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DISSERTAÇÃO DE MESTRADO IFT–D.008/15

Electric-Magnetic duality in $N = 2$ supersymmetric gauge theory

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Setembro de 2015

Acknowledgements

First, I want to thank to my supervisor Nathan Berkovits for giving me the opportunity to study here at IFT under his guidance and patience along these two years. This thesis could not be written without his help. To my family and its unconditional support despite of the distance. To my friends at IFT for the many experiences we shared, most of them enjoyable, along the academical and non-academical activities. Specially to: Segundo, Max, Ypanaque, Luan, Fernando, Prieslei, Adriana, Dennis.

Finally, I want to thank to CAPES for the financial support.

Resumo

Nesta dissertação apresentamos uma descrição da dualidade elétrica-magnética e seus aspectos clássicos e quânticos. Nosso análise se inicia com os monopolos magnéticos sugeridos por Dirac em 1931[1] e vai até o trabalho do Seiberg e Witten em 1994 [27]. Na descrição clássica, precisamos introduzir os monopolos magnéticos a fim de obter a dualidade elétrica-magnética manifesta. Mais tarde, a origem dos monopolos se mais torna mais clara quando começamos com uma teoria de Yang-Mills. Os aspectos clássicos da teoria foram explicados pela conjetura de Montonen e Olive 1977 [7]. Explorando os aspectos quânticos da teoria, notamos a importância de introduzir supersimetria, principalmente supersimetria estendida, onde tiramos vantagem da propiedade de holomorficidade, a qual nos leva aos teoremas não renormalizáveis, onde o cálculo é mais simples. Focamos na teoria de gauge supersimétrica $N = 2$ $SU(2)$. A teoria é completamente resolvível para baixas energias. A maior parte do conteúdo deste trabalho é baseada nas várias revisões da dualidade de Seiberg-Witten [30],[31],[32].

Palavras Chaves: Monopolos, campos de Yang-Mills, supersimetria, dualidade.

Áreas do conhecimento: Supersimetria e supercordas.

Abstract

In this dissertation we present a description of the electric-magnetic duality and their classical and quantum aspects. Our analysis starts from the suggested magnetic monopoles by Dirac in 1931 [1] and goes until the work of Seiberg and Witten in 1994 [27]. In the classical description, we need to introduce the magnetic monopoles in order to make manifest the electric-magnetic duality. Later, the origin of monopoles becomes clear when we start from a Yang-Mills theory. The classical aspects of the E-M duality are covered in the Montonen-Olive conjecture 1977 [7]. Working on the quantum aspects of the theory, we note the importance of introducing supersymmetry. Specially for extended supersymmetry, where we take advantage of the holomorphicity property, which leads us to the non-renormalizable theorems, where the computation is easier. We focus on the $N = 2$ $SU(2)$ supersymmetric gauge theories. It turns out that the theory is fully solvable at the low energies regime[27]. Most of this work is based on reviews about the Seiberg and Witten duality [30],[31],[32].

Contents

Chapter 1

Introduction

The electric-magnetic duality (EM duality) is a symmetry in electromagnetism whose transformation consists in the exchange of the electric field $\vec{E}$ and the magnetic field $\vec{B}$ in such a way that the theory remains invariant. The simplest case of EM duality appears in the free EM where its Maxwell equations make the duality manifest since the equations do not change after exchanging the fields $\vec{E}$ and $\vec{B}$.

This duality is not studied in more detail in undergraduate courses because in that case one is interested in describing the electromagnetic interactions when we introduce sources (electric charge and electric current), and these insertions break the symmetry in description. So we will introduce the concepts of magnetic charge and magnetic current in such a way that the theory preserves the initial symmetry (duality). Magnetic monopoles (let us say, they are objects with magnetic charge, later we will be more precisely) have not been detected experimentally yet. However, we will keep the duality and we will suppose the existence of those magnetic monopoles. We will see which theoretical advantages we can obtain from this.

We attempt to extend the EM duality to a theory with interactions and with this come new difficulties. These issues are cured if we consider a larger gauge group than the Abelian gauge group (the electromagnetism has the Abelian gauge group as internal symmetry). We will see that the standard electromagnetism is restored by using the spontaneous symmetry breaking and the remaining gauge group is an Abelian group.

In addition we will find new solutions which satisfy the equations of motion. The new extended objects (solutions of the equations of motion different than the known particles of the theory) in a four dimensional non-abelian gauge theory were found by 't Hooft [5] and Polyakov (1974) [6]. Those are solutions with magnetic charge (magnetic monopole). Also Julia and Zee (1975) [12] found a solution with both magnetic and electric charge (dyon).

The 't Hooft-Polyakov monopoles corresponds to solutions for the Georgi-Glashow model which are not fundamental particles as the photon, the Higgs scalar field and W-bosons. Montonen and Olive in 1977 [7] suggested the idea of duality as a equivalence of two lagrangians. The original theory with its fundamental particles described by fields in the Georgi-Glashow lagrangian, more precisely the massive gauge bosons $W^{\pm}$ and in the dual theory the fundamental particles are replaced by the monopole which are solitons in the original theory.

The equivalence between the lagrangians can be understood as follows, when a unique lagrangian describe both theories, the original one theory and its dual one, with their coupling constants related by a duality transformation. Montonen and Olive gave a number of arguments to support their ideas, as for instance, the masses of all particles were given by the same duality invariant. However, this is not the case of the Georgi-Glashow model. So our original system present some problems which need to be fixed to obtain a correct electric-magnetic duality description.

* There was no evidence to keep the form of the classical mass formula when we start a quantum treatment.

* How to obtain solitonic solutions of spin 1, because the fundamental particles of the original theory, gauge bosons W, have spin 1.

In order to fix the problems that appears in the formulation of the Montonen-Olive conjecture, we add supersymmetry. in this description, the Bogomonlyi bound and the BPS states appear naturally from the algebra which describes supersymmetry and not as "hidden" relations in a classical field theory. Working in the extended supersymmetry, $N = 2$ super Yang-Mills theory we are able to solve the first problem. i.e. we obtain a electric-magnetic duality description at quantum level. However, turns out that the magnetic monopole, the dual solution of the massive gauge bosons, does not have spin 1 then the Montonen-Olive conditions for electric-magnetic duality is not fully satisfied for $N = 2$ SYM theory.

Osborn in 1977 [13] considered $N = 4$ SYM theory where the BPS mass formula does not change by quantum corrections and this time the multiplet which contains the monopole solution has spin 1 as was expected in analogy to the W-bosons. Supersymmetric theories that have potentials with "flat directions" are characterized of manifolds of inequivalent vacuum and such as degeneracies sometimes are modified by quantum corrections.

The present thesis is divided in two big parts, roughly speaking. First, we give a classical description of the electric-magnetic duality. Later, in the second part we introduce supersymmetry to give a semiclassical and quantum treatment of the electric-magnetic duality.

More explicitly the outline of the thesis is: In the first part, we start with section 2, where we give an analysis of the electric-magnetic duality present in the free Maxwell's electromagnetism. Later we consider the charged matter and it is necessary to introduce monopoles to preserve the initial duality also we give a description of the Dirac charge quantization [1]. After that we analyze the Derrick's theorem to see the possibility to obtain a solitonic solution from a classical field theory. This is useful for the section 3, where we study the Georgi-Glashow model, a gauge theory with spontaneous symmetry breaking. The 't Hooft-Polyakov monopole and the Julia-Zee dyon arise as solitonic solutions for the Georgi-Glashow model. And we give a topological description for monopoles made for Corrigan. Later in section 4, we introduce the concept of Bogomolnyi bound and the BPS state. In section 5, we discuss the existence of Instantons as solutions for the Yang-Mills theory in an euclidean space and give a description of the θ-parameter for gauge theories. At the end of the first part, in section 6, we give a general description of the Montonen-Olive conjecture.

For the second part, we start with section 7, based on the appendix where we give a description of the supersymmetry algebra and its representations and the unextended supersymmetry $N = 1$, we start with a description for the $N = 2$ supersymmetric lagrangian in the $N = 1$ superspace. Later, we analyze the central charges that appears in the theory and we calculate them from the lagrangian. We introduce the $N = 2$ superspace to express the theory in the $N = 2$ language. We discuss the Global symmetries in $N = 2$ theories and the R-symmetry breaking. In section 8 we study the Low-energy effective action for $N = 2$ gauge theories. the first subsection we give a semiclassical description of the moduli space in $SU(2)$. Later, we study the metric for the moduli space and the Physical interpretation of the singularities which appears in the moduli space. Finally, we discuss the duality which appears in supersymmetric theories.

From sections 2 to 6 we mainly used the review by Goddard-Olive (1978) [34], "Magnetic monopoles in gauge field theories" and for the rest of the thesis we have used reviews on Seiberg-Witten duality as Di Vecchia "Duality in N=2,4 supersymmetric gauge theories" [31] as well as the review by Luis Alvarez Gaumé "Introduction to S-duality in N=2 supersymmetric gauge theory" [30]. Other historical papers have been used as well and there are listed in references.

Chapter 2

Classical Electric-magnetic duality

The duality symmetry is manifest directly in the Maxwell equations without sources. We can express these equations in its covariant form:

$$\partial_\mu F^{\mu\nu} = 0, \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{2.0.0.1}$$

where $F^{\mu\nu}$ is the strength field tensor and $\tilde{F}^{\mu\nu}$ is the dual of $F^{\mu\nu}$. It is $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

Immediately we note that the same equation holds for both $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$. The transformation $F^{\mu\nu} \to \tilde{F}^{\mu\nu}$ is a symmetry of the theory. As $F^{\mu\nu}$ is constructed in terms of the components of the fields $\vec{E}$ and $\vec{B}$. So the duality transformations for the fields are given by:

$$\mathbf{E} \to \mathbf{B}, \qquad \mathbf{B} \to \textbf{-E}. \tag{2.0.0.2}$$

This discrete transformation can be expressed in a matrix transformation:

$$\begin{pmatrix} \vec{E}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}. \tag{2.0.0.3}$$

We can generalize the result and say that this discrete transformation (2.0.0.3) can be written as a continuous transformation.

$$\begin{pmatrix} \vec{E}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}. \tag{2.0.0.4}$$

We can obtain (2.0.0.2) from (2.0.0.4) for $\phi = \pi/2$. The continuous transformation in this case corresponds to rotations of the fields $\vec{E}$ and $\vec{B}$ on the plane. The symmetry described for rotations in a plane correspond to the $SO(2)$ symmetry

[1]. So we can describe the pair of fields $\vec{E}$ and $\vec{B}$, $(\vec{E}, \vec{B})$ as vectors of $SO(2)$ or elements of $U(1)$.

However, we are interested in describing an EM theory with sources. So we introduce them to the free EM theory. In standard electromagnetism the interactions appear in terms of the electric charge and current, as we mentioned before the duality is broken once we include these sources.

In order to preserve the duality we add magnetic sources (magnetic charge density and magnetic current). This allows us to keep the theory invariant under dual transformation.

Which can be translated in a covariant way in terms of the field strength tensor to

$$\partial_\mu F^{\mu\nu} = -j_e^\nu, \tag{2.0.0.5}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = -j_m^\nu, \tag{2.0.0.6}$$

where j_e^μ and j_m^μ are the electric and magnetic four-currents.

We can make the duality transformation explicity by writing it in component fields $\vec{j}_e$ and $\vec{j}_m$ as follows:

$$\nabla \cdot (\vec{E} + i\vec{B}) = \rho_e + \imath\rho_m, \tag{2.0.0.7}$$

$$\nabla \times (\vec{E} + i\vec{B}) + i\frac{\partial}{\partial t}(\vec{E} + i\vec{B}) + i(\vec{j}_e + \imath\vec{j}_m), \tag{2.0.0.8}$$

We denote electric and magnetic current to $\vec{j}_e$ and $\vec{j}_m$ and electric and magnetic charge densities to ρ_e and ρ_m. We can write the Maxwell equations in a more suitable way,

Indeed, the electric and magnetic currents and charge densities have the same $U(1)$ transformation. It means,

$$\rho_e + \imath\rho_m \to e^{i\phi}(\rho_e + \imath\rho_m), \tag{2.0.0.9}$$

$$j_e + \imath j_m \to e^{i\phi}(j_e + \imath j_m), \tag{2.0.0.10}$$

where ϕ is a phase that denotes the $U(1)$ transformation.

For point-like particles the charge are the integral over the space of its charge densities, so it behave in the same way that the charge density transforms under duality transformation. Here q and g denote the electric and magnetic charge respectively.

$$q + \imath g \to e^{\imath\phi}(q + \imath g) \tag{2.0.0.11}$$

[1]It is a mathematical fact in group theory that there is an isomorphism between the groups $SO(2)$ and $U(1)$ ($SO(2) \cong U(1)$).

There is no experimental evidence of the existence of magnetic charges, but we have introduce them to the standard electromagnetic theory to preserve the duality.

So far, what we have done is to add an extra source (magnetic source). From this assumption the requirement of duality preservation is solved for a theory with interactions.

In the standard electromagnetism, the divergence of $\vec{B}$ is zero everywhere, but now we have added the magnetic four-current (j^{μ}_{m}) and the differential Gauss law for $\vec{B}$ is not the same, $\vec{\nabla} \cdot \vec{B} \neq 0$. This leads us to do a modification in the theory. We are going to describe the EM theory in terms of the electromagnetic fields $A_{\mu} = (A_0, \vec{A})$ instead of the electric and magnetic fields. A_0 corresponds to the scalar potential and $\vec{A}$ to the vector potential. It is known that the magnetic field is written as the rotational of the vector potential $\vec{A}$ to satisfy the vanishing divergence of $\vec{B}$,

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0. \tag{2.0.0.12}$$

However, now the divergence of $\vec{B}$ is not zero for all point in space. Those points, where $\vec{A}$ does not make the divergence of $\vec{B}$ be equals to zero, are called "singular points" of $\vec{A}$. Because of it, we will say that $\vec{A}$ is not well defined everywhere.

So when we make the extension to the electromagnetic theory introducing magnetic charges we find that A_{μ} has singularities.

Electric and magnetic currents generated by point like sources can be described by the following relations,

$$j^{\mu}_{e} = \sum q_a \int dx^{\mu} \delta^4(x - x_a), \tag{2.0.0.13}$$

$$j^{\mu}_{m} = \sum_{a} g_a \int dx^{\mu} \delta^4(x - x_a), \tag{2.0.0.14}$$

where a correspond to the index for each source localized at x_a.

Also for these point-like sources with mass m we can write the new Lorentz force which is an extension of the usual Lorentz force equation.

$$m \frac{d^2 x^{\mu}}{d\tau^2} = (qF^{\mu\nu} + g\tilde{F}^{\mu\nu}) \frac{dx_{\nu}}{d\tau}, \tag{2.0.0.15}$$

The equation of motion specifies completely the dynamics of a classical system of electric and magnetic charges (q, g) interacting with electric and magnetic fields $(\vec{E}, \vec{B})$ in such a way that it preserves the duality transformation.

