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Scattering Amplitudes using Twistor Strings

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Abstract

In this work we review the maximal helicity violating (MHV) scattering amplitude in the context of super-Yang-Mills theory. We study the symmetries of the MHV amplitude in the twistor space as a motivation to introduce the twistor string theory.

The twistor string action introduced by Nathan Berkovits [3] is reviewed and also a general formula is given for the scattering amplitude with n gluons. In the end, the MHV amplitude is derived from this formula.

Key Words: Super Strings, Twistors, MHV Amplitudes .

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Resumo

Neste trabalho revisamos as amplitudes de MHV no contexto da teoria de super-Yang-Mills. Nós estudamos as simetrias das amplitudes de MHV no espaço de twistors como uma motivação para introduzir a teoria de cordas com twistors.

A teoria de cordas com twistors feita por Nathan Berkovits [3] é revisada e uma fórmula geral é dada para calcular amplitudes de espalhamento com n gluons. No final, a partir desta fórmula deduzimos a amplitude de MHV.

Palavras-chave: Super Cordas, Twistors, Amplitude de MHV

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Contents

To my parents

Chapter 1

Introduction

The Large Hadron Collider (LHC) is the largest scientific lab ever constructed to conduct high-energy particle physics research. By smashing particles physicists can test different theories and discover new physics. The theoretical tools used to understand the behavior of subatomic particles include calculating scattering amplitudes using the Feynman diagrams. This method is straightforward for small systems but it can become increasingly complicated when adding multiple particles or loop corrections. For example, in the Yang-Mills theory the 5-gluon tree scattering amplitude gives almost $10,000$ terms. As an alternative, using the helicity formalism Park and Taylor [1] were able to conjecture an amazing simple formula for the maximal helicity violating (MHV) amplitude for n number of particles. Using this approach, the $10,000$ terms needed to calculate the 5-gluon tree scattering can be compressed to a single term. Aside from the numerical efficiency, we want a deeper understanding of the structure of multi-gluon amplitudes.

One big step towards this deeper understanding was done in 2004 by Witten [2] who proposed a string theory that is dual to a weakly coupled $\mathcal{N} = 4$ gauge theory. He constructed a topological B-model from twistor worldsheet variables and argued that D-instanton contribution in this model reproduce the perturbative super-YM amplitudes. In the same year an alternative string theory [3] was proposed by Nathan Berkovits. Berkovits' string theory is also based on twistor worldsheet variables, but use ordinary open string tree amplitudes instead of D-instantons in the B-model. Later giving support on this an alternative twistor string theory, a cubic open string field theory was constructed by [4] N.Berkovits and L.Motl. After the Witten original idea of twistor string theory [2], further development was done in many of papers [5–9] and drastically simplify the conventional Feynman diagram techniques for SYM and Super gravity.

Despite the success of twistor string theory at tree level, there are still many open questions. One issue is that in both Witten's and Berkovits' twistor strings gives $\mathcal{N} = 4$ conformal supergravity [10]. At tree level, it is possible to recover the Yang-Mills amplitudes by extracting the single-trace amplitude. But at loop level, the single trace gluon amplitude receive contributions from internal supergravity states. Since Yang-Mills theory makes sense without conformal supergravity, it is likely that there is a twistor string theory that is dual to pure Yang-Mills theory.

In chapter 2 we will review the spinor helicity formalism, used to describe the Park & Taylor formula for the MHV amplitude. $\mathcal{N} = 4$ super Yang-Mills theory is an important test case for perturbative gauge theory. It is the simplest case and has the same tree gluonic amplitudes as the pure Yang-Mills theory. In addition, we are also going to present a review of the MHV amplitudes and Super-Yang Mills theory.

After that in chapter 3 we present the MHV amplitudes in twistor space by Fourier transforming the spinor variables introduced in chapter 2. We also review on conformal symmetry and super conformal symmetry, and check that super Yang-Mills amplitudes have the same symmetries.

Finally in chapter 4 we switch our focus to present some aspects of $2d$ conformal field theory(CFT), the bc system, which is going to be used in chapter 5. chapter 5 will be dedicated to introduce the Berkovits' Twistor string theory. In the end of the chapter the MHV amplitude is found from the Twistor string theory. The chapter 6 is dedicated to motivate the study done here, and also shed some lights in the problems and future work. In Appendix A we explain the notation and present a spinor review.

Chapter 2

Helicity Amplitudes

In the past 5 years the helicity formalism has been used to describe most of the recent discoveries in massless scattering amplitudes. For this reason, it is a fundamental tool if one wants to understand what has been done in the field of the scattering amplitudes. With this in mind, this chapter is dedicated to introduce the helicity formalism and also give a review on MHV amplitudes and Super Yang-Mills. The notation and some background on spinors are given on Appendix A.

2.1 Null momenta - The spinor helicity formalism

We begin by introducing Appendix A that the Lorentz Group $SO(3,1)$ is isomorphic to $SL(2,\mathbb{C})$. A Null Lorentz vector p_m $(m = 0,1,2,3)$ can be constructed by the product of two spinors, one from $(1/2,0)$ representation and one from $(0,1/2)$. The spinor λ_α, $\alpha = 1,2$, that transforms under $(1/2,0)$ are called left-handed Chiral spinors and $\bar{\lambda}_{\dot{\alpha}}$, $\dot{\alpha} = \dot{1},\dot{2}$ that transforms under$(0,1/2)$ are called the right-handed Chiral spinors.

The map that takes the Lorentz vector index m to two spinors indices $(\alpha\dot{\alpha})$ is done by a linear combination of Pauli Matrices and the identity $[\sigma^m = (-I,\vec{\sigma})]$:

$$p_{\alpha\dot{\alpha}} = p_m {\sigma^m}_{\alpha\dot{\alpha}} = -p_0 I + \vec{\sigma}\cdot\vec{p} \tag{2.1}$$

As a consequence of (2.1) we get:

$$\det(p_{\alpha\dot{\alpha}}) = -p^m p_m \tag{2.2}$$

the minus sign is due to the metric $\eta_{mn} = (-+++)$. Note that we are using p to denote different objects. Because we are interested in massless particles, $p^2 = 0$ implies

$\det(p_{\alpha\dot{\alpha}}) = 0$. So p has one zero eigenvalue and the rank of the 2×2 matrix goes down by one. We can write the matrix p as a product of two component spinors:

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} \tag{2.3}$$

these spinors are usually called helicity spinors. The $\lambda, \bar{\lambda}$ are not sufficient to determinate p, there is a scale freedom:

$$(\lambda, \bar{\lambda}) \to (t\lambda, t^{-1}\bar{\lambda}) \quad t \in \mathbb{C}^* \tag{2.4}$$

And since the momentum p_m is real

$$p_{\alpha\dot{\alpha}} = p^*_{\alpha\dot{\alpha}} \Rightarrow \bar{\lambda}^*_{\dot{\alpha}} = \lambda_\alpha, \tag{2.5}$$

and the scale parameter t becomes just a phase $e^{i\theta}$. The spinors $(\lambda, \bar{\lambda})$ are not independent. This is the real world, but we are theoretical physicists and we can do almost anything, for example consider the case where the momentum is complex, then we get rid off the constrain $\bar{\lambda}^*_{\dot{\alpha}} = \lambda_\alpha$. Your mathematician friend would say that you are complexing the Lorentz group and that it is locally isomorphic to $SO(3,1,\mathbb{C}) \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. Another way to make the spinors $(\lambda, \bar{\lambda})$ be independent, is to consider the Lorentz group with a different signature: $\eta = (-+++) \to (--++)$. We can use the fact that $SO(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, and write $(\lambda, \bar{\lambda})$ as two real independent spinors.

A quick exercise is to count the degrees of freedom for each case. In complex momentum we have: $4 \times 2 - 2$ ($\det p = 0$ constrain) $= 6$ real parameters. For complex $(\lambda, \bar{\lambda})$ minus the scale freedom $2 \times 2 - 1 = 3$ complex or 6 real parameters. In the real momentum case we have $4 - 1 = 3$ real parameters.

Given two spinors λ, μ the Lorentz invariant object is:

$$\langle \lambda, \mu \rangle \equiv \varepsilon^{\alpha\beta} \lambda_\alpha \mu_\beta = \lambda_\alpha \mu^\alpha \tag{2.6}$$

where we use the antisymmetric tensor $\varepsilon^{\alpha\beta}$ defined as $\varepsilon_{12} = \varepsilon^{21} = -1$ and $\varepsilon^{\alpha\beta}\varepsilon_{\beta\rho} = \delta^\alpha{}_\rho$, to lower and raise the indices. From the definition (2.6) the product $\langle \lambda, \mu \rangle = -\langle \mu, \lambda \rangle$ is antisymmetric. In particular, if $\langle \lambda, \mu \rangle = 0$, it implies $\mu \sim \lambda$. This is true because λ, μ are commuting variables, note that if they were anti-commuting $\langle \lambda, \mu \rangle = \langle \mu, \lambda \rangle$ as seen Appendix A.

The same thing is valid for two dotted spinors:

$$[\lambda, \mu] \equiv \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\lambda}_{\dot{\alpha}} \bar{\mu}_{\dot{\beta}} = \bar{\lambda}_{\dot{\alpha}} \bar{\mu}^{\dot{\alpha}} \tag{2.7}$$

Note that for the real momentum we have the complex conjugation

$$\langle\lambda, \mu\rangle^* = [\lambda, \mu] \tag{2.8}$$

Let's see how others objects look in the spinor formalism. Given two null momentum $p^{\dot{\alpha}\alpha} = \lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$ and $q_{\alpha\dot{\alpha}} = \mu_\alpha \bar{\mu}_{\dot{\alpha}}$:

$$(p+q)^2 = 2p \cdot q = 2\lambda^\alpha \bar{\lambda}^{\dot{\alpha}} \mu_\alpha \bar{\mu}_{\dot{\alpha}} = 2\langle\lambda, \mu\rangle[\lambda, \mu] \tag{2.9}$$

For instance if we label different momentum by a number $p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)}$... we can simplify even more the notation. Using the numbers to identify the momentum write $(p^{(1)} + p^{(2)})^2 = 2\langle 1, 2\rangle[1, 2]$. This notation is common in the literature, and very elegant.

Recall that the scattering amplitudes for gluons are described by momentum p_m and polarization vectors ϵ^m. We already have p in spinors, now we need the polarization vector. From the Yang-Mills Equation of motion in momentum space:

$$p \cdot \epsilon = 0 \tag{2.10}$$

it represents the fact that the polarization vector ϵ^m does not have longitudinal components and it has a gauge symmetry $\epsilon^m \to \epsilon^m + bp^m$. To construct ϵ^m as a bi-spinor we can guess the result using its symmetries. Let us define $\epsilon^+_{\alpha\dot{\alpha}} = d^{-1}\mu_\alpha \bar{\lambda}_{\dot{\alpha}}$ where d is a Lorentz invariant object, and μ_α is an arbitrary spinor. The polarization has to be invariant under a scale $\mu \to a\mu$ because μ is arbitrary. Thus $d \sim \mu_\alpha$, but d is Lorentz invariant, and the only object that we have to construct is λ^α. The result is:

$$\epsilon^+_{\alpha\dot{\alpha}} = \frac{\mu_\alpha \bar{\lambda}_{\dot{\alpha}}}{\langle\mu, \lambda\rangle} \tag{2.11}$$

Just by looking at (2.11) we see that $\epsilon^+_{\alpha\dot{\alpha}} p^{\dot{\alpha}\alpha} = 0$ due to $[\lambda, \lambda] = 0$. The gauge transformation ($\epsilon^m \to \epsilon^m + bp^m$) is now translated to a spinor shift ($\mu_\alpha \to a\mu_\alpha + b\lambda_\alpha$). Then we get:

$$\epsilon^+_{\alpha\dot{\alpha}} \to \epsilon^+_{\alpha\dot{\alpha}} + b\frac{\lambda_\alpha \lambda_{\dot{\alpha}}}{\langle\mu, \lambda\rangle} \tag{2.12}$$

We know that the polarization vector (the photon) has two degrees of freedom. So we have another polarization. Doing the same process we find:

$$\epsilon^-_{\alpha\dot{\alpha}} = \frac{\bar{\mu}_{\dot{\alpha}} \lambda_\alpha}{[\mu, \lambda]} \tag{2.13}$$

Note that this vectors are normalized $\epsilon^- \cdot \epsilon^+ = 1$. Another thing that is important to mention is that under complex conjugation the helicity is flipped

$$(\epsilon^-_{\alpha\dot{\alpha}})^* = \epsilon^+_{\alpha\dot{\alpha}} \tag{2.14}$$

Now we return to the scaling and make contact with to helicity. The helicity is a Lorentz conserved quantity and it is define as the projection of the spin operator $\vec{S}$ into the 3-momentum $\vec{p}$

$$h \equiv \frac{\vec{S} \cdot \vec{p}}{|\vec{p}|} \tag{2.15}$$

In the massless case the particle is moving at the speed of light. So we can not do a boost that invert the direction of the rotation.

Let us write the massless Dirac equation in terms of Weyl spinors:

$$(\bar{\sigma}^m)^{\dot{\alpha}\alpha}\partial_m\psi_\alpha = 0 \tag{2.16}$$

$$(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}} = 0 \tag{2.17}$$

we get the massless Klein-Gordon equation if we multiply it by $\sigma^n_{\dot{\alpha}\alpha}\partial_n$ (using $\mathrm{Tr}\sigma^n\bar{\sigma}^m = -2\eta^{mn}$):

$$\partial^m\partial_m\psi_\alpha = 0 \tag{2.18}$$

It has a plane wave solution $\psi_\alpha = L_\alpha e^{ip\cdot x}$, for a constant L_α. From (2.16) L_α must satisfy $L_\alpha p^{\dot{\alpha}\alpha} = \langle L, \lambda\rangle\bar{\lambda}^{\dot{\alpha}} = 0$ that implies $L_\alpha = c\lambda_\alpha$. Thus the wave function is

$$\psi_\alpha = c\lambda_\alpha e^{ip^{\dot{\alpha}\alpha}x_{\alpha\dot{\alpha}}} \tag{2.19}$$

The spinor transforms as a rotation by angle θ around the $\vec{n}$ direction as described in Appendix A

$$\psi_\alpha = e^{i\frac{\vec{\theta}\cdot\vec{n}}{2}}\psi_\alpha \tag{2.20}$$

this implies that λ carries half units of angular momentum. Now if we define the scalar parameter $t \equiv e^{i\frac{\vec{\theta}\cdot\vec{n}}{2}}$ we see that the wave function scales as t^{-2h} if $h = -1/2$. So we say that λ has negative helicity $h = -1/2$.

We can go on and find the wave function for $\bar{\psi}^{\dot{\alpha}}$. This will define a wave function for helicity $h = +1/2$. Because $\bar{\psi}_{\dot{\alpha}}$ transforms as the complex conjugate of ψ_α. Thus the i on the rotation give a minus sign, and flips the h.

$$\bar{\psi}_{\dot{\alpha}} = c\bar{\lambda}_{\dot{\alpha}} e^{ip^{\dot{\alpha}\alpha}x_{\alpha\dot{\alpha}}} \tag{2.21}$$

Now we can understand why the polarization vector were label by $\epsilon^+,\ \epsilon^-$. Under the scaling $(\lambda, \bar{\lambda}) \to (t\lambda, t^{-1}\bar{\lambda})$ the polarization vectors scale as

$$(\epsilon^+,\ \epsilon^-) \to (t^{-2}\epsilon^+,\ t^{+2}\epsilon^-) = (t^{-2h}\epsilon^+,\ t^{-2h}\epsilon^-) \tag{2.22}$$

then ϵ^+ has helicity $+1$ and ϵ^- has helicity $h = -1$. Consider a function that transforms as $f(e^{i\theta}x) = e^{i\theta h}f(x)$. We can think of the function as $f(x) = x^h$ and this satisfy $x\partial_x f(x) = hf(x)$. A wave function $\psi(\lambda, \bar{\lambda})$ will satisfy a similar equation

$$\left(\lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} - \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}}\right)\psi(\lambda, \bar{\lambda}) = -2h\psi(\lambda, \bar{\lambda}) \tag{2.23}$$

sometimes this constraint is called the auxiliary condition. We can define the helicity operator as

$$h = -\frac{1}{2}\left(\lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} - \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}}\right) \tag{2.24}$$

2.2 Scattering Amplitudes

Scattering amplitudes correspond to the probability amplitude for the a specific scattering process. The amplitude for the scattering of an initial state at $t \to -\infty$ of n_i particles into a final state at $t \to +\infty$ of n_f particles is determined by the S-matrix element $\langle out|S|in\rangle$. The asymptotic states are given $|p_i, \epsilon_i\rangle$, where p_i is the on-shell external momentum and ϵ_i is the polarization vector. Since we can specify a particle with spin by its momentum $p^{(i)}_{\alpha\dot{\alpha}} = \lambda^{(i)}_\alpha \bar{\lambda}^{(i)}_{\dot{\alpha}}$ and helicity $h^{(i)}$, the amplitude can be written as function of Lorentz invariant objects in term of $(\lambda, \bar{\lambda}, h)$

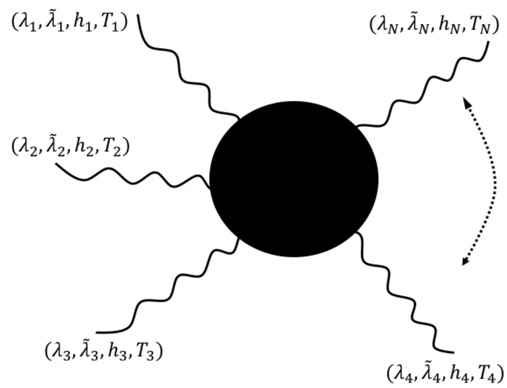

FIGURE 2.1: Scattering Amplitude with N external gluons in Yang-Mills Theory. Each external gluon is label by a color factor T_i, spinors $\lambda_i, \bar{\lambda}_i$ and helicity h_i

$$A_n = A_n(\lambda^{(1)}_\alpha, \bar{\lambda}^{(1)}_{\dot{\alpha}}, h^{(1)}; \ldots; \lambda^{(n)}_\alpha, \bar{\lambda}^{(n)}_{\dot{\alpha}}, h^{(n)}) \tag{2.25}$$

Recall that under crossing we have $p \to -p$ and $\epsilon \to \epsilon^*$ or $\epsilon^+ \to \epsilon^-$. Then we can treat as all particles as outgoing and then use crossing to find the other helicity amplitudes. Since the amplitude is a function of the helicity spinors, it must also satisfy a auxiliary condition (2.23) for each particle $\lambda^{(i)}, \bar{\lambda}^{(i)}$:

$$\left(\lambda^{\alpha(i)}\frac{\partial}{\partial\lambda^{\alpha(i)}} - \bar{\lambda}^{\dot{\alpha}(i)}\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}(i)}}\right)A(\lambda^{(i)}, \bar{\lambda}^{(i)}, h^{(i)}) = -2h^{(i)}A(\lambda^{(i)}, \bar{\lambda}^{(i)}, h^{(i)}) \tag{2.26}$$

We see that under the scaling the amplitude transforms homogeneously with weight $-2h^{(i)}$, where $h^{(i)}$ is the helicity of the particle i

$$A(t\lambda^{(i)}, t^{-1}\bar{\lambda}^{(i)}, h^{(i)}) = t^{-2h^{(i)}}A(\lambda^{(i)}, \bar{\lambda}^{(i)}, h^{(i)}) \tag{2.27}$$

2.3 MHV Amplitudes

To appreciate the power of the spinor helicity formalism let us consider tree level scattering amplitudes in Yang-Mills Theory. The Yang Mills action is

$$S_{YM} = -\frac{1}{2}\int dx^4 Tr(F^{mn}F_{mn}) \tag{2.28}$$

with

$$F_{mn} \equiv F^A_{mn}T^A = \partial_{[m}A^A_{n]}T^A - igA^B_m A^C_n[T^B, T^C] \tag{2.29}$$

and T^A are generators of the gauge group $SU(N)$ with $A = 1, ...N^2-1$. These generators satisfy the algebra with the normalization:

$$[T^A, T^B] = if^{ABC}T^C \quad ; \quad Tr(T^AT^B) = \frac{1}{2}\delta^{AB} \tag{2.30}$$

where $f^{ABC} = -2iTr[T^A, T^B]T^C$ are the structure constants and being antisymmetric in all of its indices.

The first step of simplification of the Yang-Mills amplitude is the technique used to separate the color and kinematics degrees of freedom, it is called color decomposition. Amplitudes for the gluons will have products of color indices, due to Feynman rules for the vertex, see figure 2.2. In tree level the color decomposition lets us write the amplitude in terms of a single trace factor. To convince you that this really happens, recall that for the gauge group $SU(N)$ we have this identity (G, H are combination of

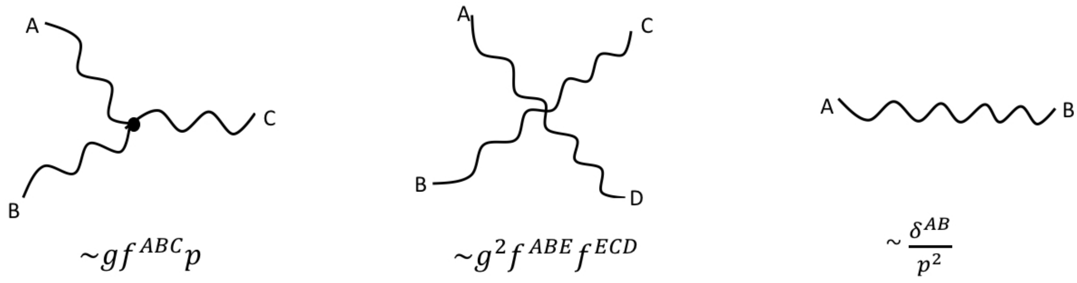

FIGURE 2.2: Color index dependence for Yang-Mills Feynman rules.

$T's$ matrices):

$$Tr(GT^E)Tr(HT^E) = \frac{1}{2}Tr(GH) \tag{2.31}$$

where we used the fact that $Tr(T^A) = 0$ and $E's$ are summed. From $Tr(F^2)$ we get two types of self interaction terms $A^A A^B \partial_m A^C \sim gp_m Tr([T^A, T^B]T^C)$ and $A^A A^B A^C A^D \sim Tr([T^A, T^B][T^C, T^D]$. The propagator glues two vertices by a δ^{AB}. Then color indices are summed over and the double trace term reduces to a single trace.

