hierarchy.

2.2.1 The supersymmetric mKdV hierarchy

We consider the affine super Lie algebra $\widehat{sl}(2,1)$, with the principal gradation $Q = 2d + \frac{1}{2}h_1^{(0)}$ and the bosonic generators $\{h_1, h_2, E_{\pm\alpha_1}\}$, and the fermionic generators $\{E_{\pm\alpha_2}, E_{\pm(\alpha_1+\alpha_2)}\}$. Moreover, the model is described by using the following combinations

$$
\begin{aligned}
F_1^{(2n+3/2)} &= (E_{\alpha_1+\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n+1)}) + (E_{-\alpha_1-\alpha_2}^{(n+1)} - E_{-\alpha_2}^{(n+1/2)}),\\
F_2^{(2n+1/2)} &= -(E_{\alpha_1+\alpha_2}^{(n)} - E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} - E_{-\alpha_2}^{(n)}),\\
G_1^{(2n+1/2)} &= (E_{\alpha_1+\alpha_2}^{(n)} + E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1-\alpha_2}^{(n+1/2)} + E_{-\alpha_2}^{(n)}),\\
G_2^{(2n+3/2)} &= -(E_{\alpha_1+\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}) + (E_{-\alpha_1-\alpha_2}^{(n+1)} + E_{-\alpha_2}^{(n+1/2)}),\\
K_1^{(2n+1)} &= -E_{-\alpha_1}^{(n+1)} - E_{\alpha_1}^{(n)},\\
K_2^{(2n+1)} &= 2H_2^{(n+1/2)},\\
M_1^{(2n+1)} &= E_{-\alpha_1}^{(n+1)} - E_{\alpha_1}^{(n)},\\
M_2^{(2n)} &= 2H_1^{(n)}.
\end{aligned}
\tag{2.25}
$$

Now we can set the constant element as $E = K^{(1)} + K^{(2)}$ in such a way that the generators are decomposed as follows,

$$
\begin{aligned}
\mathcal{M}_{bos} &= \left\{ M_2^{(2n)},\, M_1^{(2n+1)} \right\}, \qquad \mathcal{M}_{fer} = \left\{ G_1^{(2n+\frac{1}{2})},\, G_2^{(2n+\frac{3}{2})} \right\},\\
\mathcal{K}_{bos} &= \left\{ K_1^{(2n+1)},\, K_2^{(2n+1)}) \right\}, \qquad \mathcal{K}_{fer} = \left\{ F_1^{(2n+\frac{3}{2})},\, F_2^{(2n+\frac{1}{2})} \right\}.
\end{aligned}
\tag{2.26}
$$

Besides that $A_0 = u(x,t)M_2^{(0)}$, $A_{1/2} = \sqrt{i}\bar{\psi}(x,t)G_1^{(1/2)}$, where u is the bosonic field and $\bar{\psi}$ is the fermionic field.

The first equation in the system (2.21) shows us that $D_N^{(N)} \in \mathcal{K}(E)$, then if we consider that $D_N^{(N)} \in \mathcal{K}_{bos}$ we have $N = 2n+1$. Now, it is possible to expand each term $D_N^{(m)}$ by using the generators in eq. (2.26), as follows

$$
\begin{aligned}
&D_N^{(2n+\frac{3}{2})} = \gamma_{2n+\frac{3}{2}} F_1^{\left(2n+\frac{3}{2}\right)} + \delta_{2n+\frac{3}{2}} G_2^{\left(2n+\frac{3}{2}\right)},\\
&D_N^{(2n+1)} = a_{2n+1} K_1^{(2n+1)} + b_{2n+1} K_2^{(2n+1)} + c_{2n+1} M_1^{(2n+1)},\\
&D_N^{(2n+\frac{1}{2})} = \alpha_{2n+\frac{1}{2}} F_2^{\left(2n+\frac{1}{2}\right)} + \beta_{2n+\frac{1}{2}} G_1^{\left(2n+\frac{1}{2}\right)},\\
&D_N^{(2n)} = d_{2n} M_2^{(2n)},\\
&D_N^{(2n-\frac{1}{2})} = \gamma_{2n-\frac{1}{2}} F_1^{\left(2n-\frac{1}{2}\right)} + \delta_{2n-\frac{1}{2}} G_2^{\left(2n-\frac{1}{2}\right)},\\
&D_N^{(2n-1)} = a_{2n-1} K_1^{(2n-1)} + b_{2n-1} K_2^{(2n-1)} + c_{2n-1} M_1^{(2n-1)},
\end{aligned}
$$

$$
\begin{aligned}
&D_N^{(2n-\frac{3}{2})} = \alpha_{2n-\frac{3}{2}} F_2^{\left(2n-\frac{3}{2}\right)} + \beta_{2n-\frac{3}{2}} G_1^{\left(2n-\frac{3}{2}\right)},\\
&D_N^{(2n-2)} = d_{2n-2} M_2^{(2n-2)},\\
&\vdots\\
&D_N^{(\frac{3}{2})} = \gamma_{\frac{3}{2}} F_1^{\left(\frac{3}{2}\right)} + \delta_{\frac{3}{2}} G_2^{\left(\frac{3}{2}\right)},\\
&D_N^{(1)} = a_1 K_1^{(1)} + b_1 K_2^{(1)} + c_1 M_1^{(1)},\\
&D_N^{(\frac{1}{2})} = \alpha_{\frac{1}{2}} F_2^{\left(\frac{1}{2}\right)} + \beta_{\frac{1}{2}} G_1^{\left(\frac{1}{2}\right)}\\
&D_N^{(0)} = d_0 M_2^{(0)}.
\end{aligned} \tag{2.27}
$$

where $\{a_m, b_m, c_m, d_m\}$ and $\{\alpha_m, \beta_m, \gamma_m, \delta_m\}$ are functionals of the fields u and $\bar{\psi}$.

Substituting such parameterization in eq. (2.21), one can recursively solve for all $D^{(m)}, m = 0, \cdots N$. Starting from the equation of degree $N+1$,

$$[K_1^{(1)} + K_2^{(1)}, a_{2n+1} K_1^{(2n+1)} + b_{2n+1} K_2^{(2n+1)} + c_{2n+1} M_1^{(2n+1)}] = 0. \tag{2.28}$$

We obtain, after using the matrix representation in the appendix A, that $c_{2n+1} = 0$. Now substituting this result in the next equation in (2.21), i.e, the equation for $N+1/2$, we get

$$\beta_{2n+\frac{1}{2}} = \frac{\sqrt{i}}{2}\bar{\psi}(a_{2n+1} + b_{2n+1}). \tag{2.29}$$

From the equation for N we find that a_{2n+1}, b_{2n+1} are constants and

$$d_{2n} = u a_{2n+1} + \sqrt{i}\bar{\psi}\alpha_{2n+\frac{1}{2}}. \tag{2.30}$$

Proceeding in this way until the equation for $N-2$, we get

$$
\begin{aligned}
(N-1/2): \quad & \partial_x \alpha_{2n+\frac{1}{2}} - u\beta_{2n+\frac{1}{2}} + \sqrt{i}\bar{\psi} d_{2n} = 0\\
& \partial_x \beta_{2n+\frac{1}{2}} - u\alpha_{2n+\frac{1}{2}} + 2\delta_{2n-\frac{1}{2}} = 0 \qquad (2.31)\\
(N-1): \quad & \partial_x d_{2n} - 2c_{2n-1} + 2\sqrt{i}\bar{\psi}\gamma_{2n-\frac{1}{2}} = 0 \qquad (2.32)\\
(N-3/2): \quad & \partial_x \gamma_{2n-\frac{1}{2}} - u\delta_{2n-\frac{1}{2}} + \sqrt{i}\bar{\psi} c_{2n-1} = 0\\
& \partial_x \delta_{2n-\frac{1}{2}} - u\gamma_{2n-\frac{1}{2}} + 2\beta_{2n-\frac{3}{2}} - \sqrt{i}\bar{\psi}(a_{2n-1} + b_{2n-1}) = 0 \qquad (2.33)
\end{aligned}
$$

$$
\begin{aligned}
(N-2): \quad & \partial_x a_{2n-1} + 2uc_{2n-1} - 2\sqrt{i}\bar{\psi}\beta_{2n-\frac{3}{2}} = 0 \\
& \partial_x b_{2n-1} + 2\sqrt{i}\bar{\psi}\beta_{2n-\frac{3}{2}} = 0 \\
& \partial_x c_{2n-1} - 2d_{2n-2} + 2ua_{2n-1} + 2\sqrt{i}\bar{\psi}\alpha_{2n-\frac{3}{2}} = 0.
\end{aligned} \tag{2.34}
$$

The subsequent equations are all similar to the above set, in the sense that the equations for even grade will be similar to (2.32), the odd will be similar to the set in (2.34). For the semi-integer degree equations the following combinations are allowed if ($N - \frac{1}{2} - even$) then it corresponds to the set (2.31) and if the grade can be written as ($N - \frac{1}{2} - odd$) it seems like (2.33).

Then for a specific $n \in Z_+$ these results are resumed in the following way,

$$
\begin{aligned}
& c_{2n+1} = 0, \qquad \beta_{2n+\frac{1}{2}} = \frac{\sqrt{i}}{2}\bar{\psi}(a_{2n+1} + b_{2n+1}), \\
& a_{2n+1} = constant \qquad b_{2n+1} = constant \\
& d_{2n} = ua_{2n+1} + \sqrt{i}\bar{\psi}\alpha_{2n+\frac{1}{2}} \\
& \partial_x \alpha_{2n+\frac{3}{2}-j} - u\beta_{2n+\frac{3}{2}-j} + \sqrt{i}\bar{\psi}d_{2n+1-j} = 0 && \textit{(odd j)} \\
& \partial_x \beta_{2n+\frac{3}{2}-j} - u\alpha_{2n+\frac{3}{2}-j} + 2\delta_{2n+\frac{1}{2}-j} = 0 && \textit{(odd j)} \\
& \partial_x d_{2n+1-j} - 2c_{2n-j} + 2\sqrt{i}\bar{\psi}\gamma_{2n+\frac{1}{2}-j} = 0 && \textit{(odd j)} \\
& \partial_x \gamma_{2n+\frac{3}{2}-j} - u\delta_{2n+\frac{3}{2}-j} + \sqrt{i}\bar{\psi}c_{2n+1-j} = 0 && \textit{(even j)} \\
& \partial_x \delta_{2n+\frac{3}{2}-j} - u\gamma_{2n+\frac{3}{2}-j} + 2\beta_{2n+\frac{1}{2}-j} - \sqrt{i}\bar{\psi}(a_{2n+1-j} + b_{2n+1-j}) = 0 && \textit{(even j)} \\
& \partial_x a_{2n+1-j} + 2uc_{2n+1-j} - 2\sqrt{i}\bar{\psi}\beta_{2n+\frac{1}{2}-j} = 0 && \textit{(even j)} \\
& \partial_x b_{2n+1-j} + 2\sqrt{i}\bar{\psi}\beta_{2n+\frac{1}{2}-j} = 0 && \textit{(even j)} \\
& \partial_x c_{2n+1-j} - 2d_{2n-j} + 2ua_{2n+1-j} + 2\sqrt{i}\bar{\psi}\alpha_{2n+\frac{1}{2}-j} = 0 && \textit{(even j)}
\end{aligned} \tag{2.35}
$$

where $j = 1, ..., 2n$.

As we observed in (2.22)-(2.24) the grade equation (1/2) in (2.21) yields

$$\partial_x \alpha_{\frac{1}{2}} = u\beta_{\frac{1}{2}} - \sqrt{i}\bar{\psi}d_0, \tag{2.36}$$

$$\partial_{t_{2n+1}} \bar{\psi} = \frac{1}{\sqrt{i}}\left(\partial_x \beta_{\frac{1}{2}} - u\alpha_{\frac{1}{2}}\right), \tag{2.37}$$

$$\partial_{t_{2n+1}} u = \partial_x d_0. \tag{2.38}$$

Therefore the problem is to recursively solve the set of equations (2.35) varying j, and after finding their respective coefficients together with (2.36) we obtain from (2.38) and (2.37) the time evolution of the fields u, $\bar{\psi}$ in a super integrable model. The index n fixes one superintegrable equation among the infinite equations within the hierarchy. Hereafter, we consider some examples.

$\mathbf{n = 0}$

For $n = 0$ we get from (2.35) that a_1 and b_1 are constant and

$$\beta_{\frac{1}{2}} = \frac{\sqrt{i}}{2}\bar{\psi}(a_1 + b_1). \tag{2.39}$$

$$d_0 = u + \sqrt{i}\bar{\psi}\alpha_{1/2} \tag{2.40}$$

From (2.36)-(2.38)

$$\partial_x \alpha_{\frac{1}{2}} = 0, \tag{2.41}$$

$$\partial_{t_1}\bar{\psi} = \partial_x\bar{\psi} - \frac{1}{\sqrt{i}}u\alpha_{\frac{1}{2}}, \tag{2.42}$$

$$\partial_{t_1}u = \partial_x u + \sqrt{i}\alpha_{\frac{1}{2}}\partial_x\bar{\psi}. \tag{2.43}$$

By choosing $\alpha_{\frac{1}{2}} = 0$ the equations for $n = 0$ are

$$\partial_{t_1}\bar{\psi} = \partial_x\bar{\psi}, \qquad \partial_{t_1}u = \partial_x u. \tag{2.44}$$

$\mathbf{n = 1}$

For $n = 1$ we obtain from (2.35) that a_3 and b_3 are constant and the following set of equations,

$$
\begin{aligned}
d_2 = ua_3 + \bar{\psi}\alpha_{\frac{5}{2}}, \quad \beta_{\frac{5}{2}} &= \frac{\sqrt{i}}{2}\bar{\psi}\,(a_3 + b_3) \\
\partial_x\alpha_{\frac{5}{2}} - u\beta_{\frac{5}{2}} + \sqrt{i}\bar{\psi}d_2 &= 0 \qquad \textit{(j=1)} \\
\partial_x\beta_{\frac{5}{2}} - u\alpha_{\frac{5}{2}} + 2\delta_{\frac{3}{2}} &= 0 \qquad \textit{(j=1)} \\
\partial_x d_2 - 2c_1 + 2\sqrt{i}\bar{\psi}\gamma_{\frac{3}{2}} &= 0 \qquad \textit{(j=1)} \\
\partial_x\gamma_{\frac{3}{2}} - u\delta_{\frac{3}{2}} + \sqrt{i}\bar{\psi}c_1 &= 0 \qquad \textit{(j=2)} \\
\partial_x\delta_{\frac{3}{2}} - u\gamma_{\frac{3}{2}} + 2\beta_{\frac{1}{2}} - \sqrt{i}\bar{\psi}(a_1 + b_1) &= 0 \qquad \textit{(j=2)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_x a_1 + 2uc_1 - 2\sqrt{i}\bar{\psi}\beta_{\frac{1}{2}} &= 0 && \textit{(j=2)} \\
\partial_x b_1 + 2\sqrt{i}\bar{\psi}\beta_{\frac{1}{2}} &= 0 && \textit{(j=2)} \\
\partial_x c_1 - 2d_0 + 2ua_1 + 2\sqrt{i}\bar{\psi}\alpha_{\frac{1}{2}} &= 0 && \textit{(j=2)}
\end{aligned}
\tag{2.45}
$$

In addition, with (2.36), namely

$$
\partial_x \alpha_{\frac{1}{2}} = u\beta_{\frac{1}{2}} - \sqrt{i}\bar{\psi}d_0. \tag{2.46}
$$

We get the following solution,

$$
\begin{aligned}
\gamma_{\frac{3}{2}} &= -\frac{\sqrt{i}}{2}u\bar{\psi} \\
\delta_{\frac{3}{2}} &= -\frac{\sqrt{i}}{2}\partial_x\bar{\psi} \\
c_1 &= \frac{1}{2}\partial_x u \\
a_1 &= \frac{1}{2}\left(i\bar{\psi}\partial_x\bar{\psi} - u^2\right) \\
\beta_{\frac{1}{2}} &= \frac{\sqrt{i}}{4}\partial_x^2\bar{\psi} - \frac{\sqrt{i}}{2}u^2\bar{\psi} \\
b_1 &= -\frac{i}{2}\bar{\psi}\partial_x\bar{\psi} \\
\alpha_{\frac{1}{2}} &= \frac{\sqrt{i}}{4}\left(u\partial_x\bar{\psi} - \bar{\psi}\partial_x u\right) \\
d_0 &= \frac{1}{4}\partial_x^2 u - \frac{1}{2}u^3 + \frac{3i}{4}u\bar{\psi}\partial_x\bar{\psi}
\end{aligned}
\tag{2.47}
$$

where we chose $a_3 = b_3 = 1$ and $\alpha_{\frac{5}{2}} = 0$. Now, substituting these results in (2.37)-(2.38) we find the $\mathcal{N} = 1$ supersymmetric mKdV equation

$$
4\partial_{t_3} u = \partial_x^3 u - 6u^2\partial_x u + 3i\bar{\psi}\partial_x\left(u\partial_x\bar{\psi}\right), \tag{2.48}
$$

$$
4\partial_{t_3}\bar{\psi} = \partial_x^3\bar{\psi} - 3u\partial_x\left(u\bar{\psi}\right). \tag{2.49}
$$

where setting the fermions to zero we recover the mKdV equation (2.17).

Moreover, we can find the explicit solution for the Lax component $A_{t_3} = D^{(3)} + D^{(5/2)} + D^{(2)} + D^{(3/2)} + D^{(1)} + D^{(1/2)} + D^{(0)}$, namely

$$A_{t_3} = \left(\begin{array}{cc|c} a_0 + \lambda^{1/2}a_{1/2} + \lambda u + \lambda^{3/2} & a_+ - \lambda & \mu_+ + \lambda^{1/2}\nu_+ + \lambda\sqrt{i}\bar{\psi} \\ -\lambda a_- - \lambda^2 & -a_0 + \lambda^{1/2}a_{1/2} - \lambda u + \lambda^{3/2} & \lambda^{1/2}\mu_- + \lambda\nu_- + \lambda^{3/2}\sqrt{i}\bar{\psi} \\ \hline \lambda^{1/2}\mu_- - \lambda\nu_- + \lambda^{3/2}\sqrt{i}\bar{\psi} & \mu_+ - \lambda^{1/2}\nu_+ + \lambda\sqrt{i}\bar{\psi} & 2\lambda^{1/2}a_{1/2} + 2\lambda^{3/2} \end{array}\right), \tag{2.50}$$

where

$$\begin{aligned} a_0 &= \frac{1}{4}\left(\partial_x^2 u - 2u^3 + 3iu\bar{\psi}\partial_x\bar{\psi}\right), \qquad a_{1/2} = -\frac{i}{2}\bar{\psi}\partial_x\bar{\psi}, \\ a_\pm &= \frac{1}{2}\left(-\partial_x u \pm u^2 \mp i\bar{\psi}\partial_x\bar{\psi}\right), \qquad \nu_\pm = \frac{\sqrt{i}}{2}\big(\partial_x\bar{\psi} \mp \bar{\psi}u\big), \\ \mu_\pm &= \frac{\sqrt{i}}{4}\big(\partial_x^2\bar{\psi} \mp u\partial_x\bar{\psi} \pm \bar{\psi}\partial_x u - 2\bar{\psi}u^2\big). \end{aligned} \tag{2.51}$$

$\mathbf{n = 2}$

And for $n = 2$ the set of equations in (2.35) and the equation (2.36) we have

$$\begin{aligned} c_5 = 0, \qquad \beta_{\frac{9}{2}} &= \sqrt{i}\bar{\psi}, \\ d_4 &= u + \sqrt{i}\bar{\psi}\alpha_{\frac{9}{2}} \\ \partial_x\alpha_{\frac{9}{2}} - u\beta_{\frac{9}{2}} + \sqrt{i}\bar{\psi}d_4 &= 0 && \textit{(j=1)} \\ \partial_x\beta_{\frac{9}{2}} - u\alpha_{\frac{9}{2}} + 2\delta_{\frac{7}{2}} &= 0 && \textit{(j=1)} \\ \partial_x d_4 - 2c_3 + 2\sqrt{i}\bar{\psi}\gamma_{\frac{7}{2}} &= 0 && \textit{(j=1)} \\ \partial_x\gamma_{\frac{7}{2}} - u\delta_{\frac{7}{2}} + \sqrt{i}\bar{\psi}c_3 &= 0 && \textit{(j=2)} \\ \partial_x\delta_{\frac{7}{2}} - u\gamma_{\frac{7}{2}} + 2\beta_{\frac{5}{2}} - \sqrt{i}\bar{\psi}(a_3 + b_3) &= 0 && \textit{(j=2)} \\ \partial_x a_3 + 2uc_3 - 2\sqrt{i}\bar{\psi}\beta_{\frac{5}{2}} &= 0 && \textit{(j=2)} \\ \partial_x b_3 + 2\sqrt{i}\bar{\psi}\beta_{\frac{5}{2}} &= 0 && \textit{(j=2)} \end{aligned}$$

$$
\begin{aligned}
\partial_x c_3 - 2d_2 + 2ua_3 + 2\sqrt{i}\bar{\psi}\alpha_{\frac{5}{2}} &= 0 && \textit{(j=2)}\\
\partial_x \alpha_{\frac{5}{2}} - u\beta_{\frac{5}{2}} + \sqrt{i}\bar{\psi}d_2 &= 0 && \textit{(j=3)}\\
\partial_x \beta_{\frac{5}{2}} - u\alpha_{\frac{5}{2}} + 2\delta_{\frac{3}{2}} &= 0 && \textit{(j=3)}\\
\partial_x d_2 - 2c_1 + 2\sqrt{i}\bar{\psi}\gamma_{\frac{3}{2}} &= 0 && \textit{(j=3)}\\
\partial_x \gamma_{\frac{3}{2}} - u\delta_{\frac{3}{2}} + \sqrt{i}\bar{\psi}c_1 &= 0 && \textit{(j=4)}\\
\partial_x \delta_{\frac{3}{2}} - u\gamma_{\frac{3}{2}} + 2\beta_{\frac{1}{2}} - \sqrt{i}\bar{\psi}(a_1+b_1) &= 0 && \textit{(j=4)}\\
\partial_x a_1 + 2uc_1 - 2\sqrt{i}\bar{\psi}\beta_{\frac{1}{2}} &= 0 && \textit{(j=4)}\\
\partial_x b_1 + 2\sqrt{i}\bar{\psi}\beta_{\frac{1}{2}} &= 0 && \textit{(j=4)}\\
\partial_x c_1 - 2d_0 + 2ua_1 + 2\sqrt{i}\bar{\psi}\alpha_{\frac{1}{2}} &= 0 && \textit{(j=4)}\\
\partial_x \alpha_{\frac{1}{2}} - u\beta_{\frac{1}{2}} + \sqrt{i}\bar{\psi}d_0 &= 0
\end{aligned}
\tag{2.52}
$$

The solution of this system is given by

$$
\begin{aligned}
\gamma_{\frac{7}{2}} &= -\frac{\sqrt{i}}{2}u\bar{\psi}\\
\delta_{\frac{7}{2}} &= -\frac{\sqrt{i}}{2}\partial_x\bar{\psi}\\
c_3 &= \frac{1}{2}\partial_x u\\
a_3 &= \frac{1}{2}\left(i\bar{\psi}\partial_x\bar{\psi} - u^2\right)\\
\beta_{\frac{5}{2}} &= \frac{\sqrt{i}}{4}\partial_x^2\bar{\psi} - \frac{\sqrt{i}}{2}u^2\bar{\psi}\\
b_3 &= -\frac{i}{2}\bar{\psi}\partial_x\bar{\psi}\\
\alpha_{\frac{3}{2}} &= \frac{\sqrt{i}}{4}\left(u\partial_x\bar{\psi} - \bar{\psi}\partial_x u\right)\\
d_2 &= \frac{1}{4}\partial_x^2 u - \frac{1}{2}u^3 + \frac{3i}{4}u\bar{\psi}\partial_x\bar{\psi}\\
\delta_{\frac{3}{2}} &= \frac{\sqrt{i}}{8}\left(3u\partial_x u\bar{\psi} - \partial_x^3\bar{\psi} + 3u^2\partial_x\bar{\psi}\right)
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \frac{1}{8}\partial_x^3 u - \frac{3}{4}u^2\partial_x u + \frac{i}{2}\partial_x u\bar{\psi}\partial_x\bar{\psi} + \frac{i}{4}u\bar{\psi}\partial_x^2\bar{\psi} \\
\gamma_{\frac{3}{2}} &= \frac{\sqrt{i}}{8}\left(3u^3\bar{\psi} - u\partial_x^2\bar{\psi} - \partial_x^2 u\bar{\psi} + \partial_x u\partial_x\bar{\psi}\right) \\
a_1 &= \frac{3}{8}u^4 - \frac{1}{4}u\partial_x^2 u + \frac{1}{8}(\partial_x u)^2 - iu^2\bar{\psi}\partial_x\bar{\psi} + \frac{i}{8}\bar{\psi}\partial_x^3\bar{\psi} - \frac{i}{8}\partial_x\bar{\psi}\partial_x^2\bar{\psi} \\
b_1 &= \frac{i}{2}u^2\bar{\psi}\partial_x\bar{\psi} + \frac{i}{8}\partial_x\bar{\psi}\partial_x^2\bar{\psi} - \frac{i}{8}\bar{\psi}\partial_x^3\bar{\psi} \\
\beta_{\frac{1}{2}} &= \frac{\sqrt{i}}{8}\left(3u^4\bar{\psi} - 3u\partial_x^2 u\bar{\psi} - (\partial_x u)^2\bar{\psi} - 2u^2\partial_x^2\bar{\psi} + \frac{1}{2}\partial_x^4\bar{\psi} - \frac{1}{2}u\partial_x u\partial_x\bar{\psi}\right) \\
\alpha_{\frac{1}{2}} &= \frac{\sqrt{i}}{16}\left(4u^2\partial_x u\bar{\psi} - 4u^3\partial_x\bar{\psi} - \bar{\psi}\partial_x^3 u + u\partial_x^3\bar{\psi} - \partial_x u\partial_x^2\bar{\psi} + \partial_x\bar{\psi}\partial_x^2 u\right) \\
d_0 &= \frac{1}{16}\partial_x^4 u + \frac{3}{8}u^5 - \frac{5}{8}u^2\partial_x^2 u - \frac{5}{8}u(\partial_x u)^2 + \frac{5i}{16}\partial_x^2 u\bar{\psi}\partial_x\bar{\psi} - \frac{5i}{4}u^3\bar{\psi}\partial_x\bar{\psi} + \frac{5i}{16}u\bar{\psi}\partial_x^3\bar{\psi} \\
&\quad + \frac{5i}{16}\partial_x u\bar{\psi}\partial_x^2\bar{\psi}
\end{aligned}
\tag{2.53}
$$

where $a_5 = b_5 = 1$ and $a_{\frac{9}{5}} = 0$. Finally, the equations of motion provided by (2.37)-(2.38) are

$$
\begin{aligned}
16\partial_{t_5} u &= \partial_x^5 u - 10(\partial_x u)^3 - 40u(\partial_x u)(\partial_x^2 u) - 10u^2(\partial_x^3 u) + 30u^4(\partial_x u) \\
&\quad + 5i\partial_x\bar{\psi}\partial_x(u\partial_x^2\bar{\psi}) + 5i\bar{\psi}\partial_x(u\partial_x^3\bar{\psi} - 4u^3\partial_x\bar{\psi} + \partial_x u\partial_x^2\bar{\psi} + \partial_x^2 u\partial_x\bar{\psi}),
\end{aligned}
\tag{2.54}
$$

$$
16\partial_{t_5}\bar{\psi} = \partial_x^5\bar{\psi} - 5u\partial_x(u\partial_x^2\bar{\psi} + 2\partial_x u\partial_x\bar{\psi} + \partial_x^2 u\bar{\psi}) + 10u^2\partial_x(u^2\bar{\psi}) - 10(\partial_x u)\partial_x(\partial_x u\bar{\psi}).
\tag{2.55}
$$

The corresponding Lax component $A_{t_5} = D^{(5)} + D^{(9/2)} + \cdots + D^{(0)}$ is explicitly given in the appendix B.

Notice that these equations seem to be related, since their structure is the same. In fact, due to the symmetries of these equations we can relate two consecutive equations of a hierarchy through the recursion operator.

