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On equivalence of Scalar Quantum Electrodynamics via Duffin-Kemmer-Petiau and Klein-Gordon-Fock formalism using Causal Perturbation Theory approach

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Dedicated to Victorino and Ana.

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Abstract

In this Thesis we use Causal Perturbation Theory to study Scalar Quantum Electrodynamics with Duffin-Kemmer-Petiau fields. We determine the differential cross sections at the tree level, the vacuum polarization tensor, self energy function and the normalizability of the theory. After that, we compare our results with those ones obtained via Klein-Gordon-Fock fields determining that they are not completely equivalent.

Keyword: Causal Perturbation Theory; Scalar Quantum Electrodynamics; DKP.

Research field: 1.05.01.01-0;1.05.02.01-7; 1.05.03.01-3

Resumo

Nesta tese utilizamos a Teoria de Perturbação Causal para estudar a Eletrodinâmica Quântica Escalar com os campos de Duffin-Kemmer-Petiau. Determinamos as seções de choque diferenciais no nível da árvore, o tensor de polarização do vácuo, a função de auto energia e a renormalizabilidade da teoria. Depois disso, comparamos nossos resultados com os obtidos através dos campos de Klein-Gordon-Fock, determinando que eles não são completamente equivalentes.

Palavras Chaves: Teoria de Perturbação Causal; Eletrodinâmica Quântica Escalar; DKP.

Áreas do conhecimento: 1.05.01.01-0;1.05.02.01-7; 1.05.03.01-3

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Chapter 1

Introduction

The Duffin-Kemmer-Petiu (DKP) theory is based on the idea of obtaining a first order relativistic equation to model photons. This idea was implemented during 1934 by L. de Broglie, who considered that the photon was composed of two leptons and used a product of Dirac γ-matrices to construct a similar equation but with square β-matrices of order 16 [1, 2].

During the years 1936 to 1939 G. Petiau, R. J. Duffin and N. Kemmer [3–5] individually found that the 16×16 β-matrices had three irreducible representations of dimensions 1, 5 and 10. The representation of order 1 is trivial, the order 5 and 10 representation allow modeling scalar and spin-1 particles respectively[1].

After World War II, many calculations were performed on scalar quantum electrodynamics using the DKP (SDKP) and Klein-Gordon-Fock (SQED) fields. The main intention was to determine the differences between the two approaches, but up to 1-loop corrections all differential cross sections were the same [7–10]. Therefore, the belief on the total equivalence between both approaches was established in the scientific community.

However, in May 1971 the doubt about the equivalence between the two scalar particle theories was revived. In reference [11], E. Fischbach and collaborators found different results for the broken-symmetry parameter in the kaon semi-leptonic decay $k \to \pi + l + \nu$. The difference comes from the presence of two mesons with different masses and from the fact that the $SU(3)$ broken-symmetry process is sensitive to the field dimensions which takes the value 3/2 for DKP and 1 for KGF fields. Furthermore,

[1] A good historical development of the Duffin-Kemmer-Petiau equation is in [6].

the result obtained via DKP was surprisingly closer to the experimental data than that obtained via KGF formalism. Currently, the calculation is performed considering the compositional nature of k and π confirming the values obtained using DKP fields [12].

In 2000, V. Ya. Fainberg and B. M. Pimentel did a systematic study of S-matrices obtained from SDKP and SQED via minimal coupling procedure with an external or quantized electromagnetic field. They constructed the functional generator of the Green functions to quantize the DKP theory. After that, they used the LSZ reduction formula to determine the matrix elements of the S-matrix [13].

The results of V. Ya. Fainberg and B. M. Pimentel were positive but not conclusive. The equivalence between SDKP and SQE does not include the sector of diagrams generated by the self-interaction term $\sim (\phi^*\phi)^2$ and diagrams without the presence of external photons. The authors suggested the inclusion of analogous self-interacting term with DKP fields proportional to $(\bar{\psi}P\psi)^2$, where P is a projector that eliminates the DKP vectorial sector [14].

In the new millennium a rebirth in the interest of DKP theory comes from its advantages compared with the KGF fields. For example, the greater number of combinations of the DKP fields to generate self-interacting terms [14] has been used to determine analytical solutions of the DKP equation in presence of different kinds of potentials, see for example [15]. In addition, the DKP theory has been used to study their interactions in Riemann and Riemann-Cartan spaces [16–20], to study confinement in QCD [21], as well as applied to covariant Hamiltonian dynamics [22] and to the study of spin-1 particles in the Abelian monopole field [23].

The inclusion of the missing sectors in the article [13] of V. Ya. Fainberg and B. M. Pimentel is the main objective of this thesis. For this goal, we are going to use an axiomatic formalism known as Causal Perturbation Theory (CPT). The decision to use CPT has been taken because of the results obtained by M. Dütsch, F. Krahe and G. Scharf about SQED [26]. In the framework of CPT they demonstrated the unitarity, gauge invariance and normalizability without the second order self-interaction term $\sim (\phi^*\phi)^2$. The latter does not mean that the interaction term is missing in CPT, on the contrary the approach recovers all that sector using a physical property called *perturbative gauge invariance*. We must mention that the study of SDKP was initiated by J. T. Lunardi et al. [24, 25], therefore this thesis could be seen as an extension of those papers.

CPT is an approach that treats the quantized fields as *operator value distributions* and constructs the S-matrix as a formal series using two fundamental physical principles: Causality and perturbative gauge invariance. Each step is mathematically well defined within the framework of distributions theory. The main point of this formalism is to avoid the ill-defined product of distributions at the same point such as those that floods the formalism based on Feynman diagrams and that we believe are the ones that generate UV divergences.

The origin of CPT started in 1973 when H. Epstein and V. Glaser wrote their article entitled "The role of locality in perturbation theory" [27] where they developed an iterative construction of the S-matrix taking as advantage the causal support of the propagators to determine their advanced and retarded part. Ten years later, G. Scharf began to apply the approach to study Quantum Electrodynamics (QED) obtaining a finite theory, in other words, free UV and infrared divergences! [28–37]. From the striking results in QED, G. Scharf and collaborators applied CPT to study other quantum field theories as Yang-Mills [38–43], Higgs boson [44], Electroweak theory [45, 46], Super-symmetry [47–50] and Quantum Gravity [51–61]. On the other side of the Atlantic ocean, B. M. Pimentel and Collaborators applied CPT in General Quantum Electrodynamics (GQED) [62], Light front Dynamics [63, 64], SDKP [65], gauge Thirring model [66, 67], and QED_3 [68, 69].

This thesis is organized in the following form. In the second chapter, we summarize the concepts of *Distribution Theory* which we believe necessary to understand CPT. In Chapter 3, CPT is introduced in generality to be applied to any quantum field theory. In the fourth chapter, we develop the quantum properties of free DKP, electromagnetic and fermionic scalar ghost fields to be applied in SDKP and to develop gauge invariance at the quantum level. In the fifth chapter, we use perturbative gauge invariance to determine the base term of S-Matrix, after that, we determine the differential cross section of a DKP particle scattered by external electromagnetic field and for the Moller and Compton scattering process. In the sixth chapter, we compute the vacuum polarization tensor and the self energy function. In the seventh chapter, we will study the renormalizability of the theory. Finally, in the eight chapter, we write our conclusions and perspectives.

Chapter 2

Elementary Theory of Distributions

> Mathematical discovery is subversive and always ready to overthrow taboos, and it depends very little on established powers.
>
> *Laurent Schwartz*

In 1950-51 Laurent Schwartz published *Théorie des Distributions* [70], a treatise in two volumes where he constructs systematically the concept of *Distribution*[1]. Although this mathematical tool defines correctly many "functions" used in physics allowing consistent calculation, it has not yet been adopted by the community in all its potentiality.

Following the subversive spirit of L. Schwartz, in this thesis we will use the *Bogoliubov-Epstein-Glaser* or CPT approach to solve Quantum Field Theory. CPT uses the theory of distributions framework in the construction of S-matrix. For this reason, we dedicate this chapter to present the necessary concepts about Schwartz's theory.

2.1 The necessity of distribution and its definition

In 1927 P. A. M. Dirac introduce the delta symbol $\delta(x)$ [72] with the following properties

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \quad , \quad \int dx \delta(x) = 1. \tag{2.1}$$

[1]The Theory of Distributions is also known as *Theory of Generalized Functions* which was the name that S. L. Sobolev proposed in his study of Cauchy's problem in hyperbolic equations [71].

Taking into consideration functional analysis we can prove that the two properties in (2.1) are in contradiction. Knowing the latter, Dirac said "Strictly, of course, $\delta(x)$ is not a proper function of x, but can be regarded only as a limit of a certain sequence of functions", but again, in the context of function analysis, this limit does not exist.

The necessity of the Dirac delta function $\delta(x)$ is a consequence to fix a physical quantity in a point of space. For example consider the density $\rho(\mathbf{x})$ of a point particle of mass 1. We can understand this quantity as the limit of a sequence of spheres densities $\rho_\epsilon(\mathbf{x})$ with less and less radius ϵ but same mass 1. This sequence of densities have the values

$$\rho_\epsilon(\mathbf{x}) = \begin{cases} \frac{1}{\frac{4}{3}\pi\epsilon^3}, & \|\mathbf{x}\| \leqslant \epsilon \\ 0, & \|\mathbf{x}\| > \epsilon \end{cases} \tag{2.2}$$

and we can note that in the limit $\epsilon \to 0$ we have $\rho_\epsilon(\mathbf{x}) \to \delta(\mathbf{x}) = \rho(\mathbf{x})$, but the integral in all space is null, which have no physical sense for a density [73, 74]. To solve this problem, we define the *weak limit*.

Definition 2.1 *Consider the continuous $f(x)$, with $x \in \mathbb{R}^n$. For a sequence of functions $\rho_\epsilon(x)$, the function $\rho(x)$ is called the* ***weak limit*** *of that sequence if $\forall f(x)$*

$$\lim_{\epsilon\to 0} \int dx \rho_\epsilon(x) f(x) = \int dx \rho(x) f(x). \tag{2.3}$$

It is straightforward to show that for the sequence (2.2) we have

$$\lim_{\epsilon\to 0} \int dx \rho_\epsilon(x) f(x) = \int dx \delta(x) f(x) = f(0). \tag{2.4}$$

The functional result in (2.4) is in concordance with the second Dirac condition in (2.1), because in that case $\delta(x)$ must be understood multiplied by a constant function $f(x) = 1$. Consequently, the Dirac delta function $\delta(x)$ must not be used as a function in the sense of functional analysis because its mathematical behavior is to map the function $f(x)$ to its functional $f(0)$

$$\delta : f(x) \to f(0). \tag{2.5}$$

The Dirac delta function is not the unique mathematical entity that acts as functional on definite space of functions. For example we have the Heaviside step function or the principal value operator. This kind of functionals are called *distributions* or *generalized functions*.

Definition 2.2 *A distribution T is a continuous linear functional on space functions $\mathcal{T}$ where the elements $f \in \mathcal{T}$ are called* test functions

$$T : \mathcal{T} \to \mathbb{C}. \tag{2.6}$$

The definition 2.2 implies the fulfillment of the following conditions:

1. For each test function $f(x) \in \mathcal{T}$, the complex functional value associated is denoted by $\langle T, f(x) \rangle$.
2. $\forall\ \{\lambda_1, \lambda_2\} \in \mathbb{C}$, $\forall\ \{f_1(x), f_2(x)\} \in \mathcal{T}$, $\langle T, \lambda_1 f_1 + \lambda_2 f_2 \rangle = \lambda_1 \langle T, f_1 \rangle + \lambda_2 \langle T, f_2 \rangle$
3. If a sequence $f_i \in \mathcal{T}$ converge to a function $f(x) \in \mathcal{T}$, then the sequence $\langle T, f_i \rangle$ converge to $\langle T, f \rangle$

Another important concept, associated with the nature of distribution, is the *support*. We will define two kinds of support, one that belongs to test functions and another that belongs to distributions.

Definition 2.3 *The support of a test function $f(x)$ is the compact set of points supp(f) where $f(x) \neq 0$.*

Definition 2.4 *The support of a distribution T is the complement of reunion of open set points supp(T) where $\langle T, f \rangle = 0$ for all test functions f.*

In general, the distributions T represents the physical law that we want to investigate and to test. The test functions $f(x)$ are the representation of the external agent which fluctuate around the point x (supp(f)) where we want to test the physical law.

2.2 Properties of distributions and the space of test functions $\mathcal{T}$

The space of test functions $\mathcal{T}$ appear naturally to give a mathematically well defined definition of a distribution. The linearity condition of the distribution mapping implies that $\mathcal{T}$ must be a *vector space*, which means that a distribution T is an element of the *continuous dual space* $\mathcal{T}'$. The continuity property for the mapping $T : \mathcal{T} \to \mathbb{C}$

point out that $\mathcal{T}$ must have an inner product $\langle , \rangle$ to define a norm to use the Cauchy condition[2] [75].

The inner product, necessary to have a well defined theory, guides us to choose L^2 space[3] as our first option to construct $\mathcal{T}$. Actually, every function $g(x) \in L^2$ is a distribution over the test function space $\mathcal{T} = L^2$. But there is one problem, the Dirac delta function do not belongs to L^2 space [76]. Furthermore, non continuous functions belongs to L^2, which means that if we take $\mathcal{T} = L^2$ then $\delta(x) \notin \mathcal{T}'$.

Taking into account the inner product in L^2, we can classify the distributions $T \in \mathcal{T}'$ in *regulars* and *singulars.*

Definition 2.5 *A distribution T is regular if $\langle T, f\rangle$ could be written as the inner product of L^2 space*

$$\langle T, f\rangle = \int T(x) f(x) d^n x, \tag{2.7}$$

in other case the distribution is singular.

The latter definition means that $\delta(x)$ is a singular distribution and the notation (2.4) is just symbolic. Nevertheless, all properties of distributions could be obtained from the integral representation (2.7).

To include all the singular distributions, we could use the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ as space of test functions.

Definition 2.6 *The test function space $\mathcal{S}(\mathbb{R}^n)$ is the set of infinite differentiable functions $f(x) \in \mathcal{C}^\infty$ that fulfill the following property*

$$\lim_{\|x\|\to\infty} \|x\|^k \|\mathbf{D}^l f(x)\| = 0 \tag{2.8}$$

for all $k \in \mathbb{N}$ and $l \in \mathbb{N}_0^m$.[4]

[2]We say that a sequence of functions$f_n(x)$ converges to $f(x)$ if for every $\epsilon > 0$, exist $N \in \mathbb{N}$ that $\|f_m - f_n\| < \epsilon$ for every $m, n > N$.

[3]The square-integrable space L^2 is that where the inner product is define as $\langle g(x), f(x)\rangle = \int d^n x g^*(x) f(x) < \infty$.

[4]The differential operator $\mathbf{D}^l$ is defined as

$$\mathbf{D}^l f = \frac{\partial^{l_1+\ldots+l_m}}{\partial x^{l_1} \partial x^{l_2} \ldots \partial x^{l_m}} f(x), \quad l = (l_1, \ldots, l_m).$$

The property (2.8) tell us that the elements of $\mathcal{S}$ decrease faster than any power of $\|x\|^{-1}$. Furthermore, $\mathcal{S}$ has the important property that the Fourier transformation of its elements also belongs to $\mathcal{S}$ [77]. This is important because in Quantum Mechanics any state function must be independent of working in configuration or momentum space.

But the "nice" behavior of $\mathcal{S}$ is not enough to define the derivative of a distribution. It is necessary to define a sub space of S called the *close support space* $C_0^\infty(\mathbb{R}^m)$.

Definition 2.7 *The space $C_0^\infty(\mathbb{R}^m)$ is the set of functions $f(x) \in C^\infty$ with a compact support $\Omega \subset \mathbb{R}^m$.*

It is straightforward to demonstrate that $C_0^\infty(\mathbb{R}^m) \subset \mathcal{S}$ and, because of that, $\mathcal{S}' \subset C_0^{\infty\prime}(\mathbb{R}^m)$. From a physical point of view the distributions $T \in C_0^{\infty\prime}(\mathbb{R}^m)$ express the fact that *it is not possible to define a physical quantity at a point, but in a region around that point.*

Now, considering two distributions $T_1, T_2 \in C_0^{\infty\prime}(\mathbb{R}^m)$, they have the following properties:

- Addition
$$\langle T_1 + T_2, f\rangle = \langle T_1, f\rangle + \langle T_2, f\rangle.$$
- Multiplication by a complex α
$$\langle \alpha T, f\rangle = \langle T, \alpha f\rangle = \alpha\langle T, f\rangle.$$
- Translation by a vector $x_a \in \mathbb{R}^m$
$$\langle \alpha T(x + x_a), f(x)\rangle = \langle \alpha T(x), f(x - x_a)\rangle.$$
- Linear transformation of the independent variables $x \mapsto \Lambda x$ where $\mathbb{R}^m$
$$\langle T(\Lambda x), f(x)\rangle = \frac{1}{|det[\Lambda]|}\langle T(x), f(\Lambda^{-1}x)\rangle$$
- Derivative
$$\langle \frac{\partial^n T}{\partial x_i^n}, f(x)\rangle = \langle T, (-1)^n \frac{\partial^n f(x)}{\partial x_i^n}\rangle$$

It is possible to extend all these properties to distributions in $\mathcal{S}'$, considering $f(\pm\infty) = 0$ for $f(x) \in \mathcal{S}$. The product of distributions is an important point and we will present in the next section.

2.3 Product of two distributions

To define the product of two distribution, we have to take one of them as a reference to investigate the characteristics of the other distribution. Therefore, consider a distribution $T \in C_0^{\infty\prime}$, after that, if we multiply T with a distribution $g(x)$ we want that the complex value $\langle Tg, f\rangle$ exist. For the latter objective is necessary that

$$\langle Tg, f\rangle = \langle T, fg\rangle. \tag{2.9}$$

To fulfill the condition (2.9), it is sufficient that the product $f(x)g(x)$ belongs to C_0^∞. Consequently, if $f(x) \in C_0^\infty$, we need the function $g(x)$ to be infinitively differentiable to guarantee a well define product. In this thesis we will work with this type of products.

Another kind of product, which is well defined, is the tensorial product. For a two distributions T_1 and T_2 and a test function $f(x, y)$ over $\mathcal{S} \times \mathcal{S}$ we define the product $T_1(x) \times T_2(y)$ as

$$\langle T_1(x) \times T_2(y), f(x, y)\rangle \equiv \langle T_1(x), \langle T_2(y), f(x, y)\rangle\rangle = \langle T_2(y), \langle T_1(x), f(x, y)\rangle\rangle \tag{2.10}$$

2.4 Fourier transform and convolution of distributions

For an unidimensional function $f(t)$, the direct $\hat{f}(t)$ and inverse $\check{f}(t)$ Fourier transform are defined as

$$\hat{f}(p) = (2\pi)^{-\frac{1}{2}} \int dt e^{ip.t} f(t), \tag{2.11}$$

$$\check{f}(p) = (2\pi)^{-\frac{1}{2}} \int dt e^{-ip.t} f(t). \tag{2.12}$$

For a distribution T, its Fourier transformation $\hat{T}$ goes to the test function as follows

$$\langle \hat{T}, f\rangle \equiv \langle T, \hat{f}\rangle. \tag{2.13}$$

From (2.13), we can see that the following property is true

$$\langle T, f\rangle = \langle \hat{T}, \check{f}\rangle. \tag{2.14}$$

The convolution product $f * g$ in one dimension is defined as

$$\{f * g\}(t) \equiv \int dx f(t - x) g(x) = \int dx f(x) g(t - x). \tag{2.15}$$

Regarding the definition (2.15), it is straightforward to determine the relation with the Fourier transform of a product

$$\widehat{fg}(p) = (2\pi)^{-\frac{1}{2}}\{\hat{f} * \hat{g}\}(p), \tag{2.16}$$

$$\widecheck{fg}(p) = (2\pi)^{-\frac{1}{2}}\{\check{f} * \check{g}\}(p), \tag{2.17}$$

$$\widehat{f * g}(p) = (2\pi)^{-\frac{1}{2}}\hat{f}(p)\hat{g}(p), \tag{2.18}$$

$$\widecheck{f * g}(p) = (2\pi)^{-\frac{1}{2}}\check{f}(p)\check{g}(p). \tag{2.19}$$

For all distributions, again we generalize the following property of regular distributions

$$\begin{aligned}\langle F * G, f\rangle &= \int dx\{F * G\}(x)f(x) = \int dx f(x)\int dyF(y)G(x-y)\\ &= \int dyF(y)\int dxG(x-y)f(x) = \int dyF(y)\int d\xi G(\xi)f(\xi+y)\end{aligned} \tag{2.20}$$

Then, with the help of (2.10), the convolution of two distributions $T_1 * T_2$ is defined as

$$\langle T_1 * T_2, f\rangle \equiv \langle T_1(x) \times T_2(y), f(x+y)\rangle \tag{2.21}$$

In m dimensions, all these properties and definitions are

$$\hat{f}(p) = (2\pi)^{-\frac{m}{2}}\int dte^{ip.t}f(t), \tag{2.22}$$

$$\check{f}(p) = (2\pi)^{-\frac{m}{2}}\int dte^{-ip.t}f(t), \tag{2.23}$$

$$\langle \hat{T}, f\rangle \equiv \langle T, \hat{f}\rangle, \tag{2.24}$$

$$\langle T, f\rangle = \langle \hat{T}, \check{f}\rangle, \tag{2.25}$$

$$\{f * g\}(t) \equiv \int d^m x f(t-x)g(x) = \int d^m x f(x)g(t-x), \tag{2.26}$$

$$\widehat{fg}(p) = (2\pi)^{-\frac{m}{2}}\{\hat{f} * \hat{g}\}(p), \tag{2.27}$$

$$\widecheck{fg}(p) = (2\pi)^{-\frac{m}{2}}\{\check{f} * \check{g}\}(p), \tag{2.28}$$

$$\widehat{f * g}(p) = (2\pi)^{-\frac{m}{2}}\hat{f}(p)\hat{g}(p), \tag{2.29}$$

$$\widecheck{f * g}(p) = (2\pi)^{-\frac{m}{2}}\check{f}(p)\check{g}(p), \tag{2.30}$$

$$\langle T_1 * T_2, f\rangle \equiv \langle T_1(x) \times T_2(y), f(x+y)\rangle. \tag{2.31}$$

Chapter 3

Causal Perturbation Theory

> The latter is one of the most important papers in quantum field theory. However, for a long time, only a few specialists noticed this important approach to quantum field theory.
>
> *Eberhard Zeidler*, writing about the seminal paper of H. Epstein and V. Glaser [27] in his book [78]

CPT is an axiomatic approach for solving QFT where ill-defined mathematical quantities or computations are avoided due to the use of *the theory of distributions* (or *generalized function theory*) to give the correct mathematical nature to quantum fields as *operator value distributions* (OVD). [79, 80]

It focuses in the *causality* property to construct the S-Matrix as a formal perturbative power series in the coupling constant [27, 37, 81], leaving other physical properties for the end of computation. Even more, the CPT methodology is free of ultraviolet divergencies as a consequence of not containing ill-defined product of *operator value distributions* (OPV) in the same Minkowski space-time point!

In this Chapter we develop the fundamental tools to construct the S-matrix following the references [37] and [81].

3.1 Axioms of Causal Perturbation Theory

CPT works directly constructing the *scattering operator*[1] S. In this sense, it follows the Heisenberg program for QFT [82] which consider the *in* and *out* Fock spaces $\mathcal{F}_{\text{in}}$ and $\mathcal{F}_{\text{out}}$ respectively. The space $\mathcal{F}_{\text{in}}$ is the set of all multi-particle states $|\phi\rangle_{\text{in}}$ before the scattering process and $\mathcal{F}_{\text{out}}$ is the set of all multi-particle states $|\psi\rangle_{\text{out}}$ after. Then, the operator S is defined as the bijective application

$$S : \mathcal{F}_{\text{in}} \to \mathcal{F}_{\text{out}}. \tag{3.1}$$

Consequently, the transition from the *in* to *out* state is

$$S : |\phi\rangle_{\text{in}} \to |\psi\rangle_{\text{out}} \equiv S|\phi\rangle_{\text{in}}, \tag{3.2}$$

and the transition amplitude $\mathcal{A}$ from the state $|\phi\rangle_{\text{in}}$ to $|\psi\rangle_{\text{out}}$ is computed as

$$\mathcal{A}(|\phi\rangle_{\text{in}}, |\psi\rangle_{\text{out}}) \equiv (|\psi\rangle_{\text{out}}, S|\phi\rangle_{\text{in}}) =_{\text{out}} \langle\psi|S|\phi\rangle_{\text{in}}. \tag{3.3}$$

But in contrast to the usual construction of S via temporal order product, H. Epstein and V. Glaser used the formalism developed by Bogoliubov [83] where a test function is introduced to give the correct mathematical nature to the quantum fields as OVD which appear in the temporal product[2].

Bogoliubov uses a function $g(x) \in [0, 1]$ to control the long range interaction which causes the infrared divergences. The function $g(x)$ is named *switching on-off* function. If in a space-time region $g(x) = 0$ the interaction is switched-off, if $0 < g(x) < 1$ the interaction is partially switched-on, and if $g(x) = 1$ the interaction is fully switched-on.

We choose $g(x) = 0$ for $x^0 = \pm\infty$, and $g(x) \neq 0$ for a time interval $x^0 \in [t_{on}, t_{off}]$ where the interaction scattering is stronger. Furthermore, $g(x)$ must belong to C_0^∞ or $\mathcal{S}$ to allow the derivatives of singular OVDs. Via this reasoning, we conclude that the operator S must be a functional of g

$$S = S[g]. \tag{3.4}$$

[1]The S-Matrix and the scattering operator are different concepts but intimately related. The S-Matrix is the collection of all possibles transition amplitudes in a scattering process, but in this work we will use the two concepts as the same as usual.

[2]Remember from Chap. 2. that a distribution needs to be applicated in a test function space.

Now, considering a perturbative construction of S, H. Epstein and V. Glaser [27] propose the following ansatz as a formal series expansion

$$\begin{aligned} S[g] &\equiv \mathbb{1} + \int d^4x_1 T_1(x_1)g(x_1) + \frac{1}{2!}\int d^4x_2 d^4x_1 T_2(x_1,x_2)g(x_1)g(x_2) + \dots \\ &\equiv \sum_0^\infty \frac{1}{n!}\int d^4x_1 \dots d^4x_n T_n(x_1,\dots,x_n)g(x_1)\dots g(x_n) \\ &\equiv \mathbb{1} + T, \end{aligned} \tag{3.5}$$

where $\hbar = 1 = c$ as usual and $T_n(x_1, \dots, x_n)$ is called *n-point distribution* which are the terms that we need to find. The factor $n!$ is used to indicate the symmetry property of T_n when a permutation of coordinates is done.

Using unitarity, Bogoliubov found that the T_n distributions were the temporal product of interaction Lagrangian [83]. Instead of using the unitarity property of S-matrix, H. Epstein and V. Glaser postulate four axioms to constrain T_n and then develop an iterative construction with the guide of causality condition [27]. G. Scharf modernize the approach and applicated it to QED [37].

3.1.1 Axiom I: Boundary Condition

This axiom constrains the spaces $\mathcal{F}_{\text{in}}$ and $\mathcal{F}_{\text{out}}$. We postulate that in the temporal limits $t \to \pm\infty$ the particle systems are *asymptotically free*, inclusive in the adiabatic limit $g(x) \to 1$. Consequently, the two Fock spaces, in and out, are free multi-particle spaces.

We can comment that in the adiabatic limit $g(x) \to 1$, (3.5) can be written as

$$S[g] = 1 + \sum_{n=1}^{\infty} \lambda^n S_n \tag{3.6}$$

where λ is the coupling constant of the gauge theory and the convergence of the series depends on its intensity.

The most important consequence from this axiom is that the theory is determined via the free field operators, therefore *the mass and charge in the free wave equations are the physical quantities.*

3.1.2 Axiom II: Base Term and Perturbative Gauge Invariance

We postulate that the construction of the S-matrix, as a formal series (3.5), will be done inductively from the definition of the first term $T_1(x)$ which will be different for every gauge theory.

The systematized methodology to construct the term $T_1(x)$ is the major contribution of G. Scharf and collaborators to CPT. In a series of articles [30, 36, 38–61], they determine the conditions for the gauge invariant transformation for every term of the series (3.5), and apply it to construct Yang-Mills and Electroweak theories. They call this formalism *Perturbative Gauge Invariance* (PGI), and it complements CPT. PGI will be developed in Chapter 5.

3.1.3 Axiom III: Poincaré Invariance

Similar to the usual framework, CPT must be invariant under translation and Lorentz transformation. These transformations must be done on the test functions $g(x)$ because of the functional nature of S-matrix.

If an observator O uses the test function $g(x)$ to study a physical phenomenon, then an observator O' translated to $x + a$, must use a test function $g_a = g(x - a)$. Or if O' is boosted or rotated to Λx, the test function must be $g_\Lambda = g(\Lambda^{-1} x)$. In both cases the S-matrix must be invariant.

If $U(a, \Lambda)$ is a representation of translation a or Lorentz transformation Λ in Fock space $\mathcal{F}$, then the transformation rule of S-matrix is

$$S' = U(a, \Lambda) S U^{-1}(a, \Lambda), \tag{3.7}$$

Therefore, the Poincaré invariance implies

$$S = U(a, \Lambda) S U^{-1}(a, \Lambda), \tag{3.8}$$

3.1.4 Axiom IV: Causality

This is the principal axiom to construct the S-matrix. CPT postulates that there exists a parameter which order the evolution of events in space-time.

In this thesis we will use the temporal parameter x^0 to order the S-matrix scattering events. Then, because of the functional dependence on the switch on-off test functions $g(x)$, we will time order S regarding $g(x)$.

Definition 3.1 *By considering two test functions $g_1(x)$ and $g_2(x)$ with disjoint supports, then if $\forall x_1 \in supp[g_1]$ and $\forall x_2 \in supp[g_2]$ we can define the time-ordering rule*

$$x_1^0 < x_2^0 \Rightarrow supp[g_1] < supp[g_2]. \tag{3.9}$$

Now, if for a reference system, the S-matrix depends on two test functions g_1 and g_2, the time-ordering rule $\text{supp}[g_1] < \text{supp}[g_2]$ implies the following *causal decomposition*

$$S[g_1 + g_2] = S[g_2]S[g_1]. \tag{3.10}$$

3.2 Iterative construction of S-Matrix

Regarding the four axioms of CPT, we proceed to construct term by term the perturbative series (3.5). Of course, the first step is to define the one point distribution $T_1(x)$. As mentioned in the axiom II, each gauge theory presents its own term $T_1(x)$. We will describe the construction of T_1 for scalar QED in Chapter 5. In this section we describe the second step which focuses in determining the term T_n from the knowledge of the previous terms $\{T_{n-1}, \dots, T_1\}$.

3.2.1 Properties of the n-point distributions

Because the main elements to be computed are the n-point distributions T_n, it will be useful to determine some of their properties that come from the properties of the S-matrix:

1. The inverse S^{-1} will be determined in two forms, by inverting (3.5) as

$$S^{-1} = (\mathbb{1} + T)^{-1} = 1 + \sum_{r=1}^{\infty} (-T)^r, \tag{3.11}$$

and as formal series

$$S^{-1} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \widetilde{T}_n(x_1, \dots, x_n) g(x_1) \dots g(x_n), \tag{3.12}$$

where $\widetilde{T}_n(x_1, \dots, x_n)$ is an n-point distribution that is symmetric under the permutations of x_i and $\mathbb{1}$ is the identity matrix. $\widetilde{T}_n$ is not the inverse of T_n, but it can be determined as a function of the set $\{T_1, \ldots, T_n\}$ using the fact that the right-hand sides of equations (3.11) and (3.12) are equal

$$\widetilde{T}_n(x_1, \ldots, x_n) = \sum_{r=1}^{n} (-1)^r \sum_{P_r} T_{n_1}(X_1) \ldots T_{n_r}(X_r), \tag{3.13}$$

where the sum is over all partitions P_r of the set $X = \{x_1, \ldots, x_n\}$ in r disjoints and not empty sub-sets X_i.

2. Making use of

$$\mathbb{1} = S[g]S^{-1}[g] = S^{-1}[g]S[g], \tag{3.14}$$

we obtain

$$\sum_{P_2^0} T_{n_1}(X)\tilde{T}_{n-n_1}(Z \backslash X) = 0, \tag{3.15}$$

$$\sum_{P_2^0} T_{n-n_2}(Z \backslash Y)\tilde{T}_{n_2}(Y) = 0, \tag{3.16}$$

$$\sum_{P_2^0} \tilde{T}_{n-n_1}(X)T_{n_1}(Z \backslash X) = 0, \tag{3.17}$$

$$\sum_{P_2^0} \tilde{T}_{n_2}(Z \backslash Y)T_{n-n_2}(Y) = 0, \tag{3.18}$$

where the sums run over all two partitions P_2^0 of the set $Z = \{x_1, \ldots, x_n\}$ in two disjoints sub-sets X and Y allowing the cases where $X = \varnothing$ or $Y = \varnothing$.

3. From Poincaré invariance, we determine

$$T_n(x_1, \ldots, x_n) = T_n(x_1 + a, \ldots, x_n + a), \tag{3.19}$$

$$T(x_1, \ldots, x_n) = T(\Lambda x_1, \ldots, \Lambda x_n). \tag{3.20}$$

4. From causality, we can determine that the n-point distributions are well defined time ordered product. If $\{x_1^0, \ldots, x_m^0\} > \{x_{m+1}^0, \ldots, x_n^0\}$, thus

$$T_n(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) = T_m(x_1, \ldots, x_m)T_{n-m}(x_{m+1}, \ldots, x_n), \tag{3.21}$$

and for $\widetilde{T}_n$, we have

$$\widetilde{T}_n(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) = \widetilde{T}_{n-m}(x_{m+1}, \ldots, x_n)\widetilde{T}_m(x_1, \ldots, x_m). \tag{3.22}$$

3.2.2 From T_{n-1} to T_n

In the computation of T_n, the main objective is to avoid the naive procedure to determine the advanced and retarded parts of a causal propagator via the multiplication to a Heaviside step function $\Theta(t)$. Because $\Theta(t) \notin C^\infty$, the naive product could not exist as mentioned in Chapter 2.

First of all, from the knowledge of $\{T_{n-1}, \dots, T_1, \tilde{T}_{n-1}, \dots, \tilde{T}_1\}$, we define the ***intermediate distributions*** A'_n and R'_n as

$$A'_n(x_1, \dots, x_n) \equiv \sum_{P_2} \widetilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n), \tag{3.23}$$

$$R'_n(x_1, \dots, x_n) \equiv \sum_{P_2} T_{n-n_1}(Y, x_n) \widetilde{T}_{n_1}(X), \tag{3.24}$$

where the sum runs over all partitions P_2 of the set $\{x_1, \dots, x_{n-1}\}$ in two non-empty and disjoints sub-sets X and Y. This product is well define because it is done with distributions defined in different space-points.

The next step is to extend the sums (3.23) and (3.24) allowing for the empty sub-set $X = \varnothing$

$$A_n(x_1, \dots, x_n) \equiv \sum_{P_2^0} \widetilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n), \tag{3.25}$$

$$R_n(x_1, \dots, x_n) \equiv \sum_{P_2^0} T_{n-n_1}(Y, x_n) \widetilde{T}_{n_1}(X), \tag{3.26}$$

where $T_0 = 1 = \tilde{T}_0$ and P_2^0 represents the inclusion of empty sets. We will show that the distributions A_n and R_n are the retarded and advanced distributions which we want to determine. Furthermore, it is straightforward to rewrite the sums (3.25) and (3.26) as

$$A_n(x_1, \dots, x_n) = A'_n(x_1, \dots, x_n) + T_n(x_1, \dots, x_n), \tag{3.27}$$

$$R_n(x_1, \dots, x_n) = R'_n(x_1, \dots, x_n) + T_n(x_1, \dots, x_n). \tag{3.28}$$

In equations (3.27) and (3.28) just R'_n and A'_n are known. If we determine A_n or R_n through the use of another methodology, then we can determine the T_n by

$$T_n(x_1, \dots, x_n) = \begin{cases} A_n(x_1, \dots, x_n) - A'_n(x_1, \dots, x_n), \\ R_n(x_1, \dots, x_n) - R'_n(x_1, \dots, x_n). \end{cases} \tag{3.29}$$

The latter is possible in the framework of distribution theory.

3.2.3 Supports of the retarded R_n and advanced A_n distributions

The most important property of the distributions R_n and A_n is their support. To identify this property, we are going to invoke the following theorem[3]

Theorem 3.1 *Consider three sets of space points Y, P and Q such as $Y = P \cup Q$, $P \neq \varnothing$, $P \cap Q = \varnothing$, $|Y| = n-1$, and the point x such that $x \notin Y$, then:*

- *If $\{Q, x\} > P$, $|Q| = n_1$, therefore*

$$R'_n(Y, x) = -T_{n_1+1}(Q, x) T_{n-(n_1+1)}(P) \tag{3.30}$$

- *If $\{Q, x\} < P$, $|Q| = n_1$, therefore*

$$A'_n(Y, x) = -T_{n-(n_1+1)}(P) T_{n_1+1}(Q, x) \tag{3.31}$$

Now, we can study the support of R_n. If $Y = \{x_1, \ldots, x_{n-1}\}$, then we can write (3.28) as

$$R_n(Y, x_n) = R'_n(Y, x_n) + T_n(Y, x_n). \tag{3.32}$$

Now, we have three cases for time ordering the whole set $\{Y, x_n\}$[4]:

- Case one: $Y > x_n$.
- Case two: $x_n > Y$.
- Case three: $Q > x_n > P$, where $Y = P \cup Q$.

In the second and third cases, we can use the theorem 3.1 to rewrite (3.32) in the following form

$$R_n(Y, x_n) = -T_{n_1+1}(Q, x_n) T_{n-(n_1+1)}(P) + T_n(P \cup Q, x_n), \tag{3.33}$$

where if we use the causal decomposition for the n-point distribution, we get

$$\begin{aligned} R_n(Y, x_n) &= -T_{n_1+1}(Q, x_n) T_{n-(n_1+1)}(P) + T_n(P \cup Q, x_n) \\ &= -T(Q, x_n) T(P) + T(Q, x_n) T(P) \\ &= 0 \end{aligned} \tag{3.34}$$

[3]The proof can be seen in Appendix A.1.

[4]For simplicity, we will write all causality conditions obviating the zero super-index.

In conclusion, from (3.34) the unique case where $R_n \neq 0$ is for the time order

$$x_n < \{x_1, \ldots, x_{n-1}\}, \tag{3.35}$$

and for A_n, we can get the time ordering condition $x_n > \{x_1, \ldots, x_{n-1}\}$ in the non-null case.

Considering the *Lorentz invariance* of the n-point distributions T_n, it is not difficult to extend it to R_n and A_n. This is important because the causal condition (3.35) must be the same for all reference system, and of course, the set $\{x_1, \ldots, x_n\}$ must be in the light-cone with origin in x_n.

To formalize the last deduction, we will define the $4n$-dimension light-cone centered in y as

$$\Gamma_n^{\pm}(y) \equiv \{(x_1, \ldots, x_n)/x_i \in \bar{V}^{\pm}(y)\}, \tag{3.36}$$

where $\bar{V}^{\pm}(y)$ are the closed forward and backward light-cone centered in y

$$\bar{V}^{+}(y) \equiv \left\{x/(x-y)^2 \geqslant 0, \quad x^0 \geqslant y^0\right\}, \tag{3.37}$$

$$\bar{V}^{-}(y) \equiv \left\{x/(x-y)^2 \geqslant 0, \quad x^0 \leqslant y^0\right\}, \tag{3.38}$$

respectively.

Regarding (3.36), we conclude for the supports of R_n and A_n distributions the following two properties

$$supp[R_n(x_1, \ldots, x_n)] \subseteq \Gamma_{n-1}^{+}(x_n), \tag{3.39}$$

$$supp[A_n(x_1, \ldots, x_n)] \subseteq \Gamma_{n-1}^{-}(x_n), \tag{3.40}$$

respectively.

3.2.4 The causal distribution D_n

The results (3.39) and (3.40), tell us that R_n and A_n are the retarded and advanced parts of a subtraction

$$D_n(x_1, \ldots, x_n) \equiv R_n(x_1, \ldots, x_n) - A_n(x_1, \ldots, x_n), \tag{3.41}$$

where D_n is called *causal distribution* because its support will be the union of the supports of R_n and A_n

$$supp[D_n(x_1, \ldots, x_n)] \subseteq \{\Gamma_{n-1}^{+}(x_n) \cup \Gamma_{n-1}^{-}(x_n)\}. \tag{3.42}$$

The causal distribution is fully computable from (3.27) and (3.28) as

$$D_n(x_1,\ldots,x_n) = R'_n(x_1,\ldots,x_n) - A'_n(x_1,\ldots,x_n) \tag{3.43}$$

The result (3.43), is the starting point for the computation of T_n. Because $supp[R_n] \cap supp[A_n] = \{x_n\}$, it is possible to determine R_n or A_n splitting D_n with the specific supports in the framework of distribution theory. The splitting process is called ***causal splitting*** and will be developed in the next section.

3.3 Causal splitting Procedure

In the usual framework, the splitting of a causal distribution in its advanced or retarded part is done by the naive multiplication by the Heaviside step function [84]. However, this product is not always well defined because in quantum field theory there exist causal singular distributions. As demonstrated by G. Scharf, in QED [37] this naive procedure was the origin of ultraviolet divergences.

Then, we need to determine how to split correctly a causal distribution. First of all, we must remember that, in the usual framework, the UV divergence is related with two sources: the short distance behavior of the causal propagators and the bare physical parameters as mass and charge of particles [85–92].

Because in CPT it is postulated that the mass and charge are the physical quantities, this implies that we need to study the behavior of D_n in a vicinity of x_n. The latter is possible just in the numerical parts of the causal distribution D_n.

3.3.1 Numerical distribution d_n

From the properties of the n-point distributions T_n and $\tilde{T}_n$, it is not difficult to note that the intermediate distributions R' and A' could be written as products of normal order operators. Therefore, we have

$$D_2(x_1,\ldots,x_n) = \sum_k : \prod_j \mathcal{O}(x_j) : d_n^k(x_1,\ldots,x_n), \tag{3.44}$$

where $\mathcal{O}(x_j)$ represents all operator value distributions (OVD) and $d_n^k(x_1,\ldots,x_n)$ is the numerical part of each term in the sum (3.44) obtained via contractions of Wick

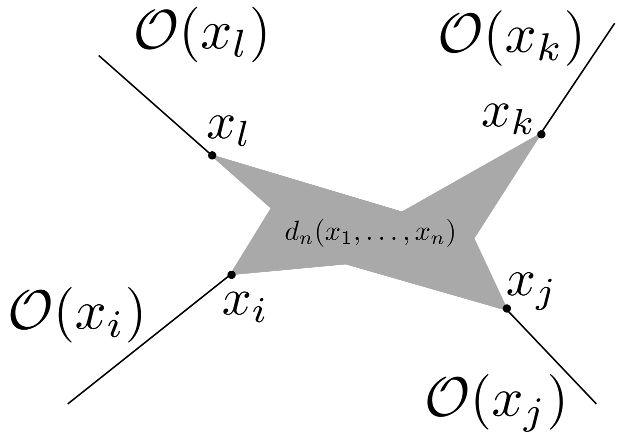


Figure 3.1: Graph with four external legs and connected n points represented by d_n.

theorem[5]. In general, we can represent each term of the summation (3.44) graphically, the uncontracted operator value distribution fields represent external legs and the numerical distributions $d_n^k(x_1,\ldots,x_n)$ represent the connection of these legs as in Fig. (3.1).