Classical electromagnetic theory with magnetic charges seems to be complete and well defined. The modification of Maxwell's theory made by Dirac looks quite trivial at classical level, but this modification leads us to highly non-trivial consequences in the quantum theory.

A practical way to visualize the problems that appears at the quantum level is by analyzing that the vector potential which is indispensable for the quantum formulation of the theory. Since the canonical variables for the electromagnetic field are not the components of $F^{\mu\nu}$ (E^i and B^i) anymore, but the components of the four vector potential. We express it as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. It implies that $\partial_\nu \tilde{F}^{\mu\nu}$ vanish, because the dual symmetry is destroyed when we introduce a magnetic current $\partial_\nu \tilde{F}^{\mu\nu} = -j^\mu_m$.

We have a magnetic monopole that lies in the origin and the divergence of the magnetic field is different to zero. So we can not express the magnetic field as a rotational of a vector potential for every point iin the space, like

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{2.0.0.16}$$

However, we can introduce a vector potential for the different regions of the space where it is well defined. $\vec{A}_N$ for the northern hemisphere and $\vec{A}_S$ for the southern hemisphere. The relation (2.0.0.16) is true for $\vec{A} = \vec{A}_N$ in the northern hemisphere and for $\vec{A} = \vec{A}_S$ in the southern hemisphere. These two solutions must match up for a gauge transformation from $\vec{A}_N$ to $\vec{A}_S$.

$$\vec{A}_N = \vec{A}_S + \vec{\nabla}\chi(\theta), \tag{2.0.0.17}$$

where θ is the angle along the equator.

In quantum mechanics, the particles with electric charge are described by wave functions, as we are differentiating the regions north and south. We label the wave functions as $\Psi_N(x)$ and $\Psi_S(x)$ respectively and in the equator these two functions should equal up to a gauge transformation.

$$\Psi_N(x) = e^{-iq\chi(\theta)/\hbar}\Psi_S(x), \tag{2.0.0.18}$$

where q is the electric charge of the particle. Since the wave function Φ has to take a unique value along the equator, it requires that the the parameter of the gauge transformation satisfy.

$$\chi(\theta + 2\pi) = \chi(\theta) + \frac{2\pi\hbar}{q}n, \tag{2.0.0.19}$$

we replace the 2π shift of χ on the gauge transformation on (2.0.0.18), we have

$$\Psi_{\theta+2\pi}(x) = e^{-iq\chi/\hbar}e^{-2\pi i n}\Psi(\theta) \tag{2.0.0.20}$$

$\Psi(\theta)$ is a periodic function.

Let us calculate the variation of the curl integral of the vector potential on the two hemispheres by using eq (2.4.0.24) we obtain.

$$\int_{\text{equator}} d\mathbf{l}\cdot\mathbf{A}_N - \int_{\text{equator}} d\mathbf{l}\cdot\mathbf{A}_S = \int_0^{2\pi} d\theta\frac{d\chi(\theta)}{d\theta} = \chi(2\pi)-\chi(0) = \frac{2\pi\hbar n}{q} \quad (2.0.0.21)$$

Also from the Stokes' theorem applied for the curl integral of $\vec{A}$, we have

$$\int_N d\mathbf{S}.\mathbf{B} + \int_S d\mathbf{S}.\mathbf{B} = \int_{\text{sphere}} d\mathbf{S}.\mathbf{B} = g \quad (2.0.0.22)$$

For the last step we used the fact that there exist a magnetic monopole with magnetic charge g. From these two equations we obtain the Dirac quantization condition. [1]

$$g = \frac{2\pi\hbar n}{q} \quad (2.0.0.23)$$

2.1 A classical description of charge quantization

Let us consider the simplest case of an electric charge q and mass m moving in a magnetic field $\vec{B}$. It is for a radial field which is generated by a magnetic monopole.

$$\vec{B} = \frac{g}{4\pi r^3}\vec{r}. \quad (2.1.0.24)$$

And the equation of motion for the particle is:

$$m\ddot{\vec{r}} = q\dot{\vec{r}} \times \vec{B}. \quad (2.1.0.25)$$

By replacing the magnetic field of the monopole we obtain

$$m\ddot{\vec{r}} = \frac{qg}{4\pi r^3}\dot{\vec{r}} \times \vec{r}. \quad (2.1.0.26)$$

From the eq(2.1.0.26), we can build the conserved angular momentum. We can argue that the orbital momentum is not conserved since the force which rules the equation of motion is not a central force. However, we can take the rate of change of the orbital momentum.

As we know; $\vec{L} = \vec{r} \times \vec{p} = m\vec{r} \times \dot{\vec{r}}$, from the equation of motion the rate of change of $\vec{L}$,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\dot{\vec{r}}) = \vec{r} \times m\ddot{\vec{r}} \quad (2.1.0.27)$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{qg}{4\pi}\frac{\vec{r}}{r^3} \times \vec{r} \Rightarrow \frac{d}{dt}(\vec{r} \times m\dot{\vec{r}}) = \frac{d}{dt}(\frac{qg}{4\pi}\frac{\vec{r}}{r}) \quad (2.1.0.28)$$

Where we have used eq(2.4.0.24). the last expression can be reduce to

$$\frac{d}{dt}(\vec{r} \times m\dot{\vec{r}} - \frac{qg}{4\pi}\frac{\vec{r}}{r}) = 0. \tag{2.1.0.29}$$

So the conserved quantity is:

$$\vec{J} = \vec{r} \times m\dot{\vec{r}} - \frac{qg}{4\pi}\frac{\vec{r}}{r}. \tag{2.1.0.30}$$

And $\frac{qg}{4\pi}\frac{\vec{r}}{r}$ is called electromagnetic angular momentum and is denoted by $\vec{J}_{em}$.

Besides of this result we can obtain the electromagnetic angular momentum from the electromagnetic momentum $\vec{p}_{em}$, which corresponds to the integral over the space of the Poynting vector $\vec{S} = \vec{E} \times \vec{B}$. So we have:

$$\vec{J}_{em} = \vec{r} \times \vec{p}_{em} = \vec{r} \times \int d^3x\vec{S} = \int d^3x\vec{r} \times (\vec{E} \times \vec{B}). \tag{2.1.0.31}$$

We can replace the magnetic fields $\vec{B} = \frac{qg}{4\pi}\frac{\vec{r}}{r}$.

$$\vec{J}_{em} = \frac{g}{4\pi}\int d^3x\vec{r} \times \vec{E} \times \frac{\vec{r}}{r^3}. \tag{2.1.0.32}$$

In components of the trivector $\vec{J}_{em}$

$$J^i_{em} = \frac{g}{4\pi}\int d^3xE_j(\delta^{ij} - \frac{x^ix^j}{r^2})\frac{1}{r} \to \frac{g}{4\pi}\int d^3xE^j\partial_j(\hat{x}^i) \tag{2.1.0.33}$$

we obtain two terms:

$$J^i_{em} = \frac{g}{4\pi}\int \hat{x}^i\vec{E} \cdot d\vec{S} - \frac{g}{4\pi}\int d^3x\vec{\nabla} \cdot \vec{E}\hat{x}^i. \tag{2.1.0.34}$$

Considering the integral surface over all the space, $\vec{E}$ goes to zero when $\vec{r} \to \infty$. So the distance between the electric and magnetic charge is neglectable respect to the distance to the boundary S^2. Then the contribution of the first integral is zero and we only have.

$$\vec{J}_{em} = -\frac{qg}{4\pi}\frac{\vec{r}}{r}. \tag{2.1.0.35}$$

So far, the discussion has been in the context of classical mechanics, we have not considered relativistic effect, it means we have neglect radiation effects. In quantum mechanics is expected that the components of $\vec{J}$ satisfy the commutation rules, and its eingenvalues correspond to integer multiplets of $n\hbar/2$.

$$\hat{r} \cdot \vec{J} = -\frac{qg}{4\pi} = \frac{1}{2}n\hbar, \tag{2.1.0.36}$$

this is the Dirac quantization condition. In principle, the criterion of quantization suggest that we have integer multiples of $n\hbar$, integer rather than half-integer multiples of $\hbar$. Because of the derivation of the eq (2.1.0.36) there is no fermions involved. we would expect to find a quantization rule with integer multiples of $\hbar$ besides than half-integer of $\hbar$, $(\frac{1}{2}n\hbar)$.We will see that the first one is the correct quantization.

The Dirac quantization condition tell us if there is a magnetic charge somewhere in the universe, then the electric charge is quantized by units of $q = \frac{2\pi\hbar}{g}$. Also, if there exist some number of purely electric charges q_i and purely magnetic charge g_j. They satisfy the relation.

$$\frac{q_i g_j}{4\pi} = \frac{1}{2} n_{ij}, \tag{2.1.0.37}$$

where n_{ij} corresponds to a matrix of integer numbers.

We can write a fundamental electric magnetic charge q_0 and g_0 respectively.

$$q_i = n_i q_0, \qquad\qquad g_j = n_j g_0. \tag{2.1.0.38}$$

So $q_i g_j = n_i n_j q_0 g_0$ where q_0 and g_0 satisfy $q_0 g_0 = 2\pi n_0 \hbar$, where n_0 depends of the theory in consideration.

Let us consider now particles which have both electric and magnetic charges. i.e. they are called dyons (q_1, g_1) and (q_2, g_2) we can see $\vec{E} = \vec{E}_1 + \vec{E}_2$ electric field and $\vec{B} = \vec{B}_1 + \vec{B}_2$. Where $\vec{2}_{em}$ is given by:

$$\vec{J}_{em} = -\frac{1}{4\pi}(q_1 g_2 - q_2 g_1)\hat{r}. \tag{2.1.0.39}$$

We can generalized the Dirac quantization condition to:

$$\frac{q_1 g_2 - q_2 g_1}{4\pi} = \frac{1}{2} n_{ij}. \tag{2.1.0.40}$$

This is called the Dirac-Schwinger-Zwansinger (DSZ) quantization condition [2] and [3]. And it is invariant under $U(1)$ transformation.

$$q + ig \to e^{i\phi}(q + ig), \tag{2.1.0.41}$$

which is the symmetry of eq.(2.0.0.5) eq.(2.0.0.6).The Dirac quantization tell us that electric and magnetic charge are inversely proportional. So in a theory where the electric charge q_0 is small, the magnetic charge g_0 is big. However, this condition does not give any restriction to the mass of the monopole. In the next sections we shall develop a description for gauge theories where monopoles appear naturally and a relation for their mass is determined.

2.2 Derrick theorem

For some models in (d+1)-dimensional space-time with $d > 1$ is possible to show that there are non-trivial static solutions of the equations of motion by applying scale arguments (Derrick 1964) [4].

Let us first consider a field theory with n-scalar fields $\varphi^a, a = 1 \cdots \cdot n$ in a (d+1)-dimensional space-time. The most general Lagrangian is:

$$\mathcal{L} = \frac{1}{2} F_{ab}(\varphi) \partial^\mu \varphi^a \partial_\mu \varphi^b - V(\varphi), \tag{2.2.0.42}$$

where $F_{ab}(\varphi)$ and $V(\varphi)$ are functions of the scalar field φ^a.

If we suppose that $\varphi_c^a(x)$ are the static solutions of the classical field equations with finite energy. This solution is obtained by taken the extremal of the energy functional.

$$E[\varphi] = \int d^d x [\frac{1}{2} F_{ab}(\varphi) \partial_1 \varphi^a \partial_i \varphi^b + V(\varphi)] \tag{2.2.0.43}$$

We suppose that $F_{ab}(\varphi)$ is a quadratic form defined possitive, it means that all the eingenvalues of the matrix F_{ab} are defined positive, too.

Then:

$$F_{ab} \partial_1 \varphi^a \partial_i \varphi^b \geq 0 \tag{2.2.0.44}$$

This relation is a equality when the value φ^a does not depend on x. Also we will assume that $V(\varphi)$ is bounded below by the zero-point energy.

$$V(\varphi^{(v)}) = 0, \tag{2.2.0.45}$$

here $\varphi^{(v)}$ is the scalar field in the classical vacuum in such a way the potential evaluated in that value is the absolute vacuum.

If $\varphi_c^a(x)$ is the static solution for the equations of motion of finite energy. Then the energy functional $E[\varphi]$ must be extremal for $\varphi^a = \varphi_c^a$ with respect to any energy variation of the fields which vanish at space infinite.

Let us consider the following field configuration as a result of a scale transformation.

$$\varphi_\lambda(x) = \varphi_c(\lambda x), \tag{2.2.0.46}$$

and the variation

$$\varphi_\lambda - \varphi_c(x) = \varphi_c(\lambda x) - \varphi_c(x). \tag{2.2.0.47}$$

It is a small variation and it vanishes at spatial infinite, for large x there is no difference with the scale transformation in the coordinates. $\varphi_c(x)$ goes to a constant value at $\mid x \mid \rightarrow \infty$, (otherwise the energy diverge). It means,

$$E[\varphi(x)] = E[\varphi_\lambda(x)], \tag{2.2.0.48}$$

energy is invariant under scale transformations. As the energy functional is extremal at $\lambda = 1$, E only depends on λ.

$$\frac{dE}{d\lambda} |_{\lambda=1}= 0 \tag{2.2.0.49}$$

Now we will calculate the energy functional for the field configuration (2.2.0.46)

$$E(\lambda) = \int d^dx[\frac{1}{2}F_{ab}(\varphi_c(\lambda x))\frac{\partial}{\partial x^i}\varphi^a(\lambda x)\frac{\partial}{\partial x^i}\varphi^b(\lambda x) + V(\varphi(\lambda x))]. \tag{2.2.0.50}$$

We define $y = \lambda x$ so $d^dx = \lambda^{-d}dy$. Under this change of variable.

$$\lambda\frac{\partial}{\partial y^i}\varphi^a_c(y) = \frac{\partial}{\partial x^i}\varphi^a_c(\lambda x), \qquad V(\varphi(\lambda x)) = V(\varphi(y)). \tag{2.2.0.51}$$

The energy functional can be expressed as

$$E(\lambda) = \lambda^{2-d}\Gamma + \lambda^{-d}\Pi, \tag{2.2.0.52}$$

where

$$\Gamma = \int d^dx\frac{1}{2}F_{ab}(\varphi(y))\partial_i\varphi^a(y)\partial_i\varphi^b(y) \tag{2.2.0.53}$$

and

$$\Pi = \int d^dxV(\varphi(y)). \tag{2.2.0.54}$$

The extremal of the energy functional using the condition (2.2.0.49) on (2.2.0.52)

$$\frac{dE}{d\lambda} |_{\lambda=1}= 0 = (2-d)\Gamma - d\Pi \tag{2.2.0.55}$$

This condition leads to strong constraints on the existence of classical solutions in scalar theories.

1. $d > 2$: condition (2.2.0.52) satisfy only if

$$\Gamma = \Pi = 0. \tag{2.2.0.56}$$

It means that $\partial_i\varphi^c_a = 0$ and φ^c_a is the absolute minimum of the potential $V(\varphi)$. It is the solution of the classical vacuum.