It is known that for the bosonic mKdV hierarchy the equations of motion can be obtained from a recursion operator R by

$$
\frac{\partial u}{\partial t_{2n+3}} = R\frac{\partial u}{\partial t_{2n+1}} = \left(\frac{1}{4}\mathbb{D}^2 - u^2 - \partial_x u\mathbb{D}^{-1}u\right)\frac{\partial u}{\partial t_{2n+1}}
\tag{2.56}
$$

where $\mathbb{D} = \partial_x$ and $\mathbb{D}^{-1}$ its inverse [37]. The supersymmetric case will be considered in the chapter 4.

Moreover, we verified that the equations of motion for all members of the hierarchy are invariant under the following supersymmetric transformations,

$$\delta u = -\sqrt{i}\bar{\epsilon}\,\partial_x\bar{\psi}, \qquad \delta\bar{\psi} = -\frac{1}{\sqrt{i}}\bar{\epsilon}\,u, \tag{2.57}$$

where $\bar{\epsilon}$ is a Grassmannian parameter.

Note that the Lax component $A_x = E^{(1)} + A_0 + A_{1/2}$ does not depend on the index n, and will be the same for the entire hierarchy, and it is given by

$$A_x = \left(\begin{array}{cc|c} \lambda^{1/2}+u & -1 & \sqrt{i}\,\bar{\psi} \\ -\lambda & \lambda^{1/2}-u & \sqrt{i}\,\lambda^{1/2}\,\bar{\psi} \\ \hline \sqrt{i}\,\lambda^{1/2}\,\bar{\psi} & \sqrt{i}\,\bar{\psi} & 2\,\lambda^{1/2} \end{array}\right). \tag{2.58}$$

Now, it is worth pointing out that the negative integrable hierarchy can be also constructed by considering the following zero curvature condition,

$$\left[\partial_x + E^{(1)} + A_0 + A_{1/2}, \partial_{t_{-M}} + D^{(-M)} + D^{(-M+1/2)} + \cdots + D^{(-1)} + D^{(-1/2)}\right] = 0. \tag{2.59}$$

The solutions are in general non-local, however, for the simplest case of $N = -M = -1$, we find that the Lax component $A_{t_{-1}} = D^{(-1)} + D^{(-1/2)}$ is

$$A_{t_{-1}} = \left(\begin{array}{cc|c} \lambda^{-1/2} & -\lambda^{-1}\,e^{2\phi} & -\lambda^{-1/2}\sqrt{i}\,\psi e^{\phi} \\ -e^{-2\phi} & \lambda^{-1/2} & -\sqrt{i}\,\psi e^{-\phi} \\ \hline \sqrt{i}\,\psi e^{-\phi} & \lambda^{-1/2}\sqrt{i}\,\psi e^{\phi} & 2\lambda^{-1/2} \end{array}\right). \tag{2.60}$$

Together with (2.58) and the parametrization $u = -\partial_x \phi$ we get the following equations of motion from the zero curvature condition

$$\begin{aligned} \partial_{t_{-1}}\partial_x\phi &= 2\sinh 2\phi + 2i\bar{\psi}\psi\sinh\phi, & (2.61a)\\ \partial_{t_{-1}}\bar{\psi} &= 2\psi\cosh\phi, & (2.61b)\\ \partial_x\psi &= 2\bar{\psi}\cosh\phi. & (2.61c) \end{aligned}$$

These are the equations of motion of the $\mathcal{N} = 1$ sshG (supersymmetric sinh-Gordon) model in the light-cone coordinates (x, t_{-1}) [31, 33].

2.3 Soliton Solutions

Another interesting feature of an integrable hierarchy is that its solutions have an universal structure and can be systematically obtained by the dressing method. Here, we only show these solutions for mKdV and super mKdV hierarchies.

For the mKdV hierarchy the soliton solutions can be written as

$$\begin{aligned} \phi_{1-sol} &= \ln\left(\frac{1+\rho_1}{1-\rho_1}\right)\\ \phi_{2-sol} &= \ln\left(\frac{1+b_1\rho_1+b_2\rho_2+\alpha_{12}b_1b_2\rho_1\rho_2}{1-b_1\rho_1-b_2\rho_2+\alpha_{12}b_1b_2\rho_1\rho_2}\right)\\ \phi_{3-sol} &= \ln\left(\frac{1+R_1+R_2+R_3+\alpha_{12}R_1R_2+\alpha_{13}R_1R_3+\alpha_{23}R_2R_3+\alpha_{123}R_1R_2R_3}{1-R_1-R_2-R_3+\alpha_{12}R_1R_2+\alpha_{13}R_1R_3+\alpha_{23}R_2R_3-\alpha_{123}R_1R_2R_3}\right)\\ &\vdots \end{aligned} \quad (2.62)$$

where $\rho_i = \exp\left(2k_i x + 2k_i^{2n+1}t\right)$ and $R_j = a_j\rho_j$, $j = 1, 2, 3$.

In order to satisfy the respective evolution equation within the hierarchy we have,

$$\begin{aligned} \alpha_{12} = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2,\ \alpha_{23} &= \left(\frac{k_2-k_3}{k_2+k_3}\right)^2,\ \alpha_{13} = \left(\frac{k_1-k_3}{k_1+k_3}\right)^2,\\ \alpha_{123} &= \alpha_{12}\alpha_{13}\alpha_{23}. \end{aligned} \quad (2.63)$$

These are the solutions for all equations in the mKdV hierarchy.

Now, considering the smKdV hierarchy the one-soliton solution is given by,

$$\phi_{1-sol} = \ln\left(\frac{1+\rho_1}{1-\rho_1}\right), \qquad \bar{\psi}_{1-sol} = \epsilon\rho_1\left(\frac{1}{1+\rho_1} + \frac{1}{1-\rho_1}\right) \tag{2.64}$$

where ϵ is a fermionic parameter.

In the next chapter we will discuss an useful method to obtain solutions for nonlinear equations, namely Bäcklund transformation as well as we will constructed systematically BT for the smKdV hierarchy.

Chapter 3

Construction of the super Bäcklund transformations for the smKdV hierarchy

In this chapter we will derive a systematic method to generate the super-Bäcklund transformations (sBT) for all members of the smKdV hierarchy. The method is based on the gauge invariance of the zero curvature condition. First of all, we will obtain the so called defect matrix associated to the super sinh-Gordon (sshG) model (which is the $n = -1$ member of the hierarchy), and then we use this result to construct the sBT for the two first flows, namely, $n = 1$ (smKdV) equation, and for the $n = 2$ super-equation.

3.1 Bäcklund transformations

Bäcklund transformation (BT) is a set of partial differential equations that relates two different solutions of a nonlinear partial differential equation. In particular, the equations in this set have a lower order than the original equation of motion.

The Bäcklund transformations for the sinh-Gordon equation are one of the most well-known in literature [8] and is defined as

$$\partial_x(\phi_1 - \phi_2) = \frac{4}{\omega^2}\sinh(\phi_1 + \phi_2), \tag{3.1}$$
$$\partial_{t_{-1}}(\phi_1 + \phi_2) = \omega^2 \sinh(\phi_1 - \phi_2). \tag{3.2}$$

where ω is the Bäcklund parameter.

The compatibility between these equations provides that ϕ_1 and ϕ_2 satisfy the sinh-Gordon equation $\partial_x\partial_{t_{-1}}\phi_i = 2\sinh 2\phi_i$, $i = 1, 2$. Thus Bäcklund transformation relates two different solutions of a nonlinear equation. This property is useful to generate solutions of nonlinear equations from an already known solution. For example, substituting the trivial solution $\phi_1 = 0$ in (3.1) and (3.2) we obtain for ϕ_2 the 1-soliton solution (2.62), namely

$$\phi_2 = \ln\left(\frac{1+\rho_1}{1-\rho_1}\right), \qquad \rho_1 = \exp\left(\frac{4}{\omega^2}x + \omega^2 t_{-1}\right) \tag{3.3}$$

where we identify the Bäcklund condition $k_1 = \frac{2}{\omega^2}$.

This procedure can be repeated several times yielding the multi-soliton solutions. Besides that, BT are important to describe integrable models in the presence of defects, as we will see in the chapter 6.

The supersymmetric extension of the BT for the $\mathcal{N} = 1$ supersymmetric sinh-Gordon (sshG) model was introduced in [40]. In this case the super Bäcklund transformations are given by,

$$\partial_x\phi_- = \frac{4}{\omega^2}\sinh(\phi_+) - \frac{2i}{\omega}\sinh\left(\frac{\phi_+}{2}\right) f_1\bar{\psi}_+, \tag{3.4}$$
$$\bar{\psi}_- = \frac{4}{\omega}\cosh\left(\frac{\phi_+}{2}\right) f_1, \tag{3.5}$$
$$\partial_x f_1 = \frac{1}{\omega}\cosh\left(\frac{\phi_+}{2}\right)\bar{\psi}_+, \tag{3.6}$$
$$\partial_{t_{-1}}\phi_+ = \omega^2\sinh(\phi_-) + i\omega\sinh\left(\frac{\phi_-}{2}\right) f_1\psi_-, \tag{3.7}$$
$$\psi_+ = 2\omega\cosh\left(\frac{\phi_-}{2}\right) f_1, \tag{3.8}$$

$$\partial_{t_{-1}} f_1 = \frac{\omega}{2} \cosh\left(\frac{\phi_-}{2}\right) \psi_-. \tag{3.9}$$

where we define $\phi_\pm = \phi_1 \pm \phi_2$, $\bar{\psi}_\pm = \bar{\psi}_1 \pm \bar{\psi}_2$, $\psi_\pm = \psi_1 \pm \psi_2$ and f_1 is an auxiliary fermionic field.

In the next section we will show how to obtain the Bäcklund transformation for super sinh-Gordon via gauge transformation.

3.2 Obtaining the K matrix for the sshG equation

At the end of the chapter 2 we saw that the $\mathcal{N} = 1$ sshG equation can be derived from the zero curvature equation,

$$\left[\partial_x + A_x, \partial_{t_{-1}} + A_{t_{-1}}\right] = 0, \tag{3.10}$$

where the temporal part of the Lax is given in (2.60) and we write the spatial part in terms of $\partial_x \phi$

$$A_x = \left(\begin{array}{cc|c} \lambda^{1/2} - \partial_x\phi & -1 & \sqrt{i}\,\bar{\psi} \\ -\lambda & \lambda^{1/2} + \partial_x\phi & \sqrt{i}\,\lambda^{1/2}\,\bar{\psi} \\ \hline \sqrt{i}\,\lambda^{1/2}\,\bar{\psi} & \sqrt{i}\,\bar{\psi} & 2\,\lambda^{1/2} \end{array}\right). \tag{3.11}$$

Now, the construction of the BT is based on the gauge invariance of the zero curvature equation. We consider a different solution for (3.10) as,

$$A_x^{(2)} = K A_x^{(1)} K^{-1} - \partial_x K K^{-1} \tag{3.12}$$

$$A_{t_{-1}}^{(2)} = K A_{t_{-1}}^{(1)} K^{-1} - \partial_{t_{-1}} K K^{-1} \tag{3.13}$$

where $A_{x,t_{-1}}^{(p)}$ represents the Lax connections depending on the respective fields ϕ_p, ψ_p, and $\bar{\psi}_p$, K is a matrix that connects two different field configurations and satisfies

$$\partial_x K = K A_x^{(1)} - A_x^{(2)} K, \tag{3.14}$$

$$\partial_{t_{-1}} K = K A_{t_{-1}}^{(1)} - A_{t_{-1}}^{(2)} K. \tag{3.15}$$

Let us consider the following ansatz for the λ-expansion of the K matrix,

$$K_{ij} = \tau_{ij} + \lambda^{-1/2}\eta_{ij} + \lambda^{1/2}\kappa_{ij}, \tag{3.16}$$

with τ_{ij}, η_{ij}, and κ_{ij} being the entries of 3×3 graded matrices. First of all, by considering the λ-expansion in order to solve the differential equations (3.14) and (3.15), we find that the $\lambda^{+3/2}$ and λ^{+1} terms lead to

$$\tau_{12} = \kappa_{12} = \kappa_{13} = \kappa_{32} = 0, \qquad \kappa_{11} = \kappa_{22} = c_{11}, \qquad \kappa_{33} = c_{33}, \tag{3.17}$$

and

$$\begin{aligned} \tau_{11} - \tau_{22} &= \sqrt{i}\left(\bar{\psi}_2\,\kappa_{31} - \kappa_{23}\,\bar{\psi}_1\right), && (3.18)\\ \tau_{13} + \kappa_{23} &= \sqrt{i}\left(\bar{\psi}_2\,\kappa_{33} - \kappa_{11}\,\bar{\psi}_1\right), && (3.19)\\ \kappa_{31} + \tau_{32} &= \sqrt{i}\left(\bar{\psi}_1\,\kappa_{33} - \kappa_{11}\,\bar{\psi}_2\right), && (3.20) \end{aligned}$$

where c_{ij} denotes arbitrary constants*. Analogously, for the degrees $\lambda^{-3/2}$ and λ^{-1} we get

$$\tau_{21} = \eta_{21} = \eta_{23} = \eta_{31} = 0, \qquad \eta_{22} = \eta_{11}e^{2\phi_-}, \qquad \eta_{33} = b_{33}, \tag{3.21}$$

and the following constraints

$$\begin{aligned} \tau_{11}\,e^{\phi_-} - \tau_{22}\,e^{-\phi_-} &= \sqrt{i}\,e^{-\frac{\phi_+}{2}}\big(\eta_{13}\,\psi_1\,e^{\frac{\phi_-}{2}} + e^{-\frac{\phi_-}{2}}\,\psi_2\,\eta_{32}\big), && (3.22)\\ \tau_{31}\,e^{(\phi_+ + \phi_-)} + \eta_{32} &= \sqrt{i}\,e^{\frac{\phi_+}{2}}\big(\eta_{33}\,\psi_1\,e^{\frac{\phi_-}{2}} - \eta_{22}\,\psi_2\,e^{-\frac{\phi_-}{2}}\big), && (3.23)\\ \tau_{23}\,e^{(\phi_+ - \phi_-)} + \eta_{13} &= \sqrt{i}\,e^{\frac{\phi_+}{2}}(\eta_{11}\,\psi_1 e^{\frac{\phi_-}{2}} - \eta_{33}\,\psi_2 e^{-\frac{\phi_-}{2}}). && (3.24) \end{aligned}$$

Notice that after suitable parameterizations, the constraints (3.19), (3.20), (3.23) and (3.24) could reproduce the Bäcklund equations (3.5) and (3.8) respectively, by introducing properly the auxiliary field f_1. The other Bäcklund equations will be derived from the differential equations coming from the degrees λ^0 and $\lambda^{\pm 1/2}$, which are fully presented in appendix C. Now, considering the equations involving η_{11} (C.40) and η_{22} (C.43), namely,

$$\partial_x\eta_{11} = -\eta_{11}\partial_x\phi_-, \qquad \partial_x\eta_{22} = \eta_{22}\partial_x\phi_-. \tag{3.25}$$

*In what follows we will denote all the constants with Latin letters.

For the constraint in (3.21), we find the simple solution $\eta_{11} = b_{11}\, e^{-\phi_-}$ and $\eta_{22} = b_{11}\, e^{\phi_-}$.

Now, by setting $\eta_{33} = b_{33} = -b_{11}$ and $c_{33} = c_{11}$, we get from (3.19) and (3.20) that

$$\tau_{13} + \tau_{32} = -(\kappa_{23} + \kappa_{31}). \tag{3.26}$$

From the eqs (C.28), (C.32), (C.48) and (C.49), involving both sides of the relation (3.26), we find that

$$\partial_{t_{-1}}(\tau_{13} + \tau_{32}) = 0, \qquad \partial_{t_{-1}}(\kappa_{23} + \kappa_{31}) = 0. \tag{3.27}$$

Then, we will consider the simple solution when the components satisfy that $\tau_{32} = -\tau_{13}$, and $\kappa_{23} = -\kappa_{31}$. Using

$$\eta_{11} = b_{11}e^{-\phi_-}, \quad \text{and} \quad \eta_{33} = -b_{11}, \tag{3.28}$$

eqn. (3.24) gives

$$\eta_{13}e^{-(\phi_+ - \phi_-)} + \tau_{23} \;\; = \;\; \sqrt{i}b_{11}\, e^{-\frac{(\phi_+ - \phi_-)}{2}}(\psi_1 + \psi_2), \tag{3.29}$$

Introducing the auxiliary field f_1,

$$f_1 = \frac{1}{2\omega}\text{sech}\Big(\frac{\phi_-}{2}\Big)\psi_+ = \frac{\omega}{4}\text{sech}\Big(\frac{\phi_+}{2}\Big)\bar{\psi}_-, \tag{3.30}$$

we find that eqn. (3.29) becomes

$$\tau_{23}e^{\frac{(\phi_+ - \phi_-)}{2}} + \eta_{13}e^{-\frac{(\phi_+ - \phi_-)}{2}} \;\; = \;\; 2\sqrt{i}\omega b_{11} \cosh\left(\frac{\phi_-}{2}\right) f_1. \tag{3.31}$$

After making the choice

$$\eta_{13} = \tau_{23}e^{\phi_+} \tag{3.32}$$

with

$$\tau_{23} = \sqrt{i}\omega b_{11}e^{-\frac{\phi_+}{2}} f_1. \tag{3.33}$$

In the same way, by taking eq (3.18)–(3.23) we obtain

$$\tau_{13} = -\frac{2\sqrt{i}}{\omega} c_{11}\, e^{\frac{\phi_+}{2}} f_1, \quad \tau_{31} = -\sqrt{i}\omega b_{11} e^{-\frac{\phi_+}{2}} f_1, \quad \eta_{32} = \tau_{31}\, e^{\phi_+}, \quad \kappa_{23} = \tau_{13}\, e^{-\phi_+}, \tag{3.34}$$

and

$$\tau_{11} = \tau_{22} = \frac{i\omega b_{11}}{4} \mathrm{sech}\left(\frac{\phi_-}{2}\right) f_1 \psi_- \,. \tag{3.35}$$

From the eqs (C.20) and (C.23), we also have

$$\begin{aligned} 2(\partial_x \tau_{11}) &= \sqrt{i}(\eta_{13}+\tau_{23})\bar{\psi}_1 - \sqrt{i}\bar{\psi}_2(\tau_{31}+\eta_{32}) \\ &= -\frac{i\omega^2 b_{11}}{2}\left(\bar{\psi}_2\bar{\psi}_1 - \bar{\psi}_2\bar{\psi}_1\right) = 0 \\ &= 0, \end{aligned} \tag{3.36}$$

where we have used for η_{13} and η_{32} eqns. (3.32) and (3.34) respectively.

Similarly, from (C.27) and (C.29) we obtain $\partial_{t_{-1}}\tau_{11} = \partial_{t_{-1}}\tau_{33} = \partial_x \tau_{33} = 0$. Therefore, it is suitable to set $\tau_{11} = a_{11}$ and $\tau_{33} = a_{33}$.

Now, let us consider the eqs (C.34), (C.37), (C.51), and (C.52) involving derivatives of the elements $\eta_{11}, \eta_{22}, \kappa_{11}$, and κ_{22},

$$\begin{aligned} \partial_x \phi_- &= \frac{1}{c_{11}}\left[\kappa_{21} - \eta_{12} - ic_{11}\bar{\psi}_2\bar{\psi}_1 + \sqrt{i}\tau_{13}\bar{\psi}_1 - \sqrt{i}\bar{\psi}_2\tau_{13}\right] \\ &= \frac{1}{c_{11}}(\kappa_{21}-\eta_{12}) - \frac{2i}{\omega}\sinh\left(\frac{\phi_+}{2}\right) f_1 \bar{\psi}_+\,, \end{aligned} \tag{3.37}$$

$$\begin{aligned} \partial_{t_{-1}}\phi_- &= \frac{1}{b_{11}}\left[\eta_{12}e^{-\phi_+} - \kappa_{21}e^{\phi_+} - ib_{11}\psi_2\psi_1 + \sqrt{i}\tau_{23}\psi_1 e^{\frac{(\phi_+-\phi_-)}{2}} - \sqrt{i}\psi_2\tau_{31}e^{\frac{(\phi_+-\phi_-)}{2}}\right] \\ &= \frac{1}{b_{11}}(\eta_{12}e^{-\phi_+} - \kappa_{21}e^{\phi_+}), \end{aligned} \tag{3.38}$$

and eqs (C.35), (C.41), (C.47) and (C.53) for η_{12} and κ_{21} respectively,

$$\partial_x \eta_{12} = \eta_{12}(\partial_x\phi_+) + 2b_{11}\sinh\phi_-, \tag{3.39}$$

$$\partial_x \kappa_{21} = -\kappa_{21}(\partial_x\phi_+) - 2b_{11}\sinh\phi_-, \tag{3.40}$$

$$\partial_{t_{-1}}\eta_{12} = -2c_{11}e^{\phi_+}\sinh\phi_- - \frac{2ic_{11}}{\omega}\, e^{\phi_+}\sinh\left(\frac{\phi_-}{2}\right) f_1\psi_-\,, \tag{3.41}$$

$$\partial_{t_{-1}}\kappa_{21} = 2c_{11}e^{-\phi_+}\sinh\phi_- + \frac{2ic_{11}}{\omega}\, e^{-\phi_+}\sinh\left(\frac{\phi_-}{2}\right) f_1\psi_-\,. \tag{3.42}$$

By introducing the following parameterizations,

$$\eta_{12} = b_{12}\, e^{\phi_+}, \qquad \kappa_{21} = b_{12}\, e^{-\phi_+}, \qquad b_{12} = -\frac{2c_{11}}{\omega^2}, \qquad b_{11} = 0, \tag{3.43}$$

the eqs (3.37), (3.41) and (3.42) become exactly the Bäcklund equations for the bosonic field $\phi_\pm$, namely,

$$\partial_x\phi_- = \frac{4}{\omega^2}\sinh\phi_+ - \frac{2i}{\omega}\sinh\left(\frac{\phi_+}{2}\right) f_1\bar{\psi}_+\,, \tag{3.44}$$

$$\partial_{t_{-1}}\phi_+ = \omega^2\sinh\phi_- + i\omega\sinh\left(\frac{\phi_-}{2}\right) f_1\psi_-\,. \tag{3.45}$$

Since $b_{11} = 0$, we also find that the set of elements $\{\tau_{11}, \tau_{22}, \tau_{23}, \tau_{31}, \eta_{11}, \eta_{13}, \eta_{22}, \eta_{32}, \eta_{33}\}$ completely vanish and then do not contribute at all to the K matrix.

Finally, if we consider equation (C.31) involving the element τ_{33}, we obtain

$$\begin{aligned}
\partial_{t_{-1}}\tau_{31} &= -\kappa_{31} - \tau_{32}e^{-(\phi_+ + \phi_-)} - \sqrt{i}\psi_2\big(e^{-\frac{(\phi_+ - \phi_-)}{2}}\tau_{11} + e^{\frac{(\phi_+ - \phi_-)}{2}}\kappa_{21}\big), \\
&+ \sqrt{i}\tau_{33}\psi_1\, e^{-\frac{(\phi_+ + \phi_-)}{2}} \\
0 &= e^{-\frac{\phi_+}{2}}\left[-\frac{2\sqrt{i}}{\omega}c_{11}\big(1 + e^{-\phi_-}\big) f_1 + \sqrt{i}\, e^{-\frac{\phi_-}{2}}\left(\tau_{33}\psi_1 - b_{12}\psi_2\right)\right],
\end{aligned} \tag{3.46}$$

from where we conclude that $\tau_{33} = \frac{2c_{11}}{\omega^2}$.

Therefore, we have found a suitable solution for the K matrix, which can be written in the following form [30],

$$K = \left(\begin{array}{cc|c}
\lambda^{1/2} & -\frac{2}{\omega^2}e^{\phi_+}\lambda^{-1/2} & -\frac{2\sqrt{i}}{\omega}e^{\frac{\phi_+}{2}} f_1 \\
-\frac{2}{\omega^2}e^{-\phi_+}\lambda^{1/2} & \lambda^{1/2} & -\frac{2\sqrt{i}}{\omega}e^{-\frac{\phi_+}{2}} f_1\lambda^{1/2} \\
\hline
\frac{2\sqrt{i}}{\omega}e^{-\frac{\phi_+}{2}} f_1\lambda^{1/2} & \frac{2\sqrt{i}}{\omega}e^{\frac{\phi_+}{2}} f_1 & \frac{2}{\omega^2} + \lambda^{1/2}
\end{array}\right) \tag{3.47}$$

where ω represents the Bäcklund parameter and we choose $c_{11} = 1$.

We have obtained the matrix that generates the type-I Bäcklund transformations for the $\mathcal{N} = 1$ supersymmetric sinh Gordon equation (the $n = -1$ member of the smKdV hierarchy). In the next section we will use this result in order to obtain the Bäcklund transformation for the first members of the smKdV hierarchy.

In the appendix D we consider a different solution for the K matrix by including a bosonic auxiliary field (λ_0) and another fermionic auxiliary field ($\tilde{f}_1$). This new solution generates the type-II Bäcklund transformations for the sshG, these transformations have the fields of the theory further addicional auxiliary fields.

Moreover, we will see that Bäcklund transformations describe integrable defects and this K matrix is called defect matrix and it will be used to derive modified conserved quantities in the chapter 7.

3.3 Bäcklund transformations for the smKdV hierarchy

Based upon the fact that the spatial Lax operator is common to all members of the mKdV hierarchy, it was recently shown that the spatial component of the Bäcklund transformation, and consequently the associated K matrix, are also common and henceforth universal within the entire hierarchy [34, 35]. Here, we will extend these results to the supersymmetric mKdV hierarchy starting from the K matrix (3.47) derived in the last section for the ($n = -1$ member), namely the super sinh-Gordon equation.