The numerical part d_n^k is what we will causal-split. Using the Poincaré invariance, we can translate d_n by x_n obtaining

$$d_n^k(x_1,\ldots,x_n) = d_n^k(x_1 - x_n,\ldots,x_{n-1} - x_n, 0) \equiv d(\tilde{x}), \tag{3.45}$$

where we define $d(\tilde{x})$ as the general notation to denote each numerical distribution to be split and $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_{n-1})$ where $\tilde{x}_i = x_i - x_n$.

From (3.45) we can note that *the short distance behavior* means the mathematical behavior of $d(\tilde{x})$ in the limit $\tilde{x}_j \to 0$. Furthermore, we can see that the UV divergence problem is the ill-defined product with the Heaviside step functions $\Theta(x_j^0 - x_n^0)$ where $j = 1,\ldots,n-1$ due to its ill defined limit $\lim\limits_{x_j^0 \to x_n^0} \Theta(x_j^0 - x_n^0)$.

3.3.2 Singular and Regular distributions

Following section (2.3), to causal split $d(\tilde{x})$, we will construct the function $\chi(t) \in C^\infty$ over $\mathbb{R}^1$

$$\chi(t) \equiv \begin{cases} 0 & \text{when} \quad t \leqslant 0, \\ [0,1] & \text{when} \quad 0 < t < 1, \\ 1 & \text{when} \quad t \geqslant 1. \end{cases} \tag{3.46}$$

[5]The Wick theorem is developed in Appendix (A.2).

There is no mathematical ill definitions in the product between $\chi(t)$ and any distribution $T \in C_0^{\infty\prime}$. Furthermore, it is not difficult to note that $\Theta(t)$ will be constructed as the limit

$$\Theta(t) = \lim_{\alpha \to 0^+} \chi(\frac{t}{\alpha}), \tag{3.47}$$

this last property was the reason to construct $\chi(t)$, because with its help we will multiply $\chi(\frac{t}{\alpha})$ by a causal distribution, then take the limit $\alpha \to 0^+$ to obtain its retarded part.

We will generalize the definition (3.46) to $m = 4n - 4$ dimensions with the help of a retarded vector $v \in \Gamma_{n-1}^+(0)$ and define the function $\chi_\alpha(\tilde{x})$ as

$$\chi_\alpha(\tilde{x}) \equiv \chi(\frac{v.\tilde{x}}{\alpha}), \tag{3.48}$$

where $v = (v_1, \ldots, v_{n-1})$, and the product $v.\tilde{x}$ is defined as

$$v.\tilde{x} \equiv \sum_{i=1}^{n-1} g_{\mu\nu} v_i^\mu \tilde{x}_i^\nu. \tag{3.49}$$

Regarding (3.49), we can see that the space-like hyperplane

$$v.\tilde{x} = 0, \tag{3.50}$$

split the causal support as show in Fig. (3.2).

From (3.48) and (3.49), we can compute that for all $\tilde{x}_i \in \bar{V}^-$ we have $\lim_{\alpha \to 0^+} \chi_\alpha(\tilde{x}) = 0$, and for all $\tilde{x}_i \in \bar{V}^+$ we get $\lim_{\alpha \to 0^+} \chi_\alpha(\tilde{x}) = 1$. This is the desired behavior to obtain the retarded part $r_n(\tilde{x})$ of $d_n(\tilde{x})$ via the multiplication $r_n(\tilde{x}) = \chi_\alpha(\tilde{x}) d_n(\tilde{x})$. The problem is to determine in which cases the following *weak limit* exists

$$\langle r_n(\tilde{x}), f(\tilde{x}) \rangle = \lim_{\alpha \to 0^+} \langle \chi_\alpha(\tilde{x}) d(\tilde{x}), f(\tilde{x}) \rangle, \tag{3.51}$$

for all test functions $f(\tilde{x}) \in C_0^\infty$.

As a Cauchy sequence labeled by α, we need to demonstrate that for all real value $\epsilon > 0$, there exists a real value $\delta < \epsilon$ such that for all α and β, with values on the interval $0 < \{\alpha, \beta\} < \delta$, the following inequality is fulfilled

$$\| \langle \chi_\beta(\tilde{x}) d(\tilde{x}), f(\tilde{x}) \rangle - \langle \chi_\alpha(\tilde{x}) d(\tilde{x}), f(\tilde{x}) \rangle \| < \epsilon. \tag{3.52}$$

Taking β as $\beta = \alpha/a$, where $a \in \mathbb{R}$ is fixed, and defining the function $\psi(x/a)$ as

$$\psi\left(\frac{\tilde{x}}{\alpha}\right) = \chi\left(\frac{v.\tilde{x}}{\frac{\alpha}{a}}\right) - \chi\left(\frac{v.\tilde{x}}{\alpha}\right), \tag{3.53}$$

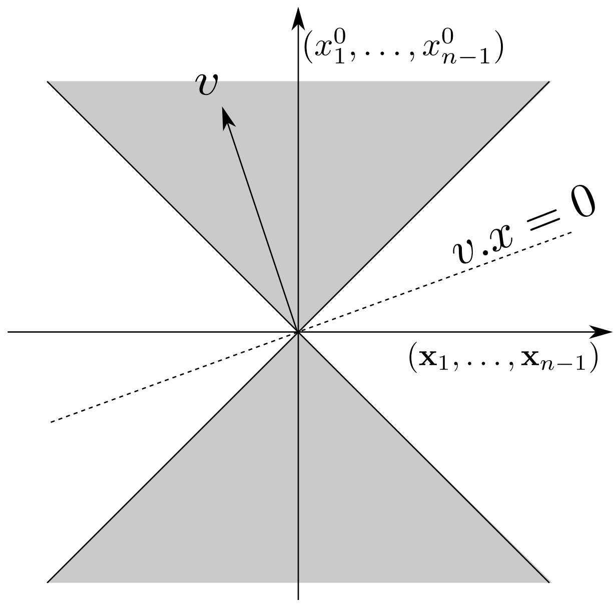


Figure 3.2: Split of the causal support by the hyperplane $v.x = 0$.

we can rewrite (3.52) in the following form

$$\|\langle \psi(\frac{\tilde{x}}{\alpha}) d(\tilde{x}), f(\tilde{x}) \rangle\| < \epsilon. \tag{3.54}$$

Because $\psi(\frac{\tilde{x}}{\alpha}) \in C^\infty$, it could be interchanged with $f(\tilde{x})$

$$\|\langle f(\tilde{x}) d(\tilde{x}), \psi(\frac{\tilde{x}}{\alpha}) \rangle\| < \epsilon. \tag{3.55}$$

In order to eliminate the α dependence of the new test function ψ, we can re-scale the variable as $\tilde{x} \to \alpha\tilde{x}$

$$\|\langle f(\alpha\tilde{x}) \alpha^m d(\alpha\tilde{x}), \psi(\tilde{x}) \rangle\| < \epsilon. \tag{3.56}$$

In the limit $\alpha \to 0^+$, we could think that the left hand side of (3.56) is null, but this is not true for distributions $d(\alpha\tilde{x})$ which increase faster than α^m in the neighborhood of $\alpha = 0$. For this reason, we introduce the function $\rho(\alpha)$ to characterize the increase behavior of $d(\alpha\tilde{x})$ and define the quasi-asymptotic distribution $d_0(\alpha\tilde{x})$.

Definition 3.2 *A distribution $d(x) \in C_0'^\infty(\mathbb{R}^m)$ has a quasi-asymptotic $d_0(x)$ over $x = 0$, if for a positive function $\rho(\alpha)$ $(\alpha > 0)$ the limit*

$$\lim_{\alpha \to 0^+} \langle \rho(\alpha) \alpha^m d(\alpha x), \psi(x) \rangle = \langle d_0(x), \psi(x) \rangle \neq 0, \tag{3.57}$$

exists

With the help of (3.57), we can multiply and divide the left hand side of (3.56) by $\rho(\alpha)$

$$\begin{aligned}\langle f(\alpha\tilde{x})\alpha^m d(\alpha\tilde{x}),\psi(\tilde{x})\rangle &=\\ &= \frac{1}{\rho(\alpha)}\langle f(\alpha\tilde{x})\rho(\alpha)\alpha^m d(\alpha\tilde{x}),\psi(\tilde{x})\rangle\\ &= \frac{1}{\rho(\alpha)}\Big[f(0)\langle\rho(\alpha)\alpha^m d(\alpha\tilde{x}),\psi(\tilde{x})\rangle + D^{(1)}f(0)\langle x\rho(\alpha)\alpha^{m+1}d(\alpha\tilde{x}),\psi(\tilde{x})\rangle + \dots\Big],\end{aligned} \tag{3.58}$$

where, in the last equality, we did the Taylor series expansion for $f(\alpha x)$ around $x = 0$.

In (3.58), after the first term, we have factors proportional to α^{m+i}, with $i = 1, 2, \dots$, which decrease more rapidly than $\rho(\alpha)d(\alpha x)$. Then, in the limit $\alpha \to 0^+$, we have

$$\langle f(\alpha\tilde{x})\alpha^m d(\alpha\tilde{x}),\psi(\tilde{x})\rangle \approx \frac{f(0)}{\rho(\alpha)}\langle d_0(\tilde{x}),\psi(\tilde{x})\rangle. \tag{3.59}$$

As shown in Appendix (A.3), we could use the result (A.13) to replace $\rho(\alpha) = \alpha^\omega L(\alpha)$ in (3.59)

$$\langle f(\alpha\tilde{x})\alpha^m d(\alpha\tilde{x}),\psi(\tilde{x})\rangle \approx \frac{f(0)}{\alpha^\omega L(\alpha)}\langle d_0(\tilde{x}),\psi(\tilde{x})\rangle, \tag{3.60}$$

where $\omega \in \mathbb{R}^+$, and $L(\alpha)$ is a slow varying or quasi-constant function of α in the neighborhood of $\alpha = 0$.

From (3.60), we can conclude that the condition (3.56) is fulfilled, for all test functions $f(x)$, just in the case where $\omega < 0$. For $\omega \geqslant 0$, the condition is fulfilled for a finite subgroup of C_0^∞. To show the latter we go back to the Taylor expansion of the test function $f(x)$ in (3.58)

$$\begin{aligned}\langle f(\alpha\tilde{x})\alpha^m d(\alpha\tilde{x}),\psi(\tilde{x})\rangle &=\\ &= \frac{1}{\alpha^\omega L(\alpha)}\Big[\sum_{|l|=0}^{\omega}\frac{1}{l!}[\mathbf{D}^l f](0)\langle x^l\rho(\alpha)\alpha^{m+l}d(\alpha\tilde{x}),\psi(\tilde{x})\rangle\\ &\quad+\sum_{|l|=\omega+1}^{\infty}\frac{1}{l!}[\mathbf{D}^l f](0)\langle x^l\rho(\alpha)\alpha^{m+l}d(\alpha\tilde{x}),\psi(\tilde{x})\rangle\Big]\\ &= \sum_{|l|=0}^{\omega}\frac{1}{\alpha^{\omega-l}L(\alpha)}\frac{1}{l!}[\mathbf{D}^l f](0)\langle x^l\rho(\alpha)\alpha^m d(\alpha\tilde{x}),\psi(\tilde{x})\rangle\\ &\quad+\sum_{|l|=\omega+1}^{\infty}\frac{\alpha^{l-\omega}}{L(\alpha)}\frac{1}{l!}[\mathbf{D}^l f](0)\langle x^l\rho(\alpha)\alpha^m d(\alpha\tilde{x}),\psi(\tilde{x})\rangle,\end{aligned} \tag{3.61}$$

where $x^l = x_1^{l_1} \dots x_n^{l_n}$ with $l = l_1 + \dots + l_n$, and the sum runs over all possible l.

From (3.61), we can note that in the case $\omega \geqslant 0$, only the test functions where $[\mathbf{D}^l f](0) = 0$ with $l = 1, \dots, \omega$ allow the condition (3.56).

Therefore, in order to causal-split $d(\tilde{x})$ valid for all test functions f, we could define the projection

$$\mathcal{W} : f \to \mathcal{W}f, \tag{3.62}$$

$$\mathcal{W}f = f(x) - w(x) \sum_{|l|=0}^{\omega} \frac{1}{l!} [\mathbf{D}^l f](0) x^l, \tag{3.63}$$

where $w(x) \in \mathcal{S}$ has the following properties $w(0) = 1$ and $[D^\nu w](0) = 0$ for all $\nu = 1, \dots, \omega$. Over the new test functions $\mathcal{W}f$ the existence condition (3.56) is valid.

In conclusion, in order to determine the retarded part r_n via (3.51), we need to compute the quantity ω first. For its importance and nature, ω is knowing as ***order of singularity*** because it could be used to classify the distributions as ***regular*** or ***singular*** in the cases where $\omega < 0$ or $\omega \geqslant 0$, respectively. As demonstrated by G. Scharf et al., ω is the formal form of the superficial degree of divergence used in the standard formalism based on Feynman diagrams.

In summary, to obtain the retarded part $r(\tilde{x})$ of a causal distribution $d(\tilde{x})$, we need to follow these steps:

1. Determine the power counting function $\rho(x)$ to obtain the quasi-asymptotic distribution $d_0(x)$ defined in (3.57).

2. Determine the order of singularity ω via

$$\lim_{\alpha \to 0^+} \frac{\rho(a\alpha)}{\rho(\alpha)} = a^\omega. \tag{3.64}$$

3. If $\omega < 0$, the retarded part $r(\tilde{x})$ is obtained from

$$\langle r(\tilde{x}), f(\tilde{x}) \rangle = \lim_{\alpha \to 0^+} \langle \chi(\frac{v.\tilde{x}}{\alpha}) d(\tilde{x}), f(\tilde{x}) \rangle = \langle \Theta(v.\tilde{x}) d(\tilde{x}), f(\tilde{x}) \rangle. \tag{3.65}$$

4. If $\omega \geqslant 0$, the retarded part $r(\tilde{x})$ is obtained from

$$\langle r(\tilde{x}), f(\tilde{x}) \rangle = \lim_{\alpha \to 0^+} \langle \chi(\frac{v.\tilde{x}}{\alpha}) d(\tilde{x}), \mathcal{W}f(\tilde{x}) \rangle = \langle \Theta(v.\tilde{x}) d(\tilde{x}), \mathcal{W}f(\tilde{x}) \rangle. \tag{3.66}$$

3.3.3 Uniqueness of the retarded part $r(x)$

In the regular case, the solution for $r(x)$ is unique. In this section we want to show that in the singular case it is not.

First of all, we want to emphasize the characteristics of projected test functions $\mathcal{W}f(x)$. From (3.63), it is not difficult to rewrite $\mathcal{W}f(x)$ as

$$\mathcal{W}f = x^{\omega+1}g(x). \tag{3.67}$$

Then, the following property is fulfilled

$$\mathbf{D}^l(\mathcal{W}f)\Big|_{x=0} = 0, \quad \text{for all } l = 0, \ldots, \omega. \tag{3.68}$$

Now, we can define the retarded part $\tilde{r}(x)$

$$\tilde{r}(x) \equiv r(x) + \sum_{l=0}^{\omega} C_l \mathbf{D}^l \delta(x), \tag{3.69}$$

where C_l are constants. By construction, we can show that $\tilde{r}(x)$ generates the same result as r

$$\begin{aligned} \langle \tilde{r}, f(x) \rangle &= \langle \Theta(x) d(x) + \sum_{l=0}^{\omega} C_l \mathbf{D}^l \delta(x), \mathcal{W}f(x) \rangle \\ &= \langle \Theta(x) d(x), \mathcal{W}f(x) \rangle + \sum_{l=0}^{\omega} C_l \langle \mathbf{D}^l \delta(x), \mathcal{W}f(x) \rangle \\ &= \langle \Theta(x) d(x), \mathcal{W}f(x) \rangle = \langle r(x), f(x) \rangle \end{aligned} \tag{3.70}$$

The result (3.70) demonstrated that in the singular case $\omega \geqslant 0$, the most general solution for the retarded part of $d(x)$ is (3.69).

3.4 Causal-splitting procedure in momentum space

As in the standard framework, we are going to present the computation of the retarded part $r(x)$ in momentum space using the properties described in section 2.4.

3.4.1 Regular distribution Case

In a regular distribution case, using (3.65) we have

$$\begin{aligned} \langle \hat{r}(p), \check{f}(p)\rangle &= \langle \widehat{\Theta(v.\tilde{x})d(\tilde{x})}(p), \check{f}(p)\rangle \\ &= (2\pi)^{-\frac{m}{2}}\langle \hat{\Theta}(p) * \hat{d}(p), \check{f}(p)\rangle \\ &= \langle (2\pi)^{-\frac{m}{2}} \int d^m k \hat{\Theta}(p-k)\hat{d}(k), \check{f}(p)\rangle, \end{aligned} \tag{3.71}$$

then

$$\hat{r}(p) = (2\pi)^{-\frac{m}{2}} \int d^m k \hat{\Theta}(p-k)\hat{d}(k). \tag{3.72}$$

In order to determine $\hat{\Theta}(q)$, we choose a vector $v = (1, \mathbf{0}, 0, \ldots)$ where $\mathbf{0}$ tell us that the first three spacial coordinates of v are null. Then, we have

$$\hat{\Theta}(q) = (2\pi)^{\frac{m}{2}-1}\delta(\mathbf{q}_1, q_2, \ldots, q_{n-1})\frac{i}{q_1^0 + i0^+}. \tag{3.73}$$

Replacing (3.73) into (3.72), we obtain

$$\hat{r}(p) = (2\pi)^{-1} \int dk_1^0 \frac{i\hat{d}(k_1^0, \mathbf{p}_1, \ldots, p_{n-1})}{p_1^0 - k_1^0 + i0^+}. \tag{3.74}$$

Regarding that $p_i \in \{\Gamma_1^+ \cup \Gamma_1^-\}$ and making the substitution $k_1^0 = t_1 p_1^0$ in (3.74), we obtain

$$\hat{r}(p) = (2\pi)^{-1} Sgn(p_1^0) \int dt_1 \frac{i\hat{d}(t_1 p_1^0, \mathbf{p}_1, \ldots, p_{n-1})}{1 - t_1 + iSgn(p_1^0)0^+}, \tag{3.75}$$

where Sgn represents the sign function.

Apparently, in (3.75), we lost the covariance but because $\hat{d}(p)$ is Lorentz invariant we can make the computation in a reference system where $\mathbf{p}_1 = 0$, then making the boost $(tp_1^0, \mathbf{0}) \to (tp_1^0, t\mathbf{p}_1)$ we will obtain

$$\hat{r}(p) = (2\pi)^{-1} Sgn(p_1^0) \int dt_1 \frac{i\hat{d}(t_1 p_1, \ldots, p_{n-1})}{1 - t_1 + iSgn(p_1^0)0^+}. \tag{3.76}$$

The result (3.76) shows that in the computation of $\hat{r}(p)$ we could choose the variables $\{p_2, p_3, \ldots, p_{n-1}\}$ arbitrarily. Of course, if we take $v = (0, \ldots, v_j^0 = 1, \mathbf{v}_j = 0, \ldots, 0)$, we finally obtain a momentum dependence on $p_j \in \{\Gamma_1^+ \cup \Gamma_1^-\}$.

To obtain $\hat{r}(p)$ independent of a specific variable p_j, we must multiply (3.71) by $n-1$ step functions $\prod_{j=0}^{n-1}\Theta(x_j^0)$ giving us the following formula

$$\begin{aligned}\hat{r}(p) = \left(\frac{i}{2\pi}\right)^{n-1} Sgn(\prod_{j=0}^{n-1} p_j^0)\times \\ \times \int \prod_{j=0}^{n-1} dt_j \left[\prod_{j=0}^{n-1} \frac{1}{1-t_j+iSgn(p_j^0)0^+}\right]\hat{d}(t.p),\end{aligned} \tag{3.77}$$

where $t.p = t_1.p_1 + \ldots + t_{n-1}.p_{n-1}$ and $p \in \{\Gamma_{n-1}^+ \cup \Gamma_{n-1}^-\}$.

For a two-point retarded part, the formula (3.77) will be

$$\hat{r}(p) = \frac{i}{2\pi} Sgn(p^0)\int_{-\infty}^{\infty} dt \frac{\hat{d}(tp)}{1-t+iSgn(p^0)0^+}, \quad p \in \Gamma_1^+ \cup \Gamma_1^-. \tag{3.78}$$

3.4.2 Singular distribution Case

Similarly to (3.71), in the singular case we have

$$\begin{aligned}\langle \hat{r}(p), \check{f}(p)\rangle &= \langle \widehat{\Theta(v.\tilde{x})d(\tilde{x})}(p), \widetilde{\mathcal{W}f}(p)\rangle \\ &= (2\pi)^{-\frac{m}{2}}\langle \hat{\Theta}(p) * \hat{d}(p), \widetilde{\mathcal{W}f}(p)\rangle.\end{aligned} \tag{3.79}$$

From (3.63), we can compute the term $\widetilde{\mathcal{W}f}(p)$

$$\begin{aligned}\widetilde{\mathcal{W}f}(p) &= \left[\widetilde{f(x) - w(x)\sum_{|l|=0}^{\omega}\tfrac{1}{l!}[\mathbf{D}^l f](0)x^l}\right](p) \\ &= \check{f}(p) - \sum_{|l|=0}^{\omega}\frac{1}{l!}[\mathbf{D}^l f](0)\left[\widetilde{w(x)x^l}\right](p) \\ &= \check{f}(p) - \sum_{|l|=0}^{\omega}\frac{1}{l!}[\mathbf{D}^l f](0)[i^l\mathbf{D}_p^l\check{w}(p)].\end{aligned} \tag{3.80}$$

The term $[\mathbf{D}^l f](0)$ could be written as

$$\begin{aligned}[\mathbf{D}^l f](0) &= \langle \delta(x), \mathbf{D}^l f(x)\rangle = (-1)^l\langle \mathbf{D}^l\delta(x), f(x)\rangle \\ &= (-1)^l\langle (2\pi)^{-\frac{m}{2}}(-ip')^l, \check{f}(p')\rangle = (2\pi)^{-\frac{m}{2}}\langle (ip')^l, \check{f}(p')\rangle,\end{aligned} \tag{3.81}$$

and replacing into (3.80), we obtain

$$\begin{aligned}\widetilde{\mathcal{W}f}(p) &= \check{f}(p) - \sum_{|l|=0}^{\omega} \frac{1}{l!}(2\pi)^{-\frac{m}{2}} \langle (ip')^l, \check{f}(p')\rangle [i^l \mathbf{D}_p^l \breve{w}(p)] \\ &= \check{f}(p) - (2\pi)^{-\frac{m}{2}} \sum_{|l|=0}^{\omega} \frac{(-1)^l}{l!} \langle p'^l, \check{f}(p')\rangle [\mathbf{D}_p^l \breve{w}(p)].\end{aligned} \tag{3.82}$$

Now, replacing (3.82) into (3.79), we get for the term $\langle \hat{r}(p), \check{f}(p)\rangle$ the following result

$$\begin{aligned}&\langle \hat{r}(p), \check{f}(p)\rangle \\ &= (2\pi)^{-\frac{m}{2}} \langle \hat{\Theta}(k), \langle \hat{d}(p-k), \check{f}(p)\rangle\rangle \\ &\quad - (2\pi)^{-m} \sum_{|l|=0}^{\omega} \frac{(-1)^l}{l!} \langle \hat{\Theta}(k), \langle \hat{d}(p-k), \mathbf{D}_p^l \breve{w}(p)\rangle\rangle \langle p'^l, \check{f}(p')\rangle \\ &= (2\pi)^{-\frac{m}{2}} \langle \hat{\Theta}(k), \langle \hat{d}(p-k), \check{f}(p)\rangle\rangle \\ &\quad - (2\pi)^{-m} \sum_{|l|=0}^{\omega} \frac{(-1)^l}{l!} \langle \hat{\Theta}(k), \langle \hat{d}(p'-k), \mathbf{D}_{p'}^l \breve{w}(p')\rangle\rangle \langle p^l, \check{f}(p)\rangle,\end{aligned} \tag{3.83}$$

where in the last line we interchange p and p'.

In (3.83), the distribution result with the step function could be written as an integral. Also we could factorize the test function $\check{f}(p)$ and obtain

$$\begin{aligned}\langle \hat{r}(p), \check{f}(p)\rangle &= (2\pi)^{-\frac{m}{2}} \int dk \hat{\Theta}(k) \langle \hat{d}(p-k), \check{f}(p)\rangle \\ &\quad - (2\pi)^{-m} \sum_{|l|=0}^{\omega} \frac{(-1)^l}{l!} \int dk \hat{\Theta}(k) \langle \hat{d}(p'-k), \mathbf{D}_{p'}^l \breve{w}(p')\rangle \langle p^l, \check{f}(p)\rangle \\ &= \langle (2\pi)^{-\frac{m}{2}} \int dk \hat{\Theta}(k) \hat{d}(p-k), \check{f}(p)\rangle \\ &\quad - \langle (2\pi)^{-m} \int dk \hat{\Theta}(k) \sum_{|l|=0}^{\omega} \frac{(-1)^l}{l!} \langle \hat{d}(p'-k), \mathbf{D}_{p'}^l \breve{w}(p')\rangle p^l, \check{f}(p)\rangle.\end{aligned} \tag{3.84}$$

Again, comparing the two sides of equation (3.84), we get the formula

$$
\begin{aligned}
\hat{r}(p) &= (2\pi)^{-\frac{m}{2}} \int dk \hat{\Theta}(k) \hat{d}(p-k) \\
&\quad - (2\pi)^{-m} \int dk \hat{\Theta}(k) \sum_{|l|=0}^{\omega} \frac{p^l(-1)^l}{l!} \langle \hat{d}(p'-k), \mathbf{D}^l_{p'} \check{w}(p') \rangle \\
&= (2\pi)^{-\frac{m}{2}} \int dk \hat{\Theta}(k) \hat{d}(p-k) \\
&\quad - (2\pi)^{-m} \int dk \hat{\Theta}(k) \sum_{|l|=0}^{\omega} \frac{p^l}{l!} \langle \mathbf{D}^l_{p'} \hat{d}(p'-k), \check{w}(p') \rangle \\
&= (2\pi)^{-\frac{m}{2}} \int dk \hat{\Theta}(k) \hat{d}(p-k) \\
&\quad - (2\pi)^{-m} \int dk \hat{\Theta}(k) \sum_{|l|=0}^{\omega} \frac{p^l}{l!} \int dp' [\mathbf{D}^l_{p'} \hat{d}(p'-k)] \check{w}(p'),
\end{aligned}
\tag{3.85}
$$

where in the second equality we used the definition of distribution's derivative, and in the second term of third equality we wrote the distribution result as an integral over p'.

The formula (3.85) depend on function $\check{w}(p)$. We could eliminate the latter dependence regarding the non-uniqueness of $r(x)$. In the momentum space, the most general solution $\hat{r}(p)$ will be obtained from the Fourier transformation of (3.69)

$$
\hat{\tilde{r}}(p) = \hat{r}(p) + \sum_{l=0}^{\omega} \hat{C}_l p^l. \tag{3.86}
$$

The formula (3.86) tells us that we can add to $\hat{r}(p)$ any polynomial, of degree equal or less than ω, to obtain an equivalent solution. The latter property allows us to define the *normalized solution* $\hat{r}_q(p)$ in the following form

$$
\hat{r}_q(p) = \hat{r}(p) - \sum_{b=0}^{\omega} \frac{(p-q)^b}{b!} [\mathbf{D}^b \hat{r}](q) \leftrightarrow [\mathbf{D}^b \hat{r}_q](q) = 0 \quad \text{for all } b \leqslant \omega, \tag{3.87}
$$

where $q \in \mathbb{R}^m$ is a fixed point.

In Appendix A.4, we show the computation to get the following explicit form for $\hat{r}_q(p)$

$$
\hat{r}_q(p) = (2\pi)^{-\frac{m}{2}} \int dk \hat{\Theta}(k) \Big[\hat{d}(p-k) - \sum_{b=0}^{\omega} \frac{(p-q)^b}{b!} \mathbf{D}^b_q \hat{d}(q-k) \Big]. \tag{3.88}
$$

Because q could be any point of $\mathbb{R}^m$, we define the ***central splitting solution***

$\hat{r}_0(p)$ when we choose $q = 0$

$$\hat{r}_0(p) = (2\pi)^{-\frac{m}{2}} \int dk \hat{\Theta}(k) \Big[\hat{d}(p-k) - \sum_{b=0}^{\omega} \frac{p^b}{b!} \mathbf{D}_q^b \hat{d}(q-k) \Big|_{q=0} \Big]. \tag{3.89}$$

As show in Appendix A.5, we could find a nice formula for $\hat{r}_0(p)$ by taking into account the indetermination of vector v in the construction of the Heaviside step function. For a two-point retarded distribution, the solution takes the following form

$$\hat{r}_0(p) = \frac{i}{2\pi} Sgn(p^0) \int dt \frac{\hat{d}(tp)}{(t - i0^+)^{\omega+1}(1 - t + iSgn(p^0)0^+)}. \tag{3.90}$$

Summarizing, in the momentum space, the procedure to obtain the retarded part of a causal distribution is:

1. Compute the Fourier transform of numerical causal distribution $\hat{d}(p)$.
2. Determine the *power counting function* $\rho(\alpha)$ via (3.57), which has a momentum space version as follows

   Definition 3.3 *A distribution $\hat{d}(p) \in C_0'^{\infty}(\mathbb{R}^m)$ has a quasi-asymptotic $\hat{d}_0(p)$ over $p = \infty$, if for a positive function $\rho(\alpha)$ $(\alpha > 0)$ there exists the limit*

   $$\lim_{\alpha \to 0} \langle \rho(\alpha) \hat{d}(\frac{p}{\alpha}), \check{\psi}(p) \rangle = \langle \hat{d}_0(p), \check{\psi}(p) \rangle \neq 0. \tag{3.91}$$

3. Obtain the order of singularity ω via

   $$\lim_{\alpha \to 0^+} \frac{\rho(a\alpha)}{\rho(\alpha)} = a^{\omega}. \tag{3.92}$$

4. If the numerical causal distribution $\hat{d}(p)$ is *regular* $\omega < 0$, the retarded part, normalized in the origin and at second order, is given by the following formula

   $$\hat{r}_0(p) = \frac{i}{2\pi} Sgn(p^0) \int_{-\infty}^{\infty} dt \frac{\hat{d}(tp)}{1 - t + iSgn(p^0)0^+}, \quad p \in \Gamma_1^+ \cup \Gamma_1^-. \tag{3.93}$$

5. For the singular case $\omega \geqslant 0$, the most general solution for the retarded part in second order is

   $$\hat{\tilde{r}}_0(p) = \frac{i}{2\pi} Sgn(p^0) \int dt \frac{\hat{d}(tp)}{(t - i0^+)^{\omega+1}(1 - t + iSgn(p^0)0^+)} + \sum_{l=0}^{\omega} \hat{C}_l p^l, \tag{3.94}$$

   where the constants $\hat{C}_l$ are not defined by the causal splitting procedure.

Of course, the uniqueness of S-matrix implies that one of the solution families (3.94) is the real physical one. The physical solution will be obtained by fixing the constants $\hat{C}_l$ regarding physical properties of the theory such as gauge invariance, charge invariance, particle masses, etc.

Chapter 4

Quantized free Fields and Perturvative Gauge Invariance

> Elementary particles are complicated real objects; free fields are simpler mathematical ones. Nevertheless, free fields are the basis of quantum field theory because the really interesting quantities like interacting fields and scattering matrix (S-matrix) can be expanded in terms of free fields.
>
> *Günter Scharf*

Free fields are solutions to the relativistic covariant homogeneous field equations with a quantization rule. They are not physical objects because they do not model all the properties of particles, but they are all we know how to solve. Fortunately, in the case of electromagnetic interaction, the value of coupling constant is small enough to allow for the expansion of the S-matrix in terms of free fields.

In this thesis we will study scalar QED as a Duffin-Kemmer-Petiau gauge theory (SDKP) via CPT. Therefore, as mentioned in Chapter 3, we need to determine the quantized electromagnetic and DKP free fields. In this Chapter we develop the latter. Also, we show the properties of a fermionic scalar (Ghost) field to introduce the physical principle of *Perturbative Gauge Invariance* (PGI) in order to complement CPT. Specifically, PGI is used to define the first term T_1 of S-matrix expansion (3.5).

4.1 Electromagnetic Field

The quantized electromagnetic field is modeled by the 4-potential $A^\mu(x)$ which obeys the relativistic wave equation

$$\Box A^\mu = 0, \quad \Box = g^{\mu\nu}\partial_\nu\partial_\mu, \quad g^{\mu\nu} = diag(+,-,-,-), \tag{4.1}$$

which is related with the Lorenz gauge condition $\partial_\mu A^\mu_{class} = 0$ for a classical electromagnetic 4-potential. We will see that the latter is related to the physical Fock space for transversal photons.

Taking into account (4.1) just as four massless Klein-Gordon-Fock equations, we define the solutions as

$$A^0(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\omega}} \left(c^0(\mathbf{k}) e^{-ikx} - c^0(\mathbf{k})^\dagger e^{ikx} \right), \tag{4.2}$$

$$A^i(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2\omega}} \left(c^i(\mathbf{k}) e^{-ikx} + c^i(\mathbf{k})^\dagger e^{ikx} \right), \tag{4.3}$$

where the operators $c^\mu(k)^\dagger$ and $c^\mu(k)$ are the creation and annihilation operators, respectively, which follows the commutation relations

$$[c^\mu(\mathbf{k}), c^\nu(\mathbf{k}')^\dagger] = \begin{cases} \delta(\mathbf{k}-\mathbf{k}') & \text{for} \quad \mu = \nu \\ 0 & \text{for} \quad \mu \neq \nu \end{cases} . \tag{4.4}$$

The minus sign in (4.2) has been chosen to lead to a mathematical consistent result for the commutation of two electromagnetic 4-potentials[1] components

$$\left[A^\alpha(x), A^\beta(y) \right] = g^{\alpha\beta} i D_0(x-y), \tag{4.5}$$

where $D_0(x-y)$ is the massless ($m = 0$) Lorentz invariant Jordan-Pauli distribution

$$D_m(x) \equiv \frac{i}{(2\pi)^3} \int d^4p \delta(p^2 - m^2) sgn(p^0) e^{-ipx}. \tag{4.6}$$

[1] If we do not use the minus sign we will obtain

$$\left[A^\alpha(x), A^\beta(y) \right] = \delta^\alpha_\beta i D_0(x-y),$$

which is not correct because we have a second rank Lorentz tensor in the left hand side of the equation and a scalar in the right. We could use the "indefinite metric" prescription to remedy the incoherence, but in that case we would have negative states in the Hilbert space.

In order to determine how the Lorenz condition works at the operator level, note that the covariant derivative of A^ν gives the following result

$$\begin{aligned}\partial_\nu A^\nu &= (2\pi)^{-3}\int d^3k\sqrt{\frac{\omega}{2}}[-i\left(c^0(\mathbf{k})+\frac{k_j}{\omega}c^j(\mathbf{k})\right)e^{-ikx}+\\ &\quad +i\left(-c^0(\mathbf{k})^\dagger+\frac{k_j}{\omega}c^j(\mathbf{k})^\dagger\right)e^{ikx}]\\ &= (2\pi)^{-3}\int d^3k\sqrt{\frac{\omega}{2}}[-i\left(c^0(\mathbf{k})+c^j_{||}(\mathbf{k})\right)e^{-ikx}+\\ &\quad +i\left(-c^0(\mathbf{k})^\dagger+c^j_{||}(\mathbf{k})^\dagger\right)e^{ikx}],\end{aligned} \tag{4.7}$$

where $c^j_{||}(\mathbf{k})=\frac{k_j}{\omega}c^j(\mathbf{k})$ is the annihilation operator for longitudinal photons.

Then, states $|\Phi\rangle \in \mathcal{F}_{phys}$, which have neither longitudinal nor scalar polarized modes photons, fulfill the following condition

$$\langle\Phi|\partial_\nu A^\nu|\Phi\rangle = 0, \tag{4.8}$$

The constraint (4.8) is the quantum equivalent of the classical Lorentz condition.

We define the negative and positive frequency solution for A^μ as

$$A^{\mu(+)} = (2\pi)^{-3/2}\int\frac{d^3k}{\sqrt{2\omega}}c^\mu(\mathbf{k})^\dagger e^{ikx}\times\begin{cases}1, & \text{for } \mu=1,2,3\\ -1, & \text{for } \mu=0,\end{cases} \tag{4.9}$$

$$A^{\mu(-)} = (2\pi)^{-3/2}\int\frac{d^3k}{\sqrt{2\omega}}c^\mu(\mathbf{k})e^{-ikx}. \tag{4.10}$$

From (4.9), (4.10) and (4.4), we compute the following commutation relations

$$\overline{A^\mu(x)A^\nu}(y) = [A^{\mu(-)}(x),A^{\nu(+)}(y)] = g^{\mu\nu}iD_0^{(+)}(x-y), \tag{4.11}$$

$$[A^{\nu(+)}(x),A^{\mu(-)}(y),] = g^{\mu\nu}iD_0^{(-)}(x-y), \tag{4.12}$$

where $\overline{A^\mu(x)A^\nu}(y)$ is the Wick contraction of two electromagnetic 4-field potentials (see Appendix A.2), and where $D_0^{(+)}(x-y)$ and $D_0^{(-)}(x-y)$ are the positive and negative part of Jordan-Pauli distribution

$$D_m^{(+)}(x) \equiv \frac{i}{(2\pi)^3}\int d^4p\delta(p^2-m^2)\Theta(p^0)e^{-ipx} = \frac{i}{(2\pi)^3}\int\frac{d^3p}{2p^0}e^{-ipx}, \tag{4.13}$$

$$D_m^{(-)}(x) \equiv \frac{-i}{(2\pi)^3}\int d^4p\delta(p^2-m^2)\Theta(p^0)e^{ipx} = \frac{-i}{(2\pi)^3}\int\frac{d^3p}{2p^0}e^{ipx}, \tag{4.14}$$

$$D_m(x) = D_m^{(+)}(x)+D_m^{(-)}(x). \tag{4.15}$$

4.2 Duffin-Kemmer-Petiau fields

DKP fields fulfill the Dirac like equation [3–5]

$$(i\beta^\mu \partial_\mu - m)\psi(x) = 0, \tag{4.16}$$

where β^μ represent four matrices which obey the following algebra

$$\beta^\mu \beta^\nu \beta^\rho + \beta^\rho \beta^\nu \beta^\mu = \beta^\mu g^{\nu\rho} + \beta^\rho g^{\mu\nu}. \tag{4.17}$$

The algebra (4.17) has three irreducible representation of order 1, 5 and 10. The representation of order 1 is trivial, the next order 5 represent scalar particles and the order 10 represents spin-1 particles. For more details of historical development of the DKP equation we refer to references [6, 14].

The equation (4.16) can be obtained from the Lagrangian density

$$\mathcal{L}_{\mathrm{DKP}} = \frac{i}{2}\overline{\psi}(x)\beta^\mu \overleftrightarrow{\partial_\mu}\psi(x) - m\overline{\psi}(x)\psi(x), \tag{4.18}$$

where the conjugate DKP field $\bar{\psi}(x)$ is obtained by

$$\bar{\psi}(x) = \psi^\dagger(x)\eta^0, \quad \eta^0 = 2(\beta^0)^2 - 1, \tag{4.19}$$

and it obeys the equation

$$\overline{\psi}(x)(i\beta^\mu \overleftarrow{\partial_\mu} + m) = 0. \tag{4.20}$$

A particular solution for the β^μ-matrices in its irreducible representation of order 5 is

$$\beta^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \beta^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\beta^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}, \ \beta^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{4.21}$$

.

A solution for the DKP field $\psi(x)$ in the scalar representation is given by

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} a(\mathbf{p}) u^-(\mathbf{p}) e^{-ipx} + \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} b^\dagger(\mathbf{p}) u^+(\mathbf{p}) e^{ipx}, \tag{4.22}$$

$$\overline{\psi}(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} a^\dagger(\mathbf{p}) \overline{u^-}(\mathbf{p}) e^{ipx} + \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} b(\mathbf{p}) \overline{u^+}(\mathbf{p}) e^{-ipx}, \tag{4.23}$$

where $a(\mathbf{p})$ is the annihilation operator of a scalar particle and $b(\mathbf{p})$ is the annihilation operator of an antiparticle. They obey the commutation relations

$$\begin{cases} [a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta(\mathbf{p} - \mathbf{p}'), \\ [b(\mathbf{p}), b^\dagger(\mathbf{p}')] = \delta(\mathbf{p} - \mathbf{p}'), \end{cases} \tag{4.24}$$

and null for other commutations.

The factors $u^-(\mathbf{p})$ and $u^+(\mathbf{p})$ are five elements column vector normalized to get a positive energy system as follows

$$\overline{u^\pm} \beta^0 u^\pm = \mp 1. \tag{4.25}$$

From solution (4.22) we can define the positive and negative frequency solutions $\psi^{(+)}$ and $\psi^{(-)}$

$$\psi^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} b^\dagger(\mathbf{p}) u^+(\mathbf{p}) e^{ipx}, \tag{4.26}$$

$$\psi^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} a(\mathbf{p}) u^-(\mathbf{p}) e^{-ipx}, \tag{4.27}$$

and by conjugation

$$\overline{\psi}^{(+)}(x) \equiv \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} a^\dagger(\mathbf{p}) \overline{u^-}(\mathbf{p}) e^{ipx}, \tag{4.28}$$

$$\overline{\psi}^{(-)}(x) \equiv \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} b(\mathbf{p}) \overline{u^+}(\mathbf{p}) e^{-ipx}. \tag{4.29}$$

For a global $U(1)$ transformation $\delta\psi(x) = ie\alpha\psi(x)$, the conserved Noether current j^μ is

$$j^\mu(x) = e : \overline{\psi}(x) \beta^\mu \psi(x) :, \tag{4.30}$$

where e is the unit charge of a scalar particle and the double dots $: \ldots :$ mean a normal ordering product, as usual, to normalize the vacuum expectation value of the current as $\langle 0 | j^\mu(x) | 0 \rangle = 0$.