2. $d = 2$: condition (2.2.0.52) is satisfied for

$$\Pi = 0. \tag{2.2.0.57}$$

If the potential is non-trivial, this condition means that the only static solution is the classical vacuum. The scalar models with two spatial dimensions

$$V(\varphi) = 0 \text{ for all } \varphi, \tag{2.2.0.58}$$

it means that there is no potential term in the lagrangian. For this case, the kinetic term has a complicated form.

3. $d = 1$: condition (2.2.0.52) gives the Virial theorem

$$\Gamma = \Pi. \tag{2.2.0.59}$$

It does not impose constraints on the choice iof the model.

Let us consider now a gauge theory. We will restrict ourselves to the case of gauge group representation. (A_μ is the gauge field and φ the scalar field. which transforms under reducible unitary representation T of the group G) The lagrangian density of the theory has the following form.

$$\mathcal{L} = \frac{1}{2g^2}TrF_{\mu\nu}^2 + (D^\mu\varphi)^\dagger(D_\mu\varphi) - V(\varphi), \tag{2.2.0.60}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{2.2.0.61}$$

$$D_\mu\varphi = \partial_\mu\varphi + T(A_\mu)\varphi. \tag{2.2.0.62}$$

Similarly as we have studied for a theory with only scalars fields. We have the energy functional.

$$E[A_\mu, \varphi] = \int d^dx[-\frac{1}{2g^2}TrF_{ij}F_{ij} + (D_i\varphi)^\dagger(D_i\varphi) + V(\varphi)], \tag{2.2.0.63}$$

where each of the three terms of E is possitive defined.

From the scale transformation of the fields

$$\begin{array}{rcl} \varphi_\lambda(x), & = & \varphi_c(x), \\ A_{\lambda\mu}(x) & = & \lambda A_{c\mu}(x). \end{array} \tag{2.2.0.64}$$

The gauge field transform similarly to the partial derivative under scale transformation.

Now the covariant derivative transforms

$$D_x^{(\lambda)}\varphi_\lambda(x) = (\frac{\partial}{\partial x^i} + TA_\lambda(x))\varphi_\lambda(x) = \lambda D_y^{(\lambda)}\varphi(y), \tag{2.2.0.65}$$

where:

$$D_y^{(\lambda)}\varphi_c(y) = (\frac{\partial}{\partial y^i} + tA_c(y))\varphi_c(y) \tag{2.2.0.66}$$

It is the covariant derivative respect to y for the initial field configuration.

Also we have:

$$F_{ij}^{(\lambda)} = \frac{\partial}{\partial x^i}A_\lambda^j(x) - \frac{\partial}{\partial x^j}A_\lambda^i(x) + [A_\lambda^i(x), A_\lambda^j(x)] = \lambda^2 F_{ij}^c(y). \tag{2.2.0.67}$$

where $F_{ij}^{c}(x)$ is the field strength tensor for the spatial coordinates in the initial field configuration. Replacing these expression in the energy functional (2.2.0.63)

$$E(\lambda) = \lambda^{4-d}G + \lambda^{2-d}\Gamma + \lambda^{-d}\Pi, \tag{2.2.0.68}$$

where:

$$\begin{aligned} G &= \int d^d y(-\frac{1}{2g^2}TrF_{ij}^{(c)}(y)F_{ij}^{(c)}(y)), \\ \Gamma &= \int d^d y(D_y\varphi_c)^\dagger(D_y\varphi_c), \\ \Pi &= \int d^d yV(\varphi_c). \end{aligned} \tag{2.2.0.69}$$

The extremality condition for the energy functional $E(\lambda)$ on $\lambda = 1$ we obtain

$$(4-d)G + (2-d)\Gamma - d\Pi = 0 \tag{2.2.0.70}$$

We have strong constraints for the for the theory with gauge fields and scalars fields.

The condition 2.2.0.70 is weaker than the condition (2.2.0.52), it does not prohibit the existence of non-trivial classical solutions for $d = 2$ and $d = 3$. The solitonic solutions for gauge theories in two and three dimensions have been well studied by Skyrme in 1962. The condition 2.2.0.70 prohibits the existence of non-trivial static classical solutions 1. in theories with scalar fields for $d \geq 4$. 2. in purely gauge theories (without scalar fields) for $d \neq 4$. A pure gauge theory of physical interest in (3+1)-dimensional space-time, there is no solitons.

2.3 't Hooft-Polyakov monopole and Julia-Zee dyon

We will make an extension of the abelian gauge group $U(1)$ from the standard EM to a non-abelian gauge group $SU(N)$ [2] and couple it to scalar fields. We will restore the standard EM by using the Higgs mechanism. We will break the gauge group $SU(N)$ spontaneously to a $U(1)$ abelian group.

We will find new solutions to the equations of motion as a result of considering a larger gauge group than the abelian one. These solution is in terms of the scalar field ϕ^a and the gauge field W_μ^a. These new solutions are called Solitons and they are time independent, stable and with non-zero finite energy. As a result, in

[2] We will work on the simplest case of the non-abelian gauge group $SU(2)$. Also for $SO(N)$ the simplest case will be $SO(3)$.

our description these solutions have non-zero magnetic charge and they are called magnetic monopole and dyons.

On the previous section we described the Derrick theorem and the strong constraints it impose to the field theories. There, we analyzed two cases: a field theory with only scalar fields and a gauge theory with scalar fields and gauge fields. For instance, the Sine-Gordon model in 1+1-dimensional space-time (Coleman 1975) a classical field theory build up with only scalar fields. There is possible to find solitonic solutions from these scalar fields. Derrick showed that it is not possible to construct simple solitonic solutions only with scalar fields in $d > 1$ spatial dimensions. The case of our interest is (3+1)-dimensional space-time. If we want to find non-trivial solutions from the equations of motion we need to add gauge fields. In the following subsections we will analyze and derive these solutions as well as their symmetries and their asymptotic behavior.

2.3.1 The Georgi-Glashow model

We will study the gauge theory with the simplest non-abelian gauge group with a $SO(3)$ gauge group. This internal symmetry will be spontaneously broken to the abelian gauge group $SO(2)$ by using the Higgs mechanism. In this way standard electric-magnetic duality symmetry is restored.

$$\mathcal{L} = -\frac{1}{4}G_a^{\mu\nu}G_{a\mu\nu} + \frac{1}{2}D^\mu\phi^a D_\mu\phi^a - V(\phi), \tag{2.3.1.1}$$

where:

$$G_a^{\mu\nu} = \partial^\mu W_a^\nu - \partial^\nu W_a^\mu - e\epsilon_{abc}W^{b\mu}W^{c\nu} \tag{2.3.1.2}$$

is the strength field tensor and W_a^μ is a gauge field. $W_\mu = W_\mu^a T^a$, where T_a are the generators of the gauge group. They satisfy the commutator conditions of the elements of a Lie algebra.

$$[T_a, T_b] = if_{abc}T_c. \tag{2.3.1.3}$$

For the algebra $so(3)$, the structure constants f_{abc} are totally antisymmetric tensor, ϵ_{abc}.

Also the covariant derivative of the scalar field is:

$$D_\mu\phi^a = \partial_\mu\phi^a - e\epsilon_{abc}W_\mu^b\phi^c \tag{2.3.1.4}$$

and

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - a^2)^2, \tag{2.3.1.5}$$

where the scalar 3-vector field $\phi^a = (\phi^1, \phi^2, \phi^3)$

The potential $V(\phi)$ is a Higgs-like potential, it means, we introduce it to obtain spontaneous symmetry breaking by Higgs mechanism. The scalar fields in the vacuum configuration ϕ_{vac} form a smooth surface in the field configuration space, a manifold, $\mathcal{M}_0 = \{\phi : \phi_1^2 + \phi_2^2 + \phi_3^2\}$. We also can note that ϕ^a, $G_a^{\mu\nu}$ and $(D^\mu\phi)_a$ transform like vectors respect to local $SO(3)$ rotations. The index a denotes the components of a vector in $SO(3)$.

Let us find the equation of motion for the fields ϕ^a and W_a^μ. Using the Euler-Lagrange equation.

$$\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} = 0 \tag{2.3.1.6}$$

$$\frac{\partial \mathcal{L}}{\partial W_\nu^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu W_\nu^a)} = 0 \tag{2.3.1.7}$$

Therefore, we have

$$(D_\nu G^{\mu\nu})_a = -e\epsilon_{abc}\phi^b(D^\mu\phi)^c \tag{2.3.1.8}$$

and

$$(D^\mu D_\mu \phi)_a = -\lambda\phi_a(\phi^2 - a^2) \tag{2.3.1.9}$$

Also we can write the Bianchi identity in its covariant way,

$$D_\nu \tilde{G}_a^{\mu\nu} = 0, \tag{2.3.1.10}$$

where $\tilde{G}_a^{\mu\nu}$ is the dual strength field tensor.

First, we are going to find the vacuum configuration for the present theory. Let us denote $G_a^{0i} = -\mathcal{E}_a^i$ and $G_a^{ij} = -\epsilon_{ijk}\mathcal{B}_a^k$ as the electric-like and magnetic-like fields for the non-Abelian gauge fields in analogy to the Abelian case.

Now we analyze the energy of the solutions in this system by integrating over the space the energy density, which is the 00-component of the energy momentum tensor $\Theta_{\mu\nu}$.

The energy-momentum tensor is given by

$$\Theta_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_a)}\partial_\nu\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial^\mu W_a^\sigma)}\partial_\nu W_a^\sigma - \eta_{\mu\nu}\mathcal{L}. \tag{2.3.1.11}$$

The 00-component of the energy momentum tensor corresponds to the energy density,

$$\Theta_{00} = \frac{1}{2}(D_0\phi^a)^2 + \frac{1}{2}(D^i\phi^a)^2 + \frac{1}{2}(G_a^{\mu\nu})^2 + \frac{1}{2}(G_a^{ij})^2 + V(\phi). \tag{2.3.1.12}$$

All terms in Θ_{00} are quadratic unless the potential $V(\phi)$, but we can define $V(\phi)$ possitive. So $\Theta_{00} \geqslant 0$ and it only vanishes for

$$G_a^{\mu\nu} = 0, \qquad (D^\mu\phi)_a = 0, \qquad V(\phi) = 0. \tag{2.3.1.13}$$

The equation (2.3.1.13) is called vacuum configuration for $\Theta_{00} = 0$. A particular case is obtained when we set the direction of the scalar field vector in the vacuum, $\phi_a = a\delta_{a3}$, where we have projected the vacuum to the third direction of the vector ϕ_a. Also, we set the gauge field as $W_a^\mu = 0$.

The choice for the scalar field with non-zero value in the third component direction satisfy the condition for the vacuum configuration, $\phi_a = a\delta_{a3}$. Since $|\phi_{vac}^a|^2 = a^2$ satisfy $V(\phi) = 0$. However, as we have taken a preferential direction for ϕ_{vac} then the gauge group $SO(2)$ is broken spontaneously.

It is enough for the scalar field to satisfy $|\phi_a|^2 = a^2$ to belong to the vacuum configuration. This condition leads us to define a 2-sphere, S^2, of radii a. The 2-sphere is the manifold $\mathcal{M}_0$ defined after eq(2.3.1.5) for the vacuum configuration of the scalar fields.

After spontaneous symmetry breaking by the Higgs mechanism, there remains the unbroken part of the initial internal symmetry which keeps the vacuum invariant with its corresponding generator. So we have broken the group $SO(3)$ to $SO(2)$, the abelian group $SO(2)$ is isomorphic to $U(1)$, ($SO(2) \cong U(1)$). Also its generator is $\phi_{vac}^t a T^a/a$ and the gauge boson associated with this generator is $A_\mu = \phi_{vac}^a W_\mu^a/a$.

It is important to understand the concept of the Higgs vacuum. We shall say that the fields in a certain region of the space-time are in the Higgs vacuum when the last two conditions of (2.3.1.13) are satisfied, but not necessarily the first one. Also the condition of finite energy enforce the equations asymptotically at large distances.

$$V(\phi) = 0, \qquad (D^\mu\phi)_a = 0, \tag{2.3.1.14}$$

After symmetry breaking, there is only left the $U(1)$ gauge theory with all the characteristics of the Maxwell's electromagnetic theory, where $U(1)$ gauge group is identified with the gauge group of the electromagnetism. $T^a, a = 1, 2, 3$ are the generators of $SO(3)$ and the generator of $U(1)$ is $\vec{\phi} \cdot \vec{T}/a$ and it is proportional to the charge Q.

We emphasize the searching of solutions with attributes of particle so we will treat the equations purely classical. We can obtain by expanding the scalar field around the vacuum. So some particles get mass after spontaneous symmetry breaking.

From $(D_\mu\phi)^2$

$$(D_\mu\phi)^2 = (\partial_\mu(\phi + a) - eA_\mu(\phi + a))^2 \Rightarrow e^2a^2A_\mu A^\mu = m_W^2 A_\mu A^\mu. \tag{2.3.1.15}$$

The mass of the massive gauge bosons is $m = ea\hbar$.

From the potential term we have:

$$\frac{\lambda}{4}a^2\phi^2 \Rightarrow \frac{m_H^2}{2}\phi^2. \tag{2.3.1.16}$$

The mass of the Higgs particle is $m_H = (2\lambda)^{1/2} a\hbar$.

We can write the field content of the perturbative spectrum: Massive Higgs H, a photon γ and massive gauge bosons $W^{\pm}$. in a table as follows:

	Mass	Spin	Charge
H	$a(2\lambda)^{\frac{1}{2}}\hbar$	0	0
γ	0	$\hbar$	0
$W^{\pm}$	$ae\hbar = aq$	$\hbar$	$\pm q = \pm e\hbar$

Figure 2.1:

2.3.2 Searching for solutions using a simplifying ansatz

So far, we have seen how particles arise as solutions in the Georgi-Glashow model after the non-abelian gauge group is broken. The particle content of the model is the Higgs scalar field, the photon and the massive gauge bosons. However, we are interested in the solitonic solutions with magnetic charges. They are extra solutions to the equations of motion.

There are many types of solitonic solutions for a field theory. In this specific case, the solution for a gauge theory in four dimensions are the magnetic monopoles and the dyons. A solitonic solution has finite energy which is given by the integral over the 3-dimensional volume of the 00-component of the energy-momentum tensor, this is the energy density. In order to guarantee the convergence of the integral of Θ_{00}, we need Θ_{00} goes to zero faster than $1/r^3$, because the integral of Θ_{00} is over all the space and $r \to \infty$.

At first, we can suggest a spherical symmetric solution, but let us analyze the symmetry conditions. The symmetries of the non-trivial solution form a group $\mathcal{G}_0$, which is a subgroup of $\mathcal{G}$. $\mathcal{G}$ is the symmetry group only for trivial solutions with zero energy. So we will impose the $\mathcal{G}$ is the symmetry group for the solutions with lower non-trivial zero energy.