The main point here is that the spatial part of the gauge transformation (3.15) should be satisfied for all members within the hierarchy, so the x part of the Bäcklund represented by (3.4), (3.5) and (3.6) is the same for all integrable equations. Remaining to derive the time component of the BT, which satisfies the following equation

$$\partial_{t_{2n+1}} K = K A_{t_{2n+1}}(\phi_1, \bar{\psi}_1) - A_{t_{2n+1}}(\phi_2, \bar{\psi}_2) K. \tag{3.48}$$

This gauge condition generates the respective BT for the corresponding equation of motion.

n = 0

For $n = 0$ we have that $A_x = A_{t_1}$ so the temporal part of the Bäcklund is

$$\partial_{t_1}\phi_+ = \partial_x\phi_+, \tag{3.49}$$
$$\partial_{t_1}f_1 = \partial_x f_1, \tag{3.50}$$

and hence,

$$\partial_{t_1}\phi_- = \partial_x\phi_-. \tag{3.51}$$

Then, for $n = 0$ the x and t components of the Bäcklund are the same.

n = 1

The next non-trivial example is the smKdV equation ($n = 1$), we get by substituting (2.50) and (3.47) in the gauge transformation (3.48),

$$\begin{aligned}
4\partial_{t_3}\phi_- &= \frac{i}{\omega}\left[\partial_x^2\phi_+\cosh\left(\frac{\phi_+}{2}\right) - (\partial_x\phi_+)^2\sinh\left(\frac{\phi_+}{2}\right)\right]\bar{\psi}_+ f_1 \\
&- \frac{i}{\omega}\left[\partial_x\phi_+\cosh\left(\frac{\phi_+}{2}\right)\partial_x\bar{\psi}_+ - 2\sinh\left(\frac{\phi_+}{2}\right)\partial_x^2\bar{\psi}_+\right] f_1 \\
&+ \frac{2}{\omega^2}\left[2(\partial_x^2\phi_+)\cosh\phi_+ - (\partial_x\phi_+)^2\sinh\phi_+ + i\bar{\psi}_+(\partial_x\bar{\psi}_+)\sinh\phi_+\right] \\
&- \frac{96i}{\omega^5}\left[\sinh\left(\frac{\phi_+}{2}\right) + 4\sinh^3\left(\frac{\phi_+}{2}\right) + 3\sinh^5\left(\frac{\phi_+}{2}\right)\right]\bar{\psi}_+ f_1 - \frac{32}{\omega^6}\sinh^3\phi_+
\end{aligned} \tag{3.52}$$

$$\begin{aligned}
4\partial_{t_3}f_1 &= \frac{1}{2\omega}\cosh\left(\frac{\phi_+}{2}\right)\left[2\partial_x^2\bar{\psi}_+ - \bar{\psi}_+(\partial_x\phi_+)^2\right] - \frac{12}{\omega^4}\sinh\phi_+\cosh^2\left(\frac{\phi_+}{2}\right)(\partial_x\phi_+)f_1 \\
&+ \frac{1}{2\omega}\sinh\left(\frac{\phi_+}{2}\right)\left[\bar{\psi}_+\partial_x^2\phi_+ - \partial_x\phi_+\partial_x\bar{\psi}_+\right] - \frac{12}{\omega^5}\sinh^2\phi_+\cosh\left(\frac{\phi_+}{2}\right)\bar{\psi}_+.
\end{aligned} \tag{3.53}$$

Equations (3.4)–(3.6), and (3.52) and (3.53) correspond to the super-Bäcklund transformations for the smKdV. It can be easily verified that they are consistent by cross-differentiating any of them. Notice also that by setting all the fermions to zero we recover the bosonic case, i.e., the Bäcklund transformation of the mKdV [34],

$$\partial_x\phi_- = \frac{4}{\omega^2}\sinh\phi_+, \tag{3.54a}$$

$$4\partial_{t_3}\phi_- = \frac{4}{\omega^2}\partial_x^2\phi_+\cosh\phi_+ - \frac{2}{\omega^2}(\partial_x\phi_+)^2\sinh\phi_+ - \frac{32}{\omega^6}\sinh^3\phi_+. \tag{3.54b}$$

n = 2

Now, we derive the temporal part of the super-Bäcklund transformation for the $n=2$ member of the hierarchy. We consider the corresponding Lax operator A_{t_5} in such a way that the gauge condition reads

$$\partial_{t_5}K = KA_{t_5}(\phi_1,\bar{\psi}_1) - A_{t_5}(\phi_2,\bar{\psi}_2)K. \tag{3.55}$$

By solving this condition for A_{t_5} given in appendix B, we obtain

$$\begin{aligned}
16\partial_{t_5}\phi_- \;=\;& -\frac{i}{\omega}\left[c_0\,\bar{\psi}_+ + c_1\,\partial_x\bar{\psi}_+ + c_2\,\partial_x^2\bar{\psi}_+ + c_3\,\partial_x^3\bar{\psi}_+ + c_4\,\partial_x^4\bar{\psi}_+\right]f_1\\
&+\frac{1}{\omega^2}\left[c_5 + ic_6\,\bar{\psi}_+\partial_x\bar{\psi}_+ + ic_7\,\bar{\psi}_+\partial_x^2\bar{\psi}_+ + ic_8\big(\bar{\psi}_+\partial_x^3\bar{\psi}_+ - (\partial_x\bar{\psi}_+)(\partial_x^2\bar{\psi}_+)\big)\right]\\
&-\frac{i}{\omega^5}\left[\,c_9\,\bar{\psi}_+ + c_{10}\,\partial_x\bar{\psi}_+ + c_{11}\,\partial_x^2\bar{\psi}_+\,\right]f_1 + \frac{1}{\omega^6}\left[\,c_{12} + i\,c_{13}\,\bar{\psi}_+\partial_x\bar{\psi}_+\,\right]\\
&+\frac{i}{\omega^9}\,c_{14}f_1\bar{\psi}_+ + \frac{c_{15}}{\omega^{10}},
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
16\partial_{t_5}f_1 \;=\;& \frac{1}{\omega}\left[\,g_0\,\bar{\psi}_+ + g_1\,\partial_x\bar{\psi}_+ + g_2\,\partial_x^2\bar{\psi}_+ + g_3\,\partial_x^3\bar{\psi}_+ + g_4\,\partial_x^4\bar{\psi}_+\right]\\
&+\frac{1}{\omega^4}\left[g_6 + i\,g_5\,\bar{\psi}_+\partial_x\bar{\psi}_+\right]f_1 + \frac{1}{\omega^5}\left[\,g_7\,\bar{\psi}_+ + g_8\,\partial_x\bar{\psi}_+ + g_9\,\partial_x^2\bar{\psi}_+\,\right]\\
&+\frac{g_{10}}{\omega^8}\,f_1 + \frac{g_{11}}{\omega^9}\,\bar{\psi}_+,
\end{aligned} \tag{3.57}$$

where c_i, $i=0,..,15$ and g_j, $j=0,..,11$ are functions depending on ϕ_+ and its derivatives, and their explicit forms are given by (E.1)-(E.16) and (E.17)-(E.28), respectively. The equations (3.4)–(3.6), and (3.56) and (3.57) correspond to the super-Bäcklund transformations for the $n=2$ super equation. Cross differentiating (3.56) and (3.57) with respect to x we recover the equations of motion (2.54) and (2.55) after using equations. (3.4)–(3.6).

It can be also shown that the Bäcklund equations (3.4)–(3.6), (3.52), (3.53), (3.56) and (3.57), are invariant under the supersymmetry transformations (2.57) if the auxiliary fermionic field f_1 transforms in the following way,

$$\delta f_1 = \frac{2\bar{\epsilon}}{\omega\sqrt{i}} \sinh\left(\frac{\phi_+}{2}\right). \tag{3.58}$$

3.4 Superspace formalism

Let us now discuss the super Bäcklund transformations from the superfield point of view. We start by introducing the fermionic superfield $\Psi(x,\theta) = \sqrt{i}\bar{\psi}(x) + \theta u(x)$ to describe the supersymmetric extension of the mKdV equation [38],

$$D_{t_3}\Psi = D^6\Psi - 3(D\Psi)D^2(\Psi D\Psi), \tag{3.59}$$

where θ is a Grassmannian coordinate, $D = \partial_\theta + \theta\partial_x$ is the covariant super derivative, and we have defined $D_{t_3} = 4\partial_{t_3}$. In components, we recover equations (2.48) and (2.49), namely,

$$4\partial_{t_3}u \;=\; \partial_x^3 u - 6u^2\partial_x u + 3i\bar{\psi}\partial_x(u\partial_x\bar{\psi}), \tag{3.60}$$

$$4\partial_{t_3}\bar{\psi} \;=\; \partial_x^3\bar{\psi} - 3u\partial_x(u\bar{\psi}). \tag{3.61}$$

Let us now define a new bosonic superfield $\Phi(x,\theta) = \phi(x) - \sqrt{i}\theta\bar{\psi}(x)$ [39], such that $\Psi = -D\Phi$, or equivalently $u = -\partial_x\phi$. Substituting in eq (3.59), we get

$$D_{t_3}\Phi = D^6\Phi - 2(D^2\Phi)^3 + 3(D\Phi)(D^2\Phi)(D^3\Phi). \tag{3.62}$$

It is well-known that the spatial part of the super Bäcklund transformations for the hierarchy equations (3.4)–(3.6) can be derived from the following equations [40, 41],

$$D\Phi_- \;=\; \frac{4i}{\omega}\cosh\left(\frac{\Phi_+}{2}\right)\Sigma, \tag{3.63}$$

$$D\Sigma \;=\; -\frac{2i}{\omega}\sinh\left(\frac{\Phi_+}{2}\right), \tag{3.64}$$

where $\Phi_\pm = \Phi_1 \pm \Phi_2$, and $\Sigma = -\frac{1}{\sqrt{i}} f_1 + \theta b_1$ is a fermionic superfield, with f_1 and b_1 being auxiliary fermionic and bosonic fields, respectively. In components, we find

$$\begin{aligned}
\bar{\psi}_- &= \frac{4}{\omega}\cosh\left(\frac{\phi_+}{2}\right) f_1, && (3.65)\\
b_1 &= -\frac{2i}{\omega}\sinh\left(\frac{\phi_+}{2}\right), && (3.66)\\
\partial_x \phi_- &= \frac{4}{\omega^2}\sinh\phi_+ + \frac{2i}{\omega}\sinh\left(\frac{\phi_+}{2}\right)\bar{\psi}_+ f_1, && (3.67)\\
\partial_x f_1 &= \frac{1}{\omega}\cosh\left(\frac{\phi_+}{2}\right)\bar{\psi}_+. && (3.68)
\end{aligned}$$

Now, we propose the following supersymmetric extension to the "temporal" part of the super Bäcklund transformation for the smKdV ($n = 1$) equation in the superspace,

$$\begin{aligned}
D_{t_3}\Phi_- &= \frac{i}{\omega}\cosh\left(\frac{\Phi_+}{2}\right)\left[D^4\Phi_+ D\Phi_+ - D^2\Phi_+ D^3\Phi_+\right]\Sigma\\
&+ \frac{i}{\omega}\sinh\left(\frac{\Phi_+}{2}\right)\left[2D^5\Phi_+ - (D^2\Phi_+)^2(D\Phi_+)\right]\Sigma\\
&+ \frac{2}{\omega^2}\sinh\Phi_+\left[(D\Phi_+)(D^3\Phi_+) - (D^2\Phi_+)^2\right] + \frac{4}{\omega^2}\cosh\Phi_+(D^4\Phi_+)\\
&- \frac{96i}{\omega^5}\left[\sinh\left(\frac{\Phi_+}{2}\right) + 4\sinh^3\left(\frac{\Phi_+}{2}\right) + 3\sinh^5\left(\frac{\Phi_+}{2}\right)\right](D\Phi_+)\Sigma\\
&- \frac{32}{\omega^6}\sinh^3\Phi_+, \qquad (3.69)\\
D_{t_3}\Sigma &= \frac{i}{2\omega}\cosh\left(\frac{\Phi_+}{2}\right)\left[(D\Phi_+)(D^2\Phi_+)^2 - 2(D^5\Phi_+)\right]\\
&+ \frac{i}{2\omega}\sinh\left(\frac{\Phi_+}{2}\right)\left[(D^2\Phi_+)(D^3\Phi_+) - (D\Phi_+)(D^4\Phi_+)\right]\\
&- \frac{12}{\omega^4}\sinh\Phi_+\cosh^2\left(\frac{\Phi_+}{2}\right)(D^2\Phi_+)\Sigma + \frac{12i}{\omega^5}\sinh^2\Phi_+\cosh\left(\frac{\Phi_+}{2}\right)(D\Phi_+). \qquad (3.70)
\end{aligned}$$

By cross-differentiating eqs. (3.63) and (3.69) we find that if Φ_1 satisfies the smKdV equation (3.62), then Φ_2 also satisfies it. Explicitly, in components we get

$$\begin{aligned} 4\partial_{t_3}\phi_- &= \frac{i}{\omega}\left[\partial_x^2\phi_+\cosh\left(\frac{\phi_+}{2}\right)-(\partial_x\phi_+)^2\sinh\left(\frac{\phi_+}{2}\right)\right]\bar{\psi}_+f_1 \\ &-\frac{i}{\omega}\left[\partial_x\phi_+\cosh\left(\frac{\phi_+}{2}\right)\partial_x\bar{\psi}_+-2\sinh\left(\frac{\phi_+}{2}\right)\partial_x^2\bar{\psi}_+\right]f_1 \\ &+\frac{2}{\omega^2}\Big[2(\partial_x^2\phi_+)\cosh\phi_+-(\partial_x\phi_+)^2\sinh\phi_++i\bar{\psi}_+(\partial_x\bar{\psi}_+)\sinh\phi_+\Big] \\ &-\frac{96i}{\omega^5}\left[\sinh\left(\frac{\phi_+}{2}\right)+4\sinh^3\left(\frac{\phi_+}{2}\right)+3\sinh^5\left(\frac{\phi_+}{2}\right)\right]\bar{\psi}_+f_1-\frac{32}{\omega^6}\sinh^3\phi_+, \end{aligned} \tag{3.71}$$

$$\begin{aligned} 4\partial_{t_3}\bar{\psi}_- &= -\frac{1}{\omega}\sinh\left(\frac{\phi_+}{2}\right)\left[(\partial_x\phi_+)^3-2\partial_x^3\phi_+-\frac{3}{2}(\partial_x\phi_+)(\bar{\psi}_+\partial_x\bar{\psi}_+)\right]f_1 \\ &+\frac{3}{\omega^2}\sinh\phi_+\left[\bar{\psi}_+\partial_x^2\phi_+-(\partial_x\phi_+)(\partial_x\bar{\psi}_+)\right] \\ &+\frac{2}{\omega^2}\left[1+3\sinh^2\left(\frac{\phi_+}{2}\right)\right]\left[2\partial_x^2\bar{\psi}_+-\bar{\psi}_+(\partial_x\phi_+)^2\right] \\ &-\frac{96}{\omega^5}\left[\sinh\left(\frac{\phi_+}{2}\right)+4\sinh^3\left(\frac{\phi_+}{2}\right)+3\sinh^5\left(\frac{\phi_+}{2}\right)\right](\partial_x\phi_+)f_1 \\ &+\frac{24}{\omega^6}\sinh^2\phi_+\big[1-7\cosh\phi_+\big]\bar{\psi}_+, \end{aligned} \tag{3.72}$$

$$\begin{aligned} 4\partial_{t_3}f_1 &= \frac{1}{2\omega}\cosh\left(\frac{\phi_+}{2}\right)\left[2\partial_x^2\bar{\psi}_+-\bar{\psi}_+(\partial_x\phi_+)^2\right]+\frac{1}{2\omega}\sinh\left(\frac{\phi_+}{2}\right)\bar{\psi}_+\partial_x^2\phi_+ \\ &-\frac{1}{2\omega}\sinh\left(\frac{\phi_+}{2}\right)\partial_x\phi_+\partial_x\bar{\psi}_+-\frac{12}{\omega^4}\sinh\phi_+\cosh^2\left(\frac{\phi_+}{2}\right)(\partial_x\phi_+)f_1 \\ &+\frac{12}{\omega^5}\sinh^2\phi_+\cosh\left(\frac{\phi_+}{2}\right)\bar{\psi}_+, \end{aligned} \tag{3.73}$$

$$\begin{aligned} 4\partial_{t_3}b_1 &= \frac{i}{2\omega}\cosh\left(\frac{\phi_+}{2}\right)\left[(\partial_x\phi_+)^3-2\partial_x^3\phi_+-\frac{3i}{2}(\partial_x\phi_+)(\bar{\psi}_+\partial_x\bar{\psi}_+)\right] \\ &+\frac{12}{\omega^4}\sinh\phi_+\cosh^2\left(\frac{\phi_+}{2}\right)f_1\partial_x\bar{\psi}_++\frac{6}{\omega^4}\Big[\cosh\phi_++\cosh(2\phi_+)\Big]\bar{\psi}_+f_1\partial_x\phi_+. \end{aligned} \tag{3.74}$$

We note that eqs. (3.72) and (3.74) can be derived from eqs. (3.71) and (3.73), and appear here only for consistency.

Chapter 4

Recursion operator for the smKdV hierarchy

In the chapter 2 we considered the construction of a set of super integrable equations, which constitutes the smKdV hierarchy. We have observed that these equations present a similar structure which makes interesting to try to relate them. In this chapter we will show that the connection of the equations within the smKdV hierarchy is achieved through the recursion operator. In addition, we propose an alternative derivation for the Bäcklund transformation obtained in the previous chapter by employing a recursion operator.

4.1 Rercursion operator for the super integrable equations of motion

A super integrable equation is constructed by solving the system in (2.35). Since the solution of this set of equations is similar for all n, it is expected the existence of a connection among the time evolution equations. The recursion operator is the mathematical object responsible for such connection and it will be constructed in this section.

In order to see this we consider the equations in (2.35)-(2.38) for $N = 2n + 1$

and $N = 2n + 3$

$$
\begin{array}{lll}
\mathbf{2n+1} & \mathbf{2n+3} & \\
c_{2n+1} = 0, & c_{2n+3} = 0 & (4.1) \\
a_{2n+1} = b_{2n+1} = 1, & a_{2n+3} = b_{2n+3} = 1, & (4.2) \\
\beta_{2n+\frac{1}{2}} = \bar{\psi} & \beta_{2n+\frac{5}{2}} = \bar{\psi} & (4.3) \\
d_{2n} = u + \bar{\psi}\alpha_{2n+\frac{1}{2}}, & d_{2n+2} = u + \bar{\psi}\alpha_{2n+\frac{5}{2}} & (4.4) \\
\partial_x \alpha_{2n+\frac{1}{2}} - u\beta_{2n+\frac{1}{2}} + \bar{\psi} d_{2n} = 0, & \partial_x \alpha_{2n+\frac{5}{2}} - u\beta_{2n+\frac{5}{2}} + \bar{\psi} d_{2n+2} = 0 & (4.5) \\
\partial_x \beta_{2n+\frac{1}{2}} - u\alpha_{2n+\frac{1}{2}} + 2\delta_{2n-\frac{1}{2}} = 0, & \partial_x \beta_{2n+\frac{5}{2}} - u\alpha_{2n+\frac{5}{2}} + 2\delta_{2n+\frac{3}{2}} = 0 & (4.6) \\
\partial_x d_{2n} - 2c_{2n-1} + 2\bar{\psi}\gamma_{2n-\frac{1}{2}} = 0, & \partial_x d_{2n+2} - 2c_{2n+1} + 2\bar{\psi}\gamma_{2n+\frac{3}{2}} = 0 & (4.7) \\
\vdots & \vdots & \\
\partial_x \alpha_{1/2} - u\beta_{1/2} + \bar{\psi} d_0 = 0, & \partial_x \alpha_{5/2} - u\beta_{5/2} + \bar{\psi} d_2 = 0 & (4.8) \\
 & \partial_x \beta_{5/2} - u\alpha_{5/2} + 2\delta_{3/2} = 0 & (4.9) \\
 & \partial_x d_2 - 2c_1 + 2\bar{\psi}\gamma_{3/2} = 0 & (4.10) \\
 & \partial_x \gamma_{3/2} - u\delta_{3/2} + \bar{\psi} c_1 = 0 & (4.11) \\
 & \partial_x \delta_{3/2} - u\gamma_{3/2} - \bar{\psi}(a_1 + b_1) + 2\beta_{1/2} = 0 & (4.12) \\
 & \partial_x a_1 - 2\bar{\psi}\beta_{1/2} + 2uc_1 = 0 & (4.13) \\
 & \partial_x b_1 + 2\bar{\psi}\beta_{1/2} = 0 & (4.14) \\
 & \partial_x c_1 - 2d_0 + 2ua_1 + 2\bar{\psi}\alpha_{1/2} = 0 & (4.15) \\
 & \partial_x \alpha_{1/2} - u\beta_{1/2} + \bar{\psi} d_0 = 0 & (4.16) \\
\partial_{t_{2n+1}} \bar{\psi} = \partial_x \beta_{1/2} - u\alpha_{1/2}, & \partial_{t_{2n+3}} \bar{\psi} = \partial_x \beta_{1/2} - u\alpha_{1/2} & (4.17) \\
\partial_{t_{2n+1}} u = \partial_x d_0, & \partial_{t_{2n+3}} u = \partial_x d_0 & (4.18)
\end{array}
$$

Notice that until the equation (4.9) the aligned equations have the same solution, in such a way that we can make the following useful identifications,

$$
d_2\Big|_{2n+3} = d_0\Big|_{2n+1}, \qquad \beta_{5/2}\Big|_{2n+3} = \beta_{1/2}\Big|_{2n+1}, \qquad \alpha_{5/2}\Big|_{2n+3} = \alpha_{1/2}\Big|_{2n+1}. \tag{4.19}
$$

The case for $N = 2n+3$ has eight additional equations (4.9)-(4.16), which can be solved in terms of the coefficients for $N = 2n+1$ by the relations (4.19). Then we will be able to relate the time evolution equations for t_{2n+3} to the time evolution equations for t_{2n+1}.

Starting with the equation (4.9) by using (4.19) we get

$$\delta_{3/2}\Big|_{2n+3} = -\frac{1}{2}\partial_{t_{2n+1}}\bar{\psi} \tag{4.20}$$

Substituting in (4.10) and (4.11)

$$c_1\Big|_{2n+3} = \frac{1}{2}\partial_{t_{2n+1}}u + \bar{\psi}\gamma_{3/2}\Big|_{2n+3} \tag{4.21}$$

$$\gamma_{3/2}\Big|_{2n+3} = -\frac{1}{2}\int dx\, \partial_{t_{2n+1}}(u\bar{\psi}) \tag{4.22}$$

Recursively solving the equations (4.12)-(4.16) we get the following coefficients,

$$\beta_{1/2}\Big|_{2n+3} = \frac{1}{4}\partial_x\partial_{t_{2n+1}}\bar{\psi} - \frac{u}{4}\int dx\, \partial_{t_{2n+1}}(u\bar{\psi}) + \frac{1}{2}\bar{\psi}(a_1+b_1)\Big|_{2n+3} \tag{4.23}$$

$$a_1\Big|_{2n+3} = -\int dx\, u\partial_{t_{2n+1}}u + \frac{1}{2}\int dx\, \bar{\psi}\partial_x\partial_{t_{2n+1}}\bar{\psi} + \frac{1}{2}\int dx'\, u\bar{\psi}\int dx\, \partial_{t_{2n+1}}(u\bar{\psi}) \tag{4.24}$$

$$b_1\Big|_{2n+3} = -\frac{1}{2}\int dx\, \bar{\psi}\partial_x\partial_{t_{2n+1}}\bar{\psi} + \frac{1}{2}\int dx'\, u\bar{\psi}\int dx\, \partial_{t_{2n+1}}(u\bar{\psi}) \tag{4.25}$$

$$\begin{aligned} d_0\Big|_{2n+3} = {} & \frac{1}{4}\partial_x\partial_{t_{2n+1}}u - \frac{1}{4}\bar{\psi}\partial_{t_{2n+1}}(u\bar{\psi}) - u\int dx\, u\partial_{t_{2n+1}}u + \frac{u}{2}\int dx\, \bar{\psi}\partial_x\partial_{t_{2n+1}}\bar{\psi} \\ & - \frac{\partial_x\bar{\psi}}{4}\int dx\, \partial_{t_{2n+1}}(u\bar{\psi}) + \frac{u}{2}\int dx'\, u\bar{\psi}\int dx\, \partial_{t_{2n+1}}(u\bar{\psi}) + \bar{\psi}\alpha_{1/2}\Big|_{2n+3} \end{aligned} \tag{4.26}$$

$$\begin{aligned} \alpha_{1/2}\Big|_{2n+3} = {} & \frac{1}{4}\int dx\, (u\partial_x\partial_{t_{2n+1}}\bar{\psi} - \bar{\psi}\partial_x\partial_{t_{2n+1}}u) + \frac{1}{4}\int dx'\, \bar{\psi}\partial_x\bar{\psi}\int dx\, \partial_{t_{2n+1}}(u\bar{\psi}) \\ & + \frac{1}{2}\int dx'\, u\bar{\psi}\int dx\, (u\partial_{t_{2n+1}}u - \bar{\psi}\partial_x\partial_{t_{2n+1}}\bar{\psi}) - \frac{1}{4}\int dx'\, u^2\int dx\, \partial_{t_{2n+1}}(u\bar{\psi}) \end{aligned} \tag{4.27}$$

Finally putting these coefficients in the equations of motion (4.17) and (4.18) we obtain that the t_{2n+3} equation of the smKdV hierarchy is given by

$$\frac{\partial u}{\partial t_{2n+3}} = R_1\frac{\partial u}{\partial t_{2n+1}} + R_2\frac{\partial\bar{\psi}}{\partial t_{2n+1}}, \qquad \frac{\partial\bar{\psi}}{\partial t_{2n+3}} = R_3\frac{\partial u}{\partial t_{2n+1}} + R_4\frac{\partial\bar{\psi}}{\partial t_{2n+1}} \tag{4.28}$$

where $\{R_1, R_4\}, \{R_2, R_3\}$ are the bosonic and fermionic recursion operators, respectively, and are given by

$$\begin{aligned} R_1 = {} & \frac{1}{4}\mathbb{D}^2 - u^2 - \partial_x u\mathbb{D}^{-1}u + \frac{i}{4}\bar{\psi}\partial_x\bar{\psi} + \frac{i}{4}u^2\bar{\psi}\mathbb{D}^{-1}\bar{\psi} + \frac{i}{2}\partial_x u\mathbb{D}^{-1}u\bar{\psi}\mathbb{D}^{-1}\bar{\psi} - \frac{i}{4}\partial_x^2\bar{\psi}\mathbb{D}^{-1}\bar{\psi} \\ & - \frac{i}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\mathbb{D} - \frac{i}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}u^2\mathbb{D}^{-1}\bar{\psi} + \frac{i}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}u\bar{\psi}\mathbb{D}^{-1}u - \frac{1}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi}, \end{aligned} \tag{4.29}$$

$$\begin{aligned} R_2 = {} & \frac{i}{2}u\bar{\psi}\mathbb{D} - \frac{i}{2}u\partial_x\bar{\psi} - \frac{i}{4}\partial_x u\bar{\psi} + \frac{i}{4}u^2\bar{\psi}\mathbb{D}^{-1}u + \frac{i}{2}\partial_x u\mathbb{D}^{-1}\bar{\psi}\mathbb{D} + \frac{i}{2}\partial_x u\mathbb{D}^{-1}u\bar{\psi}\mathbb{D}^{-1}u \\ & - \frac{i}{4}\partial_x^2\bar{\psi}\mathbb{D}^{-1}u + \frac{i}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}u\mathbb{D} - \frac{i}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}u^2\mathbb{D}^{-1}u + \frac{1}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}u\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\mathbb{D} \\ & - \frac{1}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\partial_x\bar{\psi}\mathbb{D}^{-1}u, \end{aligned} \tag{4.30}$$

$$\begin{aligned} R_3 = {} & -\frac{3}{4}u\bar{\psi} - \frac{1}{4}\partial_x u\mathbb{D}^{-1}\bar{\psi} - \frac{1}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}u + \frac{i}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}u\bar{\psi}\mathbb{D}^{-1}\bar{\psi} + \frac{1}{4}u\mathbb{D}^{-1}\bar{\psi}\mathbb{D} \\ & + \frac{1}{4}u\mathbb{D}^{-1}u^2\mathbb{D}^{-1}\bar{\psi} - \frac{1}{2}u\mathbb{D}^{-1}u\bar{\psi}\mathbb{D}^{-1}u - \frac{i}{4}u\mathbb{D}^{-1}\bar{\psi}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi}, \end{aligned} \tag{4.31}$$

$$\begin{aligned} R_4 = {} & \frac{1}{4}\mathbb{D}^2 - \frac{1}{4}u^2 - \frac{1}{4}\partial_x u\mathbb{D}^{-1}u - \frac{1}{4}u\mathbb{D}^{-1}u\mathbb{D} + \frac{1}{4}u\mathbb{D}^{-1}u^2\mathbb{D}^{-1}u + \frac{i}{2}u\mathbb{D}^{-1}u\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\mathbb{D} \\ & - \frac{i}{4}u\mathbb{D}^{-1}\bar{\psi}\partial_x\bar{\psi}\mathbb{D}^{-1}u + \frac{i}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}u\bar{\psi}\mathbb{D}^{-1}u, \end{aligned} \tag{4.32}$$

where $\mathbb{D} = \partial_x$ and $\mathbb{D}^{-1}$ is its inverse.

In the bosonic limit, when the fermions vanish, the (4.29) corresponds to the recursion operator obtained in [37] for mKdV equation.

We have explicitly checked that by employing equation (4.28) for $n = 0$ we recover the smKdV equation (2.48), (2.49). Also it was verified that (4.28) for $n = 1$, yields the t_5 flow of the hierarchy (2.54), (2.55) as predicted.