In the end, for n particles we have cyclic permutation of $Tr(T^{A_1}T^{A_2}...T^{A_n})$ and the amplitude factorizes as:

$$A_n = g^{n-2}(2\pi)^4\delta(\sum_{i=1}^{n} p_i)\mathcal{A}(p_1, h_1, ..., p_n, h_n)Tr(T^{A_1}T^{A_2}...T^{A_n}) + permutations \tag{2.32}$$

and we can concentrate in the color free Amplitude $\mathcal{A}(1, 2, \ldots, n)$.

Dimensional analysis is a powerful tool. Let us use it to have a feeling how the tree-amplitude behaves. From (2.29) we have $[dx^4] = M^{-4}$; $[F] = M^2$; $[\partial] = M^1$; $[A] = M^1$; $[g] = M^0$, where M is the mass dimension.

We have two vertices, the cubic vertex $V_3 \sim gf^{ABC}p$, from $AA\partial A$ that has mass dimension 1 ($[V_3] = M^1$). And the quartic vertex $V_4 \sim g^2 f^2$, from A^4 that has mass dimension zero ($[V_4] = M^0$). From the figure 2.3 is easy to see that the number of V_3 is always greater by one the number of propagators P (please do not get confuse with the momentum P). Also for n particle scattering, the number of V_3 is $n-2$. The mass dimension of the propagator is -2 ($[P] = M^{-2}$).

In conclusion the mass dimension of n particle scattering amplitude is

$$[\mathcal{A}_n] = \frac{M^{n-2}}{(M^2)^{n-3}} = M^{4-n} \tag{2.33}$$

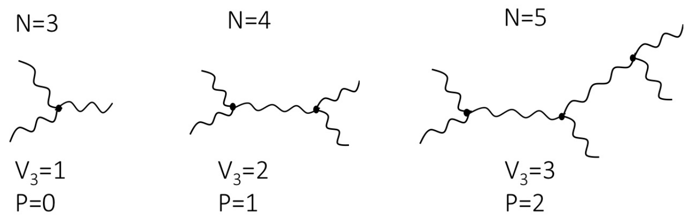

FIGURE 2.3: Sketch to count the number of vertices V_3 and Propagators P for three, four and five gluons diagrams. The mass dimension for n gluon amplitude is $[\mathcal{A}_n] = M^{4-n}$

An important note is that, in the numerator, the powers of momenta (M) can not be greater than $n-2$. Note that in the five gluons diagram 2.3 you also have a diagram with quartic vertex V_4. But you still have $[\mathcal{A}_5] = M^{4-5}$

A general tree amplitude is a product of Lorentz scalars

$$[\mathcal{A}_n] = \sum_{diagrams} \frac{\sum \prod(\epsilon_i \cdot \epsilon_j) \prod(\epsilon_i \cdot p_j) \prod(p_i \cdot p_j)}{\prod P^2} \tag{2.34}$$

in terms of spinors (2.11)-(2.13) the polarizations products take the form:

$$\epsilon_i^+ \cdot \epsilon_j^+ \propto \langle \mu_i \mu_j \rangle \quad ; \quad \epsilon_i^- \cdot \epsilon_j^- \propto [\mu_i \mu_j] \quad ; \quad \epsilon_i^- \cdot \epsilon_j^+ \propto \langle \lambda_i \mu_j \rangle [\mu_i \lambda_j] \tag{2.35}$$

where μ_i is an arbitrary spinor that represents gauge freedom (2.11). Finally we can attack the amplitude with all the tools so far developed.

Let us start with all plus polarizations amplitudes $\mathcal{A}_n(1^+2^+ \dots n^+)$. For n particles we have n polarizations ϵ^+. From (2.35) we see that if we choose $\mu_1 = \mu_2 = \dots = \mu$ the product $\prod(\epsilon_i^+ \cdot \epsilon_j^+) = 0$ due to $\langle \mu\mu \rangle = 0$. Then the only way for the amplitude not to be zero is to have $\prod(\epsilon_i \cdot p_j)$. But as we saw the amplitude in the numerator has n polarizations and we need n momenta to create a Lorentz scalar. But it can have only $n-2$ powers of momenta $p's$ and thus it is zero. In few lines we were able to evaluate ALL the plus helicity amplitudes. That's powerful.

We can continue, and see what else we learn about the amplitude with only the k particle with negative helicity, $\mathcal{A}_n(1^+ \dots k^- \dots n^+)$. Again we use the freedom that we have and choose $\mu_1 = \mu_2 = \dots \mu_n = \lambda_k$, then all the terms $\epsilon_i^+ \cdot \epsilon_j^+$ vanish where $i, j \neq k$. And from $\epsilon_k^- \cdot \epsilon_j^+ \propto \langle \lambda_k \mu_j \rangle [\mu_k \lambda_j]$ we also see that it is zero. Thus we get the same problem as before. We need n powers of momenta and we can only have $n-2$. We conclude that $\mathcal{A}_n(1^+ \dots k^- \dots n^+) = 0$.

Now that we have got this far let us do more, and calculate $\mathcal{A}_n(1^-, 2^-, 3^+ \dots \dots n^+)$.By Choosing $\mu_i = \lambda_1$ for $i \geq 3$, then $\epsilon_i^+ \cdot \epsilon_j^+ = 0$ for $i, j \geq 3$. Set $\mu_1 = \mu_2 = \lambda_k$. Then the only non zero polarizations are $\epsilon_2^- \cdot \epsilon_i^+ \propto \langle\lambda_2\mu_i\rangle[\mu_2\lambda_i]$ for $i \geq 3 \quad \& \quad i \neq k$. Now we see that we can have a term with two polarization, and we can fill in with $n-2$ momentum contraction $\epsilon_i \cdot p_j$. Therefore, we don't have a vanishing amplitude.

This Amplitude $\mathcal{A}_n(1^-, 2^-, 3^+ \dots n^+)$ is called the Maximally Helicity Violating (MHV). Before we continue let us understand this name. Because we choose all the particles to be outgoing, to see the physical ($2 \rightarrow n-2$) scattering, we have to use crossing. Recall that crossing symmetry interchange the helicity between the incoming particle and the outgoing particle. Then the first amplitude $\mathcal{A}_n(1^+, 2^+ \dots n^+)$ describes $1^-2^- \rightarrow 3^+ \dots n^+$ scattering, that's two particle incoming and $n-2$ particles outgoing. This 'violates' the helicity conservation. We can do the same thing for the other amplitudes. Then $\mathcal{A}_n(1^+, 2^- \dots n^+)$ describes $1^-2^+ \rightarrow 3^+ \dots n^+$ also violate but it is zero. Finally $\mathcal{A}_n(1^+, 2^+, 3^-, 4^- \dots n^+)$ describes $1^-2^- \rightarrow 3^-, 4^-, 5^+ \dots n^+$ is the processes that can be maximal violated that is not zero.

Now that we understand the name, let us jump to the final result. The MHV amplitude has a nice closed formula that was postulated by Parke & Taylor [1] and proved by Berends and Giele [11]. The prof uses off-shell recursion methods. This formula can be proven [12] by the BCFW recursion Relations that use on-shell methods and complex momenta. Then the tree-level amplitude for two negative helicity particles $(\{p^r, h^r = -\}, \{p^s, h^s = -\})$ and $n-2$ positive helicity particles is (without the color factor and momentum conservation)

$$\mathcal{A}_n(r^-, s^-) = \frac{\langle\lambda_r, \lambda_s\rangle^4}{\prod_{i=1}^n \langle\lambda_i, \lambda_{i+1}\rangle} \tag{2.36}$$

This is a very elegant and simple formula. Unfortunately we can are not going to prove this formula here, but we can check if the symmetries found on it hold . The Lorentz symmetry is trivial because (2.36) is made of Lorentz Invariant objects $\langle\lambda_i, \lambda_j, \rangle$. For the scaling, note that the denominator has two spinors for each particle (i), *i.e*, $\prod\langle\lambda_i, \lambda_{i+1}, \rangle \sim \lambda^{(i)}\lambda^{(i)}$. Thus under the scaling $\lambda^{(i)} \rightarrow t\lambda^{(i)}$ the amplitude gets a factor of t^{-2}. And in the numerator a factor of t^4, but note that the numerator gets this factor only for negative helicity r, s.

Then for a particle with positive helicity $h_i = +1$ $(i) \neq r, s$ the amplitude scales as

$$\mathcal{A}_n(r^-, s^-, ti^+) = t^{-2}\mathcal{A}_n(r^-, s^-, i^+) = t^{-2h^{(i)}}\mathcal{A}_n(r^-, s^-, i^+) \tag{2.37}$$

and for a particle with negative helicity $h_s = -1$

$$\mathcal{A}_n(r^-, ts^-) = t^{+2}\mathcal{A}_n(r^-, s^-) = t^{-2h_s}\mathcal{A}_n(r^-, s^-) \tag{2.38}$$

As we expected from the auxiliary condition (2.27). The googly-amplitude is defined as the MHV-amplitude with all fliped helicities . The googly amplitude can be obtained by the complex conjugation (2.8)-(2.14) on the MHV-amplitude. The amplitude with particles r, s with positive helicities and $n-2$ particles with negative helicities is given by

$$\mathcal{A}_n(r^+, s^+) = \frac{[\lambda_r, \lambda_s]^4}{\prod_{i=1}^{n}[\lambda_i, \lambda_{i+1}]}. \tag{2.39}$$

Another symmetry that Yang-Mills Theory have is conformal symmetry. Wait until the next chapter for an explain about conformal transformations and a proof that the Yang-Mills theory and MHV amplitude are conformal invariant. Now we are going to see that the reason for the amplitudes $\mathcal{A}_n(1^+2^+\ldots n^+)$ and $\mathcal{A}_n(1^+\ldots k^-\ldots n^+)$ are zero, is due to supersymmetry in the Super Yang-Mills theory.

2.4 $\mathcal{N}$ =4 Super Yang-Mills Theory

We are interesting in the supersymmetric version of Yang-Mills, but to get there I will make a review of supersymmetry, but a very brief one. The review on supersymmetry will be based on [13], and for super Yang-Mills [14, 15]. Supersymmetry is a very interesting subject and we hope to find it in the LHC soon. The supersymmetry algebra is the only graded Lie algebra of symmetries of the S-matrix in a relativistic Quantum Field Theory, this result was proven by Haag, Sohnius and Lopuszanki. We are enlarging the Poicaré algebra by adding fermionic generators $(Q, \bar{Q})$. These fermionic generators (Grassmann) map fermions to bosons. The supersymmetry algebra is given by

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} = 2\sigma^m_{\alpha\dot{\alpha}}P_m\delta^A{}_B \qquad \{Q^A_\alpha, Q^B_\beta\} = \varepsilon_{\alpha\beta}Z^{AB} \tag{2.40}$$

where m and $\alpha\dot{\alpha}$ are the usual vector and spinor index respectively. The index A labels the number of supersymmetries, it runs from $1\ldots\mathcal{N}$. Thus the total number of supercharges is $l\times\mathcal{N}$, l is the dimension of spinor representation. The $\bar{Q}_{\dot{\alpha}} \equiv (Q_\alpha)^\dagger$ is the adjoint of the operator Q_α. The $Z^{AB} = -Z^{BA}$ are called the central charges and only exist for $\mathcal{N} > 1$, because they are antisymmetric. But they are zero in the massless case[1].

Note that the algebra is preserved under a $U(\mathcal{N})$ transformation called R-Symmetry:

$$Q'^A_\alpha = R^A{}_B Q^B_\alpha \quad \bar{Q}'_{\dot{\alpha}A} = R^{-1B}_{A}\bar{Q}_{\dot{\alpha}B} \qquad R^A{}_B \in U(\mathcal{N}) \tag{2.41}$$

[1] $\{Q^A_2, \bar{Q}^B_2\} = 0 \Rightarrow \langle\psi|\bar{Q}^B_2 Q^A_2|\psi\rangle = 0$ which implies that $Q^A_2|\psi\rangle = 0$. Then Z^{AB} is zero since $Q^A_2 = 0$

Let us analyze the massless spectrum in four dimensions. In the frame $P_m = (-E, 0, 0, E)$ we have the algebra (2.40)

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \delta^A{}_B \tag{2.42}$$

In the upper side of the matrix (2.42) we can define the creation and annihilation operators $a^\dagger_A, a^A$ by

$$a^A = \frac{Q^A_1}{2\sqrt{E}} \quad ; \quad a^\dagger_A = (a^A)^\dagger = \frac{\bar{Q}_{\dot{1}A}}{2\sqrt{E}} \tag{2.43}$$

with this normalization we get $\{a^A, a^\dagger_B\} = \delta^A{}_B$. We can use $(a^\dagger_A, a^A)$ to raise and lower the helicity of a state by $\frac{1}{2}$. So we know how to create the Hilbert space. We start with the highest helicity h_{max} state $a^\dagger_A|h\rangle = 0$ and use the annihilation operator to decrease the helicity:

$$|h\rangle \quad , a^A|h\rangle \quad , a^{A_1} \cdots a^{A_\mathcal{N}}|h\rangle \tag{2.44}$$

The highest helicity is $h_{max} = h_{min} + \frac{\mathcal{N}}{2}$, thus we have $2^\mathcal{N}$ states. We can see that by looking the string of $a^{A_1}...a^{A_\mathcal{N}}$ and noting that in each space we can have or not an a^{A_i}. Each state is $\binom{\mathcal{N}}{n}$ [2] degenerate, where n is the number of a^{A_i} in the state.

If we want to keep the helicity $|h| < 1$ then the $\mathcal{N} = 4$ is the maximal amount of supersymmetry that we can have in four dimensions. And that is the Theory that we want to study. One interesting fact is that CPT theories usually double the number of states, because one needs the opposite helicity. But our theory is self-dual CPT, *i.e.* it has $h = \pm 1$; $\pm 1/2$. The spectrum can be summarize in the table 2.1.

states $\lvert h\rangle$	Number of states	Name
$\lvert 1\rangle$	1	gluon
$a^A\lvert h\rangle = \lvert 1/2\rangle$	4	gluion
$a^Aa^B\lvert h\rangle = \lvert 0\rangle$	6	scalar
$a^Aa^Ba^C\lvert h\rangle = \lvert -1/2\rangle$	4	anti-gluion
$a^1a^2a^3a^4\lvert h\rangle = \lvert -1\rangle$	1	gluon

TABLE 2.1: $\mathcal{N} = 4$ Super Yang-Mills Spectrum

The spectrum can be obtained by dimension reduction from the $d = 10, \mathcal{N} = 1$ action

$$S_{YM} = -\frac{1}{4g^2_{YM}} \int dx^{10} Tr \left(F_{MN}F^{MN} - 2i\bar{\psi}\Gamma^M D_M \psi\right) \tag{2.45}$$

M, N run from $(0, \ldots, 9)$ and ψ is a $10d$ Majorana-Weyl spinor that has the minimal representation 16 real components. The Γ^M are the gamma matrices in $10d$. By dimension reduction $(M) \to (m, i)$ we get the $\mathcal{N} = 4 \quad d = 4$ Super Yang-Mills theory, where

[2] $\binom{\mathcal{N}}{n} = \frac{\mathcal{N}!}{n!(\mathcal{N}-n)}$

m runs from 0 to 3, a $4d$ Lorentz index and i runs $4\ldots 10$ is the 6 compacted dimensions. To see that is true, note that (2.45) has $\mathcal{N}\times l = 1\times 16$ super charges (l is the spinor dimension), and no fields are higher than spin 1. The only theory in 4 dimensions that has the same properties is $\mathcal{N}=4$ SYM. For a review and more details, see [16].

After the dimension reduction, the action for $\mathcal{N}=4$ SYM in $d=4$ become

$$S_{YM} = -\frac{1}{2g_{YM}^2}\int dx^4 \left(\frac{1}{2}(F^{mn})^2 + (D_m X^i)^2 - [X^i, X^j]^2 + i\bar{\psi}^I\gamma^m D_m\psi^I + \right. \\ \left. + \bar{\psi}^I\Gamma^i_{IJ}[X^i,\psi^J]\right) \tag{2.46}$$

Here $(field)^2$ means the proper contraction of indices. The ψ^I are four $4d$ Weyl spinors $(I = 1\ldots 4)$, γ^m the $4d$ gamma matrices, and Γ^i_{IJ} the gamma matrices for $SO(6)$. Also the X^i are six real scalars, which i labels the $SO(6)$ global R-symmetry. This scalars come from the gauge field under the dimension reduction $A^M \to (A^m, X^i)$. All the fields (A^m, X^i, ψ^I) are in the adjoint of the gauge group (color group) which we choose to be $SU(N)$, *i.e.* $A^m \equiv A^{m(a)}T^{(a)}$ where $a = 1,\ldots,N^2-1$. Please do not get confused with N- number of the gauge group & $\mathcal{N}$- number of supersymmetries.

It is convenient to rewrite the X^i six real scalars as six complex field $\phi^{IJ} = -\phi^{JI}$ with $I, J = 1,\ldots 4$ transforming in the fully anti-symmetric 2-index representation of $SU(4)$. It also has to satisfy the condition $\bar{\phi}_{IJ} = \frac{1}{2}\epsilon_{IJKL}\phi^{KL}$.

From the figure 2.4 a tree level amplitude with only external gluons the scalars and fermions can not appear. This change in loop level, because we can form loop with scalar interaction and only external gluons (see the fourth vertex in figure 2.4).

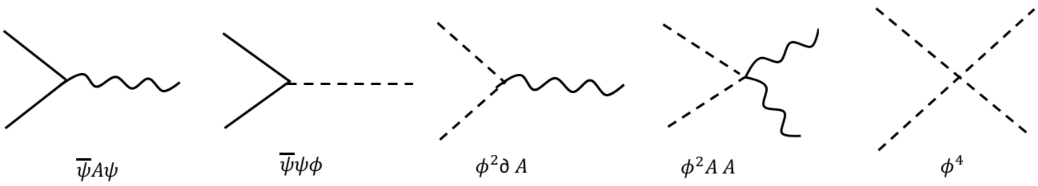

FIGURE 2.4: Interaction vertices for super Yang-Mills. Note that in amplitudes with only external gluons, the gluon-scalar, gluon-fermion interactions only appears in loop level.

2.4.1 On-shell Superspace and Superamplitudes

The on-shell degrees of freedom of $\mathcal{N}=4$ SYM can be group together on an on-shell chiral superfield Φ. If we introduce the Grassmann odd variables η_A labeled by the

$SU(4)$ index $A = 1 \dots 4$, the table 2.2 can be grouped in

$$\Phi(p,\eta) = g^+(p) + \eta_A \psi^A(p) - \frac{1}{2!}\eta_A\eta_B S^{AB}(p) - \frac{1}{3!}\eta_A\eta_B\eta_C\psi^{ABC}(p) + \\ + \frac{1}{4!}\eta_A\eta_B\eta_C\eta_D\epsilon^{ABCD}g^-(p) \tag{2.47}$$

Fields	Helicity	Name(Type)	$SU(4)_R$ representation
g^+	1	gluon (B)	singlet
ψ^A	1/2	gluion (F)	fudamental (**4**)
S^{AB}	0	scalar (B)	anti-symmetric (**6**)
$\psi^{ABC} \sim \bar{\psi}_A$	$-1/2$	anti-gluion (F)	anti-fund ($\bar{\mathbf{4}}$)
g^-	-1	gluon (B)	singlet

TABLE 2.2: Super Yang-Mills spectrum and its global R-symmetry transformation

The super-amplitude is parametrized by the external super legs that depends on the usual helicity spinors plus the new grassmann parameter $\{\lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, \eta_A\}$. We can think of $\Phi_i \equiv \Phi(p_{(i)}, \eta_{(i)})$ as the super-wavefunction for the $i'th$ external particle of the super amplitude $\mathbb{A}_n(\Phi_1, \Phi_2, \dots, \Phi_n)$. In the spirit of super space formalism, the super amplitude is to be understood as the power series expansion in η. The $SU(4)_R$-symmetry requires that the external states form a $SU(4)$ singlet.

The $SU(4)_R$ symmetry is generate by

$$R_A^{\ B} = \eta_A \frac{\partial}{\partial \eta_B} - \frac{1}{4}\delta_A^{\ B}\left(\eta_C \frac{\partial}{\partial \eta_C}\right) \tag{2.48}$$

and we can see that 1 and $\eta^4 \equiv \eta_1\eta_2\eta_3\eta_4$ are singlets, *i.e*, $R_A^{\ B}1 = R_A^{\ B}\eta^4 = 0$. For example, the amplitudes from the previous section can be extract as

$$\mathbb{A}_n\big|_{\eta^0} = A_n(1^+, 2^+, \dots, n^+) \tag{2.49}$$

$$\mathbb{A}_n\big|_{\eta_1^4} = A_n(1^-, 2^+, \dots, n^+) \qquad\qquad \eta_i^4 = \eta_{i1}\eta_{i2}\eta_{i3}\eta_{i4} \tag{2.50}$$

$$\mathbb{A}_n\big|_{\eta_1^4\eta_2^4} = A_n(1^-, 2^-, 3^+ \dots, n^+) \tag{2.51}$$

We could also have extracted them using Grassmann integrals. If we define the helicity operator

$$h_i = 1 - \frac{1}{2}\eta_A^i \frac{\partial}{\partial \eta_A^i} \tag{2.52}$$

when h_i acts on each component of the superfield Φ_i it gives the helicity of each particle. For example:

$$h_i g^+ = +1 g^+(p) \quad ; \quad h_i \eta_A \psi^A(p) = +\frac{1}{2}\eta_A\psi^A(p) \tag{2.53}$$

Using the same logic, when h_i acts on the super-amplitude it gives the helicity of the particle. One can see this because $h_i \mathbb{A}_n(\Phi_i) = (1 - \frac{K}{2})\mathbb{A}_n(\Phi_i)$ where K counts the degree in η_i. When $K = 0$ it represents the positive gluon g^+ and for $K = 4$ it represents the negative gluon g^-. The homogeneity condition on super-amplitude becomes

$$\mathbb{A}_n(t\eta_i) = t^{-2h_i+2}\mathbb{A}_n(\eta_i) \tag{2.54}$$

Let us see how the supersymmetry of the super-amplitude imposes that the two amplitudes (2.49)-(2.50) are zero. The supersymmetry generators in the on-shell superspace are

$$Q^A_\alpha = \sqrt{2}\sum_{i=1}^n \lambda^{(i)}_\alpha \eta^A_{(i)} \quad ; \quad \bar{Q}_{\dot{\alpha}B} = \sqrt{2}\sum_{i=1}^n \bar{\lambda}^{(i)}_{\dot{\alpha}} \frac{\partial}{\partial \eta^B_{(i)}} \tag{2.55}$$

here we are summing over all the external particle ($i = 1, \dots n$). These generators satisfy the algebra

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\} = 2P_{\alpha\dot{\alpha}}\ \delta^A{}_B \quad \text{with} \quad P_{\alpha\dot{\alpha}} = \sum_{i=1}^n \lambda^{(i)}_\alpha \bar{\lambda}^{(i)}_{\dot{\alpha}} \tag{2.56}$$

where $P_{\alpha\dot{\alpha}}$ is the generator of translations for the external particles. Recall that Amplitudes have the delta of momentum conservation $\delta^4(\sum_{i=1}^n \lambda^{(i)}_\alpha \bar{\lambda}^{(i)}_{\dot{\alpha}})$ on it. Thus

$$\{Q^A_\alpha, \bar{Q}_{\dot{\alpha}B}\}\mathbb{A}_n = 2P_{\alpha\dot{\alpha}}\delta^A{}_B\mathbb{A}_n \quad \Rightarrow \quad Q^A_\alpha \mathbb{A}_n = \bar{Q}_{\dot{\alpha}B}\mathbb{A}_n = 0 \tag{2.57}$$

We have that the supersymmetry generators annihilate the super-amplitude. Note that the generator Q^A_α acts multiplicatively, and we can write it as a Grassmann delta function, from the property [3] $\delta(\eta - \eta_0) = \eta - \eta_0$.