On the other hand, the time independence of the solution means that we are making a choice in the Lorentz frame, fields are at rest. As a consequence, the equations of motion have a $SO(3)$ rotational symmetry, as we suggested before the solution will be spherically symmetric.

Also the discrete symmetry, parity symmetry for the fields ϕ_a and W_a^μ and for $F^{\mu\nu}$, the field strength tensor in the Higgs vacuum.

Due to the time independence the solution is localized and the translation invariance is broken. This leaves the symmetry product $SO(3) \times SO(3)$. For spatial rotations for r^i and isotopic rotations ϕ_a and W_a^μ. However, the symmetry group product is too big. It has a covariant $SO(3)$ subgroup. They can be either isotopic rotation or spatial rotation. Invariance respect rotation in the real space enforce to ϕ_a to be constant asymptotically and the boundary conditions are satisfy. Invariance respect to isotopic rotations enforces ϕ_a to vanish everywhere and the boundary conditions are not satisfied at all.

The general ansatz after these specifications is:

$$\begin{aligned}
\phi_a(r) &= H(aer)r^a/er^2, \qquad W_a^0(r) = J(aer)r^a/er^2 \\
W_a^i(r) &= -\epsilon_{aij}\frac{r^j}{er^2}[1 - K(aer)]. \qquad (2.3.2.1)
\end{aligned}$$

The solution has as invariant group $\mathcal{G}_0 \cong SO(3) \times \mathbb{Z}_2$. $SO(3)$ corresponds to the diagonal group in the product of the spatial and isotropic rotations and $\mathbb{Z}_2$ corresponds to the group generated by parity symmetry.

Remember a is the vacuum expectation value of the scalar field, e corresponds to the coupling constant i.e. electric charge, $\vec{r}$ is a tri-vector in the three space.

We will consider a particular case where W_0^a, it means, there is no contribution of the electric fields. This particular case will leads us to the magnetic monopole solution.

First, we will replace the ansatz to the energy momentum density eq(2.3.1.11). We obtain:

$$E = \frac{4\pi a}{e}\int_0^\infty \frac{d\xi}{\xi^2}\Big[\xi^2\big(\frac{dK}{d\xi}\big)^2 + \frac{1}{2}\big(\xi\frac{dH}{d\xi} - H\big)^2 + \frac{1}{2}(K^2-1)^2 + K^2H^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2\Big] \tag{2.3.2.2}$$

where $\xi = aer$ is the argument for K and H. We need the energy E be stationary for variations of H and K.

The energy must be invariant under variation of the functions K and H, it is, we are finding the minimum energy for the system. We later will see that this configuration corresponds to the BPS condition. We can also understand these two relations as the equations of motion of the fields in terms of the 't Hooft-Polyakov ansatz [5],[6] eqs(2.3.2.1)

Let us compute which equation of motion K and H satisfy.

$$\frac{\delta E}{\delta H} = 0 \Rightarrow \frac{\partial E}{\partial H} - \partial'\big(\frac{\partial E}{\partial H'}\big) = 0 \tag{2.3.2.3}$$

We only care about the terms which depend explicity on H or H'

$$\begin{aligned}
&\frac{\partial}{\partial H}\Big[\frac{1}{2}\big(\xi\frac{dH}{d\xi} - H\big)^2 K^2H^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2\Big] \\
&-\partial'\Big(\frac{\partial}{\partial H'}\Big[\frac{1}{2}\big(\xi\frac{dH}{d\xi} - H\big)^2 K^2H^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2\Big]\Big) \quad = \quad 0 \\
&\Rightarrow 2\xi\frac{dH}{d\xi} - \xi^2\frac{d^2H}{d\xi^2} - 2H - 2\xi\frac{dH}{d\xi} + 2H + 2K^2H + \frac{\lambda}{e^2}H(H^2-\xi^2) \quad = 0
\end{aligned} \tag{2.3.2.4}$$

where $\partial' = \frac{\partial}{\partial\xi}$. Finally, we have:

$$\xi^2\frac{d^2H}{d\xi^2} = 2K^2H + \frac{\lambda}{e^2}H(H^2-\xi^2). \tag{2.3.2.5}$$

Now, let us compute for K function

$$\frac{\delta E}{\delta K} = 0 \Rightarrow \frac{\partial E}{\partial K} - \frac{d}{d\xi}\big(\frac{\partial E}{\partial K'}\big) = 0 \tag{2.3.2.6}$$

where K' is the derivative of K respect to ξ.

For this calculation we only care about the terms whcih contains K. So we obtain:

$$\begin{aligned} \partial'(\frac{\partial E}{\partial K'}) &= 2\xi^2 K'' \\ \frac{\partial E}{\partial K} &= 2K(K^2-1) + 2KH^2 \end{aligned} \tag{2.3.2.7}$$

We obtain from eq(2.3.2.6)

$$\xi^2 \frac{d^2 K}{d\xi^2} = KH^2 + K(K^2-1) \tag{2.3.2.8}$$

We can solve the equations of motion for K and H under the BPS condition, it means, we set $\lambda \to 0$. We can find the asymptotic behavior of the functions for $r \to 0$ and $r \to \infty$ and with that information is posible to construct the dependence of K and H respect to ξ.

So from BPS condition we have:

$$\xi^2 \frac{d^2 H}{d\xi^2} = 2K^2 H \tag{2.3.2.9}$$

The asymptotic behavior of K is: $K \to 1 - \epsilon$ when $r \to 0$ and $K \to 0$ when $r \to \infty$. We can obtain the real behavior of the function after applying the

$$K(\xi) = \frac{\xi}{\sinh \xi} \tag{2.3.2.10}$$

Similarly, the asymptotic behavior of the H function is: $H \to 0$ for $r \to 0$ and $H \to \xi$ for $r \to \infty$. We can obtain after

$$H(\xi) = \frac{\xi}{\tanh \xi} - 1 \tag{2.3.2.11}$$

2.3.3 Topological Nature of the Magnetic charges

On the last section we have described the Georgi-Glashow model with the $SO(3)$ gauge group. From the lagrangian (2.3.1.1), we are able to compute the equations of motion for the field configuration. Also we set the vacuum configuration for a vanishing Θ_{00} at the infinite. Let us rewrite the equations of motion by expressing the field in the fundamental representation of $SO(3)$. So we will have

$$\vec{\phi}_{vac} \cdot \vec{\phi}_{vac} = a^2 \tag{2.3.3.1}$$

$$\partial_\mu \vec{\phi}_{vac} - e\vec{W}_\mu \times \vec{\phi}_{vac} = 0, \tag{2.3.3.2}$$

where $\vec{\phi}_{vac}$ denote the scalar field in the Higg-vacuum.

We can solve $\vec{W}_\mu$ in terms of $\vec{\phi}_{vac}$, using the following calculation

$$\begin{aligned}(\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac}) \times \vec{\phi}_{vac} &= \partial_\mu \vec{\phi}_{vac}(\vec{\phi}_{vac} \cdot \vec{\phi}_{vac}) - \vec{\phi}_{vac}(\vec{\phi}_{vac} \cdot \partial_\mu \vec{\phi}_{vac}) \\ &= a^2 \partial_\mu \vec{\phi}_{vac}. \end{aligned} \tag{2.3.3.3}$$

We have used the fact that $\vec{\phi}_{vac} \cdot \vec{\phi}_{vac} = 0$ since they are orthogonal. Now from (2.3.3.1) we have:

$$(\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac}) \times \vec{\phi}_{vac} = ea^2 \vec{W}_\mu \times \vec{\phi}_{vac}. \tag{2.3.3.4}$$

It can be written as:

$$(\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac} - ea^2 \vec{W}_\mu) \times \vec{\phi}_{vac} = 0 \tag{2.3.3.5}$$

A general solution for $\vec{W}_\mu$ [11] comes by making the expression in the parenthesis zero. Also is the parenthesis is a vector proportional to $\vec{\phi}_{vac}$ is a solution for the equation. So, we have:

$$\vec{W}_\mu = \frac{1}{ea^2} \vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac} + \frac{1}{a} \vec{\phi}_{vac} A_\mu. \tag{2.3.3.6}$$

Let us compute the field strength tensor in vectorial notation $\vec{G}_{\mu\nu}$ and construct $F_{\mu\nu}$ the field strength tensor for the unbroken part of the gauge group.

$$\begin{aligned}\vec{G}_{\mu\nu} &= \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + e\vec{W}_\mu \times \vec{W}_\nu \\ &= \frac{1}{a}\partial_\mu(\frac{1}{ea}\vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_{vac} + \vec{\phi}_{vac} A_\nu) - \frac{1}{a}\partial_\nu(\frac{1}{ea}\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac} + \vec{\phi}_{vac} A_\mu) \\ &- e[\frac{1}{e^2a^4}(\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac}) \times \vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_{vac} + \frac{2}{ea^3}(\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac}) \times \vec{\phi}_{vac} A_\nu \\ &+ \frac{2}{ea^3}(\vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_{vac}) \times \vec{\phi}_{vac} A_\mu + \frac{1}{a^2}\frac{1}{a^2}\vec{\phi}_{vac} \times \vec{\phi}_{vac} A_\mu A^\mu]. \end{aligned} \tag{2.3.3.7}$$

We found that

$$\vec{G}_{\mu\nu} = \frac{1}{a}\vec{\phi}_{vac}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{ea^2}(\partial_\mu \vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_\nu), \tag{2.3.3.8}$$

we can rearrange the equation and we obtain:

$$F_{\mu\nu} = \frac{1}{a}\vec{\phi}_{vac} \cdot \vec{G}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{ea^3}\vec{\phi}_{vac} \cdot (\partial_\mu \vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_{vac}) \tag{2.3.3.9}$$

Let us replace our solution eq(2.3.3.9) in the equations of motion eq(2.3.1.8) and eq(2.3.1.9). As we are working in the Higgs vacuum both equations of motion are trivial. the covariant derivative for $G_{\mu\nu}$ or ϕ are zero.

$$\begin{aligned}(D_\mu \vec{G}^{\mu\nu}) &= 0 && (2.3.3.10)\\ D^\mu D_\mu \vec{\phi}_{vac} &= 0. && (2.3.3.11)\end{aligned}$$

As we have:

$$\begin{aligned}D_\mu &= \partial_\mu - e\vec{W}_\mu \times \\ &= \partial_\mu - \frac{1}{a^2}(\vec{\phi}_{vac} \times \partial_\mu \vec{\phi}_{vac}) \times - \frac{e}{a}(\vec{\phi}_{vac} A_\mu) \times && (2.3.3.12)\end{aligned}$$

We apply this derivative operator on the field strength tensor

$$\vec{G}_{\mu\nu} = \frac{\vec{\phi}_{vac}}{a}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{a}(\partial_\mu \vec{\phi}_{vac} A_\nu - \partial_\nu \vec{\phi}_{vac} A_\mu) + \frac{1}{ea^2}\partial_\mu \vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_{vac} \quad (2.3.3.13)$$

It means,

$$\begin{aligned}D_\nu \vec{G}^{\mu\nu} &= \partial_\nu \vec{G}^{\mu\nu} - \frac{1}{a^2}(\vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_{vac}) \times \vec{G}^{\mu\nu} - \frac{e}{a}(\vec{\phi}_{vac} A_\nu) \times \vec{G}^{\mu\nu} \\ &= \partial_\nu [\frac{1}{a}\vec{\phi}_{vac} F^{\mu\nu} + \frac{1}{ea^2}\partial^\mu \vec{\phi}_{vac} \times \partial^\nu \vec{\phi}_{vac}] \\ &- \frac{1}{a^2}(\vec{\phi}_{vac} \times \partial_\nu \vec{\phi}_{vac}) \times [\frac{1}{a}\vec{\phi}_{vac} F^{\mu\nu} + \frac{1}{ea^2}\partial^\mu \vec{\phi}_{vac} \times \partial^\nu \vec{\phi}_{vac}] \\ &- \frac{e}{a}(\vec{\phi}_{vac} A_\nu) \times [\frac{1}{a}\vec{\phi}_{vac} F^{\mu\nu} + \frac{1}{ea^2}\partial^\mu \vec{\phi}_{vac} \times \partial^\nu \vec{\phi}_{vac}]. && (2.3.3.14)\end{aligned}$$

Most terms cancel and we finally have:

$$D_\nu \vec{G}^{\mu\nu} = \frac{1}{a}\vec{\phi}_{vac}\partial_\nu F^{\mu\nu} = 0. \quad (2.3.3.15)$$

This equation tell us that $\partial_\mu F^{\mu\nu} = 0$ which corresponds to one of the Maxwell equations. Also, working on the Bianchi identity for non-Abelian gauge theory we can also obtain a Bianchi identity for $F_{\mu\nu}$. Finally, from we obtain the second Maxwell equation.

Since we have a explicit solution for the field strength tensor depending on the scalar field and the gauge field in the vacuum configuration. We can construct the magnetic field $\vec{B}$ in term of it. In components $B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}$. So the Gauss law for the magnetic field is written as:

$$\begin{aligned}g_\Sigma &= \int B^i dS^i = -\frac{1}{2}\epsilon^{ijk}\int dS^i(\frac{1}{ea^3}\vec{\phi} \cdot (\partial_j \vec{\phi} \times \partial_k \vec{\phi}) + \partial_j A_k - \partial_k A_j) \\ &= -\frac{1}{2ea^3}\epsilon^{ijk}\int \vec{\phi} \cdot (\partial_j \vec{\phi} \times \partial_k \vec{\phi}) dS^i - \epsilon^{ijk}\int \partial_j A_k dS^i && (2.3.3.16)\end{aligned}$$

There is an important conclusion that we can obtain when we work in the Higgs vacuum configuration. the only non-zero component of the gauge field tensor is the one associated to the $U(1)$ group, i.e. $F_{\mu\nu}$ which satisfies the Maxwell´s equations. In perspective we can say that outside the regions of the monopole, the $SO(3)$ gauge theory is indistinguishable from the standard electromagnetic theory.

Studying the magnetic flux, g_Σ, through the closed surface Σ and considering the global attributes of the Higgs vacuum. Following the Maxwell's equations, g_Σ will be non-zero only if Σ encloses solutions that belong to the Higgs vacuum configuration, more precisely Σ encloses finite energy solutions.

In a closer look the derivatives $\partial^j\vec{\phi}$ in eq(2.3.3.16) are tangential to Σ so the magnetic charge inside Σ depends only on the values of the Higgs field in Σ.

It can be shown that small variations of the Higgs field $\vec{\phi}$ which are in the Higgs vacuum configuration does not produce any change in the flux g_Σ.