In order to make the notation useful for the next section (where we will deal with Bäcklund transformation) we will rewrite (4.28) in terms of $\partial_x\phi$, so

$$\frac{\partial\phi}{\partial t_{2n+3}} = \mathbb{R}_1\frac{\partial\phi}{\partial t_{2n+1}} + \mathbb{R}_2\frac{\partial\bar{\psi}}{\partial t_{2n+1}}, \qquad \frac{\partial\bar{\psi}}{\partial t_{2n+3}} = \mathbb{R}_3\frac{\partial\phi}{\partial t_{2n+1}} + \mathbb{R}_4\frac{\partial\bar{\psi}}{\partial t_{2n+1}} \tag{4.33}$$

where $\mathbb{R}_1 = \mathbb{D}^{-1}\mathcal{R}_1\mathbb{D}$, $\mathbb{R}_2 = \mathbb{D}^{-1}\mathcal{R}_2$, $\mathbb{R}_3 = \mathcal{R}_3\mathbb{D}$, $\mathbb{R}_4 = \mathcal{R}_4$, with

$$\begin{aligned}\mathcal{R}_1 = {} & \frac{1}{4}\mathbb{D}^2 - (\partial_x\phi)^2 - \partial_x^2\phi\mathbb{D}^{-1}\partial_x\phi + \frac{i}{4}\bar{\psi}\partial_x\bar{\psi} + \frac{i}{4}(\partial_x\phi)^2\bar{\psi}\mathbb{D}^{-1}\bar{\psi} + \frac{i}{2}\partial_x^2\phi\mathbb{D}^{-1}\partial_x\phi\bar{\psi}\mathbb{D}^{-1}\bar{\psi} \\ & - \frac{i}{4}\partial_x^2\bar{\psi}\mathbb{D}^{-1}\bar{\psi} - \frac{i}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\mathbb{D} - \frac{i}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}(\partial_x\phi)^2\mathbb{D}^{-1}\bar{\psi} + \frac{i}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}\partial_x\phi\bar{\psi}\mathbb{D}^{-1}\partial_x\phi \\ & - \frac{1}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi},\end{aligned} \tag{4.34}$$

$$\begin{aligned}\mathcal{R}_2 = {} & \frac{i}{2}\partial_x\phi\bar{\psi}\mathbb{D} - \frac{i}{2}\partial_x\phi\partial_x\bar{\psi} - \frac{i}{4}\partial_x^2\phi\bar{\psi} + \frac{i}{4}(\partial_x\phi)^2\bar{\psi}\mathbb{D}^{-1}\partial_x\phi + \frac{i}{2}\partial_x^2\phi\mathbb{D}^{-1}\bar{\psi}\mathbb{D} \\ & - \frac{i}{4}\bar{\psi}_{2x}\mathbb{D}^{-1}\partial_x\phi + \frac{i}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}\partial_x\phi\mathbb{D} - \frac{i}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}(\partial_x\phi)^2\mathbb{D}^{-1}\partial_x\phi \\ & + \frac{1}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}\partial_x\phi\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\mathbb{D} - \frac{1}{4}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\partial_x\bar{\psi}\mathbb{D}^{-1}\partial_x\phi + \frac{i}{2}\partial_x^2\phi\mathbb{D}^{-1}\partial_x\phi\bar{\psi}\mathbb{D}^{-1}\partial_x\phi,\end{aligned} \tag{4.35}$$

$$\begin{aligned}\mathcal{R}_3 = {} & -\frac{3}{4}\partial_x\phi\bar{\psi} - \frac{1}{4}\partial_x^2\phi\mathbb{D}^{-1}\bar{\psi} - \frac{1}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}\partial_x\phi + \frac{i}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}\partial_x\phi\bar{\psi}\mathbb{D}^{-1}\bar{\psi} \\ & + \frac{1}{4}\partial_x\phi\mathbb{D}^{-1}\bar{\psi}\mathbb{D} - \frac{1}{2}\partial_x\phi\mathbb{D}^{-1}\partial_x\phi\bar{\psi}\mathbb{D}^{-1}\partial_x\phi - \frac{i}{4}\partial_x\phi\mathbb{D}^{-1}\bar{\psi}\partial_x\bar{\psi}\mathbb{D}^{-1}\bar{\psi} \\ & + \frac{1}{4}\partial_x\phi\mathbb{D}^{-1}(\partial_x\phi)^2\mathbb{D}^{-1}\bar{\psi},\end{aligned} \tag{4.36}$$

$$\begin{aligned}\mathcal{R}_4 = {} & \frac{1}{4}\mathbb{D}^2 - \frac{1}{4}(\partial_x\phi)^2 - \frac{1}{4}\partial_x^2\phi\mathbb{D}^{-1}\partial_x\phi + \frac{1}{4}\partial_x\phi\mathbb{D}^{-1}(\partial_x\phi)^2\mathbb{D}^{-1}\partial_x\phi - \frac{1}{4}\partial_x\phi\mathbb{D}^{-1}\partial_x\phi\mathbb{D} \\ & - \frac{i}{4}\partial_x\phi\mathbb{D}^{-1}\bar{\psi}\partial_x\bar{\psi}\mathbb{D}^{-1}\partial_x\phi + \frac{i}{2}\partial_x\bar{\psi}\mathbb{D}^{-1}\partial_x\phi\bar{\psi}\mathbb{D}^{-1}\partial_x\phi + \frac{i}{2}\partial_x\phi\mathbb{D}^{-1}\partial_x\phi\bar{\psi}\mathbb{D}^{-1}\bar{\psi}\mathbb{D} \\ & .\end{aligned} \tag{4.37}$$

4.2 Recursion operator for the Bäcklund transformations

In this section we will extend the idea of recursion operator to generate the Bäcklund transformation for smKdV hierarchy in an alternative way.

In order to construct the recursion operator for the Bäcklund transformations we consider two different solutions of the equation (4.33) as

$$\frac{\partial \phi_1}{\partial t_{2n+3}} = \mathbb{R}_1^{(1)} \frac{\partial \phi_1}{\partial t_{2n+1}} + \mathbb{R}_2^{(1)} \frac{\partial \bar{\psi}_1}{\partial t_{2n+1}}, \qquad \frac{\partial \bar{\psi}_1}{\partial t_{2n+3}} = \mathbb{R}_3^{(1)} \frac{\partial \phi_1}{\partial t_{2n+1}} + \mathbb{R}_4^{(1)} \frac{\partial \bar{\psi}_1}{\partial t_{2n+1}}, \tag{4.38}$$

$$\frac{\partial \phi_2}{\partial t_{2n+3}} = \mathbb{R}_1^{(2)} \frac{\partial \phi_2}{\partial t_{2n+1}} + \mathbb{R}_2^{(2)} \frac{\partial \bar{\psi}_2}{\partial t_{2n+1}}, \qquad \frac{\partial \bar{\psi}_2}{\partial t_{2n+3}} = \mathbb{R}_3^{(2)} \frac{\partial \phi_2}{\partial t_{2n+1}} + \mathbb{R}_4^{(2)} \frac{\partial \bar{\psi}_2}{\partial t_{2n+1}} \tag{4.39}$$

where $\mathbb{R}_i^{(p)} = \mathbb{R}_i \left(\partial_x \phi_p, \partial_x^2 \phi_p, \bar{\psi}_p, \partial_x \bar{\psi}_p, \partial_x^2 \bar{\psi}_p\right)$, $i = 1, ..., 4$, $p = 1, 2$. And take the following combination of these solutions

$$\begin{aligned} 2\partial_{t_{2n+3}} \phi_- = & \left(\mathbb{R}_1^{(1)} + \mathbb{R}_1^{(2)}\right) \partial_{t_{2n+1}} \phi_- + \left(\mathbb{R}_2^{(1)} + \mathbb{R}_2^{(2)}\right) \partial_{t_{2n+1}} \bar{\psi}_- \\ & + \left(\mathbb{R}_1^{(1)} - \mathbb{R}_1^{(2)}\right) \partial_{t_{2n+1}} \phi_+ + \left(\mathbb{R}_2^{(1)} - \mathbb{R}_2^{(2)}\right) \partial_{t_{2n+1}} \bar{\psi}_+, \end{aligned} \tag{4.40}$$

$$\begin{aligned} 2\partial_{t_{2n+3}} \bar{\psi}_- = & \left(\mathbb{R}_3^{(1)} + \mathbb{R}_3^{(2)}\right) \partial_{t_{2n+1}} \phi_- + \left(\mathbb{R}_4^{(1)} + \mathbb{R}_4^{(2)}\right) \partial_{t_{2n+1}} \bar{\psi}_- \\ & + \left(\mathbb{R}_3^{(1)} - \mathbb{R}_3^{(2)}\right) \partial_{t_{2n+1}} \phi_+ + \left(\mathbb{R}_4^{(1)} - \mathbb{R}_4^{(2)}\right) \partial_{t_{2n+1}} \bar{\psi}_+ \end{aligned} \tag{4.41}$$

where we introduced the new variables $\phi_\pm = \phi_1 \pm \phi_2$ and $\bar{\psi}_\pm = \bar{\psi}_1 \pm \bar{\psi}_2$.

At this point, we conjecture that the equations (4.40) and (4.41) correspond to the temporal part of the super Bäcklund transformation for a super integrable equation especified by n. We note that as well as the consecutive equations of motion within the hierarchy are connected by the same recursion operator, here the same occurs to the Bäcklund transformations. In order to clarify such hypothesis we will consider next some examples.

For $n = 0$ we have

$$
\begin{aligned}
2\partial_{t_3}\phi_- &= \left(\mathbb{R}_1^{(1)} + \mathbb{R}_1^{(2)}\right)\partial_{t_1}\phi_- + \left(\mathbb{R}_2^{(1)} + \mathbb{R}_2^{(2)}\right)\partial_{t_1}\bar{\psi}_- \\
&+ \left(\mathbb{R}_1^{(1)} - \mathbb{R}_1^{(2)}\right)\partial_{t_1}\phi_+ + \left(\mathbb{R}_2^{(1)} - \mathbb{R}_2^{(2)}\right)\partial_{t_1}\bar{\psi}_+, \qquad (4.42)\\
2\partial_{t_3}\bar{\psi}_- &= \left(\mathbb{R}_3^{(1)} + \mathbb{R}_3^{(2)}\right)\partial_{t_1}\phi_- + \left(\mathbb{R}_4^{(1)} + \mathbb{R}_4^{(2)}\right)\partial_{t_1}\bar{\psi}_- \\
&+ \left(\mathbb{R}_3^{(1)} - \mathbb{R}_3^{(2)}\right)\partial_{t_1}\phi_+ + \left(\mathbb{R}_4^{(1)} - \mathbb{R}_4^{(2)}\right)\partial_{t_1}\bar{\psi}_+. \qquad (4.43)
\end{aligned}
$$

By using (3.49)-(3.51) the equations above are

$$
\begin{aligned}
4\partial_{t_3}\phi_- &= \partial_x^3\phi_- - \frac{3}{2}\left(\partial_x\phi_+\right)^2\partial_x\phi_- - \frac{1}{2}(\partial_x\phi_-)^3 + \frac{3i}{4}\partial_x\phi_+\bar{\psi}_+\partial_x\bar{\psi}_- + \frac{3i}{4}\partial_x\phi_-\bar{\psi}_+\partial_x\bar{\psi}_+ \\
&+ \frac{3i}{4}\partial_x\phi_+\bar{\psi}_-\partial_x\bar{\psi}_+ + \frac{3i}{4}\partial_x\phi_-\bar{\psi}_-\partial_x\bar{\psi}_-, \qquad (4.44)\\
4\partial_{t_3}\bar{\psi}_- &= \partial_x^3\bar{\psi}_- - \frac{3}{2}\partial_x\phi_+\partial_x\phi_-\partial_x\bar{\psi}_+ - \frac{3}{4}(\partial_x\phi_+)^2\partial_x\bar{\psi}_- - \frac{3}{4}(\partial_x\phi_-)^2\partial_x\bar{\psi}_- \\
&- \frac{3}{4}\partial_x^2\phi_-\partial_x\phi_+\bar{\psi}_+ - \frac{3}{4}\partial_x^2\phi_+\partial_x\phi_-\bar{\psi}_+ - \frac{3}{4}\partial_x^2\phi_+\partial_x\phi_+\bar{\psi}_- - \frac{3}{4}\partial_x^2\phi_-\partial_x\phi_-\bar{\psi}_-. \\
&\qquad (4.45)
\end{aligned}
$$

Finally using the x-part of the Bäcklund transformations (3.4)-(3.6) we recovered the equations (3.52) and (3.53), i.e the time component of the Bäcklund transformantions for $n = 2$ (smKdV).

Next we consider the case for $n = 2$ and using again (3.52)-(3.53) we obtain from

(4.40),

$$\begin{aligned}
16\partial_{t_5}\phi_- &= \partial_x^5\phi_- + \frac{3}{8}\left(\partial_x\phi_-\right)^5 - \frac{5}{2}\partial_x\phi_-\left(\left(\partial_x^2\phi_-\right)^2 + \left(\partial_x^2\phi_+\right)^2\right) - 5\partial_x^2\phi_-\partial_x^2\phi_+\partial_x\phi_+ \\
&+ 5\partial_x\phi_-\partial_x\phi_+\left(\frac{3}{8}\left(\partial_x\phi_+\right)^3 - \partial_x^3\phi_+\right) - \frac{5}{2}\partial_x^3\phi_-\left(\left(\partial_x\phi_-\right)^2 + \left(\partial_x\phi_+\right)^2\right) \\
&+ \frac{5i}{4}\left(\bar{\psi}_-\partial_x\bar{\psi}_- + \bar{\psi}_+\partial_x\bar{\psi}_+\right)\left[\partial_x^3\phi_- - \partial_x\phi_-\left(\left(\partial_x\phi_-\right)^2 + 3\left(\partial_x\phi_+\right)^2\right)\right] \\
&+ \frac{5i}{4}\left(\bar{\psi}_-\partial_x\bar{\psi}_+ + \bar{\psi}_+\partial_x\bar{\psi}_-\right)\left[\partial_x^3\phi_+ - \partial_x\phi_+\left(\left(\partial_x\phi_+\right)^2 + 3\left(\partial_x\phi_-\right)^2\right)\right] \\
&+ \frac{5i}{4}\bar{\psi}_+\left(\partial_x^3\bar{\psi}_-\partial_x\phi_+ + \partial_x^3\bar{\psi}_+\partial_x\phi_-\right) + \frac{5i}{4}\bar{\psi}_+\left(\partial_x^2\bar{\psi}_-\partial_x^2\phi_+ + \partial_x^2\bar{\psi}_+\partial_x^2\phi_-\right) \\
&+ \frac{15}{4}\left(\partial_x\phi_-\right)^3\left(\partial_x\phi_+\right)^2 + \frac{5i}{4}\bar{\psi}_-\left(\partial_x^2\phi_-\partial_x^2\bar{\psi}_- + \partial_x^2\phi_+\partial_x^2\bar{\psi}_+\right) \\
&+ \frac{5i}{4}\bar{\psi}_-\left(\partial_x\phi_-\partial_x^3\bar{\psi}_- + \partial_x\phi_+\partial_x^3\bar{\psi}_+\right)
\end{aligned} \tag{4.46}$$

And for the equation (4.41) we get

$$\begin{aligned}
16\partial_{t_5}\bar{\psi}_- &= \partial_x^5\bar{\psi}_- - \frac{5}{4}\bar{\psi}_-\left(\partial_x\phi_-\partial_x^4\phi_- + \partial_x\phi_+\partial_x^4\phi_+\right) - \frac{5}{4}\bar{\psi}_+\left(\partial_x\phi_-\partial_x^4\phi_+ + \partial_x\phi_+\partial_x^4\phi_-\right) \\
&- \frac{5}{4}\left(\bar{\psi}_-\partial_x^2\phi_+ + \bar{\psi}_+\partial_x^2\phi_-\right)\left[2\partial_x^3\phi_+ - \partial_x\phi_+\left(\left(\partial_x\phi_+\right)^2 + 3\left(\partial_x\phi_-\right)^2\right)\right] \\
&- \frac{5}{4}\left(\bar{\psi}_-\partial_x^2\phi_- + \bar{\psi}_+\partial_x^2\phi_+\right)\left[2\partial_x^3\phi_- - \partial_x\phi_-\left(\left(\partial_x\phi_-\right)^2 + 3\left(\partial_x\phi_+\right)^2\right)\right] \\
&- \frac{5}{8}\partial_x\bar{\psi}_-\partial_x\phi_-\left[6\partial_x^3\phi_- - \partial_x\phi_-\left(\left(\partial_x\phi_-\right)^2 + 6\left(\partial_x\phi_+\right)^2\right)\right] \\
&- \frac{5}{8}\partial_x\bar{\psi}_-\partial_x\phi_+\left(6\partial_x^3\phi_+ - \left(\partial_x\phi_+\right)^3\right) - \frac{5}{4}\partial_x\bar{\psi}_+\left(4\partial_x^2\phi_-\partial_x^2\phi_+ + 3\partial_x^3\phi_+\partial_x\phi\right) \\
&- \frac{5}{4}\partial_x\bar{\psi}_+\partial_x\phi_+\left[3\partial_x^3\phi_- - 2\left(\left(\partial_x\phi_-\right)^2 + \left(\partial_x\phi_+\right)^2\right)\right] \\
&- \frac{15}{4}\partial_x^2\bar{\psi}_+\left(\partial_x^2\phi_+\partial_x\phi_- + \partial_x^2\phi_-\partial_x\phi_+\right) - \frac{5}{4}\partial_x^3\bar{\psi}_-\left(\left(\partial_x\phi_-\right)^2 + \left(\partial_x\phi_+\right)^2\right) \\
&- \frac{5}{2}\partial_x^3\bar{\psi}_+\partial_x\phi_-\partial_x\phi_+ - \frac{5}{2}\partial_x\bar{\psi}_-\left(\left(\partial_x^2\phi_-\right)^2 + \left(\partial_x^2\phi_+\right)^2\right) \\
&- \frac{15}{4}\partial_x^2\bar{\psi}_-\left(\partial_x^2\phi_-\partial_x\phi_- + \partial_x^2\phi_+\partial_x\phi_+\right)
\end{aligned} \tag{4.47}$$

Now, using the x-part of the Bäcklund transformation (3.4)-(3.6) in these two equations we end up with the corresponding Bäcklund transformation for $n = 3$, namely (3.56) and (3.57).

Chapter 5

Conserved charges for the smKdV hierarchy

In this chapter we will explicitly construct generating functions for an infinite set of independent conserved quantities for the smKdV hierarchy.

5.1 Conservation laws

An integrable evolution equation within the smKdV hierarchy can be formulated as the compatibility condition of the following linear problem in the (x, t_{2n+1}) coordinates,

$$\partial_x \Omega(x, t_{2n+1}; \lambda) = -A_x(x, t_{2n+1}; \lambda)\Omega(x, t_{2n+1}; \lambda), \tag{5.1}$$

$$\partial_{t_{2n+1}} \Omega(x, t_{2n+1}; \lambda) = -A_{t_{2n+1}}(x, t_{2n+1}; \lambda)\Omega(x, t_{2n+1}; \lambda) \tag{5.2}$$

where $\Omega = (\Omega_1, \Omega_2, \varepsilon\, \Omega_3)^T$, with Ω_j bosonic components and ε is a fermionic parameter, and λ is the spectral parameter. The compatibility of the above linear system yields the zero curvature equation,

$$\partial_x A_{t_{2n+1}} - \partial_{t_{2n+1}} A_x + \left[A_x, A_{t_{2n+1}}\right] = 0. \tag{5.3}$$

Now, in order to construct a generating function for the conservation laws, we define the auxiliary functions $\Gamma_{21} = \Omega_2 \Omega_1^{-1}$ and $\Gamma_{31} = \varepsilon \Omega_3 \Omega_1^{-1}$. Then, by considering the auxiliary problem (5.1) and (5.2), we find the following conservation equation,

$$\partial_{t_{2n+1}}\Big[V_{11}+V_{12}\Gamma_{21}+V_{13}\Gamma_{31}\Big]=\partial_x\Big[U_{11}+U_{12}\Gamma_{21}+U_{13}\Gamma_{31}\Big], \tag{5.4}$$

where we have redefined $V=-A_x$ and $U=-A_{t_{2n+1}}$ for simplicity, and V_{ij}, U_{ij} are the matrix elements of the respective Lax pairs. The functions Γ_{21} and Γ_{31} satisfy the following Ricatti equations,

$$\partial_x\Gamma_{21}=V_{21}+(V_{22}-V_{11})\Gamma_{21}-V_{12}(\Gamma_{21})^2+V_{23}\Gamma_{31}-V_{13}\Gamma_{31}\Gamma_{21}, \tag{5.5}$$

$$\partial_x\Gamma_{31}=V_{31}+(V_{33}-V_{11})\Gamma_{31}+V_{32}\Gamma_{21}-V_{12}\Gamma_{21}\Gamma_{31}, \tag{5.6}$$

$$\partial_{t_{2n+1}}\Gamma_{21}=U_{21}+(U_{22}-U_{11})\Gamma_{21}-U_{12}(\Gamma_{21})^2+U_{23}\Gamma_{31}-U_{13}\Gamma_{31}\Gamma_{21}, \tag{5.7}$$

$$\partial_{t_{2n+1}}\Gamma_{31}=U_{31}+(U_{33}-U_{11})\Gamma_{31}+U_{32}\Gamma_{21}-U_{12}\Gamma_{21}\Gamma_{31}. \tag{5.8}$$

Therefore, the corresponding first generating function of the conserved charges is given by,

$$I_1=\int_{-\infty}^{\infty}dx\,[V_{11}+V_{12}\Gamma_{21}+V_{13}\Gamma_{31}]=\int_{-\infty}^{\infty}dx\,\Big[-\lambda^{1/2}+\partial_x\phi+\Gamma_{21}-\sqrt{i}\bar{\psi}\Gamma_{31}\Big]. \tag{5.9}$$

In order to get the explicit form for the conserved quantities, we consider the expansion of Γ_{21} and Γ_{31} in powers of the spectral parameter λ in such way that we can recursively solve the Riccati equations. Let us expand Γ_{21} and Γ_{31} as follow,

$$\Gamma_{21}=\sum_{n=-1}^{\infty}\lambda^{-n/2}\,\Gamma_{21}^{(n/2)},\qquad \Gamma_{31}=\sum_{n=0}^{\infty}\lambda^{-n/2}\,\Gamma_{31}^{(n/2)}. \tag{5.10}$$

By substituting these expansions into the Riccati equations (5.5) and (5.6) we find that the first coefficients are given by,

$$\Gamma_{21}^{(-1/2)}=1,\quad \Gamma_{21}^{(0)}=-\partial_x\phi,\quad \Gamma_{21}^{(1/2)}=\frac{1}{2}\partial_x^2\phi+\frac{1}{2}(\partial_x\phi)^2,\quad \Gamma_{31}^{(0)}=-\sqrt{i}\bar{\psi}, \tag{5.11}$$

$$\Gamma_{21}^{(1)}=-\frac{1}{2}\partial_x\phi\,\partial_x^2\phi-\frac{1}{4}\partial_x^3\phi-\frac{i}{4}\bar{\psi}\,\partial_x\bar{\psi}\,\partial_x\phi,\quad \Gamma_{31}^{(1/2)}=\frac{\sqrt{i}}{2}(\partial_x\bar{\psi}+\bar{\psi}\,\partial_x\phi), \tag{5.12}$$

$$\begin{aligned}\Gamma_{21}^{(3/2)}=\;&\frac{1}{8}(\partial_x^2\phi)^2+\frac{1}{8}\partial_x^4\phi+\frac{1}{4}\partial_x^3\phi\,\partial_x\phi-\frac{1}{4}\partial_x^2\phi\,(\partial_x\phi)^2-\frac{1}{8}(\partial_x\phi)^4+\frac{i}{4}\bar{\psi}\,\partial_x^2\bar{\psi}\,\partial_x\phi\\&+\frac{i}{4}\bar{\psi}\,\partial_x\bar{\psi}\,\partial_x^2\phi+\frac{i}{4}\bar{\psi}\,\partial_x\bar{\psi}\,(\partial_x\phi)^2,\end{aligned} \tag{5.13}$$

$$\Gamma_{31}^{(1)}=-\frac{\sqrt{i}}{4}(\partial_x^2\bar{\psi}+\partial_x\bar{\psi}\,\partial_x\phi+\bar{\psi}\,\partial_x^2\phi), \tag{5.14}$$

$$\begin{aligned}\Gamma_{21}^{(2)} &= -\frac{1}{8}\partial_x^2\phi\,\partial_x^3\phi - \frac{1}{16}\partial_x^5\phi + \frac{1}{2}(\partial_x^2\phi)^2\,\partial_x\phi - \frac{1}{8}\partial_x^4\phi\,\partial_x\phi + \frac{1}{4}(\partial_x\phi)^2\partial_x^3\phi \\ &\quad +\frac{1}{2}(\partial_x\phi)^3\,\partial_x^2\phi - \frac{5i}{16}\bar{\psi}\partial_x^2\bar{\psi}\partial_x^2\phi - \frac{3i}{16}\bar{\psi}\partial_x\bar{\psi}\partial_x^3\phi - \frac{3i}{16}\bar{\psi}\partial_x^3\bar{\psi}\partial_x\phi \\ &\quad -\frac{i}{8}\partial_x\bar{\psi}\partial_x^2\bar{\psi}\partial_x\phi - \frac{3i}{8}\bar{\psi}\partial_x\bar{\psi}\partial_x^2\phi\partial_x\phi + \frac{i}{8}\bar{\psi}\,\partial_x\bar{\psi}\,(\partial_x\phi)^3 - \frac{i}{4}\bar{\psi}\,\partial_x^2\bar{\psi}\,(\partial_x\phi)^2, \qquad (5.15)\\ \Gamma_{31}^{(3/2)} &= \frac{\sqrt{i}}{8}\left[\partial_x^3\bar{\psi} + \partial_x^2\bar{\psi}\partial_x\phi + \partial_x\bar{\psi}\partial_x^2\phi - \partial_x\bar{\psi}(\partial_x\phi)^2 + \bar{\psi}\partial_x^3\phi - \bar{\psi}\partial_x^2\phi\partial_x\phi - \bar{\psi}(\partial_x\phi)^3\right] \end{aligned} \tag{5.16}$$

By substituting the coefficients of the expansion of the auxiliary functions in eq. (5.9), we obtain the lowest non-trivial conserved charges $I_1^{(-n/2)}$, namely,

$$I_1^{(-1/2)} = \frac{1}{2}\int_{-\infty}^{\infty} dx\left[\partial_x^2\phi + (\partial_x\phi)^2 - i\bar{\psi}\,\partial_x\bar{\psi}\right], \tag{5.17}$$

$$I_1^{(-1)} = -\frac{1}{4}\int_{-\infty}^{\infty} dx\left[2\partial_x\phi\partial_x^2\phi + \partial_x^3\phi - i\bar{\psi}\,\partial_x^2\bar{\psi}\right], \tag{5.18}$$

$$\begin{aligned} I_1^{(-3/2)} = \frac{1}{8}\int_{-\infty}^{\infty} dx\,\Big[&(\partial_x^2\phi)^2 + \partial_x^4\phi + 2\partial_x^3\phi\partial_x\phi - 2\partial_x^2\phi(\partial_x\phi)^2 - (\partial_x\phi)^4 + i\bar{\psi}\,\partial_x\bar{\psi}\,\partial_x^2\phi \\ &+3i\bar{\psi}\,\partial_x\bar{\psi}\,(\partial_x\phi)^2 + i\bar{\psi}\,\partial_x^2\bar{\psi}\,\partial_x\phi - i\bar{\psi}\,\partial_x^3\bar{\psi}\Big]. \end{aligned} \tag{5.19}$$

Analogously, we can construct a second set of conserved quantities from the second conservation law,

$$\partial_{t_{2n+1}}\Big[V_{22} + V_{21}\Gamma_{12} + V_{23}\Gamma_{32}\Big] = \partial_x\Big[U_{22} + U_{21}\Gamma_{12} + U_{23}\Gamma_{32}\Big], \tag{5.20}$$

where the auxiliary fields $\Gamma_{12} = \Omega_1\Omega_2^{-1}$ and $\Gamma_{32} = \epsilon\Omega_3\Omega_2^{-1}$ now satisfy the Ricatti equations

$$\partial_x\Gamma_{12} = V_{12} - (V_{22} - V_{11})\Gamma_{12} - V_{21}(\Gamma_{12})^2 + V_{13}\Gamma_{32} - V_{23}\Gamma_{32}\Gamma_{12}, \tag{5.21}$$

$$\partial_x\Gamma_{32} = V_{32} + (V_{33} - V_{22})\Gamma_{32} + V_{31}\Gamma_{12} - V_{21}\Gamma_{12}\Gamma_{32}, \tag{5.22}$$

$$\partial_{t_{2n+1}}\Gamma_{12} = U_{12} - (U_{22} - U_{11})\Gamma_{12} - U_{21}(\Gamma_{12})^2 + U_{13}\Gamma_{32} - U_{23}\Gamma_{32}\Gamma_{12}, \tag{5.23}$$

$$\partial_{t_{2n+1}}\Gamma_{32} = U_{32} + (U_{33} - U_{22})\Gamma_{32} + U_{31}\Gamma_{12} - U_{21}\Gamma_{12}\Gamma_{32}. \tag{5.24}$$

Now, the second generating function of conserved charges reads,

$$I_2 = \int_{-\infty}^{\infty} \left[V_{22} + V_{21}\Gamma_{12} + V_{23}\Gamma_{32} \right] = \int_{-\infty}^{\infty} \left[-\lambda^{1/2} - \partial_x\phi + \lambda\Gamma_{12} - \sqrt{i}\lambda^{1/2}\bar{\psi}\Gamma_{32} \right]. \tag{5.25}$$