4.2.1 $S(x)$ function

Now, we want to compute the commutation $[\psi_a(x), \overline{\psi}_b(y)]$. Using

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x), \tag{4.31}$$

we have

$$[\psi_a(x), \overline{\psi}_b(y)] = [\psi_a^{(+)}(x), \overline{\psi}_b^{(-)}(y)] + [\psi_a^{(-)}(x), \overline{\psi}_b^{(+)}(y)]. \tag{4.32}$$

The second commutation in the right hand side of (4.32) is

$$[\psi_a^{(-)}(x), \overline{\psi}_b^{(+)}(y)] = \int \frac{d^3p}{(2\pi)^3} \overline{u^-}_b(\mathbf{p}) u_a^-(\mathbf{p}) e^{-ip(x-y)}. \tag{4.33}$$

In order to simplify the expression (4.33), we will determine the product $\overline{u^-}_b(\mathbf{p}) u_a^-(\mathbf{p})$. Replacing $\overline{\psi}^{(+)}(x)$ from (4.28) into (4.20), we can obtain the following identity

$$\begin{aligned} \overline{\psi}^{(+)}(x)(i\beta^\mu \overleftarrow{\partial_\mu} + m) &= 0 \\ \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} a^\dagger(\mathbf{p}) \overline{u^-}(\mathbf{p}) e^{ipx}(i\beta^\mu \overleftarrow{\partial_\mu} + m) &= 0, \\ -\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} a^\dagger(\mathbf{p}) \overline{u^-}(\mathbf{p})(\beta^\mu p_\mu - m) e^{ipx} &= 0, \\ \overline{u^-}_b(\mathbf{p})(\beta^\mu p_\mu - m)_{bc} &= 0. \end{aligned} \tag{4.34}$$

Multiplying (4.34) by $u_a^-(\mathbf{p})$, we get

$$u_a^-(\mathbf{p}) \overline{u^-}_b(\mathbf{p})(\beta^\mu p_\mu - m)_{bc} = 0. \tag{4.35}$$

On the other hand, with the use of (4.17), we could obtain the identities

$$\begin{aligned} \beta^\mu p_\mu(\beta^\nu p_\nu \beta^\theta p_\theta - m^2) &= 0, \\ \beta^\mu p_\mu(\beta^\nu p_\nu + m)(\beta^\theta p_\theta - m) &= 0. \end{aligned} \tag{4.36}$$

Comparing (4.36) and (4.35), we finally have for $u_a^-(\mathbf{p}) \overline{u^-}_b(\mathbf{p})$

$$u_a^-(\mathbf{p}) \overline{u^-}_b(\mathbf{p}) = C[\beta^\mu p_\mu(\beta^\nu p_\nu + m)]_{ab} \tag{4.37}$$

where C is a constant that we could compute using the normalization condition (4.25) as follows

$$\begin{aligned} Tr[\overline{u^-}\beta^0 u^-] &= 1 \\ Tr[\beta^0 u^- \overline{u^-}] &= 1 \\ CTr[\beta^0\beta^\mu p_\mu(\beta^\nu p_\nu + m)] &= 1 \\ Cmp_\mu Tr[\beta^0\beta^\mu] &= 1 \\ Cmp_\mu 2g^{\mu 0} &= 1 \\ C &= \frac{1}{2mp^0} \end{aligned} \tag{4.38}$$

where the following properties are used

$$Tr[\beta^{\mu_1}\beta^{\mu_2}\dots\beta^{\mu_{2n-1}}] = 0, \tag{4.39}$$

$$Tr[\beta^{\mu_1}\beta^{\mu_2}\dots\beta^{\mu_{2n}}] = g^{\mu_1\mu_2}g^{\mu_3\mu_4}\dots g^{\mu_{2n-1}\mu_{2n}} + g^{\mu_2\mu_3}g^{\mu_4\mu_5}\dots g^{\mu_{2n}\mu_1}. \tag{4.40}$$

Replacing (4.38) into (4.37), we have

$$u_a^-(\mathbf{p})\overline{u^-}_b(\mathbf{p}) = \frac{1}{2mp^0}[\beta^\mu p_\mu(\beta^\nu p_\nu + m)]_{ab}, \tag{4.41}$$

and replacing the latter result into (4.33), we obtain

$$[\psi_a^{(-)}(x), \overline{\psi}_b^{(+)}(y)] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2mp^0}[\not{p}(\not{p}+m)]_{ab}e^{-ip(x-y)}, \tag{4.42}$$

where we use the notation $\not{p} = \beta^\mu p_\mu$. Following the same path, the first commutation on the right hand side of (4.32) takes the following form

$$[\psi_a^{(+)}(x), \overline{\psi}_b^{(-)}(y)] = -\int \frac{d^3p}{(2\pi)^3}\frac{1}{2mp^0}[\not{p}(\not{p}-m)]_{ab}e^{ip(x-y)}. \tag{4.43}$$

We can rewrite (4.42) as

$$[\psi_a^{(-)}(x), \overline{\psi}_b^{(+)}(y)] = \frac{1}{i}[\frac{1}{m}[i\not{\partial}(i\not{\partial}+m)]_{ab}][i\int \frac{d^3p}{(2\pi)^3}\frac{e^{-ip(x-y)}}{2E}], \tag{4.44}$$

where we can identify the positive frequency part of Jordan-Pauli causal distribution (4.13). Using the latter result (4.44), we define the positive frequency function $S^{(+)}$ in the following form

$$S_{ab}^{(+)}(x) \equiv \frac{1}{m}[i\not{\partial}(i\not{\partial}+m)]_{ab}D_m^{(+)}(x). \tag{4.45}$$

Replacing (4.45) into (4.44) we obtain

$$[\psi_a^{(-)}(x), \overline{\psi}_b^{(+)}(y)] = \frac{1}{i}S_{ab}^{(+)}(x-y). \tag{4.46}$$

Working in the same way with the commutation $[\psi_a^{(+)}(x), \overline{\psi}_b^{(-)}(y)]$, it is possible to write it as follow

$$\begin{aligned}
&[\psi_a^{(+)}(x), \overline{\psi}_b^{(-)}(y)] = \\
&= -\int \frac{d^3p}{(2\pi)^3} \frac{1}{2mp^0} [\not{p}(\not{p} - m)]_{ab} e^{ip(x-y)} \\
&= \frac{1}{i} [\frac{1}{m} [i\not{\partial}(i\not{\partial} + m)]_{ab}] [-i \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{ip(x-y)}],
\end{aligned} \tag{4.47}$$

where we identify the negative frequency part of Jordan-Pauli causal distribution (4.14). Then, we define the negative frequency function $S^{(-)}(x)$ as

$$S_{ab}^{(-)}(x) \equiv \frac{1}{m} [i\not{\partial}(i\not{\partial} + m)]_{ab} D_m^{(-)}(x). \tag{4.48}$$

Replacing (4.48) into (4.47), we obtain

$$[\psi_a^{(+)}(x), \overline{\psi}_b^{(-)}(y)] = \frac{1}{i} S_{ab}^{(-)}(x - y). \tag{4.49}$$

Finally, we define the function $S(x)$ as

$$S(x) \equiv S^{(+)}(x) + S^{(-)}(x) = \frac{1}{m} [i\not{\partial}(i\not{\partial} + m)] D_m(x), \tag{4.50}$$

and the commutation (4.32) takes the following form

$$[\psi_a(x), \overline{\psi}_b(y)] = \frac{1}{i} S_{ab}(x - y). \tag{4.51}$$

For future use, we will compute the Fourier transform of the Jordan-Pauli and $S(x)$ distributions. From (4.6), (4.13) and (4.14), the following formulas are clear

$$\hat{D}_m(p) = \frac{i}{2\pi} \delta(p^2 - m^2) Sgn(p^0), \tag{4.52}$$

$$\hat{D}_m^{(+)}(p) = \frac{i}{2\pi} \delta(p^2 - m^2) \Theta(p^0), \tag{4.53}$$

$$\hat{D}_m^{(-)}(p) = -\frac{i}{2\pi} \delta(p^2 - m^2) \Theta(-p^0). \tag{4.54}$$

From (4.45), (4.48) and (4.50) it is straightforward to determine

$$\hat{S}^{\pm}(p) = \frac{\pm i}{(2\pi)} \Theta(\pm p^0) \delta(p^2 - m^2) \frac{1}{m} [\not{p}(\not{p} + m)], \tag{4.55}$$

$$\hat{S}(p) = \frac{i}{(2\pi)} Sgn(p^0) \delta(p^2 - m^2) \frac{1}{m} [\not{p}(\not{p} + m)]. \tag{4.56}$$

4.3 Fermionic Scalar (Ghost) Fields

In this section we will follow reference [93]. We define here two scalar fields $u(x)$ and $\tilde{u}(x)$

$$u(x) \equiv (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} \left(d_2(\mathbf{p})e^{-ipx} + d_1(\mathbf{p})^\dagger e^{ipx}\right), \tag{4.57}$$

$$\tilde{u}(x) \equiv (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} \left(-d_1(\mathbf{p})e^{-ipx} + d_2(\mathbf{p})^\dagger e^{ipx}\right), \tag{4.58}$$

where the operators d_i and $d_j^\dagger$ are the annihilation and creation operators which satisfy the following anticommutation relations

$$\{d_j(\mathbf{p}), d_k^\dagger(\mathbf{q})\} = \delta_{jk}\delta(\mathbf{p}-\mathbf{q}). \tag{4.59}$$

The positive and negative part of $u(x)$ and $\tilde{u}(x)$ are

$$u^{(+)}(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} d_1(\mathbf{p})^\dagger e^{ipx}, \tag{4.60}$$

$$u^{(-)}(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} d_2(\mathbf{p}) e^{-ipx}, \tag{4.61}$$

$$\tilde{u}^{(+)}(x) = (2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} d_2(\mathbf{p})^\dagger e^{ipx}, \tag{4.62}$$

$$\tilde{u}^{(-)}(x) = -(2\pi)^{-3/2} \int \frac{d^3p}{\sqrt{2\omega}} d_1(\mathbf{p}) e^{-ipx}. \tag{4.63}$$

From (4.59), the non-null anticommutaors are

$$\{u^{(-)}(x), \tilde{u}^{(+)}(y)\} = (2\pi)^{-3} \int \frac{d^3p}{2E} e^{-ip(x-y)} = -iD_m^{(+)}(x-y), \tag{4.64}$$

$$\{u^{(+)}(x), \tilde{u}^{(-)}(y)\} = -(2\pi)^{-3} \int \frac{d^3p}{2E} e^{ip(x-y)} = -iD_m^{(-)}(x-y). \tag{4.65}$$

The need for the introduction of the fields $u(x)$ and $\tilde{u}(x)$ is to construct a quantum gauge theory in the next section.

4.4 Perturbative Gauge Invariance

As mentioned in Chapter 3, in order to begin the construction of the S-matrix, we need to define the first nontrivial distribution term $T_1(x)$ in (3.5). In the usual approach

$T_1(x) = i : \mathcal{L}_{\text{int}} :$, where $\mathcal{L}_{\text{int}}$ is the interaction Lagrangian. In causal perturbation theory this is not true.

As an example, we mention the case of SQED constructed with a complex scalar field $\varphi(x)$ which obeys the Klein-Gordon-Fock equation $(\Box + m^2)\varphi(x) = 0$. In order to obtain a gauge invariant theory, we consider $\varphi(x)$ as a classical field that will be coupled to a classical electromagnetic field $A^\mu(x)$. Using the minimal coupling prescription, we substitute the partial derivative in the free Lagrangian for $\varphi(x)$ with the covariant derivative $D^\mu = \partial^\mu + ieA^\mu$, obtaining

$$\mathcal{L}_{\text{int}} = -ieA^\mu(\varphi * \overleftrightarrow{\partial_\mu}\varphi) + e^2\varphi^*\varphi A^\mu A_\mu, \tag{4.66}$$

where e represents the electric charge of the scalar particle.

The problem of using (4.66) to construct T_1 is the second order term $e^2\varphi^*\varphi A^\mu A_\mu$ which by construction must belong to T_2 because in CPT the unit charge e represents the physical charge and not a simple parameter. What is unquestionable is that T_1 must be defined from the gauge invariance property but at the quantum level.

In general, a gauge transformation $A^\mu(x) \to A'^\mu(x)$, implies that $A^\mu(x)$ and $A'^\mu(x)$ obey the same equation of motion. The latter is equivalent to obtain a transformation where $A'^\mu(x)$ obey the same commutation relation as $A^\mu(x)$. This is possible with the following transformation

$$A'^\mu(x) = e^{-i\lambda Q}A^\mu(x)e^{i\lambda Q}, \tag{4.67}$$

where Q is called ***gauge charge***.

By expanding the exponential operators, we obtain

$$A'^\mu(x) = A^\mu(x) - i\lambda[Q, A^\mu(x)] + O(\lambda^2). \tag{4.68}$$

On the other hand, consider the following classical gauge transformation but at the operator level

$$A'^\mu(x) = A^\mu(x) + \lambda\partial^\mu u(x) + O(\lambda^2), \tag{4.69}$$

where u is a free quantum field which obeys the massless Klein-Gordon-Fock equation

$$\Box u(x) = 0. \tag{4.70}$$

For an infinitesimal parameter λ, by comparing (4.68) and (4.69), we can obtain an equation which defines Q uniquely

$$[Q, A^\mu(x)] = i\partial^\mu u(x). \tag{4.71}$$

The solution for Q from (4.71) is

$$Q = \int d^3x[\partial_\nu A^\nu \partial_0 u - (\partial_0 \partial_\nu A^\nu)u] = \int d^3x \partial_\nu A^\nu \overleftrightarrow{\partial}_0 u, \tag{4.72}$$

where the integral is evaluated over a hyperplane $x^0 = \text{constant}$.

If $u(x)$ is a *fermionic scalar ghost field*, we can obtain a nilpotent Q in the form

$$Q^2 = \frac{1}{2}\{Q, Q\} = 0, \tag{4.73}$$

which can be used to construct a physical Fock space as we will show bellow.

Using (4.2), (4.3) and (4.57) we can obtain Q as

$$\begin{aligned} Q &= \int d^3k\omega(k)[(a_\parallel(\mathbf{k})^\dagger - a_0(\mathbf{k})^\dagger)d_2(\mathbf{k}) + d_1(\mathbf{k})^\dagger(a_\parallel(\mathbf{k}) + a_0(\mathbf{k}))] \\ &= \int d^3k\omega(k)[c_2(\mathbf{k})^\dagger d_2(\mathbf{k}) + d_1(\mathbf{k})^\dagger c_1(\mathbf{k})], \end{aligned} \tag{4.74}$$

where

$$c_1 = a_\parallel(\mathbf{k}) + a_0(\mathbf{k}), \quad c_2 = a_\parallel(\mathbf{k}) - a_0(\mathbf{k}), \tag{4.75}$$

are new operators which satisfy the usual commutation rule

$$[c_i(\mathbf{k}), c_j^\dagger(\mathbf{k}')] = \delta_{ij}\delta(\mathbf{k} - \mathbf{k}'). \tag{4.76}$$

An important result for Q, stemming from (4.71), is the following identity

$$\{Q^\dagger, Q\} = 2\int d^3k\omega^2(\mathbf{k})[c_1^\dagger c_1 + c_2^\dagger c_2 + d_1^\dagger d_1 + d_2^\dagger d_2], \tag{4.77}$$

where we can identify the number operators of non-physical particles. Consequently, we could use the anticommutation $\{Q^\dagger, Q\}$ to define the physical Fock space $\mathcal{F}_{phys}$. Every physical Fock state $|\Phi\rangle \in \mathcal{F}_{phys}$ must fulfill the following condition

$$\{Q^\dagger, Q\}|\Phi\rangle = 0. \tag{4.78}$$

Now, returning to the quantum gauge invariance principle, we can see that the gauge charge Q represents an infinitesimal gauge transformation generator. This allows us to define the *gauge derivative* d_Q for a product F of Bose fields and even number of ghost fields and for a product G of Bose fields and odd number of ghost fields as follow

$$d_Q F \equiv [Q, F], \quad d_Q G \equiv \{Q, G\}. \tag{4.79}$$

In order to obtain a gauge invariant theory, we demand that all n-point distributions T_n must fulfill the following property

$$d_Q T_n(x_1, \ldots, x_n) = i \sum_{l=1}^{n} \frac{\partial}{\partial x_l^\mu} T^\mu_{n/l}(x_1, \ldots, x_n), \tag{4.80}$$

where $T^\mu_{n/l}(x_1, \ldots, x_n)$ is the following time ordering product constructed by causal perturbation theory

$$T^\mu_{n/l}(x_1, \ldots, x_n) = T\{T_1(x_1) \ldots T^\mu_{1/1}(x_l) \ldots T_1(x_n)\}, \tag{4.81}$$

and $T^\mu_{1/1}$ is called the Q-vertex. The property (4.80) is called ***perturbative Gauge invariance***.

Chapter 5

Scattering processes of Scalar Quantum Electrodynamics at the tree-level

> Hence most physicists are very satisfied with the situation. They say: "Quantum electrodynamics is a good theory, and we do not have to worry about it any more." I must say that I am very dissatisfied with the situation, because this so-called "good theory" does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small—not neglecting it just because it is infinitely great and you do not want it!
>
> *P. A. M. Dirac*

Here we begin to determine the equivalence between the two approaches to study scalar QED. The first one via Klein-Gordon-Fock fields (SQED) and the second one via Duffin-Kemmer-Petiau fields (SDKP). For this goal, we will follow the same spirit that Fainberg and Pimentel used in [13] comparing the elements of the S-matrix.

As we mentioned in Chapter 1, we will use CPT to consider all sectors of SQED and SDKP. As demonstrated by Scharf and collaborators in [26], it is not necessary to add by hand the sectors generated by the vertices proportional to $\phi^*(x)\phi(x)A^\mu(x)A_\mu(x)$ and $(\phi^*(x)\phi(x))^2$. In CPT, these terms appears naturally from *perturbative gauge invariance* and the *causal splitting process*. This is the power of CPT approach.

In this chapter we compute the following process at the tree level only:

1. Scattering of scalar particle by external electromagnetic field.
2. Moller scattering.
3. Compton scattering.

Bhabha scattering can be studied by crossing properties from Moller process.

5.1 Definition of term T_1 for SDKP via Perturbation Gauge Invariance at first order

For a massless gauge field $A^\mu(x)$ we have Q in the form (4.72) and the following gauge transformations

$$d_Q A^\mu(x) = i\partial^\mu u(x), \tag{5.1}$$

$$d_Q u(x) = 0, \tag{5.2}$$

$$d_Q \tilde{u}(x) = -i\partial_\mu A^\mu(x). \tag{5.3}$$

First of all, in order to determine $T_1(x)$ we can use (4.80) for $n = 1$

$$d_Q T_1(x_1) = i\partial_\mu T^\mu_{1/1}(x_1). \tag{5.4}$$

Secondly, due to the adiabatic limit $g(x) \to 1$, the term $T_1(x)$ must contain all kinds of interactions between gauge and matter fields. As shown by Scharf and collaborators [38–41], for massless gauge fields $A^\mu(x)$ only in the case with a collection $A^\mu_a(x)$, where $a = 1, \dots, N$, there are self interactions between gauge and ghost fields.

For SDKP we have only one massless gauge field, therefore T_1 contain just the interaction between electromagnetic and matter current j^μ in the form

$$T_1^{(\mathrm{SDKP})}(x_1) = ij^\mu(x_1)A_\mu(x_1). \tag{5.5}$$

As usual, $j^\mu(x)$ contains DKP fields $\psi(x)$ and $\bar{\psi}(x)$. Now, with the help of (5.1), (5.2) and ((5.3)), the gauge derivative of (5.5) will take the following form

$$d_Q T_1^{(\mathrm{SDKP})}(x_1) = d_Q\{ij^\mu(x_1)A_\mu(x_1)\} = ij^\mu(x_1)d_Q A_\mu(x_1) = -j^\mu(x_1)\partial_\mu u(x_1). \tag{5.6}$$

From (5.5) and (5.6), we can conclude that in order to obtain (5.4), the matter current $j^{\mu}(x)$ must have null divergence. The latter is fulfilled if j^{μ} is the DKP Noether current (4.30) as follows

$$\partial_{\mu} j^{\mu}(x) = 0 \Longrightarrow d_Q T_1^{(\mathrm{SDKP})}(x_1) = i\partial_{\mu} T_{1/1}^{\mu(\mathrm{SDKP})} = i\partial_{\mu}[ij^{\mu}(x_1)u(x_1)]. \tag{5.7}$$

Finally, replacing (4.30) in the expressions for the Q-vertex $T_{1/1}^{\mu(\mathrm{SDKP})}$, we obtain the following form for T_1

$$T_{1/1}^{\mu(\mathrm{SDKP})} = ij^{\mu}(x_1)u(x_1) = ie : \overline{\psi}(x_1)\beta^{\mu}\psi(x_1) : u(x_1) \tag{5.8}$$

$$T_1^{(\mathrm{SDKP})}(x) = ie : \overline{\psi}(x_1)\beta^{\mu}\psi(x_1) : A_{\mu}(x_1). \tag{5.9}$$

5.2 Scattering of DKP scalar particle by static external field

As a first application, we are going to determine the differential cross section $d\sigma/d\Omega$ in the scattering of scalar by an external electromagnetic field A_{μ}^{ext}.

If the system includes an external field A_{μ}^{ext}, we need to make the substitution $A_{\mu} \rightarrow A_{\mu} + A_{\mu}^{ext}$, then the perturbative expansion of S Matrix includes a term

$$S = \ldots + \int d^4x ie : \overline{\psi}(x)\beta^{\mu}\psi(x) : A_{\mu}^{ext}(x) + \ldots, \tag{5.10}$$

this term is important in the case of a scattered scalar particle by this external field. Because the initial and final states do not include creation (or annihilation) of photons, these states are

$$|in\rangle = |\Psi_i\rangle = \int d^3p_1 \Phi_i(\mathbf{p}_1) a^{\dagger}(\mathbf{p}_1)|0\rangle, \tag{5.11}$$

$$|out\rangle = |\Psi_f\rangle = \int d^3p_2 \Phi_f(\mathbf{p}_2) a^{\dagger}(\mathbf{p}_2)|0\rangle, \tag{5.12}$$

where $\mathrm{a}^{\dagger}(\mathbf{p}_{1,2})$ are the creation operators for scalar particles with momenta $\mathbf{p}_{1,2}$ and $\Phi_{i,f}$ are wave packets sharply picked at $\mathbf{p}_i$ and $\mathbf{p}_f$ which are the initial and final momentum of the bunch of particles before and after the scattering, respectively.

Now, computing the scattering amplitude $S_{if} = \langle \Psi_f | S | \Psi_i \rangle$, it is not difficult to see that the unique non-null result is

$$\begin{aligned} S_{if} &= ie\int d^4x \langle\Psi_f| : \overline{\psi}(x)\beta^\mu\psi(x) : |\Psi_i\rangle A_\mu^{ext}(x), \\ &= ie\int d^4x \int d^3p_2 \int d^3p_1 \Phi_f^*(\mathbf{p}_2)\Phi_i(\mathbf{p}_1)\times \\ &\quad \times\langle 0|a(\mathbf{p}_2) : \overline{\psi}(x)\beta^\mu\psi(x) : a^\dagger(\mathbf{p}_1)|0\rangle A_\mu^{ext}(x). \end{aligned} \tag{5.13}$$

In order to compute the term $\langle 0|a(\mathbf{p}_2) : \overline{\psi}(x)\beta^\mu\psi(x) : a^\dagger(\mathbf{p}_1)|0\rangle$, we can use the Wick theorem (see Appendix A.2). Therefore, in the Wick expansion only the term with the two simultaneous contractions $\overbrace{a(\mathbf{p}_2)\overline{\psi}}(x)$ and $\overbrace{\psi(x)a}^\dagger(p_1)$ is not null. These contractions are

$$\begin{aligned} \overbrace{a(\mathbf{p}_2)\overline{\psi}}(x) &= \overbrace{a(\mathbf{p}_2)\overline{\psi}}^{(+)}(x) = \overbrace{a(\mathbf{q}_2)\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}a}^\dagger(\mathbf{p})\overline{u^-}(\mathbf{p})e^{ipx} \\ &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\delta(\mathbf{p}_2-\mathbf{p})\overline{u^-}(\mathbf{p})e^{ipx} = \frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}(\mathbf{p}_2)e^{ip_2x}, \end{aligned} \tag{5.14}$$

$$\begin{aligned} \overbrace{\psi(x)a}^\dagger(\mathbf{p}_1) &= \overbrace{\psi^{(-)}(x)a}^\dagger(\mathbf{p}_1) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\overbrace{a(\mathbf{p})u^-(\mathbf{p})e^{-ipx}a}^\dagger(\mathbf{p}_1) \\ &= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}}\delta(\mathbf{p}_1-\mathbf{p})u^-(\mathbf{p})e^{-ipx} = \frac{1}{(2\pi)^{\frac{3}{2}}}u^-(\mathbf{p}_1)e^{-ip_1x}. \end{aligned} \tag{5.15}$$

With the help of (5.14) and (5.15), we obtain

$$\begin{aligned} \langle 0|a(\mathbf{p}_2) : \overline{\psi}(x)\beta^\mu\psi(x) : a^\dagger(\mathbf{p}_1)|0\rangle &= \langle 0|\overbrace{a(\mathbf{p}_2)\overline{\psi}}(x)\beta^\mu\overbrace{\psi(x)a}^\dagger(\mathbf{p}_1)|0\rangle \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}(\mathbf{p}_2)e^{ip_2x}\beta^\mu\frac{1}{(2\pi)^{\frac{3}{2}}}u^-(\mathbf{p}_1)e^{-ip_1x} \\ &= \frac{1}{(2\pi)^3}\overline{u^-}(\mathbf{p}_2)\beta^\mu u^-(\mathbf{p}_1)e^{-i(p_1-p_2)x}. \end{aligned} \tag{5.16}$$

Replacing (5.16) into (5.13), the scattering amplitude takes the following form

$$\begin{aligned} S_{if} &= ie\int d^4x\int d^3p_2\int d^3p_1\Phi_f^*(\mathbf{p}_2)\Phi_i(\mathbf{p}_1)\frac{1}{(2\pi)^3}\overline{u^-}(\mathbf{p}_2)\beta^\mu u^-(\mathbf{p}_1)e^{-i(p_1-p_2)x}A_\mu^{ext}(x) \\ &= ie\int d^3p_2\int d^3p_1\Phi_f^*(\mathbf{p}_2)\Phi_i(\mathbf{p}_1)\frac{1}{(2\pi)^3}\overline{u^-}(\mathbf{p}_2)\beta^\mu u^-(\mathbf{p}_1)\int d^4x e^{-i(p_1-p_2)x}A_\mu^{ext}(x). \end{aligned} \tag{5.17}$$

Considering a static electromagnetic field, we can replace $A_\mu^{ext}(x) = A_\mu^{ext}(\mathbf{x})$ and evaluate the integral in x^0 to obtain

$$\begin{aligned} S_{if} &= \frac{ie}{(2\pi)^2} \int d^3p_2 \int d^3p_1 \Phi_f^*(\mathbf{p}_2)\Phi_i(\mathbf{p}_1)\overline{u^-}(\mathbf{p}_f)\beta^\mu u^-(\mathbf{p}_1)\delta(E_1 - E_2) \int d^3x e^{-i(\mathbf{p}_2-\mathbf{p}_1)\mathbf{x}} A_\mu^{ext}(\mathbf{x}) \\ &= \frac{ie}{(2\pi)^2} \int d^3p_2 \int d^3p_1 \Phi_f^*(\mathbf{p}_2)\Phi_i(\mathbf{p}_1)\overline{u^-}(\mathbf{p}_2)\beta^\mu u^-(\mathbf{p}_1)\delta(E_1 - E_2)(2\pi)^{\frac{3}{2}} \hat{A}_\mu(\mathbf{p}_2 - \mathbf{p}_1) \\ &= \int d^3p_2 \int d^3p_1 \Phi_f^*(\mathbf{p}_2)\Phi_i(\mathbf{p}_1) M_{if}(\mathbf{p}_1, \mathbf{p}_2)\delta(E_1 - E_2), \end{aligned} \tag{5.18}$$

where

$$\hat{A}(\mathbf{p}_2 - \mathbf{p}_1) = (2\pi)^{-\frac{3}{2}} \int d^3x e^{-i(\mathbf{p}_2-\mathbf{p}_1)\mathbf{x}} A_\mu^{ext}(\mathbf{x}), \tag{5.19}$$

$$M_{if}(\mathbf{p}_1, \mathbf{p}_2) = \frac{ie}{(2\pi)^{\frac{1}{2}}} \overline{u^-}(\mathbf{p}_2)\beta^\mu u^-(\mathbf{p}_1)\hat{A}(\mathbf{p}_2 - \mathbf{p}_1). \tag{5.20}$$

With the computation of S_{if}, we will determine the probability transition P_{if} defined as

$$P_{if} = |S_{if}|^2. \tag{5.21}$$

Replacing (5.18) into (5.21), we have

$$P_{if} = \int d^3p_1 d^3p_2 \Phi_f(\mathbf{p}_2)\tilde{S}_{if}^*(\mathbf{p}_1, \mathbf{p}_2)\Phi_i^*(\mathbf{p}_1) \int d^3p_1' d^3p_2' \Phi_f^*(\mathbf{p}_2')\tilde{S}_{if}(\mathbf{p}_1', \mathbf{p}_2')\Phi_i(\mathbf{p}_1'), \tag{5.22}$$

where

$$\tilde{S}_{if} = M_{if}(\mathbf{p}_1, \mathbf{p}_2)\delta(E_1 - E_2). \tag{5.23}$$

Summing over all possible final states

$$\begin{aligned} \sum_f P_{if} &= \int d^3p_1 d^3p_2 \tilde{S}_{if}^*(\mathbf{p}_1, \mathbf{p}_2)\Phi_i^*(\mathbf{p}_1) \int d^3p_1' d^3p_2' \tilde{S}_{if}(\mathbf{p}_1', \mathbf{p}_2')\Phi_i(\mathbf{p}_1') \sum_f \Phi_f(\mathbf{p}_2)\Phi_f^*(\mathbf{p}_2') \\ &= \int d^3p_1 d^3p_2 \tilde{S}_{if}^*(\mathbf{p}_1, \mathbf{p}_2)\Phi_i^*(\mathbf{p}_1) \int d^3p_1' d^3p_2' \tilde{S}_{if}(\mathbf{p}_1', \mathbf{p}_2')\Phi_i(\mathbf{p}_1')\delta(\mathbf{p}_2 - \mathbf{p}_2') \\ &= \int d^3p_1 d^3p_2 \tilde{S}_{if}^*(\mathbf{p}_1, \mathbf{p}_2)\Phi_i^*(\mathbf{p}_1) \int d^3p_1' \tilde{S}_{if}(\mathbf{p}_1', \mathbf{p}_2)\Phi_i(\mathbf{p}_1') \\ &= \int d^3p_1 d^3p_2 M_{if}^*(\mathbf{p}_1, \mathbf{p}_2)\delta(E_1 - E_2)\Phi_i^*(\mathbf{p}_1) \int d^3p_1' M_{if}(\mathbf{p}_1', \mathbf{p}_2)\delta(E_1' - E_2)\Phi_i(\mathbf{p}_1') \end{aligned} \tag{5.24}$$

where in the last line we used (5.23).

Using the fact that the function $\Phi_i(\mathbf{p}_1)$ is sharply peaked around p_i and considering that its width is too small compared with the scale of varying of M_{if}, we can rewrite

(5.24) as follows

$$\sum_f P_{if} = \int d^3p_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2)|^2 \int d^3p_1 d^3p_1' \delta(E_1 - E_2) \Phi_i^*(\mathbf{p}_1) \delta(E_1' - E_2) \Phi_i(\mathbf{p}_1'). \quad (5.25)$$

Computing the $\mathbf{p}_2$ integral in spherical coordinates, we obtain

$$\begin{aligned}
\sum_f P_{if} &= \int d\Omega_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2)|^2 \int |\mathbf{p}_2|^2 d|\mathbf{p}_2| \int d^3p_1 d^3p_1' \delta(E_1 - E_2) \Phi_i^*(\mathbf{p}_1) \delta(E_1' - E_2) \Phi_i(\mathbf{p}_1') \\
&= \int d\Omega_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2) \int |\mathbf{p}_2| E_2 dE_2 \int d^3p_1 d^3p_1' \delta(E_1 - E_2) \Phi_i^*(\mathbf{p}_1) \delta(E_1' - E_2) \Phi_i(\mathbf{p}_1') \\
&= \int d\Omega_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2)|^2 \int d^3p_1 d^3p_1' \Phi_i^*(\mathbf{p}_1) \Phi_i(\mathbf{p}_1') \int |\mathbf{p}_2| E_2 dE_2 \delta(E_1 - E_2) \delta(E_1' - E_2) \\
&= \int d\Omega_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2)|^2 \int d^3p_1 d^3p_1' \Phi_i^*(\mathbf{p}_1) \Phi_i(\mathbf{p}_1') |\mathbf{p}_i| E_i \delta(E_1' - E_1).
\end{aligned} \quad (5.26)$$

Replacing the integral form of the delta function $\delta(E_1' - E_1) = (2\pi)^{-1} \int dt e^{-i(E_1' - E_1)t}$ into (5.26), we can rewrite it as

$$\begin{aligned}
\sum_f P_{if} &= \int d\Omega_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2)|^2 |\mathbf{p}_i| E_i \int d^3p_1 d^3p_1' \Phi_i^*(\mathbf{p}_1) \Phi_i(\mathbf{p}_1') \delta(E_1' - E_1) \\
&= \int d\Omega_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2)|^2 |\mathbf{p}_i| E_i \int d^3p_1 d^3p_1' \Phi_i^*(\mathbf{p}_1) \Phi_i(\mathbf{p}_1') (2\pi)^{-1} \int dt e^{-i(E_1' - E_1)t} \\
&= \int d\Omega_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2)|^2 |\mathbf{p}_i| E_i (2\pi)^2 \int dt (2\pi)^{-\frac{3}{2}} \int d^3p_1 \Phi_i^*(\mathbf{p}_1) e^{iE_1 t} (2\pi)^{-\frac{3}{2}} \int d^3p_1' \Phi_i(\mathbf{p}_1') e^{-iE_1' t} \\
&= \int d\Omega_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2)|^2 |\mathbf{p}_i| E_i (2\pi)^2 \int dt \\
&\quad \left[(2\pi)^{-\frac{3}{2}} \int d^3p_1 \Phi_i^*(\mathbf{p}_1) e^{i(E_1 t - \mathbf{p}_1 \mathbf{x})} \right]_{\mathbf{x}=\mathbf{0}} \left[(2\pi)^{-\frac{3}{2}} \int d^3p_1' \Phi_i(\mathbf{p}_1') e^{-i(E_1' t - \mathbf{p}_1' \mathbf{x})} \right]_{\mathbf{x}=\mathbf{0}} \\
&= \int d\Omega_2 |M_{if}(\mathbf{p}_i, \mathbf{p}_2)|^2 |\mathbf{p}_i| E_i |(2\pi)^2 \int dt |\Phi(t, \mathbf{x} = \mathbf{0})|^2,
\end{aligned} \quad (5.27)$$

where $\Phi(t, \mathbf{x})$ is the following free wave packet in $\mathbf{x}$-space

$$\Phi(t, \mathbf{x}) = (2\pi)^{-\frac{3}{2}} \int d^3q \Phi_i(\mathbf{q}) e^{-i(E_q t - \mathbf{q}\mathbf{x})}. \quad (5.28)$$

Considering that the velocity of the scattered particles is $\mathbf{v}$, the wave packet have the form

$$\Phi(t, \mathbf{x}) = \Phi_0(\mathbf{x} + \mathbf{v}t). \quad (5.29)$$

Now, averaging (5.27) in a cylinder of radius R parallel to $\mathbf{v}$ using the wave packet (5.29), we have

$$\sum_f P_{if}(R) = \frac{1}{\pi R^2} \int\limits_{x_\perp \leqslant R} d^2x_\perp \int dt |\Phi_0(\mathbf{x}+\mathbf{v}t)|^2 \int d\Omega_2 |M_{if}(\mathbf{p}_i,\mathbf{p}_2)|^2 |\mathbf{p}_i| E_i (2\pi)^2. \quad (5.30)$$

The cross section is defined as

$$\sigma = \lim_{R\to\infty} \pi R^2 \sum_f P_{if}(R). \quad (5.31)$$

Then, replacing (5.30) into (5.31), we get

$$\begin{aligned}
\sigma &= \int\limits_{x_\perp \leqslant R} d^2x_\perp \int dt |\Phi_0(\mathbf{x}_\perp+\mathbf{v}t)|^2 \int d\Omega_2 |M_{if}(\mathbf{p}_i,\mathbf{p}_2)|^2 |\mathbf{p}_i| E_i (2\pi)^2 \\
&= \frac{1}{|\mathbf{v}|} \int d^3x |\Phi_0|^2 \int d\Omega_2 |M_{if}(\mathbf{p}_i,\mathbf{p}_2)|^2 |\mathbf{p}_i| E_i (2\pi)^2 \\
&= \frac{1}{|\mathbf{v}|} \int d\Omega_2 |M_{if}(\mathbf{p}_i,\mathbf{p}_2)|^2 |\mathbf{p}_i| E_i (2\pi)^2 \\
&= \frac{|\mathbf{p}_i| E_i (2\pi)^2}{(\frac{|\mathbf{p}_i|}{E_i})} \int d\Omega_2 |M_{if}(\mathbf{p}_i,\mathbf{p}_2)|^2 \\
&= E_i^2 (2\pi)^2 \int d\Omega_2 |M_{if}(\mathbf{p}_i,\mathbf{p}_2)|^2,
\end{aligned} \quad (5.32)$$

which tell us that the differential cross section will take the following form

$$\frac{d\sigma}{d\Omega} = (2\pi)^2 E_i^2 |M(\mathbf{p}_f,\mathbf{p}_i)|^2. \quad (5.33)$$

Replacing (5.20) into (5.33), we get

$$\begin{aligned}
&\frac{d\sigma}{d\Omega} = \\
&= (2\pi)^2 E_i^2 [\frac{ie}{(2\pi)^{\frac{1}{2}}} \overline{u^-}(\mathbf{p}_f)\beta^\mu u^-(\mathbf{p}_i)\hat{A}_\mu(\mathbf{p}_f-\mathbf{p}_i)][-\frac{ie}{(2\pi)^{\frac{1}{2}}} \overline{u^-}(\mathbf{p}_i)\beta^\nu u^-(\mathbf{p}_f)\hat{A}_\nu(\mathbf{p}_i-\mathbf{p}_f)] \\
&= (2\pi) E_i^2 e^2 [u_m^-(\mathbf{p}_f)\overline{u^-}_a(\mathbf{p}_f)]\beta^\mu_{ad}[u_d^-(\mathbf{p}_i)\overline{u^-}_l(\mathbf{p}_i)]\beta^\nu_{lm}\hat{A}_\nu(\mathbf{p}_f-\mathbf{p}_i)\hat{A}_\mu(\mathbf{p}_i-\mathbf{p}_f) \\
&= \frac{(2\pi)E_i^2 e^2}{4m^2 {p_i}^0 {p_f}^0} Tr[\not{p}_f(\not{p}_f+m)\beta^\mu \not{p}_i(\not{p}_i+m)\beta^\nu]\hat{A}_\nu(\mathbf{p}_f-\mathbf{p}_i)\hat{A}_\mu(\mathbf{p}_i-\mathbf{p}_f) \\
&= \frac{(2\pi)E_i^2 e^2}{4 {p_i}^0 {p_f}^0} [p_i^\mu p_i^\nu + p_f^\nu p_f^\mu + p_f^\mu p_i^\nu + p_f^\nu p_i^\mu]\hat{A}_\nu(\mathbf{p}_f-\mathbf{p}_i)\hat{A}_\mu(\mathbf{p}_i-\mathbf{p}_f),
\end{aligned}$$

(5.34)

where (4.37) was used in the second line, and the trace identities (4.39) and (4.40) were used in next ones.

For the potential A^μ, we will use the Coulomb potential. Therefore, only the 0-component is not null and takes the following form

$$A^0(\mathbf{x}) = \frac{Ze}{|\mathbf{x}|}, \quad \hat{A}^0(\mathbf{p}) = \sqrt{\frac{2}{\pi}}\frac{Ze}{|\mathbf{p}|^2}, \tag{5.35}$$

replacing this potential into (5.34), we have

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{(2\pi)E_i^2e^2}{4{p_i}^0{p_f}^0}[p_i^0p_i^0 + p_f^0p_f^0 + p_f^0p_i^0 + p_f^0p_i^0]\hat{A}_0(\mathbf{p}_f - \mathbf{p}_i)\hat{A}_0(\mathbf{p}_i - \mathbf{p}_f) \\
&= \frac{(2\pi)E_i^2e^2}{4E_iE_f}[E_i + E_f]^2[\frac{2}{\pi}\frac{Z^2e^2}{|\mathbf{p}_i - \mathbf{p}_f|^4}] \\
&= \frac{Z^2E_ie^4}{E_f}\frac{[E_i + E_f]^2}{|\mathbf{p}_i - \mathbf{p}_f|^4} \\
&= Z^2e^4\frac{4E^2}{|\mathbf{p}_i - \mathbf{p}_f|^4} \\
&= Z^2e^4\frac{4E^2}{16|\mathbf{p}|^4\sin^4(\vartheta/2)} \\
&= Z^2e^4\frac{E^2}{4|\mathbf{p}|^4\sin^4(\vartheta/2)}.
\end{aligned} \tag{5.36}$$

The latter result is equivalent to that obtained in [10] using the usual approach.

5.3 Causal distribution in the second order $D_2(x, y)$

After setting $T_1(x)$ for SDKP, the next step is to compute the causal distribution $D_2(x, y)$. Following (3.23) and (3.24), the intermediate distributions in second order A_2' and R_2' take the following forms

$$A_2'(x, y) = \tilde{T}_1(x)T_1(y) = -T_1(x)T_1(y), \tag{5.37}$$

$$R_2'(x, y) = T_1(y)\tilde{T}_1(x) = -T_1(y)T_1(x), \tag{5.38}$$

where (3.13) was used.