It turns out that the expression at the r.h.s. of eq(2.3.3.16) is topological quantity: Since $\phi^2 = a^2$, the manifold of the Higgs vacua $\mathcal{M}_0$ has the topology of the sphere S^2 and $\vec{\phi}_{vac}$ is the map from Σ to $\mathcal{M}_0$, but Σ is also a S^2 then the map $\vec{\phi}_{vac} : \Sigma \to \mathcal{M}_0$ is the homotopic group $\pi_2(S^2)$. Another way to say this, $\vec{\phi}_{vac}$ is characterized by an integer number n which is named winding number.

$$n = \frac{1}{4\pi a^3}\epsilon^{ijk}\int_\Sigma \frac{1}{2}\vec{\phi}\cdot(\partial_j\vec{\phi}\times\partial_k\vec{\phi})dS^i. \tag{2.3.3.17}$$

Comparing it with the expression for the magntic charge in eq(2.3.3.16), we obtain:

$$g_\Sigma = -\frac{4\pi n}{e}. \tag{2.3.3.18}$$

So the winding number determines the magnetic charge of the soliton solution and we notice that this quantization condition differs from the Dirac quantization condition by a factor of 2. Then the smallest possible electric charge that we can obtain is $q_0 = e\hbar/2$. then eq(2.3.3.18) reduces to the Dirac condition.

2.4 The Bogomolnyi bound and the BPS State

One important characteristic of the monopole solution, as well as the dyonic solution obtained from the non-Abelian gauge theories, is that they have a smooth internal structure. Something different happens in the case of the Dirac monopole, where we introduce an external source arbitrarily and as a consequence the mass is also arbitrary. This arbitrariness is due to that the mass renormalization from quantum corrections is divergent since the inverse square law for the magnetic field diverges at its origin.

In the Higgs-vacuum the field strength tensor $F^{\mu\nu} = \vec{\phi}_{vac}\vec{G}^{\mu\nu}/a$ because of the gauge group $SO(3)$ is broken to $U(1)$ and there is only left one unbroken generator.

So we can express the magnetic charge

$$\begin{aligned} g = \int \vec{B}\cdot D\vec{S} &= \frac{1}{a}\int \mathcal{B}_a^k \phi_a dS^k \\ &= \frac{1}{a}\int \mathcal{B}_a^k (D^k\phi)_a d^3\vec{r} \end{aligned} \tag{2.4.0.19}$$

Here we have used the Bianchi identity for non-abelian gauge fields $D^k\mathcal{B} = 0$. Also by using Gauss' law we have integrated over a spherical surface taking a limit of the sphere radius to be infinite.

In a similar way, we can apply this equation for the electric field and express the electric charge for a non-Abelian gauge group as

$$q = \int \mathcal{E}_a^k (D^k\phi)_a d^3\vec{r}. \tag{2.4.0.20}$$

We can extend our analysis also for dyonic solutions. i.e. for a solution with electric charge as well as magnetic charge. Once we have these two result for magnetic and electric charge eqs (2.4.0.19) and (2.4.0.20). Let us now analyze in a different way the energy for the dyon. At the end of the analysis we will set q to zero to obtain the monopole case.

The mass corresponds to the energy at rest then the mass for the Georgi-Glashow model is:

$$M = \int d^3\vec{r}\{\frac{1}{2}[(\mathcal{E}_a^k)^2 + (\mathcal{B}_a^k)^2 + (D^0\phi_a)^2 + (D^i\phi_a)^2] + V(\phi)\} \tag{2.4.0.21}$$

In vacuum we can set the potential to zero, then every term in (2.4.0.21) is quadratic. We can take $\vec{\phi}$ time independent and by a gauge transformation we can set $A_0 \to 0$ so $D_0\vec{\phi} = 0$. Then we have the inequality:

$$M \geq \int d^3\vec{r}\frac{1}{2}[(\mathcal{E}_a^i)^2 + (\mathcal{B}_a^i)^2 + (D^i\phi_a)^2]. \tag{2.4.0.22}$$

Now let us introduce a parameter θ which we are going to eliminate later, but will be useful to rearrange our expression (2.4.0.22) in such a way that it describes the bound of the mass by an expression which depends only on the parameters of the theory and not on their fields.

Introduce θ angle as a parameter.

$$\begin{aligned} M &\geq (D^i\phi_a)^2 + (\mathcal{E}_a^i)^2 + (\mathcal{B}_a^i)^2 \\ &= (D^i\phi_a)^2(\sin^2\theta + \cos^2\theta) + (\mathcal{E}_a^i)^2 + (\mathcal{B}_a^i)^2 \end{aligned} \tag{2.4.0.23}$$

We can form quadratic terms by combining the electric and magnetic fields with the covariant derivatives of the scalar field as follows

$$\begin{aligned} M \geq\ & \tfrac{1}{2} \int d^3\{(\mathcal{E}_a^i - (D^i\phi)_a \sin\theta)^2 + (\mathcal{B}_a^i - (D^i\phi)_a \sin\theta)^2\} \\ +\ & \int \mathcal{E}_a^i (D^i\phi)_a d^3\vec{r} \sin\theta + \int \mathcal{B}_a^i (D^i\phi)_a d^3\vec{r} \cos\theta \end{aligned} \tag{2.4.0.24}$$

The r.h.s. of eq(2.4.0.24) takes a minimum value when we take the quadratic terms equal to zero. Also by using eq(2.4.0.19) and eq(2.4.0.20) the inequality (2.4.0.24) is reduced to

$$M \geq a(q \sin\theta + g\cos\theta) \tag{2.4.0.25}$$

First, we want to find a bound for the mass independent on the fields. This value has to be restricted by the parameters of the theory. Few lines above we had introduced a free parameter θ to rearrange our calculations. Now, we are going to maximize the expression in the r.h.s. of (2.4.0.25) depending only on θ and it is going to give us the bound for the mass.

So we take the derivative respect to θ and we find that $\tan\theta_0 = q/g$, where θ_0 is the extremal value of θ, reinserting this value of the parameter θ in (2.4.0.25), we obtain.

$$M \geq a\sqrt{q^2 + g^2} \tag{2.4.0.26}$$

This is the bound for the mass for dyon solution (if we take $q = 0$. We obtain the bound for the monopole mass). This is the so called Bogomolnyi bound. [9]

Let us describe in more detail the monopole case. As we have seen before from the Dirac quantization condition there is a relation between the electric and magnetic charge given by $g = -4\pi/q$. If we replace it into the Bogomolnyi bound (2.4.0.26), we have

$$M \geq \frac{4\pi a}{e} = \frac{4\pi a e}{e^2} = 137ae = 137M_W, \tag{2.4.0.27}$$

where M_W is the mass for the massive gauge bosons and $\frac{1}{137}$ is the fine structure constant $\alpha = \frac{e^2}{4\pi}$. The eq(2.4.0.27) tell us that the monopole mass for a field configuration is much greater than the massive gauge bosons. In quantum theory, we use the fine structure constant to perform perturbations due to α is small respect to 1. However, in this case we can not make a perturbative treatment of the theory since the coupling constant for monopole description g is no longer small. So the monopole solutions remains classically.

As we expected (2.4.0.26) allows us to know the lower bound for the mass of the dyon (or $q = 0$ monopole) solution. The bound only depends on the parameters of the theory. In our case a, the v.e.v. of the scalar field, q and g.

Let us continue with our analysis of the Bogomonlyi bound when we set the inequality to a equation. We have taken the quadratic term to zero and the constraints we obtain for the fields are.

$$\begin{aligned} \mathcal{E}_a^i &= (D^i\phi)_a \sin\theta \\ \mathcal{B}_a^i &= (D^i\phi)_a \cos\theta \\ (D^0\phi)_a &= 0 \end{aligned} \tag{2.4.0.28}$$

In the BPS limit we can use the 't Hooft-Polyakov monopole or the Julia-Zee dyon solution and they fit (2.4.0.28)

The BPS limit is obtained when the inequality (2.4.0.26) is saturated. It is we have take lower value for the mass M, by state we means the field configuration which is described by the ϕ, $\mathcal{E}$ and $\mathcal{B}$. On the specific BPS state the mass is completely determined by the coupling constants q and g of the theory.

In a quantum mechanical analysis, we will see the importance of this states since quantum corrections for the mass mean quantum corrections of the coupling constant. Further in supersymmetry theory, the BPS states [10] are special due that their mass is fixed by the supersymmetry algebra (We will see the importance of central charges which appears in our future discussion) Also we will see how the BPS states does not receive quantum corrections in some supersymmetric theories.

2.5 Instantons and Witten effect

2.5.1 Instantons in Yang-Mills gauge theory

We have seen how monopoles and dyons appears as extra solutions for the Georgi-Glashow model and how they behave as new "particles"' . We also note that they can not be described by any of the fields of the model, but as a configuration that relates the scalar and gauge fields.

In the minkowski theory, the non-trivial solutions that we have found for the Yang-Mills theory are the monopoles and dyons. Instantons are the classical solutions that we found for a pure Yang-Mills theory in a Euclidean space. We can undesrtand instantons as field configurations that appears when we extremize the integral functional of the Yang-Mills action. We restrict ourselves to the simplest non-Abelian gauge group $SU(2)$ which we have studied to obtain the monopole solutions. Those solutions for instantons were studied by Belavin and 't Hooft in 1976 [16]. The Euclidean pure Yang-Mills Lagrangian for this theory is:

$$S_E = \frac{1}{4}\int d^4x F^a_{\mu\nu}F^a_{\mu\nu}, \tag{2.5.1.1}$$

where $a = 1, \cdots, N^2 - 1$ a sum on each generator of the gauge group $SU(N)$. this actions works for a general Yang-Mills theory with a $SU(N)$ gauge group, in our case $SU(2)$, $a = 1, 2, 3$. We manly write this sum as a trace, $TrF_{\mu\nu}F_{\mu\nu}$.

We need to find the extremal value of the action so we form a quadratic term from eq(2.5.1.1) as follows:

$$S_E = \frac{1}{8}\int d^4x\{(F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu})^2 \mp 2F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}\}. \tag{2.5.1.2}$$

Now, we can obtain the extremal minimum value of the action by taking the quadratic term to zero. Then the action is bounded from below as it is shown in the following relation:

$$S_E \geq \mp\frac{1}{4}\int d^4x F^{\mu\nu}_a \tilde{F}^a_{\mu\nu} \tag{2.5.1.3}$$

With the constraint for the gauge fields $F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu} = 0$. Working in the r.h.s. of the relation (2.5.1.3). We will obtain a well-known mathematical relation form topology, homotopy, in similarity to the monopole case.

$$K = \frac{1}{4}\int d^4x F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}. \tag{2.5.1.4}$$

We can expand it on its gauge fields, A_{mu}. And fromthe defition of the dual field strength tensor $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, we obtain:

$$\begin{aligned} K &= \pm\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\int d^4x F_{\mu\nu}F_{\rho\sigma} \\ &= \pm\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\int d^4x[\partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]][\partial_\rho A_\sigma - \partial_\sigma A_\rho - g[A_\rho, A_\sigma]] \end{aligned} \tag{2.5.1.5}$$

Using the property of the antisymmetric tensor ϵ we reduce eq(2.5.1.5) to:

$$K = \epsilon^{\mu\nu\rho\sigma}\int d^4x(\partial_\mu A_\nu\partial_\rho A_\sigma - 2g\partial_\mu A_\nu A_\rho A_\sigma). \tag{2.5.1.6}$$

We can extract a total derivative to the integral in eq(2.5.1.6), we will have an extra term with a double derivative which will cancel since it is multiplied by the total antisymmetric tensor. Then we obtain:

$$K = \int d^4x\epsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu[\partial_\rho A_\sigma - \frac{2g}{3}A_\rho A_\sigma]). \tag{2.5.1.7}$$

We define a conserved topological current J_μ that respects,

$$\partial_\mu J_\mu = \epsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu[\partial_\rho A_\sigma - \frac{2g}{3}A_\rho A_\sigma]). \tag{2.5.1.8}$$

And the topological charge conserved from the topological current J_μ is

$$Q = \frac{1}{8\pi^2}\int d^4x \partial_\mu J_\mu \tag{2.5.1.9}$$

Integrating the total derivative. We have a surface integral in $\mathbb{R}^4$

$$S_E = \frac{1}{8\pi^2}\int_{S^3} J_\mu d^3\sigma_\mu. \tag{2.5.1.10}$$

A finite action requires that the potential A_μ becomes pure gauge when we take the limit $x \to \infty$ in $\mathbb{R}^4$ as follows:

$$A_\mu \to ig(x)^{-1}\partial_\mu g(x), \tag{2.5.1.11}$$

where $g(x)$ is an element of the gauge group. Any finite action configuration provides a map from $\partial\mathbb{R}^4 \cong S^3$ into the group $SU(2)$ which is also isomorphic with S^3. Similarly, when we studied monopoles we obtained a similar mapping. i.e. we insert a S^2 in S^2. This new case for instantons, the new mapping corresponds to wrap 2-spheres in 2-spheres. It is the homotopic group $\pi_3(S^3)$. From Homotopy theory is a known result $\pi_n(S^n)$ which maps n-spheres in n-spheres and it is isomorphic to $\mathbb{Z}$, $\pi_3(S^3) \cong \mathbb{Z}$. In the present case, we have $n = 3$. It is called the third homotopic group. Where each integer number corresponds to a equivalent class. In topology (homotopy) it is called Pontryagin number. In instanton theory, it is called instanton number it makes a distinction of the homotopic classes correspond to different field configurations $A^{(1)}_\mu$, $A^{(2)}_\mu$, which can not be continuously deformed one to another.

$$S_E = \pm\frac{8\pi^2}{g^2}Q \tag{2.5.1.12}$$

Belavin and 't Hooft in 1077 found a exact solution for the $SU(2)$ gauge group and it has the following form:

$$\imath g A_\mu = \frac{x^2}{x^2+\lambda^2}g^{-1}(x)\partial_\mu g(x) \qquad g(x) = \frac{\mathbb{I}x_4 + \imath\vec{\sigma}\cdot\vec{x}}{\sqrt{x^2}} \tag{2.5.1.13}$$

2.5.2 θ-parameter and Witten effect

In the subsection 2.1 we showed a generalization of the Dirac quantization condition given by DSZ for a treatment of dyons with charges (q_1, g_1) and (q_2, g_2). For dyons with charge q_0, we have as quantization condition $q_0 g_n = 2\pi n\hbar$. The smallest magnetic charge is $g_0 = 2\pi\hbar/q_0$. If we have two dyons with the same magnetic charge g_0 and with electric charge q_1 and q_2 from the DSZ condition. We obtain

$$(q_1 - q_2)g_0 = 2\pi\hbar n \Rightarrow q_1 - q_2 = nq_0 \tag{2.5.2.1}$$

The eq(2.5.2.1) shows that the difference of the electric charges is quantized, but it does not says anything about the quantization of the charges separately. The arbitrariness of the electric charge of the dyons can be found if we consider that the theory is CP invariant. If the theory is invariant under CP transformation i.e. $(q, g) \to (-q, g)$, then the existence of a dyon with charges (q, g_0) make necessary to exist a dyon with charge $(-q, g_0)$. From DSZ condition we have:

$$2qg_0 = 2\pi n\hbar \to 2q = nq_0 \tag{2.5.2.2}$$

which leads us to two possible situations.

$$q = nq_0 \qquad n, \text{integer} \tag{2.5.2.3}$$

$$q = (n + \frac{1}{2})q_0 \qquad n, \text{odd} \tag{2.5.2.4}$$

So we can have dyons with half-integer electric charge or with integer electric charge, but not both.