The statement that $Q^A_\alpha \mathbb{A}_n = 0$ translates into $\mathbb{A}_n \propto \delta^8(Q^A_\alpha)$ because $Q^A_\alpha \delta^8(Q^A_\alpha) = 0$. The explicit expression for the Grassmann delta is

$$\delta^8(Q^A_\alpha) = \prod_{\alpha=1}^2 \prod_{A=1}^4 \left(\sum_{i=1}^n \lambda^{(i)}_\alpha \eta^A_{(i)}\right) = \frac{1}{2}\prod_{A=1}^4 \sum_{i,j=1}^n \langle ij\rangle \eta^A_{(i)}\eta^A_{(j)} \text{ [4]} \tag{2.58}$$

we used $\langle \lambda_i, \lambda_j\rangle = \langle ij\rangle$. Then the superamplitude can be decomposed as

$$\mathbb{A}_n(\lambda_i, \bar{\lambda}_i, \eta_i) = \delta^8(Q^A_\alpha)\delta^4(P)\mathbb{A}^{(K)}_n(\lambda_i, \bar{\lambda}_i, \eta_i) \tag{2.59}$$

[3] To see that use a test function $F(\eta) = F_0 + F_1\eta$ and the integration rules $\int d\eta\eta = 1 \quad \int d\eta 1 = 0$ and prove $\int d\eta\delta(\eta - \eta_0)F(\eta) = F(\eta_0)$.

[4] for Grassmann variables we have $\theta_1\theta_2 = \frac{1}{2}\theta^\alpha\theta_\alpha$, then $Q^A_1 Q^A_2 = \frac{1}{2}Q^{\alpha A}Q^A_\alpha$

K is the degree of η, where $\mathbb{A}_n^{(K)} \sim \mathcal{O}(\eta^K)$. Note that the $SU(4)_R$ symmetry imposes on $\mathbb{A}_n^{(K)}$ an expansion of the form

$$\mathbb{A}_n^{(K)} = \mathbb{A}_n^{(0)} + \mathbb{A}_n^{(4)} + \mathbb{A}_n^{(8)} + \cdots + \mathbb{A}_n^{(4n-16)} \tag{2.60}$$

Finally we can see that the expansion on the super-amplitude starts at order η^8 from the Grassmann delta. Then the amplitudes (2.49)-(2.50) are zero because they are order zero and forth in η. As I promised in the end of section 2.3, these two amplitudes are zero due to a hidden supersymmetry.

The MHV amplitude with particles r, s with negative helicity, can be extracted from the superamplidute as

$$\begin{aligned} A_n(r^-, s^-) &= \int d^4\eta_s d^4\eta_r \mathbb{A}_n^K \qquad (2.61)\\ &= \delta^4(P) \int d^4\eta_s d^4\eta_r \prod_{A=1}^{4} \sum_{i,j=1}^{n} \langle ij\rangle \eta_{(i)}^A \eta_{(j)}^A \mathbb{A}_n^K \\ &= \delta^4(P)\langle sr\rangle^4 \mathbb{A}_n^0 \end{aligned}$$

and we can read from (2.36) that $\mathbb{A}_n^0$ (zeroth order expansion in η) is given by

$$\mathbb{A}_n^0 = \frac{1}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle} \tag{2.62}$$

We can also see that the power in $\langle sr\rangle^4$ is due to $\mathcal{N} = 4$ supercharges.

Chapter 3

MHV in Twistors variables

In four dimensions, twistor variables were introduced in 1967 by Penrose as an alternative description of spacetime in which light-like lines replace points as the fundamental objects [17]. These $d = 4$ twistor variables transform linearly under $SO(4, 2)$ conformal transformations and provide a compact description of massless states.

We will start with a conformal transformations and then see that the generators in twistor variables are easier to deal with than the spinors variables $(\lambda, \bar{\lambda})$. Then we will see how the MHV-amplitudes looks like in twistor variables, which is going to be related to the twistor string theory.

3.1 Conformal Invariance

Conformal transformations on an space are those that locally preserve angles between two lines, mathematically:

$$g_{lk}(x)\frac{\partial x^l}{\partial x'^m}\frac{\partial x^k}{\partial x'^n} = \Lambda(x^{'})g_{mn}(x^{'}) \tag{3.1}$$

In our case will be sufficient to consider the flat metric in d-dimensions $\eta_{mn} = (-, ++ ...+)$. Then the equation that we want to solve is:

$$\eta_{lk}\frac{\partial x^l}{\partial x'^m}\frac{\partial x^k}{\partial x'^n} = \Lambda(x^{'})\eta_{mn} \tag{3.2}$$

We could also write this equation in infinitesimal form using $x^{'m} = x^m + \epsilon\xi^m(x) + \mathcal{O}(\epsilon^2)$, where ϵ is small. So if we just do a Taylor expansion in $\Lambda(x^{'}) = 1 - \epsilon K(x) + \mathcal{O}(\epsilon^2)$ and

$\frac{\partial x^k}{\partial x'^n} = \delta^k_n - \epsilon\partial_n \xi^k + \mathcal{O}(\epsilon^2)$ we see that in the infinitesimal form is given by

$$\partial_m \xi_n + \partial_n \xi_m = K(x)\eta_{mn} \tag{3.3}$$

we can solve the $K(x)$ by taking the trace in both sides and to get:

$$\partial_m \xi_n + \partial_n \xi_m = \frac{2}{d}(\partial \cdot \xi)\eta_{mn} \tag{3.4}$$

Before we continue let us show that Yang-Mills has conformal symmetry. The Yang-Mills in d-flat dimensions is given by the action

$$S_{YM} = -\frac{1}{4g}\int d^d x Tr(F^{mn}F_{mn}) \tag{3.5}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$ is the usual field strength, and g is the coupling constant and $A_m = A^A_m T^A$. So let us see how this action transform under a conformal transformation. The potential transforms as $A'_m(x') = \frac{\partial x^l}{\partial x'^m}A_l(x)$ then under $x'_n = x_n - \epsilon\xi_n$:

$$A'_m(x') = A'_m(x) - \epsilon\xi^n\partial_n A'_m(x) = A_m(x) + \partial_m(\epsilon\xi^l)A_l(x)$$

Then the infinitesimal transformation is given by

$$\delta_\xi A_m \equiv A'_m(x) - A_m(x) = \epsilon\xi^n\partial_n A_m(x) + \partial_m(\epsilon\xi^l)A_l(x) \tag{3.6}$$

this infinitesimal transformation is the Lie derivative $\mathcal{L}_\xi A_m$. Now if we do the same thing for the tensor F_{mn} we get

$$\delta_\xi F_{mn} = \epsilon\xi^r\partial_r F_{mn}(x) + \partial_m(\epsilon\xi^l)F_{ln}(x) + \partial_n(\epsilon\xi^l)F_{ml}(x) \tag{3.7}$$

Now we have all the ingredients we can calculate the variation of the action

$$\begin{aligned}
\delta_\xi S_{YM} &= -\frac{1}{2g}\int d^d x Tr[F^{mn}\{\xi^r\partial_r F_{mn} + \partial_m(\xi^l)F_{ln} + \partial_n(\xi^l)F_{ml}\}] \\
&= -\frac{1}{2g}\int d^d x Tr[\{\frac{1}{2}\partial_r(\xi^r F^2) - \frac{1}{2}\partial\cdot\xi(F^2) + F^{mr}F^n{}_r(\partial_m\xi_n + \partial_n\xi_m)\}] \\
&= -\frac{1}{4g}\int d^d x Tr[\{\partial_r(\xi^r F^2) + (\frac{4}{d} - 1)F^2\}]
\end{aligned} \tag{3.8}$$

here $F^2 \equiv F^{mn}F_{mn}$ and to go from the second line to third I used the equation (3.4). The Yang-Mills theory is conformal invariant only in $d = 4$.

3.1.1 Generators of conformal algebra

Let us continue the study of conformal transformations and find the generators. It will be based on two good books [18, 19]. To find the generator we have to solve the (3.4).

First let us manipulate (3.4) by acting with ∂^m and then ∂^n we get:

$$d\partial^2\xi_n = (2-d)\partial_n(\partial\cdot\xi) \quad ; \quad (d-1)\partial^2(\partial\cdot\xi) = 0 \tag{3.9}$$

we see that $d = 2$ things become simpler. If you write $\partial_1 = \partial_z + i\partial_{\bar{z}}$ and $\partial_2 = \partial_z - i\partial_{\bar{z}}$ in complex variables $z, \bar{z}$, you find (in $\eta = (1,1)$) the Laplacian $\partial_z\partial_{\bar{z}}\xi(z,\bar{z}) = 0$. This implies that $\xi(z,\bar{z})$ has infinity solutions, for example, $\xi(z,\bar{z}) = f(z)$ or $g(\bar{z})$. From the second equation in (3.9) when $d \neq 1$ the vector is $\xi \sim x^2$. From equation (3.2) if we set $\Lambda(x) = 1$ then the equation (3.3) becomes

$$\partial_m\xi_n + \partial_n\xi_m = 0 \tag{3.10}$$

which implies $\xi_m = a_m + b_m{}^n x_n$ with $b_{mn} = -b_{nm}$. This is Translation and Lorentz transformation, i.e the Poicaré transformations. Now if we make rescaling on $x' = \lambda x$ we see from (3.9) that $\Lambda = \lambda^2$ so it's a valid transformation, these are called dilations. We still can have order quadratic in x. There are not some many ways to write a second order term. A good start could be $\xi_m = a_m x^2 + (b\cdot x)x_m$. If one plug this in the equations (3.9) and do some manipulations, you get the right answer $\xi_m = d^n(\eta_{mn}x^2 - 2x_n x_m)$ this are called the Special Conformal Transformations (SCTs). To summarize the conformal vector is given by:

$$\xi_m = \underbrace{a_m}_{Translation} + \underbrace{b_m{}^n x_n}_{Lorentz} + \underbrace{c x_m}_{Dilation} + \underbrace{d^n(\eta_{mn}x^2 - 2x_n x_m)}_{SCT} \tag{3.11}$$

One thing to recall is that all this analysis was done for the Minkowski metric. Now we can read the generator G for each transformation from (3.11), if we use the differential operator representation, from the definition $\delta_\xi x_m = \xi\cdot G x_m$, we get:

$$P_m = -i\partial_m \quad \text{and} \quad M_{mn} = i(x_m\partial_n - x_n\partial_m) \tag{3.12a}$$

$$D = -ix^m\partial_m \quad \text{and} \quad K^m = -i(\eta^{mn}x^2 - 2x^m x^n)\partial_n \tag{3.12b}$$

with these generators we can go and calculate the algebra they form. With $[P_m, X_n] = i\eta_{mn}$ and $[A, BC] = [A, B]C + B[A, C]$ we get

$$[D, P^m] = iP^m \quad ; \quad [D, K^m] = -iK^m \quad ; \quad [D, M^{mn}] = 0 \tag{3.13}$$

$$[M^{mn}, P^l] = -\eta^{ml}P^n + \eta^{nl}P^m \quad ; \quad [M^{mn}, K^l] = -\eta^{ml}K^n + \eta^{nl}K^m \tag{3.14}$$

$$[M^{mn}, M^{lr}] = i(-\eta^{ml}M^{nr} - \eta^{nr}M^{ml} + \eta^{nl}M^{mr} + \eta^{mr}M^{nl}) \tag{3.15}$$

$$[K^m, P^n] = 2i(-M^{mn} + \eta^{mn}D) \tag{3.16}$$

We can just look the generator and count how many they are. In d-dimension $M : \frac{d(d-1)}{2}$, $P : d$, $K : d$ and $D : 1$ a total of $\frac{(d+1)(d+2)}{2}$. What is this group? If we look for the number of generators in the Lorentz group(M) and compare with the conformal group is just a shift on $d \to d+2$, $i.e$, $\frac{d(d-1)}{2} \to \frac{(d+1)(d+2)}{2}$, this is a hint[1] that the Conformal group is $SO(q, p)$ with $q + p = d + 2$. The conformal group in $d = 4$ is $SO(4, 2)$

3.2 Conformal invariance of the MHV amplitudes

As we just saw, the Yang-Mills theory has a bigger group that the Poicaré symmetry, it is also invariant under conformal transformations. Thus, the amplitude should also be invariant under these transformations. To be able to calculate this invariance we have to find the generators (3.12a)-(3.12b) in the spinor $(\lambda, \bar{\lambda})$ representation. The momentum generator are just the multiplication operator

$$P_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} \tag{3.17}$$

The Lorentz Generators can be found by looking how a spinor transform (Appendix A):

$$\delta\lambda_\alpha = \frac{i}{2} \underbrace{\omega_{mn}(\sigma^{mn})_\alpha{}^\beta}_{\Omega_\alpha{}^\beta} \lambda_\beta \tag{3.18}$$

the variation is define as

$$\delta\lambda_\rho = \Omega^{\alpha\beta} m_{\alpha\beta} \lambda_\rho \tag{3.19}$$

where $\Omega^{\alpha\beta}$ are the coefficients of the transformation and $m_{\alpha\beta}$ is the generator of the transformation. Thus, the Lorentz generator is a first order differential operator given by

$$m_{\alpha\beta} = \frac{i}{2} \left(\lambda_\alpha \frac{\partial}{\partial\lambda^\beta} + \lambda_\beta \frac{\partial}{\partial\lambda^\alpha} \right) \tag{3.20}$$

[1]This just shows that the number of generators are the same, to prove that this is indeed the group one should show that the algebra is the same.

The same thing for the dotted indices:

$$\bar{m}_{\dot{\alpha}\dot{\beta}} = \frac{i}{2}\left(\bar{\lambda}_{\dot{\alpha}}\frac{\partial}{\partial\bar{\lambda}^{\dot{\beta}}} + \bar{\lambda}_{\dot{\beta}}\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}}\right) \tag{3.21}$$

They are symmetric in (α, β) and $(\dot{\alpha}, \dot{\beta})$ which gives $4 - 1 = 3$ generators in a total of 6 for both $m, \bar{m}$. They can be thought as the projection $m_{\alpha\beta} = M^{mn}\sigma_{\alpha\beta}$ and $\bar{m}_{\dot{\alpha}\dot{\beta}} = M^{mn}\bar{\sigma}_{\dot{\alpha}\dot{\beta}}$ where $\sigma_{\alpha\beta}$ & $\bar{\sigma}_{\dot{\alpha}\dot{\beta}}$ are the Lorentz generators (Appendix A). We have to find the generators of dilation and special conformal transformation. From the $[D, P^m] = iP^m$ and $[D, K^m] = -iK^m$, we can associate a dilation weight $+1$ and a dilation weight -1 for K^m. A natural guess for the dilation is

$$d = \frac{i}{2}\left(\lambda^{\alpha}\frac{\partial}{\partial\lambda^{\alpha}} + \bar{\lambda}^{\dot{\alpha}}\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}} + c\right) \tag{3.22}$$

we see that $[d, \lambda_{\alpha}] = \frac{i}{2}\lambda_{\alpha}$ and $[d, \bar{\lambda}_{\dot{\alpha}}] = \frac{i}{2}\bar{\lambda}_{\dot{\alpha}}$, for any constant c. Thus is natural to associate the dilation weight $1/2$ for the $(\lambda, \bar{\lambda})$ spinors.

Is not so trivial to guess the special conformal transformation operator:

$$K_{\alpha\dot{\alpha}} = \frac{\partial^2}{\partial\lambda^{\alpha}\partial\bar{\lambda}^{\dot{\alpha}}} \tag{3.23}$$

but is the simplest operator that has the right dilation weight -1 and the commutator with d

$$[d, K_{\alpha\dot{\alpha}}] = -iK_{\alpha\dot{\alpha}} \tag{3.24}$$

If it works is fine, but keep in mind that the second order differential operator is not so trivial to work with, and this may be seen as a motivation to introduce Twistors. One way to fix the constant c in the dilation operator is using the commutation relation

$$[K_{\alpha\dot{\alpha}}, P^{\dot{\beta}\beta}] = -i(\delta_{\alpha}{}^{\beta}\bar{m}_{\dot{\alpha}}{}^{\dot{\beta}} + \delta_{\dot{\alpha}}{}^{\dot{\beta}}m_{\alpha}{}^{\beta} + \delta_{\alpha}{}^{\beta}\delta_{\dot{\alpha}}{}^{\dot{\beta}}d) \tag{3.25}$$

doing the calculation on the left side by just plugging the definition of K and P we find

$$[K_{\alpha\dot{\alpha}}, P^{\dot{\beta}\beta}] = \delta_{\alpha}{}^{\beta}\delta_{\dot{\alpha}}{}^{\dot{\beta}} + \delta_{\alpha}{}^{\beta}\bar{\lambda}^{\dot{\beta}}\frac{\partial}{\partial\bar{\lambda}^{\dot{\alpha}}} + \delta_{\dot{\alpha}}{}^{\dot{\beta}}\lambda^{\beta}\frac{\partial}{\partial\lambda^{\alpha}} \tag{3.26}$$

using the antisymmetric property of two spinors $(A_{\alpha}B_{\beta} - A_{\beta}B_{\alpha}) = \varepsilon_{\alpha\beta}A^{\rho}B_{\rho}$. (To see this just plug the numbers). Thus we can split $\lambda^{\beta}\frac{\partial}{\partial\lambda^{\alpha}}$ in symmetric plus antisymmetric part

$$\lambda^{\beta}\frac{\partial}{\partial\lambda^{\alpha}} = -im_{\alpha}{}^{\beta} + \frac{1}{2}\delta_{\alpha}{}^{\beta}\lambda^{\rho}\frac{\partial}{\partial\lambda^{\rho}} \tag{3.27}$$

the same is true for the dotted index

$$\bar{\lambda}^{\dot{\beta}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} = -i\bar{m}_{\dot{\alpha}}{}^{\dot{\beta}} + \frac{1}{2}\delta_{\dot{\alpha}}{}^{\dot{\beta}} \lambda^{\dot{\rho}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\rho}}} \tag{3.28}$$

here we used the fact that the symmetric part is proportional to the Lorentz generator. Finally we can identify with the left-side of (3.25) and the dilation operator is

$$d = \frac{i}{2}\left(\lambda^{\alpha} \frac{\partial}{\partial \lambda^{\alpha}} + \bar{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} + 2\right) \tag{3.29}$$

Now we are ready to prove that the MHV amplitude is invariant under conformal transformations. First the generators that we found were for one particle. For n particle is just the sum of the individual particle.

Let me remind you the form of the MHV-Amplitude (without the color trace):

$$A_n(r^-, s^-) = g^{n-2}(2\pi)^4 \delta^4 \left(\sum_{i}^{n} \lambda_{\alpha}^{(i)} \bar{\lambda}_{\dot{\alpha}}^{(i)}\right) \frac{\langle \lambda_r, \lambda_s, \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle} \tag{3.30}$$

- Translation operator:

$$P_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \lambda_{\alpha}^{(i)} \bar{\lambda}_{\dot{\alpha}}^{(i)} \tag{3.31}$$

then

$$P_{\alpha\dot{\alpha}} A_n = 0 \tag{3.32}$$

as a distribution function.

- Lorentz operator:

$$m_{\alpha\beta} = \frac{i}{2} \sum_{i=1}^{n} \left(\lambda_{\alpha}^{(i)} \frac{\partial}{\partial \lambda^{\beta(i)}} + \lambda_{\beta}^{(i)} \frac{\partial}{\partial \lambda^{\alpha(i)}}\right) \tag{3.33}$$

the Lorentz is manifest in the MHV amplitude because it only depends on Lorentz Invariant objects $\langle \lambda^{(i)}, \lambda^{(j)} \rangle$.