Replacing (5.9) into (5.37) and (5.38), and using Wick theorem to obtain normal ordered terms, we have

$$
\begin{aligned}
A_2'(x,y) &= -T_1(x)T_1(y) \\
&= e^2 :\overline{\psi}_a(x)\beta^{\mu}_{ab}\psi_b(x)::\overline{\psi}_c(y)\beta^{\nu}_{cd}\psi_d(y): A_\mu(x)A_\nu(y) \\
&= e^2[:\overline{\psi}_a(x)\beta^{\mu}_{ab}\psi_b(x)\overline{\psi}_c(y)\beta^{\nu}_{cd}\psi_d(y): + :\overline{\psi}_a(x)\beta^{\mu}_{ab}\psi_b(x)\overline{\psi}_c(y)\beta^{\nu}_{cd}\psi_d(y): \\
&+ :\overline{\psi}_a(x)\beta^{\mu}_{ab}\psi_b(x)\overline{\psi}_c(y)\beta^{\nu}_{cd}\psi_d(y): + :\overline{\psi}_a(x)\beta^{\mu}_{ab}\psi_b(x)\overline{\psi}_c(y)\beta^{\nu}_{cd}\psi_d(y):]\times \\
&\times [:A_\mu(x)A_\nu(y): + A_\mu(x)A_\nu(y)],
\end{aligned}
\tag{5.39}
$$

where the field contractions are

$$A^\mu(x)A^\nu(y) = [A^{\mu(-)}(x), A^{\nu(+)}(y)] = g^{\mu\nu} iD_0^{(+)}(x-y), \tag{5.40}$$

$$\psi_a(x)\bar{\psi}_b(y) = [\psi^{(-)}(x), \bar{\psi}^{(+)}(y)] = \frac{1}{i}S^{(+)}_{ab}(x-y), \tag{5.41}$$

$$\bar{\psi}_c(x)\psi_d(y) = [\bar{\psi}^{(-)}(x), \psi^{(+)}(y)] = -\frac{1}{i}S^{(-)}_{dc}(y-x). \tag{5.42}$$

Replacing (5.40), (5.41) and (5.42) into (5.39), we obtain

$$
\begin{aligned}
A_2'(x,y) &= +e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} :\overline{\psi}_a(x)\psi_b(x)\overline{\psi}_c(y)\psi_d(y): ig_{\mu\nu}D_0^{(+)}(x-y) \\
&- e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} :\psi_b(x)\overline{\psi}_c(y): \frac{1}{i}S^{(-)}_{da}(y-x) :A_\mu(x)A_\nu(y): \\
&+ e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} :\overline{\psi}_a(x)\psi_d(y): \frac{1}{i}S^{(+)}_{bc}(x-y) :A_\mu(x)A_\nu(y): \\
&- e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} \frac{1}{i}S^{(-)}_{da}(y-x)\frac{1}{i}S^{(+)}_{bc}(x-y) :A_\mu(x)A_\nu(y): \\
&- e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} :\psi_b(x)\overline{\psi}_c(y): \frac{1}{i}S^{(-)}_{da}(y-x) ig_{\mu\nu}D_0^{(+)}(x-y) \\
&+ e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} :\overline{\psi}_a(x)\psi_d(y): \frac{1}{i}S^{(+)}_{bc}(x-y) ig_{\mu\nu}D_0^{(+)}(x-y) \\
&- e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} \frac{1}{i}S^{(-)}_{da}(y-x)\frac{1}{i}S^{(+)}_{bc}(x-y) ig_{\mu\nu}D_0^{(+)}(x-y) \\
&+ e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} :\overline{\psi}_a(x)\psi_b(x)\overline{\psi}_c(y)\psi_d(y)::A_\mu(x)A_\nu(y):,
\end{aligned}
\tag{5.43}
$$

which can be rewritten as

$$
\begin{aligned}
A_2'(x,y) &= A_2'^{(1)}(x,y) + A_2'^{(2)}(x,y) + A_2'^{(3)}(x,y) + A_2'^{(4)}(x,y) + A_2'^{(5)}(x,y) \\
&+ e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} :\overline{\psi}_a(x)\psi_b(x)\overline{\psi}_c(y)\psi_d(y)::A_\mu(x)A_\nu(y):,
\end{aligned}
\tag{5.44}
$$

where

$$A_2'^{(1)}(x,y) = +e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} :\overline{\psi}_a(x)\psi_b(x)\overline{\psi}_c(y)\psi_d(y): ig_{\mu\nu}D_0^{(+)}(x-y), \tag{5.45}$$

$$\begin{aligned} A_2'^{(2)}(x,y) = &-e^2\beta^\mu_{ab}\beta^\nu_{cd} :\psi_b(x)\overline{\psi}_c(y): \frac{1}{i}S^{(-)}_{da}(y-x) : A_\mu(x)A_\nu(y): \\ &+ e^2\beta^\mu_{ab}\beta^\nu_{cd} :\overline{\psi}_a(x)\psi_d(y): \frac{1}{i}S^{(+)}_{bc}(x-y) : A_\mu(x)A_\nu(y):, \end{aligned} \quad (5.46)$$

$$A_2'^{(3)}(x,y) = -e^2\beta^\mu_{ab}\beta^\nu_{cd}\frac{1}{i}S^{(-)}_{da}(y-x)\frac{1}{i}S^{(+)}_{bc}(x-y) : A_\mu(x)A_\nu(y):, \quad (5.47)$$

$$\begin{aligned} A_2'^{(4)}(x,y) = &-e^2\beta^\mu_{ab}\beta^\nu_{cd} :\psi_b(x)\overline{\psi}_c(y): \frac{1}{i}S^{(-)}_{da}(y-x) ig_{\mu\nu}D_0^{(+)}(x-y) \\ &+ e^2\beta^\mu_{ab}\beta^\nu_{cd} :\overline{\psi}_a(x)\psi_d(y): \frac{1}{i}S^{(+)}_{bc}(x-y) ig_{\mu\nu}D_0^{(+)}(x-y), \end{aligned} \quad (5.48)$$

$$A_2'^{(5)}(x,y) = -e^2\beta^\mu_{ab}\beta^\nu_{cd}\frac{1}{i}S^{(-)}_{da}(y-x)\frac{1}{i}S^{(+)}_{bc}(x-y) ig_{\mu\nu}D_0^{(+)}(x-y). \quad (5.49)$$

Similarly for $R_2'(x,y)$, we have

$$\begin{aligned} R_2'(x,y) = &R_2'^{(1)}(y,x) + R_2'^{(2)}(y,x) + R_2'^{(3)}(y,x) + R_2'^{(4)}(y,x) + R_2'^{(5)}(y,x) \\ &+ e^2\beta^\mu_{ab}\beta^\nu_{cd} :\overline{\psi}_a(y)\psi_b(y)\overline{\psi}_c(x)\psi_d(x)::A_\mu(y)A_\nu(x): \end{aligned} \quad (5.50)$$

where

$$R_2'^{(1)}(x,y) = +e^2\beta^\mu_{ab}\beta^\nu_{cd} :\overline{\psi}_a(y)\psi_b(y)\overline{\psi}_c(x)\psi_d(x): ig_{\mu\nu}D_0^{(+)}(y-x), \quad (5.51)$$

$$\begin{aligned} R_2'^{(2)}(x,y) = &-e^2\beta^\mu_{ab}\beta^\nu_{cd} :\psi_b(y)\overline{\psi}_c(x): \frac{1}{i}S^{(-)}_{da}(x-y) : A_\mu(y)A_\nu(x): \\ &+ e^2\beta^\mu_{ab}\beta^\nu_{cd} :\overline{\psi}_a(y)\psi_d(x): \frac{1}{i}S^{(+)}_{bc}(y-x) : A_\mu(y)A_\nu(x):, \end{aligned} \quad (5.52)$$

$$R_2'^{(3)}(y,x) = -e^2\beta^\mu_{ab}\beta^\nu_{cd}\frac{1}{i}S^{(-)}_{da}(x-y)\frac{1}{i}S^{(+)}_{bc}(y-x) : A_\mu(y)A_\nu(x):, \quad (5.53)$$

$$\begin{aligned} R_2'^{(4)}(y,x) = &-e^2\beta^\mu_{ab}\beta^\nu_{cd} :\psi_b(y)\overline{\psi}_c(x): \frac{1}{i}S^{(-)}_{da}(x-y) ig_{\mu\nu}D_0^{(+)}(y-x) \\ &+ e^2\beta^\mu_{ab}\beta^\nu_{cd} :\overline{\psi}_a(y)\psi_d(x): \frac{1}{i}S^{(+)}_{bc}(y-x) ig_{\mu\nu}D_0^{(+)}(y-x), \end{aligned} \quad (5.54)$$

$$R_2'^{(5)}(y,x) = -e^2\beta^\mu_{ab}\beta^\nu_{cd}\frac{1}{i}S^{(-)}_{da}(x-y)\frac{1}{i}S^{(+)}_{bc}(y-x) ig_{\mu\nu}D_0^{(+)}(y-x). \quad (5.55)$$

The causal distribution D_2 is obtained by the subtraction (3.43) as follows

$$D_2(x,y) = R_2'(x,y) - A_2'(x,y) = D_2^{(1)} + D_2^{(2)} + D_2^{(3)} + D_2^{(4)} + D_2^{(5)} \quad (5.56)$$

where

$$D_2^{(1)} = ie^2 g_{\mu\nu} :\overline{\psi}_a(y)\beta^\mu_{ab}\psi_b(y)\overline{\psi}_c(x)\beta^\nu_{cd}\psi_d(x): \Big(D_0^{(+)}(y-x) - D_0^{(+)}(x-y)\Big), \quad (5.57)$$

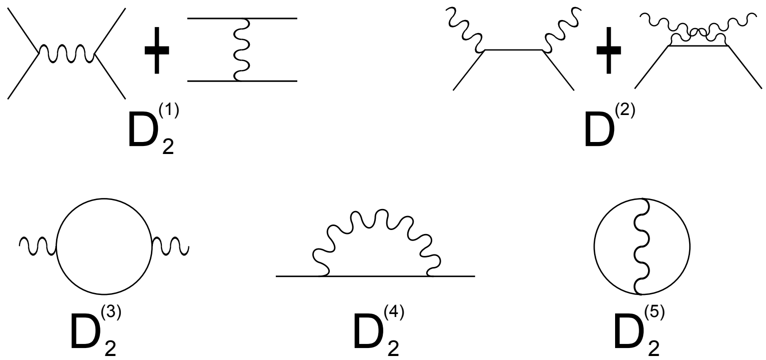


Figure 5.1: General graph for processes in the causal distribution $D_2(x,y)$.

$$\begin{aligned} D_2^{(2)} =& ie^2 : \overline{\psi}_c(x)\beta^{\nu}_{cd}\left(S^{(+)}_{da}(x-y)+S^{(-)}_{da}(x-y)\right)\beta^{\mu}_{ab}\psi_b(y) :: A_\mu(y)A_\nu(x) : \\ &- ie^2 : \overline{\psi}_a(y)\beta^{\mu}_{ab}\left(S^{(+)}_{bc}(y-x)+S^{(-)}_{bc}(y-x)\right)\beta^{\nu}_{cd}\psi_d(x) :: A_\mu(y)A_\nu(x) :, \end{aligned} \tag{5.58}$$

$$\begin{aligned} D_2^{(3)} =& e^2 Tr[\beta^\mu S^{(+)}(y-x)\beta^\nu S^{(-)}(x-y) - \beta^\nu S^{(+)}(x-y)\beta^\mu S^{(-)}(y-x)]\times \\ &\times : A_\mu(y)A_\nu(x) :, \end{aligned} \tag{5.59}$$

$$\begin{aligned} D_2^{(4)} =& -e^2 g_{\mu\nu} : \overline{\psi}(x)\beta^\nu[S^{(-)}(x-y)D_0^{(+)}(y-x) + S^{(+)}(x-y)D_0^{(+)}(x-y)]\beta^\mu\psi(y) : \\ &+ e^2 g_{\mu\nu} : \overline{\psi}(y)\beta^\mu[S^{(+)}(y-x)D_0^{(+)}(y-x) + S^{(-)}(y-x)D_0^{(+)}(x-y)]\beta^\nu\psi(x) :, \end{aligned} \tag{5.60}$$

$$\begin{aligned} D_2^{(5)} =& +e^2\beta^\mu S^{(+)}(y-x)\beta^\nu S^{(-)}(x-y) i g_{\mu\nu} D_0^{(+)}(y-x) \\ &- e^2\beta^\mu S^{(+)}(x-y)\beta^\nu S^{(-)}(y-x) i g_{\mu\nu} D_0^{(+)}(x-y). \end{aligned} \tag{5.61}$$

Each term $D_2^{(i)}$ represents different processes in the S-matrix and their diagrams are represented in Fig. 5.1.

5.4 Moller scattering

Now, we will determine the differential cross section of Moller process which consist in the elastic scattering of two scalar particles $b(p_i)+b(q_i)\rightarrow b(p_f)+b(q_f)$ where $p_{i,f}$ and $q_{i,f}$ are the initial and final momentum of particles after and before the interaction as usual. Therefore, the in and out states take on the following form

$$|in_{\text{Moller}}\rangle = |\Psi_i\rangle\otimes|\Phi_i\rangle = \int d^3p_1 d^3q_1 \Psi_i(\mathbf{p}_1)\Phi_i(\mathbf{q}_1)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1)|0\rangle, \tag{5.62}$$

$$|out_{\text{Moller}}\rangle = |\Psi_f\rangle\otimes|\Phi_f\rangle = \int d^3p_2 d^3q_2 \Psi_f(\mathbf{p}_2)\Phi_f(\mathbf{q}_2)a^\dagger(\mathbf{p}_2)a^\dagger(\mathbf{q}_2)|0\rangle, \tag{5.63}$$

where $\{\Psi_i(p_1), \Psi_f(p_2), \Phi_i(q_1), \Phi_f(q_2)\}$ are the wave packet functions sharply peaked in $\{p_i, p_f, q_i, q_f\}$, respectively.

Taking into account that *the causal splitting* procedure does not transform the quantized fields, in the computation of scattering amplitude $\langle out_{\text{Moeller}}|S|in_{\text{Moeller}}\rangle$ just the term coming from $D_2^{(1)}$ will be no-null. Consequently, we will determine the contribution to $T_2(x,y)$ coming from $D_2^{(1)}$.

Using the property $D_0^{(+)}(x) = -D_0^{(-)}(-x)$, we can rewrite $D_2^{(1)}(x,y)$ in the following form

$$\begin{aligned}
D_2^{(1)}(x,y) &= e^2 i g_{\mu\nu} : \overline{\psi}(y)\beta^\mu\psi(y)\overline{\psi}(x)\beta^\nu\psi(x) : (D_0^{(+)}(y-x) - D_0^{(+)}(x-y)) \\
&= e^2 i g_{\mu\nu} : \overline{\psi}(y)\beta^\mu\psi(y)\overline{\psi}(x)\beta^\nu\psi(x) : (-D_0^{(-)}(x-y) - D_0^{(+)}(x-y)) \\
&= -e^2 i g_{\mu\nu} : \overline{\psi}(y)\beta^\mu\psi(y)\overline{\psi}(x)\beta^\nu\psi(x) : D_0(x-y),
\end{aligned} \tag{5.64}$$

where we can see that the *numerical part* of $D_2^{(1)}$ is $D_0(x-y)$.

5.4.1 Causal splitting of D_0

We will begin the *causal splitting* in momentum space. From (4.52), the Fourier transformation of $D_0(x-y)$ has the following form

$$\widehat{D}_0(p) = \frac{i}{2\pi}\delta(p^2)Sgn(p^0) \tag{5.65}$$

The power counting function $\rho(\alpha)$ for $\widehat{D}_0(p)$ is determined using (3.91). With this goal, we first have to note that the form of $\widehat{D}_0(\frac{p}{\alpha})$ is

$$\begin{aligned}
\widehat{D}_0(\frac{p}{\alpha}) &= \frac{i}{2\pi}\delta(p^2\alpha^{-2})Sgn(p^0\alpha^{-1}) \\
&= \frac{i\alpha^2}{2\pi}\delta(p^2)Sgn(p^0\alpha^{-1}).
\end{aligned} \tag{5.66}$$

After that, it is not difficult to conclude that for $\rho(\alpha) = \alpha^{-2}$ we obtain the following no null limit

$$\lim_{\alpha\to 0}\rho(\alpha)\left\langle \widehat{D}_0(\frac{p}{\alpha}), \check{f}(p)\right\rangle = \left\langle \widehat{D}_0(p), \check{f}(p)\right\rangle \neq 0. \tag{5.67}$$

From (3.92), the *singular order* of $\hat{D}_0$ is

$$\omega[\hat{D}_0] = -2, \tag{5.68}$$

which means that $\widehat{D}_0(p)$ is a *regular* distribution and its splitting in retarded and advanced part is

$$D_0(x-y) = \theta(x^0-y^0)D_0(x-y) - \theta(y^0-x^0)D_0(x-y) \\ = D_0^{\mathrm{ret}}(x-y) - D_0^{\mathrm{adv}}(x-y), \tag{5.69}$$

where $D_0^{\mathrm{ret}}(x-y) = \theta(x^0-y^0)D_0(x-y)$ and $D_0^{\mathrm{adv}}(x-y) = \theta(y^0-x^0)D_0(x-y)$.

From the splitting (5.69) and (5.64), the second order retarded distribution $R_2^{(1)}(x,y)$ is

$$R_2^{(1)}(x,y) = -e^2 i g_{\mu\nu}\beta^{\mu}_{ab}\beta^{\nu}_{cd} : \overline{\psi}_a(y)\psi_b(y)\overline{\psi}_c(x)\psi_d(x) : D_0^{\mathrm{ret}}. \tag{5.70}$$

Finally, using (3.29), (5.70) and (5.45) we are able to determine the contribution $T_2^{(1)}(x,y)$ for S-matrix coming from $D_2^{(1)}$ in the following form

$$\begin{aligned} T_2^{(1)}(x,y) &= R_2^{(1)} - R_2^{\prime(1)} \\ &= -e^2 i g_{\mu\nu}\beta^{\mu}_{ab}\beta^{\nu}_{cd} : \overline{\psi}_a(y)\psi_b(y)\overline{\psi}_c(x)\psi_d(x) : D_0^{\mathrm{ret}}(y-x) \\ &- [e^2\beta^{\mu}_{ab}\beta^{\nu}_{cd} : \overline{\psi}_a(y)\psi_b(y)\overline{\psi}_c(x)\psi_d(x) : i g_{\mu\nu} D_0^{(+)}(y-x)] \\ &= -e^2 i g_{\mu\nu}\beta^{\mu}_{ab}\beta^{\nu}_{cd} : \overline{\psi}_a(y)\psi_b(y)\overline{\psi}_c(x)\psi_d(x) : (D_0^{\mathrm{ret}}(x-y) + D_0^{(+)}(y-x)) \\ &= -e^2 i g_{\mu\nu}\beta^{\mu}_{ab}\beta^{\nu}_{cd} : \overline{\psi}_a(y)\psi_b(y)\overline{\psi}_c(x)\psi_d(x) : (D_0^{\mathrm{ret}}(x-y) - D_0^{(-)}(x-y)) \\ &= -e^2 i g_{\mu\nu}\beta^{\mu}_{ab}\beta^{\nu}_{cd} : \overline{\psi}_a(y)\psi_b(y)\overline{\psi}_c(x)\psi_d(x) : D_0^{F}(x-y), \end{aligned} \tag{5.71}$$

where $D^F(x) \equiv D_0^{\mathrm{ret}}(x) - D_0^{(-)}(x)$ is the well-known Feynman propagator for massless scalar field.

5.4.2 Computation of differential cross section

The S-matrix term $S^{(1)}(g)$, which contributes in the computation of the differential cross section for Moller scattering, takes the following form

$$S^{(1)}(g) = \frac{1}{2!}\int d^4y d^4x T_2^{(1)}(x,y)g(x)g(y). \tag{5.72}$$

Recalling the equations (5.62) and (5.63), and taking the adiabatic limit for the computations, we have the scattering amplitude as

$$\begin{aligned} S_{fi}^{(Mo)} &= \langle out_{Mo}|S|in_{Mo}\rangle \\ &= \int d^3p_2 d^3q_2 \int d^3p_1 d^3q_1 \Psi_f^*(\mathbf{p}_2)\Phi_f^*(\mathbf{q}_2)\tilde{S}_{if}^{(Mo)}\Psi_i(\mathbf{p}_1)\Phi_i(\mathbf{q}_1) \end{aligned} \tag{5.73}$$

where

$$\begin{aligned}
\tilde{S}_{if}^{(Mo)} &= \langle 0|a(\mathbf{p}_2)a(\mathbf{q}_2)S^{(1)}a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1)|0\rangle \\
&= \langle 0|a(\mathbf{p}_2)a(\mathbf{q}_2)\frac{1}{2!}\int d^4yd^4xT_2^M(x,y)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1)|0\rangle \\
&= -\frac{1}{2!}\int d^4yd^4xe^2ig_{\mu\nu}\beta^\mu_{ab}\beta^\nu_{cd}D_0^F(x-y)\times \\
&\times \langle 0|a(\mathbf{p}_2)a(\mathbf{q}_2) : \overline{\psi}_a(y)\psi_b(y)\overline{\psi}_c(x)\psi_d(x) : a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1)|0\rangle \\
&= -\frac{1}{2!}\int d^4yd^4xe^2ig_{\mu\nu}\beta^\mu_{ab}\beta^\nu_{cd}D_0^F(x-y)\times \\
&\times \langle 0|a(\mathbf{p}_2)a(\mathbf{q}_2) : \overline{\psi}_a^{(+)}(y)\psi_b^{(-)}(y)\overline{\psi}_c^{(+)}(x)\psi_d^{(-)}(x) : a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1)|0\rangle \\
&= -\frac{1}{2!}\int d^4yd^4xe^2ig_{\mu\nu}\beta^\mu_{ab}\beta^\nu_{cd}D_0^F(x-y)\times \\
&\times \langle 0|a(\mathbf{p}_2)a(\mathbf{q}_2)\overline{\psi}_a^{(+)}(y)\overline{\psi}_c^{(+)}(x)\psi_b^{(-)}(y)\psi_d^{(-)}(x)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1)|0\rangle.
\end{aligned} \tag{5.74}$$

Using Wick theorem and the contractions (5.14) and (5.15), we could reduce the expression (5.74) as follows

$$\begin{aligned}
\tilde{S}_{fi}^{(Mo)} &= -\frac{1}{2!}\int d^4yd^4xe^2ig_{\mu\nu}\beta^\mu_{ab}\beta^\nu_{cd}D_0^F(x-y)\times \\
&\times \langle 0|\Big[+ a(\mathbf{p}_2)a(\mathbf{q}_2)\overline{\psi}_a^{(+)}(y)\overline{\psi}_c^{(+)}(x)\psi_b^{(-)}(y)\psi_d^{(-)}(x)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1) + \\
&+ a(\mathbf{p}_2)a(\mathbf{q}_2)\overline{\psi}_a^{(+)}(y)\overline{\psi}_c^{(+)}(x)\psi_b^{(-)}(y)\psi_d^{(-)}(x)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1) + \\
&+ a(\mathbf{p}_2)a(\mathbf{q}_2)\overline{\psi}_a^{(+)}(y)\overline{\psi}_c^{(+)}(x)\psi_b^{(-)}(y)\psi_d^{(-)}(x)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1) \\
&+ a(\mathbf{p}_2)a(\mathbf{q}_2)\overline{\psi}_a^{(+)}(y)\overline{\psi}_c^{(+)}(x)\psi_b^{(-)}(y)\psi_d^{(-)}(x)a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{q}_1)\Big] \times |0\rangle \\
&= -\frac{1}{2!}\int d^4yd^4xe^2ig_{\mu\nu}\beta^\mu_{ab}\beta^\nu_{cd}D_0^F(x-y)\Big[\\
&\frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}_a(\mathbf{p}_2)e^{ip_2y}\frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}_c(\mathbf{q}_2)e^{iq_2x}\frac{1}{(2\pi)^{\frac{3}{2}}}u_b^-(\mathbf{p}_1)e^{-ip_1y}\frac{1}{(2\pi)^{\frac{3}{2}}}u_d^-(\mathbf{q}_1)e^{-iq_1x} \\
&+ \frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}_a(\mathbf{p}_2)e^{ip_2y}\frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}_c(\mathbf{q}_2)e^{iq_2x}\frac{1}{(2\pi)^{\frac{3}{2}}}u_d^-(\mathbf{p}_1)e^{-ip_1x}\frac{1}{(2\pi)^{\frac{3}{2}}}u_b^-(\mathbf{q}_1)e^{-iq_1y} \\
&+ \frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}_c(\mathbf{p}_2)e^{ip_2x}\frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}_a(\mathbf{q}_2)e^{iq_2y}\frac{1}{(2\pi)^{\frac{3}{2}}}u_b^-(\mathbf{p}_1)e^{-ip_1y}\frac{1}{(2\pi)^{\frac{3}{2}}}u_d^-(\mathbf{q}_1)e^{-iq_1x} \\
&+ \frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}_c(\mathbf{p}_2)e^{ip_2x}\frac{1}{(2\pi)^{\frac{3}{2}}}\overline{u^-}_a(\mathbf{q}_2)e^{iq_2y}\frac{1}{(2\pi)^{\frac{3}{2}}}u_d^-(\mathbf{p}_1)e^{-ip_1x}\frac{1}{(2\pi)^{\frac{3}{2}}}u_b^-(\mathbf{q}_1)e^{-iq_1y}\Big].
\end{aligned} \tag{5.75}$$

The four integrals in (5.75) can be evaluated by the following formula

$$\begin{aligned}
&\int d^4x d^4y D_0^F(x-y)e^{iAx+iBy} \\
&\quad = \int d^4r e^{i(A+B)r}\int d^4u D_0^F(u)e^{i(\frac{A-B}{2})u} \\
&\quad = (2\pi)^4\delta(A+B)\int d^4u[-(2\pi)^{-4}\int d^4k\frac{e^{-iku}}{k^2+i0}]e^{i(\frac{A-B}{2})u} \\
&\quad = (2\pi)^4\delta(A+B)[-(2\pi)^{-4}\int d^4k\frac{1}{k^2+i0}\int d^4u e^{i(-k+\frac{A-B}{2})u}] \\
&\quad = (2\pi)^4\delta(A+B)[-\int d^4k\frac{1}{k^2+i0}\delta(-k+\frac{A-B}{2})] \\
&\quad = (2\pi)^4\delta(A+B)[-\frac{1}{A^2+i0}],
\end{aligned} \tag{5.76}$$

with the substitutions

$$x = r+\frac{u}{2},\quad y = r-\frac{u}{2},\quad r=\frac{x+y}{2},\quad u = x-y. \tag{5.77}$$

Therefore, (5.75) can be rewritten as

$$\tilde{S}_{fi}^{(Mo)} = \delta(q_2-q_1+p_2-p_1)\mathcal{M}, \tag{5.78}$$

where

$$\begin{aligned}
\mathcal{M} = \frac{e^2 i g_{\mu\nu}}{(2\pi)^2}\Big[&\overline{u^-}_a(\mathbf{p}_2)\beta^\mu_{ab}u^-_b(\mathbf{p}_1)\overline{u^-}_c(\mathbf{q}_2)\beta^\nu_{cd}u^-_d(\mathbf{q}_1)\frac{1}{(q_2-q_1)^2+i0}+ \\
&+\overline{u^-}_a(\mathbf{p}_2)\beta^\mu_{ab}u^-_b(\mathbf{q}_1)\overline{u^-}_c(\mathbf{q}_2)\beta^\nu_{cd}u^-_d(\mathbf{p}_1)\frac{1}{(q_2-p_1)^2+i0}\Big].
\end{aligned} \tag{5.79}$$

In Appendix B, we perform the computation of the differential cross section for a general value of $\mathcal{M}$. Here we will use its form in the center-of-mass reference

$$\frac{d\sigma_{c.m}}{d\Omega} = (2\pi)^2\frac{E^2}{4}|\mathcal{M}|^2, \tag{5.80}$$

where $\sigma_{c.m}$ is the differential cross section in the center of mass reference.

The factor $|\mathcal{M}|^2$ for the Moller scattering is computed using (5.79) as follows

$$
\begin{aligned}
&|\mathcal{M}|^2 = \\
&= \frac{e^4}{(2\pi)^4}\Big[\overline{u^-}(\mathbf{p}_f)\beta_\mu u^-(\mathbf{p}_i)\overline{u^-}(\mathbf{q}_f)\beta^\mu u^-(\mathbf{q}_i)\frac{1}{(q_f-q_i)^2}+ \\
&\quad + \overline{u^-}(\mathbf{p}_f)\beta_\omega u^-(\mathbf{q}_i)\overline{u^-}(\mathbf{q}_f)\beta^\omega u^-(\mathbf{p}_i)\frac{1}{(q_f-p_i)^2}\Big]\times \\
&\quad \times \Big[\overline{u^-}(\mathbf{p}_i)\beta_\nu u^-(\mathbf{p}_f)\overline{u^-}(\mathbf{q}_i)\beta^\nu u^-(\mathbf{q}_f)\frac{1}{(q_f-q_i)^2}+ \\
&\quad + \overline{u^-}(\mathbf{q}_i)\beta_\alpha u^-(\mathbf{p}_f)\overline{u^-}(\mathbf{p}_i)\beta^\alpha u^-(\mathbf{q}_f)\frac{1}{(q_f-p_i)^2}\Big] \\
&= \frac{e^4}{(2\pi)^4}\Big[\frac{1}{16m^4p_f^0p_i^0q_f^0q_i^0}\frac{g_{\mu\alpha}g_{\nu\omega}}{(q_f-q_i)^4}\times \\
&\quad \times Tr\{\not{p}_f(\not{p}_f+m)\beta^\alpha\not{p}_i(\not{p}_i+m)\beta^\omega\}Tr\{\not{q}_f(\not{q}_f+m)\beta^\mu\not{q}_i(\not{q}_i+m)\beta^\nu\}+ \\
&\quad + \frac{1}{16m^4p_f^0p_i^0q_f^0q_i^0}\frac{g_{\mu\alpha}g_{\nu\omega}}{(q_f-q_i)^2(q_f-p_i)^2}\times \\
&\quad \times Tr\{\not{p}_f(\not{p}_f+m)\beta^\alpha\not{p}_i(\not{p}_i+m)\beta^\nu\not{q}_f(\not{q}_f+m)\beta^\mu\not{q}_i(\not{q}_i+m)\beta^\omega\}+ \\
&\quad + \frac{1}{16m^4p_f^0q_i^0q_f^0p_i^0}\frac{g_{\mu\alpha}g_{\nu\omega}}{(q_f-p_i)^2(q_f-q_i)^2}\times \\
&\quad \times Tr\{\not{p}_f(\not{p}_f+m)\beta^\alpha\not{q}_i(\not{q}_i+m)\beta^\nu\not{q}_f(\not{q}_f+m)\beta^\mu\not{p}_i(\not{p}_i+m)\beta^\omega\}+ \\
&\quad + \frac{1}{16m^4p_f^0q_i^0q_f^0p_i^0}\frac{g_{\mu\alpha}g_{\nu\omega}}{(q_f-p_i)^4}\times \\
&\quad \times Tr\{\not{p}_f(\not{p}_f+m)\beta^\alpha\not{q}_i(\not{q}_i+m)\beta^\omega\}Tr\{\not{q}_f(\not{q}_f+m)\beta^\mu\not{p}_i(\not{p}_i+m)\beta^\nu\}\Big].
\end{aligned}
\tag{5.81}
$$

The traces in (5.81) could be re-expressed with the help of properties (4.39) and (4.40). We obtain from (5.81)

$$
|\mathcal{M}|^2 = \frac{e^4}{(2\pi)^4}\frac{1}{16E^4}[\frac{p_i.q_i+p_i.q_f+p_f.q_i+p_f.q_f}{(q_f-q_i)^2}+\frac{q_i.p_i+q_i.q_f+p_f.p_i+p_f.q_f}{(q_f-p_i)^2}]^2.
\tag{5.82}
$$

We can reduce this result further by taking into account the center-of-mass reference frame. The result is

$$
|\mathcal{M}|^2 = \frac{e^4}{(2\pi)^4}\frac{1}{4E^4}\left|\frac{(p_iq_i)+(q_fq_i)}{(q_f-p_i)^2}+\frac{(q_ip_i)+(p_fq_i)}{(p_f-p_i)^2}\right|^2.
\tag{5.83}
$$

Using the Mandelstam variables

$$
s=(p_i+q_i)^2=(p_f+q_f)^2, \quad \frac{s}{2}-m^2=p_iq_i=p_fq_f,
\tag{5.84}
$$

$$t = (p_i - p_f)^2 = (q_i - q_f)^2, \quad m^2 - \frac{t}{2} = p_i p_f = q_i q_f, \tag{5.85}$$

$$u = (p_i - q_f)^2 = (q_i - p_f)^2, \quad m^2 - \frac{u}{2} = p_i q_f = q_i p_f, \tag{5.86}$$

we can rewrite (5.83) as

$$|\mathcal{M}|^2 = \frac{e^4}{(2\pi)^4} \frac{1}{16E^4} \left| \frac{s-t}{u} + \frac{s-u}{t} \right|^2. \tag{5.87}$$

Replacing (5.87) into (5.80), we have

$$\frac{d\sigma_{c.m}}{d\Omega} = \frac{\alpha^2}{4s} \left| \frac{s-t}{u} + \frac{s-u}{t} \right|^2, \tag{5.88}$$

where α is the fine structure constant.

The result (5.88) is identical to that obtained by C. Itzykson and J. B. Zuber in [94] using the usual approach, and by J. Beltran in [95] using CPT with Klein-Gordon-Fock fields.

5.5 Compton scattering

The Compton scattering is the following process

$$b + \gamma \to b + \gamma, \tag{5.89}$$

where b represents an scalar particle and γ a photon.

Using the creation and annihilation operator formalism, the in and out states will take the following form

$$\begin{aligned} |in_{\text{Comp}}\rangle &= |\Psi_i\rangle \otimes |\Phi_i\rangle \\ &= \int d^3p_1 d^3k_1 \Psi_i(\mathbf{p}_1)\Phi_i(\mathbf{k}_1) a^\dagger(\mathbf{p}_1)\varepsilon_{i\nu}(\mathbf{k}_1) c_\nu^\dagger(\mathbf{k}_1)|0\rangle, \end{aligned} \tag{5.90}$$

$$\begin{aligned} |out_{\text{Comp}}\rangle &= |\Psi_f\rangle \otimes |\Phi_f\rangle \\ &= \int d^3p_2 d^3k_2 \Psi_f(\mathbf{p}_2)\Phi_f(\mathbf{k}_2) a^\dagger(\mathbf{p}_2)\varepsilon_{f\mu}(\mathbf{k}_2) c_\mu^\dagger(\mathbf{k}_2)|0\rangle, \end{aligned} \tag{5.91}$$

where $\Psi_{i,f}(\mathbf{p}_1)$ and $\Phi_{i,f}(\mathbf{k}_2)$ are wave packet sharply peaked in $\mathbf{p}_{i,f}$ and $\mathbf{k}_{i,f}$. Besides $\varepsilon_{i\nu}$ and $\varepsilon_{f\mu}$ are the initial and final vector polarization for photons.

The transition matrix element $S_{fi} = \langle out_{\text{comp}}|S|in_{\text{comp}}\rangle$ is then

$$S_{fi}^{Comp} = \int d^3p_2 d^3k_2 \int d^3p_1 d^3k_1 \Psi_f^*(\mathbf{p}_2)\Phi_f^*(\mathbf{k}_2)\tilde{S}_{fi}^{Comp}\Psi_i(\mathbf{p}_1)\Phi_i(\mathbf{k}_1), \tag{5.92}$$

where

$$\tilde{S}_{fi}^{Comp} = \langle 0|a(\mathbf{p}_2)\varepsilon_{f\mu}(\mathbf{k}_2)c_\mu(\mathbf{k}_2)Sa^\dagger(\mathbf{p}_1)\varepsilon_{i\nu}(\mathbf{k}_1)c_\nu^\dagger(\mathbf{k}_1)|0\rangle. \tag{5.93}$$

Because of the creation and annihilation operators in (5.93), only the contribution for S coming from $D_2^{(2)}$ will produce transition matrix element $\tilde{S}_{fi}^{Comp}$ non-null. Therefore, we will focus on determining the contribution of T_2 from $D_2^{(2)}$ which we rewrite as follows

$$\begin{aligned} D_2^{(2)} =& e^2 i : \overline{\psi}(x)\beta^\nu S(x-y)\beta^\mu\psi(y) :: A_\mu(y)A_\nu(x) : \\ & - e^2 i : \overline{\psi}(y)\beta^\mu S(y-x)\beta^\nu\psi(x) :: A_\mu(y)A_\nu(x) : . \end{aligned} \tag{5.94}$$

To begin the causal splitting procedure, we can see in (5.94) that the numerical of $D_2^{(2)}$ is $S(x-y)$.

5.5.1 Causal splitting of $S(x-y)$

In momentum space, the function $\hat{S}(p)$ is given by the following formula

$$\begin{aligned} \widehat{S}_{ab}(p) &= \frac{1}{m}[\not p(\not p + m)]_{ab}\widehat{D}_m(p) \\ &= \frac{i}{2\pi m}[\not p(\not p + m)]_{ab}\delta(p^2 - m^2)Sgn(p^0). \end{aligned} \tag{5.95}$$

In order to determine the order of singularity, we will compute the form of $\widehat{S}_{ab}(p/\alpha)$

$$\begin{aligned} \widehat{S}_{ab}(\frac{p}{\eta}) &= \frac{i}{2\pi m}[\not p\alpha^{-1}(\not p\alpha^{-1} + m)]_{ab}\delta(p^2\alpha^{-2} - m^2)Sgn(p^0\alpha^{-1}) \\ &= \frac{i}{2\pi m}[\not p(\not p + m\alpha)]_{ab}\delta(p^2 - \alpha^2 m^2)Sgn(p^0\alpha^{-1}). \end{aligned} \tag{5.96}$$

Using (5.96) and (3.91), we can see that for a power counting function $\rho(\eta) = 1$, we obtain the following non-null quasi-asymptotic distribution

$$\lim_{\alpha\to 0}\rho(\alpha)\langle S(\frac{p}{\alpha}), \check{f}(p)\rangle = \langle \frac{i}{2\pi m}[\not p\not p]_{ab}\delta(p^2)Sgn(p^0), \check{f}(p)\rangle \neq 0. \tag{5.97}$$

Therefore, from (3.92), we can obtain the order of singularity

$$\omega[\hat{S}(p)] = 0, \tag{5.98}$$

which means that, unpredictably, $\hat{S}(p)$ is a singular distribution.

The retarded part of $\hat{S}(p)$, is given by (3.94). Replacing (5.95) into (3.94), we have

$$\begin{aligned}
\hat{r}_0(p) &= \frac{i}{2\pi} Sgn(p^0) \int_{-\infty}^{\infty} dt \frac{\widehat{S}(tp)}{t^{\omega+1}(1-t+Sgn(p^0)i0^+)} \\
&= \frac{i}{2\pi} Sgn(p^0) \int_{-\infty}^{\infty} dt \frac{1}{t(1-t+Sgn(p^0)i0^+)} \frac{i}{2\pi m} [t\not{p}(t\not{p}+m)]_{ab} \delta(t^2p^2-m^2) Sgn(tp^0) \\
&= \frac{i}{2\pi} Sgn(p^0) \int_{-\infty}^{\infty} dt \frac{1}{t(1-t+Sgn(p^0)i0^+)} \frac{i}{2\pi m} [t^2\not{p}\not{p}]_{ab} \delta(t^2p^2-m^2) Sgn(tp^0) \\
&+ \frac{i}{2\pi} Sgn(p^0) \int_{-\infty}^{\infty} dt \frac{1}{t(1-t+Sgn(p^0)i0^+)} \frac{i}{2\pi m} [t\not{p}m]_{ab} \delta(t^2p^2-m^2) Sgn(tp^0).
\end{aligned} \tag{5.99}$$

We can reduce the two integrals in (5.99) using the symmetry properties presented in Appendix A.6. For the first term in the right hand side of (5.99) we use (A.31), and (A.30) for the second, obtaining the following result

$$\begin{aligned}
\hat{r}_0(p) &= \frac{i}{2\pi} Sgn(p^0) \int_{0}^{\infty} dt \frac{2}{t(1-t+Sgn(p^0)i0^+)} \frac{i}{2\pi m} [t^2\not{p}\not{p}]_{ab} \delta(t^2p^2-m^2) Sgn(tp^0) \\
&+ \frac{i}{2\pi} Sgn(p^0) \int_{0}^{\infty} dt \frac{2t}{t(1-t+Sgn(p^0)i0^+)} \frac{i}{2\pi m} [t\not{p}m]_{ab} \delta(t^2p^2-m^2) Sgn(tp^0) \\
&= \frac{1}{m} [\not{p}(\not{p}+m)]_{ab} \{ \frac{i}{2\pi} Sgn(p^0) \int_{0}^{\infty} dt \frac{2t^2}{t(1-t+Sgn(p^0)i0^+)} \frac{i}{2\pi} \delta(t^2p^2-m^2) Sgn(tp^0) \} \\
&= \frac{1}{m} [\not{p}(\not{p}+m)]_{ab} \{ \frac{i}{2\pi} Sgn(p^0) \int_{0}^{\infty} dt \frac{2t\widehat{D}_m(pt)}{(1-t+Sgn(p^0)i0^+)} \}.
\end{aligned} \tag{5.100}$$

From (4.52), we can compute that the order of singularity of $\hat{D}_m(p)$ is $\omega[\hat{D}_m(p)] = -2$. In addition, because $\hat{D}_m(p)$ is odd, using (A.28) we can see that the factor between braces in the last line of (5.100) is $D_m^{\text{ret}}(p)$. Consequently, the retarded part that we are computing $S^{\text{ret}}(p) = \hat{r}_0(p)$ is equal to

$$S^{\text{ret}}(p) = \frac{1}{m} [\not{p}(\not{p}+m)] D_m^{\text{ret}}(p). \tag{5.101}$$

Furthermore, as expected for the tree level, the causal splitting for $S(x-y)$ could be done as usual

$$S(x-y) = S^{ret}(x-y) - S^{adv}(x-y), \tag{5.102}$$

but, ***CPT tell us that this splitting is not unique*** because the order of singularity of $S(x-y)$ implies that the more general solution for $S^{ret}(x-y)$ (which is given by

(3.69)) has the following form

$$\tilde{S}^{ret}(x-y) = S^{ret}(x-y) + C\delta(x-y), \tag{5.103}$$

where C is a 5×5 constant matrix which is not fixed by the causal-split procedure.