However, we do not have to take CP invariance so rigorously. Since CP symmetry is violated in nature, we will consider a weak CP violation in the theory. Let us consider the Georgi-Glashow described in section 2.3.1 and we add to it a θ-term. which is going to be the source of the CP violation. So we have the lagrangian

$$\mathcal{L} = \mathcal{L}_{GG} + \frac{\theta e^2}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}. \tag{2.5.2.5}$$

In principle, the new term in the lagrangian does not affect the equations of motion of the Georgi-Glashow model, because we can express this term as a total derivative eq(2.5.1.8), i.e. it is a shift of the lagrangian. However, we will see that the θ-term has a physical implication due to it violates the CP symmetry [17].

We will construct a electric charge operator to apply CP transformation. N operator of a gauge rotation in the direction $\hat{\phi}$ and the gauge transformation is $\Lambda^a = \phi^a/a$. The operator N acts over vectors on the gauge fields and respect the following transformation rule.

$$\delta\vec{v} = \frac{1}{a}\hat{\phi} \times \vec{v}. \tag{2.5.2.6}$$

$$\delta\vec{A}_\mu = \frac{1}{ea} D_\mu\vec{\phi}. \tag{2.5.2.7}$$

We have the Noether current for the lagrangian eq(2.5.2.5):

$$N = \int d^3 \Big(\frac{\delta \mathcal{L}}{\delta(\partial_0 A_i^a)} \delta A_i^a + \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi^a)} \delta \phi^a \Big) \tag{2.5.2.8}$$

Since ϕ^a is invariant under eq(2.5.2.6) we only have gauge field contribution and the important terms to compute are:

$$\frac{\delta}{\delta(\partial_0 A_i^a)} F_{\mu\nu}^a F^{a\mu\nu} = 4F^{a0i} = -4\mathcal{E}^{ai}, \tag{2.5.2.9}$$

$$\frac{\delta}{\delta(\partial_0 A_i^a)} F_{\mu\nu}^a \tilde{F}^{a\mu\nu} = 2\epsilon^{ijk} F_{jk}^a = -4\mathcal{B}^{ai}. \tag{2.5.2.10}$$

We can replace eqs(2.5.2.9) on eq(2.5.2.8). Then we have:

$$N = \frac{1}{ea} \int d^3x D_i \vec{\phi} \cdot \vec{\mathcal{E}}^i - \frac{\theta e}{8\pi^2 a} \int d^3x D_i \vec{\phi} \cdot \vec{\mathcal{B}}^i. \tag{2.5.2.11}$$

N is a operator with integer eingenvalues

$$N = \frac{1}{e} Q - \frac{\theta e}{8\pi^2} M, \tag{2.5.2.12}$$

where Q and M are the electric and magnetic charge operators with eingenvalues q and g respectively.

From eq(2.5.2.12)

$$en = q - \frac{\theta e^2}{8\pi^2} g \to q = ne + \frac{\theta e^2}{8\pi^2} g. \tag{2.5.2.13}$$

From (DQC) we have $g_0 = 4\pi/e$ if we replace it on eq(2.5.2.13)

$$q = ne + \frac{\theta e^2}{8\pi^2} (\frac{4\pi}{e} m) \to q = ne + \frac{\theta e}{2\pi} m. \tag{2.5.2.14}$$

$$q = (n + \frac{\theta}{2\pi} m) e. \tag{2.5.2.15}$$

Notice that we can make a shift on θ "generating" more electric charges by taking $\theta \to \theta + 2\pi$:

$$q = (n + m + \frac{\theta m}{2\pi}) e = (n' + \frac{\theta m}{2\pi}) e, \tag{2.5.2.16}$$

with $n' = n + m$. We can define a complex coupling constant:

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{e^2}, \tag{2.5.2.17}$$

by taking $q + ig = e(n + m\tau)$.

The shift $\theta \to \theta + 2\pi$ can be translated in term of τ as: $\tau \to \tau + 1$. Also we have already seen this quantity τ on the previous section. Where we can express the BPS bound eq(2.4.0.28) as

$$M \geq \sqrt{2} |ae(n + m\tau)| \tag{2.5.2.18}$$

2.6 Montonen-Olive conjecture

Let us leave aside for a moment the description of dyons solutions and focus on the solitons solutions only with magnetic charge, the magnetic monopole, as dual solution of the gauge bosons $W_\mu^\pm$. Such as solutions were obtained from the Georgi-Glashow model in section 2.3.1 in the BPS limit, where we have two central charges: electric charge zero for the photon γ and e for the gauge boson W_μ and $-4\pi/e$ for the magnetic monopole.

From the BPS condition (2.4.0.28) the dual transformation which preserve the mass relation is

$$\begin{aligned} q_0 &\to g_0 \\ g_0 &\to -q_0 \end{aligned} \tag{2.6.0.19}$$

Which implies:

$$q_0 \to \frac{2\pi\hbar}{q_0}. \tag{2.6.0.20}$$

Montonen and Olive conjectured in 1977 [7] that there are two equivalent descriptions of the same theory, dual to each other. In the electrical description, the gauge bosons are elementary particles and the magnetic monopoles are solitons while in the magnetic dual description the magnetic monopoles are elementary particles and the gauge bosons are solitons of the theory.

In other words, in the electric description when the electric charge tends to zero $e \to 0$ the theory is weakly couple and in the magnetic description when the magnetic charge tends to zero $e \to \infty$ the theory is weakly couple. We say that such as theories with this behavior present a particular case of strong-weak duality (S-duality). If we have two dual theories T and T_D, T is weakly couple then it is possible to make a perturbative description of the theory at quantum level, then the dual theory T_D is strongly couple and its analysis is purely classical and viceverse.

Montonen and Olive gave some reasons to support their conjecture:

1. The eq(2.4.0.28) is valid for all particles of the theory and this relation is duality invariant.

2. There are no monopole-monopole interaction we can see from the conditions obtained of the BPS limit that the gauge bosons from the electric part do not interact with those of the same charge signature. However, monopoles and anti-monopoles do interact as well as gauge bosons of the same charge signature.

On the other hand, the conjecture has also some unanswered questions:

1. In the Georgi-Glashow theory the gauge bosons have spin 1. The magnetic monopoles must also have spin 1.

2. We have considered the mass formula eq(2.4.0.28) along our duality analysis in a classical description. How this relation is modify after quantum corrections?

3. We have not talked about dyons in our description of the conjecture. How do we implement it?

Chapter 3

Supersymmetric electric-magnetic duality

3.1 N=2 Supersymmetry Yang-Mills theory in $4d$

In the section 2.4, we found solitonic solutions in a Yang-Mills field theory which have electric-magnetic duality, such as solutions were called BPS states. We have mentioned that electric-magnetic duality is preserved as long as we respect the relation $M = \sqrt{2}|Z|$, were Z is the central charge. So far, what we have done is a purely classical description of the theory. We expect to appear some problems at the moment to consider a quantum description, i.e. to consider corrections from quantum perturbations.

In principle, these corrections will affect the coupling constant and the bound of the mass for the BPS states. i.e. that the electric-magnetic duality would break. To avoid this, we need to find a theory that fix this problem or that gives an extension of the classical model in such a way that it preserves the E-M duality at quantum level.

In order to preserve the EM duality, it is the same that we need to keep the relation for the coupling constants, $\alpha(e) \rightarrow \frac{1}{\alpha_D(e_D)}$, where α_D is the coupling constant for the dual theory. In terms of the electric and magnetic charges this relation corresponds to $g \rightarrow -\frac{1}{q}$.

In quantum field theory, the beta function $\beta(g)$, determines the variation of the coupling constant respect to the scale of energy, $\beta(g) = -\frac{\partial g}{\partial \mu}$, where g is the coupling constant and μ is the energy scale. From this point of view, the electric-magnetic duality is broken when we have a beta function different from zero. An ideal theory which preserves the EM duality will have $\beta = 0$ and would respect the relation for BPS states.

At this point, we include supersymmetry to make calculations easier at quantum level. Now, we give some features for supersymmetric field theories.

Non-renormalizability properties: Corrections from quantum perturbations are less violent for a SUSY theory than a QFT and it is due to

Holomorphic structure: It leads to vacuum degenerations. It allows us to use powerful methods of complex analysis.

Duality symmetries: Electric-magnetic duality and strong-weak duality is more or less manifest depending on which SUSY we use.

Supersymmetry by construction has central charges in its algebra and they respect the Bogomolnyi bound (2.4.0.26). This restriction is manifest in the massive representation of the extended supersymetry.

If we restrict ourselves to a global supersymmetry theories, then the maximum number of supersymmetries is four. So we have the following possibilities, $N = 1, 2, 4$.

N=4 SYM: It is conjectured to be self-dual and it is full invariant under the exchange of the electric and magnetic sectors. In this theory the electric-magnetic duality is too simple when we extend the analysis to the quantum level, because there is no quantum corrections for the coupling constant $\beta = 0$. EM duality is well described at classical level for $N = 4$ SYM.

N=1 SYM: Moreover, this is not solvable exactly and its quantum correction are no under control. Only certain sectors are described by holomorphic objects.(chiral superfield), which are protected from perturbative quantum corrections.

N=2 SYM: This theory is in the middle term between the trivial and not fully solvable. This theory can be solve exactly in the low energy limit. It is governed by a holomorphic function $\mathcal{F}$ called prepotential, which determines the perturbative quantum corrections. These corrections are under control and are at most to 1-loop. We will describe in some detail the $N = 2$ SYM theory. For the simplest gauge group $SU(2)$ or $SO(3)$. It is an extension of the theory described in the classical level of the EM duality in section 2.3.1.

We will see that the BPS relations (2.4.0.26) is found directly from the supersymmetry algebra. Also we will find the same relation when we will work with the N=2 SYM lagrangian. Later, we will study the global symmetries for this lagrangian, R-symmetry as well as holomorphism and the behavior of the theory at low energies. We will analyze the behavior of the theory around the vacuum and find important properties for the inequivalent vacuum. Later, we will called it up as moduli space. We will write a semi-classical treatment of the moduli space to go to the quantum description of it. Because of quantum perturbation on the vacuum there is going to appear singularities which are studied by using a strong mathematical tool called monodromies. this was the work for Seiberg-Witten [27],

[28] for $N = 2$ SYM theories. They solved the theory at low energie completely. So we will start by describing the massive representation of extended supersymmetry.

3.1.1 Supersymmetry algebra with central charges

We have already written the supersymmetry algebra with central charges in (A.2.0.30) [20]. Now, we will see how central charges are useful to describe correctly the massive representation of the extended supersymmetry.

Central charges are antisymmetric in the indices I and J. We will focus on the case for even N. We can use the unitary transformation for Z and it can be skew-symetric diagonalize as:

$$Z^{IJ} = U^I_A U^J_B Z^{AB}, \qquad U^{I,J}_{A,B} \quad \text{are unitary matrices.,} \tag{3.1.1.1}$$

in such a way that Z has the form $Z = \epsilon \otimes D$. ϵ is an antisymmetric tensor in two indices and D is a diagonal matrix of dimension $N/2$. The number I counts the number of supersymmetry and can be decomposed in the pair (p, m), $p = 1, 2$ for the tensor ϵ and $m = 1, 2, \cdots, N$ for the matrix D. Using this decomposition we have the supersymmetry algebra (A.2.0.30) in the new following form:

$$\{Q^p_\alpha, \bar{Q}_{\dot{\alpha}q}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^p_q, \tag{3.1.1.2}$$
$$\{Q^p_\alpha, Q^q_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab} Z, \tag{3.1.1.3}$$
$$\{\bar{Q}_{\dot{\alpha}p}, \bar{Q}_{\dot{\beta}q}\} = 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{pq}\bar{Z}. \tag{3.1.1.4}$$

Now, we are going to define a general linear combination of the supersymmetry generators $Q's$ and $\bar{Q}'s$ and they are denoted by.

$$\tilde{Q}_\alpha = \frac{1}{2}(Q^1_\alpha + w\epsilon_{\alpha\beta}(Q^2_\beta)^\dagger), \tag{3.1.1.5}$$

where w is a complex number of unitary modulo. Also, in the rest frame we have the following relation.

$$\begin{aligned}
\{\tilde{Q}_\alpha, \tilde{Q}^\dagger_\gamma\} &= \frac{1}{4}\{Q^1_\alpha + w\epsilon_{\alpha\beta}(Q^2_\beta)^\dagger, (Q^1_\gamma)^\dagger + w^*\epsilon_{\gamma\delta}Q^2_\delta\} \\
&= \frac{1}{2}(\sigma_0^{\alpha\gamma} M + w\sqrt{2}\delta_{\alpha\gamma}\epsilon_{21}Z^* + w^*\sqrt{2}\epsilon_{\alpha\gamma}\epsilon^{21}Z + ww^*\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\sigma^0_{\beta\delta}M) \\
&= \delta_{\alpha\gamma}(M + \sqrt{2}Re(w^*Z)).
\end{aligned} \tag{3.1.1.6}$$

Here w^*Z defines rotations in the complex plane since $|w| = 1$. So $Re(w^*Z)$ has two extremal values.

$$-|Z| \leq Re(w^*Z) \leq |Z|. \tag{3.1.1.7}$$

For these extremal values we have:

$$\{\tilde{Q}_\alpha, \tilde{Q}^\dagger_\gamma\} = \delta_{\alpha\gamma}(M \pm \sqrt{2}|Z|). \tag{3.1.1.8}$$

The expression (3.1.1.8) is zero for $M = \sqrt{2}|Z|$. For $\alpha = \gamma$ the anticommutator must be positive defined. Since all physical states have positive defined norm, we obtain:

$$M \geq \sqrt{2}|Z|. \tag{3.1.1.9}$$

We have already seen the relation (3.1.1.9) in classical theory and it is known as the Bogomolnyi bound or BPS bound.

In this case we have obtained the Bogomonlyi bound as a consequence of the supersymmetric algebra construction

3.1.2 $N = 2$ Lagrangian in the $N = 1$ superspace

Even though the relation (2.4.0.26) is more relevant for massive representation of $N = 2$ multiplet(for matter Hypermultiplet) The relation (2.4.0.26) is trivial for the gauge multiplet because there the mass is zero and the central charge is zero too.