- Dilation operator:

$$d = \frac{i}{2} \sum_{i=1}^{n} \left(\lambda^{\alpha(i)} \frac{\partial}{\partial \lambda^{\alpha(i)}} + \bar{\lambda}^{\dot{\alpha}(i)} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}(i)}} + 2\right) \tag{3.34}$$

As we saw the the dilation operator measures the weight in mass units. Then the operator will give the mass weight plus n when act on the amplitude

$$dA_n = ([A_n] + n)A_n = \left([\langle \lambda^{(r)}, \lambda^{(s)} \rangle^4] + [\delta^4(p)] + [\frac{1}{\prod_{i=1}^n \langle \lambda_i, \lambda_{i+1} \rangle}] + n \right) A_n \qquad (3.35)$$

thus we have $[\lambda] = 1/2$; $[\langle \lambda^{(i)}, \lambda^{(j)} \rangle^4] = 4$; and $[\delta^4(p)] = -4$. Also$[\frac{1}{\langle \lambda_i, \lambda_{i+1} \rangle}] = -1$. We have

$$dA_n = (4 - 4 + n(-1) + n)A_n = 0 \qquad (3.36)$$

- Special Conformal operator:

$$K_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial^2}{\partial \lambda^{\alpha(i)} \partial \bar{\lambda}^{\dot{\alpha}(i)}} \qquad (3.37)$$

To prove that the invariance of the MHV amplitude we note that $\bar{\lambda}$ appears on the delta function only so

$$K_{\alpha\dot{\alpha}} A_n = \sum_{i=1}^{n} \left[\frac{\partial}{\partial \lambda^{\alpha(i)}} \left(\frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}(i)}} \delta^4(p) \right) \mathcal{A}_n + \left(\frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}(i)}} \delta^4(p) \right) \frac{\partial}{\partial \lambda^{\alpha(i)}} \mathcal{A}_n \right] \qquad (3.38)$$

using the chain rule $\frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}(i)}} = \frac{\partial P^{\dot{\beta}\beta}}{\partial \bar{\lambda}^{\dot{\alpha}(i)}} \frac{\partial}{\partial P^{\dot{\beta}\beta}} = \lambda^{\beta(i)} \frac{\partial}{\partial P^{\dot{\alpha}\beta}}$ we get

$$\begin{aligned} K_{\alpha\dot{\alpha}} A_n = \left(n \frac{\partial}{\partial P^{\dot{\alpha}\alpha}} \delta^4(p) + P^{\dot{\beta}\beta} \frac{\partial}{\partial P^{\dot{\alpha}\beta}} \frac{\partial}{\partial P^{\alpha\dot{\beta}}} \delta^4(p) \right) \mathcal{A}_n + \\ + \sum_{i=1}^{n} \left(\frac{\partial}{\partial P^{\dot{\alpha}\beta}} \delta^4(p) \right) \lambda^{\beta(i)} \frac{\partial}{\partial \lambda^{\alpha(i)}} \mathcal{A}_n \end{aligned} \qquad (3.39)$$

The last term we use the decomposition of symmetric and antisymmetric (3.27). Keeping in mind that ${m_\alpha}^\beta \mathcal{A}_n = 0$. The antisymmetric piece is the λ part of the dilation operator, thus

$$\lambda^\beta \frac{\partial}{\partial \lambda^\alpha} \mathcal{A}_n = [\mathcal{A}_n] = (4 - n)\mathcal{A}_n \qquad (3.40)$$

The last piece of information is to note that

$$P^{\dot{\beta}\beta} \frac{\partial}{\partial P^{\dot{\alpha}\beta}} \frac{\partial}{\partial P^{\alpha\dot{\beta}}} \delta^4(p) = -4 \frac{\partial}{\partial P^{\dot{\alpha}\alpha}} \delta^4(p) \qquad (3.41)$$

this can be seen as a property of delta function $\delta^{'}(x)f(x) = -f^{'}(x)\delta(x)$ and $x\delta^{''}(x)f(x) = -2\delta^{'}(x)f(x)$. The extra 2 in the formula is due to $\frac{\partial P^{\beta\dot{\beta}}}{\partial P^{\beta\dot{\alpha}}} = 2{\delta^{\dot{\beta}}}_{\dot{\alpha}}$ compare to the simple example. Plugging (3.40)-(3.41) in to (3.39), we see that the MHV amplitude is indeed invariant under special conformal transformations.

3.3 Twistor Space as a conformal representation

The space where the conformal generators are simpler (transforms linearly) is called Twistor space. Consider the wave function $\psi(\lambda, \bar{\lambda})$ and let us make a Fourier transformation in the i particle

$$\tilde{\psi}(Z_i) = \int d^2\bar{\lambda}_i \psi(\lambda_i, \bar{\lambda}_i) \exp(i\bar{\lambda}^{\dot{\alpha}}_{(i)}\mu_{\dot{\alpha}(i)}) \tag{3.42}$$

where $Z_i^I \equiv (\lambda_i^{\alpha}, \mu_i^{\dot{\alpha}})$ denotes a 4-component vector, and $I = (\alpha, \dot{\alpha})$. The Z^I is called the twistor variable and lives on Twistor space. If we consider the metric $(--++)$ the spinors $(\lambda, \bar{\lambda})$ are real and independent. Thus (3.42) is the usual Fourier transformation, and the variable μ can be interpreted as the conjugate of $\bar{\lambda}$, in the same sense p is the conjugate of x in Quantum Mechanics.

As you probably noted the scaling $(\lambda, \bar{\lambda}) \to (t\lambda, t^{-1}\bar{\lambda})$ plays a important role, it connects the helicity of the particle and how the amplitude changes under this scaling. Just to remind you

$$\mathcal{A}(t\lambda_i, t^{-1}\bar{\lambda}_i) = t^{-2h_i}\mathcal{A}(\lambda_i, \bar{\lambda}_i) \tag{3.43}$$

If we transform the amplitude to the Twistor space

$$\tilde{\mathcal{A}}(Z_i) = \int d^2\bar{\lambda}_i \mathcal{A}(\lambda_i, \bar{\lambda}_i) \exp(i\bar{\lambda}^{\dot{\alpha}}_{(i)}\mu_{\dot{\alpha}(i)}) \tag{3.44}$$

Under $Z_i(\lambda_i, \mu_i) \to tZ_i(\lambda_i, \mu_i)$ the amplitude change as

$$\tilde{\mathcal{A}}(tZ_i) = \int d^2\bar{\lambda}_i \mathcal{A}(t\lambda_i, \bar{\lambda}_i) \exp(i\bar{\lambda}^{\dot{\alpha}}_{(i)}t\mu_{\dot{\alpha}(i)}) = t^{-2(h_i+1)}\tilde{\mathcal{A}}(Z_i) \tag{3.45}$$

where the t^{-2} came from redefining the $d^2\bar{\lambda}_i \to t^{-2}d^2\bar{\lambda}_i$. From (3.45) the $\tilde{\mathcal{A}}(Z_i)$ transforms homogeneously with weight $-2(h_i+1)$ in the Z_i variable. We can identify the sets of Z^I that differ by the scaling $Z^I \to tZ^I$ and throw away the point $Z^I = 0$. We get the projective twistor space $\mathbb{RP}^3$. From (3.45) we can write the helicity operator (2.24) as

$$h_i = -\frac{1}{2}\left(Z_i^I \frac{\partial}{\partial Z_i^I} + 2\right) \tag{3.46}$$

To find the generators $G(\lambda, \bar{\lambda})$ in the twistor space $\tilde{G}(\lambda, \mu)$ we use the Fourier transformation

$$\tilde{G}(\lambda, \mu)\tilde{\psi}(\lambda, \mu) = \int d^2\bar{\lambda}_i G(\lambda, \bar{\lambda})\psi(\lambda, \bar{\lambda}) \exp(i\bar{\lambda}^{\dot{\alpha}}\mu_{\dot{\alpha}}) \tag{3.47}$$

Doing carefully you find that

$$\bar{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\lambda}^{\dot{\beta}}} \to -\varepsilon_{\dot{\alpha}\dot{\beta}} + \mu_{\dot{\beta}} \frac{\partial}{\partial \mu^{\dot{\alpha}}} \tag{3.48}$$

and the remaining transformations are trivial. The generators in twistor variables become homogeneous first order operators

- Lorentz operator:

$$m_{\alpha\beta} = \frac{i}{2}\sum_{i=1}^{n}\left(\lambda_{\alpha}^{(i)}\frac{\partial}{\partial\lambda^{\beta(i)}} + \lambda_{\beta}^{(i)}\frac{\partial}{\partial\lambda^{\alpha(i)}}\right) \tag{3.49}$$

$$\bar{m}_{\dot{\alpha}\dot{\beta}} = \frac{i}{2}\sum_{i=1}^{n}\left(\bar{\mu}_{\dot{\alpha}}^{(i)}\frac{\partial}{\partial\mu^{\dot{\beta}(i)}} + \mu_{\dot{\beta}}^{(i)}\frac{\partial}{\partial\mu^{\dot{\alpha}(i)}}\right) \tag{3.50}$$

- Dilation operator:

$$d = \frac{i}{2}\sum_{i=1}^{n}\left(\lambda^{\alpha(i)}\frac{\partial}{\partial\lambda^{\alpha(i)}} - \mu^{\dot{\alpha}(i)}\frac{\partial}{\partial\mu^{\dot{\alpha}(i)}}\right) \tag{3.51}$$

- Special Conformal operator:

$$K_{\alpha\dot{\alpha}} = i\sum_{i=1}^{n}\mu_{\dot{\alpha}}\frac{\partial}{\partial\lambda^{\alpha(i)}} \tag{3.52}$$

- Translation operator :

$$P_{\alpha\dot{\alpha}} = i\sum_{i=1}^{n}\lambda_{\alpha}^{(i)}\frac{\partial}{\partial\mu^{\dot{\alpha}(i)}} \tag{3.53}$$

The conformal group in the signature $(--++)$ is $SO(3,3) \cong SL(4,\mathbb{R})$. Note that the Z^I in this metric is a 4 real component vector that transforms naturally under $SL(4,\mathbb{R})$. That is why the generator of the conformal group looks more naturally (just first order differential operators) in the Twistor space. In this signature the Twistor space is a copy of $\mathbb{RP}^3$ instead $\mathbb{R}^4$ due to the equivalence $Z^I \to tZ^I$. If we consider the $(-+++)$ metric the conformal group is $SO(4,2)$ that is locally isomorphic to $SU(2,2)$.

The conformal generators in the twistor variables can be written compactly as

$$G^I{}_J = iZ^I\frac{\partial}{\partial Z^J} - \frac{i}{4}\delta^I{}_J\left(Z^K\frac{\partial}{\partial Z^K}\right) \quad ; \qquad I,J = (\alpha,\dot{\alpha}) \tag{3.54}$$

then we identify the generators as:

$$G^{\dot{\alpha}}{}_{\alpha} = K^{\dot{\alpha}}{}_{\alpha} \quad ; \quad G^{\alpha}{}_{\alpha} = d \quad ; \quad G^{\alpha}{}_{\dot{\alpha}} = P^{\alpha}{}_{\dot{\alpha}}$$
$$G^{\alpha}{}_{\beta} = m^{\alpha}{}_{\beta} + \frac{1}{2}\delta^{\alpha}{}_{\beta}d \quad ; \quad G^{\dot{\alpha}}{}_{\dot{\beta}} = \bar{m}^{\dot{\alpha}}{}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}}d$$

3.3.1 The MHV amplitude in Twistor space

Now that we know how the conformal generators look in Twistor variables, let us go one step further and transform the MHV-amplitude (3.30) to twistor space.

The MHV-amplitude can be written as

$$A_n(r^-, s^-) = g^{n-2}(2\pi)^4\delta^4\left(\sum_i^n \lambda_\alpha^{(i)}\bar{\lambda}_{\dot{\alpha}}^{(i)}\right) f(\lambda) \tag{3.55}$$

where $f(\lambda)$ is the $\langle rs\rangle^4/\langle 12\rangle \ldots \langle n1\rangle$ piece. First replace the momentum delta function for its integral representation:

$$(2\pi)^4\delta^4\left(\sum_i^n \lambda_\alpha^{(i)}\bar{\lambda}_{\dot{\alpha}}^{(i)}\right) = \int d^4x_{\alpha\dot{\alpha}} \exp\left(ix_{\alpha\dot{\alpha}}\sum_i^n \lambda_{(i)}^{\alpha}\bar{\lambda}_{(i)}^{\dot{\alpha}}\right) \tag{3.56}$$

and them perform the Fourier transformation to twistor space $A(\lambda^{(i)}, \bar{\lambda}^{(i)}) \to \tilde{A}(\lambda^{(i)}, \mu^{(i)})$ for each particle

$$\tilde{A}_n(\lambda^{(i)}, \mu^{(i)}) = g^{n-2}\int d^2\bar{\lambda}^{(1)} \ldots \int d^2\bar{\lambda}^{(n)} e^{\left(i\sum_j^n \bar{\lambda}_{(j)}^{\dot{\alpha}}\mu_{\dot{\alpha}(j)}\right)} \int d^4x_{\alpha\dot{\alpha}} e^{\left(ix_{\alpha\dot{\alpha}}\sum_i^n \lambda_{(i)}^{\alpha}\bar{\lambda}_{(i)}^{\dot{\alpha}}\right)} f(\lambda) \tag{3.57}$$

we can write the $\bar{\lambda}$ integrals using the delta function $\delta^2(C_{\dot{\alpha}}) = \int d^2\bar{\lambda} e^{iC_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}}$. We get

$$\tilde{A}_n(\lambda^{(i)}, \mu^{(i)}) = g^{n-2}\int d^4x_{\alpha\dot{\alpha}} \prod_{i=1}^n \delta^2(\mu_{\dot{\alpha}} + x_{\alpha\dot{\alpha}}\lambda^\alpha) f(\lambda) \tag{3.58}$$

and

$$\mu_{\dot{\alpha}} + x_{\alpha\dot{\alpha}}\lambda^\alpha = 0 \quad \text{with} \quad \dot{\alpha} = 1, 2. \tag{3.59}$$

If we pick some real $x_{\alpha\dot{\alpha}}$ we can solve the $\mu_{\dot{\alpha}}$ in term of λ^α, so (λ^1, λ^2) serve as homogeneous coordinates. We say that the curve has degree one, because the equation are linear in λ^α. The two equations (3.59) defines a real curve in $\mathbb{RP}^3$. The curve can be described as a straight line. Throwing away the set $\lambda^1 = 0$ in $\mathbb{RP}^3$, we can describe the rest of $\mathbb{RP}^3$ by coordinates by $x = \lambda_2/\lambda_1$,$y = \mu_1/\lambda_1$ and $z = \mu_2/\lambda_1$ in $\mathbb{R}^3$. The delta functions in (3.58) mean that all the n points $(\mu_j^{\dot{\alpha}}, \lambda_j^\alpha)$, $j = 1, \ldots, n$ are in the curve defined by $x^{\alpha\dot{\alpha}}$. The points are collinear in $\mathbb{R}^3$. The MHV amplitudes with mostly plus helicities are thus supported, in twistor space, on configurations of n points that all lie on a curve in $\mathbb{RP}^3$ of degree one and genus zero.

The Twistor equation (3.59) or the Incidence relation has a nice geometrical picture 3.1. A point in Twistor space represents a null line in spacetime. And a Line in twistor space represents a point in spacetime. Note is that given $x_{\alpha\dot{\alpha}}$ the solution for (3.59) does not change by a shift in momentum direction $x_{\alpha\dot{\alpha}} \to x_{\alpha\dot{\alpha}} + c\lambda_\alpha\bar{\lambda}_{\dot{\alpha}}$. Being more explicit, given two points in space time $x_{\alpha\dot{\alpha}}, y_{\alpha\dot{\alpha}}$ that satisfy (3.59), we get the equation $(x_{\alpha\dot{\alpha}} - y_{\alpha\dot{\alpha}})\lambda^\alpha = 0$. This implies that the matrix $x_{\alpha\dot{\alpha}} - y_{\alpha\dot{\alpha}}$ has zero eigenvalue or $det(x_{\alpha\dot{\alpha}} - y_{\alpha\dot{\alpha}}) = (x-y)^2 = 0$. Thus the set of points that are null/lightlike separated in spacetime (ST) represents a point in Twistor Space (TS).

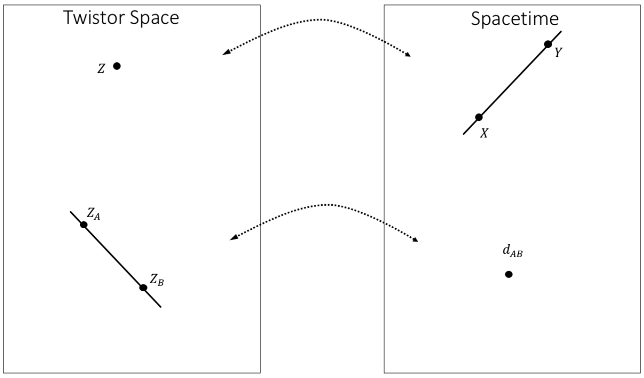

FIGURE 3.1: Upper side: A point in twistor space represents a null line in spacetime. A line in twistor space correspond to a point in space time

Given two points $(a, b) \equiv (Z_a, Z_b)$ in TS that defines a straight line. Each Z represents a null line in spacetime. A natural question is do they intersect? If yes, where? The answer is yes. The solution is given by the point $d_{\alpha\dot{\alpha}}$

$$d_{\alpha\dot{\alpha}} = \frac{-\lambda^a_\alpha \mu^b_{\dot{\alpha}} + \lambda^b_\alpha \mu^a_{\dot{\alpha}}}{\langle \lambda^a, \lambda^b \rangle} \tag{3.60}$$

The solution can be checked by direct substitution in $\mu^{(a)}_{\dot{\alpha}} + d_{\alpha\dot{\alpha}}\lambda^{\alpha(a)} = 0$ and $\mu^{(b)}_{\dot{\alpha}} + d_{\alpha\dot{\alpha}}\lambda^{\alpha(b)} = 0$. The two null lines in spacetime intersect in one point $d_{\alpha\dot{\alpha}}$. Thus a line in twistor space represent a point in spacetime.

3.3.2 The Ambitwistor

In (3.44) we choose to Fourier transform $\bar{\lambda}$ but we could have done the opposite and Fourier transform λ

$$\tilde{\mathcal{A}}(W_i) = \int d^2\lambda_i \mathcal{A}(\lambda_i, \bar{\lambda}_i) \exp(i\bar{\mu}^\alpha_{(i)}\lambda_{\alpha(i)}) \tag{3.61}$$

here we define $W_I = (\bar{\mu}_\alpha, \bar{\lambda}_{\dot{\alpha}})$ called the ambitwistor variable. Than we can use the scaling to see how the amplitude behaves in the ambtwistor space

$$\tilde{\mathcal{A}}(tW_i) = \int d^2\lambda_i \mathcal{A}(\lambda_i, t\bar{\lambda}_i) \exp(it\bar{\mu}^\alpha_{(i)}\lambda_{\alpha(i)}) = t^{2(h-1)}\tilde{\mathcal{A}}(W_i) \tag{3.62}$$

using $\mathcal{A}(t^{-}\lambda_i, t\bar{\lambda}_i) = t^{2h_i}\mathcal{A}(\lambda_i, \bar{\lambda}_i)$. The W can be interpreted as the conjugate of Z. In the same way we did for the μ and $\bar{\lambda}$. We can construct a invariant Lorentz object $Z^I W_I = \lambda^\alpha \bar{\mu}_{\dot{\alpha}} + \mu^\alpha \bar{\lambda}_{\dot{\alpha}}$.

We can write the MHV with ambitwistor variables W_I if we Fourier transform $\bar{\lambda}$ instead λ. Then we get a similar equation for the MHV-amplitude (with the momentum delta function)

$$\tilde{A}_n(\bar{\lambda}^{(i)}, \bar{\mu}^{(i)}) = g^{n-2} \int d^4 x_{\alpha\dot{\alpha}} \prod_{i=1}^{n} \delta^2(\bar{\mu}_\alpha + x_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}) g(\bar{\lambda}) \tag{3.63}$$

Note that if the particle i has positive helicity then $\mathcal{A}(tW_i) = \mathcal{A}(W_i)$ and the same is true for a particle with negative helicity $\mathcal{A}(tZ_i) = \mathcal{A}(Z_i)$. With these constrains we can calculate for example the 4 gluon scattering $\mathcal{A}(1^+, 2^+, 3^-, 4^-)$. Using the fact that

$$\begin{aligned}\mathcal{A}(W_1^+, W_2^+, Z_3^-, Z_4^-) =& \mathcal{A}(tW_1^+, W_2^+, Z_3^-, Z_4^-) = \mathcal{A}(W_1^+, tW_2^+, Z_3^-, Z_4^-) = \\ =& \mathcal{A}(W_1^+, W_2^+, tZ_3^-, Z_4^-) = \mathcal{A}(W_1^+, W_2^+, Z_3^-, tZ_4^-)\end{aligned} \tag{3.64}$$

it seems to not depend on either $W's$ and $Z's$ variables. Therefore, in twistor/ambitwistor variables the amplitude is just a constant!. Actually the amplitude will depend the sign of each Lorentz product.

$$\mathcal{A}(W_1^+, W_2^+, Z_3^-, Z_4^-) = sign(Z_3 \cdot W_1) sign(Z_3 \cdot W_2) sign(Z_4 \cdot W_1) sign(Z_4 \cdot W_2) \tag{3.65}$$

This is amazing! We were able to find the 4 gluon scattering in very few lines using just scaling and the right variables.

3.4 Super conformal Transformation

The $\mathcal{N} = 4$ SUSY YM theory has the superconformal symmetry defined by the supergroup $PSU(2,2|4)$ [20]. It has 30 bosonic generators and 32 fermionic generators. We have 15 bosonic generators from the $SU(2,2)$, the conformal group. The other 15 bosonic come from R_A^B, the $SU(4)_R$ symmetry. Supersymmetry generators give 16 fermionic generators. We have two new sets of generators, $S_{\alpha A}$ and $\bar{S}_{\dot{\alpha}}^A$ that give $8+8$ fermionic generators. We are going to find the algebra of the group, first in the $\lambda, \bar{\lambda}, \eta$ variables then we are going to Fourier Transform to Twistors variables λ, μ, ψ where ψ is going to be the supertwistor variable that will be define later.