By using the expansions of Γ_{12} and Γ_{32} as

$$\Gamma_{12} = \sum_{n=1}^{\infty} \lambda^{-n/2}\,\Gamma_{12}^{(n/2)}, \qquad \Gamma_{32} = \sum_{n=0}^{\infty} \lambda^{-n/2}\,\Gamma_{32}^{(n/2)}, \tag{5.26}$$

we find

$$\Gamma_{12}^{(1/2)} = 1, \qquad \Gamma_{12}^{(1)} = \partial_x\phi, \qquad \Gamma_{32}^{(0)} = 0, \qquad \Gamma_{32}^{(1/2)} = -\sqrt{i}\bar{\psi}, \tag{5.27}$$

$$\Gamma_{12}^{(3/2)} = -\frac{1}{2}\partial_x^2\phi + \frac{1}{2}(\partial_x\phi)^2, \qquad \Gamma_{12}^{(2)} = \frac{1}{4}\partial_x^3\phi - \frac{1}{2}\partial_x\phi\partial_x^2\phi + \frac{i}{4}\bar{\psi}\,\partial_x\bar{\psi}\,\partial_x\phi, \tag{5.28}$$

$$\Gamma_{32}^{(1)} = \frac{\sqrt{i}}{2}(\partial_x\bar{\psi} - \bar{\psi}\,\partial_x\phi), \qquad \Gamma_{32}^{(3/2)} = \frac{\sqrt{i}}{4}(\bar{\psi}\,\partial_x^2\phi + \partial_x\bar{\psi}\,\partial_x\phi - \partial_x^2\bar{\psi}), \tag{5.29}$$

$$\begin{aligned}\Gamma_{12}^{(5/2)} = {} & \frac{1}{8}(\partial_x^2\phi)^2 - \frac{1}{8}\partial_x^4\phi + \frac{1}{4}\partial_x^3\phi\,\partial_x\phi + \frac{1}{4}\partial_x^2\phi\,(\partial_x\phi)^2 - \frac{1}{8}(\partial_x\phi)^4 - \frac{i}{4}\bar{\psi}\,\partial_x\bar{\psi}\,\partial_x^2\phi \\ & + \frac{i}{4}\bar{\psi}\,\partial_x\bar{\psi}\,(\partial_x\phi)^2 - \frac{i}{4}\bar{\psi}\,\partial_x^2\bar{\psi}\,\partial_x\phi,\end{aligned} \tag{5.30}$$

$$\Gamma_{32}^{(2)} = \frac{\sqrt{i}}{8}\left[\partial_x^3\bar{\psi} - \partial_x^2\bar{\psi}\partial_x\phi - \partial_x\bar{\psi}\partial_x^2\phi - \partial_x\bar{\psi}(\partial_x\phi)^2 - \bar{\psi}\partial_x^3\phi - \bar{\psi}\partial_x\phi\partial_x^2\phi + \bar{\psi}(\partial_x\phi)^3\right]. \tag{5.31}$$

Then, by substituting the above coefficients for the expansion of the auxiliary functions, we get the second set of non-trivial conserved quantities $I_2^{(-n/2)}$,

$$I_2^{(-1/2)} = -\frac{1}{2}\int_{-\infty}^{\infty} dx\,\left[\partial_x^2\phi - (\partial_x\phi)^2 + i\bar{\psi}\,\partial_x\bar{\psi}\right], \tag{5.32}$$

$$I_2^{(-1)} = -\frac{1}{4}\int_{-\infty}^{\infty} dx\,\left[2\partial_x\phi\,\partial_x^2\phi - \partial_x^3\phi - i\bar{\psi}\,\partial_x^2\bar{\psi}\right], \tag{5.33}$$

$$\begin{aligned}I_2^{(-3/2)} = {} & \frac{1}{8}\int_{-\infty}^{\infty} dx\,\left[(\partial_x^2\phi)^2 - \partial_x^4\phi + 2\partial_x^3\phi\partial_x\phi + 2\partial_x^2\phi(\partial_x\phi)^2 - (\partial_x\phi)^4 - i\bar{\psi}\,\partial_x\bar{\psi}\,\partial_x^2\phi \right. \\ & \left. + 3i\bar{\psi}\,\partial_x\bar{\psi}\,(\partial_x\phi)^2 - i\bar{\psi}\,\partial_x^2\bar{\psi}\,\partial_x\phi - i\bar{\psi}\,\partial_x^3\bar{\psi}\right].\end{aligned} \tag{5.34}$$

The canonical energy and momentum in the bulk theory for $n = 1$ member of the hierarchy are recovered by simple combinations of the conserved quantities derived above, namely

$$P = I_1^{(-1/2)} + I_2^{(-1/2)} = \int_{-\infty}^{\infty} dx \left[(\partial_x \phi)^2 - i\bar{\psi}\, \partial_x \bar{\psi} \right], \tag{5.35}$$

and

$$\begin{aligned} E &= I_1^{(-3/2)} + I_2^{(-3/2)} \\ &= \frac{1}{4}\int_{-\infty}^{\infty} dx \left[(\partial_x^2 \phi)^2 - (\partial_x \phi)^4 + 2\partial_x^3 \phi\, \partial_x \phi - i\bar{\psi}\, \partial_x^3 \bar{\psi} + 3i(\partial_x \phi)^2 \bar{\psi} \partial_x \bar{\psi} \right]. \end{aligned} \tag{5.36}$$

Note that if the fermions vanish we recover the canonical momentum and energy for the mKdV equation[6].

It is worth to point out that these two set of conserved quantities only depend on the x-part of the Lax ($V = -A_x$), which is commom for the entire hierarchy. Therefore, we should expect that these charges are conserved for all flows, in fact we will check its conservation with respect t_1, t_3 and t_5 in the next section.

5.2 Inspecting the conservation

In order to verify that the conservation is valid for the entire hierarchy we take the $n = 0$, $n = 1$ and $n = 2$ time derivatives of the charge (5.35) and (5.36).

$n = 0$

The t_1 derivative of (5.35) yields

$$\frac{dP}{dt_1} = \int_{-\infty}^{\infty} dx \left[2\partial_x \phi \partial_x^2 \phi - i\bar{\psi} \partial_x \bar{\psi} \right] \tag{5.37}$$

and using the equations of motion (2.44), namely $\partial_x \phi = \partial_{t_1} \phi$, $\partial_x \bar{\psi} = \partial_{t_1} \bar{\psi}$ we get

$$\frac{dP}{dt_1} = \left[(\partial_x \phi)^2 - i\bar{\psi} \partial_x \bar{\psi} \right]_{-\infty}^{+\infty} \tag{5.38}$$

Now, considering the t_1 derivative of (5.36) and after using the equations of motion we have,

$$\frac{dE}{dt_1} = \left[\frac{1}{4}(\partial_x^2 \phi)^2 - \frac{1}{4}(\partial_x \phi)^4 + \frac{1}{2}\partial_x \phi \partial_x^3 \phi - \frac{i}{4}\bar{\psi} \partial_x^3 \bar{\psi} + \frac{3i}{4}(\partial_x \phi)^2 \bar{\psi} \partial_x \bar{\psi} \right]_{-\infty}^{+\infty} \tag{5.39}$$

which vanish since we are considering sufficiently smooth decaying fields at $\pm\infty$.

$n = 1$

For $n = 1$ the time derivative of (5.35) yields by using the equations of motion (2.48), (2.49),

$$\frac{dP}{dt_3} = \left[\frac{1}{2}\partial_x\phi\partial_x^3\phi - \frac{3}{4}(\partial_x\phi)^4 - \frac{1}{4}(\partial_x^2\phi)^2 + \frac{9i}{4}(\partial_x\phi)^2\bar{\psi}\partial_x\bar{\psi} + \frac{i}{2}\partial_x\bar{\psi}\partial_x^2\bar{\psi} - \frac{i}{4}\bar{\psi}\partial_x^3\bar{\psi}\right]_{-\infty}^{+\infty}. \tag{5.40}$$

Analogously, for the canonical energy (5.36),* taking the t_3 derivative and using the equations of motion (2.48), (2.49), we obtain

$$\begin{aligned} \frac{dE}{dt_3} &= \Bigg[\frac{1}{8}\partial_x\phi\,\partial_x^5\phi + \frac{1}{4}(\partial_x\phi)^6 + \frac{1}{16}(\partial_x^3\phi)^2 - \frac{3}{2}(\partial_x\phi)^2(\partial_x^2\phi)^2 - (\partial_x\phi)^3\partial_x^3\phi \\ &- \frac{3i}{16}\left((\partial_x\phi)^4 - 5(\partial_x^2\phi)^2 - 9\partial_x\phi\,\partial_x^3\phi\right)\bar{\psi}\partial_x\bar{\psi} + \frac{3i}{2}\partial_x\phi\,\partial_x^2\phi\,\bar{\psi}\partial_x^2\bar{\psi} + \frac{3i}{4}(\partial_x\phi)^2\,\bar{\psi}\partial_x^3\bar{\psi} \\ &- \frac{i}{16}\bar{\psi}\partial_x^5\bar{\psi} + \frac{i}{16}\partial_x\bar{\psi}\partial_x^4\bar{\psi} - \frac{i}{16}\partial_x^2\bar{\psi}\partial_x^3\bar{\psi}\Bigg]_{-\infty}^{+\infty}, \end{aligned} \tag{5.41}$$

Again these terms vanish at $x = \pm\infty$.

$n = 2$

For $n = 2$ the time derivative of (5.35) and (5.36) yields by using the equations of motion (2.54), (2.55) the following surface terms,

$$\begin{aligned} \frac{dP}{dt_5} &= \Bigg[\frac{5}{8}(\partial_x\phi)^6 + \frac{1}{16}(\partial_x^3\phi)^2 - \frac{1}{8}\partial_x^2\phi\,\partial_x^4\phi + \frac{1}{8}\partial_x\phi\,\partial_x^5\phi - \frac{5}{8}(\partial_x\phi)^2(\partial_x^2\phi)^2 \\ &- \frac{5}{4}(\partial_x\phi)^3\partial_x^3\phi + i\bar{\psi}\partial_x\bar{\psi}\left(\frac{35}{16}\partial_x\phi\,\partial_x^3\phi + \frac{5}{8}(\partial_x^2\phi)^2 - \frac{25}{8}(\partial_x\phi)^4\right) + \frac{15i}{16}\partial_x\phi\,\partial_x^2\phi\bar{\psi}\partial_x^2\bar{\psi} \\ &+ \frac{15i}{16}(\partial_x\phi)^2\bar{\psi}\partial_x^3\bar{\psi} - \frac{5i}{8}(\partial_x\phi)^2\partial_x\bar{\psi}\partial_x^2\bar{\psi} - \frac{i}{16}\bar{\psi}\partial_x^5\bar{\psi} + \frac{i}{8}\partial_x\bar{\psi}\partial_x^4\bar{\psi} - \frac{i}{8}\partial_x^2\bar{\psi}\partial_x^3\bar{\psi}\Bigg]_{-\infty}^{+\infty}, \end{aligned} \tag{5.42}$$

* Notice that although the charges (5.35) and (5.36) are conserved with respect to all flows, they are interpreted as canonical momentum and canonical energy, respectively, only for the t_3 model.

and

$$\frac{dE}{dt_5} = \Bigg[\frac{1}{64}(\partial_x^2\phi)^4 - \frac{15}{64}(\partial_x\phi)^8 - \frac{1}{64}(\partial_x^4\phi)^2 - \frac{3}{2}(\partial_x\phi)^2(\partial_x^3\phi)^2 + \frac{1}{32}\partial_x\phi\,\partial_x^7\phi + \frac{1}{32}\partial_x^3\phi\,\partial_x^5\phi$$
$$+ \ \frac{125}{32}(\partial_x\phi)^4(\partial_x^2\phi)^2 + \frac{25}{16}(\partial_x\phi)^5\partial_x^3\phi - \frac{41}{16}\partial_x\phi(\partial_x^2\phi)^2\partial_x^3\phi - \frac{3}{8}(\partial_x\phi)^3\partial_x^5\phi$$
$$- \ \frac{27}{16}(\partial_x\phi)^2\partial_x^2\phi\,\partial_x^4\phi + i\bar{\psi}\partial_x^2\bar{\psi}\Big(\frac{69}{64}\partial_x\phi\,\partial_x^4\phi - \frac{335}{64}(\partial_x\phi)^3\partial_x^2\phi + \frac{127}{64}\partial_x^2\phi\,\partial_x^3\phi\Big)$$
$$+ \ \frac{59i}{64}\partial_x\phi\,\partial_x^2\phi\bar{\psi}\partial_x^4\bar{\psi} + i\bar{\psi}\partial_x\bar{\psi}\left(\frac{23}{32}\partial_x\phi\,\partial_x^5\phi + \frac{55}{32}(\partial_x\phi)^6 - \frac{285}{32}(\partial_x\phi)^2(\partial_x^2\phi)^2\right.$$
$$- \left. \ \frac{375}{64}(\partial_x\phi)^3\partial_x^3\phi + \frac{89}{64}\partial_x^2\phi\,\partial_x^4\phi + \frac{73}{64}(\partial_x^3\phi)^2\right) + i\bar{\psi}\partial_x^3\bar{\psi}\left(\frac{111}{64}\partial_x\phi\,\partial_x^3\phi - \frac{85}{64}(\partial_x\phi)^4\right.$$
$$+ \left. \ \frac{7}{8}(\partial_x^2\phi)^2\right) - \frac{i}{64}\bar{\psi}\partial_x^7\bar{\psi} + \frac{i}{64}\partial_x\bar{\psi}\partial_x^6\bar{\psi} + \frac{33i}{64}\partial_x\phi\,\partial_x^2\phi\,\partial_x\bar{\psi}\partial_x^3\bar{\psi} - \frac{i}{64}\partial_x^2\bar{\psi}\partial_x^5\bar{\psi}$$
$$- \ i\partial_x\bar{\psi}\partial_x^2\bar{\psi}\left(\frac{9}{64}\partial_x\phi\,\partial_x^3\phi + \frac{5}{32}(\partial_x\phi)^4 + \frac{39}{64}(\partial_x^2\phi)^2\right) + \frac{21i}{64}(\partial_x\phi)^2\partial_x^2\bar{\psi}\partial_x^3\bar{\psi}$$
$$+ \ \frac{9i}{64}(\partial_x\phi)^2\partial_x\bar{\psi}\partial_x^4\bar{\psi} + \frac{i}{32}\partial_x^3\bar{\psi}\partial_x^4\bar{\psi} + \frac{9i}{32}(\partial_x\phi)^2\bar{\psi}\partial_x^5\bar{\psi}\Bigg]_{-\infty}^{+\infty}, \tag{5.43}$$

Here, we have explicitly used the t_1, t_3 and t_5 time evolutions to show the conservation of the charges.

However, it was already shown in [32] that for every isospectral flow t_{2n+1} of the supersymmetric hierarchy, the charges derived from the spatial part of the Lax operator $I_k^{(-m)}$ are conserved. This is a novel property of the integrable hierarchy, and we will use it to consider in the chapter 7 the modified conserved quantities after introducing a defect in the theory.

Chapter 6

Integrable defects in the smKdV hierarchy

An interesting topic in the study of integrable system is to analyse it in the presence of defects or impurities. Accordingly, in two dimensional integrable field theories that discontinuity is introduced as an internal boundary condition, located at a point in the x-axis, which connects a field theory of both sides of it. In particular, notice that with the introduction of this kind of defect, the spatial translation invariance is broken once we impose a restriction on the variable x, and hence it is expected a violation of the momentum conservation. However, it was verified that to preserve the integrability of the model, the fields of the theory at the defect point should satisfy a kind of Bäcklund transformations [4]. This introduces a defect potential that allows us to build modified conserved quantities.

In the past few years several integrable field theories, that allow integrable defects have been studied using the Lagrangian formalism, for instance sine-Gordon, Liouville, non-linear Schördinger and affine toda models [4, 6, 5]. The supersymmetric extensions were considered in [11, 27, 28, 30, 31] for Liouville and sinh-Gordon models.

The defects can be classified as type-I if the interaction at the defect point only depends on the fields of the theory. And it is called type-II if they interact through additional degrees of freedom present only at the defect point [9]. However, it is important to point out that the type-I super Bäcklund transformations that we are considering in this thesis should be treated as a "partial" type-II defect, in the sense that it contains intrinsically an auxiliary fermionic field (f_1). A real type-II defect will contain one bosonic auxiliary field and two fermionic auxiliary fields [31]. In the appendix D we show the type-II BT obtained for the super sinh-Gordon and by using the universality argument we can extend this result to the entire hierarchy.

Therefore, the classical integrability is guaranteed if the conditions at the defect correspond to Bäcklund transformations. In the Lax approach that we are using in this thesis the Bäcklund transformations (or defect conditions) are encoded within the defect matrix (3.47).

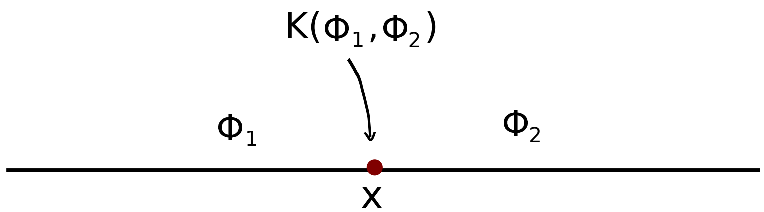

Figure 6.1: Defect representation.

where $\Phi_i = (\phi_i, \bar{\psi}_i)$, $i = 1, 2$, and we consider a defect at $x = 0$.

The whole system can be interpreted as a junction of two domains, each one, satisfying the same integrable equation. This condition is assured by the K matrix via a gauge transformation as we shown in the chapter 3.

In this chapter, we will construct the defect contributions for the momentum and energy (obtained in the last chapter) by using the BT for $n = 0$, $n = 1$ and $n = 2$ from the chapter 3. Next we will consider some Bäcklund solutions and its interaction with the defect.

6.1 Defect contribution for the momentum

Let us start by considering the modification of the momentum, which can be written as follows

$$P=\int_{-\infty}^{0}dx\left[(\partial_x\phi_1)^2-i\bar{\psi}_1\,\partial_x\bar{\psi}_1\right]+\int_{0}^{+\infty}dx\left[(\partial_x\phi_2)^2-i\bar{\psi}_2\,\partial_x\bar{\psi}_2\right]. \tag{6.1}$$

Note that the integral was splitted corresponding to a theory in $x<0$ by $(\phi_1,\bar{\psi}_1)$ and other in the region $x>0$ by $(\phi_2,\bar{\psi}_2)$. The fields remain vanish at $x=\pm\infty$, however in this case we still have the contributions at the defect point $x=0$.

n = 0

For $n=0$ we have found the surface term in (5.38), however now we need to include the contributions from the fields ϕ_1, ϕ_2, $\bar{\psi}_1$ and $\bar{\psi}_2$ at the defect point, namely

$$\frac{dP}{dt_1}=\left[(\partial_x\phi_1)^2-i\bar{\psi}_1\,\partial_x\bar{\psi}_1-(\partial_x\phi_2)^2+i\bar{\psi}_2\,\partial_x\bar{\psi}_2\right]_{x=0} \tag{6.2}$$

The equation above can be written in terms of the variables $\phi_\pm=\phi_1\pm\phi_2$ and $\bar{\psi}_\pm=\bar{\psi}_1\pm\bar{\psi}_2$, as follows

$$\frac{dP}{dt_1}=\left[\partial_x\phi_+\partial_x\phi_--\frac{i}{2}\bar{\psi}_+\,\partial_x\bar{\psi}_--\frac{i}{2}\bar{\psi}_-\,\partial_x\bar{\psi}_+\right]_{x=0} \tag{6.3}$$

We point out that the momentum is no longer conserved, since we have field contributions at the defect point $x=0$. By using the defect conditions (3.4)–(3.6), (3.51) and (3.50), we get

$$\frac{dP}{dt_1}=\left[\frac{4}{\omega^2}\sinh\phi_+\partial_{t_1}\phi_+-\frac{2i}{\omega}\cosh\left(\frac{\phi_+}{2}\right)f_1\partial_{t_1}\bar{\psi}_+-\frac{i}{\omega}\sinh\left(\frac{\phi_+}{2}\right)\partial_{t_1}\phi_+f_1\bar{\psi}_+\right]_{x=0} \tag{6.4}$$

which is a total derivative in t_1, yielding

$$\mathcal{P}=P-\left[\frac{4}{\omega^2}\cosh(\phi_+)-\frac{2i}{\omega}\cosh\left(\frac{\phi_+}{2}\right)f_1\bar{\psi}_+\right]_{x=0}, \tag{6.5}$$

the modified conserved momentum including a defect contribution. In the next example we will see that for $n=1$ (smKdV) this contribution is the same even the defect contributions being so different.

n = 1

Since we have already obtained the surface term in (5.40), we have now non-zero contributions at the defect point, namely

$$\frac{dP}{dt_3} = \frac{1}{4}\Big[2\partial_x\phi_1\partial_x^3\phi_1 - 3(\partial_x\phi_1)^4 - (\partial_x^2\phi_1)^2 + 9i(\partial_x\phi_1)^2\bar{\psi}_1\partial_x\bar{\psi}_1 + 2i\partial_x\bar{\psi}_1\partial_x^2\bar{\psi}_1 - i\bar{\psi}_1\partial_x^3\bar{\psi}_1 \\ - 2\partial_x\phi_2\partial_x^3\phi_2 + 3(\partial_x\phi_2)^4 + (\partial_x^2\phi_2)^2 - 9i(\partial_x\phi_2)^2\bar{\psi}_2\partial_x\bar{\psi}_2 - 2i\partial_x\bar{\psi}_2\partial_x^2\bar{\psi}_2 + i\bar{\psi}_2\partial_x^3\bar{\psi}_2\Big]_{x=0}. \tag{6.6}$$

From eqs. (2.48) and (2.49), we also have that

$$\begin{aligned} \partial_x^3\phi_i &= 4\partial_{t_3}\phi_i + 2(\partial_x\phi_i)^3 - 3i\,\partial_x\phi_i\,\bar{\psi}_i\partial_x\bar{\psi}_i, \\ \partial_x^3\bar{\psi}_i &= 4\partial_{t_3}\bar{\psi}_i + 3\partial_x\phi_i\,\partial_x(\partial_x\phi_i\,\bar{\psi}_i), \qquad i = 1, 2, \end{aligned} \tag{6.7}$$

and then eq. (6.6) becomes

$$\begin{aligned} \frac{dP}{dt_3} &= \Big[2\partial_x\phi_1\,\partial_{t_3}\phi_1 + \frac{1}{4}(\partial_x\phi_1)^4 - \frac{1}{4}(\partial_x^2\phi_1)^2 + i\bar{\psi}_1\partial_{t_3}\bar{\psi}_1 + \frac{i}{2}\partial_x\bar{\psi}_1\partial_x^2\bar{\psi}_1 \\ &\quad -2\partial_x\phi_2\,\partial_{t_3}\phi_2 - \frac{1}{4}(\partial_x\phi_2)^4 + \frac{1}{4}(\partial_x^2\phi_2)^2 - i\bar{\psi}_2\partial_{t_3}\bar{\psi}_2 - \frac{i}{2}\partial_x\bar{\psi}_2\partial_x^2\bar{\psi}_2\Big]_{x=0} \end{aligned} \tag{6.8}$$

Here the canonical momentum is no longer conserved. However, we still can use the defect conditions (3.4)–(3.6), (3.52) and (3.53), to show that this contribution is a total t_3-derivative. Again, we rewrite eq. (6.8) in terms of the variables $\phi_\pm$ and $\bar{\psi}_\pm$, as follows

$$\begin{aligned} \frac{dP}{dt_3} &= \Big[\partial_x\phi_-\,\partial_{t_3}\phi_+ + \partial_x\phi_+\,\partial_{t_3}\phi_- - \frac{1}{4}\partial_x^2\phi_-\,\partial_x^2\phi_+ + \frac{1}{8}(\partial_x\phi_-)^3\partial_x\phi_+ + \frac{1}{8}(\partial_x\phi_+)^3\partial_x\phi_- \\ &\quad -\frac{i}{2}\bar{\psi}_-\partial_{t_3}\bar{\psi}_+ - \frac{i}{2}\bar{\psi}_+\partial_{t_3}\bar{\psi}_- + \frac{i}{4}\partial_x\bar{\psi}_-\partial_x^2\bar{\psi}_+ + \frac{i}{4}\partial_x\bar{\psi}_+\partial_x^2\bar{\psi}_-\Big]_{x=0}, \\ &= \Big[\partial_{t_3}\left(\frac{4}{\omega^2}\cosh\phi_+ - \frac{2i}{\omega}\cosh\left(\frac{\phi_+}{2}\right)f_1\bar{\psi}_+\right) + \frac{2i}{\omega}\cosh\left(\frac{\phi_+}{2}\right)\partial_{t_3}f_1\bar{\psi}_+ \\ &\quad + \frac{i}{2\omega^2}\left(\cosh^2\left(\frac{\phi_+}{2}\right)\bar{\psi}_+\partial_x^2\bar{\psi}_+ - \frac{1}{4}\sinh(\phi_+)\partial_x\phi_+\bar{\psi}_+\partial_x\bar{\psi}_+\right) \\ &\quad + \frac{6i}{\omega^5}\sinh(\phi_+)\cosh^3\left(\frac{\phi_+}{2}\right)\partial_x\phi_+ f_1\bar{\psi}_+\Big]_{x=0}. \end{aligned} \tag{6.9}$$

Then, by using eq. (3.53), we find that

$$\mathcal{P} = P - \Big[\frac{4}{\omega^2}\cosh(\phi_+) - \frac{2i}{\omega}\cosh\left(\frac{\phi_+}{2}\right) f_1\bar{\psi}_+\Big]_{x=0}, \tag{6.10}$$

is the modified conserved momentum, which includes defect contributions in order to preserve the original integrability, i.e. $\frac{d\mathcal{P}}{dt_3} = 0$. This is the same contribution we have obtained for $n = 0$.