Mostly, we will restrict our analysis to the pure gauge multiplet of a $N = 2$ supersymmetric lagrangian. Also, we have mentioned on the appendix the eq(A.3.1) that the pure gauge multiplet in the $N = 1$ language corresponds to a chiral multiplet and a vector multiplet, with the field content (ϕ, ψ_α) and (λ_α, A_μ) for each multiplet respectively. The corresponding lagrangian for this multiplet is similar to the (A.5.1.1) they share the same field content. However, the multiplets in the $N = 1$ language belongs to the same multiplet for $N = 2$. So all fields transform under the same representation of the gauge fields, which is the adjoint representation of the gauge group. The number of generators for the case of $SU(2)$ gauge group is three and the fields are written as ϕ^a, $\psi^{\alpha a}$, $\lambda^{\alpha a}$, A^a_μ, where $a = 1, 2, 3$ correspond to the $SU(2)$ generators.

Following the appendix, the general lagrangians for $N = 1$ SYM for the scalar and vector multiplet are

$$\mathcal{L}_S = \int d^2\theta d^2\bar{\theta} K(\Phi, \Phi^\dagger) + \int d^2\theta \mathcal{W} + \int d^2\bar{\theta}\bar{\mathcal{W}}. \tag{3.1.2.1}$$

and

$$\mathcal{L}_V = \frac{1}{4g^2} Tr[\int d^2\theta W^\alpha W_\alpha + \int d^2\bar{\theta}\bar{W}_\alpha \bar{W}^\alpha]. \tag{3.1.2.2}$$

The general $N=1$ lagrangian in field components has the form:

$$\begin{aligned}\mathcal{L} = & - \frac{1}{4g^2}F^a_{\mu\nu}F^{\mu\nu a} + \frac{\theta}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{\mu\nu a} - \frac{1}{g^2}\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + \frac{1}{2g^2}D^aD^a \\ & + (\partial_\mu\phi - iA^a_\mu T^a\phi)^\dagger(\partial^\mu\phi - iA^{\mu a}T^a\phi) - i\bar{\psi}\bar{\sigma}^\mu(\partial_\mu\psi - iA^a_\mu T^a\psi) \\ & - D^a\psi^\dagger T^a\phi - i\sqrt{2}\phi^\dagger T^a\lambda^a\psi + i\sqrt{2}\bar{\psi}T^a\phi\bar{\lambda}^a + F_i^\dagger F_i \\ & + \frac{\partial\mathcal{W}}{\partial\phi_i}F_i + \frac{\partial\bar{\mathcal{W}}}{\partial\phi^\dagger}F_i^\dagger - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial\phi_i\partial\phi_j}\psi_i\psi_j - \frac{\partial^2\bar{\mathcal{W}}}{\partial\phi_i^\dagger\phi_j^\dagger}\bar{\psi}_i\bar{\psi}_j \end{aligned} \tag{3.1.2.3}$$

Now, we are going to give a restriction to the form of the theory in (3.1.2.3). The superpotentials of (3.1.2.3) must be zero since they couple only to the fermions of the chiral multiplet of $N=1$, ψ^α. So all terms that depends on $\mathcal{W}$ and $\bar{\mathcal{W}}$ break the R-symmetry. Also, we require that the kinetic terms for the fermionic field are equal normalized to allow the R-symmetry. This can be achieved by rescaling the Chiral field. $\Phi \to \Phi/g$. We can rewrite the lagrangian (3.1.2.3) and eliminate the auxiliary fields D^a and F_i. We can do it by considering the on-shell lagrangian, i. e. we are going to substitute the fields D^a and F_i in terms of the scalar and vector fields by using the equations of motion. We concentrate on the term which corresponds to the potential of the lagrangian.

$$V = \frac{1}{g^2}Tr[\frac{1}{2}D^aD^a + D^a[\phi^\dagger,\phi] + F_i^\dagger F_i], \tag{3.1.2.4}$$

the equations of motion for the auxiliary fields are:

$$F_i = 0 \qquad\qquad D^a = -[\phi^\dagger,\phi]. \tag{3.1.2.5}$$

Now, we substitute these relations on the potential, we obtain:

$$V = -\frac{1}{2g^2}\mathrm{Tr}([\phi^\dagger,\phi]^2). \tag{3.1.2.6}$$

With these restriction we obtain the pure Yang-Mills lagrangian for $N=2$ supersymmetry

$$\begin{aligned}\mathcal{L} &= \frac{1}{8\pi}\mathrm{Im\ Tr}\big[\tau\big(\int d^2\theta W^\alpha W_\alpha + 2\int d^4\theta e^{2V}\Phi^\dagger e^{-2V}\Phi\big)\big] \\ &= \frac{1}{g^2}\mathrm{Tr}\big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^2\frac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \lambda\sigma^\mu D_\mu\bar{\lambda} - \bar{\psi}\bar{\sigma}^\mu D_\mu\psi \\ &\qquad + (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{2}[\phi,\phi^\dagger]^2 - i\sqrt{2}\phi^\dagger[\lambda,\psi] + i\sqrt{2}[\bar{\lambda},\bar{\psi}]\phi\big) \end{aligned} \tag{3.1.2.7}$$

Notice that the lagrangian we are writing eq(3.1.4.6), $N=2$ SYM $SU(2)$, is the generalization of the Yang-Mills theory coupled to a Higgs-like potential scalar field. In the classical description we have made we used the Georgi-Glashow model. Also, we considered the θ-term. We have discussed in section 2.3.1 that the vacuum configuration for the condition $D_\mu\phi = 0$ and $V(\phi) = 0$. In order to preserve supersymmetry, we demanded condition for the Higgs vacuum is $V(\phi) = 0$. The potential is zero for the ground state of the scalar field otherwise supersymmetry is broken. As the potential is zero, then the commutator of ϕ_{vac} and $\phi^\dagger_{vac}$ is zero. but not necessarily their expectation values. In analogy to the classical case, the model in (2.3.1.13) admits magnetic monopole and dyon as non-trivial solutions and there are massive gauge bosons. Because of spontaneous symmetry breaking of the gauge group, $SU(2)$ or $SO(3)$ break to $U(1)$ or $SO(2)$ respectively.

3.1.3 Central charges in $N = 2$ Pure gauge theory

We have seen that the $N = 2$ supersymmetry algebra with central charge Z implies the bound (2.4.0.28) we have already studied in sec.2.4 on the particle masses. Also, this bound corresponds to the same bound of the BPS state on the masses which is determined in terms of electric and magnetic charges. In this subsection we will probe the statement from above by calculating explicitly the central charge of the theory. This was made by Witten and Olive in [23]

In the supersymmetry algebra, the central charge appears as an anticommutator of the supercharges Q^p_α, which are space integrals of J^{p0}_α, $J^{p\mu}_\alpha$ is the supercurrent. To achieve this, we have to compute the supercurrent from the field components of the lagrangian, integrating it over the space and take the anticommutators. The central charge will corresponds to a surface term in the space integral of the commutator, which is non-zero if the field configuration corresponds to the electric and magnetic charges.

We can start by taking two multiplets: a chiral multiplet and a vector multiplet in $N = 1$ to construct a vector multiplet in $N = 2$ and calculate the supercurrent contribution from each of them to a total supercurrent in $N = 2$. Also, we saw that the lagrangian for the chiral multiplet in $N = 1$ should not have a superpotential to be part of the vector multiplet in $N = 2$.

There is another important requirement, we need the field components be in the adjoint representation. Then we have:

$$\mathcal{L}_{chiral} = \int d^4\theta e^V \Phi^\dagger e^{-V} \Phi \tag{3.1.3.1}$$

Defining $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, and the superfield is expanded in θ components with the supercharges Q_α and $\bar{Q}_\alpha$. Then a supersymmetric variation is given by

$\delta_\epsilon = \epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}$ of the fields given by:

$$
\begin{aligned}
\delta\phi &= \sqrt{2}\epsilon\psi, & \delta\bar{\phi} &= \sqrt{2}\bar{\epsilon}\bar{\psi}, \\
\delta\psi &= \sqrt{2}\epsilon F + i\sqrt{2}\sigma^\mu\bar{\epsilon}D_\mu\phi, & \delta\bar{\psi} &= -i\sqrt{2}\epsilon\sigma^\mu D_\mu\bar{\phi} + \sqrt{2}\bar{F}\bar{\epsilon}, \\
\delta F &= i\sqrt{2}\bar{\epsilon}\sigma^\mu D_\mu\psi, & \delta F &= i\sqrt{2}\epsilon\sigma^\mu D_\mu\bar{\psi}.
\end{aligned}
\tag{3.1.3.2}
$$

Using these variation, we can obtain the supersymmetric variation of the lagrangian for the chiral multiplet.

$$
\begin{aligned}
\delta\mathcal{L}_D = & - \frac{i}{\sqrt{2}}D_\mu(F\bar{\psi}\bar{\sigma}^\mu\epsilon) + \frac{i}{\sqrt{2}}D_\mu(\epsilon\psi D^\mu\phi^\dagger - \epsilon\sigma^{\nu\mu}\psi D_\nu\phi^\dagger) \\
& + \frac{i}{\sqrt{2}}D_\mu(F^\dagger\bar{\epsilon}\bar{\sigma}^\mu\psi) + \frac{i}{\sqrt{2}}D_\mu(\bar{\epsilon}\bar{\psi}D^\mu\phi - \bar{\psi}\bar{\sigma}^{\nu\mu}\bar{\epsilon}D_\nu\phi),
\end{aligned}
\tag{3.1.3.3}
$$

From the definitions:

$$
\sigma^{\mu\nu} = \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\nu), \qquad \bar{\sigma}^{\mu\nu} = \frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu). \tag{3.1.3.4}
$$

We obtain the supercurrent:

$$
J^\mu_{chiral} = \sqrt{2}\epsilon\sigma^\nu\bar{\sigma}^\rho\psi D_\nu\phi^\dagger + \sqrt{2}\bar{\psi}\bar{\sigma}^\rho\sigma^\nu\bar{\epsilon}D_\nu\phi. \tag{3.1.3.5}
$$

For convenience, we have included the supersymmetry transformation parameters ϵ and $\bar{\epsilon}$ for the supercurrent definition. Expanding it in components we have:

$$
J^\rho = \epsilon^\alpha J^\rho_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{J}^{\rho\dot{\alpha}}. \tag{3.1.3.6}
$$

Now, we consider the vector multiplet. The supersymmetric variations are given by:

$$
\begin{aligned}
\delta A^a_\mu &= -i\bar{\epsilon}\bar{\sigma}_\mu\lambda^a + i\bar{\lambda}^a\bar{\sigma}_\mu\epsilon, \\
\delta D^a &= \bar{\epsilon}\sigma^\mu D_\mu\lambda^a + D_\mu\lambda^a\bar{\sigma}^\mu\epsilon, \\
\delta\lambda^a &= \frac{1}{2}\sigma^{\mu\nu}\epsilon F^a_{\mu\nu} + i\epsilon D^a, \\
\delta\bar{\lambda}^a &= \frac{1}{2}\bar{\epsilon}\bar{\sigma}^{\nu\mu}F^a_{\mu\nu} - i\bar{\epsilon}D^a
\end{aligned}
\tag{3.1.3.7}
$$

Following a similar procedure to obtain the variation of the lagrangian of the vector multiplet, we obtain the supercurrent for this multiplet.

$$
J^\rho_{gauge} = -\frac{i}{2g^2}\left(\bar{\lambda}\bar{\sigma}^\rho\sigma^{\mu\nu}\epsilon F^a_{\mu\nu} + \bar{\epsilon}\bar{\sigma}^{\mu\nu}\sigma^\rho\lambda^a F^a_{\mu\nu}\right) \tag{3.1.3.8}
$$

Our theory is coupled to matter, Then the supercurrent is not only the sum the two supercurrents for the chiral and vector multiplet. there is an extra contribution from the terms $D^a\phi^\dagger T^a\phi$ and $i\phi^\dagger T^a\lambda^a\psi + h.c.$

Then the supercurrent for the $N = 2$ lagrangian is:

$$\begin{aligned} J^\rho = & - \frac{i}{2}\big(\bar\lambda\bar\sigma^\rho\sigma^{\mu\nu}\epsilon + \bar\epsilon\bar\sigma^{\mu\nu}\sigma^\rho\lambda^a\big)F^a_{\mu\nu} - \big(\bar\epsilon\bar\sigma^\rho\lambda^a + \bar\lambda^a\bar\sigma^\rho\epsilon\big)\phi^\dagger T^a\phi \\ & + \sqrt{2}\epsilon\sigma^\nu\bar\sigma^\rho\psi D_\nu\phi^\dagger + \sqrt{2}\bar\psi\bar\sigma^\rho\sigma^\nu\bar\epsilon D_\nu\phi. \end{aligned} \tag{3.1.3.9}$$

We have scaling the chiral superfield to $\Phi \to \Phi/g$. Now, Φ is a vector in the adjoint representation of the gauge group. Relabeling (λ,ψ) to (λ^1,λ^2). We can write down a $N = 2$ supercurrent more properly.

$$\begin{aligned} g^2 J^\rho_{(1)} = & - \frac{i}{2}\big(\bar\lambda^a_1\bar\sigma^\rho\sigma^{\mu\nu}\epsilon + \bar\epsilon\bar\sigma^{\mu\nu}\sigma^\rho\lambda^{1a}\big)F^a_{\mu\nu} - \big(\bar\epsilon\bar\sigma^\rho\lambda^{1a} + \bar\lambda^a_1\bar\sigma^\rho\epsilon\big)\phi^\dagger T^a\phi \\ & + \sqrt{2}\epsilon\sigma^\mu\bar\sigma^\rho\lambda^{2a}D_\mu\phi^{a\dagger} + \sqrt{2}\bar\lambda^a_2\bar\sigma^\rho\sigma^\mu\bar\epsilon D_\mu\phi^a. \end{aligned} \tag{3.1.3.10}$$

The $N = 2$ supersymmetry theory is also invariant under a second set of supersymmetry transformations, with parameter ϵ'. It is obtained from the eq(3.1.3.2) and eq(3.1.3.7) by exchanging $\lambda \to \psi$ and $\psi \to -\lambda$. This corresponds to the transformation of the doublet (λ,ψ) of $SU(2)_R$. The conserved associated supercurrent is obtained from J^ρ_1 by the replacement $\lambda_1 \to \lambda_2$, $\lambda_2 \to -\lambda_1$:

$$\begin{aligned} g^2 J^\rho_{(2)} = & - \frac{i}{2}\big(\bar\lambda^a_2\bar\sigma^\rho\sigma^{\mu\nu}\epsilon' + \bar\epsilon'\bar\sigma^{\mu\nu}\sigma^\rho\lambda^a_2\big)F^a_{\mu\nu} - \big(\bar\epsilon'\bar\sigma^\rho\lambda^a_2 + \bar\lambda^a_2\bar\sigma^\rho\epsilon'\big)\phi^\dagger T^a\phi \\ & - \sqrt{2}\epsilon'\sigma^\mu\bar\sigma^\rho\lambda^a_1 D_\mu\phi^{a\dagger} + \sqrt{2}\bar\lambda^a_1\bar\sigma^\rho\sigma^\mu\bar\epsilon' D_\mu\phi^a. \end{aligned} \tag{3.1.3.11}$$