First let me redefine the supersymmetry generators (2.55) just to removes the annoying factor of $\sqrt{2}$ and the sum over particles ($\sum_{i=1}^{n}$). So the supersymmetry generators become

$$q^{\alpha A} = \lambda^{\alpha}\eta^{A} \quad ; \quad \bar{q}^{\dot\alpha}_{A} = \bar\lambda^{\dot\alpha}\partial_A \qquad \text{where} \quad \partial_A = \frac{\partial}{\partial\eta^A} \tag{3.66}$$

The two new sets of fermionic generatos arise from the commutators

$$\begin{aligned}[] [K_{\alpha\dot\alpha}, q^{\beta A}] &= \delta_{\alpha}{}^{\beta}\bar{S}^{A}_{\dot\alpha} \quad \text{with} \quad \bar{S}^{A}_{\dot\alpha} = \eta^{A}\frac{\partial}{\partial\bar\lambda^{\dot\alpha}} \\ [K_{\alpha\dot\alpha}, \bar{q}^{\dot\beta}_{A}] &= \delta_{\dot\alpha}{}^{\dot\beta} S_{\alpha A} \quad \text{with} \quad S_{\alpha A} = \frac{\partial}{\partial\lambda^{\alpha}}\frac{\partial}{\partial\eta^{A}} \end{aligned} \tag{3.67}$$

The $S_{\alpha A}$ and $\bar{S}^{A}_{\dot\alpha}$ are called the super conformal generators in the same sense the $q^{\alpha A}$ and $\bar{q}^{\alpha A}$ are called the super translation :

$$\{q^{\alpha A}, \bar{q}^{\dot\alpha}_{B}\} = \delta^{A}{}_{B}P^{\alpha\dot\alpha} \quad ; \quad \{S_{\alpha A}, \bar{S}^{B}_{\dot\alpha}\} = \delta_{A}{}^{B}K_{\alpha\dot\alpha}{}^{2} \tag{3.68}$$

The non-trivial anti-commutators are:

$$\{q^{\alpha A}, S_{\beta B}\} = m^{\alpha}{}_{\beta}\delta^{A}{}_{B} + \delta^{\alpha}{}_{\beta}R^{A}{}_{B} + \frac{1}{2}\delta^{A}{}_{B}\delta^{\alpha}{}_{\beta}\,(d+c) \tag{3.69}$$

$$\{\bar{q}^{\dot\alpha}{}_{A}, \bar{S}_{\dot\beta}{}^{B}\} = \bar{m}^{\dot\alpha}{}_{\dot\beta}\delta^{A}{}_{B} - \delta^{\alpha}{}_{\beta}R^{A}{}_{B} + \frac{1}{2}\delta^{A}{}_{B}\delta^{\alpha}{}_{\beta}\,(d-c) \tag{3.70}$$

where d is the dilation operator and c is the central charge (because it commutes with all the other generators)

$$c = \left(1 - \frac{1}{2}\eta^{C}\partial_{C}\right) + \left(\frac{1}{2}\lambda^{\alpha}\frac{\partial}{\partial\lambda^{\alpha}} - \frac{1}{2}\lambda^{\dot\alpha}\frac{\partial}{\partial\bar\lambda^{\dot\alpha}}\right) = h_f - h_b \tag{3.71}$$

where h_f is the helicity fermionic operator (2.52) and h_b is the helicity bosonic operator (2.24). So we can see that $c\mathbb{A}_n = 0$ for every particle. Before we go to Twistor space, let us show that the super-amplitude it is $PSU(2,2|4)$ invariant. To do this we just have to check that $S_{\beta B}\mathbb{A}_n = \bar{S}_{\dot\beta}{}^{B}\mathbb{A}_n = 0$. The super-amplitude is given by

$$\mathbb{A}_n(\lambda_i, \bar\lambda_i, \eta_i) = \delta^{8}(Q^{A}_{\alpha})\delta^{4}(P)\mathbb{A}^{(K)}_{n}(\lambda_i, \bar\lambda_i, \eta_i) \tag{3.72}$$

but let us restrict to the MHV amplitude (K=0), where $\mathbb{A}^{0}_{n} = \frac{1}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}$.

First note that

$$\bar{S}_{\dot\beta}{}^{B}\delta^{4}(P) = \eta^{B}\frac{\partial}{\partial\bar\lambda^{\dot\beta}}\delta^{4}(P) = \eta^{B}\lambda_{\alpha}\frac{\partial}{\partial P^{\dot\beta}_{\alpha}}\delta^{4}(P) = q^{B}_{\alpha}\delta^{4}(P) \tag{3.73}$$

[2] Just to be clear $\{A, B\} = AB + BA$ is the anti-commutator.

then $\bar{S}_{\dot{\beta}}{}^{B}\mathbb{A}_n = 0$ due to the delta of super charge $\delta^8(Q)$. The second operator is more complicated :

$$\begin{aligned} S_{\beta B}\mathbb{A}_n &= \sum_{k=1}^{n} \frac{\partial}{\partial \lambda_k^\beta}\frac{\partial}{\partial \eta_k^B}\left[\delta^8(Q)\delta^4(P)\mathbb{A}_n^0\right] \\ &= \sum_{k=1}^{n} \frac{\partial}{\partial \lambda_k^\beta}\left(\lambda_k^\alpha \frac{\partial}{\partial q_k^{B\alpha}}\delta^8(Q)\right)\delta^4(P)\mathbb{A}_n^0 + \frac{\partial}{\partial q^{B\alpha}}\delta^8(Q)\sum_{k=1}^{n}\lambda_k^\alpha\frac{\partial}{\partial \lambda_k^\beta}\left(\delta^4(P)\mathbb{A}_n^0\right) \\ &= \left[(n-1+3)\frac{\partial}{\partial q^{B\alpha}}\delta^8(Q) + \frac{\partial}{\partial q^{B\alpha}}\delta^8(Q)(-n-2)\right]\delta^4(P)\mathbb{A}_n^0 \end{aligned}$$

using

$$\sum_{k=1}^{n}\lambda_k^\alpha\frac{\partial}{\partial \lambda_k^\beta}\delta^4(P) = P^{\alpha\dot{\gamma}}\frac{\partial}{P^{\beta\dot{\gamma}}}\delta^4(P) = -2\delta_\beta^\alpha\delta^4(P)$$

and

$$S_{\beta B}\delta^8(Q) = (n-1+3)\frac{\partial}{\partial q^{B\alpha}}\delta^8(Q) \tag{3.74}$$

in the sence of distribution function as we did on $K^{\alpha\dot{\alpha}}$. The amplitude is zero.

3.4.1 Super-Twistor Space

In the same spirit we can transform the Grassmann parameter to Sueper-Twistor space by performing a Fourier transformation

$$\tilde{f}(\psi) = \int d^4\eta f(\eta)e^{(i\eta^A\psi_A)} \tag{3.75}$$

with this transformation we get

$$\eta^A \to i\frac{\partial}{\partial \psi_A} \quad ; \quad \frac{\partial}{\partial \eta^A} \to i\psi_A \quad ; \quad \eta^A\frac{\partial}{\partial \eta^B} \to \delta_B^A - \psi_B\frac{\partial}{\partial \psi_A} \tag{3.76}$$

Now we can use these transformations (plus the bosonic twistor transformation) to get make all the generators first order operators:

$$q^{\alpha A} = \lambda^\alpha\eta^A \quad \to \quad \lambda^\alpha\frac{\partial}{\partial \psi_A} \tag{3.77}$$

$$\bar{q}^{\alpha A} = \bar{\lambda}^{\dot{\alpha}}\partial_A \quad \to \quad \psi_A\frac{\partial}{\partial \mu_{\dot{\alpha}}} \tag{3.78}$$

$$\bar{S}^A_{\dot{\alpha}} = \eta^A \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} \quad \rightarrow \quad \bar{\mu}_{\dot{\alpha}} \frac{\partial}{\partial \psi_A} \tag{3.79}$$

$$S_{\alpha A} = \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \eta^A} \quad \rightarrow \quad \psi_A \frac{\partial}{\partial \lambda^\alpha} \tag{3.80}$$

$$h_f = 1 - \frac{1}{2}\eta^A \frac{\partial}{\partial \eta^A} \quad \rightarrow \quad -1 + \frac{1}{2}\psi_A \frac{\partial}{\partial \psi_A} \tag{3.81}$$

and

$$R^A{}_B = \eta^A \frac{\partial}{\partial \eta^B} - \delta^A{}_B \frac{1}{4}\eta^C \frac{\partial}{\partial \eta^C} \quad \rightarrow \quad \psi_B \frac{\partial}{\partial \psi_A} - \delta^A{}_B \frac{1}{4}\psi_C \frac{\partial}{\partial \psi_C} \tag{3.82}$$

$$c = h_f - h_b \quad \rightarrow \quad \frac{1}{2}\left(\lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} + \mu^{\dot{\alpha}} \frac{\partial}{\partial \mu^{\dot{\alpha}}} + \psi_C \frac{\partial}{\partial \psi_C}\right) \tag{3.83}$$

If we define an extention for the twistor variable $Z^I(\lambda^\alpha, \mu^{\dot{\alpha}})$ called SuperTwistor $\mathcal{Z}^R = (\lambda^\alpha, \mu^{\dot{\alpha}}, \psi_A)$ with $R = (\alpha, \dot{\alpha}, A)$. Then we can write the generators of $PSU(2,2|4)$ group in a compact form

$$\mathcal{G}^R{}_S = \mathcal{Z}^R \frac{\partial}{\partial \mathcal{Z}^S} - \frac{1}{4}\delta^R{}_S \mathcal{Z}^T \frac{\partial}{\partial \mathcal{Z}^T} \tag{3.84}$$

and the generators can be organize in a nice matrix:

$$\mathcal{G}^R{}_S = \left[\begin{array}{c|c} SU(2,2) & Mix \\ \hline Mix & SU(4)_R \end{array}\right] = \left[\begin{array}{cc|c} m^\alpha{}_\beta & P^\alpha{}_{\dot{\beta}} & Q^\alpha{}_B \\ K^{\dot{\alpha}}{}_\beta & \bar{m}^{\dot{\alpha}}{}_{\dot{\beta}} & \bar{S}^{\dot{\alpha}}{}_B \\ \hline S^A{}_\beta & \bar{Q}^A{}_{\dot{\beta}} & R^A{}_B \end{array}\right]$$

The $SU(2,2)$ block are the 15 bosonic generators of conformal transformation. The $SU(4)_R$ block are the 15 bosonic generators of the global R-Symmetry. The Mix block are the 32 fermionic generators of supersymmetry and superconformal transformation. We can study the homogeneity of superamplitude in the supertwistor variables. Recall that in the bosonic part the amplitude in twistor space satisfies $\tilde{A}(tZ_i) = t^{-2(h_i+1)}\tilde{A}(Z_i)$. For the fermionic part we do the supertwistor transfomation in the superamplitude

$$\tilde{\mathbb{A}}_n(t\psi_i) = \int d^4\eta \mathbb{A}(\eta_i) e^{(i\eta^A t\psi_A)} = t^{2h_i+2}\tilde{\mathbb{A}}_n(\psi_i) \tag{3.85}$$

using $\eta \rightarrow t^{-1}\eta$, $d^4\eta \rightarrow t^{+4}d^4\eta$ and also recall (2.54) $\mathbb{A}(t^{-1}\eta_i) = t^{2h_i-2}\mathbb{A}(\eta_i)$. Thus, the homogeneity condition of the superamplitude in the supertwistor space become

$$\tilde{\mathbb{A}}_n(t\mathcal{Z}_i) = t^{-2(h_i+1)} t^{2h_i+2} \tilde{\mathbb{A}}_n(\mathcal{Z}_i) = \tilde{\mathbb{A}}_n(\mathcal{Z}_i) \tag{3.86}$$

or

$$\mathcal{Z}_i^R \frac{\partial}{\partial \mathcal{Z}_i^R} \tilde{\mathbb{A}}_n(\mathcal{Z}_i) = \underbrace{\left(Z_i^I \frac{\partial}{\partial Z_i^I} + \psi_{iA} \frac{\partial}{\partial \psi_{iA}} \right)}_{CentralCharge\ (c)} \tilde{\mathbb{A}}_n(\mathcal{Z}_i) = 0 \tag{3.87}$$

Note that (3.87) is the same observation that we found that the central charge annihilates the superamplitude $c\mathbb{A}_n = 0$.

Chapter 4

First order Lagrangian

Let us switch gears completely and review some properties of 2d Conformal Field Theory (CFT), the bc system. It will be based on [21] and Polchinski's book [19]. The goal for this chapter is to become familiar with the bc CFT that is similar to the Berkovits's Twistor String in the next Chapter.

4.1 bc Conformal Field Theory

Consider the fields $\mathbf{b}$ and $\mathbf{c}$ with the action

$$S = \frac{1}{2\pi} \int d^2z \mathbf{b}(z,\bar{z}) \partial_{\bar{z}} \mathbf{c}(z,\bar{z}) \tag{4.1}$$

This action is conformal invariant if $\mathbf{b}$ and $\mathbf{c}$ transform as tensor/primary operators with weight $(\lambda, 0)$ and $(1-\lambda, 0)$ respectively. The fields $\mathbf{b}$ and $\mathbf{c}$ can have Bose or Fermi statistics, we are going to treat both cases in parallel. The tracker device is given by ϵ such that $\mathbf{bc} = -\epsilon \mathbf{cb}$, and so $\epsilon = +1$ for Fermi statistics and $\epsilon = -1$ for Bose statistics.

Just to make sure that we are in the same page a primary operator $O'(z', \bar{z}')$ with weight $(h, \bar{h})$ transforms as

$$O'(z', \bar{z}') = (\partial_z z')^{-h} (\partial_{\bar{z}} \bar{z}')^{-\bar{h}} O(z, \bar{z}) \tag{4.2}$$

under a conformal transformation ($z' = f(z)$) recall the paragraph above (3.10). The action is invariant using the transformations $dz' d\bar{z}' = (\partial_z z')(\partial_{\bar{z}} \bar{z}') dz d\bar{z}$, $\partial_{\bar{z}'} = (\partial_{\bar{z}} z')^{-1} \partial_{\bar{z}}$ and (4.2) for $\mathbf{b}, \mathbf{c}$.

The operator equation of motion are given by

$$\bar{\partial}\mathbf{b}(z) = \bar{\partial}\mathbf{c}(z) = 0 \tag{4.3}$$

The $\mathbf{b}, \mathbf{c}$ are holomorphic under the E.O.M. To find the OPE we use the Path Integral technique. We insert the operator $\mathbf{b}$ inside the path integral to obtain

$$0 = \int [d\mathbf{b}d\mathbf{c}]\frac{\delta}{\delta\mathbf{b}(z_1)}\left[e^{-S}\mathbf{b}(z_2)\right] = \int [d\mathbf{b}d\mathbf{c}]\left[-\frac{1}{2\pi}\bar{\partial}_1\mathbf{c}(z_1)\mathbf{b}(z_2) + \delta^2(z_{12}, \bar{z}_{12})\right]e^{-S} \tag{4.4}$$

which tell us that

$$\bar{\partial}_1\mathbf{c}(z_1)\mathbf{b}(z_2) = 2\pi\delta^2(z_{12}, \bar{z}_{12}) \tag{4.5}$$

The normal order $\mathbf{bc}$ is defined as:

$$: \mathbf{c}(z_1)\mathbf{b}(z_2) := \mathbf{c}(z_1)\mathbf{b}(z_2) - \frac{1}{z_{12}} \tag{4.6}$$

such that this satisfies[1] the naive equation of motion (4.5):

$$\bar{\partial}_1 : \mathbf{c}(z_1)\mathbf{b}(z_2) := 0 \tag{4.7}$$

The Operator Product Expansion (OPE) are given by

$$\mathbf{b}(z_1)\mathbf{c}(z_2) \sim \frac{\epsilon}{z_1 - z_2} \quad ; \quad \mathbf{c}(z_1)\mathbf{b}(z_2) \quad \sim \frac{1}{z_1 - z_2} \tag{4.8}$$

4.1.1 Properties

The stress energy tensor is defined as the conserved currents which arise from translational invariance. Consider the infinitesimal change in the coordinates where $r(z), l(\bar{z})$ are small

$$z' = z + r(z) \quad ; \quad \bar{z}' = \bar{z} + l(\bar{z}) \tag{4.9}$$

To find the holomorphic $T(z)$ and antiholomorphic $\tilde{T}(\bar{z})$ stress energy tensor we make $r(z)$ be $\bar{z}$ depended, thus $r(z) \to r(z, \bar{z})$, and similar for $l(\bar{z})$. The variation of the action becomes

$$\delta S \sim \int d^2z[T(z)\bar{\partial}r(z, \bar{z}) + \tilde{T}(\bar{z})\partial l(z, \bar{z})] \tag{4.10}$$

Note that when we return to $r(z, \bar{z}) \to r(z)$, $\delta S = 0$. So (4.9) is a symmetry as we expect. With (4.2) the variation for $\mathbf{b}, \mathbf{c}$ become

$$\delta\mathbf{b} \equiv \mathbf{b}'(z, \bar{z}) - \mathbf{b}(z, \bar{z}) = [r(z, \bar{z})\partial\mathbf{b}(z, \bar{z}) + \mathbf{b}(z, \bar{z})\lambda\partial r(z, \bar{z})] - l(z, \bar{z})\bar{\partial}\mathbf{b}(z, \bar{z}) \tag{4.11}$$

$$\delta\mathbf{c} \equiv \mathbf{c}'(z, \bar{z}) - \mathbf{c}(z, \bar{z}) = [r(z, \bar{z})\partial\mathbf{c}(z, \bar{z}) + \mathbf{c}(z, \bar{z})(1 - \lambda)\partial r(z, \bar{z})] - l(z, \bar{z})\bar{\partial}\mathbf{c}(z, \bar{z}) \tag{4.12}$$

[1] using $\partial\frac{1}{\bar{z}} = \bar{\partial}\frac{1}{z} = 2\pi\delta^2(z, \bar{z})$

Plugging these on the action we find the stress-energy tensor

$$T(z) =: (\partial \mathbf{b})\mathbf{c} : -\lambda\partial(: \mathbf{bc} :) \tag{4.13}$$

$$\tilde{T}(\bar{z}) = 0 \tag{4.14}$$

The $\tilde{T}(\bar{z})$ is easy to see that is zero just by looking that the terms with $l(z,\bar{z})$ form a total derivative $\bar{\partial}(l(z,\bar{z})\mathbf{bc})$ inside the action. This is because the action only has $\bar{\partial}$.

The OPE's are:

$$T(z)\mathbf{b}(w) \sim \frac{\lambda}{(z-w)^2}\mathbf{b}(w) + \frac{1}{(z-w)}\partial_w \mathbf{b}(w) \tag{4.15}$$

$$T(z)\mathbf{c}(w) \sim \frac{1-\lambda}{(z-w)^2}\mathbf{c}(w) + \frac{1}{(z-w)}\partial_w \mathbf{c}(w) \tag{4.16}$$

and

$$T(z)T(w) \sim \frac{-2\epsilon(6\lambda^2 - 6\lambda + 1)}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \tag{4.17}$$

Then the central charge (c) is

$$c = -2\epsilon(6\lambda^2 - 6\lambda + 1) = \epsilon(1 - 3Q^2) \quad ; \quad Q = \epsilon(1 - 2\lambda) \tag{4.18}$$

The usual b, c Faddev-Popov ghosts from the gauge-fixing the Polyakov string has $\lambda = 2$, Fermi statistics ($\epsilon = +1$) and central charge $c = -26$. Another example are the β, γ ghosts from gauge fixing the super string, that has $\lambda = \frac{3}{2}$, Bose statistics($\epsilon = -1$) and $c = -11$.

The $\mathbf{b}, \mathbf{c}$ action has one more symmetry, a ghost number, $U(1)$ charge. Under $\delta\mathbf{b} = -i\varepsilon\mathbf{b}$, $\mathbf{c} = +i\varepsilon\mathbf{c}$ and the Noether current is

$$j(z) = - : \mathbf{b}(z)\mathbf{c}(z) : \tag{4.19}$$

with OPE's

$$j(z)\mathbf{b}(w) \sim \frac{-\mathbf{b}(w)}{z-w} \quad ; \quad j(z)\mathbf{c}(w) \sim \frac{+\mathbf{c}(w)}{z-w} \quad ; \quad j(z)j(w) \sim \frac{\epsilon}{(z-w)^2} \tag{4.20}$$

We call the number in front of the single pole of $j(z)O(w)$ OPE the Ghost number N_g or $U(1)$ charge of operator $O(w)$. From (4.20) $N_g(\mathbf{b}) = -1$ and $N_g(\mathbf{c}) = +1$. The Current j is not a tensor (has cubic pole):

$$T(z)j(w) \sim \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial_w j(w)}{z-w} \tag{4.21}$$

So the transformation under $z' = z + r(z)$ is

$$\delta j(z) = -r(z)\partial j(z) - j(z)\partial r(z) - \frac{Q}{2}\partial^2 r(z) \tag{4.22}$$

The $U(1)$ current can be used to built a stress-energy tensor such that it has the same OPE (4.21):

$$T_j(z) = \epsilon\left(\frac{1}{2}j(z)j(z) - \frac{1}{2}Q\partial_z j(z)\right) \tag{4.23}$$

The OPE's :

$$T_j(z)j(w) \sim \frac{Q}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial_w j(w)}{(z-w)} \tag{4.24}$$

$$T_j(z)T_j(w) \sim \frac{\frac{1}{2}(1-3\epsilon Q^2)}{(z-w)^4} + \dots \tag{4.25}$$

If we compare the central charge ($c_j = 1 - 3\epsilon Q^2$) for this new stress-tensor (T_j) with the original central charge $c = \epsilon(1 - 3Q^2)$ we have these relations

$$c = \begin{cases} c_j, & \text{if } \epsilon = +1 \text{ Fermi} \\ c_j - 2, & \text{if } \epsilon = -1 \text{ Bose} \end{cases} \tag{4.26}$$

In the case of Fermi Theory the $U(1)$ current $j(z)$ can be used to construct the Total stress-energy tensor, the Total means it has the same central charge. For the Bose theory we need another stress-energy tensor that has central charge -2 denoted by T_{-2}. If the central charge is -2, from (4.18), we can choose $Q = -1 \to \lambda = 1$ for a Fermi theory. For a Bose theory (4.18) does not have a solution. So we can write this auxiliary fermi theory as

$$S_{\eta\xi} = \frac{1}{2\pi}\int d^2z\ \eta\bar{\partial}\xi \qquad \text{with weight } h_\eta = 1\ \ h_\xi = 0 \tag{4.27}$$

$$T_{-2} \equiv T_{\eta\xi}(z) =: (\partial\eta)\xi : -\partial : (\eta\xi) : \tag{4.28}$$

Thus the Bose $\mathbf{b}, \mathbf{c}$ theory can be rewritten with $T_j + T_{\eta\xi}$.

4.2 Linear Dilaton and bc are the same?

There is a very nice relation to the Linear Dilaton Theory and the $\mathbf{b}, \mathbf{c}$ theory. To see this we need something called bosonization.

4.2.1 Bosonization

Note that we can reproduce the OPE (4.20) if we define

$$j(z) = \epsilon \partial_z \phi(z) \quad \text{with} \quad \phi(z)\phi(w) \sim \epsilon \ln(z-w) \tag{4.29}$$

Consider the operator $e^{q\phi(z)}$. To calculate its OPE's with $j(z)$ and $T_j(z)$ we need two basic OPE's

$$\partial_z \phi(z) : e^{q\phi}(w) := \frac{\epsilon q}{(z-w)} : e^{q\phi}(w) : \tag{4.30}$$

and the double contraction

$$: \partial_z \phi(z) \partial_z \phi(z) :: e^{q\phi}(w) := \frac{q^2}{(z-w)^2} : e^{q\phi}(w) : \tag{4.31}$$

One way to obtain these results is to use [2] $e^{q\phi}(w) = \sum_{n=0}^{\infty} \frac{1}{n!} q^n \phi^n(w)$. Then with (4.30)-(4.31) we find

$$j(z) : e^{q\phi}(w) :\sim \frac{q}{(z-w)} e^{q\phi}(w) \tag{4.32}$$

$$T_j(z) : e^{q\phi}(w) :\sim \left(\frac{\frac{1}{2}\epsilon q(q+Q)}{(z-w)^2} + \frac{1}{z-w}\partial_w \right) e^{q\phi}(w) \tag{4.33}$$

The conformal weight of $e^{q\phi}(w)$ is $\frac{\frac{1}{2}\epsilon q(q+Q)}{(z-w)^2}$. Note that

$$: e^{q\phi}(z) :: e^{-q\phi}(w) :\sim \frac{1}{(z-w)^{\epsilon q^2}} \tag{4.34}$$

For a Fermi theory $\epsilon = +1$ if choose $q = 1$ then (4.34) has the same OPE as (4.8). So we can identify $\mathbf{b}, \mathbf{c}$ with $e^{q\phi}(z)$:

$$\mathbf{b}(z) = e^{-\phi}(z) \quad ; \quad \mathbf{c}(z) = e^{\phi} \qquad \text{Fermi} \tag{4.35}$$

But For Boson theory $\epsilon = -1$ we have to add more operators, in the same way we add in the stress energy tensor. Using the same variables η, ξ we can identify

$$\mathbf{b}(z) = e^{-\phi}(z)\partial_z \xi(z) \quad ; \quad \mathbf{c}(z) = e^{\phi}\eta(z) \qquad \text{Boson} \tag{4.36}$$

[2] Given a set of m operators $: O_1 \dots O_m(z) :$ the total number of maximal contraction (all operators are contracted) with a set with n operators is given by $\binom{n}{m} = \frac{n!}{m!(n-m)!}$. In the first case we have $\binom{n}{1} = \frac{n!}{(n-1)}!$ and the second case $\binom{n}{2} = \frac{n!}{(n-2)}!$.