6.2 Defect contribution for the energy

Now, the canonical energy in the presence of the defect is given by

$$\begin{aligned} E &= \frac{1}{4}\int_{-\infty}^{0} dx\,\big[(\partial_x^2\phi_1)^2 - (\partial_x\phi_1)^4 + 2\partial_x\phi_1\,\partial_x^3\phi_1 - i\bar{\psi}_1\,\partial_x^3\bar{\psi}_1 + 3i(\partial_x\phi_1)^2\bar{\psi}_1\partial_x\bar{\psi}_1\big] \\ &+ \frac{1}{4}\int_{0}^{+\infty} dx\,\big[(\partial_x^2\phi_2)^2 - (\partial_x\phi_2)^4 + 2\partial_x\phi_2\,\partial_x^3\phi_2 - i\bar{\psi}_2\,\partial_x^3\bar{\psi}_2 + 3i(\partial_x\phi_2)^2\bar{\psi}_2\partial_x\bar{\psi}_2\big]. \end{aligned} \tag{6.11}$$

Considering only contributions at the defect as before, we get

$$\frac{dE}{dt_3} = [E_1 - E_2]_{x=0} \tag{6.12}$$

where

$$\begin{aligned} E_i &= \frac{1}{8}\partial_x\phi_i\,\partial_x^5\phi_i + \frac{1}{4}(\partial_x\phi_i)^6 + \frac{1}{16}(\partial_x^3\phi_i)^2 - \frac{3}{2}(\partial_x\phi_i)^2(\partial_x^2\phi_i)^2 - (\partial_x\phi_i)^3\partial_x^3\phi_i \\ &- \frac{21i}{16}(\partial_x\phi_i)^4\bar{\psi}_i\partial_x\bar{\psi}_i + \frac{15i}{16}(\partial_x^2\phi_i)^2\bar{\psi}_i\partial_x\bar{\psi}_i + \frac{27i}{16}\partial_x\phi_i\partial_x^3\phi_i\bar{\psi}_i\partial_x\bar{\psi}_i + \frac{3i}{2}\partial_x\phi_i\partial_x^2\phi_i\bar{\psi}_i\partial_x^2\bar{\psi}_i \\ &+ \frac{3i}{4}(\partial_x\phi_i)^2\bar{\psi}_i\partial_x^3\bar{\psi}_i - \frac{i}{16}\bar{\psi}_i\partial_x^5\bar{\psi}_i + \frac{i}{16}\partial_x\bar{\psi}_i\partial_x^4\bar{\psi}_i - \frac{i}{16}\partial_x^2\bar{\psi}_i\partial_x^3\bar{\psi}_i, \quad i = 1, 2. \end{aligned} \tag{6.13}$$

In order to obtain the defect contribution for the energy we take its t_3-derivative and use the equations of motion to get

$$\begin{aligned} \frac{dE}{dt_3} = \Bigg[&\partial_{t_3}\phi_-\partial_{t_3}\phi_+ + \frac{1}{4}\partial_{t_3}\partial_x^2\phi_-\partial_x\phi_+ + \frac{1}{4}\partial_{t_3}\partial_x^2\phi_+\partial_x\phi_- + \frac{i}{8}\partial_x\bar{\psi}_-\partial_{t_3}\partial_x\bar{\psi}_+ - \frac{i}{8}\partial_x^2\bar{\psi}_+\partial_{t_3}\bar{\psi}_- \\ &+ \frac{3i}{32}\left(\bar{\psi}_-\partial_{t_3}\bar{\psi}_+ + \bar{\psi}_+\partial_{t_3}\bar{\psi}_-\right)\left((\partial_x\phi_-)^2 + (\partial_x\phi_+)^2\right) - \frac{i}{8}\partial_x^2\bar{\psi}_-\partial_{t_3}\bar{\psi}_+ + \frac{i}{8}\partial_x\bar{\psi}_+\partial_{t_3}\partial_x\bar{\psi}_- \\ &+ \frac{3i}{16}\partial_x\phi_-\partial_x\phi_+\left(\bar{\psi}_-\partial_{t_3}\bar{\psi}_- + \bar{\psi}_+\partial_{t_3}\bar{\psi}_+\right) - \frac{i}{8}\bar{\psi}_-\partial_{t_3}\partial_x^2\bar{\psi}_+ - \frac{i}{8}\bar{\psi}_+\partial_{t_3}\partial_x^2\bar{\psi}_-\Bigg]_{x=0}. \end{aligned} \tag{6.14}$$

Now, by using the Bäcklund equations (3.4)–(3.6), (3.52) and (3.53) we obtain

$$\begin{aligned}\frac{dE}{dt_3} = \Big[&\partial_{t_3}\left(\frac{1}{\omega^2}\partial_x^2\phi_+\sinh\phi_+ + \frac{1}{2\omega^2}(\partial_x\phi_+)^2\cosh\phi_+ + \frac{6i}{\omega^5}\cosh\left(\frac{\phi_+}{2}\right)\sinh^2\phi_+\, f_1\bar{\psi}_+\right.\\ &\left.-\frac{i}{4\omega}\sinh\left(\frac{\phi_+}{2}\right)\partial_x\phi_+ f_1\partial_x\bar{\psi}_+ -\frac{i}{4\omega}\sinh\left(\frac{\phi_+}{2}\right)\partial_x^2\phi_+ f_1\bar{\psi}_+ -\frac{i}{2\omega}\cosh\left(\frac{\phi_+}{2}\right) f_1\partial_x^2\bar{\psi}_+\right)\\ &+\ \frac{i}{2\omega}\sinh\left(\frac{\phi_+}{2}\right)\partial_x^2\phi_+\partial_{t_3}f_1\bar{\psi}_+ - \frac{i}{2\omega}\sinh\left(\frac{\phi_+}{2}\right)\partial_x\phi_+\partial_{t_3}f_1\partial_x\bar{\psi}_+\\ &-\ \frac{i}{2\omega}\cosh\left(\frac{\phi_+}{2}\right)(\partial_x\phi_+)^2\partial_{t_3}f_1\bar{\psi}_+ - \frac{8}{\omega^6}\sinh^3\phi_+\partial_{t_3}\phi_+ + \frac{i}{\omega}\cosh\left(\frac{\phi_+}{2}\right)\partial_{t_3}f_1\partial_x^2\bar{\psi}_+\\ &+\ \frac{12i}{\omega^4}\cosh^2\left(\frac{\phi_+}{2}\right)\sinh\phi_+\partial_x\phi_+ f_1\partial_{t_3}f_1 - \frac{12i}{\omega^5}\cosh\left(\frac{\phi_+}{2}\right)\sinh^2\phi_+\partial_{t_3}f_1\bar{\psi}_+\Big]_{x=0}.\end{aligned} \tag{6.15}$$

Finally using (3.53), we find that the modified conserved energy is given by

$$\begin{aligned}\mathcal{E} = E - \Big[&\frac{1}{\omega^2}\left(\partial_x^2\phi_+\sinh\phi_+ + \frac{1}{2}(\partial_x\phi_+)^2\cosh\phi_+\right) + \frac{1}{\omega^6}\left(6\cosh\phi_+ - \frac{2}{3}\cosh(3\phi_+)\right)\\ &-\frac{i}{2\omega}\cosh\left(\frac{\phi_+}{2}\right) f_1\partial_x^2\bar{\psi}_+ - \frac{i}{4\omega}\sinh\left(\frac{\phi_+}{2}\right)\left(\partial_x\phi_+ f_1\partial_x\bar{\psi}_+ + \partial_x^2\phi_+ f_1\bar{\psi}_+\right)\\ &+\frac{6i}{\omega^5}\cosh\left(\frac{\phi_+}{2}\right)\sinh^2(\phi_+) f_1\bar{\psi}_+\Big]_{x=0},\end{aligned} \tag{6.16}$$

i.e., $\frac{d\mathcal{E}}{dt_3} = 0$.

Although we have calculated the defect contributions to the modified conserved charges by considering conservation under t_1, t_3-derivatives, it is natural to expect that this defect contributions are the same for all members of the integrable hierarchy and then, they are conserved under any t_{2n+1}-derivative. In the appendix F, we explicitly check this statement for the t_5-derivative too.

6.3 Bäcklund solutions for super mKdV hierarchy

In this section we will consider some solutions for (2.44), (2.48), (2.49), (2.54) and (2.55).

Vacuum → one soliton solution

Starting with $\phi_1 = 0$ and $\bar{\psi}_1 = 0$ and the one soliton solution (2.64)

$$\phi_2 = \ln\left(\frac{1+\rho_1}{1-\rho_1}\right), \qquad \bar{\psi}_2 = \epsilon\rho_1\left(\frac{1}{1+\rho_1} + \frac{1}{1-\rho_1}\right) \tag{6.17}$$

where $\rho_1 = exp\left(2xk_1 + 2tk_1^{2n+1}\right)$. The defect conditions (3.4)–(3.6), (3.50)–(3.53), (3.56) and (3.57) are satisfied for

$$k_1 = -\frac{2}{\omega^2}. \tag{6.18}$$

One soliton → one soliton solution

Now we will consider the scaterring of two one soliton solution by the defect, let the solutions for the one soliton

$$\phi_1 = \ln\left(\frac{1+r_1\rho_1}{1-r_1\rho_1}\right), \qquad \bar{\psi}_1 = \epsilon s_1\rho_1\left(\frac{1}{1+r_1\rho_1} + \frac{1}{1-r_1\rho_1}\right), \tag{6.19}$$

$$\phi_2 = \ln\left(\frac{1+r_2\rho_2}{1-r_2\rho_2}\right), \qquad \bar{\psi}_2 = \epsilon s_2\rho_2\left(\frac{1}{1+r_2\rho_2} + \frac{1}{1-r_2\rho_2}\right) \tag{6.20}$$

where $\rho_2 = exp\left(2xk_2 + 2tk_2^{2n+1}\right)$, r_i, s_i, $i = 1, 2$ are the bosonic and fermionic delay, respectively.

The Bäcklund conditons are

$$k_2 = k_1, \quad r_2 = r_1\left(\frac{k_1\omega^2 - 2}{k_1\omega^2 + 2}\right), \quad s_2 = s_1\left(\frac{k_1\omega^2 - 2}{k_1\omega^2 + 2}\right). \tag{6.21}$$

Chapter 7

Modified conserved charges for the smKdV hierarchy

In the chapter 5 we have constructed the conserved charges for the smKdV hierarchy in the bulk theory. In particular, we have examined the conservation of the momentum and energy with respect to the first flows of the hierarchy. Now, in this chapter we will investigate the modified conserved quantities due to the introduction of a defect at $x = 0$. We construct a set of modified conserved charges via defect matrix.

7.1 Defect contributions from the defect matrix

In general, to compute higher order modified conserved charges we can use the defect matrix in order to derive a generating function of the defect contributions. To do that, let us consider the K matrix (3.47), linking two different solutions of the linear problem (5.1), (5.2) by

$$\Omega^{(2)} = K\Omega^{(1)}. \tag{7.1}$$

Then, considering a defect located at $x = 0$ we have for the first set of conserved quantities,

$$\mathcal{I}_1 = \int_{-\infty}^{0} dx \left[V_{11}^{(1)} + V_{12}^{(1)}\Gamma_{21}^{(1)} + V_{13}^{(1)}\Gamma_{31}^{(1)}\right] + \int_{0}^{+\infty} dx \left[V_{11}^{(2)} + V_{12}^{(2)}\Gamma_{21}^{(2)} + V_{13}^{(2)}\Gamma_{31}^{(2)}\right], \tag{7.2}$$

where $V_{ij}^{(p)}$ with $p = 1, 2$ are the spatial part of the Lax for each auxiliary problem (5.1) in the region $x < 0$ and $x > 0$, and $\Gamma_{21}^{(p)} = \Omega_2^{(p)}(\Omega_1^{(p)})^{-1}$ and $\Gamma_{31}^{(p)} = \epsilon\, \Omega_3^{(p)}(\Omega_1^{(p)})^{-1}$ are the respective auxiliary functions for each region. Taking the time t_{2n+1}-derivative of (7.2) and using the conservation equation (5.4), we get

$$\frac{d\mathcal{I}_1}{dt_{2n+1}} = \left[U_{11}^{(1)} + U_{12}^{(1)}\Gamma_{21}^{(1)} + U_{13}^{(1)}\Gamma_{31}^{(1)}\right]_{x=0} - \left[U_{11}^{(2)} + U_{12}^{(2)}\Gamma_{21}^{(2)} + U_{13}^{(2)}\Gamma_{31}^{(2)}\right]_{x=0} \tag{7.3}$$

It is easy to see from (7.1) that the auxiliary fields $\Gamma_{21}^{(2)}$ and $\Gamma_{31}^{(2)}$ satisfy the following relations,

$$\Gamma_{21}^{(2)} = \frac{K_{21} + K_{22}\Gamma_{21}^{(1)} + K_{23}\Gamma_{31}^{(1)}}{K_{11} + K_{12}\Gamma_{21}^{(1)} + K_{13}\Gamma_{31}^{(1)}}, \qquad \Gamma_{31}^{(2)} = \frac{K_{31} + K_{32}\Gamma_{21}^{(1)} + K_{33}\Gamma_{31}^{(1)}}{K_{11} + K_{12}\Gamma_{21}^{(1)} + K_{13}\Gamma_{31}^{(1)}}. \tag{7.4}$$

Besides that, we have the gauge condition (3.48), namely

$$\partial_{t_N} K = K A_{t_N}^{(1)} - A_{t_N}^{(2)} K = -K U^{(1)} + U^{(2)} K, \tag{7.5}$$

Now, substituting these relations in (7.3) and using (7.5), (5.7), (5.8), we get

$$\frac{d\mathcal{I}_1}{dt_N} = \left[-\frac{\partial_{t_N}\left(K_{11} + K_{12}\Gamma_{21}^{(1)} + K_{13}\Gamma_{31}^{(1)}\right)}{K_{11} + K_{12}\Gamma_{21}^{(1)} + K_{13}\Gamma_{31}^{(1)}}\right]_{x=0}, \tag{7.6}$$

that the first generating function for the modified conserved quantities is $\mathcal{I}_1 - \mathcal{D}_1$, where

$$\mathcal{D}_1 = -\ln\left[K_{11} + K_{12}\Gamma_{21}^{(1)} + K_{13}\Gamma_{31}^{(1)}\right], \tag{7.7}$$

is the defect contribution and depends on the elements of the defect matrix K, which is common for the entire hierarchy (see section 3.3). Then using the corresponding coefficients of expansions of the Γ_{21} and Γ_{31}, that we have obtained in the chapter 5, we find that the lower terms can be written as follows,

$$\mathcal{D}_1^{(-1/2)} = \frac{2i}{\omega}\, e^{\frac{\phi_1+\phi_2}{2}}\,\bar{\psi}_1 f_1 + \frac{2}{\omega^2}\, e^{\phi_1+\phi_2}, \tag{7.8}$$

$$\mathcal{D}_1^{(-1)} = \frac{i}{\omega}\, e^{\frac{\phi_1+\phi_2}{2}}\, f_1\left(\bar{\psi}_1\,\partial_x\phi_1 + \partial_x\bar{\psi}_1\right) - \frac{2}{\omega^2}\, e^{\phi_1+\phi_2}\,\partial_x\phi_1 - \frac{4i}{\omega^3}\, e^{\frac{3(\phi_1+\phi_2)}{2}}\, f_1\bar{\psi}_1 + \frac{2}{\omega^4}\, e^{2(\phi_1+\phi_2)}, \tag{7.9}$$

$$\begin{aligned}\mathcal{D}_1^{(-3/2)} = {} & \frac{i}{2\omega}\, e^{\frac{\phi_1+\phi_2}{2}}\,(\bar{\psi}_1\,\partial_x^2\phi_1 + \partial_x\bar{\psi}_1\partial_x\phi_1 + \partial_x^2\bar{\psi}_1) f_1 + \frac{1}{\omega^2}\, e^{\phi_1+\phi_2}\left[\partial_x^2\phi_1 + (\partial_x\phi_1)^2\right] \\ & - \frac{2i}{\omega^3}\, e^{\frac{3(\phi_1+\phi_2)}{2}}\,(3\bar{\psi}_1\,\partial_x\phi_1 + \partial_x\bar{\psi}_1) f_1 - \frac{4}{\omega^4}\, e^{2(\phi_1+\phi_2)}\,\partial_x\phi_1 - \frac{8i}{\omega^5}\, e^{\frac{5(\phi_1+\phi_2)}{2}}\, f_1\bar{\psi}_1 \\ & + \frac{8}{3\omega^6}\, e^{3(\phi_1+\phi_2)}.\end{aligned} \tag{7.10}$$

Analogously, we have the defect contributions for the second set of modified conserved quantities,

$$\mathcal{D}_2 = -\ln\left[K_{22} + K_{21}\Gamma_{12}^{(1)} + K_{23}\Gamma_{32}^{(1)}\right], \tag{7.11}$$

from which we explicitly find the following coefficients,

$$\mathcal{D}_2^{(-1/2)} = \frac{2i}{\omega}\, e^{\frac{-(\phi_1+\phi_2)}{2}}\,\bar{\psi}_1 f_1 + \frac{2}{\omega^2}\, e^{-(\phi_1+\phi_2)}, \tag{7.12}$$

$$\mathcal{D}_2^{(-1)} = \frac{i}{\omega}\, e^{\frac{\phi_1+\phi_2}{2}}\,(\bar{\psi}_1\,\partial_x\phi_1 - \partial_x\bar{\psi}_1) f_1 - \frac{2}{\omega^2}\, e^{-(\phi_1+\phi_2)}\,\partial_x\phi_1 + \frac{4i}{\omega^3}\, e^{\frac{-3(\phi_1+\phi_2)}{2}}\,\bar{\psi}_1 f_1 + \frac{2}{\omega^4}\, e^{-2(\phi_1+\phi_2)}, \tag{7.13}$$

$$\begin{aligned}\mathcal{D}_2^{(-3/2)} = {} & \frac{i}{2\omega}\, e^{\frac{-(\phi_1+\phi_2)}{2}}\, f_1(\bar{\psi}_1\,\partial_x^2\phi_1 + \partial_x\bar{\psi}_1\partial_x\phi_1 - \partial_x^2\bar{\psi}_1) - \frac{1}{\omega^2}\, e^{-(\phi_1+\phi_2)}\left[\partial_x^2\phi_1 - (\partial_x\phi_1)^2\right] \\ & + \frac{2i}{\omega^3}\, e^{\frac{-3(\phi_1+\phi_2)}{2}}\,(3\bar{\psi}_1\,\partial_x\phi_1 - \partial_x\bar{\psi}_1) f_1 + \frac{4}{\omega^4}\, e^{-2(\phi_1+\phi_2)}\,\partial_x\phi_1 + \frac{8i}{\omega^5}\, e^{\frac{-5(\phi_1+\phi_2)}{2}}\,\bar{\psi}_1 f_1 \\ & + \frac{8}{3\omega^6}\, e^{-3(\phi_1+\phi_2)}.\end{aligned} \tag{7.14}$$

We note that the defect contributions for the momentum

$$P_D = \mathcal{D}_1^{(-1/2)} + \mathcal{D}_2^{(-1/2)} = \left[\frac{4}{\omega^2}\cosh\phi_+ - \frac{2i}{\omega}\cosh\left(\frac{\phi_+}{2}\right) f_1\bar{\psi}_+\right]_{x=0}, \tag{7.15}$$

and for the energy

$$\begin{aligned} E_D &= \mathcal{D}_1^{(-3/2)} + \mathcal{D}_2^{(-3/2)} \\ &= \Big[\frac{1}{\omega^2} \left(\partial_x^2 \phi_+ \sinh \phi_+ + \frac{1}{2} (\partial_x \phi_+)^2 \cosh \phi_+ \right) + \frac{1}{\omega^6} \left(6 \cosh \phi_+ - \frac{2}{3} \cosh(3\phi_+) \right) \\ &\quad - \frac{i}{2\omega} \cosh \left(\frac{\phi_+}{2} \right) f_1 \partial_x^2 \bar{\psi}_+ - \frac{i}{4\omega} \sinh \left(\frac{\phi_+}{2} \right) \left(\partial_x \phi_+ f_1 \partial_x \bar{\psi}_+ + \partial_x^2 \phi_+ f_1 \bar{\psi}_+ \right) \\ &\quad + \frac{6i}{\omega^5} \cosh \left(\frac{\phi_+}{2} \right) \sinh^2(\phi_+) f_1 \bar{\psi}_+ \Big]_{x=0}, \end{aligned} \tag{7.16}$$

can be properly recovered as linear combinations of the defect contributions $\mathcal{D}_k^{(-m)}$, with $m = 1/2$ and $m = 3/2$ respectively.

For sake of compactness, we have only considered contributions for the canonical energy and momentum, however this method can be applied systematically to obtain higher modified conserved charges of the smKdV hierarchy. Furthermore, the generalization to other supersymmetric integrable hierarchies seems to be straightforward, and certainly deserve further investigations.

Chapter 8

Conclusions and further developments

In this thesis we have examined the presence of defects in the $\mathcal{N} = 1$ supersymmetric mKdV integrable hierarchy. Starting with the algebraic structure shared by the members (equations of motion) of the hierarchy, it was possible to systematically construct the corresponding Bäcklund transformations as well as the conserved quantities and their respective defect contributions in order to ensure the integrability of the whole system.

Firstly, we have studied the systematic construction of a set of nonlinear super integrable equations of motion based on the zero curvature condition and a super affine Lie algebra $\hat{sl}(2,1)$. In particular, we have considered the supersymmetric mKdV hierarchy. We have shown that these equations of motion are related by a recursion operator that maps subsequent time flows, for instance, we have verified that for $n = 0, 1$ the recursion equation (4.28) provides the smKdV equation (t_3) and the t_5 flow within the super mKdV hierarchy, respectively. The Bäcklund recursion operator was obtained and we have verified that it also relates two consecutive time flows within the hierarchy.

Next, we have constructed the Bäcklund transformations for the supersymmetric mKdV hierarchy based on the invariance of the zero curvature representation under gauge transformation. Such characteristic allowed the construction of a defect-gauge matrix (K) connecting two different field configurations of the same integrable model and hence generating the Bäcklund transformation. The virtue of the method is that all models within the hierarchy are constructed from the zero curvature representation and consequently are all invariant under the defect-gauge transformation (3.14), (3.48). It therefore leads to a systematic construction of Bäcklund transformation for all models within the hierarchy in a universal manner.

The classical integrability of the model is associated with the existence of a sufficient (infinity) number of conserved charges*. We have constructed these charges solving the Ricatti equations associated to the elements of the x-part of the Lax, which is common to all equations. This provides a method to systematically obtain the conserved quantities for the entire hierarchy. In particular, we have written the energy and momentum for the smKdV as a linear combination of these charges, and we checked its conservation under the flows t_1, t_3 and t_5.

Moreover, the introduction of defects in the supersymmetric mKdV hierarchy requires, in order to ensure the integrability, additional contributions to the conserved quantities which were constructed from the defect matrix in a systematic way. We should point out that the arguments stated in this thesis are general and the modified charges are conserved with respect to all flows. To illustrate, we have explicitly worked out examples of Bäcklund transformation and checked the conservation for the first few charges under the time flows t_1, t_3 and t_5.

The Lagrangian formalism of the $\mathcal{N} = 1$ supersymmetric mKdV model in the presence of defects, which corresponds to the construction of the defect potentials by preserving the momentum conservation, remains for future investigation. Furthermore, in [31] we have considered the solutions of the type-II Bäcklund transformation for the $\mathcal{N} = 1$ supersymmetric sinh-Gordon model, and we expect that solutions will have the same structure for the entire hierarchy based on the universality arguments exposed in this thesis.

Another interesting future application of this framework would be to generalize the construction to the super KdV hierarchy as proposed in [36] for the pure bosonic

*Here we focus in this condition however a strict treatment should consider the involution of such charges.

case where a Miura transformation was realized in terms of gauge transformation.

Appendix A

Representation of the $\widehat{sl}(2,1)$ affine Lie superalgebra

In this paper we are considering the following representation of the $\widehat{sl}(2,1)$ affine superalgebra,

$$K_1^{(2n+1)} = \begin{pmatrix} 0 & -\lambda^n & 0 \\ -\lambda^{n+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2^{(2n+1)} = \begin{pmatrix} \lambda^{n+\frac{1}{2}} & 0 & 0 \\ 0 & \lambda^{n+\frac{1}{2}} & 0 \\ 0 & 0 & 2\lambda^{n+\frac{1}{2}} \end{pmatrix}, \tag{A.1}$$

$$M_1^{(2n+1)} = \begin{pmatrix} 0 & -\lambda^n & 0 \\ \lambda^{n+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2^{(2n)} = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & -\lambda^n & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.2}$$

$$F_1^{\left(2n+\frac{3}{2}\right)} = \begin{pmatrix} 0 & 0 & \lambda^{n+\frac{1}{2}} \\ 0 & 0 & -\lambda^{n+1} \\ \lambda^{n+1} & -\lambda^{n+\frac{1}{2}} & 0 \end{pmatrix}, \quad F_2^{\left(2n+\frac{1}{2}\right)} = \begin{pmatrix} 0 & 0 & -\lambda^n \\ 0 & 0 & \lambda^{n+\frac{1}{2}} \\ \lambda^{n+\frac{1}{2}} & -\lambda^n & 0 \end{pmatrix}, \tag{A.3}$$

$$G_1^{\left(2n+\frac{1}{2}\right)} = \begin{pmatrix} 0 & 0 & \lambda^n \\ 0 & 0 & \lambda^{n+\frac{1}{2}} \\ \lambda^{n+\frac{1}{2}} & \lambda^n & 0 \end{pmatrix}, \quad G_2^{\left(2n+\frac{3}{2}\right)} = \begin{pmatrix} 0 & 0 & -\lambda^{n+\frac{1}{2}} \\ 0 & 0 & -\lambda^{n+1} \\ \lambda^{n+1} & \lambda^{n+\frac{1}{2}} & 0 \end{pmatrix}. \tag{A.4}$$

Appendix B

$N = 5$ Lax component

The Lax component A_{t_5} takes the following form,

$$A_{t_5} = \left(\begin{array}{cc|c} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \hline b_{31} & b_{32} & b_{33} \end{array}\right), \tag{B.1}$$

where,

$$\begin{aligned} b_{11} \;&= \lambda^{5/2} + \lambda^2\,\partial_x^2\phi - \frac{i\lambda^{3/2}}{2}\bar{\psi}\partial_x\bar{\psi} + \lambda\left(\frac{1}{2}(\partial_x\phi)^3 - \frac{1}{4}\partial_x^3\phi - \frac{3i}{4}\partial_x\phi\,\bar{\psi}\partial_x\bar{\psi}\right) \\ &+ \lambda^{1/2}\left(\frac{i}{2}(\partial_x\phi)^2\,\bar{\psi}\partial_x\bar{\psi} + \frac{i}{8}\partial_x\bar{\psi}\partial_x^2\bar{\psi} - \frac{i}{8}\bar{\psi}\partial_x^3\bar{\psi}\right) + \frac{5}{8}(\partial_x\phi)^2\partial_x^3\phi + \frac{5}{8}\partial_x\phi(\partial_x^2\phi)^2 \\ &+ \frac{5i}{4}(\partial_x\phi)^3\,\bar{\psi}\partial_x\bar{\psi} - \frac{5i}{16}\partial_x\phi\,\bar{\psi}\partial_x^3\bar{\psi} - \frac{5i}{16}\partial_x^2\phi\,\bar{\psi}\partial_x^2\bar{\psi} - \frac{5i}{16}\partial_x^3\phi\,\bar{\psi}\partial_x\bar{\psi} - \frac{3}{8}(\partial_x\phi)^5, \end{aligned} \tag{B.2}$$

$$\begin{aligned} b_{12} \;&= -\lambda^2 + \frac{\lambda}{2}\left(\partial_x^2\phi + (\partial_x\phi)^2 - i\bar{\psi}\partial_x\bar{\psi}\right) - \frac{1}{8}(\partial_x^2\phi)^2 + \frac{1}{8}\partial_x^4\phi + \frac{1}{4}\partial_x\phi\partial_x^3\phi - \frac{3}{4}(\partial_x\phi)^2\partial_x^2\phi \\ &- \frac{3}{8}(\partial_x\phi)^4 + \frac{i}{4}\partial_x\phi\bar{\psi}\partial_x^2\bar{\psi} + i(\partial_x\phi)^2\bar{\psi}\partial_x\bar{\psi} + \frac{i}{2}\partial_x^2\phi\,\bar{\psi}\partial_x\bar{\psi} + \frac{i}{8}\partial_x\bar{\psi}\partial_x^2\bar{\psi} - \frac{i}{8}\bar{\psi}\partial_x^3\bar{\psi}, \end{aligned} \tag{B.3}$$

$$
\begin{aligned}
b_{13} &= \lambda^2\sqrt{i}\bar\psi + \frac{\lambda^{3/2}\sqrt{i}}{2}\left(\partial_x\phi\,\bar\psi + \partial_x\bar\psi\right) + \frac{\lambda\sqrt{i}}{4}\left(\partial_x\phi\partial_x\bar\psi - 2(\partial_x\phi)^2\bar\psi - \partial_x^2\phi\bar\psi + \partial_x^2\bar\psi\right)\\
&+ \frac{\lambda^{1/2}\sqrt{i}}{8}\left(\partial_x\phi\partial_x^2\bar\psi - 3(\partial_x\phi)^2\partial_x\bar\psi - \partial_x^2\phi\partial_x\bar\psi + \partial_x^3\phi\bar\psi - 3\partial_x\phi\partial_x^2\phi\bar\psi - 3(\partial_x\phi)^3\bar\psi + \partial_x^3\bar\psi\right)\\
&+ \frac{\sqrt{i}}{16}\left(\partial_x\phi\,\partial_x^3\bar\psi - \partial_x^2\phi\,\partial_x^2\bar\psi + \partial_x^3\phi\,\partial_x\bar\psi - \partial_x^4\phi\,\bar\psi + \partial_x^4\bar\psi\right) - \frac{\sqrt{i}}{2}\partial_x\phi\partial_x^2\phi\partial_x\bar\psi\\
&+ \frac{\sqrt{i}}{4}\left(\partial_x^2\phi\,(\partial_x\phi)^2\,\bar\psi - (\partial_x\phi)^3\partial_x\bar\psi - (\partial_x\phi)^2\,\partial_x^2\bar\psi\right)\\
&\quad + \frac{\sqrt{i}}{8}\left(3(\partial_x\phi)^4\,\bar\psi - 3\partial_x\phi\,\partial_x^3\phi\,\bar\psi - (\partial_x^2\phi)^2\bar\psi\right),
\end{aligned} \tag{B.4}
$$

$$
\begin{aligned}
b_{21} &= -\lambda^3 + \frac{\lambda^2}{2}\left(-\partial_x^2\phi + (\partial_x\phi)^2 - i\bar\psi\partial_x\bar\psi\right) + \lambda\left(\frac{1}{4}\partial_x\phi\,\partial_x^3\phi - \frac{1}{8}(\partial_x^2\phi)^2 - \frac{1}{8}\partial_x^4\phi\right.\\
&- \frac{3}{8}(\partial_x\phi)^4 + \frac{3}{4}(\partial_x\phi)^2\partial_x^2\phi + i(\partial_x\phi)^2\bar\psi\partial_x\bar\psi - \frac{i}{4}\partial_x\phi\bar\psi\partial_x^2\bar\psi - \frac{i}{2}\partial_x^2\phi\bar\psi\partial_x\bar\psi\\
&+ \left.\frac{i}{8}\partial_x\bar\psi\partial_x^2\bar\psi - \frac{i}{8}\bar\psi\partial_x^3\bar\psi\right),
\end{aligned} \tag{B.5}
$$