On the other hand, in the causal distribution (5.94) the numerical part of the second term to split has opposite sign in the variable. The latter means that the retarded part of $S(y-x)$ has the following form

$$\tilde{S}^{ret}(y-x) = -S^{adv}(y-x) + C'\delta(x-y). \tag{5.104}$$

With the help of (5.103) and (5.104), the retarded distribution $R_2^{(2)}$ is

$$\begin{aligned} R_2^{(2)} =& e^2 i :\overline{\psi}(x)\beta^\nu(S^{ret}(x-y) + C\delta(x-y))\beta^\mu\psi(y) :: A_\mu(y)A_\nu(x) : \\ &- e^2 i :\overline{\psi}(y)\beta^\mu(-S^{adv}(y-x) + C\delta(x-y)\beta^\nu\psi(x) :: A_\mu(y)A_\nu(x) : . \end{aligned} \tag{5.105}$$

Finally, making the difference $T_2^{(2)} = R_2^{(2)} - R_2'^{(2)}$, we obtain the two point function associated with the Compton process in the following form

$$\begin{aligned} T_2^{(2)} &= R_2^{(2)} - R_2'^{(2)} \\ &= [e^2 i :\overline{\psi}(x)\beta^\nu(S^{ret}(x-y) + C\delta(x-y))\beta^\mu\psi_b(y) :: A_\mu(y)A_\nu(x) : - \\ &\quad - e^2 i :\overline{\psi}(y)\beta^\mu(-S^{adv}(y-x) + C\delta(x-y)\beta^\nu\psi(x) :: A_\mu(y)A_\nu(x) :] - \\ &\quad - [-e^2\beta^\mu\beta^\nu :\psi(y)\overline{\psi}(x) : \frac{1}{i}S^{(-)}(x-y) : A_\mu(y)A_\nu(x) : + \\ &\quad + e^2\beta^\mu\beta^\nu :\overline{\psi}(y)\psi(x) : \frac{1}{i}S^{(+)}(y-x) : A_\mu(y)A_\nu(x) :] \\ &= e^2 i :\overline{\psi}(x)\beta^\nu(S^{ret}(x-y) - S^{(-)}(x-y) + C\delta(x-y))\beta^\mu\psi(y) : \times \\ &\quad \times : A_\mu(y)A_\nu(x) : + \\ &\quad + e^2 i :\overline{\psi}_a(y)\beta^\mu_{ab}(S^{adv}_{bc}(y-x) + S^{(+)}_{bc}(y-x) - C'_{bc}\delta(x-y)\beta^\nu_{cd}\psi_d(x) : \times \\ &\quad \times : A_\mu(y)A_\nu(x) : . \end{aligned} \tag{5.106}$$

Using as usual

$$S^F(x) = S^{(-)}(x) - S^{ret}(x) = -S^{(+)}(x) - S^{adv}(x), \tag{5.107}$$

we have for (5.106)

$$\begin{aligned} T_2^{(2)}(x,y) = e^2 i &:\overline{\psi}(x)\beta^\nu(-S^F(x-y) + C\delta(x-y))\beta^\mu\psi(y) :: A_\mu(y)A_\nu(x) : \\ &+ e^2 i :\overline{\psi}(y)\beta^\mu(-S^F(y-x) - C'\delta(x-y))\beta^\nu\psi(x) :: A_\mu(y)A_\nu(x) : . \end{aligned} \tag{5.108}$$

Because the symmetry of $T_2^{(2)}(x,y)$ in the interchange of variable $x \rightleftarrows y$, we can see that $C' = -C$, and finally obtain

$$\begin{aligned} T_2^{(2)}(x,y) &= e^2 i : \overline{\psi}(x)\beta^\nu(-S^F(x-y) + C\delta(x-y))\beta^\mu\psi(y) :: A_\mu(y)A_\nu(x) : \\ &+ e^2 i : \overline{\psi}(y)\beta^\mu(-S^F(y-x) + C\delta(x-y))\beta^\nu\psi(x) :: A_\mu(y)A_\nu(x) : . \end{aligned} \tag{5.109}$$

5.5.2 Fixation of constant C

Because $D_2^{(2)}(x,y)$ is a singular distribution, the *causal splitting procedure* (based on *causality* and *gauge invariance* at first order) give us a family of 2-point causal distributions $T_2^{(2)}(x,y)$ represented in the freedom of constant C. To obtain the physical solution, we must use other physical properties of the theory.

Graphically, the Compton scattering has two external photon legs, this allow us to use ***perturbative gauge invariance*** at second order to determine C. Then, we need to compute the gauge derivative $d_Q T_2(x,y)$, this result is

$$\begin{aligned} d_Q T_2(x,y) &= d_Q\Big[e^2 i : \overline{\psi}(x)\beta^\nu(-S^F(x-y) + C\delta(x-y))\beta^\mu\psi(y) :: A_\mu(y)A_\nu(x) : \\ &\quad + e^2 i : \overline{\psi}(y)\beta^\mu(-S^F(y-x) + C\delta(x-y)\beta^\nu\psi(x) :: A_\mu(y)A_\nu(x) : \Big] \\ &= e^2 i : \overline{\psi}(x)\beta^\nu(-S^F(x-y) + C\delta(x-y))\beta^\mu\psi(y) :: i\partial_\mu u(y)A_\nu(x) : \\ &+ e^2 i : \overline{\psi}(x)\beta^\nu(-S^F(x-y) + C\delta(x-y))\beta^\mu\psi(y) :: A_\mu(y)i\partial_\nu u(x) : \\ &+ e^2 i : \overline{\psi}(y)\beta^\mu(-S^F(y-x) + C\delta(x-y)\beta^\nu\psi(x) :: A_\mu(y)i\partial_\nu u(x) : \\ &+ e^2 i : \overline{\psi}(y)\beta^\mu(-S^F(y-x) + C\delta(x-y)\beta^\nu\psi(x) :: i\partial_\mu u(y)A_\nu(x) : . \end{aligned} \tag{5.110}$$

Defining $Q^{\nu\mu}_{xy}$ as

$$\begin{aligned} Q^{\nu\mu}_{xy} =& : \overline{\psi}(x)\beta^\nu[-S^F(x-y) + C\delta(x-y)]\beta^\mu\psi(y) : \\ & : \overline{\psi}(y)\beta^\mu[-S^F(y-x) + C\delta(x-y)]\beta^\nu\psi(x) :, \end{aligned} \tag{5.111}$$

we can rewrite (5.110), as

$$\begin{aligned} d_Q T_2(x,y) &= e^2 i^2 \partial^y_\mu \left(Q^{\nu\mu}_{xy} : u(y)A_\nu(x) :\right) - e^2 i^2 \partial^y_\mu(Q^{\nu\mu}_{xy}) : u(y)A_\nu(x) : \\ &+ e^2 i^2 \partial^x_\nu \left(Q^{\nu\mu}_{xy} : A_\mu(y)u(x) :\right) - e^2 i^2 \partial^x_\nu(Q^{\nu\mu}_{xy}) : A_\mu(y)u(x) : . \end{aligned} \tag{5.112}$$

From (5.112) and (4.80) it is clear that, to get gauge invariance at second order, $Q^{\nu\mu}_{xy}$ must fulfill the following conditions

$$\partial^x_\nu(Q^{\nu\mu}_{xy}) = 0 = \partial^y_\mu(Q^{\nu\mu}_{xy}). \tag{5.113}$$

Taking the first derivative of (5.113), we obtain

$$\begin{aligned}\partial^x_\nu Q^{\mu\nu} =&: \partial^x_\nu\overline{\psi}(x)\beta^\nu[-S^F(x-y)+C\delta(x-y)]\beta^\mu\psi(y):\\&+:\overline{\psi}(x)\beta^\nu[-\partial^x_\nu S^F(x-y)+C\partial^x_\nu\delta(x-y)]\beta^\mu\psi(y):\\&+:\overline{\psi}(y)\beta^\mu[-\partial^x_\nu S^F(y-x)+C\partial^x_\nu\delta(x-y)]\beta^\nu\psi(x):\\&+:\overline{\psi}(y)\beta^\mu[-S^F(y-x)+C\delta(x-y)]\beta^\nu\partial^x_\nu\psi(x):.\end{aligned} \tag{5.114}$$

Besides, considering (5.101) and (4.50), we can write the DKP Feynman propagator $S^F(x)$ as

$$S^F(x)=S^{(-)}(x)-S^{ret}(x)=-S^{(+)}(x)-S^{adv}(x)=-\frac{1}{m}[i\partial\!\!\!/(i\partial\!\!\!/+m)]D^F(x). \tag{5.115}$$

In order to reduce (5.114) we must obtain the derivative of $S^F(x)$. The latter can be done regarding (5.115) and the β-matrix algebra (4.17) from the computation of $(i\partial\!\!\!/-m)S^F(x)$ as follows

$$\begin{aligned}(i\partial\!\!\!/-m)S^F(x)&=-\frac{1}{m}[i\partial\!\!\!/(i\partial\!\!\!/-m)(i\partial\!\!\!/+m)]D^F(x)\\&=-\frac{1}{m}[-i\partial\!\!\!/\partial\!\!\!/\partial\!\!\!/-i\partial\!\!\!/m^2]D^F(x)\\&=-\frac{1}{m}[-i(-\partial\!\!\!/\partial\!\!\!/\partial\!\!\!/+2\partial\!\!\!/\Box)-i\partial\!\!\!/m^2]D^F(x)\\&=-\frac{1}{m}[i\partial\!\!\!/\partial\!\!\!/\partial\!\!\!/+i\partial\!\!\!/m^2]D^F(x)-\frac{1}{m}[-2i\partial\!\!\!/\Box-2i\partial\!\!\!/m^2]D^F(x)\\&=-(i\partial\!\!\!/-m)S^F(x)+\frac{2i\partial\!\!\!/}{m}[\Box+m^2]D^F(x)\\i\partial\!\!\!/S^F(x)&=mS^F(x)+\frac{i}{m}\partial\!\!\!/\delta(x).\end{aligned} \tag{5.116}$$

Now, to obtain the derivative of $S^F(-x)$, firstly we will compute the conjugate transpose of $S^F(x)$ using the property $\beta^{\mu\dagger}=\eta^0\beta^\mu\eta^0$

$$\begin{aligned}(S^F(x))^\dagger&=-\frac{1}{m}[-i\partial\!\!\!/^\dagger(-i\partial\!\!\!/^\dagger+m)]D^{F\dagger}(x)\\&=-\frac{1}{m}[-i\eta^0\partial\!\!\!/\eta^0(-i\eta^0\partial\!\!\!/\eta^0+m)]((2\pi)^{-4}\int d^4p\frac{e^{-ipx}}{m^2-p^2-i0})^\dagger\\&=-\eta^0\frac{1}{m}[-i\partial\!\!\!/(-i\partial\!\!\!/+m)](2\pi)^{-4}\int d^4p\frac{e^{ipx}}{m^2-p^2+i0}\eta^0\\&=-\eta^0\frac{1}{m}[-i\partial\!\!\!/(-i\partial\!\!\!/+m)]D^F(-x)\eta^0\\&=\eta^0S^F(-x)\eta^0.\end{aligned} \tag{5.117}$$

Secondly, using (5.117), the conjugate transpose of (5.116) is

$$\begin{aligned} (i\partial\!\!\!/ S^F(x))^\dagger &= (mS^F(x) + \frac{i}{m}\partial\!\!\!/\delta(x))^\dagger \\ -i\partial_\nu\eta^0 S^F(-x)\eta^0\eta^0\beta^\nu\eta^0 &= m\eta^0 S^F(-x)\eta^0 - \frac{i}{m}\eta^0\partial\!\!\!/\delta(x)\eta^0 \\ -i\partial_\nu S^F(-x)\beta^\nu &= mS^F(-x) - \frac{i}{m}\partial\!\!\!/\delta(x). \end{aligned} \tag{5.118}$$

Finally, replacing (5.118) and (5.116) into (5.114), we have

$$\begin{aligned} \partial_{\nu,x}Q^{\mu\nu} =&: i\bar{\psi}(x)m[-S^F(x-y) + C\delta(x-y)]\beta^\mu\psi(y) : \\ &+ : \overline{\psi}(x)[miS^F(x-y) - \frac{1}{m}\partial\!\!\!/\delta(x-y) + \partial_{\nu,x}C\delta(x-y)]\beta^\mu\psi(y) : \\ &+ : \overline{\psi}(y)\beta^\mu[-imS^F(y-x) - \frac{1}{m}\partial\!\!\!/\delta(x-y) + C\partial\!\!\!/\delta(x-y)]\psi(x) : \\ &+ : \overline{\psi}(y)\beta^\mu[-S^F(y-x) + C\delta(x-y)](-im\psi(x)) : \\ =& + : \overline{\psi}(x)[-\frac{1}{m}\partial\!\!\!/\delta(x) + C\partial\!\!\!/\delta(x-y)]\beta^\mu\psi(y) : \\ &+ : \overline{\psi}(y)\beta^\mu[-\frac{1}{m}\partial\!\!\!/\delta(x) + C\partial\!\!\!/\delta(x-y)]\psi(x) :, \end{aligned} \tag{5.119}$$

which tell us that to get quantum gauge invariance, C must be

$$C = \frac{I}{m}, \tag{5.120}$$

where I is the 5×5 identity matrix.

5.5.3 Computation of the differential cross section

Replacing (5.120) into (5.109), we obtain the 2-point distribution associated with the Compton process $T_2^{(2)}(x,y)$ as follows

$$\begin{aligned} T_2^{(2)}(x,y) &= e^2 i : \overline{\psi}(x)\beta^\nu(-S^F(x-y) + \frac{I}{m}\delta(x-y))\beta^\mu\psi(y) :: A_\mu(y)A_\nu(x) : \\ &+ e^2 i : \overline{\psi}(y)\beta^\mu(-S^F(y-x) + \frac{I}{m}\delta(x-y)\beta^\nu\psi(x) :: A_\mu(y)A_\nu(x) : . \end{aligned} \tag{5.121}$$

To sort the computation, we will denote as $S_2^{(2)}$ the term of S-matrix associated with the Compton scattering. Using (5.121) $S_2^{(2)}$ can be written as

$$\begin{aligned} S_2^{(2)} &= \frac{1}{2}\int d^4x d^4y T_2^{(2)}(x,y)g(x)g(y) \\ &= -\frac{1}{2}\int d^4x d^4y e^2 i : \overline{\psi}(x)\beta^{\nu}_{cd}S^F(x-y)\beta^{\mu}\psi(y) :: A_\mu(y)A_\nu(x) :]g(x)g(y) \\ &- \frac{1}{2}\int d^4x d^4y[e^2 i : \overline{\psi}(y)\beta^{\mu}_{ab}S^F(y-x)\beta^{\nu}\psi(x) :: A_\mu(y)A_\nu(x) :]g(x)g(y) \\ &+ \frac{e^2 i}{m}\int d^4x : \overline{\psi}(x)\beta^{\mu}\beta^{\nu}\psi(x) :: A_\mu(x)A_\nu(x) : g^2(x), \end{aligned} \tag{5.122}$$

where in the last integral we have joined the two ones coming from the Dirac delta functions. **This latter term can be seen as a graph where we have two photons and scalars in the same point**. As pointed out by Akhiezer and Berestetskii in [10], an advantage of DKP theory is that this term does not appear as in that developed by the Klein-Gordon-Fock equation. But, from (5.122), it is indisputable that such term appears because of the *singular* nature of the associated causal propagator.

To continue our computation, we will decompose $S_2^{(2)}$ in the following form

$$S_2^{(2)} =_a S_2^{(2)} +_b S_2^{(2)}, \tag{5.123}$$

where

$${}_aS_2^{(2)} = \frac{e^2 i}{m}\int d^4x : \overline{\psi}_a(x)\beta^{\mu}_{ac}\beta^{\nu}_{cd}\psi_d(x) :: A_\mu(x)A_\nu(x) : g^2(x), \tag{5.124}$$

$$\begin{aligned} {}_bS_2^{(2)} &= -\frac{e^2 i}{2}\int d^4x d^4y : \overline{\psi}(x)\beta^{\nu}S^F(x-y)\beta^{\mu}\psi(y) :: A_\mu(y)A_\nu(x) : g(x)g(x) \\ &- \frac{e^2 i}{2}\int d^4x d^4y : \overline{\psi}(y)\beta^{\mu}S^F(y-x)\beta^{\nu}\psi_d(x) :: A_\mu(y)A_\nu(x) : g(x)g(x). \end{aligned} \tag{5.125}$$

The operator transition amplitude distribution $\tilde{S}_{fi}^{Comp}$ will be written as

$$\tilde{S}_{fi}^{Comp} =_a S_{if}^{(2)} +_b S_{if}^{(2)}. \tag{5.126}$$

Considering the adiabatic limit $g(x) \to 1$ and the expression (5.93), the terms ${}_aS_{if}^{(2)}$ and ${}_bS_{if}^{(2)}$ take the following forms

$$\begin{aligned} {}_aS_{if}^{(2)} &= \frac{e^2 i}{m}\int d^4x \langle 0|a(\mathbf{p}_2) : \overline{\psi}(x)\beta^{\mu}\beta^{\nu}\psi(x) : a^{\dagger}(\mathbf{p}_1)|0\rangle \times \\ &\times \langle 0|\varepsilon_{f\beta}(\mathbf{k}_2)c_\beta(\mathbf{k}_2) : A_\mu(x)A_\nu(x) : \varepsilon_{i\alpha}(\mathbf{k}_1)c_\alpha^{\dagger}(\mathbf{k}_1)|0\rangle, \end{aligned} \tag{5.127}$$

$$
\begin{aligned}
{}_bS^{(2)}_{if} = & -\frac{e^2 i}{2}\int d^4x d^4y \langle 0|a(\mathbf{p}_2) : \overline{\psi}(x)\beta^\nu S^F(x-y)\beta^\mu \psi(y) : a^\dagger(\mathbf{p}_1)|0\rangle \times \\
& \times \langle 0|\varepsilon_{f\beta}(\mathbf{k}_2)c_\beta(\mathbf{k}_2) : A_\mu(y)A_\nu(x) : \varepsilon_{i\alpha}(\mathbf{k}_1)c^\dagger_\alpha(\mathbf{k}_1)|0\rangle - \\
& -\frac{e^2 i}{2}\int d^4x d^4y \langle 0|a(\mathbf{p}_2) : \overline{\psi}(y)\beta^\mu S^F(y-x)\beta^\nu \psi(x) : a^\dagger(\mathbf{p}_1)|0\rangle \times \\
& \times \langle 0|\varepsilon_{f\beta}(\mathbf{k}_2)c_\beta(\mathbf{k}_2) : A_\mu(y)A_\nu(x) : \varepsilon_{i\alpha}(\mathbf{k}_1)c^\dagger_\alpha(\mathbf{k}_1)|0\rangle \\
= & -e^2 i\int d^4x d^4y \langle 0|a(\mathbf{p}_2) : \overline{\psi}(x)\beta^\nu S^F(x-y)\beta^\mu \psi(y) : a^\dagger(\mathbf{p}_1)|0\rangle \times \\
& \times \langle 0|\varepsilon_{f\beta}(\mathbf{k}_2)c_\beta(\mathbf{k}_2) : A_\mu(y)A_\nu(x) : \varepsilon_{i\alpha}(\mathbf{k}_1)c^\dagger_\alpha(\mathbf{k}_1)|0\rangle .
\end{aligned}
\tag{5.128}
$$

Before reducing the expressions (5.127) and (5.128), we must consider a real polarization vector ε_ν with the following properties

$$
\varepsilon_\nu = (0, \boldsymbol{\varepsilon}), \quad \boldsymbol{\varepsilon}.\boldsymbol{k} = 0, \quad \boldsymbol{\varepsilon}^2 = 1, \tag{5.129}
$$

and we need to determine the contractions between the electromagnetic potential field $A^\mu(x)$ and a creation or annihilation operator for photons. This contractions are

$$
\begin{aligned}
\varepsilon_{f\beta}(\mathbf{k}_j)c_\beta(\mathbf{k}_j)A_\mu(x) &= \varepsilon_{f\beta}(\mathbf{k}_j)c_\beta(\mathbf{k}_j)A^{(+)}_\mu(x) \\
&= \varepsilon_{f\beta}(\mathbf{k}_j)c_\beta(\mathbf{k}_j)(2\pi)^{-3/2}\int \frac{d^3k}{\sqrt{2\omega}} c_\mu(\mathbf{k})^\dagger e^{ikx} \times \begin{cases} 1, & \text{for } \mu = 1,2,3 \\ -1, & \text{for } \mu = 0, \end{cases} \\
&= \varepsilon_{f\beta}(\mathbf{k}_j)(2\pi)^{-3/2}\int \frac{d^3k}{\sqrt{2\omega}} \delta_{\mu\beta}\delta(\mathbf{k}_j - \mathbf{k}) e^{ikx} \times \begin{cases} 1, & \text{for } \mu = 1,2,3 \\ -1, & \text{for } \mu = 0, \end{cases} \\
&= (2\pi)^{-3/2}\frac{\delta_{\mu\beta}\varepsilon_{f\beta}(\mathbf{k}_j)}{\sqrt{2\omega_j}} e^{ik_jx} \times \begin{cases} 1, & \text{for } \mu = 1,2,3 \\ -1, & \text{for } \mu = 0, \end{cases} \\
&= (2\pi)^{-3/2}\frac{\varepsilon_{f\mu}(\mathbf{k}_j)}{\sqrt{2\omega_j}} e^{ik_jx},
\end{aligned}
\tag{5.130}
$$

$$
\begin{aligned}
A_\mu(x)\varepsilon_{i\beta}(\mathbf{k}_j)c^\dagger_\beta(\mathbf{k}_j) &= A^{(-)}_\mu(x)\varepsilon_{i\beta}(\mathbf{k}_j)c^\dagger_\beta(\mathbf{k}_j) \\
&= (2\pi)^{-3/2}\int \frac{d^3k}{\sqrt{2\omega}} c_\mu(\mathbf{k}) e^{-ikx}\varepsilon_{i\beta}(\mathbf{k}_j)c^\dagger_\beta(\mathbf{k}_j) \\
&= (2\pi)^{-3/2}\int \frac{d^3k}{\sqrt{2\omega}} \delta_{\mu\beta}\delta(\mathbf{k} - \mathbf{k}_j) e^{-ikx}\varepsilon_{i\beta}(\mathbf{k}_j) \\
&= (2\pi)^{-3/2}\frac{\varepsilon_{i\mu}(\mathbf{k}_j)}{\sqrt{2\omega}} e^{-ik_jx}.
\end{aligned}
\tag{5.131}
$$

Using Wick theorem and contractions (5.14), (5.15), (5.130) and (5.131); the vacuum expectation values in (5.127) are

$$\begin{aligned}\langle 0|a(\mathbf{p}_2):\overline{\psi}(x)\beta^\mu\beta^\nu\psi(x):a^\dagger(\mathbf{p}_1)|0\rangle = \\ &= \langle 0|a(\mathbf{p}_2):\overline{\psi}_a(x)\beta^\mu_{ac}\beta^\nu_{cd}\psi_d(x):a^\dagger(\mathbf{p}_1)|0\rangle \\ &= \frac{1}{(2\pi)^3}\overline{u^-}(\mathbf{p}_2)\beta^\mu\beta^\nu u^-(\mathbf{p}_1)e^{-i(p_1-p_2)x},\end{aligned} \tag{5.132}$$

$$\begin{aligned}&\langle 0|\varepsilon_{f\beta}(\mathbf{k}_2)a_\beta(\mathbf{k}_2):A_\mu(x)A_\nu(x):\varepsilon_{i\alpha}(\mathbf{k}_1)a^\dagger_\alpha(\mathbf{k}_1)|0\rangle = \\ &= \langle 0|\varepsilon_{f\beta}(\mathbf{k}_f)a_\beta(\mathbf{k}_f):A_\mu(x)A_\nu(x):\varepsilon_{i\alpha}(\mathbf{k}_i)a^\dagger_\alpha(\mathbf{k}_i)|0\rangle \\ &+ \langle 0|\varepsilon_{f\beta}(\mathbf{k}_2)a_\beta(\mathbf{k}_2):A_\mu(x)A_\nu(x):\varepsilon_{i\alpha}(\mathbf{k}_1)a^\dagger_\alpha(\mathbf{k}_1)|0\rangle \\ &= (2\pi)^{-3}\frac{\varepsilon_{f\mu}(\mathbf{k}_2)}{\sqrt{2\omega_f}}\frac{\varepsilon_{i\nu}(\mathbf{k}_1)}{\sqrt{2\omega_i}}e^{-i(k_1-k_2)x} + (2\pi)^{-3}\frac{\varepsilon_{f\nu}(\mathbf{k}_2)}{\sqrt{2\omega_f}}\frac{\varepsilon_{i\mu}(\mathbf{k}_1)}{\sqrt{2\omega_i}}e^{-i(k_1-k_2)x}.\end{aligned} \tag{5.133}$$

Replacing (5.133) and (5.132) into (5.127), we obtain

$$ {}_aS^{(2)}_{if} = \delta(p_1-p_2+k_1-k_2)\mathcal{M}_a, \tag{5.134}$$

where

$$\begin{aligned}\mathcal{M}_a &= \frac{ie^2}{m(2\pi)^2\sqrt{2\omega_1}\sqrt{2\omega_2}}\overline{u^-}(\mathbf{p}_2)\beta^\mu\beta^\nu u^-(\mathbf{p}_1)[\varepsilon_{f\mu}\varepsilon_{i\nu}+\varepsilon_{f\nu}\varepsilon_{i\mu}] \\ &= \frac{ie^2}{m(2\pi)^2\sqrt{2\omega_1}\sqrt{2\omega_2}}[\overline{u^-}(\mathbf{p}_2)\not{\varepsilon}_f\not{\varepsilon}_i u^-(\mathbf{p}_1) + \overline{u^-}_a(\mathbf{p}_2)\not{\varepsilon}_i\not{\varepsilon}_f u^-(\mathbf{p}_1)].\end{aligned} \tag{5.135}$$

The computation of ${}_bS^{(2)}_{if}$ from (5.128), is similar. The difference between the two calculation processes lies on the S^F function between the β-matrices that ${}_aS^{(2)}_{if}$ does not have. The final result for ${}_bS^{(2)}_{if}$ is

$$\begin{aligned}{}_bS^{(2)}_{if} &= -\frac{e^2 i}{2}[\frac{1}{(2\pi)^6}\overline{u^-}_c(\mathbf{p}_2)\beta^\nu_{cd}\beta^\mu_{al}u^-_l(\mathbf{p}_1)]\times \\ &\times\Big[\frac{\varepsilon_{f\mu}(\mathbf{k}_2)}{\sqrt{2\omega_2}}\frac{\varepsilon_{i\nu}(\mathbf{k}_1)}{\sqrt{2\omega_1}}\int d^4xd^4yS^F_{da}(x-y)e^{ip_2x-ip_1y}e^{ik_2y-ik_1x}+ \\ &+\frac{\varepsilon_{f\nu}(\mathbf{k}_2)}{\sqrt{2\omega_2}}\frac{\varepsilon_{i\mu}(\mathbf{k}_1)}{\sqrt{2\omega_1}}\int d^4xd^4yS^F_{da}(x-y)e^{ip_2x-ip_1y}e^{ik_2x-ik_1y}\Big] \\ &= \delta(p_2+k_2-p_1-k_1)\mathcal{M}_b,\end{aligned} \tag{5.136}$$

where

$$\begin{aligned}\mathcal{M}_b = &-\frac{e^2 i}{m(2\pi)^2\sqrt{2\omega_f}\sqrt{2\omega_1}}\Big[\frac{\overline{u^-}(\mathbf{p}_2)\not{\varepsilon}_i(\not{p}_1-\not{k}_2)(\not{p}_1-\not{k}_2+m)\not{\varepsilon}_f u^-(\mathbf{p}_1)}{(p_1-k_2)^2-m^2}+ \\ &+\frac{\overline{u^-}(\mathbf{p}_2)\not{\varepsilon}_f(\not{p}_1+\not{k}_1)(\not{p}_1+\not{k}_1+m)\not{\varepsilon}_i u^-(\mathbf{p}_1)}{(p_1+k_1)^2-m^2}\Big].\end{aligned} \tag{5.137}$$

Regarding (5.136) and (5.134), the transition amplitude $\tilde{S}_{fi}^{Comp}$ could be written as

$$\tilde{S}_{fi}^{Comp} = \delta(p_2 + k_2 - p_1 - k_1)\mathcal{M}, \quad \mathcal{M} = \mathcal{M}_a + \mathcal{M}_b. \tag{5.138}$$

Using the result (B.25) from Appendix B, the differential cross section in the laboratory system is given by

$$\frac{d\sigma}{d\Omega} = (2\pi)^2 \frac{\omega_f^3 E_f}{m\omega_i} |\mathcal{M}(p_i, k_i, p_f, k_f)|^2. \tag{5.139}$$

where

$$|\mathcal{M}|^2 = |\mathcal{M}_a|^2 + \mathcal{M}_a^* \mathcal{M}_b + \mathcal{M}_b^* \mathcal{M}_a + |\mathcal{M}_b|^2, \tag{5.140}$$

and, because the sharply peaked form of the wave packets, we have the following change of variables

$$p_1 \to p_i, \quad p_2 \to p_f, \quad k_1 \to k_i, \quad k_2 \to k_f. \tag{5.141}$$

Before computing the terms in (5.140), we must take into account two properties. Firstly, because we are working in the laboratory system, we can fix $p_i = (m, \mathbf{0})$. The latter has the following consequences

$$p_i \varepsilon_i = 0, \quad p_i \varepsilon_f = 0, \tag{5.142}$$

which complement the polarization conditions that in covariant notation have the following form

$$\varepsilon_i k_i = 0, \quad \varepsilon_f k_f = 0. \tag{5.143}$$

Furthermore, the denominators of fractions in the brackets of (5.137), will be reduced to

$$(p_i - k_f)^2 - m^2 = -2p_i k_f = -2m\omega_f, \tag{5.144}$$

$$(p_i + k_i)^2 - m^2 = 2p_i k_i = 2m\omega_i. \tag{5.145}$$

Secondly, we will find traces with the form

$$Tr[\not{A}_1 \not{A}_2 \ldots \not{A}_{2n}] = (A_1.A_2)(A_3.A_4)\ldots(A_{2n-1}.A_{2n}) + (A_2.A_3)(A_4.A_5)\ldots(A_{2n}.A_1). \tag{5.146}$$

Then, using (5.142), (5.143) and (5.146) we can construct many null traces. As an example we could mention

$$Tr[\ldots \not{\varepsilon}_i \not{p}_i \not{\varepsilon}_i \ldots] = 0, \tag{5.147}$$

$$Tr[\ldots \not{p}_i \not{\varepsilon}_f \not{k}_f \ldots] = 0, \tag{5.148}$$

$$Tr[\dots \not\epsilon_f \not p_i \not\epsilon_i \dots] = 0, \tag{5.149}$$

and other combinations.

Now, returning to (5.140), we have

$$\begin{aligned} |M_a|^2 = \frac{e^4}{16m^4(2\pi)^4\omega_i\omega_f p_i^0 p_f^0}\Big(& Tr[\not p_f(\not p_f + m)\not\epsilon_f\not\epsilon_i\not p_i(\not p_i + m)\not\epsilon_i\not\epsilon_f] + \\ & + Tr[\not p_f(\not p_f + m)\not\epsilon_f\not\epsilon_i\not p_i(\not p_i + m)\not\epsilon_f\not\epsilon_i] + Tr[\not p_f(\not p_f + m)\not\epsilon_i\not\epsilon_f\not p_i(\not p_i + m)\not\epsilon_i\not\epsilon_f] + \\ & + Tr[\not p_f(\not p_f + m)\not\epsilon_i\not\epsilon_f\not p_i(\not p_i + m)\not\epsilon_f\not\epsilon_i]\Big), \end{aligned} \tag{5.150}$$

$$\begin{aligned} \mathcal{M}_a^*\mathcal{M}_b = & \frac{e^4}{m^3(2\pi)^4 8\omega_i\omega_f^2}\frac{1}{2mp_f^0}\frac{1}{2mp_i^0}Tr\big[(\not\epsilon_i\not\epsilon_f + \not\epsilon_f\not\epsilon_i)\not p_f(\not p_f + m)\not\epsilon_i(\not p_i - \not k_f)\times \\ & \times (\not p_i - \not k_f + m)\boxed{\not\epsilon_f}\not p_i(\not p_i + m)\big] - \frac{e^4}{m^3(2\pi)^4 8\omega_i^2\omega_f}\frac{1}{2mp_f^0}\frac{1}{2mp_i^0}\times \\ & \times Tr\big[(\not\epsilon_i\not\epsilon_f + \not\epsilon_f\not\epsilon_i)\not p_f(\not p_f + m)\not\epsilon_f(\not p_i + \not k_i)(\not p_i + \not k_i + m)\boxed{\not\epsilon_i}\not p_i(\not p_i + m)\big], \end{aligned} \tag{5.151}$$

$$\begin{aligned} |\mathcal{M}_b|^2 = & \frac{e^4}{16m^4(2\pi)^4\omega_f^3\omega_i}Tr[\not p_f(\not p_f + m)\not\epsilon_i(\not p_i - \not k_f)\times \\ & \times (\not p_i - \not k_f + m)\not\epsilon_f\not p_i(\not p_i + m)\boxed{\not\epsilon_f}(\not p_i - \not k_f)(\not p_i - \not k_f + m)\not\epsilon_i] \\ & - \frac{e^4}{16m^4(2\pi)^4\omega_f^2\omega_i^2}Tr[\not p_f(\not p_f + m)\not\epsilon_i(\not p_i - \not k_f)\times \\ & \times (\not p_i - \not k_f + m)\boxed{\not\epsilon_f}\not p_i(\not p_i + m)\not\epsilon_i(\not p_i + \not k_i)(\not p_i + \not k_i + m)\not\epsilon_f] \\ & - \frac{e^4}{16m^4(2\pi)^4\omega_f^2\omega_i^2}Tr[\not p_f(\not p_f + m)\not\epsilon_f(\not p_i + \not k_i)\times \\ & \times (\not p_i + \not k_i + m)\not\epsilon_i\not p_i(\not p_i + m)\boxed{\not\epsilon_f}(\not p_i - \not k_f)(\not p_i - \not k_f + m)\not\epsilon_i] \\ & + \frac{e^4}{16m^4(2\pi)^4\omega_f\omega_i^3}Tr[\not p_f(\not p_f + m)\not\epsilon_f(\not p_i + \not k_i)\times \\ & \times (\not p_i + \not k_i + m)\not\epsilon_i\not p_i(\not p_i + m)\boxed{\not\epsilon_i}(\not p_i + \not k_i)(\not p_i + \not k_i + m)\not\epsilon_f]. \end{aligned} \tag{5.152}$$

Before continuing, we can see that the boxed terms in (5.151) and (5.152) are surrounded by other which nullifies the traces where they belong as in (5.147-5.149), in consequence

$$\mathcal{M}_a^*\mathcal{M}_b = 0 = \mathcal{M}_b^*\mathcal{M}_a, \quad |\mathcal{M}_b|^2 = 0. \tag{5.153}$$

For $|\mathcal{M}_a|^2$, there are null terms in (5.150) because the products $\not\epsilon_i\not p_i\not\epsilon_i$, $\not\epsilon_f\not p_i\not\epsilon_i$, $\not\epsilon_i\not p_i\not\epsilon_f$

and $\not\varepsilon_f \not p_i \not\varepsilon_f$. Avoiding this terms and those with odd number of β-matrices, we have

$$\begin{aligned} |M_a|^2 &= \frac{e^4}{16m^4(2\pi)^4\omega_i\omega_f p_i^0 p_f^0}\Big(Tr[\not p_f \not p_f \not\varepsilon_f \not\varepsilon_i \not p_i \not p_i \not\varepsilon_i \not\varepsilon_f] + Tr[\not p_f \not p_f \not\varepsilon_f \not\varepsilon_i \not p_i \not p_i \not\varepsilon_f \not\varepsilon_i] \\ &\quad + Tr[\not p_f \not p_f \not\varepsilon_i \not\varepsilon_f \not p_i \not p_i \not\varepsilon_i \not\varepsilon_f] + Tr[\not p_f \not p_f \not\varepsilon_i \not\varepsilon_f \not p_i \not p_i \not\varepsilon_f \not\varepsilon_i] \Big) \\ &= \frac{e^4}{16m^4(2\pi)^4\omega_i\omega_f p_i^0 p_f^0}\Big(4m^4(\varepsilon_i.\varepsilon_f)^2\Big) \\ &= \frac{e^4}{4(2\pi)^4\omega_i\omega_f m E_f}(\varepsilon_i.\varepsilon_f)^2. \end{aligned} \tag{5.154}$$

Replacing (5.154) and (5.153) into (5.139), we obtain

$$\begin{aligned} \frac{d\sigma}{d\Omega}\Big|_{\text{Lab}} &= \frac{e^4\omega_f^2}{16\pi^2 m^2\omega_i^2}(\varepsilon_i.\varepsilon_f)^2 \\ &= \frac{\alpha^2\omega_f^2}{m^2\omega_i^2}\left(\varepsilon_f.\varepsilon_i\right)^2. \end{aligned} \tag{5.155}$$

In the framework of Klein-Gordon-Fock equation, the result (5.155) was obtained in [94] and using CPT in [95].

Chapter 6

Radiative Corrections.

> In desperation I asked Fermi whether he was not impressed by the agreement between our calculated numbers and his measured numbers. He replied, "How many arbitrary parameters did you use for your calculations?" I thought for a moment about our cut-off procedures and said, "Four." He said, "I remember my friend Johnny von Neumann used to say, with four parameters I can fit an elephant, and with five I can make him wiggle his trunk." With that, the conversation was over.
>
> *Freeman John Dyson*

As we saw in the previous Chapter, in the tree level the computations of differential cross sections via Klein-Gordon-Fock and Duffin-Kemmer-Petiau frameworks are equivalent. In this chapter we will compute the *vacuum polarization tensor* and the *self energy* function of DKP scalar particle.