The Linear dilaton relation with $\mathbf{b}, \mathbf{c}$ can be viewed if we ask what action reproduces the stress-energy tensor $T_j(z)$ or, more importantly, what part of the action reproduces the Q part. The answer is given by

$$S_Q = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}(\frac{Q}{2})\phi R^{(2)} \tag{4.37}$$

The R^2 is the world-sheet Ricci scalar in $2d$. And this is the action for the Linear Dilaton where ϕ is called the dilaton field. The stress energy tensor is given by

$$T_{\alpha\beta} = -4\pi\frac{\delta S}{\delta g^{\alpha\beta}} = -\partial_\alpha\partial_\beta(\frac{Q}{2})\phi + \partial^2\delta_{\alpha\beta}(\frac{Q}{2})\phi \tag{4.38}$$

where we set $g_{\alpha\beta} = \delta_{\alpha\beta}$. In complex coordinates $\delta_{zz} = \delta_{\bar{z}\bar{z}} = 0$, thus we get the stress-energy tensor

$$T(z) = -\frac{Q}{2}\partial_z\partial_z\phi = -\epsilon\frac{Q}{2}\partial_z j(z) \tag{4.39}$$

agreeing with (4.23).

4.2.2 The twist

Consider the action

$$S = \frac{1}{2\pi}\int d^2z\mathbf{b}\nabla_{\bar{z}}\mathbf{c} = \frac{1}{2\pi}\int d^2z(\mathbf{b}\partial_{\bar{z}}\mathbf{c} - hA_{\bar{z}}\mathbf{bc}) \tag{4.40}$$

where $A_{\bar{z}}$ is a gauge field and h is some constant. Let us focus on the new part (the term with $A_{\bar{z}}$). Recall that the $U(1)$ current under bozonization is given by $j(z) = -\mathbf{bc}(z) = \epsilon\partial_z\phi(z)$. Thus we can write the action as

$$S_A \sim \int d^2zhA_{\bar{z}}\mathbf{bc} = \int d^2zhA_{\bar{z}}\epsilon\partial_z\phi(z) = -\int d^2zh(\partial_z A_{\bar{z}})\epsilon\phi(z) \tag{4.41}$$

where the second equality we integrated by parts. If we choose

$$A_{\bar{z}} = 2t\Gamma^{\bar{z}}_{\bar{z}\bar{z}} \tag{4.42}$$

where t is some other constant.[3] Then

$$-\frac{1}{2\pi}\int d^2z(\partial_z A_{\bar{z}})h\epsilon\phi(z) = \frac{1}{2\pi}\int d^2z\sqrt{g}R^{(2)}\phi(ht\epsilon) \tag{4.43}$$

where $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$ is the Christoffel symbol in conformal gauge and complex variables. Recall that in conformal gauge $g_{ab} = \delta_{ab}e^{2\omega}$ and in complex coordinates $g^{zz} = g^{\bar{z}\bar{z}} = 0$. The

[3] I add this constant so in the next chapter we can keep track on the choice of $A_{\bar{z}}$.

Christoffel symbol and Ricci scalar are define as

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{im}(\partial_l g_{km} + \partial_k g_{ml} - \partial_m g_{kl}) \tag{4.44}$$

$$R = g^{ab}(\partial_c \Gamma^c{}_{ab} - \partial_b \Gamma^c{}_{ac} + \Gamma^d{}_{ab}\Gamma^c{}_{cd} - \Gamma^d{}_{ac}\Gamma^c{}_{bd}) \tag{4.45}$$

In conformal gauge the only non-zero components of the Christoffel symbols are

$$\Gamma^z{}_{zz} = 2\partial_z \omega \quad \text{and} \quad \Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} = 2\partial_{\bar{z}}\omega \tag{4.46}$$

with

$$\sqrt{g}R = -(\partial \Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} + \partial_{\bar{z}} \Gamma^z{}_{zz}) = -2(\partial \Gamma^{\bar{z}}{}_{\bar{z}\bar{z}}) = -2\partial_{\bar{z}}\Gamma^z{}_{zz} \tag{4.47}$$

with (4.47) we see that (4.43) is true. The (4.43) is equal to (4.37) with $ht\epsilon = Q/2$ and $d^2\sigma = 2d^2z$. Then the $\mathbf{b}, \mathbf{c}$ stress energy tensor $T_{\mathbf{bc}}(z)$ is modify by

$$T_{\mathbf{bc}}(z) - ht\partial j(z) \tag{4.48}$$

The conformal weight of $\mathbf{b}$ will be shift by $-htN_g$, where N_g is the ghost number or $U(1)$ charge. From the $j(z)\mathbf{b}(w)$ and $j(z)\mathbf{c}(w)$ OPE's

$$(T_{\mathbf{bc}}(z) - ht\partial j(z))\mathbf{b}(w) \sim \frac{\lambda + htN_g(\mathbf{b})}{(z-w)^2}\mathbf{b}(w) \tag{4.49}$$

$$(T_{\mathbf{bc}}(z) - ht\partial j(z))\mathbf{c}(w) \sim \frac{1 - \lambda + htN_g(\mathbf{c})}{(z-w)^2}\mathbf{c}(w) \tag{4.50}$$

This result will be important for the twistor string action on the next chapter.

Chapter 5

Berkovits' Twistor String Action

This chapter is dedicated to the twistor string action introduced by Nathan Berkovits [3] as an alternative theory to the Witten's twistor theory[2]. In this new proposal, the tree-level amplitude for super-Yang-Mills come from open strings amplitudes. A general formula is given for the scattering amplitude with n gluons. The MHV amplitude is derived from this formula. In the end we see some of the issues that arises from the conformal supergravity.

5.1 Twistor String action

The worldsheet matter variables in string theory consist of left-moving super-twistor $\mathcal{Z}^I = (\lambda^\alpha, \mu^{\dot\alpha}, \psi^A)$ with $\alpha, \dot\alpha = 1$ to 2 and $A = 1$ to 4, and their conjugate super-twistor $\mathcal{W}_I = (\bar\lambda_\beta, \bar\mu_{\dot\beta}, \bar\psi_B)$. Also a left-moving current algebra j^k with $k = 1$ to $\dim(G)$. The complete action has right-moving variables $\tilde{\mathcal{Z}}^I, \tilde{\mathcal{W}}_I, \tilde{j}^k$ which in the open string they satisfy the boundary conditions:

$$\mathcal{Z}^I = \tilde{\mathcal{Z}}^I \quad ; \quad \mathcal{W}_I = \tilde{\mathcal{W}}_I \quad ; \quad j^k = \tilde{j}^k \tag{5.1}$$

The worldsheet action for the matter field is

$$S = \int d^2z [\mathcal{W}_I \nabla_{\bar z} \mathcal{Z}^I + \tilde{\mathcal{W}}_I \nabla_z \tilde{\mathcal{Z}}^I] + S_c \tag{5.2}$$

where S_c is the action for the right and left-moving current algebras. The covariant derivatives are

$$\nabla_{\bar z} = \partial_{\bar z} - A_{\bar z} \quad ; \quad \nabla_z = \partial_z - A_z \tag{5.3}$$

where A_a is a $GL(1)$ worldsheet gauge field. The definition of A_a is such that the $\tilde{\mathcal{Z}}^I, \mathcal{Z}^I$ have charge $+1$ and $\tilde{\mathcal{W}}^I, \mathcal{W}^I$ have charge -1. The gauge transformations are:

$$\delta\mathcal{W}_I = -i\theta_{\bar{z}}\mathcal{W}_I \qquad \delta\tilde{\mathcal{W}}_I = -i\theta_z\tilde{\mathcal{W}}_I \tag{5.4}$$

$$\delta\mathcal{Z}^I = +i\theta_{\bar{z}}\mathcal{Z}^I \qquad \delta\tilde{\mathcal{Z}}_I = +i\theta_z\tilde{\mathcal{Z}}_I \tag{5.5}$$

$$\delta A_{\bar{z}} = -i\partial_{\bar{z}}\theta_{\bar{z}} \qquad \delta A_z = -i\partial_z\theta_z \tag{5.6}$$

giving finite transformations

$$A'_{\bar{z}} = g\partial_{\bar{z}}g^{-1} + A_{\bar{z}} \quad \text{with} \quad g = e^{i\theta_{\bar{z}}} \tag{5.7}$$

$$A'_z = h\partial_{\bar{z}}h^{-1} + A_z \quad \text{with} \quad h = e^{i\theta_z} \tag{5.8}$$

Locally the conformal weight of $\mathcal{Z}$ is $(0,0)$ and the conformal weight of $\mathcal{W}$ is $(1,0)$, when we can gauge fix $A_{\bar{z}} = A_z = 0$. But we can not do this globally, we have to add instantons since the topology is not trivial[1]. We will see how to do this explicit in a moment. Note that the transformation h, g are independent, so we can think $A_{\bar{z}}, A_z$ as components $\mathbb{A}_z, \tilde{\mathbb{A}}_{\bar{z}}$ coming from different gauge fields $\mathbb{A}_m, \tilde{\mathbb{A}}_m$. The components of these fields are

$$\mathbb{A}_z = A_z, \quad \mathbb{A}_{\bar{z}} = 0 \quad \text{and} \quad \tilde{\mathbb{A}}_z = 0, \quad \tilde{\mathbb{A}}_{\bar{z}} = A_{\bar{z}} \tag{5.9}$$

The action (5.3) is just a set of $\mathbf{b}, \mathbf{c}$ variables from (4.1) where the conformal weight of $h(\mathbf{b}) = (\lambda, 0)$ and $h(\mathbf{c}) = (1-\lambda, 0)$ with $\lambda = 1$. So we can use all the results from chapter 4 if we identify $\mathbf{b} \leftrightarrow \mathcal{W}$ and $\mathbf{c} \leftrightarrow \mathcal{Z}$. The conformal transformation is given by (4.12)

$$\delta\mathcal{W} = \partial[r(z,\bar{z})\mathcal{W}(z,\bar{z})] - l(z,\bar{z})\bar{\partial}\mathcal{W}(z,\bar{z}), \tag{5.10}$$

$$\delta\mathcal{Z} = r(z,\bar{z})\partial\mathcal{Z}(z,\bar{z}) - l(z,\bar{z})\bar{\partial}\mathcal{Z}(z,\bar{z}). \tag{5.11}$$

The OPE's are

$$\mathcal{W}_J(z)\mathcal{Z}^I(w) \sim \frac{\epsilon\delta_J{}^I}{(z-w)} \quad ; \quad \mathcal{Z}^I(z)\mathcal{W}_I(w) \sim \frac{\delta_J{}^I}{(z-w)} \tag{5.12}$$

where

$$\delta_J{}^I = \begin{cases} +\delta_\beta{}^\alpha \\ +\delta_{\dot\beta}{}^{\dot\alpha} \\ +\delta_B{}^A \end{cases} \quad ; \quad \epsilon\delta_J{}^I = \begin{cases} -\delta_\beta{}^\alpha \\ -\delta_{\dot\beta}{}^{\dot\alpha} \\ +\delta_B{}^A \end{cases} \tag{5.13}$$

[1] Topologically trivial space we mean a space, which can be shrunk to a point without tearing it

Here ϵ is the tracker device defined in chapter 4:

$$\mathcal{W}_J \mathcal{Z}^I = -\epsilon \mathcal{Z}^I \mathcal{W}_J \quad ; \quad \epsilon = \begin{cases} +1 & \text{fermionic} \\ -1 & \text{bosonic} \end{cases} \tag{5.14}$$

Note that $\epsilon {\delta_I}^I = -2 - 2 + 4 = 0$. The stress-energy tensor is given by (4.14) with $\lambda = 1$

$$T_0(z) = -\mathcal{W}_I \partial_z \mathcal{Z}^I + T_c \tag{5.15}$$

where T_c is the stress energy tensor for the current algebra. The subscript in T_0 will become clear in a moment. It also have a $GL(1)$ current

$$J(z) = -\mathcal{W}_I \mathcal{Z}^I \tag{5.16}$$

Because the number of bosons $\{4\}$ and fermions $\{4\}$ are the same, the central charge for $\mathcal{W}, \mathcal{Z}$ is zero

$$T_0(z) T_0(w) \sim + \frac{\epsilon {\delta_I}^I}{(z-w)^4} + \frac{c_c}{2(z-w)^4} + O((z-w)^{-2}) \tag{5.17}$$

here c_c is the central charge for the Current algebra. Basically all quantities that depend on ϵ in chapter 4 will be zero for $\mathcal{W}, \mathcal{Z}$ because we have to sum and subtract the same quantity. For example the central charge was $c = \epsilon(1 - 3Q^2)$ so the total central charge is $4(1 - 3Q^2) - 4(1 - 3Q^2) = 0$.

5.1.1 Q-BRST & Current Algebra

The BRST ghosts can be treated as negative degrees of freedom. The BRST transformations for the matter fields $(\mathcal{Z}^I, \mathcal{W}_J)$ can be obtained by replacing the conformal parameter by a Grassmann parameter $(r(z) \to c(z))$ *i.e* the ghost. Now we have one more gauge symmetry, the $GL(1)$. Replacing the $GL(1)$ parameter by a new ghost $(\theta_{\bar{z}}(z) \to v(z))$ we get the BRST transformations

$$\delta_B \mathcal{Z}^I(w) = \xi c \partial_w \mathcal{Z}^I + \xi v \mathcal{Z}^I \qquad \delta_B \mathcal{W}_I(w) = \xi \partial_w (c \mathcal{W}_I) + (-1) \xi v \mathcal{W}_I \tag{5.18}$$

where ξ is just a Grassmann parameter, such that transformation preserves statistics.

The Virasoro ghosts are the usual bc with conformal weight $h_b = 2, h_c = -1$ and central charge -26. The $GL(1)$ ghosts are uv with conformal weight $h_u = 1, h_v = 0$ and central charge -2. So we can use (4.14) to construct the stress energy tensor for these ghosts:

$$T_{bc} = (\partial b)c - 2\partial(bc) = -(\partial b)c - 2b\partial c \tag{5.19}$$

$$T_{uv} = (\partial u)v - 1\partial(uv) = -u\partial v \tag{5.20}$$

The BRST charge is

$$Q = \int dz j_B = \int dz[c(T_0 + \frac{1}{2}T_{bc} + T_{uv}) + vJ] \tag{5.21}$$

The BRST transformation for the field $O(w)$ can be read off from the single pole in the $j_B(z)O(w)$ OPE.

$$\delta_B b(w) = \xi(T_0 + T_{bc} + T_{uv}) \qquad \delta_B c(w) = \xi c\partial_w c \tag{5.22}$$

$$\delta_B u(w) = \xi J + \xi\partial_w(cu) \qquad \delta_B v(w) = \xi c\partial_w v \tag{5.23}$$

$$\delta_B \mathcal{Z}^I(w) = \xi c\partial_w \mathcal{Z}^I + \xi v\mathcal{Z}^I \qquad \delta_B \mathcal{W}_I(w) = \xi\partial_w(c\mathcal{W}_I) + (-1)\xi v\mathcal{W}_I \tag{5.24}$$

The BRST transformation of the b ghost is the expected, since it came from the gauge fixing of the metric. It is proportional to the Total stress energy tensor, the generator of the conformal transformation. For the u ghost is almost just generator of $GL(1)$ transformation, but the other piece is necessary to make $\delta'_B\delta_B u(w) = 0$. The BRST charge is *nilpotent*, $Q^2 = 0$. The Q is used to construct physical states. The natural construction for a nilpotent operator is called the cohomology. One important result from $Q^2 = 0$ is that the central charge of the current algebra has to be $c_c = 28$ or the total central charge has to be zero $c = c_c - 26 - 2 + 0 = 0$.

Current Algebra

The current algebra is responsible to add the color factor on the amplitude. Usually this is done by the Chan-Paton factors, where you add degrees of freedom to the end points of an open string. The basis for a string state is $|N; k; ab\rangle$, where a, b labels the states of the end points of the string, and a is the group index. But here the group index comes from the currents $j^a(z)$. Let us see some properties of the current algebras. The current $j^a(z)$ is a holomorphic operator with conformal weight $(1, 0)$. The currents $j^a(z), j^b(w)$ satisfy the OPE

$$j^a(z)j^b(w) \sim \frac{kg^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c j^c(w)}{z-w} \tag{5.25}$$

and have Laurent expansion

$$j^a(z) = \sum_{m=-\infty}^{\infty} \frac{j^a_m}{z^{m+1}} \tag{5.26}$$

in which the coefficients satisfy the current algebra [2]

$$[j^a_m, j^b_n] = mkg^{ab}\delta_{m,-n} + if^{ab}{}_c j^c_{m+n} \tag{5.27}$$

The $m = 0$ modes form a Lie algebra G with $f^{ab}{}_c$ being the structure constant totally antisymmetric

$$[j^a_0, j^b_0] = if^{ab}{}_c j^c_0 \tag{5.28}$$

and g^{ab} is a symmetric tensor invariant under the algebra. The last piece of information before calculating the correlation function of these currents is that j^a_1, j^b_0, j^c_{-1} Jacobi identity requires

$$f^{bc}{}_d g^{ad} + f^{ba}{}_d g^{dc} = 0 \tag{5.29}$$

so we can think g^{ab} as the metric used to raise the index.

The correlation function for the one and two currents is given by

$$\langle j^a(z)\rangle = 0 \quad ; \quad \langle j^a(z_1) j^b(z_2)\rangle = \frac{kg^{ab}}{z_{12}^2} \tag{5.30}$$

where $z_{12} \equiv z_1 - z_2$. From this we can build the three point function,and the $SL(2,R)$ symmetry on the Disk. The tree point function is given by

$$\langle j^a(z_1) j^b(z_2) j^c(z_3)\rangle = \frac{ikf^{abc}}{z_{12}z_{23}z_{31}} \tag{5.31}$$

To calculate the $n-$point function it is useful to give a representation for $j^a_0 = T^a$. Where the matrix T^a is a element of the Lie algebra. We can represent $j^a(z)$ as

$$j^a(z) = (T^a)^i_j \psi^j(z)\bar{\psi}_i(z) \tag{5.32}$$

with the OPE

$$\psi^j(z_1)\bar{\psi}_i(z_2) \sim \frac{\delta^j_i}{(z_1 - z_2)} \tag{5.33}$$

Note that for the two point function

$$\langle j^{a_1}(z_1) j^{a_2}(z_2)\rangle = \langle (T^{a_1})^{i_1}_{j_1}\psi^{j_1}(z_1)\bar{\psi}_i(z_1)(T^{a_2})^{i_2}_{j_2}\psi^{j_2}(z_2)\bar{\psi}_2(z_2)\rangle = Tr[T^{a_1}T^{a_2}]\frac{1}{z_{12}^2} \tag{5.34}$$

The correlation function for the specific order $T^{a_1}T^{a_2}\ldots T^{a_n}$ comes from the contraction of $\psi^j(z_k)\bar{\psi}_i(z_{k+1})$ for $k = 1, \ldots, n$.

$$\langle j^{a_1}(z_1) j^{a_2}(z_2)\ldots j^{a_n}(z_n)\rangle = Tr[T^{a_1}T^{a_2}\ldots T^{a_n}] \times \frac{1}{z_{12}z_{23}\ldots z_{n1}} \tag{5.35}$$

[2]It also has the names affine algebra or (affine) Kac-Moody algebra

The total correlation function for n particles is given by permutations of n elements minus the subset of cyclic permutations, which is equivalent to the set of $n-1$ permutations. For example $n = 3$ we have $(n-1)! = 2$, which are $Tr[T^{a_1}T^{a_2}T^{a_3}], Tr[T^{a_1}T^{a_3}T^{a_2}]$. We are going to consider a fix order for the correlation function, and the rest is given by non-cyclic permutations. To be clear, the permutation also involves the points, so if the trace is $Tr[T^{a_1}T^{a_3}\dots]$ the denominator will have z_{13}.