$$
\begin{aligned}
b_{22} &= \lambda^{5/2} + \lambda^2\partial_x\phi - \frac{i\lambda^{3/2}}{2}\bar\psi\partial_x\bar\psi + \lambda\left(\frac{1}{4}\partial_x^3\phi - \frac{1}{2}(\partial_x\phi)^3 + \frac{3i}{4}\partial_x\phi\bar\psi\partial_x\bar\psi\right)\\
&+ \lambda^{1/2}\left(\frac{i}{2}(\partial_x\phi)^2\bar\psi\partial_x\bar\psi + \frac{i}{8}\partial_x\bar\psi\partial_x^2\bar\psi - \frac{i}{8}\bar\psi\partial_x^3\bar\psi\right) + \frac{1}{16}\partial_x^5\phi - \frac{5}{8}\partial_x\phi(\partial_x^2\phi)^2\\
&- \frac{5}{8}\partial_x^3\phi(\partial_x\phi)^2 + \frac{3}{8}(\partial_x\phi)^5 + \frac{5i}{16}\partial_x^2\phi\bar\psi\partial_x^2\bar\psi + \frac{5i}{16}\partial_x^3\phi\bar\psi\partial_x\bar\psi + \frac{5i}{16}\partial_x\phi\bar\psi\partial_x^3\bar\psi\\
&\quad - \frac{5i}{4}(\partial_x\phi)^3\bar\psi\partial_x\bar\psi,
\end{aligned} \tag{B.6}
$$

$$
\begin{aligned}
b_{23} &= \lambda^{5/2}\sqrt{i}\bar\psi + \frac{\lambda^2\sqrt{i}}{2}\left(\partial_x\bar\psi - \partial_x\phi\bar\psi\right) + \frac{\lambda^{3/2}\sqrt{i}}{4}\left(\partial_x^2\phi\bar\psi - 2(\partial_x\phi)^2\bar\psi - \partial_x\phi\partial_x\bar\psi + \partial_x^2\bar\psi\right)\\
&+ \frac{\lambda\sqrt{i}}{8}\left(\partial_x^2\phi\partial_x\bar\psi - \partial_x\phi\partial_x^2\bar\psi - \partial_x^3\phi\bar\psi - 3(\partial_x\phi)^2\partial_x\bar\psi + 3(\partial_x\phi)^3\bar\psi - 3\partial_x\phi\partial_x^2\phi\bar\psi + \partial_x^3\bar\psi\right)\\
&+ \frac{\lambda^{1/2}\sqrt{i}}{16}\left(\partial_x^4\phi\,\bar\psi - \partial_x^3\phi\,\partial_x\bar\psi - \partial_x\phi\,\partial_x^3\bar\psi + \partial_x^2\phi\,\partial_x^2\bar\psi + \partial_x^4\bar\psi\right) - \frac{\lambda^{1/2}\sqrt{i}}{2}\partial_x\phi\,\partial_x^2\phi\,\partial_x\bar\psi\\
&+ \frac{\lambda^{1/2}\sqrt{i}}{4}\left((\partial_x\phi)^3\partial_x\bar\psi - (\partial_x\phi)^2\partial_x^2\bar\psi - \partial_x^2\phi(\partial_x\phi)^2\bar\psi + \frac{3}{2}(\partial_x\phi)^4\,\bar\psi - 12\partial_x^3\phi\,\partial_x\phi\,\bar\psi\right.\\
&\qquad\qquad \left. -4(\partial_x^2\phi)^2\,\bar\psi\right),
\end{aligned} \tag{B.7}
$$

$$
\begin{aligned}
b_{31} &= \lambda^{5/2}\sqrt{i}\bar\psi + \frac{\lambda^2\sqrt{i}}{2}(\partial_x\phi\bar\psi - \partial_x\bar\psi) + \frac{\lambda^{3/2}\sqrt{i}}{4}\left(\partial_x^2\phi\bar\psi - \partial_x\phi\partial_x\bar\psi - 2(\partial_x\phi)^2\bar\psi + \partial_x^2\bar\psi\right)\\
&+ \frac{\lambda\sqrt{i}}{8}\left(\partial_x\phi\partial_x^2\bar\psi - \partial_x^2\phi\partial_x\bar\psi + \partial_x^3\phi\bar\psi + 3\partial_x\phi\partial_x^2\phi\bar\psi - 3(\partial_x\phi)^3\bar\psi + 3(\partial_x\phi)^2\partial_x\bar\psi - \partial_x^3\bar\psi\right)\\
&+ \frac{\lambda^{1/2}\sqrt{i}}{16}\left(\partial_x^4\phi\,\bar\psi - \partial_x^3\phi\,\partial_x\bar\psi - \partial_x\phi\partial_x^3\bar\psi + \partial_x^2\phi\,\partial_x^2\bar\psi + \partial_x^4\bar\psi\right) - \frac{\lambda^{1/2}\sqrt{i}}{2}\partial_x\phi\,\partial_x^2\phi\,\partial_x\bar\psi\\
&+ \frac{\lambda^{1/2}\sqrt{i}}{4}\Bigg((\partial_x\phi)^3\partial_x\bar\psi - (\partial_x\phi)^2\partial_x^2\bar\psi - \partial_x^2\phi(\partial_x\phi)^2\bar\psi - \frac{1}{2}(\partial_x^2\phi)^2\bar\psi - 12\partial_x\phi\partial_x^3\phi\bar\psi\\
&\qquad\qquad +12(\partial_x\phi)^4\,\bar\psi\Bigg)\,, \qquad \text{(B.8)}
\end{aligned}
$$

$$
\begin{aligned}
b_{32} &= \lambda^2\sqrt{i}\bar\psi - \frac{\lambda^{3/2}\sqrt{i}}{2}(\partial_x\phi\,\bar\psi + \partial_x\bar\psi) + \frac{\lambda\sqrt{i}}{4}\left(\partial_x\phi\partial_x\bar\psi - \partial_x^2\phi\,\bar\psi - 2(\partial_x\phi)^2\,\bar\psi + \partial_x^2\bar\psi\right)\\
&+ \frac{\lambda^{1/2}\sqrt{i}}{8}\left(3\partial_x\phi\partial_x^2\phi\bar\psi + 3(\partial_x\phi)^3\bar\psi - \partial_x^3\phi\bar\psi + \partial_x^2\phi\partial_x\bar\psi - \partial_x\phi\partial_x^2\bar\psi + 3(\partial_x\phi)^2\partial_x\bar\psi - \partial_x^3\bar\psi\right)\\
&+ \frac{\sqrt{i}}{16}\left(\partial_x\phi\,\partial_x^3\bar\psi - \partial_x^2\phi\,\partial_x^2\bar\psi - \partial_x^4\phi\,\bar\psi + \partial_x^3\phi\,\partial_x\bar\psi + \partial_x^4\bar\psi\right) - \frac{\sqrt{i}}{2}\partial_x\phi\,\partial_x^2\phi\,\partial_x\bar\psi\\
&- \frac{\sqrt{i}}{4}(\partial_x\phi)^2\,\partial_x^2\bar\psi - \frac{\sqrt{i}}{4}(\partial_x\phi)^3\,\partial_x\bar\psi - \frac{\sqrt{i}}{8}(\partial_x^2\phi)^2\,\bar\psi - \frac{3}{8}\partial_x^3\phi\,\partial_x\phi\,\bar\psi + \frac{\sqrt{i}}{4}(\partial_x\phi)^2\,\partial_x^2\phi\,\bar\psi\\
&\quad +\frac{3}{8}(\partial_x\phi)^4\,\bar\psi, \qquad \text{(B.9)}
\end{aligned}
$$

$$
b_{33} = 2\lambda^{5/2} - i\lambda^{3/2}\bar\psi\partial_x\bar\psi + i\lambda^{1/2}\left((\partial_x\phi)^2\,\bar\psi\partial_x\bar\psi + \frac{1}{4}\partial_x\bar\psi\partial_x^2\bar\psi - \frac{1}{4}\bar\psi\partial_x^3\bar\psi\right). \qquad \text{(B.10)}
$$

Appendix C

Calculation of the defect matrix

The defect matrix K is directly derived by solving the differential equations,

$$\partial_x K = K A_x^{(1)} - A_x^{(2)} K, \tag{C.1}$$
$$\partial_{t_{-1}} K = K A_{t_{-1}}^{(1)} - A_{t_{-1}}^{(2)} K \tag{C.2}$$

with the Lax connections are given

$$A_+ = \left(\begin{array}{cc|c} \lambda^{1/2} - \partial_+\phi & -1 & \sqrt{i}\bar{\psi} \\ -\lambda & \lambda^{1/2} + \partial_+\phi & \lambda^{1/2}\sqrt{i}\bar{\psi} \\ \hline \sqrt{i}\lambda^{1/2}\bar{\psi} & \sqrt{i}\bar{\psi} & 2\lambda^{1/2} \end{array}\right), \tag{C.3}$$

$$A_- = \left(\begin{array}{cc|c} \lambda^{-1/2} & -\lambda^{-1}e^{2\phi} & -\sqrt{i}\lambda^{-1/2}\psi\, e^{\phi} \\ -e^{-2\phi} & \lambda^{-1/2} & -\sqrt{i}\psi e^{-\phi} \\ \hline \sqrt{i}\psi e^{-\phi} & \sqrt{i}\lambda^{-1/2}\psi\, e^{\phi} & 2\lambda^{-1/2} \end{array}\right). \tag{C.4}$$

To find a solution for the defect matrix $\mathcal{K}$, we propose the following λ-expansion,

$$K_{ij} = \tau_{ij} + \lambda^{-1/2}\eta_{ij} + \lambda^{1/2}\kappa_{ij}. \tag{C.5}$$

Now, considering term by term we find a set of constraints coming from the $\lambda^{\pm 3/2}$ and $\lambda^{\pm 1}$ terms, which we will explictly present as follows:

$\lambda^{+3/2}$- terms :

$$\kappa_{12} = \kappa_{13} = \kappa_{32} = 0, \qquad \kappa_{11} = \kappa_{22}. \tag{C.6}$$

λ^{+1}- terms :

$$\begin{aligned}
\tau_{12} &= \sqrt{i}\kappa_{13}\,\bar{\psi}_1, && \text{(C.7)}\\
\kappa_{13} &= -\sqrt{i}\kappa_{12}\,\bar{\psi}_1, && \text{(C.8)}\\
\kappa_{32} &= -\sqrt{i}\kappa_{12}\,\bar{\psi}_2, && \text{(C.9)}\\
\tau_{11} - \tau_{22} &= \sqrt{i}\left(\bar{\psi}_2\,\kappa_{31} - \kappa_{23}\,\bar{\psi}_1\right), && \text{(C.10)}\\
\tau_{13} + \kappa_{23} &= \sqrt{i}\left(\bar{\psi}_2\,\kappa_{33} - \kappa_{11}\,\bar{\psi}_1\right), && \text{(C.11)}\\
\kappa_{31} + \tau_{32} &= \sqrt{i}\left(\bar{\psi}_1\,\kappa_{33} - \kappa_{11}\,\bar{\psi}_2\right). && \text{(C.12)}
\end{aligned}$$

$\lambda^{-3/2}$- terms :

$$\eta_{21} = \eta_{23} = \eta_{31} = 0, \qquad \eta_{22} = \eta_{11}e^{2\phi_-}. \tag{C.13}$$

λ^{-1}- terms :

$$\begin{aligned}
\tau_{21} &= -\sqrt{i}\psi_2\,\eta_{31}\,e^{-\frac{(\phi_+-\phi_-)}{2}}, && \text{(C.14)}\\
\eta_{23} &= \sqrt{i}\eta_{21}\,\psi_1\,e^{\frac{(\phi_++\phi_-)}{2}}, && \text{(C.15)}\\
\eta_{31} &= -\sqrt{i}\psi_2\,\eta_{21}\,e^{\frac{(\phi_+-\phi_-)}{2}}, && \text{(C.16)}
\end{aligned}$$

$$\begin{aligned}
\tau_{11}\,e^{\phi_-} - \tau_{22}\,e^{-\phi_-} &= \sqrt{i}\,e^{-\frac{\phi_+}{2}}\big(\eta_{13}\,\psi_1\,e^{\frac{\phi_-}{2}} + e^{-\frac{\phi_-}{2}}\,\psi_2\,\eta_{32}\big), && \text{(C.17)}\\
\tau_{31}\,e^{(\phi_++\phi_-)} + \eta_{32} &= \sqrt{i}\,e^{\frac{\phi_+}{2}}\big(\eta_{33}\,\psi_1\,e^{\frac{\phi_-}{2}} - \eta_{22}\,\psi_2\,e^{-\frac{\phi_-}{2}}\big), && \text{(C.18)}\\
\tau_{23}\,e^{(\phi_+-\phi_-)} + \eta_{13} &= \sqrt{i}\,e^{\frac{\phi_+}{2}}(\eta_{11}\,\psi_1 e^{\frac{\phi_-}{2}} - \eta_{33}\,\psi_2 e^{-\frac{\phi_-}{2}}). && \text{(C.19)}
\end{aligned}$$

We have denoted $\phi_\pm = \phi_1 \pm \phi_2$ for convenience. From eqs (C.7), (C.8) we get that $\tau_{12} = 0$, and from (C.14) and (C.15) that $\tau_{21} = 0$.

Now, we obtain a set of differential equations from the λ^0 and $\lambda^{\pm 1/2}$ terms. Here we summarize them:

λ^0- terms :

$$\begin{aligned}
\partial_x\tau_{11} &= -\tau_{11}\,\partial_x\phi_- + \sqrt{i}\left(\eta_{13}\,\bar\psi_1 - \bar\psi_2\,\tau_{31}\right), && \text{(C.20)}\\
\partial_x\tau_{13} &= \frac{\tau_{13}}{2}\,\partial_x(\phi_+ - \phi_-) + \eta_{13} + \tau_{23} - \sqrt{i}\bar\psi_2\,\tau_{33} + \sqrt{i}(\tau_{11}+\eta_{12})\bar\psi_1, && \text{(C.21)}\\
\partial_x\tau_{23} &= -\frac{\tau_{23}}{2}\partial_x(\phi_+ - \phi_-) + \sqrt{i}\left(\eta_{22}\,\bar\psi_1 - \bar\psi_2\,\eta_{33}\right), && \text{(C.22)}\\
\partial_x\tau_{22} &= \tau_{22}\,\partial_x\phi_- + \sqrt{i}\left(\tau_{23}\,\bar\psi_1 - \bar\psi_2\,\eta_{32}\right), && \text{(C.23)}\\
\partial_x\tau_{31} &= -\frac{\tau_{31}}{2}\partial_x(\phi_+ + \phi_-) + \sqrt{i}\left(\eta_{33}\,\bar\psi_1 - \bar\psi_2\,\eta_{11}\right), && \text{(C.24)}\\
\partial_x\tau_{32} &= \frac{\tau_{32}}{2}\partial_x(\phi_+ + \phi_-) - \tau_{31} - \eta_{32} + \sqrt{i}\tau_{33}\,\bar\psi_1 - \sqrt{i}\bar\psi_2(\eta_{12}+\tau_{22}), && \text{(C.25)}\\
\partial_x\tau_{33} &= \sqrt{i}(\eta_{32}+\tau_{31})\bar\psi_1 - \sqrt{i}\bar\psi_2(\tau_{23}+\eta_{13}), && \text{(C.26)}\\
\partial_{t_{-1}}\tau_{11} &= \sqrt{i}\,e^{-\frac{\phi_-}{2}}(\tau_{13}\,\psi_1 e^{-\frac{\phi_+}{2}} + e^{\frac{\phi_+}{2}}\psi_2\,\kappa_{31}), && \text{(C.27)}\\
\partial_{t_{-1}}\tau_{13} &= \sqrt{i}\,e^{\frac{\phi_+}{2}}(e^{-\frac{\phi_-}{2}}\,\psi_2\,\kappa_{33} - \kappa_{11}\,\psi_1\,e^{\frac{\phi_-}{2}}), && \text{(C.28)}\\
\partial_{t_{-1}}\tau_{22} &= \sqrt{i}\,e^{\frac{\phi_-}{2}}(e^{-\frac{\phi_+}{2}}\,\psi_2\,\tau_{32} + \kappa_{23}\,\psi_1\,e^{\frac{\phi_+}{2}}), && \text{(C.29)}\\
\partial_{t_{-1}}\tau_{23} &= \kappa_{23} + \tau_{13}\,e^{-(\phi_+-\phi_-)} - \sqrt{i}(\kappa_{21}\,e^{\frac{(\phi_++\phi_-)}{2}} + \tau_{22}\,e^{-\frac{(\phi_++\phi_-)}{2}})\,\psi_1\\
&\quad + \sqrt{i}\,e^{-\frac{(\phi_+-\phi_-)}{2}}\psi_2\,\tau_{33}, && \text{(C.30)}\\
\partial_{t_{-1}}\tau_{31} &= -\kappa_{31} - \tau_{32}e^{-(\phi_++\phi_-)} - \sqrt{i}\psi_2(e^{-\frac{(\phi_+-\phi_-)}{2}}\tau_{11} + e^{\frac{(\phi_+-\phi_-)}{2}}\kappa_{21})\\
&\quad + \sqrt{i}\tau_{33}\psi_1\,e^{-\frac{(\phi_++\phi_-)}{2}}, && \text{(C.31)}\\
\partial_{t_{-1}}\tau_{32} &= \sqrt{i}\,e^{\frac{\phi_+}{2}}(\kappa_{33}\,\psi_1\,e^{\frac{\phi_-}{2}} - e^{-\frac{\phi_-}{2}}\,\psi_2\,\kappa_{11}), && \text{(C.32)}\\
\partial_{t_{-1}}\tau_{33} &= -\sqrt{i}(\kappa_{31}e^{\frac{(\phi_++\phi_-)}{2}} + \tau_{32}\,e^{-\frac{(\phi_++\phi_-)}{2}})\psi_1 - \sqrt{i}\psi_2(e^{-\frac{(\phi_+-\phi_-)}{2}}\tau_{13} + e^{\frac{(\phi_+-\phi_-)}{2}}\kappa_{23}). && \text{(C.33)}
\end{aligned}$$

$\lambda^{-1/2}$- terms :

$$\begin{aligned}
\partial_{t_{-1}}\eta_{11} &= e^{-\phi_-}(\kappa_{21}\,e^{\phi_+} - \eta_{12}\,e^{-\phi_+}) + \sqrt{i}\,e^{-\frac{\phi_-}{2}}(\eta_{13}\,\psi_1\,e^{-\frac{\phi_+}{2}} + e^{\frac{\phi_+}{2}}\,\psi_2\,\tau_{31}), && \text{(C.34)}\\
\partial_{t_{-1}}\eta_{12} &= \kappa_{11}\,e^{\phi_+}(e^{-\phi_-} - e^{\phi_-}) + \sqrt{i}\,e^{\frac{\phi_+}{2}}(\tau_{13}\,\psi_1\,e^{\frac{\phi_-}{2}} + e^{-\frac{\phi_-}{2}}\,\psi_2\,\tau_{32}), && \text{(C.35)}
\end{aligned}$$

$$\partial_{t_{-1}}\eta_{13} = \kappa_{23}\, e^{(\phi_+-\phi_-)} + \tau_{13} - \sqrt{i}(\tau_{11}\, e^{\frac{(\phi_++\phi_-)}{2}} + \eta_{12}\, e^{-\frac{(\phi_++\phi_-)}{2}})\,\psi_1 + \sqrt{i}e^{\frac{(\phi_+-\phi_-)}{2}}\psi_2\,\tau_{33}, \quad (C.36)$$

$$\partial_{t_{-1}}\eta_{22} = e^{\phi_-}(\eta_{12}\, e^{-\phi_+} - \kappa_{21}\, e^{\phi_+}) + \sqrt{i}\, e^{\frac{\phi_-}{2}}(\tau_{23}\,\psi_1\, e^{\frac{\phi_+}{2}} + e^{-\frac{\phi_+}{2}}\,\psi_2\,\eta_{32}), \quad (C.37)$$

$$\partial_{t_{-1}}\eta_{32} = -\tau_{32} - \kappa_{31}\, e^{(\phi_++\phi_-)} - \sqrt{i}\psi_2(e^{\frac{(\phi_+-\phi_-)}{2}}\tau_{22} + e^{-\frac{(\phi_+-\phi_-)}{2}}\eta_{12}) + \sqrt{i}\tau_{33}\,\psi_1\, e^{\frac{(\phi_++\phi_-)}{2}}, \quad (C.38)$$

$$\partial_{t_{-1}}\eta_{33} = -\sqrt{i}(\eta_{32}e^{-\frac{(\phi_++\phi_-)}{2}} + \tau_{31}e^{\frac{(\phi_++\phi_-)}{2}})\psi_1 - \sqrt{i}\psi_2(e^{-\frac{(\phi_+-\phi_-)}{2}}\eta_{13} + e^{\frac{(\phi_+-\phi_-)}{2}}\tau_{23}), \quad (C.39)$$

$$\partial_x\eta_{11} = -\eta_{11}\,\partial_x\phi_-, \quad (C.40)$$

$$\partial_x\eta_{12} = \eta_{12}\,\partial_x\phi_+ + \eta_{22} - \eta_{11} - \sqrt{i}\left(\bar\psi_2\,\eta_{32} - \eta_{13}\,\bar\psi_1\right), \quad (C.41)$$

$$\partial_x\eta_{13} = \frac{\eta_{13}}{2}\,\partial_x(\phi_+ - \phi_-) - \sqrt{i}\left(\bar\psi_2\eta_{33} - \eta_{11}\bar\psi_1\right), \quad (C.42)$$

$$\partial_x\eta_{22} = \eta_{22}\,\partial_x\phi_-, \quad (C.43)$$

$$\partial_x\eta_{32} = \frac{\eta_{32}}{2}\partial_x(\phi_+ + \phi_-) - \sqrt{i}\left(\bar\psi_2\,\eta_{22} - \eta_{33}\,\bar\psi_1\right), \quad (C.44)$$

$$\partial_x\eta_{33} = 0. \quad (C.45)$$

<u>$\lambda^{+1/2}$- terms :</u>

$$\partial_{t_{-1}}\kappa_{11} = 0, \quad (C.46)$$

$$\partial_{t_{-1}}\kappa_{21} = \kappa_{11}\, e^{-\phi_+}(e^{\phi_-} - e^{-\phi_-}) + \sqrt{i}\, e^{-\frac{\phi_+}{2}}(\kappa_{23}\,\psi_1\, e^{-\frac{\phi_-}{2}} + e^{\frac{\phi_-}{2}}\,\psi_2\,\kappa_{31}), \quad (C.47)$$

$$\partial_{t_{-1}}\kappa_{23} = \sqrt{i}\, e^{-\frac{\phi_+}{2}}(e^{\frac{\phi_-}{2}}\,\psi_2\,\kappa_{33} - \kappa_{11}\,\psi_1\, e^{-\frac{\phi_-}{2}}), \quad (C.48)$$

$$\partial_{t_{-1}}\kappa_{31} = \sqrt{i}\, e^{-\frac{\phi_+}{2}}(\kappa_{33}\,\psi_1\, e^{-\frac{\phi_-}{2}} - e^{\frac{\phi_-}{2}}\psi_2\,\kappa_{11}), \quad (C.49)$$

$$\partial_{t_{-1}}\kappa_{33} = 0, \quad (C.50)$$

$$\partial_x\kappa_{11} = -\kappa_{11}\,\partial_x\phi_- + \kappa_{21} - \eta_{12} + \sqrt{i}\left(\tau_{13}\,\bar\psi_1 - \bar\psi_2\,\kappa_{31}\right), \quad (C.51)$$

$$\partial_x\kappa_{22} = \kappa_{22}\,\partial_x\phi_- - \kappa_{21} + \eta_{12} + \sqrt{i}\left(\kappa_{23}\,\bar\psi_1 - \bar\psi_2\,\tau_{32}\right), \quad (C.52)$$

$$\partial_x\kappa_{21} = -\kappa_{21}\,\partial_x\phi_+ + \eta_{11} - \eta_{22} + \sqrt{i}\left(\tau_{23}\,\bar\psi_1 - \bar\psi_2\,\tau_{31}\right), \quad (C.53)$$

$$\partial_x\kappa_{23} = -\frac{\kappa_{23}}{2}\partial_x(\phi_+ - \phi_-) + \eta_{13} + \tau_{23} - \sqrt{i}\bar\psi_2\,\tau_{33} + \sqrt{i}(\tau_{22} + \kappa_{21})\bar\psi_1 \quad (C.54)$$

$$\partial_x \kappa_{31} = -\frac{\kappa_{31}}{2}\partial_x(\phi_+ + \phi_-) - \eta_{32} - \tau_{31} + \sqrt{i}\tau_{33}\,\bar{\psi}_1 - \sqrt{i}\bar{\psi}_2(\tau_{11} + \kappa_{21}), \tag{C.55}$$

$$\partial_x \kappa_{33} = \sqrt{i}(\tau_{32} + \kappa_{31})\bar{\psi}_1 - \sqrt{i}\bar{\psi}_2(\tau_{13} + \kappa_{23}). \tag{C.56}$$

Appendix D

Type-II defect matrix for the mKdV hierarchy

Here we consider a different solution for the defect matrix [31]*, satisfying (C.1), as follows,

$$K = \left(\begin{array}{cc|c} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ \hline K_{31} & K_{32} & K_{33} \end{array}\right), \tag{D.1}$$

where

$$K_{11} = \frac{i}{2}m\sigma c_{11}\, e^{-\frac{\phi_-}{2}} f_1\tilde{f}_1 + \lambda^{-1/2} b_{11}\, e^{-\phi_-} + \lambda^{1/2} c_{11}, \tag{D.2}$$

$$K_{12} = \lambda^{-1/2} b_{12}\, e^{(\phi_+ - \lambda_0)}, \tag{D.3}$$

$$K_{13} = -\sqrt{im\sigma}\, c_{11}\, e^{\frac{(\phi_+ - \lambda_0)}{2}} f_1 + \lambda^{-1/2} \frac{2\sqrt{i}}{\sqrt{m\sigma}} b_{11}\, e^{\frac{(\phi_+ - \lambda_0)}{2}} e^{-\frac{\phi_-}{2}} \tilde{f}_1, \tag{D.4}$$

$$K_{21} = \lambda^{1/2} b_{12}\, e^{-(\phi_+ - \lambda_0)} \left[\sinh^2\left(\frac{\phi_-}{2}\right) + \cosh^2\tau + i\cosh\tau\cosh\left(\frac{\phi_-}{2}\right) f_1\tilde{f}_1\right], \tag{D.5}$$

$$K_{22} = \frac{i}{2}m\sigma c_{11}\, e^{\frac{\phi_-}{2}} f_1\tilde{f}_1 + \lambda^{-1/2} b_{11}\, e^{\phi_-} + \lambda^{1/2} c_{11} \tag{D.6}$$

*Note that in order to keep the notation consistent with this thesis we require that $m = -2$ and $\sigma = \frac{1}{\omega^2}$.

$$
\begin{aligned}
K_{23} &= \frac{2\sqrt{i}}{\sqrt{m\sigma}} b_{11}\, e^{-\frac{(\phi_+-\lambda_0)}{2}} e^{\frac{\phi_-}{2}} \left[\cosh\tau\, \tilde{f}_1 - \sinh\left(\frac{\phi_-}{2}\right) f_1\right] \\
&\quad -\lambda^{1/2}\sqrt{im\sigma}\, c_{11}\, e^{-\frac{(\phi_+-\lambda_0)}{2}} \left[\sinh\left(\frac{\phi_-}{2}\right)\tilde{f}_1 + \cosh\tau\, f_1\right], \qquad \text{(D.7)} \\
K_{31} &= \frac{2\sqrt{i}}{\sqrt{m\sigma}} b_{11}\, e^{-\frac{(\phi_+-\lambda_0)}{2}} e^{-\frac{\phi_-}{2}} \left[\cosh\tau\, \tilde{f}_1 - \sinh\left(\frac{\phi_-}{2}\right) f_1\right] \\
&\quad +\lambda^{1/2}\sqrt{im\sigma}\, c_{11}\, e^{-\frac{(\phi_+-\lambda_0)}{2}} \left[\sinh\left(\frac{\phi_-}{2}\right)\tilde{f}_1 + \cosh\tau\, f_1\right], \qquad \text{(D.8)} \\
K_{32} &= \sqrt{im\sigma}\, c_{11}\, e^{\frac{(\phi_+-\lambda_0)}{2}} f_1 + \lambda^{-1/2}\frac{2\sqrt{i}}{\sqrt{m\sigma}} b_{11}\, e^{\frac{(\phi_+-\lambda_0)}{2}} e^{\frac{\phi_-}{2}}\, \tilde{f}_1, \qquad \text{(D.9)} \\
K_{33} &= -m\sigma c_{11}\left[i\cosh\left(\frac{\phi_-}{2}\right)\tilde{f}_1 f_1 + \cosh\tau\right] + \lambda^{-1/2} b_{11} + \lambda^{1/2} c_{11}, \qquad \text{(D.10)}
\end{aligned}
$$

where the constant parameters satisfy the following relations,

$$
b_{11} = \frac{m\sigma}{4} b_{12}, \qquad b_{12} = m\sigma c_{11}. \qquad \text{(D.11)}
$$