6.1 Vacuum polarization

It is not difficult to see that the term which contributes to the *vacuum polarization* tensor is (5.59). That term is rewritten as follows

$$D_2^{(3)} = -i : A_\nu(y) D^{\mu\nu}(x,y) A_\mu(x) :, \tag{6.1}$$

where

$$D^{\mu\nu} = ie^2 Tr[\beta^\nu S^{(+)}(y-x)\beta^\mu S^{(-)}(x-y) - \beta^\mu S^{(+)}(x-y)\beta^\nu S^{(-)}(y-x)]. \tag{6.2}$$

As before, to obtain the contribution for T_2 coming from $D_2^{(3)}$, we have to split the numerical part

$$\begin{aligned} D^{\mu\nu}(z) &= ie^2 Tr[\beta^\nu S^{(+)}(y-x)\beta^\mu S^{(-)}(x-y) - \beta^\mu S^{(+)}(x-y)\beta^\nu S^{(-)}(y-x)] \\ &= ie^2[P^{\nu\mu}(z) - P^{\mu\nu}(-z)]. \end{aligned} \tag{6.3}$$

where

$$P^{\nu\mu}(z) = Tr[\beta^\nu S^{(+)}(-z)\beta^\mu S^{(-)}(z)], \quad z = x - y. \tag{6.4}$$

To obtain the order of singularity ω of $D^{\mu\nu}(z)$, we will determine its Fourier transform. From (6.3), it is clear that we just need to determine the transform of $P^{\nu\mu}(z)$ which is equal to

$$\begin{aligned} \hat{P}^{\nu\mu}(k) &= (2\pi)^{-2}\int d^4z P^{\nu\mu}(z)e^{ikz} \\ &= (2\pi)^{-2}\int d^4z Tr[\beta^\nu S^{(+)}(-z)\beta^\mu S^{(-)}(z)]e^{ikz}. \end{aligned} \tag{6.5}$$

Using the trace properties (4.39) and (4.40), and replacing the expressions of $S^{(+)}(-z)$ and $S^{(-)}(z)$ into (6.5) from (4.55), we obtain

$$\begin{aligned} \hat{P}^{\nu\mu}(k) = \frac{1}{m^2(2\pi)^4}\int d^4p\Theta(p^0)\delta(p^2-m^2)\Theta(-p^0-k^0)\delta((p+k)^2-m^2)\times \\ \times\, m^2[4p^\mu p^\nu + 2k^\mu p^\nu + 2p^\mu k^\nu + k^\mu k^\nu]. \end{aligned} \tag{6.6}$$

At this point, it is useful notice that because of the two delta functions and the expression in brackets, we have

$$k_\mu P^{\mu\nu} = 0, \tag{6.7}$$

which means that $P^{\mu\nu}(k)$ has the following form

$$P^{\mu\nu}(k) = (k^\mu k^\nu - k^2 g^{\mu\nu})B(k^2), \quad B(k^2) = \frac{-1}{3k^2}P^\mu{}_\mu, \tag{6.8}$$

where

$$\hat{P}^\mu{}_\mu(k) = \frac{1}{(2\pi)^4}4m^2[1 - \frac{k^2}{4m^2}]\int d^4p\Theta(p^0)\delta(p^2-m^2)\Theta(-p^0-k^0)\delta(k^2+2k.p). \tag{6.9}$$

To continue the calculation, we can note from the step and delta functions of (6.9) that k is time-like, the latter means that we could do the integral using a special reference frame where $k = (k^0, \mathbf{0})$. Consequently, we have for (6.9) the following

$$\begin{aligned}\int d^4p\ldots &= \int d^4p\Theta(p^0)\delta(p^2-m^2)\Theta(-p^0-k^0)\delta(k^2+2k.p)\\ &= \int d^3p\int dp^0\Theta(p^0)\delta((p^0)^2-E_{\mathbf{p}}^2)\Theta(-p^0-k^0)\delta((k^0)^2+2k^0.p^0)\\ &= \int d^3p\int dp^0\frac{1}{2E_{\mathbf{p}}}\delta(p^0-E_{\mathbf{p}})\Theta(-p^0-k^0)\delta((k^0)^2+2k^0.p^0)\\ &= \int d^3p\frac{1}{2E_{\mathbf{p}}}\Theta(-E_{\mathbf{p}}-k^0)\delta((k^0)^2+2k^0.E_{\mathbf{p}}).\end{aligned} \tag{6.10}$$

Evaluating the integral in spherical coordinates and using $E_{\mathbf{p}}dE_{\mathbf{p}} = |\mathbf{p}|d|\mathbf{p}|$, we have

$$\int d^4p\ldots = 4\pi\int E_{\mathbf{p}}dE_{\mathbf{p}}|\mathbf{p}|\frac{1}{2E_{\mathbf{p}}}\Theta(-E_{\mathbf{p}}-k^0)\delta((k^0)^2+2k^0.E_{\mathbf{p}}). \tag{6.11}$$

From the delta function we can determine $|\mathbf{p}|$ as

$$\delta((k^0)^2+2k^0.E_{\mathbf{p}}) \Rightarrow -\frac{k^0}{2} = \sqrt{\mathbf{p}^2+m^2} \Rightarrow \sqrt{\frac{(k^0)^2}{4}-m^2} = |\mathbf{p}|. \tag{6.12}$$

Replacing (6.12) into (6.11), we obtain

$$\begin{aligned}\int d^4p\ldots &= 4\pi\int E_{\mathbf{p}}dE_{\mathbf{p}}\sqrt{\frac{(k^0)^2}{4}-m^2}\Theta((k^0)^2-4m^2)\frac{1}{2E_{\mathbf{p}}}\Theta(-E_{\mathbf{p}}-k^0)\delta((k^0)^2+2k^0.E_{\mathbf{p}})\\ &= \pi\sqrt{\frac{(k^0)^2}{4}-m^2}\frac{1}{k^0}\Theta(-k^0)\Theta((k^0)^2-4m^2).\end{aligned} \tag{6.13}$$

Finally, we get for (6.9) the following form

$$P^{\mu}{}_{\mu}(k) = -\frac{1}{(2\pi)^4}k^2[1-\frac{4m^2}{k^2}]^{\frac{3}{2}}\frac{\pi}{2}\Theta(-k^0)\Theta((k^0)^2-4m^2). \tag{6.14}$$

Going to a general reference frame and replacing (6.14) into (6.8), we obtain for $\hat{P}^{\mu\nu}(k)$ the following result

$$\hat{P}^{\mu\nu}(k) = \frac{\pi}{6(2\pi)^4}(\frac{k^\mu k^\nu}{k^2}-g^{\mu\nu})k^2[1-\frac{4m^2}{k^2}]^{\frac{3}{2}}\Theta(-k^0)\Theta((k^0)^2-4m^2). \tag{6.15}$$

On the other hand, using that $\widehat{P^{\mu\nu}(-z)} = \hat{P}^{\mu\nu}(-k)$ and (6.15), we have for the

Fourier transform of $D^{\mu\nu}(x,y)$ the following form

$$\begin{aligned}\hat{D}^{\mu\nu}(k) &= ie^2\widehat{[P^{\nu\mu}(z) - P^{\mu\nu}(-z)]}(k)\\ &= ie^2[\hat{P}^{\nu\mu}(k) - \hat{P}^{\mu\nu}(-k)]\\ &= ie^2\frac{k^2\pi}{6(2\pi)^4}\Big(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu}\Big)\Big(1 - \frac{4m^2}{k^2}\Big)^{\frac{3}{2}}[\Theta(k^0) - \Theta(-k^0)]\Theta((k^0)^2 - 4m^2)\\ &= -\frac{ie^2k^2\pi}{6(2\pi)^4}\Big(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu}\Big)\Big(1 - \frac{4m^2}{k^2}\Big)^{\frac{3}{2}}Sgn(k^0)\Theta((k^0)^2 - 4m^2).\end{aligned} \tag{6.16}$$

Since the tensorial part of $D^{\mu\nu}(k)$ does not affect the causal splitting process, we will rewrite (6.16) as follows

$$D^{\mu\nu}(k) = -\frac{ie^2\pi}{6(2\pi)^4}(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu})d^{vac}, \tag{6.17}$$

where the factor d^{vac} is the numerical distribution to split and is equal to

$$d^{vac}(k) = k^2[1 - \frac{4m^2}{k^2}]^{\frac{3}{2}}Sgn(k^0)\Theta((k^0)^2 - 4m^2). \tag{6.18}$$

Now, we need to compute the order of singularity of $d^{vac}(k)$. From (6.18) it is straightforward see that

$$d^{vac}(\frac{k}{\alpha}) = k^2\alpha^{-2}[1 - \frac{4m^2\alpha^2}{k^2}]^{\frac{3}{2}}Sgn(\frac{k^0}{\alpha})\Theta((\frac{k^0}{\alpha})^2 - 4m^2). \tag{6.19}$$

In consequence, for the *power counting function* $\rho(\alpha) = \alpha^2$, the following limit exist

$$\lim_{\alpha\to 0}\rho(\alpha)d^{vac}(\frac{k}{\alpha}) = \lim_{\alpha\to 0}[k^2[1 - \frac{4m^2\alpha^2}{k^2}]^{\frac{3}{2}}Sgn(\frac{k^0}{\alpha})] = k^2Sgn(k^0). \tag{6.20}$$

Therefore, using the relation (3.92), we can see that the order of singularity of d^{vac} is

$$\omega[d^{vac}] = 2, \tag{6.21}$$

which means that the distribution is singular and its retarded part will be given by the formula (3.94). Considering the fact that $d^{vac}(-k) = -d^{vac}(k)$, we can use the

Appendix result (A.31)

$$
\begin{aligned}
r_0^{vac}(k) &= \frac{i}{2\pi} Sgn(k^0) \int_0^\infty dt \frac{2\hat{d}^{vac}(tp)}{t^{\omega+1}(1-t^2+iSgn(k^0)0^+)} \\
&= \frac{i}{2\pi} Sgn(k^0) \int_0^\infty dt \frac{2t^2k^2[1-\frac{4m^2}{t^2k^2}]^{\frac{3}{2}} Sgn(tk^0)\Theta(t^2(k^0)^2-4m^2)}{t^3(1-t^2+iSgn(k^0)0^+)} \\
&= \frac{i}{2\pi} Sgn(k^0) \int_0^\infty dt \frac{2k^2[1-\frac{4m^2}{t^2k^2}]^{\frac{3}{2}} Sgn(tk^0)\Theta(t^2(k^0)^2-4m^2)}{t(1-t^2+iSgn(k^0)0^+)}.
\end{aligned}
\tag{6.22}
$$

Now, because $t > 0$ we have $Sgn(tk^0)Sgn(k^0) = 1$. Furthermore, going to the reference system where $k = (k^0, \mathbf{0})$, and making the change of variable $s = t^2 {k^0}^2$, we can rewrite (6.22) as

$$
\begin{aligned}
r_0^{vac}(k) &= \frac{i}{2\pi} \int_0^\infty dt \frac{2{k^0}^2[1-\frac{4m^2}{t^2k^2}]^{\frac{3}{2}} \Theta(t^2(k^0)^2-4m^2)}{t(1-t^2+iSgn(k^0)0^+)} \\
&= \frac{i}{2\pi} 2{k^0}^2 \int_0^\infty ds \frac{|k^0||k^0|[1-\frac{4m^2}{s}]^{\frac{3}{2}} \Theta(s-4m^2)}{2{k^0}^2\sqrt{s}\sqrt{s}(1-\frac{s}{{k^0}^2}+iSgn(k^0)0^+)} \\
&= \frac{i}{2\pi} {k^0}^2 \int_{4m^2}^\infty ds \frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s(1-\frac{s}{{k^0}^2}+iSgn(k^0)0^+)} \\
&= \frac{i}{2\pi} {k^0}^4 \int_{4m^2}^\infty ds \frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s({k^0}^2-s+iSgn(k^0)0^+)}.
\end{aligned}
\tag{6.23}
$$

Using the Sokhotski–Plemelj formula

$$
\frac{1}{x \pm i0^+} = PV(\frac{1}{x}) \mp i\pi\delta(x),
\tag{6.24}
$$

we can rewrite (6.23) as

$$
\begin{aligned}
r_0^{vac}(k) = \frac{i(k^0)^4}{2\pi} \Big\{ &P.V \int_{4m^2}^\infty ds \frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s((k^0)^2-s)} - \\
&- i\pi Sgn(k^0)\Theta[(k^0)^2-4m^2] \frac{1}{(k^0)^2}[1-\frac{4m^2}{(k^0)^2}]^{\frac{3}{2}} \Big\}.
\end{aligned}
\tag{6.25}
$$

Now we need the intermediate retarded distribution r'^{vac}. The latter is the term coming from $P^{\mu\nu}(z)$ and its expression is

$$r'^{vac}(k) = -k^2[1 - \frac{4m^2}{k^2}]^{\frac{3}{2}}\Theta(-k^0)\Theta[(k^0)^2 - 4m^2]. \tag{6.26}$$

Finally, the numerical part of the two-point distribution $t^{vac}(k)$ is computing as the subtraction

$$\begin{aligned}
t^{vac}(k) &= r_0^{vac}(k) - r'(k) \\
&= \frac{i}{2\pi}(k^0)^4 P.V \int_{4m^2}^{\infty} ds \frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s((k^0)^2 - s)} + \frac{1}{2}(k^0)^2 Sgn(k^0)\Theta[(k^0)^2 - 4m^2][1 - \frac{4m^2}{(k^0)^2}]^{\frac{3}{2}}\Big\} \\
&\quad - [-k^{02}[1 - \frac{4m^2}{k^2}]^{\frac{3}{2}}\Theta(-k^0)\Theta[(k^0)^2 - 4m^2]] \\
&= \frac{i}{2\pi}(k^0)^4 P.V \int_{4m^2}^{\infty} ds \frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s((k^0)^2 - s)} + \\
&\quad + \frac{1}{2}(k^0)^2\{Sgn(k^0) + 2\Theta(-k^0)\}\Theta[(k^0)^2 - 4m^2][1 - \frac{4m^2}{(k^0)^2}]^{\frac{3}{2}}\Big\} \\
&= \frac{i}{2\pi}(k^0)^4 P.V \int_{4m^2}^{\infty} ds \frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s((k^0)^2 - s)} + \\
&\quad + \frac{1}{2}(k^0)^2\{\Theta(k^0) + \Theta(-k^0)\}\Theta[(k^0)^2 - 4m^2][1 - \frac{4m^2}{(k^0)^2}]^{\frac{3}{2}}\Big\} \\
&= \frac{i}{2\pi}(k^0)^4 P.V \int_{4m^2}^{\infty} ds \frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s((k^0)^2 - s)} + \frac{1}{2}(k^0)^2[1 - \frac{4m^2}{(k^0)^2}]^{\frac{3}{2}}\Theta[(k^0)^2 - 4m^2] \\
&= \frac{i}{2\pi}(k^0)^4 \int_{4m^2}^{\infty} ds \frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s((k^0)^2 - s)}.
\end{aligned} \tag{6.27}$$

With (6.27), we can write the more general solution $\widetilde{t}^{vac}$ for the two point distribution. Because of its singular order $\omega = 2$, $\widetilde{t}^{vac}$ has the following form

$$\widetilde{t}^{vac}(k) = \frac{i}{2\pi}(k^0)^4 \int_{4m^2}^{\infty} ds \frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s((k^0)^2 - s)} + c_0 + c_\alpha k^\alpha + c_2 k^2. \tag{6.28}$$

Finally, with $\widetilde{t}^{vac}$ we can write the vacuum polarization tensor $\Pi^{\mu\nu}(k)$ as

$$\begin{aligned}\hat{\Pi}^{\mu\nu}(k) &= -\frac{ie^2\pi}{6(2\pi)^4}(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu})\widetilde{t}^{vac} \\ &= \Big\{\frac{e^2(k^0)^4}{12(2\pi)^4}(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu})\int\limits_{4m^2}^{\infty} ds\frac{[1-\frac{4m^2}{s}]^{\frac{3}{2}}}{s((k^0)^2 - s)}\Big\} + C_0 + C_\alpha k^\alpha + C_2 k^2.\end{aligned} \tag{6.29}$$

To obtain the constants C_0 and C_2 we need to determine the complete photon propagator. This is possible if the order of singularity of the distributions associated with two and more vacuum polarizations insertions have the same value. We will see this in next Chapter.

6.2 Self-Energy

To study the self-energy function of scalar DKP particle, we start with the causal distribution (5.60) which we rewrite here as follows

$$\begin{aligned}D_2^{(4)} &= -e^2 g_{\mu\nu} : \overline{\psi}(x)\beta^\nu[-S^{(-)}(x-y)D_0^{(-)}(x-y) + S^{(+)}(x-y)D_0^{(+)}(x-y)]\beta^\mu\psi(y) : \\ &+ e^2 g_{\mu\nu} : \overline{\psi}(y)\beta^\mu[S^{(+)}(y-x)D_0^{(+)}(y-x) - S^{(-)}(y-x)D_0^{(-)}(y-x)]\beta^\nu\psi(x) : .\end{aligned} \tag{6.30}$$

For future uses, we rewrite too the intermediate distribution $R'^{(4)}(x,y)$

$$\begin{aligned}R_2'^{(4)}(y,x) &= e^2 g_{\mu\nu} : \overline{\psi}(x)\beta^\nu S^{(-)}(x-y)D_0^{(-)}(x-y)\beta^\mu\psi(y) : \\ &+ e^2 g_{\mu\nu} : \overline{\psi}(y)\beta^\mu S^{(+)}(y-x)D_0^{(+)}(y-x)\beta^\nu\psi(x) : .\end{aligned} \tag{6.31}$$

Again, to compute the order of singularity, we will work in the momentum space, therefore we must determine the Fourier transform of the numerical parts in (6.30). To do the computation, we introduce the functions $A(z)$ and $B(z)$ as follows

$$A(x-y) = A(z) = S^{(-)}(z)D_0^{(-)}(z), \tag{6.32}$$

$$B(x-y) = B(z) = S^{(+)}(z)D_0^{(+)}(z), \tag{6.33}$$

where it is easy to note that if we obtain the Fourier transform of these functions, then we could evaluate the Fourier transform required.

Replacing (4.14) and (4.55) in (6.32), the Fourier transform $\hat{A}(k)$ has the following form

$$\begin{aligned}
\hat{A}(k) = (2\pi)^{-2}\int d^4z\Big\{\frac{-i}{m(2\pi)^3}\int d^4p\delta(p^2-m^2)\Theta(p^0)[\not{p}(\not{p}-m)]_{ab}e^{ipz}\Big\}\times \\
\times\Big\{\frac{-i}{(2\pi)^3}\int d^4q\delta(q^2)\Theta(-q^0)e^{-iqz}\Big\}e^{izk} \\
= -\frac{m}{m(2\pi)^4}\int d^4p\delta(p^2-m^2)\Theta(-p^0)[\not{p}]\delta((k-p)^2)\Theta(p^0-k^0)- \\
-\frac{1}{m(2\pi)^4}\int d^4p\delta(p^2-m^2)\Theta(-p^0)[\not{p}\not{p}]_{ab}\delta((k-p)^2)\Theta(p^0-k^0) \\
= A_1(k)+A_2(k)
\end{aligned} \tag{6.34}$$

where

$$\hat{A}_1(k) = -\frac{m}{m(2\pi)^4}\int d^4p\delta(p^2-m^2)\Theta(-p^0)\not{p}\delta((k-p)^2)\Theta(p^0-k^0), \tag{6.35}$$

$$\hat{A}_2(k) = -\frac{1}{m(2\pi)^4}\int d^4p\delta(p^2-m^2)\Theta(-p^0)\not{p}\not{p}\delta((k-p)^2)\Theta(p^0-k^0). \tag{6.36}$$

To compute the integral $\hat{A}_1(k)$, we will separate the term proportional to p^0 in the contraction $\not{p}=\beta_0p^0+\beta_ip^i$ obtaining

$$\begin{aligned}
\hat{A}_1(k) = -\frac{m}{m(2\pi)^4}\beta_\nu\int d^4p\delta(p^2-m^2)\Theta(-p^0)p^\nu\delta((k-p)^2)\Theta(p^0-k^0) \\
= -\frac{m}{m(2\pi)^4}\beta_0\int d^4p\delta(p^2-m^2)\Theta(-p^0)p^0\delta((k-p)^2)\Theta(p^0-k^0)- \\
-\frac{m}{m(2\pi)^4}\beta_i\int d^4p\delta(p^2-m^2)\Theta(-p^0)p^i\delta((k-p)^2)\Theta(p^0-k^0).
\end{aligned} \tag{6.37}$$

The second integral in the right hand side of (6.37) is null because p^i makes anti-symmetric the integrand. To calculate the first one, we will use a reference system where $k=(k^0,\mathbf{0})$, thus we have

$$\begin{aligned}
\hat{A}_1(k) = -\frac{m}{m(2\pi)^4}\beta_0\int d^4p\delta(p^2-m^2)\Theta(-p^0)p^0\delta((k-p)^2)\Theta(p^0-k^0) \\
= \frac{m}{m(2\pi)^4}\frac{1}{2}\beta_0\int d^3p\delta((k^0)^2+2k^0E_{\mathbf{p}}+m^2)\Theta(-E_{\mathbf{p}}-k^0).
\end{aligned} \tag{6.38}$$

From the last delta and step function in (6.38), we can compute $|\mathbf{p}|$ as

$$k^0 = -E_p-|\mathbf{p}| \Rightarrow |\mathbf{p}| = \frac{m^2-(k^0)^2}{2k^0}, \tag{6.39}$$

then, using spherical coordinates, we obtain

$$\begin{aligned}\hat{A}_1(k) &= \frac{m}{m(2\pi)^4}\frac{1}{2}\beta_0 4\pi \int \mathbf{p}^2 d|\mathbf{p}| \frac{1}{2|k^0|}\delta(E_{\mathbf{p}} + \frac{m^2+(k^0)^2}{2k^0}) \\ &= \frac{m}{m(2\pi)^4}\frac{1}{2}\beta_0 4\pi \int E_{\mathbf{p}}|\mathbf{p}|dE_{\mathbf{p}} \frac{1}{2|k^0|}\delta(E_{\mathbf{p}} + \frac{m^2+(k^0)^2}{2k^0}) \\ &= \frac{m}{m(2\pi)^4}\frac{1}{2}\beta_0 4\pi \frac{1}{2|k^0|}\int E_{\mathbf{p}}(\frac{m^2-(k^0)^2}{2k^0})dE_{\mathbf{p}}\delta(E_{\mathbf{p}} + \frac{m^2+(k^0)^2}{2k^0}) \\ &= \frac{1}{(4\pi)^3}(\frac{m^2}{(k^0)^2}-1)(\frac{m^2}{(k^0)^2}+1)\not{k}\Theta(k^2-m^2)\Theta(-k^0).\end{aligned} \tag{6.40}$$

Now, we will compute $\hat{A}_2(k)$ which can be rewritten as

$$A_2(k) = -\frac{1}{m(2\pi)^4}\int d^4p\delta(p^2-m^2)\Theta(-p^0)\beta_\mu p^\mu \beta_\nu p^\nu \delta((k-p)^2)\Theta(p^0-k^0). \tag{6.41}$$

For the same parity reasons of the integrand, we can show that for $\mu \neq \nu$ the integral is null. For example

$$A_2(k) = \ldots - \frac{1}{m(2\pi)^4}\beta_1\beta_3 \int d^3p \frac{1}{2E(p)} p^1 p^3 \delta((k^0+E(p))^2 - \mathbf{p}^2)\Theta(p^0+E(p))\ldots \tag{6.42}$$

Using the particular representation (4.21) of the β-matrices, we can write the integral (6.41) as

$$\begin{aligned}A_2(k) &= -\frac{1}{m(2\pi)^4}\beta^\mu\beta_\nu \int d^4p\delta(p^2-m^2)\Theta(-p^0)p_\mu p^\nu \delta((k-p)^2)\Theta(p^0-k^0)\} \\ &= -\frac{1}{m(2\pi)^4}\int d^4p\delta(p^2-m^2)\Theta(-p^0)\begin{pmatrix} p_0p^0 & 0 & 0 & 0 & 0 \\ 0 & p_1p^1 & 0 & 0 & 0 \\ 0 & 0 & p_2p^2 & 0 & 0 \\ 0 & 0 & 0 & p_3p^3 & 0 \\ 0 & 0 & 0 & 0 & p_\nu p^\nu \end{pmatrix} \times \\ &\quad \times \delta((k-p)^2)\Theta(p^0-k^0)\}.\end{aligned} \tag{6.43}$$

Also in that representation, just the entry $(5,5)$ is relevant after multiplication with the fields Ψ and $\overline{\Psi}$. Therefore, using the following result

$$\frac{1}{4}\beta_\nu\beta^\nu p^2 = \begin{pmatrix} \frac{p^2}{4} & 0 & 0 & 0 & 0 \\ 0 & \frac{p^2}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{p^2}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{p^2}{4} & 0 \\ 0 & 0 & 0 & 0 & p^2 \end{pmatrix}, \tag{6.44}$$

we can write[1] (6.43) as follows

$$A_2(k) = -\frac{1}{m(2\pi)^4}\frac{\beta^\nu\beta_\nu}{4}\int d^4p\delta(p^2-m^2)\Theta(-p^0)p^2\delta((k-p)^2)\Theta(p^0-k^0). \tag{6.45}$$

From the delta and step functions of (6.45), we could obtain the following properties

$$E_{\mathbf{p}} = -k^0 - |\mathbf{p}| \Rightarrow |\mathbf{p}| = \frac{m^2-(k^0)^2}{2k^0}, \quad m^2-(k^0)^2 < 0. \tag{6.46}$$

With the help of (6.46), the integral (6.45) can be evaluated in spherical coordinates and a reference system where $k = (k^0, \mathbf{0})$ as follows

$$\begin{aligned}
A_2(k) &= -\frac{1}{m(2\pi)^4}\frac{\beta^\nu\beta_\nu}{4}\int d^4p\delta(p^2-m^2)\Theta(-p^0)p^2\delta(k^2-2k.p+p^2)\Theta(p^0-k^0)\\
&= -\frac{1}{m(2\pi)^4}m^2\frac{\beta^\nu\beta_\nu}{4}\int d^3p\frac{1}{2E_{\mathbf{p}}}\delta((k^0)^2+2k^0.E_{\mathbf{p}}+m^2)\Theta(-E_{\mathbf{p}}-k^0)\\
&= -\frac{1}{m(2\pi)^4}m^2\Theta((k^0)^2-m^2)\Theta(-k^0)\frac{\beta^\nu\beta_\nu}{4}(4\pi)\times\\
&\quad\times\int \mathbf{p}^2 d|\mathbf{p}|\frac{1}{2E_{\mathbf{p}}}\delta((k^0)^2+2k^0.E_{\mathbf{p}}+m^2)\\
&= -\frac{1}{m(2\pi)^4}m^2\Theta((k^0)^2-m^2)\Theta(-k^0)\frac{\beta^\nu\beta_\nu}{4}(4\pi)\frac{1}{2}\frac{1}{2|k^0|}\frac{m^2-(k^0)^2}{2k^0}\times\\
&\quad\times\int dE_{\mathbf{p}}\delta(\frac{(k^0)^2+m^2}{2k^0}+E_{\mathbf{p}})\\
&= -\frac{m}{2(4\pi)^3}\Theta((k^0)^2-m^2)\Theta(-k^0)\beta^\nu\beta_\nu(1-\frac{m^2}{(k^0)^2}).
\end{aligned} \tag{6.47}$$

Replacing (6.40) and (6.47) in (6.34), we obtain for $\hat{A}(k)$ the following result

$$\hat{A}(k) = \Theta(k^2-m^2)\Theta(-k^0)\frac{1}{(4\pi)^3}(\frac{m^2}{(k^0)^2}-1)\Big\{(\frac{m^2}{k^2}+1)\not{k}+\frac{m}{2}\beta^\nu\beta_\nu\Big\}, \tag{6.48}$$

where the return of a general reference system has been done.

Now, we will compute the Fourier transform of $B(z)$. Replacing the explicit forms

[1]Other form to solve the integral is noting that it needs to be proportional to $g^{\mu\nu}$ to have Lorentz invariance.

of $S^{(+)}(z)$ and $D_0^{(+)}(z)$ into (6.33), we have for $\hat{B}(k)$ the following form

$$\begin{aligned}\hat{B}(k) &= (2\pi)^{-2}\int d^4z S^{(+)}(z) D_0^{(+)}(z) e^{izk}, \quad x-y=z \\ &= (2\pi)^{-2}\int d^4z\{\frac{+i}{m(2\pi)^3}\int \frac{d^3p}{2p^0}[\not{p}(\not{p}+m)]_{ab}e^{-ipz}\}\{\frac{i}{(2\pi)^3}\int\frac{d^3q}{2q^0}e^{-iqz}\}e^{izk} \\ &= -\frac{1}{m(2\pi)^4}\{\int d^4p\delta(p^2-m^2)\Theta(p^0)[\not{p}(\not{p}+m)]_{ab}\}\{\delta((k-p)^2)\Theta(k^0-p^0)\} \\ &= -\frac{1}{m(2\pi)^4}\{\int d^4p\delta(p^2-m^2)\Theta(p^0)[\not{p}(m)]_{ab}\}\{\delta((k-p)^2)\Theta(k^0-p^0)\} \\ &\quad -\frac{1}{m(2\pi)^4}\{\int d^4p\delta(p^2-m^2)\Theta(p^0)[\not{p}(\not{p})]_{ab}\}\{\delta((k-p)^2)\Theta(k^0-p^0)\} \\ &= \hat{B}_1(k)+\hat{B}_2(k),\end{aligned} \tag{6.49}$$

where $B_1(k)$ and $B_2(k)$ are

$$\hat{B}_1(k) = -\frac{1}{(2\pi)^4}\beta_\mu\int d^4p p^\mu\delta(p^2-m^2)\Theta(p^0)\delta((k-p)^2)\Theta(k^0-p^0), \tag{6.50}$$

$$\hat{B}_2(k) = -\frac{1}{m(2\pi)^4}\beta_\nu\beta_\mu\int d^4p\delta(p^2-m^2)\Theta(p^0)p^\nu p^\mu\delta((k-p)^2)\Theta(k^0-p^0). \tag{6.51}$$

The integral (6.50) is zero for $\mu = 1, 2, 3$ because in those cases the integrand is odd and the integral interval is symmetric. The integral for $\mu = 0$ could be done in a reference system where $k = (k^0, \mathbf{0})$, in that case we obtain the following form to solve

$$\hat{B}_1(k) = -\frac{1}{2(2\pi)^4}\beta_0\int d^3p\delta((k^0)^2 - 2k^0E_p + m^2)\Theta(k^0 - E_p). \tag{6.52}$$

Solving the delta function of (6.52), we get that $k^0 = E_p \pm |\mathbf{p}|$. After that, using the two values of k^0 in the step function we could see that just $k^0 = E_p + |\mathbf{p}|$ is not zero, which means that $k^0 > 0$. On the other hand, we can obtain that $|\mathbf{p}| = \frac{k^{02}-m^2}{2k^0}$, therefore $k^{02} - m^2 > 0$. With these properties, we could rewrite the integral (6.52) as

$$\begin{aligned}\hat{B}_1(k) &= -\frac{1}{2(2\pi)^4}\Theta(k^0)\Theta(k^{02}-m^2)\beta_0(4\pi)\int \mathbf{p}^2 d|\mathbf{p}|\delta((k^0)^2-2k^0E_p+m^2) \\ &= -\frac{1}{2(2\pi)^4}\Theta(k^0)\Theta(k^{02}-m^2)\beta_0(4\pi)\frac{k^{02}-m^2}{2k^0}\int dE_{\mathbf{p}}E_{\mathbf{p}}\frac{1}{|2k^0|}\delta(\frac{(k^0)^2+m^2}{2k^0}-E_{\mathbf{p}}) \\ &= -\frac{1}{(4\pi)^3}\Theta(k^0)\Theta(k^2-m^2)(1-\frac{m^2}{k^2})(1+\frac{m^2}{k^2})\not{k}.\end{aligned} \tag{6.53}$$

The computation of the integral $\hat{B}_2(k))$ is similar to the one done for $\hat{A}_2(k)$. Using the criterion of Lorentz invariance, we get the following intermediate result

$$\begin{aligned}\hat{B}_2(k) &= -\frac{1}{m(2\pi)^4}\beta_\nu\beta_\mu\int d^4p\delta(p^2-m^2)\Theta(p^0)p^\nu p^\mu\delta((k-p)^2)\Theta(k^0-p^0)\\ &= -\frac{1}{m(2\pi)^4}\frac{\beta^\mu\beta_\mu}{4}\int d^4p\delta(p^2-m^2)\Theta(p^0)p^2\delta((k-p)^2)\Theta(k^0-p^0)\\ &= -\frac{1}{m(2\pi)^4}\frac{\beta^\mu\beta_\mu}{4}\int d^3p(m^2)\frac{1}{2E_{\mathbf{p}}}\delta((k^0)^2-2k^0E_{\mathbf{p}}+m^2)\Theta(k^0-E_{\mathbf{p}}).\end{aligned} \tag{6.54}$$

From the delta and step functions in (6.54), we can determine $|\mathbf{p}|$ as

$$k^0 = E_{\mathbf{p}} + |\mathbf{p}| \Rightarrow |\mathbf{p}| = \frac{m^2+(k^0)^2}{2k^0}, \quad m^2+(k^0)^2>0. \tag{6.55}$$

Using spherical coordinates, we obtain

$$\begin{aligned}\hat{B}_2(k) &= -\frac{m}{2(2\pi)^4}\frac{\beta^\mu\beta_\mu}{4}\Theta(k^0)\Theta(k^2-m^2)(4\pi)\int \mathbf{p}^2 d|\mathbf{p}|\frac{1}{E_p}\delta((k^0)^2-2k^0E_p+m^2)\\ &= -\frac{m}{2(2\pi)^4}\frac{\beta^\mu\beta_\mu}{4}\Theta(k^0)\Theta(k^2-m^2)(4\pi)\int\frac{k^{02}-m^2}{2k^0}dE_p\delta((k^0)^2-2k^0E_p+m^2)\\ &= -\frac{m}{2(2\pi)^4}\frac{\beta^\mu\beta_\mu}{4}\Theta(k^0)\Theta(k^2-m^2)(4\pi)\frac{k^{02}-m^2}{2k^0}\int dE_p\delta((k^0)^2-2k^0E_p+m^2)\\ &= -\frac{m}{2(4\pi)^3}\beta^\mu\beta_\mu\Theta(k^0)\Theta(k^2-m^2)(1-\frac{m^2}{k^2}).\end{aligned} \tag{6.56}$$

Replacing (6.53) and (6.56) into (6.49), we get

$$\begin{aligned}\hat{B}(k) &= \hat{B}_1(k)+\hat{B}_2(k)\\ &= -\frac{1}{(4\pi)^3}\Theta(k^0)\Theta(k^2-m^2)(1-\frac{m^2}{k^2})\Big\{(1+\frac{m^2}{k^2})\not{k}+\frac{m}{2}\beta^\mu\beta_\mu\Big\}.\end{aligned} \tag{6.57}$$

With the computation of $\hat{B}(k)$ and $\hat{A}(k)$, we will return to study the numerical parts of (6.30) that we rewrite here as follows

$$\begin{aligned}D_2^{(4)} &= ie^2 :\overline{\psi}(x)i\beta_\mu[-S^{(-)}(x-y)D_0^{(-)}(x-y)+S^{(+)}(x-y)D_0^{(+)}(x-y)]\beta^\mu\psi(y):\\ &+ ie^2:\overline{\psi}(y)(-i)\beta_\mu[S^{(+)}(y-x)D_0^{(+)}(y-x)-S^{(-)}(y-x)D_0^{(-)}(y-x)]\beta^\mu\psi(x):\\ &= i:\overline{\psi}(x)D_I^{Self}(x-y)\psi(y):+i:\overline{\psi}(y)D_{II}^{Self}(x-y)\psi(x):,\end{aligned} \tag{6.58}$$

where D_I^{Self} and D_{II}^{Self} are the numerical parts which we have to study in order to obtain their order of singularity. Using the expressions of $A(z)$ and $B(z)$, D_I^{Self} and D_{II}^{Self} could be written as follows

$$\begin{aligned} D_I^{Self}(x-y) &= ie^2\beta_\mu[-S^{(-)}(x-y)D_0^{(-)}(x-y)+S^{(+)}(x-y)D_0^{(+)}(x-y)]\beta^\mu \\ &= ie^2\beta_\mu[-A(x-y)+B(x-y)]\beta^\mu, \end{aligned} \tag{6.59}$$

$$\begin{aligned} D_{II}^{Self}(x-y) &= (-i)e^2\beta_\mu[S^{(+)}(y-x)D_0^{(+)}(y-x)-S^{(-)}(y-x)D_0^{(-)}(y-x)]\beta^\mu \\ &= (-i)e^2\beta_\mu[B(y-x)-A(y-x)]\beta^\mu. \end{aligned} \tag{6.60}$$

After that, we can take the Fourier transform of (6.59) and (6.60). For $\hat{D}_I^{Self}(k)$, we have

$$\begin{aligned} \hat{D}_I^{Self}(k) &= i\beta_\alpha[-\hat{A}(k)+\hat{B}(k)]\beta^\alpha \\ &= i\beta_\alpha[-\Theta(k^2-m^2)\Theta(-k^0)\frac{1}{(4\pi)^3}(\frac{m^2}{(k^0)^2}-1)\Big\{(\frac{m^2}{k^2}+1)\not{k}+\frac{m}{2}\beta^\nu\beta_\nu\Big\} \\ &\quad -\frac{1}{(4\pi)^3}\Theta(k^0)\Theta(k^2-m^2)(1-\frac{m^2}{k^2})\Big\{(1+\frac{m^2}{k^2})\not{k}+\frac{m}{2}\beta^\mu\beta_\mu\Big\}]\beta^\alpha \\ &= i\beta_\alpha[\frac{1}{(4\pi)^3}Sgn(k^0)\Theta(k^2-m^2)(\frac{m^2}{k^2}-1)\Big\{(1+\frac{m^2}{k^2})\not{k}+\frac{m}{2}\beta^\mu\beta_\mu\Big\}]\beta^\alpha, \end{aligned} \tag{6.61}$$

where (6.48) and (6.57) are used.

Now, using the follow properties of β-matrices

$$\beta^\mu\beta^\nu\beta_\mu = \beta^\nu, \tag{6.62}$$

$$\beta^\mu\beta^\nu\beta_\nu\beta_\mu = 4, \tag{6.63}$$

we finally get

$$\begin{aligned} \hat{D}_I^{Self}(k) &= i\frac{1}{(4\pi)^3}Sgn(k^0)\Theta(k^2-m^2)(\frac{m^2}{k^2}-1)\Big\{(1+\frac{m^2}{k^2})\beta_\alpha\not{k}\beta^\alpha+\frac{m}{2}\beta_\alpha\beta^\mu\beta_\mu\beta^\alpha\Big\} \\ &= i\frac{1}{(4\pi)^3}Sgn(k^0)\Theta(k^2-m^2)(\frac{m^2}{k^2}-1)\Big\{(1+\frac{m^2}{k^2})\not{k}+2m\Big\}. \end{aligned} \tag{6.64}$$

On the other hand, we can see that

$$D_{II}^{Self}(z) = -D_I^{Self}(-z), \tag{6.65}$$

then

$$\begin{aligned} \hat{D}_{II}^{Self}(k) &= -\hat{D}_I^{Self}(-k) \\ &= i[\frac{1}{(4\pi)^3}Sgn(k^0)\Theta(k^2-m^2)(\frac{m^2}{k^2}-1)\Big\{-(1+\frac{m^2}{k^2})\not{k}+2m\Big\}]. \end{aligned} \tag{6.66}$$

With $\hat{D}_I^{Self}(k)$ and $\hat{D}_{II}^{Self}(k)$ computed, we have to calculate their order of singularity. To achieve these objectives, we must analyze the form of $\hat{D}_I^{Self}(\frac{k}{\alpha})$ that we write bellow

$$\begin{aligned}\hat{D}_I^{Self}(\frac{k}{\alpha}) = i\frac{1}{(4\pi)^3}Sgn(\frac{k^0}{\alpha})\Theta(\alpha^{-2}k^2 - m^2)(\frac{m^2\alpha^2}{k^2} - 1)\times \\ \times\left\{(1 + \frac{\alpha^2 m^2}{k^2})\alpha^{-1}\not{k} + 2m\right\}.\end{aligned} \tag{6.67}$$

It is straightforward to see that with the *power counting function* $\rho(\alpha) = \alpha$, we can obtain the non null limit

$$\begin{aligned}\lim_{\alpha\to 0^+}\rho(\alpha)\hat{D}_I^{Self}(\frac{k}{\alpha}) &= \lim_{\alpha\to 0^+} i\alpha[\frac{1}{(4\pi)^3}Sgn(\frac{k^0}{\alpha})\Theta(\alpha^{-2}k^2 - m^2)(\frac{m^2\alpha^2}{k^2} - 1)\times \\ &\times\left\{(1 + \frac{\alpha^2 m^2}{k^2})\alpha^{-1}\not{k} + 2m\right\}] \\ &= -i\frac{1}{(4\pi)^3}Sgn(k^0)(\not{k} + 2m) \neq 0.\end{aligned} \tag{6.68}$$

Using (3.92), we could see that the order of singularity for $\hat{D}_I^{Self}(k)$ and $\hat{D}_{II}^{Self}(k)$ are

$$\omega[\hat{D}_I^{Self}(k)] = \omega[\hat{D}_{II}^{Self}(k)] = 1. \tag{6.69}$$

Therefore, retarded parts could be obtained using the equivalent formula of (3.94) in a special reference system where $k = (k^0, \mathbf{0})$

$$\begin{aligned}\widehat{r}^{Self}(k^0) &= \frac{i}{(2\pi)}(k^0)^{\omega+1}\int dq^0\left[\frac{1}{(q^0 - i0)^{\omega+1}(k^0 - q^0 + i0)}\right]\widehat{d}(q^0). \\ &= \frac{i}{(2\pi)}(k^0)^2\int dq^0\left[\frac{1}{(q^0 - i0)^2(k^0 - q^0 + i0)}\right]\widehat{d}(q^0).\end{aligned} \tag{6.70}$$

Working first with $D_I^{Self}(k)$, we have the following computation

$$\begin{aligned}
\widehat{r}_I^{Self}(k^0) &= \frac{i}{(2\pi)}(k^0)^2 \int dq^0 \left[\frac{1}{(q^0 - i0)^2(k^0 - q^0 + i0)}\right] \hat{D}_I^{Self}(q^0) \\
&= \frac{i}{(2\pi)}(k^0)^2 \int dq^0 \left[\frac{1}{(q^0 - i0)^2(k^0 - q^0 + i0)}\right] \{i[\frac{1}{(4\pi)^3} Sgn(q^0) \times \\
&\quad \times \Theta(q^2 - m^2)(\frac{m^2}{q^2} - 1)\Big\{(1 + \frac{m^2}{q^2})\not{q} + 2m\Big\}]\} \\
&= -\frac{mk^{0^2}}{64\pi^4} \int dq^0 \left[\frac{1}{(q^0)^2(k^0 - q^0 + i0)}\right] Sgn(q^0)\Theta(q^2 - m^2)(\frac{m^2}{q^2} - 1) - \\
&\quad - \frac{k^{0^2}}{128\pi^4} \int dq^0 \left[\frac{1}{(q^0)^2(k^0 - q^0 + i0)}\right] Sgn(q^0)\Theta(q^2 - m^2)(\frac{m^2}{q^2} - 1)(1 + \frac{m^2}{q^2})\not{q} \\
&= r_1(k^0) + r_2(k^0),
\end{aligned} \tag{6.71}$$

where

$$r_1(k^0) = -\frac{mk^{0^2}}{64\pi^4} \int dq^0 \left[\frac{1}{(q^0)^2(k^0 - q^0 + i0)}\right] Sgn(q^0)\Theta(q^2 - m^2)(\frac{m^2}{q^2} - 1), \tag{6.72}$$

$$\begin{aligned}
r_2(k^0) = -\frac{k^{0^2}}{128\pi^4} \int dq^0 \left[\frac{1}{(q^0)^2(k^0 - q^0 + i0)}\right] Sgn(q^0) \times \\
\times \Theta(q^2 - m^2)(\frac{m^2}{q^2} - 1)(1 + \frac{m^2}{q^2})\not{q}.
\end{aligned} \tag{6.73}$$

Solving the integral r_1, we have

$$\begin{aligned}
r_1(k^0) &= -\frac{mk^{0^2}}{64\pi^4} \int dq^0 \left[\frac{1}{(q^0)^2(k^0 - q^0 + i0)}\right] Sgn(q^0)\Theta(q^2 - m^2)(\frac{m^2}{q^2} - 1) \\
&= \frac{mk^{0^2}}{64\pi^4} \int_0^\infty dq^0 \frac{1}{(q^0)^2} \Theta(q^{0^2} - m^2)(1 - \frac{m^2}{q^{0^2}}) \left[\frac{2q^0}{(k^{0^2} - q^{0^2} + ik^0 0)}\right] \\
&= \frac{mk^{0^2}}{64\pi^4} \int_0^\infty ds \frac{1}{s} \Theta(s - m^2)(1 - \frac{m^2}{s}) \left[\frac{1}{(k^{0^2} - s + ik^0 0)}\right], q^{0^2} \to s \\
&= \frac{mk^{0^2}}{64\pi^4} \int_{m^2}^\infty ds \frac{1}{s}(1 - \frac{m^2}{s}) \left[\frac{1}{(k^{0^2} - s + ik^0 0)}\right] \Theta(s - m^2) \\
&= r_{11} + r_{12},
\end{aligned} \tag{6.74}$$

where

$$r_{11}(k^0) = \frac{k^{0^2} m}{64(\pi)^4} \int_{m^2}^\infty ds \frac{1}{s} \left[\frac{1}{(k^{0^2} - s + ik^0 0)}\right] \Theta(s - m^2), \tag{6.75}$$

$$r_{12}(k^0) = -\frac{k^{0^2} m}{64(\pi)^4} \int_{m^2}^\infty ds \frac{m^2}{s^2} \left[\frac{1}{(k^{0^2} - s + ik^0 0)}\right] \Theta(s - m^2). \tag{6.76}$$

To solve the integral r_{11}, we can use the Sokhotski–Plemelj formula (6.24) as follows

$$\begin{aligned}
r_{11}(k^0) &= \frac{k^{02}m}{64(\pi)^4}\int_{m^2}^{\infty} ds\frac{1}{s}\left[\frac{1}{(k^{02}-s+ik^0 0)}\right]\Theta(s-m^2) \\
&= \frac{k^{02}m}{64(\pi)^4}\int_{m^2}^{\infty} ds\frac{1}{s}\left[PV(\frac{1}{k^{02}-s}) - i\pi Sgn(k^0)\delta(k^{02}-s)\Theta(s-m^2)\right] \\
&= \frac{m}{64(\pi)^4}\left[PV\int_{m^2}^{\infty} ds[\frac{1}{s}+\frac{1}{k^{02}-s}] - i\pi Sgn(k^0)\Theta(k^{02}-m^2)\right] \\
&= \frac{m}{64(\pi)^4}\left[\log\frac{|k^{02}-m^2|}{m^2} - i\pi Sgn(k^0)\Theta(k^{02}-m^2)\right].
\end{aligned} \tag{6.77}$$

Now, solving r_{12} in the similar form, we have

$$\begin{aligned}
r_{12}(k^0) &= -\frac{k^{02}m}{64(\pi)^4}\int_{m^2}^{\infty} ds\frac{1}{s}\frac{m^2}{s}\left[\frac{1}{(k^{02}-s+ik^0 0)}\right]\Theta(s-m^2) \\
&= -\frac{k^{02}m^3}{64(\pi)^4}\int_{m^2}^{\infty} ds\frac{1}{s^2}\left[PV(\frac{1}{k^{02}-s}) - i\pi Sgn(k^0)\delta(k^{02}-s)\Theta(s-m^2)\right] \\
&= -\frac{k^{02}m^3}{64(\pi)^4}\left[PV\int_{m^2}^{\infty}\frac{ds}{k^{02}}[\frac{1}{s^2}+\frac{1}{s(k^{02}-s)}] - i\pi Sgn(k^0)\frac{\Theta(k^{02}-m^2)}{k^{04}}\right] \\
&= -\frac{m^3}{64(\pi)^4}\left[\frac{1}{m^2}+\frac{1}{k^{02}}\log\frac{|k^{02}-m^2|}{m^2} - i\pi Sgn(k^0)\frac{1}{k^{02}}\Theta(k^{02}-m^2)\right] \\
&= \frac{m}{64(\pi)^4}\left[-1+\frac{m^2}{k^{02}}\log\frac{m^2}{|k^{02}-m^2|} + i\pi Sgn(k^0)\frac{m^2}{k^{02}}\Theta(k^{02}-m^2)\right].
\end{aligned} \tag{6.78}$$