The correlation function also have double trance contributions. This can be seen from the four point function. Where we can contract $j^{a_1}(z_1)j^{a_2}(z_2)$ and $j^{a_3}(z_3)j^{a_4}(z_4)$ them the four-point function splits in two two-point functions:

$$\langle j^{a_1}(z_1)j^{a_2}(z_2)j^{a_3}(z_3)j^{a_4}(z_4)\rangle = \frac{Tr[T^{a_1}T^{a_2}]}{z_{12}^2} \times \frac{Tr[T^{a_3}T^{a_4}]}{z_{34}^2} \tag{5.36}$$

5.1.2 Twisting and instanton number

The Euler characteristic number for some genus g in a 2d surface is

$$\chi(g) = 2 - 2g = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R^{(2)} \tag{5.37}$$

so for a sphere $g = 0$ and in $z, \bar{z}$ coordinates ($d^2\sigma = 2d^2z$) we get

$$4\pi = \int d^2z\sqrt{g}R^{(2)} \tag{5.38}$$

The instanton number is given by

$$d = \frac{i}{4\pi}\int d^2\sigma\epsilon_{ab}F_{ab} \tag{5.39}$$

with $\epsilon_{12} = 1$ and $F_{ab} = \partial_a A_b - \partial_b A_a$. In $z, \bar{z}$ coordinates[3] becomes

$$2\pi d = \int d^2z(\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z) \tag{5.40}$$

Recall that (5.9) we have two gauge fields $\mathbb{A}_m, \tilde{\mathbb{A}}_m$. So we will have two intanton numbers

$$2\pi\tilde{d} = \int d^2z(\partial_z\tilde{\mathbb{A}}_{\bar{z}} - \partial_{\bar{z}}0) \tag{5.41}$$

$$2\pi d = \int d^2z(\partial_z 0 - \partial_{\bar{z}}\mathbb{A}_z) \tag{5.42}$$

[3] $A_z = \frac{1}{2}(A_1 + iA_2)$, $A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2)$, $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$

Writing the Ricci scalar in complex coordinates (4.47)

$$4\pi = -\int d^2z(\partial_z\Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} + \partial_{\bar{z}}\Gamma^{z}{}_{zz}) = -2\int d^2z(\partial_z\Gamma^{\bar{z}}{}_{\bar{z}\bar{z}}) = -2\int d^2z(\partial_{\bar{z}}\Gamma^{z}{}_{zz}) \tag{5.43}$$

then we can identify

$$\tilde{\mathbb{A}}_{\bar{z}} = -\tilde{d}\Gamma^{\bar{z}}{}_{\bar{z}\bar{z}} \quad ; \quad \mathbb{A}_z = d\Gamma^{z}{}_{zz} \tag{5.44}$$

In the boundary $\tilde{d} = -d$. Then the stress energy tensor is twisted by

$$T_d(z) = T_0 + \frac{d}{2}\partial_z J(z) + T_c = -\mathcal{W}_I\partial_z\mathcal{Z}^I + \frac{d}{2}\partial_z J(z) + T_c \tag{5.45}$$

5.2 Tree Amplitudes

The N-point tree-level scattering amplitude [4] in string theory is given by

$$A(k_1,\ldots,k_N) = \int \frac{[d\mathcal{Z}dAdg]}{V_{diff\times Weyl}} \exp(-S)\prod_{i=1}^{N}\int d^2\sigma_i g^{1/2}V_i(\sigma_i). \tag{5.46}$$

Here k_i is the momentum of the string. The way we get red of the diff$\times$ Weyl redundancy of the world-sheet metric g^{ab} in a quantum theory is the usual way with bc ghosts (the BRST quantization). Where in tree level we just add a c ghost for each fixed position. From $SL(2,R)$ invariance we can fix three positions.

We can not fix the $GL(1)$ gauge globally to zero, instead we can choose the gauge field such that we can construct a topological number, *i.e*, the instanton number d. The instanton number is a not continuous parameter, so the integration on the path integral after the gauge fixing becomes a discrete sum over the instanton number

$$[dA] \to \sum_{Instanton} \tag{5.47}$$

So the amplitude for N particles becomes

$$A_N = \sum_d \langle c_1V_1(z_1)c_1V_2(z_2)c_3V_3(z_3)\prod_{i=4}^{N}\int dz_iV_i(z_i)\rangle_d \tag{5.48}$$

where $\langle\quad\rangle_d$ represents the correlation function on a disk with instanton number d. As we saw, the intanton number changes the conformal weight of $\mathcal{Z}^I$ $h_{\mathcal{Z}} = (-d/2, 0)$. Under

[4] Recall that in the Bosonic string, the S-matrix for N particles is $S(k_1,\ldots,k_N) = \sum_{top}\int \frac{[dXdg]}{V_{diff\times Weyl}} \exp(-S_x - \lambda\chi)\prod_{i=1}^{N}\int d^2\sigma_i g^{1/2}V_i$. Where g is the world-sheet metric, and X is the matter field. The sum is over different topologies, but in tree level we consider just genus zero surface.

a coordinate transformation it transforms as

$$\mathcal{Z}'(z') = z^{-d}\mathcal{Z}(z) \quad \text{for} \quad z' = 1/z \tag{5.49}$$

To cover the sphere we have to use two coordinate patches u, z and join them by identifying the points $u = 1/z$. We can work in the z coordinate all the time and just check that things behave properly on $z \to \infty$ or $u = 0$ point. When $z \to \infty$ the equation (5.49) implies that

$$\mathcal{Z}^I(z) = \sum_{n=0}^{d} z^n a_n^I \qquad \text{where } a_n^I \text{ are the zero modes} \tag{5.50}$$

recall that $I = (\alpha, \dot{\alpha}, A)$ where $(\alpha, \dot{\alpha})$ label bosonic states and A fermionic states. This means that the correlation function involves an integration over $4d + 4$ bosonic modes and $4d + 4$ fermionic modes. We can use the $GL(1)$ to fix one bosonic mode.

Physical states are described by dimension-one fields that are $GL(1)$ invariant. The Super-Yang-Mills vertex operator is built using a dimension zero-field $\varphi(\mathcal{Z}^I)$ and a current j^k, $k = 1, \ldots, dimG$. The function φ has to be invariant under the $GL(1)$ scaling of $\mathcal{Z}^I$, *i.e*, φ is invariant under $\mathcal{Z} \to t\mathcal{Z}$.

$$V_\varphi = j^k \varphi_k(\mathcal{Z}) \tag{5.51}$$

The twistor space wavefunction $\varphi_k(\mathcal{Z})$ of massless Yang-Mills particle is

$$\varphi_k(\mathcal{Z}) = \delta\left(\frac{\lambda^2(z_r)}{\lambda^1(z_r)} - \frac{\pi_r^2}{\pi_r^1}\right) \exp\left(i\frac{\mu^{\dot{\alpha}}(z_r)}{\lambda^1(z_r)}\bar{\pi}_{r\dot{\alpha}}\pi_r^1\right)\phi_{rk} \tag{5.52}$$

where

$$\phi_{rk} = \begin{cases} \phi_{rk}^+ = (\pi_r^1)^{-2} A_{rk}^+, & \text{for } h = +1 \\ \phi_{rk}^- = (\pi_r^1)^{-2}\left(\frac{\psi_r^A \pi_r^1}{\lambda_r^1}\right)^4 A_{rk}^-, & \text{for } h = -1 \end{cases} \tag{5.53}$$

Here the external momentum is $p_r^{\alpha\dot{\alpha}} = \pi_r^\alpha \bar{\pi}_r^{\dot{\alpha}}$, and in the signature $(2, 2)$, $\pi_r^\alpha\, \bar{\pi}_r^{\dot{\alpha}}$ are real and independent quantities. The Yang-Mills gauge field for the r particle with color index k is

$$A_{rk}^{\alpha\dot{\alpha}} = \pi^\alpha s^{\dot{\alpha}} A_{rk}^+ + \bar{s}^\alpha \bar{\pi}^{\dot{\alpha}} A_{rk}^- \tag{5.54}$$

such that $\bar{s}^\alpha \pi_\alpha = 1$ and $s^{\dot{\alpha}}\bar{\pi}_{\dot{\alpha}} = 1$.

We see that under the scaling of $\mathcal{Z}^I \to t\mathcal{Z}^I$ that is ($\mu \to t\mu$, $\lambda \to t\lambda$, $\psi \to t\psi$) the twistor wavefunction (5.52) is $GL(1)$ invariant. Now under the scaling of the external momentum

$$(\pi, \tilde{\pi}) \to (t\pi, t^{-1}\tilde{\pi})$$

the twistor wavefunction scales properly with respect to its helicity (h)

$$\begin{cases} \phi^+_{rk} \to t^{-2}\phi^+_{rk} = t^{-2h}\phi^+_{rk} \\ \phi^-_{rk} \to t^{+2}\phi^-_{rk} = t^{-2h}\phi^-_{rk} \end{cases} \tag{5.55}$$

For each particle with negative helicity the wave function (5.53) contains four fermionic variables $(\psi^A)^4 = \psi^1\psi^2\psi^3\psi^4$. The correlation function involves integration over $4d+4$ fermionic modes – the integral absorbs $4d+4$ fermionic modes. Thus for $d = 1$ the fermionic integral is not zero only if you have 8 fermionic variables, that is to say, two particles with negative helicity. Hence each winding number d calculate a different helicity violating amplitude. For $N > 3$ the amplitude with N positive helicities and $N-1$ positive and 1 negative helicities gives zero. This is the result that we found in *section* 2.3 and chapter 3 that the first non-vanishing Yang-Mills amplitude contains two negative helicities, *i.e*, the MHV-Amplitude.

The tree amplitude with $(N-d-1)$ positive helicity particles and $d+1$ negative helicity particles is given by

$$\begin{aligned} A(\lambda_i, \bar{\lambda}_i, \psi_i) =& \frac{1}{VOL(GL(2))} \int d^{2d+2}a \; d^{2d+2}b \; d^{4d+4}\gamma \int dz_1 \ldots \int dz_N \times \\ & \times \prod_{r=1}^{N} \frac{1}{(z_r - z_{r+1 \; mod \; N})} Tr\left[T^{a_1}T^{a_2}\ldots T^{a_n}\right] \phi_1\phi_2\ldots\phi_N \times \\ & \times \prod_{r=1}^{N} \exp\left(i\frac{\mu^{\dot{\alpha}}(z_r)}{\lambda^1(z_r)}\bar{\pi}_{r\dot{\alpha}}\pi^1_r\right) \prod_{r=1}^{N} \delta\left(\frac{\lambda^2(z_r)}{\lambda^1(z_r)} - \frac{\pi^2_r}{\pi^1_r}\right) \end{aligned} \tag{5.56}$$

with the zero modes $(a^\alpha_k, b^{\dot{\alpha}}_k, \gamma^A_k)$ from

$$\lambda^\alpha(z) = \sum_{n=0}^{d} a^\alpha_n z_n \quad \mu^{\dot{\alpha}}(z) = \sum_{n=0}^{d} b^{\dot{\alpha}}_n z_n \quad \psi^A(z) = \sum_{n=0}^{d} \gamma^A_n z_n$$

the second line come from the current correlation function (5.35) and the Trace is taken over the Lie Algebra, which gives the color trace in one specific order.

5.2.1 The MHV amplitude from Twistor String

Let us derive the MHV-amplitude $A(r^-, s^-)$ with the particles s, r with negative helicity and $N-2$ with positive helicity.

For two negative particles the degree is one ($d = 1$) and the zero modes are given by

$$\lambda^{\alpha}(z) = a^{\alpha} + \tilde{a}^{\alpha} z \quad ; \quad \mu^{\dot{\alpha}}(z) = b^{\dot{\alpha}} + \tilde{b}^{\dot{\alpha}} z \quad ; \quad \psi^{A}(z) = \gamma^{A} + \tilde{\gamma}^{A} z \quad ; \tag{5.57}$$

It is convenient to rewrite the zero modes using the spacetime and super-space variables ($x^{\alpha\dot{\alpha}}$, $\theta^{A\alpha}$). Using the twitor equation $\mu^{\dot{\alpha}}(z) = x^{\alpha\dot{\alpha}}\lambda_{\alpha}$, $\psi^{A} = \theta^{A\alpha}\lambda_{\alpha}$ we have:

$$b^{\dot{\alpha}} = a_{\alpha} x^{\alpha\dot{\alpha}} \quad , \quad \tilde{b}x^{\dot{\alpha}} = \tilde{a}_{\alpha} x^{\alpha\dot{\alpha}} \quad , \quad \gamma^{A} = a_{\alpha}\theta^{A\alpha} \quad , \quad \tilde{\gamma}^{A} = \tilde{a}_{\alpha}\theta^{A\alpha} \tag{5.58}$$

The $SL(2)$ part of $GL(2)$ can be used to fix tree position of the z_r integrals, we usually pick $0, 1, \infty$. But it is easy to calculate if we use the $SL(2)$ plus the $GL(1)$ invariance to set $\lambda^1 = 1$ and $\lambda^2 = z$, and then integrate over all the insertions positions for the vertex operators. The gauge fixing explicitly is $a^1 = 1$, $a^2 = 0$, $\tilde{a}^1 = 0$ and $\tilde{a}^2 = 1$. Now the relation with the new variables and the zero modes becomes:

$$b^{\alpha} = x^{\alpha\dot{2}} \quad ; \quad \tilde{b}^{\dot{\alpha}} = x^{1\dot{\alpha}} \quad ; \quad \gamma^{A} = \theta^{A1} \quad ; \quad \tilde{\gamma}^{A} = \theta^{A2} \tag{5.59}$$

So the Jacobian from $b^{\alpha}, \tilde{b}^{\dot{\alpha}}, \gamma^{A}, \tilde{\gamma}^{A}$ to $(x^{\alpha\dot{\alpha}}, \theta^{\alpha\dot{\alpha}})$ is one. Note that we don't have any $a's$ because we fixed them already. Finally we have

$$\frac{1}{VOL(GL(2))} \int d^4a \; d^4b \; d^8\gamma \to \int d^4x^{\alpha\dot{\alpha}} \int d^8\theta^{A\alpha} \tag{5.60}$$

Under the support of the delta function $\delta\left(\frac{\lambda^2}{\lambda^1} - \frac{\pi^2}{\pi^1}\right)$ and the supertwistor equation $\psi^{A} = \theta^{A\alpha}\lambda_{\alpha}$, one can write the negative helicity wavefuntion as

$$\phi^{-} = (\pi^1)^{-2} \left(\frac{\psi^{A}\pi^1}{\lambda^1}\right)^4 A^{-} = (\pi^1)^{-2} (\theta^{A\alpha}\pi_{\alpha})^4 A^{-} \tag{5.61}$$

Just to be clear $(\theta^{A\alpha}\pi_{\alpha})^4 \equiv \theta^{1\alpha}\pi_{\alpha}\theta^{2\beta}\pi_{\beta}\theta^{3\gamma}\pi_{\gamma}\theta^{4\rho}\pi_{\rho}$. The wave-function part of the Amplitude (5.56) becomes

$$\begin{aligned} \phi_1\phi_2 \ldots \phi_s \ldots \phi_r \ldots \phi_N = \prod_{i=1}^{N} (\pi_i^1)^{-2} \times (\theta_s^{A\alpha}\pi_{s\alpha})^4 (\theta_r^{A\alpha}\pi_{r\alpha})^4 \times \\ \times A_1^{+} A_2^{+} \ldots A_s^{-} \ldots A_r^{-} \ldots A_N^{+} \end{aligned} \tag{5.62}$$

Using the Twistor equation $\mu^{\dot{\alpha}}(z) = x^{\alpha\dot{\alpha}}\lambda_\alpha(z)$ and the support of the delta function[5] we can write the exponential part of the Amplitude (5.56) as

$$\prod_{r=1}^{N} \exp\left(ix^{\alpha\dot{\alpha}}\bar{\pi}_{r\dot{\alpha}}\pi_{r\alpha}\right) = \exp\left(ix^{\alpha\dot{\alpha}}\sum_{r=1}^{N}\bar{\pi}_{r\dot{\alpha}}\pi_{r\alpha}\right) \tag{5.63}$$

note that this is almost the conservation of external momentum delta function, we just have to integrate over $x^{\alpha\dot{\alpha}}$. Another useful variable to introduce is

$$u_r = \frac{\lambda_r^2}{\lambda_r^1} = \frac{a^2 + \tilde{a}^2 z_r}{a^1 + \tilde{a}^1 z_r} \tag{5.64}$$

Note that I didn't use the gauge fixing for $a's, \tilde{a}'s$ here, just to show that this is independent of the gauge choice. Let us rename $a's$ and $\tilde{a}'s$ so that the manipulations becomes more clear, such that

$$u_r = \frac{b + az_r}{d + cz_r} \quad ; \quad \frac{\partial u_r}{\partial z_n} = \frac{ad - bc}{(cz_n + d)^2} \tag{5.65}$$

The $z's$ denominator in term of $u's$ are

$$\prod_{r=1}^{N} \frac{1}{(z_r - z_{r+1\ mod\ N})} = \prod_{r=1}^{N} \frac{1}{(u_r - u_{r+1\ mod\ N})} \times \prod_{i=1}^{N} \frac{(cz_i + d)^2}{(ad - bc)} \tag{5.66}$$

The Jacobian J from $dz_1 \ldots dz_N \to Jdu_1 \ldots du_N$ is $J = det(\frac{\partial u_r}{\partial z_n}) = \prod_{i=1}^{N} \frac{ad-bc}{(cz_n+d)^2}$ so the extra part (5.66) cancels out, and we get

$$du_1 \ldots du_N \prod_{r=1}^{N} \frac{1}{(u_r - u_{r+1\ mod\ N})} \tag{5.67}$$

Connecting the pieces (5.60)-(5.62)-(5.63)-(5.67) we can write the amplitude as

$$\begin{aligned} A(\lambda_i, \bar{\lambda}_i, \psi_i) = \int d^4x \int d^8\theta \int du_1 \ldots \int du_N \prod_{r=1}^{N} \frac{1}{(u_r - u_{r+1\ mod\ N})} \delta\left(u_r - \frac{\pi_r^2}{\pi_r^1}\right) \times \\ \times \exp\left(ix^{\alpha\dot{\alpha}}\sum_{r=1}^{N}\bar{\pi}_{r\dot{\alpha}}\pi_{r\alpha}\right)\prod_{i=1}^{N}(\pi_i^1)^{-2} \times (\theta_s^{A\alpha}\pi_{s\alpha})^4(\theta_r^{A\alpha}\pi_{r\alpha})^4 \\ \times Tr\left[T^{a_1}T^{a_2}\ldots T^{a_n}\right](A_1^+A_2^+\ldots A_s^-\ldots A_r^-\ldots A_N^+) \end{aligned} \tag{5.68}$$

[5] To see this just plug $x^{\alpha\dot{\alpha}}\lambda_\alpha$ and open in components and use $\pi^1\frac{\lambda^2}{\lambda^1} = \pi^2$

Now is fairly easy to do the rest of the derivation. The conservation of momentum delta function appears as expected.

$$\delta^4\left(\sum_{r=1}^{N} p_r\right) = \int d^4x \exp\left(ix^{\alpha\dot{\alpha}}\sum_{r=1}^{N}\bar{\pi}_{r\dot{\alpha}}\pi_{r\alpha}\right) \tag{5.69}$$

Recall the clean notation $\langle i,j\rangle \equiv \varepsilon_{\alpha\beta}\pi_i^\alpha\pi_j^\beta$ with $\varepsilon_{12} = -1$. Then using the delta function we get the usual MHV factor

$$\prod_{r=1}^{N}\frac{1}{(u_r - u_{r+1 \; mod \; N})} = \prod_{r=1}^{N}\frac{(\pi_r^1)^2}{\langle r, r+1\rangle} \tag{5.70}$$

where the extra term in the numerator cancels out the inverse term in the amplitude (5.68)($(\pi_r^1)^2(\pi_r^1)^{-2} = 1$). The fermionic integral gives the MHV numerator [6]

$$\int d^8\theta(\theta_s^{A\alpha}\pi_{s\alpha})^4(\theta_r^{A\alpha}\pi_{r\alpha})^4 = \langle r,s\rangle^4 \tag{5.71}$$

Collecting everything we get the familiar MHV-amplitude (without the color trace)

$$A(r^-,s^-) = \delta^4\left(\sum_{r=1}^{N} p_r\right)\frac{\langle r,s\rangle^4}{\langle 1,2\rangle\langle 2,3\rangle\ldots\langle N,1\rangle} \tag{5.72}$$

Indeed the formula (5.56) reproduces the MHV amplitude. As we saw for each negative gluon that we add we have to increase the degree d. The relation between the MHV amplitude ($n-2$ positive gluons and 2 negative gluons) and the googly MHV ($n-2$ negative gluons and 2 positive gluons) is just the complex conjugation($\langle ij\rangle^* = [ij] \equiv \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}_{\dot{\beta}}$). But the in the twistor formula this relation is not apparently. The integration is over different degrees. The googly amplitude was calculated in [22, 23], where they Fourier transform $\mu \to \bar{\lambda}$ in (5.56).

5.3 Conformal Supergravity

As we saw the correlation function (5.36) has double-trace contributions. These contributions turns out to be related to exchange of conformal supergravity states. The open twistor string also have vertex operators that represents conformal supergravity. This section is dedicated to introduce these vertex operators and show that conformal supergravity arises from the double-trace amplitudes.

[6] using that $\theta^{1\beta}\theta^{1\gamma} = (\theta^1\cdot\theta^1)\frac{1}{2}\varepsilon^{\beta\gamma}$ and $\int d\theta^{11}d\theta^{12}(\theta^1\cdot\theta^1)\frac{1}{2}\varepsilon^{\beta\gamma} = \varepsilon^{\beta\gamma}$

5.3.1 Conformal supergravity vertex

One can also construct these dimension one vertex operators

$$V_f = \mathcal{W}_I f^I(\mathcal{Z}), \qquad V_g = g_I(\mathcal{Z})\partial\mathcal{Z}^I \tag{5.73}$$

which turns out to describe conformal supergravity. They have dimension one since $\mathcal{W}_I$ and $\partial\mathcal{Z}^I$ have dimension one. The field f^I has $+1$ $GL(1)$ charge (it scales $f \to tf$ under $\mathcal{Z} \to t\mathcal{Z}$) and g_I has -1 charge (scales as $g \to t^{-1}g$). To be primary fields with respect to T and J, f^I and g_I must satisfy

$$\partial_I f^I = 0, \qquad \mathcal{Z}^I g_I = 0. \tag{5.74}$$

These field also have a gauge invariance

$$\delta f^I = \mathcal{Z}^I \Lambda, \qquad \delta g_I = \partial_I \chi \tag{5.75}$$

since

$$\delta V_f = \mathcal{W}_I \mathcal{Z}^I \Lambda = J_{-1}\Lambda \quad \text{and} \quad \delta V_g = \partial\mathcal{Z}^I \partial_I \chi = T_{-1}\chi \tag{5.76}$$

are null states.

Now we show that both the vertex operators (5.73) indeed describes the spectrum of conformal supergravity. The first thing we need is that a function of the homogeneous coordinates $\mathcal{Z}^I$ of twistor space that is homogeneous in $\mathcal{Z}^I$ of degree k describes a massless state in Minkowski spacetime of helicity $h = 1 + k/2$. As we saw the wave function $\varphi(\mathcal{Z})$ was homogeneous of degree zero and describes the gluons with helicity $h = 1 + 0/2 = 1$.

First let us start with the homogeneous degree one field $f^I(\mathcal{Z})$, and set $\psi^A = 0$. Recall that index I runs from four bosonic states $I = (\alpha, \dot\alpha)$, and four fermionic states $I = A$. For each value of I, $f^I(\mathcal{Z})$ describes four bosonic and four fermionic helicity states, with $h = 3/2$. But we have ignored the spin carried by the index I. For the indices $I = (\alpha, \dot\alpha)$ the field f^I describes two states with helicity $h = 1/2$ and two with $h = -1/2$, as we saw how a spinor transforms under a Lorentz transformation in chapter 2. Thus we get two bosonic states with $h = 3/2 + 1/2 = 2$ and two bosonic states with $h = 3/2 - 1/2 = 1$. The fermionic index A carries information how it transforms under the $SU(4)_R$ symmetry group. We get four fermionic states that transform in the fundamental of $SU(4)_R$ with $h = 3/2$.