It leads to the following equations,

$$
\begin{aligned}
\psi_- &= \sqrt{\frac{m}{\sigma}}\left[e^{\frac{\lambda_0}{2}}\sinh\left(\frac{\phi_-}{2}\right) f_1 - \left(e^{-\frac{\lambda_0}{2}} + e^{\frac{\lambda_0}{2}}\cosh\tau\right)\tilde{f}_1\right], \qquad \text{(D.12)} \\
\bar{\psi}_- &= \sqrt{m\sigma}\left(e^{\frac{(\phi_+-\lambda_0)}{2}} + e^{-\frac{(\phi_+-\lambda_0)}{2}}\cosh\tau\right) f_1 \\
&\quad + \sqrt{m\sigma}\, e^{-\frac{(\phi_+-\lambda_0)}{2}}\sinh\left(\frac{\phi_-}{2}\right)\tilde{f}_1, \\
&\qquad \text{(D.13)} \\
\partial_-(\phi_+-\lambda_0) &= -\frac{m}{2\sigma} e^{\lambda_0}\sinh\phi_- - \frac{im}{2\sigma}\left(1 + e^{\lambda_0}\cosh\tau\right)\sinh\left(\frac{\phi_-}{2}\right) f_1\tilde{f}_1 \\
&\quad -\frac{i}{2}\sqrt{\frac{m}{\sigma}} e^{\frac{\lambda_0}{2}}\cosh\left(\frac{\phi_-}{2}\right)\psi_+ f_1, \qquad \text{(D.14)} \\
\partial_-\phi_- &= \frac{m}{\sigma}\left[e^{-\lambda_0} - e^{\lambda_0}\left(\sinh^2\left(\frac{\phi_-}{2}\right) + \cosh^2\tau\right)\right] \\
&\quad -\frac{i}{2}\sqrt{\frac{m}{\sigma}}\left[\left(e^{-\frac{\lambda_0}{2}} - e^{\frac{\lambda_0}{2}}\cosh\tau\right)\psi_+\tilde{f}_1 + e^{\frac{\lambda_0}{2}}\sinh\left(\frac{\phi_-}{2}\right)\psi_+ f_1\right] \\
&\quad -\frac{im}{\sigma} e^{\lambda_0}\cosh\tau\cosh\left(\frac{\phi_-}{2}\right) f_1\tilde{f}_1, \qquad \text{(D.15)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_+\phi_- &= m\sigma\left[e^{-(\phi_+-\lambda_0)}\Big(\sinh^2\left(\frac{\phi_-}{2}\right)+\cosh^2\tau\Big)-e^{(\phi_+-\lambda_0)}\right] \\
&+\frac{i\sqrt{m\sigma}}{2}\Big(e^{\frac{(\phi_+-\lambda_0)}{2}}-e^{-\frac{(\phi_+-\lambda_0)}{2}}\cosh\tau\Big)\bar{\psi}_+f_1 \\
&-\frac{i\sqrt{m\sigma}}{2}\,e^{-\frac{(\phi_+-\lambda_0)}{2}}\sinh\left(\frac{\phi_-}{2}\right)\bar{\psi}_+\tilde{f}_1 \\
&+im\sigma\,e^{-(\phi_+-\lambda_0)}\cosh\tau\cosh\left(\frac{\phi_-}{2}\right)f_1\tilde{f}_1, \qquad (D.16)\\
\partial_+\lambda_0 &= -\frac{m\sigma}{2}\,e^{-(\phi_+-\lambda_0)}\sinh\phi_- -\frac{im\sigma}{2}\Big(1+e^{-(\phi_+-\lambda_0)}\cosh\tau\Big)\sinh\left(\frac{\phi_-}{2}\right)f_1\tilde{f}_1 \\
&+\frac{i\sqrt{m\sigma}}{2}\,e^{-\frac{(\phi_+-\lambda_0)}{2}}\cosh\left(\frac{\phi_-}{2}\right)\bar{\psi}_+\tilde{f}_1, \qquad (D.17)\\
\partial_+\tilde{f}_1 &= -\frac{\sqrt{m\sigma}}{2}e^{-\frac{(\phi_+-\lambda_0)}{2}}\sinh\left(\frac{\phi_-}{2}\right)\bar{\psi}_+ +\frac{m\sigma}{2}\cosh\left(\frac{\phi_-}{2}\right)f_1 \qquad (D.18)\\
&+\frac{m\sigma}{2}e^{-(\phi_+-\lambda_0)}\cosh\tau\ \cosh\left(\frac{\phi_-}{2}\right)f_1 \qquad (D.19)\\
\partial_-\tilde{f}_1 &= \frac{1}{2}\sqrt{\frac{m}{\sigma}}\Big(e^{-\frac{\lambda_0}{2}}+e^{\frac{\lambda_0}{2}}\cosh\tau\Big)\psi_+ -\frac{m}{2\sigma}\left[1+e^{\lambda_0}\cosh\tau\right]\cosh\left(\frac{\phi_-}{2}\right)f_1, \qquad (D.20)\\
\partial_- f_1 &= -\frac{1}{2}\sqrt{\frac{m}{\sigma}}e^{\frac{\lambda_0}{2}}\sinh\left(\frac{\phi_-}{2}\right)\psi_+ +\frac{m}{2\sigma}\left[1+e^{\lambda_0}\cosh\tau\right]\cosh\left(\frac{\phi_-}{2}\right)\tilde{f}_1, \qquad (D.21)\\
\partial_+ f_1 &= -\frac{\sqrt{m\sigma}}{2}\Big(e^{\frac{(\phi_+-\lambda_0)}{2}}+e^{-\frac{(\phi_+-\lambda_0)}{2}}\cosh\tau\Big)\bar{\psi}_+ -\frac{m\sigma}{2}\cosh\left(\frac{\phi_-}{2}\right)\tilde{f}_1 \\
&-\frac{m\sigma}{2}e^{-(\phi_+-\lambda_0)}\cosh\tau\ \cosh\left(\frac{\phi_-}{2}\right)\tilde{f}_1. \qquad (D.22)
\end{aligned}
$$

which correspond to the type-II super-Bäcklund transformation for the $N=1$ sshG model with two arbitrary parameters (σ,τ).

In [31] we also have written this type-II super-Bäcklund transformation in terms of superfields.

Appendix E

Coefficients of the Bäcklund transformations for $N = 5$ member

The coefficients c_i in the Bäcklund equations (3.56) are given by,

$$
\begin{aligned}
c_0 &= -\partial_x^4\phi_+ \cosh\left(\frac{\phi_+}{2}\right) + \left(\partial_x^2\phi_+\right)^2 \sinh\left(\frac{\phi_+}{2}\right) + 3\left(\partial_x^3\phi_+\right)\left(\partial_x\phi_+\right)\sinh\left(\frac{\phi_+}{2}\right) \\
&\quad + \left(\partial_x^2\phi_+\right)\left(\partial_x\phi_+\right)^2\cosh\left(\frac{\phi_+}{2}\right) - \frac{3}{4}\left(\partial_x\phi_+\right)^4\sinh\left(\frac{\phi_+}{2}\right), && \text{(E.1)} \\
c_1 &= \partial_x^3\phi_+ \cosh\left(\frac{\phi_+}{2}\right) - \left(\partial_x\phi_+\right)^3\cosh\left(\frac{\phi_+}{2}\right) + 4\left(\partial_x^2\phi_+\right)\left(\partial_x\phi_+\right)\sinh\left(\frac{\phi_+}{2}\right), && \text{(E.2)} \\
c_2 &= -\left(\partial_x^2\phi_+\right)\cosh\left(\frac{\phi_+}{2}\right) + 2\left(\partial_x\phi_+\right)^2\sinh\left(\frac{\phi_+}{2}\right), && \text{(E.3)} \\
c_3 &= \partial_x\phi_+ \cosh\left(\frac{\phi_+}{2}\right), && \text{(E.4)} \\
c_4 &= -2\sinh\left(\frac{\phi_+}{2}\right), && \text{(E.5)} \\
c_5 &= 4\left(\partial_x^4\phi_+\right)\cosh\phi_+ - 6\left(\partial_x^2\phi_+\right)\left(\partial_x\phi_+\right)^2\cosh\phi_+ + 2\left(\partial_x^2\phi_+\right)^2\sinh\phi_+ \\
&\quad -4\left(\partial_x^3\phi_+\right)\left(\partial_x\phi_+\right)\sinh\phi_+ + \frac{3}{2}\left(\partial_x\phi_+\right)^4\sinh\phi_+, && \text{(E.6)} \\
c_6 &= 4\left(\partial_x^2\phi_+\right)\cosh\phi_+ - 4\left(\partial_x\phi_+\right)^2\sinh\phi_+, && \text{(E.7)} \\
c_7 &= 2(\partial_x\phi_+)(\cosh\phi_+), && \text{(E.8)}
\end{aligned}
$$

$$
\begin{aligned}
c_8 &= 2\sinh\phi_+, && \text{(E.9)}\\
c_9 &= \left[-20\cosh\left(\frac{\phi_+}{2}\right)+20\cosh\left(\frac{3\phi_+}{2}\right)+80\cosh\left(\frac{5\phi_+}{2}\right)\right](\partial_x^2\phi_+)\\
&\quad+\left[35\ \sinh\left(\frac{\phi_+}{2}\right)-\frac{15}{2}\sinh\left(\frac{3\phi_+}{2}\right)+\frac{75}{2}\sinh\left(\frac{5\phi_+}{2}\right)\right](\partial_x\phi_+)^2, && \text{(E.10)}\\
c_{10} &= \left[70\cosh\left(\frac{\phi_+}{2}\right)-25\cosh\left(\frac{3\phi_+}{2}\right)+35\cosh\left(\frac{5\phi_+}{2}\right)\right]\partial_x\phi_+, && \text{(E.11)}\\
c_{11} &= -20\sinh\left(\frac{\phi_+}{2}\right)+10\sinh\left(\frac{3\phi_+}{2}\right)+30\sinh\left(\frac{5\phi_+}{2}\right), && \text{(E.12)}\\
c_{12} &= 40\partial_x^2\phi_+\left(\cosh\phi_+-\cosh(3\phi_+)\right)-20(\partial_x\phi_+)^2\left(5\sinh\phi_++\sinh(3\phi_+)\right), && \text{(E.13)}\\
c_{13} &= 30\sinh\phi_+-10\sinh(3\phi_+), && \text{(E.14)}\\
c_{14} &= -120\sinh\left(\frac{\phi_+}{2}\right)+80\sinh\left(\frac{3\phi_+}{2}\right)+240\sinh\left(\frac{5\phi_+}{2}\right)-60\sinh\left(\frac{7\phi_+}{2}\right)\\
&\quad-100\sinh\left(\frac{9\phi_+}{2}\right), && \text{(E.15)}\\
c_{15} &= 240\sinh\phi_+-120\sinh(3\phi_+)+24\sinh(5\phi_+). && \text{(E.16)}
\end{aligned}
$$

And the coefficients g_j in the Bäcklund equations (3.57) are the following,

$$
\begin{aligned}
g_0 &= \left[-\frac{1}{2}(\partial_x^2\phi_+)^2-\frac{3}{2}(\partial_x^3\phi_+)(\partial_x\phi_+)+\frac{3}{8}(\partial_x\phi_+)^4\right]\cosh\left(\frac{\phi_+}{2}\right)\\
&\quad+\left[\frac{1}{2}\partial_x^4\phi_+-\frac{1}{2}(\partial_x^2\phi_+)(\partial_x\phi_+)^2\right]\sinh\left(\frac{\phi_+}{2}\right), && \text{(E.17)}\\
g_1 &= \left[-\frac{1}{2}\partial_x^3\phi_++\frac{1}{2}(\partial_x\phi_+)^3\right]\sinh\left(\frac{\phi_+}{2}\right)-2(\partial_x^2\phi_+)(\partial_x\phi_+)\cosh\left(\frac{\phi_+}{2}\right), && \text{(E.18)}\\
g_2 &= \frac{1}{2}(\partial_x^2\phi_+)\sinh\left(\frac{\phi_+}{2}\right)-(\partial_x\phi_+)^2\cosh\left(\frac{\phi_+}{2}\right), && \text{(E.19)}\\
g_3 &= -\frac{1}{2}\partial_x\phi_+\sinh\left(\frac{\phi_+}{2}\right), && \text{(E.20)}\\
g_4 &= \cosh\left(\frac{\phi_+}{2}\right), && \text{(E.21)}\\
g_5 &= \left[10\sinh\phi_+-5\sinh(2\phi_+)\right]\partial_x\phi_+, && \text{(E.22)}
\end{aligned}
$$

$$\begin{aligned}
g_6 &= -\left[\frac{5}{2} + 20\cosh\phi_+ + \frac{35}{2}\cosh(2\phi_+)\right](\partial_x^2\phi_+)(\partial_x\phi_+) \\
&- \left[10\sinh\phi_+ + 5\sinh(2\phi_+)\right]\partial_x^3\phi_+ - \left[\frac{15}{2}\sinh\phi_+ + \frac{15}{4}\sinh(2\phi_+)\right](\partial_x\phi_+)^3, \qquad \text{(E.23)} \\
g_7 &= -\left[\frac{35}{2}\cosh\left(\frac{\phi_+}{2}\right) + \frac{45}{4}\cosh\left(\frac{3\phi_+}{2}\right) + \frac{45}{4}\cosh\left(\frac{5\phi_+}{2}\right)\right](\partial_x\phi_+)^2 \\
&- \left[15\sinh\left(\frac{3\phi_+}{2}\right) + 15\sinh\left(\frac{5\phi_+}{2}\right)\right]\partial_x^2\phi_+, \qquad \text{(E.24)} \\
g_8 &= \left[-15\sinh\left(\frac{\phi_+}{2}\right) - \frac{45}{2}\sinh\left(\frac{3\phi_+}{2}\right) - \frac{15}{2}\sinh\left(\frac{5\phi_+}{2}\right)\right](\partial_x\phi_+), \qquad \text{(E.25)} \\
g_9 &= 10\cosh\left(\frac{\phi_+}{2}\right) - 5\cosh\left(\frac{3\phi_+}{2}\right) - 5\cosh\left(\frac{5\phi_+}{2}\right), \qquad \text{(E.26)} \\
g_{10} &= \left[-120\sinh\phi_+ - 40\sinh(2\phi_+) + 40\sinh(3\phi_+) + 20\sinh(4\phi_+)\right]\partial_x\phi_+, \qquad \text{(E.27)} \\
g_{11} &= 60\cosh\left(\frac{\phi_+}{2}\right) - 40\cosh\left(\frac{3\phi_+}{2}\right) - 40\cosh\left(\frac{5\phi_+}{2}\right) + 10\cosh\left(\frac{7\phi_+}{2}\right) \\
&+ 10\cosh\left(\frac{9\phi_+}{2}\right). \qquad \text{(E.28)}
\end{aligned}$$

Appendix F

Conservation of mKdV momentum defect with respect to t_5

Here we verify the conservation of the momentum of the mKdV with respect to t_5, namely

$$\begin{aligned}
\frac{dP}{dt_5} &= \Bigg[\frac{5}{8}(\partial_x\phi_1)^6 + \frac{1}{16}(\partial_x^3\phi_1)^2 - \frac{1}{8}\partial_x^2\phi_1\,\partial_x^4\phi_1 + \frac{1}{8}\partial_x\phi_1\,\partial_x^5\phi_1 - \frac{5}{8}(\partial_x\phi_1)^2(\partial_x^2\phi_1)^2 \\
&\quad - \frac{5}{4}(\partial_x\phi_1)^3\partial_x^3\phi_1 + i\bar{\psi}_1\partial_x\bar{\psi}_1\left(\frac{35}{16}\partial_x\phi_1\,\partial_x^3\phi_1 + \frac{5}{8}(\partial_x^2\phi_1)^2 - \frac{25}{8}(\partial_x\phi_1)^4\right) \\
&\quad - \frac{i}{8}\partial_x^2\bar{\psi}_1\partial_x^3\bar{\psi}_1 - \frac{i}{16}\bar{\psi}_1\partial_x^5\bar{\psi}_1 - \frac{5i}{8}(\partial_x\phi_1)^2\partial_x\bar{\psi}_1\partial_x^2\bar{\psi}_1 + \frac{15i}{16}\partial_x\phi_1\,\partial_x^2\phi_1\bar{\psi}_1\partial_x^2\bar{\psi}_1 \\
&\quad + \frac{15i}{16}(\partial_x\phi_1)^2\bar{\psi}_1\partial_x^3\bar{\psi}_1 + \frac{i}{8}\partial_x\bar{\psi}_1\partial_x^4\bar{\psi}_1\Bigg]_{x=0} - \left[\phi_1\to\phi_2,\,\bar{\psi}_1\to\bar{\psi}_2\right]_{x=0}. \qquad \text{(F.1)}
\end{aligned}$$

Using the equations of motion (2.54) and (2.55), we can write the above expression in the following form,

$$\begin{aligned}\frac{dP}{dt_5} &= \Big[2\partial_x\phi_1\,\partial_{t_5}\phi_1 - \frac{1}{8}(\partial_x\phi_1)^6 + \frac{1}{16}(\partial_x^3\phi_1)^2 + \frac{5}{8}(\partial_x\phi_1)^2(\partial_x^2\phi_1)^2 - \frac{1}{8}\partial_x\phi_1\,\partial_x^4\phi_1 \\ &- \frac{5i}{8}(\partial_x\phi_1)^2\partial_x\bar\psi_1\partial_x^2\bar\psi_1 - \frac{5i}{8}\partial_x\phi_1\,\partial_x^2\phi_1\bar\psi_1\partial_x^2\bar\psi_1 + \frac{5i}{8}\partial_x\phi_1\,\partial_x^3\phi_1\bar\psi_1\partial_x\bar\psi_1 - i\bar\psi_1\partial_{t_5}\bar\psi_1 \\ &+ \frac{i}{8}\partial_x\bar\psi_1\partial_x^4\bar\psi_1 - \frac{i}{8}\partial_x^2\bar\psi_1\partial_x^3\bar\psi_1\Big]_{x=0} - \left[\phi_1\to\phi_2,\,\bar\psi_1\to\bar\psi_2\right]_{x=0}. \end{aligned} \tag{F.2}$$

In terms of the variables $\phi_\pm = \phi_1 \pm \phi_2$ and $\bar\psi_\pm = \bar\psi_1 \pm \bar\psi_2$, it reads

$$\begin{aligned}\frac{dP}{dt_5} &= \Big[\partial_x\phi_-\,\partial_{t_5}\phi_+ + \partial_x\phi_+\,\partial_{t_5}\phi_- + \frac{1}{16}\partial_x^3\phi_-\,\partial_x^3\phi_+ - \frac{3}{128}(\partial_x\phi_-)^5\partial_x\phi_+ \\ &- \frac{1}{16}\partial_x^2\phi_+\,\partial_x^4\phi_- - \frac{1}{16}\partial_x^2\phi_-\,\partial_x^4\phi_+ - \frac{5}{64}(\partial_x\phi_-)^3(\partial_x\phi_+)^3 + \frac{5}{32}(\partial_x\phi_-)^2\partial_x^2\phi_-\,\partial_x^2\phi_+ \\ &+ \frac{5}{32}\partial_x\phi_-\,\partial_x\phi_+(\partial_x^2\phi_-)^2 + \frac{5}{32}\partial_x\phi_-\,\partial_x\phi_+(\partial_x^2\phi_+)^2 + \frac{i}{16}\partial_x\bar\psi_-\partial_x^4\bar\psi_+ + \frac{i}{16}\partial_x\bar\psi_+\partial_x^4\bar\psi_- \\ &- \frac{5i}{32}\partial_x\phi_-\,\partial_x\phi_+\left(\partial_x\bar\psi_-\partial_x^2\bar\psi_- + \partial_x\bar\psi_+\partial_x^2\bar\psi_+\right) - \frac{i}{16}\partial_x^2\bar\psi_-\partial_x^3\bar\psi_+ - \frac{i}{16}\partial_x^2\bar\psi_+\partial_x^3\bar\psi_- \\ &- \frac{5i}{64}\left(\bar\psi_-\partial_x^2\bar\psi_- + \bar\psi_+\partial_x^2\bar\psi_+\right)\left(\partial_x\phi_+\,\partial_x^2\phi_- + \partial_x\phi_-\,\partial_x^2\phi_+\right) - \frac{i}{2}\bar\psi_-\partial_{t_5}\bar\psi_+ - \frac{i}{2}\bar\psi_+\partial_{t_5}\bar\psi_- \\ &+ \frac{5i}{64}\left(\bar\psi_-\partial_x\bar\psi_- + \bar\psi_+\partial_x\bar\psi_+\right)\left(\partial_x\phi_+\,\partial_x^3\phi_- + \partial_x\phi_-\,\partial_x^3\phi_+\right) - \frac{3}{128}(\partial_x\phi_+)^5\partial_x\phi_- \\ &+ \frac{5i}{64}\left(\bar\psi_-\partial_x\bar\psi_+ + \bar\psi_+\partial_x\bar\psi_-\right)\left(\partial_x\phi_-\,\partial_x^3\phi_- + \partial_x\phi_+\,\partial_x^3\phi_+\right) \\ &+ \frac{5i}{64}\left(\bar\psi_-\partial_x^2\bar\psi_+ + \bar\psi_+\partial_x^2\bar\psi_-\right)\left(\partial_x\phi_-\,\partial_x^2\phi_- + \partial_x\phi_+\,\partial_x^2\phi_+\right) \\ &- \frac{5i}{64}\left(\partial_x\bar\psi_-\partial_x^2\bar\psi_+ + \partial_x\bar\psi_+\partial_x^2\bar\psi_-\right)\left((\partial_x\phi_-)^2 + (\partial_x\phi_+)^2\right)\Big]_{x=0}. \end{aligned} \tag{F.3}$$

Now using the Bäcklund equations (3.4)–(3.6), (3.56), (3.57), we find

$$\begin{aligned}
\frac{dP}{dt_5} = & \left[-\frac{i}{\omega}\left(2\cosh\left(\frac{\phi_+}{2}\right) f_1\partial_{t_5}\bar{\psi}_+ + \sinh\left(\frac{\phi_+}{2}\right)\partial_{t_5}\phi_+ f_1\bar{\psi}_+\right) + \frac{4}{\omega^2}\sinh\phi_+\partial_{t_5}\phi_+\right. \\
&+ \frac{i}{8\omega^2}\left(\cosh^2\left(\frac{\phi_+}{2}\right)\bar{\psi}_+\partial_x^4\bar{\psi}_+ - \frac{1}{4}\sinh\phi_+\partial_x\phi_+\bar{\psi}_+\partial_x^3\bar{\psi}_+\right) \\
&+ \frac{i}{16\omega^2}\bar{\psi}_+\partial_x\bar{\psi}_+\left(\frac{1}{2}\sinh\phi_+\left((\partial_x\phi_+)^3 - \partial_x^3\phi_+\right) - 4\cosh^2\left(\frac{\phi_+}{2}\right)\partial_x\phi_+\partial_x^2\phi_+\right) \\
&+ \frac{i}{8\omega^2}\bar{\psi}_+\partial_x^2\bar{\psi}_+\left(\frac{1}{4}\sinh\phi_+\partial_x^2\phi_+ - \cosh^2\left(\frac{\phi_+}{2}\right)(\partial_x\phi_+)^2\right) \\
&+ \frac{5i}{8\omega^5}\cosh^3\left(\frac{\phi_+}{2}\right)\sinh\phi_+ f_1\bar{\psi}_+\left(3(\partial_x\phi_+)^3 + 4\partial_x^3\phi_+\right) \\
&- \frac{5i}{2\omega^6}\cosh^2\left(\frac{\phi_+}{2}\right)\sinh\phi_+\left(\sinh\phi_+\bar{\psi}_+\partial_x^2\bar{\psi}_+ + 3\cosh^2\left(\frac{\phi_+}{2}\right)\partial_x\phi_+\bar{\psi}_+\partial_x\bar{\psi}_+\right) \\
&+ \frac{5i}{4\omega^5}\cosh^3\left(\frac{\phi_+}{2}\right) f_1\bar{\psi}_+\partial_x\phi_+\partial_x^2\phi_+(7\cosh\phi_+ - 3) \\
&\left.- \frac{40i}{\omega^9}\cosh^3\left(\frac{\phi_+}{2}\right)\sinh^3\phi_+\partial_x\phi_+ f_1\bar{\psi}_+\right]_{x=0}.
\end{aligned} \tag{F.4}$$

Note that the first term can be written as follows,

$$\partial_{t_5}\left(2\cosh\left(\frac{\phi_+}{2}\right) f_1\bar{\psi}_+\right) - 2\cosh\left(\frac{\phi_+}{2}\right)\partial_{t_5} f_1\bar{\psi}_+, \tag{F.5}$$

and then,

$$\begin{aligned}
\frac{dP}{dt_5} &= \Bigg[\partial_{t_5}\left(\frac{4}{\omega^2}\cosh\phi_+ - \frac{2i}{\omega}\cosh\left(\frac{\phi_+}{2}\right)f_1\bar{\psi}_+\right) + \frac{2i}{\omega}\cosh\left(\frac{\phi_+}{2}\right)\partial_{t_5}f_1\bar{\psi}_+ \\
&+ \frac{i}{8\omega^2}\left(\cosh^2\left(\frac{\phi_+}{2}\right)\bar{\psi}_+\partial_x^4\bar{\psi}_+ - \frac{1}{4}\sinh\phi_+\partial_x\phi_+\bar{\psi}_+\partial_x^3\bar{\psi}_+\right) \\
&+ \frac{i}{16\omega^2}\bar{\psi}_+\partial_x\bar{\psi}_+\left(\frac{1}{2}\sinh\phi_+\left((\partial_x\phi_+)^3 - \partial_x^3\phi_+\right) - 4\cosh^2\left(\frac{\phi_+}{2}\right)\partial_x\phi_+\partial_x^2\phi_+\right) \\
&+ \frac{i}{8\omega^2}\bar{\psi}_+\partial_x^2\bar{\psi}_+\left(\frac{1}{4}\sinh\phi_+\partial_x^2\phi_+ - \cosh^2\left(\frac{\phi_+}{2}\right)(\partial_x\phi_+)^2\right) \\
&+ \frac{5i}{8\omega^5}\cosh^3\left(\frac{\phi_+}{2}\right)\sinh\phi_+ f_1\bar{\psi}_+\left(3(\partial_x\phi_+)^3 + 4\partial_x^3\phi_+\right) \\
&- \frac{5i}{2\omega^6}\cosh^2\left(\frac{\phi_+}{2}\right)\sinh\phi_+\left(\sinh\phi_+\bar{\psi}_+\partial_x^2\bar{\psi}_+ + 3\cosh^2\left(\frac{\phi_+}{2}\right)\partial_x\phi_+\bar{\psi}_+\partial_x\bar{\psi}_+\right) \\
&+ \frac{5i}{4\omega^5}\cosh^3\left(\frac{\phi_+}{2}\right)f_1\bar{\psi}_+\partial_x\phi_+\partial_x^2\phi_+(7\cosh\phi_+ - 3) \\
&- \frac{40i}{\omega^9}\cosh^3\left(\frac{\phi_+}{2}\right)\sinh^3\phi_+\partial_x\phi_+ f_1\bar{\psi}_+\Bigg]_{x=0}. \qquad \text{(F.6)}
\end{aligned}$$

By using eq. (3.57) we can write eq (F.6) as a total time (t_3) derivative, and finally we obtain that

$$\mathcal{P} = P - \left[\frac{4}{\omega^2}\cosh\phi_+ - \frac{2i}{\omega}\cosh\left(\frac{\phi_+}{2}\right)f_1\bar{\psi}_+\right]_{x=0}, \qquad \text{(F.7)}$$

is the modified conserved momentum, which includes the same defect contribution previously derived by applying the t_3 derivative.

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