Replacing (6.77) and (6.78) into (6.74), we get

$$\begin{aligned}
r_1(k^0) &= r_{11}+r_{12} \\
&= \frac{m}{64(\pi)^4}\left[\log\frac{|k^{02}-m^2|}{m^2} - i\pi Sgn(k^0)\Theta(k^{02}-m^2)\right] \\
&\quad + \frac{m}{64(\pi)^4}\left[-1+\frac{m^2}{k^{02}}\log\frac{m^2}{|k^{02}-m^2|} + i\pi Sgn(k^0)\frac{m^2}{k^{02}}\Theta(k^{02}-m^2)\right] \\
&= \frac{m}{64(\pi)^4}\left[-1+(1-\frac{m^2}{k^{02}})\log\frac{|k^{02}-m^2|}{m^2} - (1-\frac{m^2}{k^{02}})i\pi Sgn(k^0)\Theta(k^{02}-m^2)\right].
\end{aligned} \tag{6.79}$$

Evaluating $r_2(k^0)$, we obtain

$$\begin{aligned} r_2(k^0) &= -\frac{k^{02}}{128\pi^4}\int dq^0\left[\frac{1}{(q^0)^2(k^0-q^0+i0)}\right]Sgn(q^0)\Theta(q^2-m^2)(\frac{m^2}{q^2}-1)(1+\frac{m^2}{q^2})\not{q} \\ &= -\frac{k^{02}}{128\pi^4}\int_0^\infty dq^0\frac{1}{(q^0)^2}\left[\frac{2k^0}{k^{02}-q^{02}+2k^0 i0}\right]\Theta(q^2-m^2)(\frac{m^4}{q^4}-1)\not{q} \\ &= -\frac{k^{02}}{128\pi^4}\beta_0\int_0^\infty 2dq^0\frac{1}{(q^0)^2}\left[\frac{1}{k^{02}-q^{02}+2k^0 i0}\right]\Theta(q^2-m^2)(\frac{m^4}{q^4}-1)q^0, \quad q^{02}\to s \\ &= -\frac{k^{02}}{128\pi^4}\beta_0\int_{m^2}^\infty ds\frac{1}{s}\left[\frac{1}{k^{02}-s+2k^0 i0}\right](\frac{m^4}{s^2}-1) \\ &= r_{21}(k^0)+r_{22}(k^0), \end{aligned} \tag{6.80}$$

where

$$r_{21}(k^0) = -\frac{1}{(2\pi)}(k^0)^3\frac{1}{(4\pi)^3}\beta_0\int_{m^2}^\infty ds\frac{1}{s}\left[\frac{1}{k^{02}-s+2k^0 i0}\right](-1), \tag{6.81}$$

$$r_{22}(k^0) = -\frac{1}{(2\pi)}(k^0)^3\frac{1}{(4\pi)^3}\beta_0\int_{m^2}^\infty ds\frac{1}{s}\left[\frac{1}{k^{02}-s+2k^0 i0}\right](\frac{m^4}{s^2}). \tag{6.82}$$

The integral (6.81) could be solved as follows

$$\begin{aligned} r_{21}(k^0) &= -\frac{1}{(2\pi)}(k^0)^3\frac{1}{(4\pi)^3}\beta_0\int_{m^2}^\infty ds\frac{1}{s}\left[\frac{1}{k^{02}-s+2k^0 i0}\right](-1) \\ &= \frac{1}{(2\pi)}(k^0)^3\frac{1}{(4\pi)^3}\beta_0\int_{m^2}^\infty ds\frac{1}{s}\left[\frac{1}{k^{02}-s+2k^0 i0}\right] \Leftarrow (6.77) \\ &= \frac{1}{(2\pi)}(k^0)\frac{1}{(4\pi)^3}\beta_0\left[\log\frac{|k^{02}-m^2|}{m^2}-i\pi Sgn(k^0)\Theta(k^{02}-m^2)\right]. \end{aligned} \tag{6.83}$$

Working with (6.82), we obtain

$$\begin{aligned} r_{22}(k^0) &= -\frac{k^{03}m^4}{128\pi^4}\beta_0\int_{m^2}^\infty ds\frac{1}{s^3}\left[\frac{1}{k^{02}-s+2k^0 i0}\right] \\ &= -\frac{k^{03}m^4}{128\pi^4}\beta_0\int_{m^2}^\infty ds\frac{1}{s^3}\left[PV(\frac{1}{k^{02}-s})-i\pi Sgn(k^0)\delta(k^{02}-s)\Theta(s-m^2)\right] \\ &= -\frac{k^0}{128\pi^4}\beta_0\left[\frac{1}{2}+\frac{m^2}{k^{02}}+\frac{m^4}{k^{04}}(\log\frac{|k^{02}-m^2|}{m^2}-i\pi Sgn(k^0)\Theta(k^{02}-m^2))\right] \end{aligned} \tag{6.84}$$

Replacing (6.84) and (6.83) into (6.80), we have

$$\begin{aligned} r_2(k^0) &= r_{21}(k^0) + r_{22}(k^0) \\ &= -\frac{k^0}{128\pi^4}\beta_0\Bigg[\frac{1}{2} + \frac{m^2}{k^{0^2}} + \\ &\quad + (\frac{m^4}{k^{0^4}} - 1)(\log\frac{|k^{0^2} - m^2|}{m^2} - i\pi Sgn(k^0)\Theta(k^{0^2} - m^2))\Bigg] \end{aligned} \tag{6.85}$$

With r_1 and r_2 given by the expressions (6.79) and (6.85), we obtain for $\widehat{r}_I^{Self}(k^0)$ the following form

$$\begin{aligned} &\widehat{r}_I^{Self}(k^0) = \\ &= r_1(k^0) + r_2(k^0) \\ &= \frac{1}{64(\pi)^4}m\left[-1 + (1 - \frac{m^2}{k^{0^2}})\log\frac{|k^{0^2} - m^2|}{m^2} - (1 - \frac{m^2}{k^{0^2}})i\pi Sgn(k^0)\Theta(k^{0^2} - m^2)\right] \\ &\quad - \frac{1}{(2\pi)}k^0\frac{1}{(4\pi)^3}\beta_0\left[\frac{1}{2} + \frac{m^2}{k^{0^2}} + (\frac{m^4}{k^{0^4}} - 1)(\log\frac{|k^{0^2} - m^2|}{m^2} - i\pi Sgn(k^0)\Theta(k^{0^2} - m^2))\right] \\ &= \frac{1}{64(\pi)^4}\left[-m + m(1 - \frac{1}{b^2})(\log|b^2 - 1| - i\pi Sgn(k^0)\Theta(k^{0^2} - m^2))\right] \\ &\quad + \frac{1}{64(\pi)^4}\left[-\frac{\not k}{4} - \frac{m^2\not k}{2k^{0^2}} - \frac{\not k}{2}(\frac{1}{b^4} - 1)(\log|b^2 - 1| - i\pi Sgn(k^0)\Theta(k^{0^2} - m^2))\right] \\ &= \frac{1}{64(\pi)^4}\Bigg[-m - \frac{\not k}{4} - \frac{\not k}{2b^2} + \left\{m(1 - \frac{1}{b^2}) - \frac{\not k}{2}(\frac{1}{b^4} - 1)\right\}\times \\ &\quad \times\left(\log|b^2 - 1| - i\pi Sgn(k^0)\Theta(k^{0^2} - m^2)\right)\Bigg], \end{aligned} \tag{6.86}$$

where in the last line we return to a general reference system and $b^2 = \frac{k^2}{m^2}$.

The intermediate distribution for that part is the term of (6.61) proportional to $A(k)$, that term is given by

$$r_I'^{Self}(k) = -i[\frac{1}{(4\pi)^3}\Theta(-k^0)\Theta(k^2 - m^2)(\frac{m^2}{k^2} - 1)\Big\{(1 + \frac{m^2}{k^2})\not k + 2m\Big\}]. \tag{6.87}$$

Therefore, numerical part of the 2-point distribution $T_I(x, y)$ in momentum space is given by the subtraction

$$
\begin{aligned}
t_I^{Self}(k) &= \widehat{r}_I^{Self}(k) - r_I'^{Self}(k) \\
&= \frac{1}{64(\pi)^4}\Bigg[- m - \frac{\not k}{4} - \frac{\not k}{2b^2} + \Big\{ m(1-\frac{1}{b^2}) - \frac{\not k}{2}(\frac{1}{b^4}-1)\Big\} \times \\
&\times \Big(\log|b^2-1| - i\pi Sgn(k^0)\Theta({k^0}^2 - m^2)\Big)\Bigg] - \\
&- \{-i[\frac{1}{(4\pi)^3}\Theta(-k^0)\Theta(k^2-m^2)(\frac{m^2}{k^2}-1)\Big\{(1+\frac{m^2}{k^2})\not k + 2m\Big\}]\} \\
&= \frac{e^2}{64(\pi)^4}\Bigg[- m - \frac{\not k}{4} - \frac{\not k}{2b^2} + \Big\{ m(1-\frac{1}{b^2}) - \frac{\not k}{2}(\frac{1}{b^4}-1)\Big\} \times \\
&\times \Big(\log|b^2-1| - i\pi \Theta({k^0}^2 - m^2)\Big)\Bigg].
\end{aligned}
\tag{6.88}
$$

Because of the singular order $\omega = 1$, the general solution is given by

$$
\begin{aligned}
\widetilde{t}_I^{Self}(k) = \frac{e^2}{64(\pi)^4}\Bigg[- m - \frac{\not k}{4} - \frac{\not k}{2b^2} + \Big\{ m(1-\frac{1}{b^2}) - \frac{\not k}{2}(\frac{1}{b^4}-1)\Big\} \times \\
\times \Big(\log|b^2-1| - i\pi \Theta({k^0}^2 - m^2)\Big)\Bigg] + C_0 + C_1\not k.
\end{aligned}
\tag{6.89}
$$

Now, we will consider the causal splitting of D_{II}^{self}. Because its order of singularity $\omega = 1$, the retarded part is given by

$$
\begin{aligned}
\widehat{r}_{II}^{Self}(k^0) &= \frac{i}{(2\pi)}(k^0)^2 \int dq^0 \left[\frac{1}{(q^0-i0)^2(k^0-q^0+i0)}\right] \hat{D}_{II}^{self}(q^0) \\
&= \frac{i}{(2\pi)}(k^0)^2 \int dq^0 \left[\frac{1}{(q^0-i0)^2(k^0-q^0+i0)}\right] \times \\
&\times \{i[\frac{1}{(4\pi)^3}Sgn(q^0)\Theta(q^2-m^2)(\frac{m^2}{q^2}-1)\Big\{-(1+\frac{m^2}{q^2})\not q + 2m\Big\}]\} \\
&= -\frac{m{k^0}^2}{(64\pi^4)}\int dq^0 \left[\frac{1}{(q^0)^2(k^0-q^0+i0)}\right] Sgn(q^0)\Theta(q^2-m^2)(\frac{m^2}{q^2}-1) \\
&+ \frac{{k^0}^2}{128\pi^4}\int dq^0 \left[\frac{1}{(q^0)^2(k^0-q^0+i0)}\right] Sgn(q^0)\Theta(q^2-m^2)(\frac{m^2}{q^2}-1)(1+\frac{m^2}{q^2})\not q \\
&= r_1(k^0) - r_2(k^0),
\end{aligned}
\tag{6.90}
$$

where r_1 and r_2 are the same of (6.72) and (6.73). Therefore

$$\begin{aligned}\widehat{r}^{Self}_{II}(k^0) &= r_1(k^0) - r_2(k^0)\\ &= \frac{1}{64(\pi)^4}\Big[- m + \frac{\not k}{4} + \frac{\not k}{2b^2} + \Big\{ m(1-\frac{1}{b^2}) + \frac{\not k}{2}(\frac{1}{b^4}-1)\Big\}\times \\ &\times \Big(\log|b^2-1| - i\pi Sgn(k^0)\Theta(k^{0^2}-m^2)\Big)\Big].\end{aligned} \tag{6.91}$$

The intermediate distribution associated with $\widehat{r}^{Self}_{II}(k^0)$ is given by the term proportional to $B(-z)$ of (6.31). The Fourier transform of that term is

$$\begin{aligned}\widehat{r'}^{Self}_{II}(k^0) &= -D^{Self}_I(-z)\\ &= -i\beta_\alpha[B(-k)]\beta^\alpha\\ &= -i\beta_\alpha[-\frac{1}{(4\pi)^3}\Theta(-k^0)\Theta(k^2-m^2)(1-\frac{m^2}{k^2})\Big\{ - (1+\frac{m^2}{k^2})\not k + \frac{m}{2}\beta^\mu\beta_\mu\Big\}]\beta^\alpha\\ &= i\frac{1}{(4\pi)^3}\Theta(-k^0)\Theta(k^2-m^2)(1-\frac{1}{b^2})\Big\{ - (1+\frac{1}{b^2})\not k + 2m\Big\}\\ &= i\frac{1}{(4\pi)^3}\Theta(-k^0)\Theta(k^2-m^2)\Big\{ - (1-\frac{1}{b^4})\not k + 2m(1-\frac{1}{b^2})\Big\},\end{aligned} \tag{6.92}$$

where the properties (6.62) and (6.63) have been used.

The numerical part of the 2-point distribution $T_{II}(x,y)$ in the momentum space is given by the following subtraction

$$\begin{aligned}t^{Self}_{II}(k) &= \widehat{r}^{Self}_{II}(k) - r'^{Self}_{II}(k)\\ &= \frac{1}{64(\pi)^4}\Big[- m + \frac{\not k}{4} + \frac{\not k}{2b^2} + \Big\{ m(1-\frac{1}{b^2}) + \frac{\not k}{2}(\frac{1}{b^4}-1)\Big\}\times \\ &\times \Big(\log|b^2-1| - i\pi Sgn(k^0)\Theta(k^{0^2}-m^2)\Big)\Big] - \\ &- \{i\frac{1}{(4\pi)^3}\Theta(-k^0)\Theta(k^2-m^2)\Big\{ - (1-\frac{1}{b^4})\not k + 2m(1-\frac{1}{b^2})\Big\}\}\\ &= \frac{e^2}{64(\pi)^4}\Big[- m + \frac{\not k}{4} + \frac{\not k}{2b^2} + \Big\{ m(1-\frac{1}{b^2}) + \frac{\not k}{2}(\frac{1}{b^4}-1)\Big\}\times \\ &\times \Big(\log|b^2-1| - i\pi\Theta(k^{0^2}-m^2)\Big)\Big].\end{aligned} \tag{6.93}$$

Again, because of the order of singularity $\omega = 1$, the general solution is given by the following formula

$$\widetilde{t}_{II}^{Self}(k) = \frac{e^2}{64(\pi)^4}\Big[-m + \frac{\not{k}}{4} + \frac{\not{k}}{2b^2} + \Big\{m(1-\frac{1}{b^2}) + \frac{\not{k}}{2}(\frac{1}{b^4}-1)\Big\}\times \\ \times\Big(\log|b^2-1| - i\pi\Theta(k^{0^2}-m^2)\Big)\Big] + C_3 + C_4\not{k}. \tag{6.94}$$

Using (6.93) we conclude that

$$t_{II}^{Self}(k) = t_I^{Self}(-k). \tag{6.95}$$

Returning to the configuration space, we get

$$\begin{aligned} T_2^{(4)}(x,y) &= i : \overline{\psi}(x)\widetilde{t}_I^{Self}(x-y)\psi(y) : \\ &+ i : \overline{\psi}(y)\widetilde{t}_{II}^{Self}(x-y)\psi(x) : \\ &= i : \overline{\psi}(x)\{t_I^{Self}(x-y) + C_0\delta(x-y) + C_1\not{\partial}^{\mu}\delta(x-y)\}\psi(y) : \\ &+ i : \overline{\psi}(y)\{t_I^{Self}(y-x) + C_3\delta(x-y) + C_4\not{\partial}^{\mu}\delta(x-y)\}\psi(x) : . \end{aligned} \tag{6.96}$$

Using the symmetry property of T_n under the permutation of variables, we see that $C_0 = C_3$ and $C_1 = C_4$. Therefore, we rewrite $T_2^{(4)}(x,y)$ as

$$\begin{aligned} T_2^{(4)}(x,y) &= i : \overline{\psi}(x)\Sigma(x-y)\psi(y) : \\ &+ i : \overline{\psi}(y)\Sigma(y-x)\psi(x) : \end{aligned} \tag{6.97}$$

where

$$\hat{\Sigma}(k) = \frac{e^2}{64(\pi)^4}\Big[-m - \frac{\not{k}}{4} - \frac{\not{k}}{2b^2} + \Big\{m(1-\frac{1}{b^2}) - \frac{\not{k}}{2}(\frac{1}{b^4}-1)\Big\}\times \\ \times\Big(\log|b^2-1| - i\pi\Theta(k^{0^2}-m^2)\Big)\Big] + C_0 + C_1\not{k}. \tag{6.98}$$

Similar to vacuum polarization, to obtain the constants C_0 and C_1, we can determine the propagator loop corrections for a scalar DKP particle. Therefore, we need to prove that the processes with more than two self-energies loop corrections have the same order of singularity. We will see this in the next Chapter.

Chapter 7

(Re)Normalizability of SDKP

> Because it contained speculations too remote from reality to be of interest to the reader.
>
> *Nature Editors to Fermi*

As we saw in Chapter 2, the *causal splitting procedure* for singular distributions left constants that we need to find with the help of physical properties different to causality. Because this procedure occur at each order in the perturbation expansion of $S-matrix$, we can use the singular order to define when a theory is renormalizable, non-renormalizable, and super-renormalizable in the following form:

- We call a theory renormalizable when at each order of perturbation, a finite number of constants appears from the causal splitting procedure.
- The theory is non-renormalizable when the order of singularity increase at higher order of perturbation.
- A super-renormalizable theory implies a finite number of constants for the low order of perturbation expansion.

In this chapter we will investigate the renormalizability of the SDKP theory assuming that the intermediate and causal distributions have the same order of singularity.

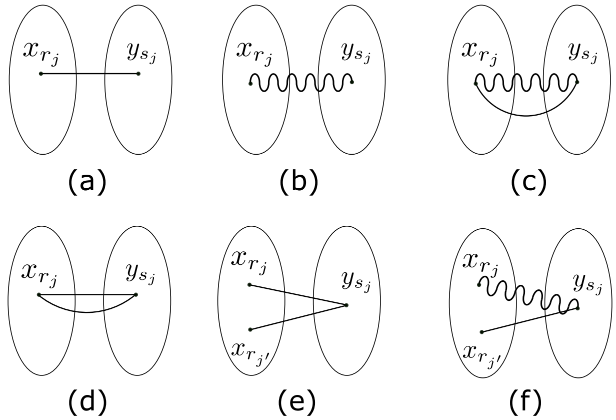


Figure 7.1:

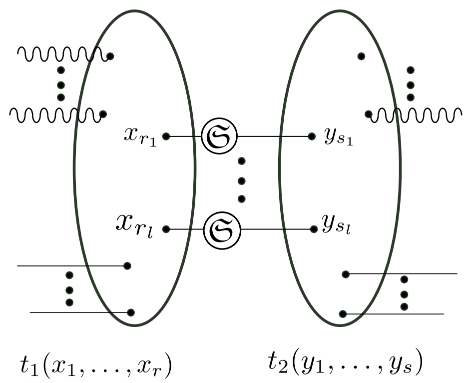


Figure 7.2:

7.1 Order of singularity of the intermediate distributions by an independent contraction

In the computation of intermediate distribution, we have the tensorial product of two n-point distributions T_r and T_s with numerical parts t_1 and t_2, respectively. From Wick theorem, many graphs emerge combining many different kind of contractions. All these graphs could be constructed as combinations of what we called *independent contractions* as shown in Fig. 7.1.

For the purpose of this thesis, we will determine in a general way the order of singularity of a graph coming from l independent contractions of the form (a), (b), (c), and (d) in Fig. 7.1. We will denote by $\mathfrak{S}(x_{r_j} - y_{s_j})$ these contractions as shown in Fig. 7.2.

Before using Wick theorem, the n-point distributions T_r and T_s have the following form

$$T_{r,s} =: \prod_{j=1}^{n} \bar{\psi}(x_{k_j}) t_{1,2}(x_1, \ldots, x_{r,s}) \prod_{j=1}^{n} \psi(x_{n_j}) :: \prod_{j=1}^{m} A(x_{m_j}) :, \tag{7.1}$$

where n represents the number of external $\bar{\psi}$ DKP fields (or ψ DKP fields) and m the number of external electromagnetic four potential fields. Note that we do not write explicitly the Lorentz index, as we will see later, that will not be necessary.

After the l contractions from the tensor product $T_r T_s$, we will obtain the following numerical part

$$t_1(x_1, \ldots, x_r)[\prod_{j}^{l} \mathfrak{S}(x_{r_j} - y_{s_j})] t_2(y_1, \ldots, y_s), \tag{7.2}$$

where $j = 1, \ldots, l$.

Using the translation invariance of t_1 and t_2, we can rewrite (7.2) as

$$t_1(x_1 - x_r, \ldots, x_{r-1} - x_r)[\prod_{j}^{l} \mathfrak{S}(x_{r_j} - y_{s_j})] t_2(y_1 - y_s, \ldots, y_{s-1} - y_s), \tag{7.3}$$

where we can see that there are $4r + 4s - 4$ independent variables.

To evaluate the calculations simpler, we will introduce a new group of variables ξ_J, ρ_I and ρ as

$$\xi_J = x_J - x_r, \quad \rho_I = y_I - y_s, \quad \rho = x_r - y_s, \tag{7.4}$$

where $J = 1, \ldots, r-1$ and $I = 1, \ldots, s-1$. The new set of variables allows us to rewrite the numerical part (7.3) in the following form

$$\begin{aligned} t_1(\xi_1, \ldots, \xi_{r-1})[\prod_{j}^{l} \mathfrak{S}(\xi_{r_j} - \rho_{s_j} + \rho)] t_2(\rho_1, \ldots, \rho_{s-1}) \\ \equiv t(\xi_1, \ldots, \xi_{r-1}, \rho_1, \ldots, \rho_{s-1}, \rho), \end{aligned} \tag{7.5}$$

where t represent the numerical part of the new graph.

Now, we will compute the Fourier transform of $\hat{t}(\vec{\xi}, \vec{\rho}, \rho)$ as

$$\begin{aligned} &\hat{t}(p_1, \ldots, p_{r-1}, q_1, \ldots, q_{s-1}, q) \\ &= \int d^{4r-4}\xi d^{4s}\rho e^{i\vec{p}.\vec{\xi} + i\vec{q}.\vec{\rho} + iq\rho} t(\vec{\xi}, \vec{\rho}, \rho) \\ &= \int \prod_{j}^{l} [d^4 k_j] \hat{t}_1(\ldots, p_i - k_{r_i}, \ldots) \prod_{j}^{l} [\mathfrak{S}(k_j)] \hat{t}_2(\ldots, q_i + k_{s_i}, \ldots) \delta(q - \sum_{j}^{l} k_j), \end{aligned} \tag{7.6}$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_{r-1})$, $\vec{\rho} = (\rho_1, \ldots, \rho_{s-1})$, $\vec{p} = (p_1, \ldots, p_{r-1})$, $\vec{q} = (q_1, \ldots, q_{s-1})$, and $\{r_i, s_i\}$ are the indices of two points joined by a contraction.

To obtain the order of singularity of $\hat{t}(\vec{p}, \vec{q}, q)$, we will apply (7.6) to a test function $\check{f} \in \mathbb{R}^m$ where $m = 4(r+s-1)$

$$\begin{aligned}\langle \hat{t}, \check{f} \rangle &= \int d^{4r-4}p \int d^{4s-4}q \int d^4q \hat{t}(\vec{p}, \vec{q}, q) \check{f}(\vec{p}, \vec{q}, q) \\ &= \int d^{4r-4}p \int d^{4s-4}q \int d^4q \int \prod_j^l [d^4k_j \mathfrak{S}(k_j)] \hat{t}_2(\ldots, q_i + k_{s_i}, \ldots) \times \\ &\quad \times \hat{t}_1(\ldots, p_i - k_{r_i}, \ldots) \delta(q - \sum_j^l k_j) \check{f}(\vec{p}, \vec{q}, q).\end{aligned} \tag{7.7}$$

In the integrals under $\vec{q}$ and $\vec{p}$ we could do the following transformation

$$\begin{cases} q_j \rightarrow q_j - k_{s_j} \\ p_j \rightarrow p_j + k_{r_j}, \end{cases} \tag{7.8}$$

then we can rewrite (7.7) as

$$\begin{aligned}\langle \hat{t}, \check{f} \rangle &= \int d^{4r-4}p \int d^{4s-4}q \hat{t}_2(\vec{q}) \hat{t}_1(\vec{p}) \int d^4q \int \prod_j^l [d^4k_j \mathfrak{S}(k_j)] \\ &\quad \delta(q - \sum_j^l k_j) \check{f}(\ldots, q_v - k_{s_v}, \ldots, p_j + k_{r_j}, \ldots, q) \\ &= \int d^{4r-4}p \int d^{4s-4}q \hat{t}_2(\vec{q}) \hat{t}_1(\vec{p}) F(\vec{p}, \vec{q}, q),\end{aligned} \tag{7.9}$$

where

$$F(\vec{p}, \vec{q}, q) = \int d^4q \int \prod_j^l [d^4k_j \mathfrak{S}(k_j)] \delta(q - \sum_j^l k_j) \check{f}(\ldots, q_v - k_{s_v}, \ldots, p_j + k_{r_j}, \ldots, q). \tag{7.10}$$

The next step is compute the form of a rescaled distribution $\langle \hat{t}(\frac{\vec{p}}{\alpha}, \frac{\vec{q}}{\alpha}, \frac{q}{\alpha}), \check{f} \rangle = \langle \hat{t}(\frac{\vec{P}}{\alpha}), \check{f} \rangle$ as follows

$$\begin{aligned}\langle \hat{t}(\frac{\vec{P}}{\alpha}), \check{f} \rangle &= \alpha^m \langle \hat{t}, \check{f}(\alpha \vec{P}) \rangle \\ &= \alpha^m \int d^{4r-4}p \int d^{4s-4}q \hat{t}_2(\vec{q}) \hat{t}_1(\vec{p}) F_\alpha(\vec{p}, \vec{q}, q),\end{aligned} \tag{7.11}$$

where $\vec{P} = (\vec{p}, \vec{q}, q)$, and

$$\begin{aligned} F_\alpha(\vec{p}, \vec{q}, q) = \int d^4 q \int \prod_j^l [d^4 k_j \mathfrak{S}(k_j)] \delta(q - \sum_j^l k_j) \times \\ \times \hat{f}(\dots, \alpha(q_j - k_{s_j}), \dots, \alpha(p_j + k_{r_j}), \dots, \alpha q). \end{aligned} \tag{7.12}$$

We introduce the rescaled variables $\tilde{k} = \alpha k$ and $\tilde{q} = \alpha q$, and we will use the order of singularity ω of the contraction $\mathfrak{S}(k_j)$, then

$$\lim_{\alpha \to 0^+} \alpha^\omega \mathfrak{S}(\frac{\tilde{k}}{\alpha}) = \mathfrak{S}_0(\tilde{k}), \tag{7.13}$$

where $\mathfrak{S}_0(\tilde{k})$ is the asymptotic distribution of $\mathfrak{S}(\tilde{k})$. Therefore, for F_α we have the following form

$$\begin{aligned} F_\alpha(\vec{p}, \vec{q}, q) &= \int \frac{d^4 \tilde{q}}{\alpha^4} \int \prod_j^l [\frac{d^4 \tilde{k}_j}{\alpha^4} \mathfrak{S}(\frac{\tilde{k}_j}{\alpha})] \alpha^4 \delta(\tilde{q} - \sum_j^l \tilde{k}_j) \times \\ &\times \check{f}(\dots, \alpha q_j - \tilde{k}_{s_j}, \dots, \alpha p_j + \tilde{k}_{r_j}, \dots, \tilde{q}) \\ &= \frac{1}{\alpha^{4l}} \int d^4 \tilde{q} \int \prod_j^l [d^4 \tilde{k}_j \mathfrak{S}(\frac{\tilde{k}_j}{\alpha})] \delta(\tilde{q} - \sum_j^l \tilde{k}_j) \times \\ &\times \check{f}(\dots, \alpha q_j - \tilde{k}_{s_j}, \dots, \alpha p_j + \tilde{k}_{r_j}, \dots, \tilde{q}). \end{aligned} \tag{7.14}$$

Now, using the rescaled variables $\alpha p_i = \tilde{p}_i$ and $\alpha q_i = \tilde{q}_i$, we can rewrite (7.11) as

$$\begin{aligned} \langle \hat{t}(\frac{p}{\alpha}), f \rangle &= \alpha^m \int \frac{d^{4-4}\tilde{p}}{\alpha^{4r-4}} \int d^{4s-4} \frac{\tilde{q}}{\alpha^{4s-4}} \hat{t}_2(\frac{\tilde{\vec{q}}}{\alpha}) \hat{t}_1(\frac{\tilde{\vec{p}}}{\alpha}) F_\alpha(\frac{\tilde{\vec{p}}}{\alpha}, \frac{\tilde{\vec{q}}}{\alpha}) \\ &= \frac{\alpha^4}{\alpha^{4l}} \int d^{4r-4}\tilde{p} \int d^{4s-4}\tilde{q} \hat{t}_2(\frac{\tilde{\vec{q}}}{\alpha}) \hat{t}_1(\frac{\tilde{\vec{p}}}{\alpha}) \int d^4 \tilde{q} \int \prod_j^l [d^4 \tilde{k}_j \mathfrak{S}(\frac{\tilde{k}_j}{\alpha})] \times \\ &\times \delta(\tilde{q} - \sum_j^l \tilde{k}_j) \hat{f}(\dots, \tilde{q}_j - \tilde{k}_{s_j}, \dots, \tilde{p}_j + \tilde{k}_{r_j}, \dots, \tilde{q}). \end{aligned} \tag{7.15}$$

Finally, considering the two orders of singularity $\omega[t_1] = \omega_1$ and $\omega[t_2] = \omega_2$, we can

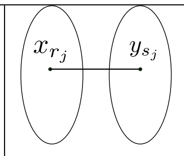
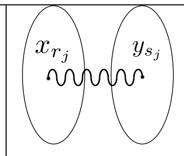
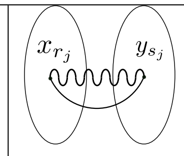
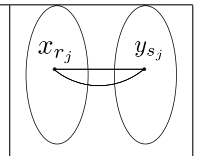
i				
ω_i	0	-2	1	2

Table 7.1: Order of singularity of the four kind of contractions that we compute in the two previous Chapters.

see that the following limit exists

$$\begin{aligned}
&\lim_{\alpha\to 0^+} \alpha^{\omega_1+\omega_2-4+4l+l\omega}\langle \hat{t}(\frac{p}{\alpha}), f\rangle \\
&= \lim_{\alpha\to 0^+} \int d^{4r-4}\tilde{p} \int d^{4s-4}\tilde{q}\alpha^{\omega_2}\hat{t}_2(\frac{\tilde{\vec{q}}}{\alpha})\alpha^{\omega_1}\hat{t}_1(\frac{\tilde{\vec{p}}}{\alpha}) \int d^4\tilde{q} \int \prod_j^l [d^4\tilde{k}_j \alpha^{\omega} \mathfrak{S}(\frac{\tilde{k}_j}{\alpha})]\times \\
&\times \delta(\tilde{q} - \sum_j^l \tilde{k}_j)\hat{f}(\ldots, \tilde{q}_j - \tilde{k}_{s_j}, \ldots, \tilde{p}_j + \tilde{k}_{r_j}, \ldots, \tilde{q}) \\
&\neq 0,
\end{aligned} \tag{7.16}$$

then, the order of singularity of $\hat{t}(p)$ is

$$\omega[t] = \omega_1 + \omega_2 - 4 + 4l + l\omega. \tag{7.17}$$

The result (7.17) could be generalized for l_i contractions of each different kind i as follows

$$\omega[t] = \omega_1 + \omega_2 - 4 + \sum_i^{l_i}(4 + \omega_i)l_i. \tag{7.18}$$

In Table 7.1, we show the order of singularity of the four kind of independent contraction that we will use in this Chapter.

7.1.1 Normalization of vacuum polarization tensor

In section 6.1, the causal splitting procedure gave us a vacuum polarization tensor $\Pi^{\mu\nu}$ with three constants that we need to determine with other physical properties. We can rewrite that result here as

$$\hat{\Pi}^{\mu\nu}(k) = \frac{1}{(2\pi)^2}(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu})\Pi(k), \tag{7.19}$$

where

$$\Pi(k) = \left\{ \frac{e^2}{12(2\pi)^2}(k^0)^4 \int_{4m^2}^{\infty} ds \frac{[1 - \frac{4m^2}{s}]^{\frac{3}{2}}}{s((k^0)^2 - s)} \right\} + C_0 + C_\alpha k^\alpha + C_2 k^2. \tag{7.20}$$

To have *parity symmetry*, it is straightforward to note that

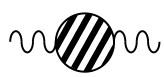
$$C_\alpha = 0, \tag{7.21}$$

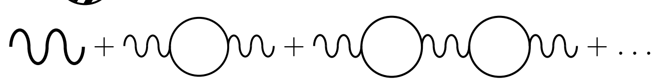
but for the others constants we must do an extra effort.

Gauge invariance in second order is not helpful, because it tell us that C_0 and C_2 must be proportional to $(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu})$.

Similar to standard formalism, we will compute the total photon propagator $D^{\mu\nu}_{Tot}$ to get a structure that allows us to fix the constants. The latter comes from the sum of loop corrections by polarization insertions in Moller or Bhabha scattering process. Therefore, $D^{\mu\nu}_{Tot}$ is given by the following expression

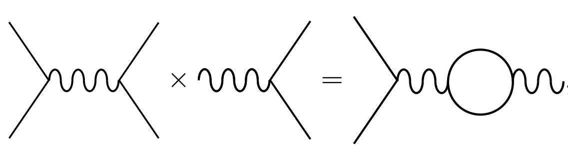
$D^{\mu\nu}_{Tot}(x-y)$

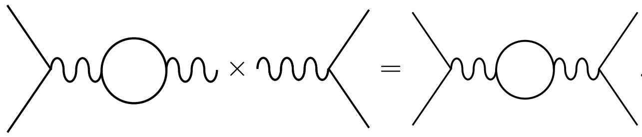
$=$ 

$=$ 

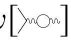
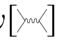
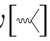
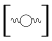
$$\begin{aligned} &= g^{\mu\nu} D_0^F(x-y) + \int d^4 z_1 d^4 z_2 D_0^F(x-z_1)\Pi^{\mu\nu}(z_1-z_2)D_0^F(z_2-y) + \\ &+ \int d^4 z_1 \dots d^4 z_4 D_0^F(x-z_1)\Pi^{\mu}{}_{\lambda}(z_1-z_2)D_0^F(z_2-z_3)\times \\ &\times \Pi^{\lambda\nu}(z_3-z_4)D_0^F(z_4-y) + \dots . \end{aligned} \tag{7.22}$$

But, before determining the sum (7.22), we must know if all these terms have the same order of singularity. Let us examine the terms derived from Bhabha processes. There are two forms to obtain the 1-loop correction for the Bhabha scattering diagram. The first one is contract two DKP fields between a Bhabha diagram and a basic vertex. After that, we must contract two electromagnetic four potential, one from the polarized vertex and the other from a basic vertex. The last process is represented in the following equations

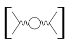
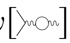
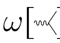
 (7.23)

 (7.24)

The corresponding orders of singularity can be computed using (7.18), then we have

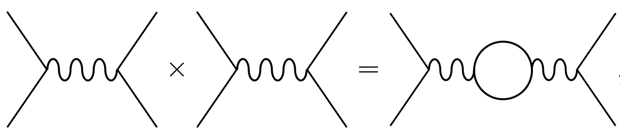
$\omega[$$] = \omega[$$] + \omega[$$] - 4 + (4 + \omega[$$]).1$

$= (-2) + (0) - 4 + (4 + (2)).1 = 0,$ (7.25)

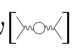
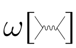
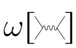
$\omega[$$] = \omega[$$] + \omega[$$] - 4 + (4 + \omega[$$]).1$

$= (0) + (0) - 4 + (4 + (-2)).1 = -2.$ (7.26)

The second form to obtain the 1-loop correction, is contracting two DKP fields from two Bhabha diagrams as follows

 (7.27)

with a order of singularity given by

$\omega[$$] = \omega[$$] + \omega[$$] - 4 + (4 + \omega[$$]).1$

$= (-2) + (-2) - 4 + (4 + (2)).1 = -2.$ (7.28)

Similarly, for the construction of 2-loop correction by vacuum polarization insertion we will have the same order of singularity $\omega = 2$, this means that the summation (7.22) is consistent with the theory.

Now, going to momentum space, the summation (7.22) becomes convolutions giving us the following expression

$$\begin{aligned} \hat{D}^{\mu\nu}_{Tot}(p) &= g^{\mu\nu}\hat{D}^F_0(p) + \hat{D}^F_0(p)\bar{\Pi}^{\mu\nu}(p)D^F_0(p) + \\ &\quad + \hat{D}^F_0(p)\bar{\Pi}^{\mu}{}_{\lambda}(p)\hat{D}^F_0(p)\bar{\Pi}^{\lambda\nu}(p)\hat{D}^F_0(p) + \dots \\ &= D^F_0(g^{\mu\nu} + \bar{\Pi}^{\mu}_{\lambda}\hat{D}^{\lambda\nu}_{Tot}), \end{aligned} \tag{7.29}$$

where

$$\bar{\Pi}^{\mu\nu}(p) = (2\pi)^4\hat{\Pi}^{\mu\nu} = (2\pi)^2(\frac{k^\mu k^\nu}{k^2} - g^{\mu\nu})\Pi(p). \tag{7.30}$$

Multiplying (7.29) by the left with $(D^F_0)^{-1}$ and by the right with $(D^{-1}_{Tot})_\nu{}^\theta$, we obtain

$$(D^F_0)^{-1}g^{\mu\theta} - \bar{\Pi}^{\mu\theta} = (D^{-1}_{Tot})^{\mu\theta}. \tag{7.31}$$

Replacing (7.30) into (7.31) and using that $(D_0^F)^{-1} = -(2\pi)^2 p^2$, we obtain

$$\begin{aligned}(D^{-1}_{Tot})^{\mu\theta}(k) &= -(2\pi)^2 k^2 g^{\mu\theta} - (2\pi)^2 (\frac{k^\mu k^\theta}{k^2} - g^{\mu\theta})\Pi \\ &= (2\pi)^2 \Big[(g^{\mu\theta} - \frac{k^\mu k^\theta}{k^2})(\Pi - k^2) - \frac{k^\mu k^\theta}{k^2} k^2\Big] \\ &= (2\pi)^2 \Big[P_1^{\mu\theta}(\Pi - k^2) - P_2^{\mu\theta} k^2\Big],\end{aligned} \tag{7.32}$$

where $P_{1,2}$ are projectors which fulfill the following property

$$(P_1)^{\mu\nu} = g^{\mu\theta} - \frac{k^\mu k^\theta}{k^2}, \tag{7.33}$$

$$(P_2)^{\mu\nu} = \frac{k^\mu k^\theta}{k^2}, \tag{7.34}$$

$$(P_i)^\mu{}_\nu (P_j)^\nu{}_\lambda = \delta_{ij}\delta^\mu{}_\lambda. \tag{7.35}$$

Then, using (7.35), we can check that $D^{\mu\theta}_{Tot}(k)$ is

$$\begin{aligned}D^{\mu\theta}_{Tot}(k) &= (2\pi)^{-2}\Big[P_1^{\mu\theta}\frac{1}{\Pi - k^2 + i0^+} - P_2^{\mu\theta}\frac{1}{k^2 + i0^+}\Big], \\ &= (2\pi)^{-2}\Big[(g^{\mu\theta} - \frac{k^\mu k^\theta}{k^2})\frac{1}{\Pi - k^2 + i0^+} - (\frac{k^\mu k^\theta}{k^2})\frac{1}{k^2 + i0^+}\Big].\end{aligned} \tag{7.36}$$

Notice that the second term in brackets of (7.36) will be null between transversal polarized photon states, therefore we will concentrate our analysis in the first one. First of all, if we separate the constant C_0 from $\Pi(k)$, it will be a pole for the propagator giving it mass. Because we know that the photon is massless, we fix this constant as

$$C_0 = 0. \tag{7.37}$$

Secondly, if we separate the term $C_2 k^2$ from $\Pi(k)$, we can rewrite (7.36) in the following form

$$D^{\mu\theta}_{Tot}(k) = (2\pi)^{-2}(g^{\mu\theta} - \frac{k^\mu k^\theta}{k^2})\frac{Z}{\Pi' - k^2 + i0^+}, \;\; Z = \frac{1}{1 - C_2}, \;\; \Pi' = Z(\pi - C_2 k^2), \tag{7.38}$$

which is equivalent to renormalize the "bare" electric charge in the standard formalism, but in CPT the electric charge is already the physical one. Therefore, C_2 must be fixed as

$$C_2 = 0. \tag{7.39}$$

With all constants fixed, we can normalize the vacuum polarization at $k = 0$ as

$$\Pi(0) = 0, \quad \frac{\Pi(k)}{k^2}\Big|_{k^2=0} = 0. \tag{7.40}$$

7.1.2 The non-renormalizability of Self Energy sector

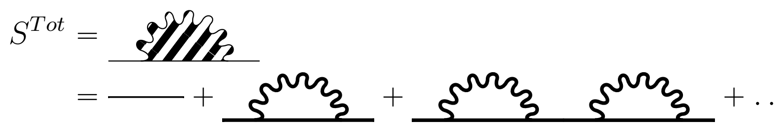
Now, let us determine the constants that appear in the causal splitting process of the self-energy function of the DKP propagator which we rewrite here as

$$T_2^{(4)}(x,y) = i : \overline{\psi}(x)\Sigma(x-y)\psi(y) : + i : \overline{\psi}(y)\Sigma(y-x)\psi(x) :, \tag{7.41}$$

where

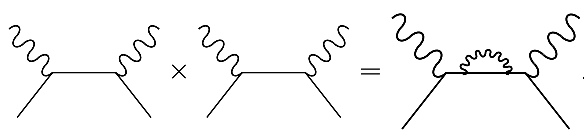
$$\begin{aligned}\Sigma(k) = \frac{e^2}{64(\pi)^4}\Big[- m - \frac{\not k}{4} - \frac{\not k}{2b^2} + \Big\{ m(1-\frac{1}{b^2}) - \frac{\not k}{2}(\frac{1}{b^4}-1)\Big\}\times \\ \times\Big(\log|b^2-1| - i\pi\Theta(k^{0^2}-m^2)\Big)\Big] + C_0 + C_1\not k.\end{aligned} \tag{7.42}$$

From (7.41), we can see that parity symmetry does not cancel the constant C_1 as in vacuum polarization. In consequence, we will try to determine the radiative correction of the DKP propagator S^{Tot} by insertions of self-energy functions as follows



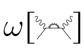
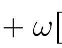
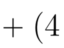
(7.43)

But the problem in the computation of (7.43) is that the order of singularity of each term is increasing. Consider the construction of the 1-loop correction coming from the contraction of two Compton diagrams



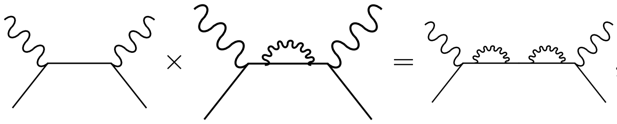
(7.44)

Using the formula (7.18), we obtain

ω[] = ω[] + ω[] $- 4 + (4 + \omega$[]$).1$

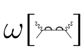
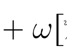
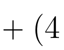
$$= (0) + (0) - 4 + (4+1).1 = 1. \tag{7.45}$$

For the 2-loop correction we have



(7.46)

with an order of singularity given by

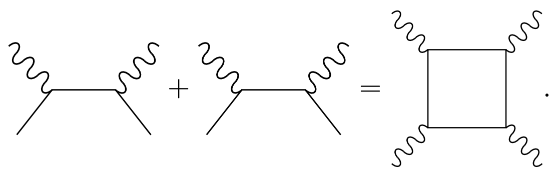
ω[] = ω[] + ω[] $- 4 + (4 + \omega$[]$).1$

$$= (0) + (1) - 4 + (4+1).1 = 2. \tag{7.47}$$

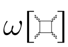
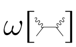
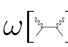
Therefore, if we increase the number of self-energy insertions by one then the order of singularity increases by one unit too. In CPT, the latter means that SDKP is a non-renormalizable theory

7.1.3 The non-renormalizability of Photon-Photon scattering

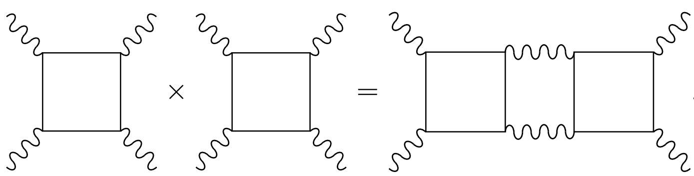
Another sector where SDKP is non-renormalizable is the photon-photon scattering. We can contract two DKP fields from two Compton diagrams to obtain the following

 (7.48)

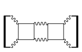
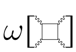
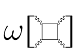
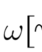
The order of singularity is

$\omega[$$]$ $= \omega[$$] + \omega[$$] - 4 + (4 + \omega[$$]).2l,$
$= 0 + 0 - 4 + (4 - 0)2 = 4.$ (7.49)

Now, we can contract two electromagnetic four potentials to obtain

 (7.50)

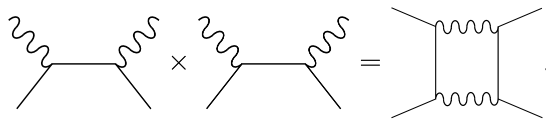
The order of singularity of the new diagram is

$\omega[$$]$ $= \omega[$$] + \omega[$$] - 4 + (4 + \omega[$$]).2,$
$= 4 + 4 - 4 + (4 - 2)2 = 8.$ (7.51)

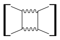
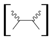
Therefore, we can see that we have an increasing order of singularity which tell us that the sector is non-renormalizable.