The gauge invariance $f^I \to \mathcal{Z}^I\Lambda$ implies that we have to remove a state described by homogeneous function Λ of degree one, that is to say, remove one bosonic $h = 1$ state.

The constrain $\partial_I f^I$ also removes another bosonic $h = 1$ state. Thus after removing these two, we are left with two bosonic states of $h = 2$ and four fermionic states that transform in the fundamental of $SU(4)_R$ with $h = 3/2$ for $\psi^A = 0$. The get the full spectrum, we expand in powers of ψ.

$$f^I(\lambda, \mu, \psi) = f_0^I(\lambda, \mu) + \cdots + f_4^I(\lambda, \mu)\psi^4 \tag{5.77}$$

where f_k^I has degree $1-k$. If we take in to account the index $I = (\alpha, \dot{\alpha})$ we get helicities $h = 2, 1, 0, -1$. The helicities $1, -1$ are removed using the constrains. So we have a massless spin $+2$ and a scalar, plus other states from the dots in (5.77).

One can do the same analysis for g_I field

$$g_I(\lambda, \mu, \psi) = g_{I0}(\lambda, \mu) + \cdots + g_{I4}(\lambda, \mu)\psi^4 \tag{5.78}$$

where g_{Ik} has degree $-1-k$. In the end we get massless spin -2 and a scalar $h = 0$. Doing the analysis more carefully one finds that the massless fields describes by g_I have opposite helicities and conjugate $SU(4)_R$ representations from those described by f^I.

5.3.2 Linearized conformal supergravity

At the lineareized level, $\mathcal{N} = 4$ conformal supergravity can be described off-shell by a chiral scalar superfield $\mathbb{W}(x^{\alpha\dot{\alpha}}, \theta_\alpha^A, \bar{\theta}_A^{\dot{\alpha}})$ which satisfies the condition

$$\epsilon^{ABCD} D_C^\alpha D_{E\alpha} D_{D\beta} D_F^\beta \mathbb{W} = \epsilon_{EFGH} \bar{D}^{\dot{\alpha}A} \bar{D}_{\dot{\alpha}}^G \bar{D}_{\dot{\beta}}^B \bar{D}^{\dot{\beta}H} \bar{\mathbb{W}} \tag{5.79}$$

Here D_A^α and $\bar{D}_{\dot{\alpha}}^A$ are the usual superspace derivatives. It is convenient to choose co-ordinates such that $\bar{D}_{\dot{\alpha}}^A = \frac{\partial}{\partial \bar{\theta}_A^{\dot{\alpha}}}$, so the chiral condition $\bar{D}_{\dot{\alpha}}^A \mathbb{W} = 0$, implies that $\mathbb{W}$ is independent of $\bar{\theta}_A^{\dot{\alpha}}$. The component field expation of $\mathbb{W}$ is

$$\mathbb{W}(x, \theta) = C + \text{a bunch of terms} + \theta^8 (\partial_m \partial^m)^2 \bar{C}. \tag{5.80}$$

Of course this is a super simplified expansion, but I am interest in the scalar field only. The linearized action for $\mathcal{N} = 4$ conformal supergravity is

$$S = \int d^4x \int d^8\theta \mathbb{W}^2 = \int d^4x \left[C(\partial_m \partial^m)^2 \bar{C} + \dots \right] \tag{5.81}$$

The scalar propagator $\propto \frac{1}{k^4}$ is going to be important on the double-trace amplitudes. To Support the idea that the supergravity states contribute to double-trace amplitudes, let us analyze the four-gluon amplitude. Consider the $A(1^+, 2^+, 3^-, 4^-)$, from the double-trace contribution of the correlation function (5.36) one finds

$$A(1^+, 2^+, 3^-, 4^-) = \langle \lambda_3, \lambda_4 \rangle^4 \times \frac{Tr[T^1T^2]}{\langle \lambda_1, \lambda_2 \rangle^2} \times \frac{Tr[T^3T^4]}{\langle \lambda_3, \lambda_4 \rangle^2} \tag{5.82}$$

Now if we consider $k^2 = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = 2\langle \lambda_1, \lambda_2 \rangle [\lambda_1, \lambda_2]$ from (2.9). The amplitude become

$$A(1^+, 2^+, 3^-, 4^-) = Tr[T^1T^2]Tr[T^3T^4] \times \frac{\langle \lambda_3, \lambda_4 \rangle^2 [\lambda_1, \lambda_2]^2}{k^4} \tag{5.83}$$

This amplitude represents the exchange of the superconformal scalar field $C, \bar{C}$ with the coupling $CTr[F_{mn}F^{mn}]$, from the channel $12 \to 34$ see figure 5.1. Indeed with the polarization contractions $\epsilon_1^+ \cdot \epsilon_2^+ = k^2[\lambda_1, \lambda_2]^2$ and $\epsilon_3^- \cdot \epsilon_4^- = \langle \lambda_3, \lambda_4 \rangle^2 / k^2$ and the scalar propagator $C\bar{C} = \frac{1}{k^4}$ we get (5.83) . There is a similar analyzes for the graviton, that contributes with the double-trace amplitude $A(1^+, 2^-, 3^+, 4^-)$, but it is a little more difficult, so for simplicity we choose to look on the scalar field only.

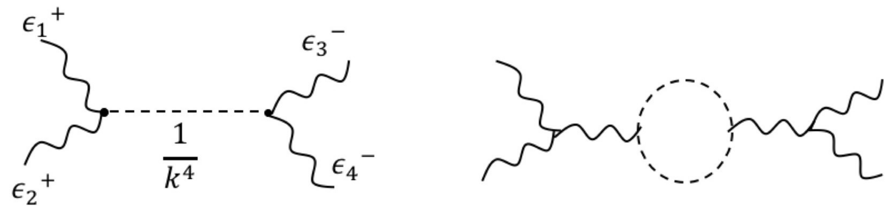

FIGURE 5.1: A double trace contribution coming from the exchange of conformal supergravity scalar in the $12 \to 34$ channel. In the right diagram, a single trace loop level contribution.

At tree level we can extract Yang-Mills amplitudes by considering only single-trace amplitudes. However at loop level single-trace diagrams can be produced by conformal supergravity states, so the presence of conformal supergravity (5.73) in the twistor string theory, is the first sign of difficult to calculate loop amplitude in SYM using twistor strings. Since Yang-Mills theory makes sense without conformal supergravity, it is likely that there is a twistor string theory that is dual to pure Yang-Mills theory. It would be very exciting to find such theory.

Chapter 6

Perspectives for the future

The reason, the goal and the difficulty

As we saw in chapter 5 super-twistors have played an important role in simplifying the computation of $d = 4$ super-Yang-Mills scattering amplitudes, where in a single formula (5.56) we have a way to calculate MHV and non-MHV super-Yang-Mills tree amplitudes. After Witten original idea of twistor string theory [2], further developed was done in a series of papers[5–9] and drastically simplified the conventional Feynman diagram techniques for SYM and super-gravity. These twistor strings are specific to its own theories and it remains unclear how to extend them to other theories, or if it is valid for loop amplitudes.

In 2013, in the remarkable series of papers [24–27] , Cachazo, He and Yuan have presented analogous formulae based on $d = 4$ twistor-inspired ideia in [7], but extended to describe scattering of massless particles of spins 0, 1 or 2 in arbitrary dimension. So we can express tree-level amplitudes for SYM and super-gravity in any dimension in an elegant form using the results of Cachazo, He and Yuan. In the same year Mason and Skinner [28] presented a family of string theories (Type II version, Heterotic version) that in the infinite tension limit (or $\alpha' \to 0$ limit) reproduces the Cachazo, He and Yuan formulae for some amplitudes in $d = 10$. Soon after Nathan Berkovits constructed [29] the pure spinor infinity tension limit of the Mason and Skinner superstring.

The pure spinor string can calculate all the $d = 10$ SYM and supergravity amplitudes. The construction mirrors the Mason-Skinner approach and starts with the pure spinor version of the $d = 10$ $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superparticle action. The advantage of the pure spinor formalism is that spacetime-supersymmetry is manifest. In principle one could use this string theory to go beyond the tree-level and calculate the loop amplitudes. Mason-Monteiro-Geyer-Tourkine have some papers [30, 31] that present some formulas for loop amplitudes. A good open problem is to calculate loop amplitudes using the pure spinor approach in this infinity tension limit. However, there are two reasons to suspect this will be difficult. First SYM and super-gravity in $d = 10$ have ultraviolet divergences. Secondly the fact that this string theory depended on only holomorphic world-sheet variables remember the same problem mention in end of chapter 5 with the open twistor string theory [10].

Furthermore the Cachazo-He-Yuan results that were based on twistor string ideas, and their SYM and super-gravity formulae in $d = 10$ can be useful to understand the relation between pure spinors and Nathan's $d = 10$ supertwistors [32]. In the supertwistor paper, he found a relation between the twistor superfield $\Phi(Z)$ and the ten-dimensional SYM vertex operator $\lambda^\alpha A_\alpha(x, \theta)$. This vertex operator is the same one that appears in the superstring pure spinor formalism, but in super-Yang-Mills case the momentum spinor play the role of pure spinor ghost. There is little discussion on super-twistors in higher dimensions [33], but it is worth noticing that superstring theory, and twistor theory have similar features [34]. Thus a future goal is to study the pure spinor and twistor strings and try to relate them using the Cachazo-He-Yuan amplitudes as links.

Appendix A

Dotted or Undotted

This appendix it's aimed to set up the conventions/notation that I will use in the rest of the thesis and to refresh the reader some topics. The metric is the mostly plus $\eta = (- + ++)$. The notation will be the same as the book Wess & Bagger [13]. If the reader is not familiar with these concepts keep going that in the end I will make a connection with the usual Dirac stuff.

Let $\mathbf{M}$ be a two-by-two matrix with $\det \mathbf{M} = 1$, i.e. $\mathbf{M} \in SL(2, C)$, this are matrices with complex values and unit determinant. One thing to note, is the number of generators of this group. We have 4 complex entries (8 real) and the constrain from the unit determinant give two equations (real part $= 1$ and imaginary $= 0$). Thus we have $8 - 2 = 6$ generators, the same as our old friend The Lorentz Group $SO(3, 1)$ with 3 boosts + 3 rotations. Now we introduce the the dotted and undotted indices. The spinor with dotted indices transform under the $(0, 1/2)$ representation of Lorentz group and spinor with undotted indices transform under $(1/2, 0)$ conjugate representation.The spinor indices take values $\alpha = 1, 2 \quad \dot{\alpha} = \dot{1}, \dot{2}$.

$$\psi'_\alpha = M_\alpha{}^\beta \psi_\beta \qquad ; \qquad \psi'^\alpha = (M^{-1})_\beta{}^\alpha \psi^\beta \tag{A.1a}$$

$$\bar{\psi}'_{\dot{\alpha}} = (M^*)_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\psi}_{\dot{\beta}} \qquad ; \qquad \bar{\psi}'^{\dot{\alpha}} = (M^*)^{-1}{}_{\dot{\beta}}{}^{\dot{\alpha}} \bar{\psi}^{\dot{\beta}} \tag{A.1b}$$

We have two indices (dotted and undotted) the two representation are inequivalent, i.e. we can not find a matrix $\mathbf{C}$ such that $\mathbf{M} = \mathbf{C}\mathbf{M}^*\mathbf{C}^{-1}$. But the two representation in (A.1a) are equivalent, thus exist a matrix ε such that $\mathbf{M} = \varepsilon \mathbf{M}^{-1T} \varepsilon^{-1}$. Hang on that we will see what it is this matrix . The same is valid for the two transformation with dotted indices.

We recall that any 2×2 matrix can be written as linear combination of the Pauli matrices plus the identity. Let me call this basis as $\sigma^m = (-I, \vec{\sigma})$, where $m = 0, ..., 3$.

$$\mathbf{P} = P_m \sigma^{\mathbf{m}} = -I P_0 + \vec{P} \cdot \vec{\sigma} = \begin{pmatrix} -P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_0 - P_3 \end{pmatrix} \tag{A.2}$$

We can see that $\mathbf{P}$ is hermitian ($\mathbf{P} = \mathbf{P}^\dagger$). A nice property of the matrix P is that $\det \mathbf{P} = P_0^2 - \vec{P} \cdot \vec{P} = -\eta^{mn} P_m P_n$. Using the fact that $\mathbf{P}$ is hermitian we can write another matrix $\mathbf{P'}$ as:

$$\mathbf{P'} = \mathbf{M P M}^\dagger \tag{A.3}$$

$$\mathbf{P'}^\dagger = (\mathbf{M P M}^\dagger)^\dagger = \mathbf{M P M}^\dagger = \mathbf{P'} \tag{A.4}$$

Both $\mathbf{P'}$ and $\mathbf{P}$ can be written as linear combination of σ^m. The determinant of $\mathbf{P'}$ (because the determinant of $\mathbf{M}$ is one and $\det[ABC] = \det[A] \det[B] \det[C]$) is equal to the determinant of $\mathbf{P'}$.

$$\det \mathbf{P'} = -\eta^{mn} P'_m P'_n = -\eta^{mn} P_m P_n \tag{A.5}$$

Now we start to see the connection between the Lorentz group and this matrices.This transformation correspond to a Lorentz transformation, that's cool. Before we continue let's appreciate what we have done. We stared defining a 2×2 matrix $\mathbf{M}$ that had determinant one (you could say unimodular), and we noted that any 2×2 hermitian matrix $\mathbf{P}$ could be expanded as a linear combination of σ^m and the determinant of this was the inner product of a Lorentz four vector, i.e, $\eta^{mn} P_m P_n$. Finally we found a transformation that is the same as the Lorentz Transformation.

Lets take a look on the index structure of $\mathbf{P}$. From (A.1a) that $\mathbf{M}^\dagger \equiv (\mathbf{M}^{\mathbf{T}})^* = ((M_\alpha{}^\beta)^T)^* = (M^\beta{}_\alpha)^* = M^{\dot\beta}{}_{\dot\alpha}$. Thus we can rewrite (A.3) as:

$$P'_{\alpha\dot\alpha} = M_\alpha{}^\beta P_{\beta\dot\beta} M^{\dot\beta}{}_{\dot\alpha} \tag{A.6}$$

The index structure of the Pauli matrices is : $\sigma^m = \sigma^m_{\alpha\dot\alpha}$. Note that we use Latin indices for vectors and tensors and Greek indices for spinors. Now we return to the matrix ε that relate the two equivalent representation (A.1a).

Let

$$\varepsilon = (\varepsilon_{\alpha\beta}) = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.7}$$

$$\varepsilon^{-1} = (\varepsilon^{\alpha\beta}) = \begin{pmatrix} \varepsilon^{11} & \varepsilon^{12} \\ \varepsilon^{21} & \varepsilon^{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.8}$$

with this matrix one can just plug in $\mathbf{M}^{-1T} = \varepsilon \mathbf{M} \varepsilon^{-1}$ and check that works. The $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ are antisymmetric tensors, and satisfy $\det \varepsilon = 1$ and $\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha{}^\gamma$. we can write $\mathbf{M}^{-1T} = \varepsilon \mathbf{M} \varepsilon^{-1}$ with indices :

$$\varepsilon^{\alpha\beta} M_\beta{}^\gamma \varepsilon_{\gamma\rho} = (M^{-1T})_\rho{}^\alpha = (M^{-1})^\alpha{}_\rho \tag{A.9}$$

$$\psi'^\alpha = (M^{-1})^\alpha{}_\rho \psi^\rho = \varepsilon^{\alpha\beta} M_\beta{}^\gamma (\varepsilon_{\gamma\rho}\psi^\rho) \tag{A.10}$$

$$(\varepsilon_{\beta\alpha}\psi'^\alpha) = M_\beta{}^\gamma (\varepsilon_{\gamma\rho}\psi^\rho) \tag{A.11}$$

Thus $\varepsilon_{\gamma\rho}\psi^\rho$ transform as ψ_γ and the ε tensor can be used to lower and raise indices:

$$\psi_\gamma = \varepsilon_{\gamma\rho}\psi^\rho \quad ; \quad \psi^\gamma = \varepsilon^{\gamma\rho}\psi_\rho \tag{A.12}$$

Every thing that we have done for undotted indices can be done similar for dotted. We are almost ready to see the connection to Dirac usual spinor. Before to do we take a look on some identities of the Pauli matrix. If we define another Pauli basis:

$$\bar{\sigma}^m = (-I, -\vec{\sigma}) \tag{A.13a}$$

with indices

$$(\bar{\sigma}^m)^{\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}(\sigma^m)_{\beta\dot{\beta}} \tag{A.13b}$$

we have some important identities:

$$\sigma^{(m}\bar{\sigma}^{n)} = (\sigma^m\bar{\sigma}^n + \sigma^n\bar{\sigma}^m)_\alpha{}^\beta = -2\eta^{mn}\delta_\alpha{}^\beta \tag{A.14a}$$

$$\bar{\sigma}^{(m}\sigma^{n)} = (\bar{\sigma}^m\sigma^n + \bar{\sigma}^n\sigma^m)_{\dot{\alpha}}{}^{\dot{\beta}} = -2\eta^{mn}\delta_{\dot{\alpha}}{}^{\dot{\beta}} \tag{A.14b}$$

and

$$\mathrm{Tr}\ \sigma^m\bar{\sigma}^n = -2\eta^{mn} \tag{A.15a}$$

$$(\sigma^m)_{\alpha\dot{\alpha}}(\bar{\sigma}_m)^{\dot{\beta}\beta} = -2\delta_{\dot{\alpha}}{}^{\dot{\beta}}\delta_\alpha{}^\beta \tag{A.15b}$$

Now we can easy go back and forth between Lorentz indices and bispinor indices ($m \leftrightarrow \alpha\dot{\alpha}$):

$$p_{\alpha\dot{\alpha}} = p_m(\sigma^m)_{\alpha\dot{\alpha}} \quad ; \quad p_m = -\frac{1}{2}(\bar{\sigma}_m)^{\dot{\beta}\beta} p_{\beta\dot{\beta}} \tag{A.16}$$

As I promise, let's see the connection with the usual Dirac matrices and spinors. The Clifford algebra is (in the $(-,+++)$ metric):

$$\{\Gamma^m, \Gamma^n\} = -2I\eta^{mn} \tag{A.17}$$

In the Weyl basis the gamma matrix is :

$$\Gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \bar{\sigma}^m & 0 \end{pmatrix} \tag{A.18}$$

One can easily see that this gamma matrix satisfy the Clifford algebra (A.17). The gamma matrix act on a 4 components spinor with index structure:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

One can write the Dirac equation:

$$(\Gamma^m \partial_m + m)\Psi = 0 \tag{A.19a}$$

and in the weyl basis

$$((\bar{\sigma}^m)^{\dot{\alpha}\alpha}\partial_m + m)\psi_\alpha = 0 \tag{A.19b}$$

$$((\sigma^m)_{\alpha\dot{\alpha}}\partial_m + m)\bar{\chi}^{\dot{\alpha}} = 0 \tag{A.19c}$$

Remember also that the Lorentz generators were given by $S^{mn} = \frac{1}{4}[\Gamma^m, \Gamma^n]$, and the Dirac component spinor transform as $\Psi \to \exp(\frac{1}{2}\omega_{mn}S^{mn})\Psi$. Then the Lorentz group generator in the spinor representation become:

$$(\sigma^{mn})_\alpha{}^\beta = \frac{1}{4}(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m)_\alpha{}^\beta \tag{A.20a}$$

$$(\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m)^{\dot{\alpha}}{}_{\dot{\beta}} \tag{A.20b}$$

These matrices seems strange when one looks for the first time, remember they are made of Pauli Matrices, in particular:

$$\sigma^{ij} = \bar{\sigma}^{ij} = -\frac{i}{2}\epsilon^{ijk}\sigma^k \quad \text{and} \quad \sigma^{0i} = -\bar{\sigma}^{0i} = \frac{1}{2}\sigma^i$$

these are rotations and boost respectively. The ϵ^{ijk} is the usual Levi-Civita symbol, and $i, j, k = 1, 2, 3$ with $\epsilon^{123} = 1$. Note that rotations act the same in booth spinors as opposed for boost. The Lorentz transformation acts as:

$$\psi'_\alpha = (e^{\frac{1}{2}\omega_{mn}\sigma^{mn}})_\alpha{}^\beta \psi_\beta \tag{A.21}$$

$$\bar{\psi}'^{\dot{\alpha}} = (e^{\frac{1}{2}\omega_{mn}\bar{\sigma}^{mn}})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}} \tag{A.22}$$

Now let's make some checks and see that with these Lorentz transformation we get the right answers. We know what a rotation does on a vector P^m. Then if we use (A.3) and multiply by $\bar{\sigma}^m$ and take the trace using (A.15a) we get:

$$P'^m = -\frac{1}{2}\,\mathrm{Tr}[\bar{\sigma}^m \mathbf{M}\sigma^n\mathbf{M}^\dagger]P_n \tag{A.23}$$

If we choose a rotation on z-axis:

$$\mathbf{M} = e^{\omega_{12}\sigma^{12}} = e^{\frac{i}{2}\theta\sigma^3} = I\cos(\theta/2) + i\sigma^3\sin(\theta/2)$$

where we used the fact $\omega_{12} = -\omega_{21}$ that kills the $1/2$ in (A.21) and then we choose $\omega_{12} = -\theta$. Now that we have all the elements we can express (A.23) as:

$$P'^m = P^m\cos^2(\theta/2) - \frac{1}{2}P_n\sin^2(\theta/2)\mathrm{Tr}[\bar{\sigma}^m\sigma^3\sigma^n\sigma^3] - \frac{i}{2}\sin(\theta/2)\cos(\theta/2)P_n\mathrm{Tr}[\bar{\sigma}^m[\sigma^3,\sigma^n]] \tag{A.24}$$

with this expression and plus some sigma Trace identities and trigonometric one find:

$$P'^0 = P^0 \quad ; \quad \vec{P}' = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{P} \tag{A.25}$$

That's our rotation matrix, it worked! Let me say the trivial fact that if $\theta = 2\pi$ then $\vec{P}' = \vec{P}$. Now we do the same trivial statement for a spinor, under a full rotation $\psi' = e^{\frac{i}{2}2\pi\sigma^3}\psi = -\psi$, that's another way to see that it's ψ has spin 1/2.

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