7.2 The $\sim (\bar{\psi}\psi)^2$ term

In this section we want to show that it is possible to obtain a proportional term to $\sim (\bar{\psi}\psi)^2$. Starting from two Compton diagrams, we can contract two pair of photons as follows

 (7.52)

The order of singularity of the new diagram is

$\omega[$$] = \omega[$$] + \omega[$$] - 4 + (4 + \omega[$$]).2,$

$$= 0 + 0 - 4 + (4 - 2)2 = 0. \tag{7.53}$$

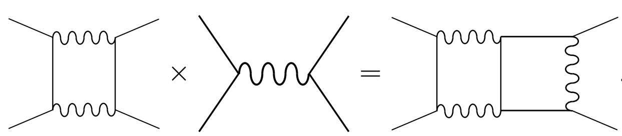
Now, because the order of singularity is $\omega = 0$, the four point distributions $T_4(x_1, \ldots, x_4)$, associated with the right hand side of (7.52), will have one term of the following form

$$\begin{aligned} T_4(x_1, \ldots, x_4) = \ldots + e^4 \lambda_4 \delta(x_1 - x_4)\delta(x_2 - x_4)\delta(x_3 - x_4) \times \\ \times \sum_{i_1 < i_2, j_1 < j_2} \bar{\psi}(x_{i_1})\psi(x_{j_1})\bar{\psi}(x_{i_2})\psi(x_{j_2}). \end{aligned} \tag{7.54}$$

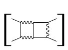
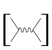
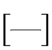
Replacing (7.54) in the expresion of the S-matrix, we will find the following term

$$\begin{aligned} S &= \ldots + \frac{1}{4!}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^4 \lambda_4 \delta(x_1 - x_4)\delta(x_2 - x_4)\delta(x_3 - x_4), \\ &= \ldots + \int d^4x_1 \frac{e^4\lambda_4}{4}\left(\bar{\psi}(x_1)\psi(x_1)\right)^2. \end{aligned} \tag{7.55}$$

Therefore, the theory contains a sector proportional to $\sim (\bar{\psi}\psi)^2$. We can show that this sector is non-renormalizable. For example, contracting two pair of DKP fields between the 4-DKP diagram and Bhabha or Moller processes, we obtain

 (7.56)

After that, the order of singularity is

$\omega[$$] = \omega[$$] + \omega[$$] - 4 + (4 + \omega[$$]).2,$

$$= 0 + (-2) - 4 + (4 - 0)2 = 2. \tag{7.57}$$

It is important to clarify that the final diagram of (7.57) contains the term proportional to $\sim (\bar{\psi}\psi)^2$.

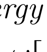
Chapter 8

Conclusions and perspectives

At the end of this Thesis it is clear that the main difference between SQED and SDKP is the non-renormalizability of the last one. Therefore, we will analyze why this happens looking back at our results.

In chapter 5 we used the principle of *perturbative gauge invariance* to determine the correct form of the base term $T_1(x_1)$ to construct the S-Matrix for SDKP gauge theory. With the term T_1 we determine the differential cross section for the scattering of a scalar particle via non-quantized electromagnetic field obtaining the same result as that in SQED. After that, we used CPT to determine the causal 2-point distribution $D_2(x_1, x_2)$ which contain many processes: Moller, Bhabha, Compton, vacuum polarization and self-energy.

The differential cross section computed for Bhabha, Moller and Compton processes are the same for the ones obtained via SQED. We must highlight the case of Compton scattering where we found a singular DKP propagator with order of singularity $\omega = 0$ because of the extra $\not{p}$ coming from the β-matrix algebra. The same happens for SQED via CPT [26, 95] when two derivatives scalar fields are contracted via Wick theorem. Furthermore, these singular propagators reproduce the second order terms $\sim e^2 A_\mu A^\mu \phi^* \phi$ and $\sim e^2 A_\mu A^\mu \bar{\psi}\psi$ for SQED and SDKP, respectively. In SQED, via CPT or Feynman diagrams, the term $\sim e^2 A_\mu A^\mu \phi^* \phi$ is really important to obtain the correct differential cross section because the contributions from the other diagrams are null [94, 95]. For SDKP the same happens as viewed in section 5.5.3.

In Chapter 6 we computed the *vacuum polarization tensor* and *self energy* finding that they are singular propagators with orders of singularity $\omega[$$] = 2$ and $\omega[$$] = 1$, respectively. The non fixed constants coming from *causal-splitting procedure* were left to be calculated in Chapter 7 because, before obtaining the complete photon and scalar propagators, we need to study the renormalizability of the theory.

So, in Chapter 7 we found that the theory is non-renormalizable. Comparing our computations with the reference [26], we can note that the renormalizability of SQED is based on the presence of derivatives in the KGF field. The latter makes the order of singularity ω of scattering processes indepent of the internal structure. The Compton diagram in SDKP is equivalent to the one in SQED with the contraction of two derivatives KGF fields, however Comptom diagrams with other kind of contractions do not appear. Nevertheless, equivalence occurs because these diagrams do not contribute to the calculation of the cross section but it will be reflected in the computation of the complete DKP propagator.

For these reasons, we believe that SDKP represents an *effective theory* for a bound state with 0-spin of two leptons coupled with an electromagnetic field. The latter could explain why, in the computation of the form factor in the semileptonic decay K_{l3} [11], the use of the DKP fields to represent k and π particles gave a result closer to the experimental value.

It is important to comment that A. A. Nogueira [96] found that in the case of general SQED[1] via DKP fields (GSDKP) the theory is renormalizable [96]. As shown by Soto et al., the use of Podolsky fields makes general QED super renormalizable, what is explained due to the smaller order of singularity in comparison to that of the Maxwellian fields. The GSDKP via CPT is a future topic that we hope to study. Furthermore, A. A. Nogueira found that the UV divergence of photon-photon scattering is solved by gauge invariance and DKP algebra, so it is possible that the non renormalizability that we found in this work could be solved in the same way.

With this idea in mind, for future projects we will investigate the possible composite behavior of Higgs boson modeled by a neutral DKP field using CPT where spontaneous symmetry breaking is not used [44]. We believe that the latter is possible considering that the DKP algebra for scalar particles is a 5-order representation of the group SO(5) [6] which is used to study composite Higgs models via spontaneous symmetry breaking [98].

Another interesting topic is to investigate the Gribov theory of quark confinement

[1]A *general* theory means that the Maxwellian electromagnetic fields are generalized as Podolsky electromagnetic fields.

[21]. In this theory, a computational tool to obtain a confinement potential is to use the 10-order DKP algebra to model gluons as spin-1 DKP field. It is possible that these DKP-gluons represent a spin-1 bound state of two quarks.

We want to highlight that the study of SDKP is not complete yet. For example, the gauge invariance of the full theory and the unitarity are sections that we have not finished yet. We hope to complete the work as soon as possible.

Appendix A

Computations for the General theory

A.1 Causality of intermediate distributions

Theorem A.1 *Consider* $Y = P \cup Q$ *where* $P \neq \varnothing$, $P \cap Q = \varnothing$, $|Y| = n-1$, *and the point* x *such that* $x \notin Y$, *then:*

- *If* $\{Q, x\} > P$, $|Q| = n_1$, *therefore*

$$R'_n(Y,x) = -T_{n_1+1}(Q,x)T_{n-(n_1+1)}(P) \tag{A.1}$$

- *If* $\{Q, x\} < P$, $|Q| = n_1$, *therefore*

$$A'_n(Y,x) = -T_{n-(n_1+1)}(P)T_{n_1+1}(Q,x) \tag{A.2}$$

Proof.

We will present the proof of (A.1). From the definition (3.24), we have

$$R'(Y,x) = \sum_{P_2} T(W,x)\widetilde{T}(X), \tag{A.3}$$

where the sub-index of distributions T_n are not written for simplicity. P_2 is all partitions of Y in the non-empty and disjoints sub-sets W and X.

The causality condition $P < \{Q, x\}$ allows to split each partition W and X such that

- $W = W_1 \cup W_2$, where $\{W_1 = W \cap P\} < \{W_2 = W \cap Q\}$,
- $X = X_1 \cup X_2$, where $\{X_1 = X \cap P\} < \{X_2 = X \cap Q\}$.

Applying the causality decompositions (3.21) and (3.22) for the n-point distributions in (A.3), we get

$$R'(Y,x) = \sum_{P_4^0} T(W_2,x)T(W_1,x)\widetilde{T}(X_1)\widetilde{T}(X_2), \tag{A.4}$$

where P_4^0 represents all partitions of Y in the four sub-sets $\{W_1, W_2, X_1, X_2\}$ allowing for an empty set but with the conditions $X_1 \cup X_2 \neq \varnothing \neq W_1 \cup W_2$. The latter means that $\{W_1, X_1\}$ and $\{W_2, X_2\}$ are all independent partitions P_2^0 of P and Q, respectively. Then, it is straightforward to rewrite (A.4) as

$$\begin{aligned} R'(Y,x) = \sum_{P_2,Q} T(W_2,x)\Big[\sum_{P_2^0,P} T(W_1,x)\widetilde{T}(X_1)\Big]\widetilde{T}(X_2) \\ + T(Q,x)\Big[\sum_{P_2,P} T(W_1,x)\widetilde{T}(X_1)\Big]\widetilde{T}(\varnothing). \end{aligned} \tag{A.5}$$

The first term of the right hand side of equation (A.5) is null because of the property (3.15), and using the same property, we get

$$\begin{aligned} R'(Y,x) &= T(Q,x)\Big[\sum_{P_2,P} T(W_1,x)\widetilde{T}(X_1)\Big]\widetilde{T}(\varnothing) \\ &= T(Q,x)\Big[\sum_{P_2^0,P} T(W_1,x)\widetilde{T}(X_1) - T(P,x)\widetilde{T}(\varnothing)\Big]\widetilde{T}(\varnothing) \\ &= -T(Q,x)T(P,x) \end{aligned} \tag{A.6}$$

which proves the theorem for R'. Analog path could be use to proof the theorem for A'. □

A.2 Wick Theorem

All products of n operator value distributions $\mathcal{O}_i = \mathcal{O}(x_i)$, are in normal order if they read

$$\begin{aligned}\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_n =: \mathcal{O}_1\ldots\mathcal{O}_n : +[\mathcal{O}_1\ldots\overbracket{\mathcal{O}_i\ldots\mathcal{O}_j}\ldots\mathcal{O}_n + \text{permutations}]+ \\ +[\mathcal{O}_1\ldots\overbracket{\mathcal{O}_i\ldots\mathcal{O}_k\ldots\mathcal{O}_j}\ldots\mathcal{O}_l\ldots\mathcal{O}_n + permutations] + \ldots,\end{aligned} \tag{A.7}$$

where the contractions $\overbracket{\mathcal{O}(x_i)\mathcal{O}}(x_j)$ are defined as the c-number

$$\overbracket{\mathcal{O}(x_i)\mathcal{O}}(x_j) = [\mathcal{O}^{(-)}(x_i), \mathcal{O}^{(+)}(x_j)]. \tag{A.8}$$

A.3 Power counting function $\rho(x)$

In this section we show some properties of $\rho(x)$. Using a rescaled test function $\psi(x/a)$, we have from (3.2)

$$\begin{aligned}\langle d_0(x), \psi(\frac{x}{a})\rangle &= \lim_{\alpha\to 0^+}\langle \alpha^m\rho(\alpha)d(\alpha x), \psi(\frac{x}{a})\rangle \\ &= \lim_{\alpha\to 0^+}\langle (a\alpha)^m\rho(\alpha)d(a\alpha x), \psi(x)\rangle \\ &= \lim_{\alpha\to 0^+}\frac{\rho(\alpha)}{\rho(a\alpha)}\langle (a\alpha)^m\rho(a\alpha)d(a\alpha x), \psi(x)\rangle \\ &= \lim_{\alpha\to 0^+}\frac{\rho(\alpha)}{\rho(a\alpha)}\langle d_0(x), \psi(x)\rangle \\ &= \rho_0(a)\langle d_0(x), \psi(x)\rangle,\end{aligned} \tag{A.9}$$

where we define the function $\rho_0(a)$ as

$$\rho_0(a) \equiv \lim_{\alpha\to 0^+}\frac{\rho(\alpha)}{\rho(a\alpha)}. \tag{A.10}$$

Via another rescaling of the test function, we have

$$\begin{aligned}\langle d_0(x), \psi(\frac{x}{ba}),\rangle &= \rho_0(b)\langle d_0(x), \psi(\frac{x}{a})\rangle \\ &= \rho_0(b)\rho_0(b) \\ \rho_0(ab) &= \rho_0(b)\rho_0(b).\end{aligned} \tag{A.11}$$

The last line of (A.11) defines the form of ρ_0 as

$$\rho_0(a) = a^\omega, \tag{A.12}$$

and, from the definition (A.10), we conclude that in the limit $\alpha \to 0^+$ the power counting $\rho(x)$ has the following form

$$\lim_{\alpha\to 0^+} \rho(\alpha) = \alpha^{\omega} L(\alpha), \tag{A.13}$$

where $\omega \in \mathbb{R}$, and $L(\alpha)$ is a slow varying or quasi-constant function of α in the neighborhood of $\alpha = 0$. In practice, $L(\alpha)$ could be omitted in the computations.

A.4 Normalized solution for the retarded numerical distribution

In this section we will determine the explicit form for the normalized solution $\hat{r}_q(p)$ defined in (3.87) as

$$\hat{r}_q(p) = \hat{r}(p) - \sum_{b=0}^{\omega} \frac{(p-q)^b}{b!}[\mathbf{D}^b \hat{r}](q). \tag{A.14}$$

From (3.85), we can compute $[\mathbf{D}^b \hat{r}](q)$

$$\begin{aligned}
[\mathbf{D}^b \hat{r}](q) = \mathbf{D}^b\Big[&(2\pi)^{-\frac{m}{2}} \int dk \hat{\Theta}(k) \hat{d}(p-k) \\
&- (2\pi)^{-m} \int dk \hat{\Theta}(k) \sum_{|l|=0}^{\omega} \frac{p^l}{l!} \int dp' [\mathbf{D}^l_{p'} \hat{d}(p'-k)] \breve{w}(p')\Big]_{p=q} \\
= (2\pi)^{-\frac{m}{2}} &\int dk \hat{\Theta}(k) \Big[\mathbf{D}^b_q \hat{d}(q-k) - \\
&- (2\pi)^{-\frac{m}{2}} \sum_{|l|=|b|}^{\omega} \frac{l! q^{l-b}}{l!(l-b)!} \int dp' [\mathbf{D}^l_{p'} \hat{d}(p'-k)] \breve{w}(p')\Big].
\end{aligned} \tag{A.15}$$

Regarding (A.15), the sum in the right hand side of (A.14) is

$$
\begin{aligned}
&\sum_{b=0}^{\omega}\frac{(p-q)^b}{b!}[\mathbf{D}^b\hat{r}](q)\\
&=(2\pi)^{-\frac{m}{2}}\int dk\hat{\Theta}(k)\Big[\sum_{b=0}^{\omega}\frac{(p-q)^b}{b!}\mathbf{D}_q^b\hat{d}(q-k)-\\
&\quad-(2\pi)^{-\frac{m}{2}}\sum_{l=0}^{\omega}\frac{1}{l!}\Big(\sum_{b=0}^{l}\frac{l!q^{l-b}}{(l-b)!}\frac{(p-q)^b}{b!}\Big)\int dp'[\mathbf{D}_{p'}^l\hat{d}(p'-k)]\breve{w}(p')\Big] \qquad (A.16)\\
&=(2\pi)^{-\frac{m}{2}}\int dk\hat{\Theta}(k)\Big[\sum_{b=0}^{\omega}\frac{(p-q)^b}{b!}\mathbf{D}_q^b\hat{d}(q-k)-\\
&\quad-(2\pi)^{-\frac{m}{2}}\sum_{l=0}^{\omega}\frac{1}{l!}p^l\int dp'[\mathbf{D}_{p'}^l\hat{d}(p'-k)]\breve{w}(p')\Big],
\end{aligned}
$$

where in the first equality we use the property $\sum_{b=0}^{\omega}\sum_{l=b}^{\omega}=\sum_{l=0}^{\omega}\sum_{b=0}^{l}$. In the second equality we use the following identity

$$
p^l=[q+(p-q)]^l=\sum_{b=0}^{l}\frac{l!q^{l-b}}{(l-b)!}\frac{(p-q)^b}{b!}. \qquad (A.17)
$$

Finally, using (3.85) and replacing (A.16) into (A.14), we obtain

$$
\hat{r}_q(p)=(2\pi)^{-\frac{m}{2}}\int dk\hat{\Theta}(k)\Big[\hat{d}(p-k)-\sum_{b=0}^{\omega}\frac{(p-q)^b}{b!}\mathbf{D}_q^b\hat{d}(q-k)\Big], \qquad (A.18)
$$

which is equation (3.88).

A.5 Central splitting solution

In this section we want to show the computation to get an explicit formula for the central splitting solution $\hat{r}_0(p)$ starting with the formula (3.89)

$$
\hat{r}_0(p)=(2\pi)^{-\frac{m}{2}}\int dk\hat{\Theta}(k)\Big[\hat{d}(p-k)-\sum_{b=0}^{\omega}\frac{p^b}{b!}\mathbf{D}_q^b\hat{d}(q-k)\Big|_{q=0}\Big], \qquad (A.19)
$$

and using $[\mathbf{D}_q^b\hat{d}(q-k)](q=0)=(-1)^b\mathbf{D}_k^b\hat{d}(-k)$, we can rewrite (A.19) as

$$
\hat{r}_0(p)=(2\pi)^{-\frac{m}{2}}\Big[\int dk\hat{\Theta}(k)\hat{d}(p-k)-\sum_{b=0}^{\omega}\frac{p^b}{b!}\int dk\hat{\Theta}(k)(-1)^b\mathbf{D}_k^b\hat{d}(-k)\Big]. \qquad (A.20)
$$

Integrating by parts the second integral in (A.20), we have

$$\begin{aligned}\hat{r}_0(p) &= (2\pi)^{-\frac{m}{2}}\Big[\int dk\hat{\Theta}(k)\hat{d}(p-k) - \sum_{b=0}^{\omega}\frac{p^b}{b!}\int dk\mathbf{D}_k^b\hat{\Theta}(k)\hat{d}(-k)\Big],\\ &= (2\pi)^{-\frac{m}{2}}\Big[\int dk\hat{\Theta}(p-k)\hat{d}(k) - \sum_{b=0}^{\omega}\frac{p^b}{b!}\int dk[D_k^b\hat{\Theta}](-k)\hat{d}(k)\Big], \qquad \text{(A.21)}\\ &= (2\pi)^{-\frac{m}{2}}\int dk\hat{d}(k)\Big[\hat{\Theta}(p-k) - \sum_{b=0}^{\omega}\frac{p^b}{b!}[D_k^b\hat{\Theta}](-k)\Big],\end{aligned}$$

where in the second line we introduced the change of variables $k \to k-p$ and $k \to -k$ in the first and second integrals, respectively.

Now, we introduce the Fourier transform of $\Theta(vx) = \Theta(x_1^0)$ for $v = (1, \mathbf{0}, 0, \ldots)$

$$\hat{\Theta}(q) = (2\pi)^{\frac{m}{2}-1}\delta(\mathbf{q}_1, q_2, \ldots, q_{n-1})\frac{i}{q_1^0 + i0^+}, \qquad \text{(A.22)}$$

and considering $q = (q_1^0, \mathbf{0}, 0, \ldots, 0)$ in the computation of (A.18), we will have in (A.21) $\mathbf{D}^b = \partial^b$ and $p^b = (p_1^0)^b$, so that

$$\begin{aligned}\hat{r}_0(p) &= (2\pi)^{-\frac{m}{2}}\int dk\hat{d}(k)\Big[\hat{\Theta}(p-k) - \sum_{b=0}^{\omega}\frac{(p_1^0)^b}{b!}[\partial_k^b\hat{\Theta}](-k)\Big],\\ &= \frac{i}{2\pi}\int dk_1^0\hat{d}(k_1^0, \mathbf{p}, \ldots, p_{n-1})\Big[\frac{1}{p_1^0 - k_1^0 + i0^+} + \sum_{b=0}^{\omega}(p_1^0)^b\Big(\frac{1}{k_1^0 - i0^+}\Big)\Big], \qquad \text{(A.23)}\\ &= \frac{i}{2\pi}\int dk_1^0\hat{d}(k_1^0, \mathbf{p}, \ldots, p_{n-1})\Big[\frac{(p_1^0)^{\omega+1}}{(k_1^0 - i0^+)^{\omega+1}(p_1^0 - k_1^0 + i0^+)}\Big].\end{aligned}$$

Introducing the change of variables $k_1^0 \to t_1 p_1^0$ and doing the same change of reference system as in the regular case, we obtain the covariant formula

$$\hat{r}_0(p) = \frac{i}{2\pi}Sgn(p_1^0)\int dt_1\frac{\hat{d}(t_1p_1, p_2, \ldots, p_{n-1})}{(t_1 - i0^+)^{\omega+1}(1 - t_1 + iSgn(p_1^0)0^+)}, \qquad \text{(A.24)}$$

which is valid for $p_1 \in \{\Gamma_1^+ \cup \Gamma_1^-\}$.

Similar to the regular distribution case, we could choose the vector v to get a formula for $\hat{r}_0(p)$ dependent on the integral of any $p_j \in \{\Gamma_1^+ \cup \Gamma_1^-\}$. But, differently here, it will be impossible to get an independent formula as (3.78).

For a second order solution we get

$$\hat{r}_0(p) = \frac{i}{2\pi}Sgn(p^0)\int dt\frac{\hat{d}(tp)}{(t - i0^+)^{\omega+1}(1 - t + iSgn(p^0)0^+)}, \qquad \text{(A.25)}$$

this is equation (3.90).

A.6 Symmetry of retarded formulas

In this section we show some properties for the central splitting solutions in the regular and singular cases.

A.6.1 Regular Case

We will determine particular forms of (3.93) taking into account parity characteristics in the integrals at second order of perturbation where the formula takes the following form

$$\hat{r}_0(p) = \frac{i}{2\pi} Sgn(p^0) \int_{-\infty}^{\infty} dt \frac{\hat{d}(tp)}{1 - t + iSgn(p^0)0^+}, \quad p \in \Gamma_1^+ \cup \Gamma_1^-. \tag{A.26}$$

If $\hat{d}(p)$ is even

$$\begin{aligned}
\hat{r}_0(p) &= \frac{i}{2\pi} Sgn(p^0) \Big[\int_{-\infty}^{0} dt \frac{\hat{d}(tp)}{1 - t + iSgn(p^0)0^+} + \int_{0}^{\infty} dt \frac{\hat{d}(tp)}{1 - t + iSgn(p^0)0^+} \Big] \\
&= \frac{i}{2\pi} Sgn(p^0) \Big[\int_{0}^{\infty} dt \frac{\hat{d}(tp)}{1 + t + iSgn(p^0)0^+} + \int_{0}^{\infty} dt \frac{\hat{d}(tp)}{1 - t + iSgn(p^0)0^+} \Big] \\
&= \frac{i}{2\pi} Sgn(p^0) \int_{0}^{\infty} dt \frac{2\hat{d}(tp)}{1 - t^2 + iSgn(p^0)0^+}.
\end{aligned} \tag{A.27}$$

If $\hat{d}(p)$ is odd

$$\begin{aligned}
\hat{r}_0(p) &= \frac{i}{2\pi} Sgn(p^0) \Big[\int_{-\infty}^{0} dt \frac{\hat{d}(tp)}{1 - t + iSgn(p^0)0^+} + \int_{0}^{\infty} dt \frac{\hat{d}(tp)}{1 - t + iSgn(p^0)0^+} \Big] \\
&= \frac{i}{2\pi} Sgn(p^0) \Big[- \int_{0}^{\infty} dt \frac{\hat{d}(tp)}{1 + t + iSgn(p^0)0^+} + \int_{0}^{\infty} dt \frac{\hat{d}(tp)}{1 - t + iSgn(p^0)0^+} \Big] \\
&= \frac{i}{2\pi} Sgn(p^0) \int_{0}^{\infty} dt \frac{2t\hat{d}(tp)}{1 - t^2 + iSgn(p^0)0^+}.
\end{aligned} \tag{A.28}$$

A.6.2 Singular Case

The same performed in the last subsections will be repeated but for the formula (3.94) in the singular case

$$\hat{r}_0(p) = \frac{i}{2\pi} Sgn(p^0) \int_{-\infty}^{+\infty} dt \frac{\hat{d}(tp)}{t^{\omega+1}(1 - t + iSgn(p^0)0^+)}. \tag{A.29}$$

If $\hat{d}(p)$ and ω are both even or odd

$$\begin{aligned} \hat{r}_0(p) &= \frac{i}{2\pi} Sgn(p^0) \Big[\int_{-\infty}^{0} dt \frac{\hat{d}(tp)}{t^{\omega+1}(1 - t + iSgn(p^0)0^+)} + \int_{0}^{\infty} dt \frac{\hat{d}(tp)}{t^{\omega+1}(1 - t + iSgn(p^0)0^+)} \Big] \\ &= \frac{i}{2\pi} Sgn(p^0) \int_{0}^{\infty} dt \frac{2t\hat{d}(tp)}{t^{\omega+1}(1 - t^2 + iSgn(p^0)0^+)}. \end{aligned} \tag{A.30}$$

If $\hat{d}(p)$ and ω are even and odd respectively or opposite

$$\begin{aligned} \hat{r}_0(p) &= \frac{i}{2\pi} Sgn(p^0) \Big[\int_{-\infty}^{0} dt \frac{\hat{d}(tp)}{t^{\omega+1}(1 - t + iSgn(p^0)0^+)} + \int_{0}^{\infty} dt \frac{\hat{d}(tp)}{t^{\omega+1}(1 - t + iSgn(p^0)0^+)} \Big] \\ &= \frac{i}{2\pi} Sgn(p^0) \int_{0}^{\infty} dt \frac{2\hat{d}(tp)}{t^{\omega+1}(1 - t^2 + iSgn(p^0)0^+)}. \end{aligned} \tag{A.31}$$

Appendix B

Calculation of differential cross sections using wave packets

In the computation of differential cross section, the first step is to determine the transition amplitude $S_{if} = \langle in|S|out\rangle$. In the case of two particle scattering, the in and out states in the Hilbert space are

$$|in\rangle = |\psi_i\rangle = \int d^3p_1 d^3q_1 \varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1)|\mathbf{p}_1, \mathbf{q}_1\rangle, \tag{B.1}$$

$$|out\rangle = |\psi_f\rangle = \int d^3p_2 d^3q_2 \psi_f(\mathbf{p}_2, \mathbf{q}_2)|\mathbf{p}_2, \mathbf{q}_2)\rangle, \tag{B.2}$$

where $\varphi_{1,2}$ and ψ_f are wave packets sharply picked in $p_1 = p_i$, $p_2 = p_f$, $q_1 = q_i$ and $q_2 = q_f$.

Using (B.1) and (B.2), the transition amplitude take the following form

$$S_{fi} = \int d^3p_1 d^3q_1 d^3p_2 d^3q_2 \psi_f^*(\mathbf{p}_2, \mathbf{q}_2)\tilde{S}_{if}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)\varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1). \tag{B.3}$$

where $\tilde{S}_{if}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)$ is "transition amplitude" computed in the standard formalism

$$\tilde{S}_{if}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2) = \langle \mathbf{p}_2, \mathbf{q}_2|S|\mathbf{p}_1, \mathbf{q}_1\rangle. \tag{B.4}$$

In the framework of distribution theory, the wave packets are the *test functions* of $\tilde{S}_{if}$. This computation with wave packets is well defined, for that reason we choose it to this thesis.

Now, the transition probability is defined as

$$P_{if} \equiv |S_{if}|^2. \tag{B.5}$$

Replacing (B.3) into (B.5), we have

$$P_{if} = \int d^3p_1 d^3q_1 d^3p_2 d^3q_2 \psi_f^*(\mathbf{p}_2, \mathbf{q}_2) S_{if}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2) \varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1) \times \int d^3p'_1 d^3q'_1 d^3p'_2 d^3q'_2 \psi_f(\mathbf{p}'_2, \mathbf{q}'_2) S^*_{if}(\mathbf{p}'_1, \mathbf{q}'_1, \mathbf{p}'_2, \mathbf{q}'_2) \varphi_1^*(\mathbf{p}'_1)\varphi_2^*(\mathbf{q}'_1). \tag{B.6}$$

Summing over all final states, as usual, we obtain

$$\begin{aligned}\sum_f P_{if} &= \int d^3p'_1 d^3q'_1 d^3p_1 d^3q_1 d^3p_2 d^3q_2 S_{if}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)\varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1)\times \\ &\int d^3p'_2 d^3q'_2 \sum_f \psi_f^*(\mathbf{p}'_2, \mathbf{q}'_2)\psi_f(\mathbf{p}_2, \mathbf{q}_2) S^*_{if}(\mathbf{p}'_1, \mathbf{q}'_1, \mathbf{p}'_2, \mathbf{q}'_2)\varphi_1^*(\mathbf{p}'_1)\varphi_2^*(\mathbf{q}'_1) \\ &= \int d^3p'_1 d^3q'_1 d^3p_1 d^3q_1 d^3p_2 d^3q_2 S_{if}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)\varphi_1^*(\mathbf{p}'_1)\varphi_2^*(\mathbf{q}'_1)\varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1)\times \\ &\int d^3p'_2 d^3q'_2 \delta(\mathbf{p}_2 - \mathbf{p}'_2)\delta(\mathbf{q}_2 - \mathbf{q}'_2) S^*_{if}(\mathbf{p}'_1, \mathbf{q}'_1, \mathbf{p}'_2, \mathbf{q}'_2) \\ &= \int d^3p'_1 d^3q'_1 d^3p_1 d^3q_1 d^3p_2 d^3q_2 S_{if}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2) S^*_{if}(\mathbf{p}'_1, \mathbf{q}'_1, \mathbf{p}_2, \mathbf{q}_2)\times \\ &\varphi_1^*(\mathbf{p}'_1)\varphi_2^*(\mathbf{q}'_1)\varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1).\end{aligned} \tag{B.7}$$

In the computation of the distribution $\tilde{S}_{fi}$, it is possible to give it in the following structure

$$\tilde{S}_{fi}(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2) = \delta(p_2 + q_2 - p_1 - q_1) M(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2), \tag{B.8}$$

where the delta function represents the conservation of energy and momentum, and $M(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)$ has information about the scattering process. Replacing (B.8) into (B.7), we have

$$\begin{aligned}\sum_f P_{if} = \int & d^3p'_1 d^3q'_1 d^3p_1 d^3q_1 d^3p_2 d^3q_2 \delta(p_1 + q_1 - p_2 - q_2) M(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)\times \\ &\times \delta(p'_1 + q'_1 - p_2 - q_2) M^*(\mathbf{p}'_1, \mathbf{q}'_1, \mathbf{p}_2, \mathbf{q}_2)\varphi_1^*(\mathbf{p}'_1)\varphi_2^*(\mathbf{q}'_1)\varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1).\end{aligned} \tag{B.9}$$

Taking into account that the wave packets φ_1 and φ_2 are sharply peaked over $\mathbf{p}_i$ and $\mathbf{q}_i$, respectively, and M takes smaller values over the same coordinates, we can rewrite (B.9) as

$$\begin{aligned}\sum_f P_{if} &= \int d^3p_2 d^3q_2 |M(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)|^2\times \\ &\times \int d^3p_1 d^3q_1 \delta(p_1 + q_1 - p_2 - q_2)\varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1)\times \\ &\times \int d^3p'_1 d^3q'_1 \delta(p'_1 + q'_1 - p_2 - q_2)\varphi_1^*(\mathbf{p}'_1)\varphi_2^*(\mathbf{q}'_1),\end{aligned} \tag{B.10}$$

where the integrals in p_1, p_1', q_1 y q_1' depend of the initial states. Now, replacing the delta functions by its integral representation

$$\delta(p) = (2\pi)^{-4} \int d^4 x e^{\pm ipx}, \tag{B.11}$$

we obtain

$$\begin{aligned}
\sum_f P_{if} &= \int d^3p_2 d^3q_2 |M(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)|^2 \times \\
&\times [(2\pi)^{-8} \int d^3p_1 d^3q_1 \varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1) \int d^4x_1 e^{-i(p_1+q_1-(p_2+q_2))x_1} \times \\
&\times \int d^3p_1' d^3q_1' \varphi_1^*(\mathbf{p}_1')\varphi_2^*(\mathbf{q}_1') \int d^4x_2 e^{i(p_1'+q_1'-(p_2+q_2))x_2}] \\
&\equiv \int d^3p_2 d^3q_2 |M(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)|^2 F(p_2+q_2),
\end{aligned} \tag{B.12}$$

where we define the function F as

$$\begin{aligned}
F(p) = (2\pi)^{-8} &\int d^3p_1 d^3q_1 \varphi_1(\mathbf{p}_1)\varphi_2(\mathbf{q}_1) \int d^4x_1 e^{-i(p_1+q_1-p)x_1} \times \\
\times &\int d^3p_1' d^3q_1' \varphi_1^*(\mathbf{p}_1')\varphi_2^*(\mathbf{q}_1') \int d^4x_2 e^{i(p_1'+q_1'-p)x_2},
\end{aligned} \tag{B.13}$$

and $p = p_f + q_f$.

From (B.13), we can construct the following free wave packets in x-space

$$\widetilde{\varphi}(x) = (2\pi)^{-3/2} \int d^3p e^{-ipx} \varphi(\mathbf{p}). \tag{B.14}$$

Replacing (B.14) into (B.13), we have

$$F(p) = (2\pi)^{-2} \int d^4x_1 d^4x_2 \widetilde{\varphi}_1(x_1)\widetilde{\varphi}_2(x_1)\widetilde{\varphi}_1^*(x_2)\widetilde{\varphi}_2^*(x_2) e^{ip(x_1-x_2)}. \tag{B.15}$$

If we integrate the positive function $F(p)$ in p, we obtain

$$\int d^4p F(p) = (2\pi)^2 \int d^4x |\widetilde{\varphi}_1(x)|^2 |\widetilde{\varphi}_2(x)|^2. \tag{B.16}$$

Furthermore, $F(p)$ must be concentrated around $p = p_2 + k_2 = p_1 + q_1 \approx p_i + q_i$ because the wave packets $\varphi_1(\mathbf{p}_1)$ and $\varphi_2(\mathbf{q}_1)$ are sharply peaked around $\mathbf{p}_i$ and $\mathbf{q}_i$, respectively. Then, we can rewrite $F(p)$ in the following form

$$F(p) = \delta(p - p_i - k_i)(2\pi)^2 \int d^4x |\widetilde{\varphi}_1(x)|^2 |\widetilde{\varphi}_2(x)|^2. \tag{B.17}$$

Replacing (B.17) into (B.12), we have

$$\sum_f P_{if} = \int d^3p_2 d^3k_2 |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2)|^2 \delta(p_2 + k_2 - p_i - q_i)(2\pi)^2 \int d^4x |\widetilde{\varphi}_1(x)|^2 |\widetilde{\varphi}_2(x)|^2, \tag{B.18}$$

where the wave packets $\widetilde{\varphi}_{1,2}(x)$ represent the movement of the two bunches of particles scattered.

Considering a laboratory frame, we will define $\widetilde{\varphi}_1(x)$ representing the bunch of particles in movement with velocity $\mathbf{v}$ and $\widetilde{\varphi}_2(x)$ the target at rest, the wave packets take the following forms

$$\widetilde{\varphi}_1(t, \mathbf{x}) = f_1(\mathbf{x} + \mathbf{x}_1 + \mathbf{v}t), \tag{B.19}$$

$$\widetilde{\varphi}_2(t, \mathbf{x}) = f_2(\mathbf{x}). \tag{B.20}$$

Replacing (B.19) and (B.20) into (B.18), and averaging over the cylinder of radius R parallel to $\mathbf{v}$, we obtain

$$\begin{aligned} \overline{\sum_f P_{if}(R)} = \frac{(2\pi)^2}{\pi R^2} \int d^3p_2 d^3q_2 |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2)|^2 \delta(p_2 + q_2 - p_i - q_i) \times \\ \times \int\limits_{|x_{1\perp}| \leqslant R} d^2x_{1\perp} \int d^4x |f_1(\mathbf{x} + \mathbf{x}_1 + \mathbf{v}t)|^2 |f_2(\mathbf{x})|^2. \end{aligned} \tag{B.21}$$

Grouping the temporal integration variable from the third integral with $d^2x_{1\perp}$ and using the normalization of wave packets functions, we obtain

$$\overline{\sum_f P_{if}(R)} = \frac{(2\pi)^2}{\pi R^2 |\mathbf{v}|} \int d^3p_2 d^3q_2 |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2)|^2 \delta(p_2 + q_2 - p_i - q_i). \tag{B.22}$$

Replacing (B.22) into the following definition of the cross section

$$\sigma_{\text{lab}} \equiv \lim_{R \to \infty} \pi R^2 \overline{\sum_f P_{if}(R)}, \tag{B.23}$$

we obtain

$$\sigma_{\text{lab}} = (2\pi)^2 \frac{E_i}{|\mathbf{p}_i|} \int d^3p_2 d^3q_2 |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2)|^2 \delta(p_2 + q_2 - p_i - q_i). \tag{B.24}$$

The formula (B.24), is written as Lorentz invariant in the following form

$$\sigma = (2\pi)^2 \frac{E_{q_i} E_{p_i}}{\sqrt{(p_i q_i)^2 - m^4}} \int d^3p_f d^3q_f |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2)|^2 \delta(p_f + q_f - p_i - q_i). \tag{B.25}$$

From ((B.24)), we could determine the cross section in the center of mass reference replacing $\mathbf{q}_i = -\mathbf{p}_i$ and $\mathbf{q}_f = -\mathbf{p}_f$

$$
\begin{aligned}
\sigma_{c.m} &= (2\pi)^2 \frac{E}{2\sqrt{E^2 - m^2}} \int d^3p_f d^3q_f |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2|^2 \delta^3(\mathbf{p}_f + \mathbf{q}_f - \mathbf{p}_i - \mathbf{q}_i) \times \\
&\times \delta(2E_f - 2E) \\
&= (2\pi)^2 \frac{E}{2\sqrt{E^2 - m^2}} \int d^3q_f |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2)|^2 \delta(2E_f - 2E) \\
&= (2\pi)^2 \frac{E}{4\sqrt{E^2 - m^2}} \int |\mathbf{q}_f|^2 d|\mathbf{q}_f| d\Omega |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2)|^2 \delta(E_f - E) \\
&= (2\pi)^2 \frac{E}{4\sqrt{E^2 - m^2}} \int |\mathbf{q}_f| E_f dE_f d\Omega |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2)|^2 \delta(E_f - E). \\
&= (2\pi)^2 \frac{E}{4} \int E d\Omega |M(\mathbf{p}_i, \mathbf{q}_i, \mathbf{p}_2, \mathbf{q}_2)|^2.
\end{aligned}
\tag{B.26}
$$

From the last result (B.26), we obtain the following differential cross section in the center-of-mass reference system

$$
\frac{d\sigma_{c.m}}{d\Omega} = (2\pi)^2 \frac{E^2}{4} |M|^2. \tag{B.27}
$$

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