Using the properties for Pauli matrices and for grassmannian variables, the supercurrent can be rewritten as:

$$\begin{aligned} g^2 J^\mu_{(1)} = & - \epsilon\sigma_\nu\bar\lambda^a_1(iF^{a\mu\nu} + \tilde F^{a\mu\nu}) + \sqrt{2}\epsilon\sigma^\nu\bar\sigma^\mu\lambda^a_2 D_\nu\phi^{\dagger a} + \epsilon\bar\lambda^a_1\phi^\dagger T^a\phi \\ & + (\bar\epsilon - \text{term dependent}). \end{aligned} \tag{3.1.3.12}$$

For the above can be read off in components $J^\mu_{(1)\alpha}$ from eq(3.1.3.6) as

$$g^2 J^\mu_{(1)\alpha} = \sigma_{\nu\alpha\dot\alpha}\bar\lambda^{a\dot\alpha}_1(iF^{a\mu\nu} + \tilde F^{a\mu\nu}) + \sqrt{2}(\sigma^\nu\bar\sigma^\mu\lambda^{2a})_\alpha D_\nu\phi^{\dagger a} + \sigma^\mu_{\alpha\dot\alpha}\bar\lambda^{a\dot\alpha}_1\phi^\dagger T^a\phi. \tag{3.1.3.13}$$

Using the vector notation for spatial coordinates of vectors, the $\mu = 0$ component of the supercurrent takes the form

$$\begin{aligned} g^2 J^0_{(1)\alpha} = & - i(\vec\sigma\sigma_2\lambda^{1\dagger a})_\alpha \cdot (i\vec F + \vec{\tilde F})^a \\ & + \sqrt{2}\lambda^a_{2\alpha}D_0\phi^{\dagger a} + \sqrt{2}(\vec\sigma\cdot\vec D\phi^{\dagger a}\lambda^{2a})_\alpha + i(\sigma_2\lambda^{1\dagger a})_\alpha\phi^\dagger T^a\phi. \end{aligned} \tag{3.1.3.14}$$

where $\vec{F}^a = F^{a0i}$ and $\vec{\tilde{F}}^a = \tilde{F}^{a0i}$. As we did after eq(3.1.3.10), the supercurrent $J^0_{(2)\alpha}$ is obtained by exchange the fermionic fields $\lambda_1 \to \lambda_2$ and $\lambda_2 \to -\lambda_1$ from the supercurrent $J^0_{(1)\alpha}$.

To evaluate the central charge, we need to evaluate the anticommutator.

$$\{Q^1_\alpha, Q^2_\beta\} = \{\int d^3x J^{10}_\alpha(0,\vec{x}), \int d^3y J^{20}_\beta(0,\vec{y})\}. \tag{3.1.3.15}$$

However, it is not necessary to evaluate all the integral, but only those terms which is going to give us boundary terms which measure the electric and magnetic charges. In order to obtain these boundary terms, we have to look for the terms with the form $\int d^3x \partial_i(\phi^{\dagger a} F^{a0i} + \phi^{\dagger a}\tilde{F}^{a0i})$. So from the supercurrent we will keep only the following relevant terms:

$$\begin{aligned} g^2 J^1_\alpha 0 &= -i(\vec{\sigma}\sigma_2\lambda^{1\dagger a})_\alpha \cdot (i\vec{F} + \vec{\tilde{F}})^a + \sqrt{2}(\vec{\sigma}\cdot\vec{D}\phi^{\dagger a}\lambda^{2a})_\alpha + \cdots \\ g^2 J^0_{(2)\alpha} &= -i(\vec{\sigma}\sigma_2\lambda^{2\dagger a})_\alpha \cdot (i\vec{F} + \vec{\tilde{F}})^a - \sqrt{2}(\vec{\sigma}\cdot\vec{D}\phi^{\dagger a}\lambda^{1a})_\alpha + \cdots \end{aligned} \tag{3.1.3.16}$$

From the last two expressions, we obtain

$$\begin{aligned} \{Q^1_\alpha, Q^2_\beta\} &= \frac{1}{g^4}\int d^3x \int d^3y [i\sqrt{2}(\sigma^i\sigma_2)_{\alpha\gamma}\sigma^j_{\beta\lambda}\{\lambda^{1\dagger a}_\gamma, \lambda^{1b}_\lambda\}(iF^a_{0i} + \tilde{F}^a_{0i})D_j\phi^{\dagger b} \\ &\quad - i\sqrt{2}(\sigma^j)_{\alpha\gamma}(\sigma^j\sigma_2)_{\beta\lambda}\{\lambda^{2\dagger a}_\gamma, \lambda^{2b}_\lambda\}(iF^a_{0i} + \tilde{F}^a_{0i})D_j\phi^{\dagger b}] \\ &= \frac{1}{g^4}\int d^3x i\sqrt{2}\big[(\sigma^i\sigma_2\sigma^{jT})_{\alpha\beta} - (\sigma^i\sigma_2\sigma^{jT})_{\beta\alpha}\big](iF^a_{0i} + \times F^a_{0i})D_j\phi^{\dagger a} \end{aligned} \tag{3.1.3.17}$$

We can simplify this expression by using the following relation for σ-matrices, $\sigma^i\sigma_2 = -\sigma_2\sigma^{iT}$, then

$$(\sigma^i\sigma_2\sigma^{jT})_{\alpha\beta} = -(\sigma_2\sigma^{iT}\sigma^{jT})_{\alpha\beta} = -[\sigma_2(\delta^{ij} - i\epsilon^{ijk}\sigma^T_k)]_{\alpha\beta}. \tag{3.1.3.18}$$

Now, we can reduce the square brackets in the eq (3.1.3.17) and it is equal to $-2(\sigma_2)_{\alpha\beta}\delta^{ij} = 2i\epsilon_{\alpha\beta}\delta^{ij}$. Then the anticommutator takes the form:

$$\{Q^1_\alpha, Q^2_\beta\} = -\frac{2\sqrt{2}}{g^2}\epsilon_{\alpha\beta}\int d^3x (iF^{a0i} + \tilde{F}^{a0i})D_j\phi^{\dagger a} \tag{3.1.3.19}$$

We use the equation of motion for the electromagnetism and the Bianchi identity and we show that eq(3.1.3.19) is equal to:

$$\{Q^1_\alpha, Q^2_\beta\} = -\frac{2\sqrt{2}}{g^2}\epsilon_{\alpha\beta}\int d^3x \partial_i[(iF^{a0i} + \tilde{F}^{a0i})\phi^{\dagger a}]. \tag{3.1.3.20}$$

Similarly,

$$\{\bar{Q}_{1\dot{\alpha}}, \bar{Q}_{2\dot{\beta}}\} = -\frac{2\sqrt{2}}{g^2}\epsilon_{\dot{\alpha}\dot{\beta}} \int d^3x \partial_i[(-iF^{a0i} + \tilde{F}^{a0i})\phi^a]. \tag{3.1.3.21}$$

Notice that from the anticommutators of the supercharges, we obtain total derivatives which corresponds to the electric and magnetic charge densities definitions given in eq(2.4.0.19) and eq(2.4.0.20):

$$Q_{ele} = -\frac{1}{ag}\int d^3x \partial_i(F^{a0i}\phi^a) = gn_e, \qquad Q_{mag} = -\frac{1}{ag}\int d^3x \partial_i(\tilde{F}^{a0i}\phi^a) = \frac{4\pi}{g}n_m, \tag{3.1.3.22}$$

where n_e and n_m are integer numbers for electric and magnetic charge quantization. Remember a is the expectation value of ϕ^a in the Higgs vacuum. The charge quantization condition corresponds to integral fundamental charges when the $SU(2)$ gauge group is broken to $U(1)$ with the fields in the adjoint representation.

If we replace (3.1.3.22) in (3.1.3.20) and (3.1.3.21) and from the relations in (3.1.1.2), we obtain.

$$Z = -ia(n_e + (4\pi i/g^2)n_m), \tag{3.1.3.23}$$

Z is the central charge for $N = 2$ supersymmetry. If we introduce the θ-term, then we can either write $\theta F\tilde{F}$ term since the beginning in the lagrangian or use the Witten effect by a shift in the coupling constant describe in sec.2.5. As we learned there, the effect of the θ parameter is a shift of the electric charge to $Q_{ele} = gn_e + (\theta g^2/8\pi^2)Q_{mag}$. We can obtain a new expression for the central charge.

$$Z = a(n_e + \tau_{cl}n_m), \qquad \text{where,} \qquad \tau_{cl} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \tag{3.1.3.24}$$

From eq (2.4.0.28)

$$M \geq \sqrt{2}|Z| = \sqrt{2}|a(n_e + \tau_{cl}n_m)|, \tag{3.1.3.25}$$

which is the BPS bound.

3.1.4 N=2 Superspace formalism

So far, we have construct a description of the lagrangian for $N = 2$ in the language of $N = 1$. We can write eq(3.1.4.6) manifest in $N = 2$ supersymmetry starting from the $N = 1$ superspace. First, we need to construct the superspace for $N = 2$ [21][22]. So we need to add two new fermionic degrees of freedom $\tilde{\theta}$ and $\bar{\tilde{\theta}}$ to the $N = 1$ superspace description. Now, our superfields in $N = 2$ in general will be written as: $F(x, \theta, \bar{\theta}, \tilde{\theta}, \bar{\tilde{\theta}})$.

A superfield corresponding to a manifest $N = 2$ lagrangian will be constrained by the conditions of chirality and reality for the superfield. Chirality condition for the superfield in $N = 2$ is written as:

$$\bar{D}_{\dot{\alpha}}\Psi = 0, \qquad \bar{\tilde{D}}_{\dot{\alpha}}\Psi = 0, \tag{3.1.4.1}$$

where $\bar{\tilde{D}}_{\dot{\alpha}}$ has the same form of $\bar{D}_{\dot{\alpha}}$, but we have exchanged θ (θ^1) by $\tilde{\theta}$ (θ^2). We can write the supersymmetric transformation on field components given in eqs (3.1.3.2) and eqs (3.1.3.7) in $N = 2$ superspace;

$$\begin{aligned} \delta A^{a\mu} &= i\epsilon^{pq}(\epsilon_p\sigma^\mu\bar{\psi}^a_q + \bar{\epsilon}\bar{\sigma}^\mu\psi^a_q), \\ \delta\phi^a &= \sqrt{2}\epsilon^{pq}\epsilon_p\psi^a_p, \\ \delta\psi^{qa}_\alpha &= i\sqrt{2}\bar{\epsilon}^{\dot{\beta}q}\sigma^\mu_{\alpha\dot{\beta}}D_\mu\phi^a - iF^{a\mu\nu}(\sigma^{\mu\nu}\epsilon_{pq}\epsilon_q)_\alpha, \end{aligned} \tag{3.1.4.2}$$

where ϵ_{pq} is an antisymmetric tensor, ϵ_p is the supersymmetric parameter for $N = 2$ supersymmetry. $a = 1, 2, 3$, $p, q = 1, 2$.

We expand Ψ in terms of the coordinates $\tilde{\theta}$ as a superchiral field:

$$\Psi = \Psi^{(1)}(\tilde{y}, \theta) + \sqrt{2}\tilde{\theta}^\alpha\Psi^{(2)}_\alpha(\tilde{y}, \theta) + \tilde{\theta}^\alpha\tilde{\theta}_\alpha\Psi^{(3)}(\tilde{y}, \theta), \tag{3.1.4.3}$$

where $\tilde{y} = x + i\theta\sigma\bar{\theta} + i\tilde{\theta}\sigma\bar{\tilde{\theta}}$. Using this expansion for y, we write again elements of $N = 2$ supersymmetry in terms of elements of $N = 1$ supersymmetry. Here $\Psi^{(1)}$ corresponds to a chiral superfield in $N = 1$ and $\Psi^{(2)}_\alpha$ and $\Psi^{(3)}$ are constrained by reality conditions. As a result $\Psi^{(2)}_\alpha$ corresponds to $W_\alpha(\tilde{y}, \theta)$, the supersymmetric field strength tensor for non-abelian gauge fields, which is in the $N = 1$ superspace. And $\Psi^{(3)}$ is given by:

$$\Psi^{(3)} = -\frac{1}{2}\int d^2\theta \exp\big[2gV(\varrho, \theta, \tilde{\theta})\big][\Phi(\varrho, \theta, \tilde{\theta})]^\dagger \exp\big[-2gV(\varrho, \theta, \tilde{\theta})\big], \tag{3.1.4.4}$$

where $\varrho = (\tilde{y} - i\theta\sigma\bar{\theta}$

We can verify that the $N = 2$ lagrangian in (3.1.2.3) is written in terms of $N = 2$ superfields in a more compact way as

$$\mathcal{L} = \frac{1}{4\pi}\mathbb{I}m\mathrm{Tr}\int d\theta^2 d\tilde{\theta}^2 \frac{1}{2}\tau\Psi^2. \tag{3.1.4.5}$$

Let us expand the lagrangian for $N = 2$ SYM in the $N = 1$ superspace:

$$\mathcal{L} = \frac{1}{4\pi}\mathbb{I}m\mathrm{Tr}\int d\theta^2 d\bar{\theta}^2 \frac{1}{2}\tau\Psi^2, \tag{3.1.4.6}$$

with the superfield expanding in the $N=1$ superspace:

$$\begin{aligned}\Psi &= \Phi(\tilde{y},\theta)+\sqrt{2}\tilde{\theta}^{\alpha}W_{\alpha}(\tilde{y},\theta)\\ &+ \tilde{\theta}^{\alpha}\tilde{\theta}_{\alpha}\big[\exp\big[2gV(\varrho,\theta,\tilde{\theta})\big]\Phi^{\dagger}(\varrho,\theta,\bar{\theta})\exp(-2gV(\varrho,\theta,\bar{\theta}))\big]|_{\bar{\theta}\bar{\theta}}\end{aligned} \quad (3.1.4.7)$$

We only keep terms on Ψ^2 which are proportional to $\tilde{\theta}^2$, we have:

$$\Psi^2|_{\tilde{\theta}^2} = -W^{\alpha}(\tilde{y},\theta)W_{\alpha}(\tilde{y},\theta)+\exp\big[2gV(\varrho,\theta,\tilde{\theta})\big]\Phi^{\dagger}(\varrho,\theta,\bar{\theta})\exp(-2gV(\varrho,\theta,\bar{\theta}))\Phi(\tilde{y},\theta) \quad (3.1.4.8)$$

So in the lagrangian in the $N=1$ language. We have

$$\begin{aligned}\mathcal{L} = \frac{1}{4\pi}\mathbb{I}m\mathrm{Tr}\int d\theta^2\tau\Big(W^{\alpha}W_{\alpha}(\tilde{y},\theta)+2\int d\bar{\theta}^2\exp\big[2gV(\varrho,\theta,\tilde{\theta})\big]\\ \Phi^{\dagger}(\varrho,\theta,\bar{\theta})\exp(-2gV(\varrho,\theta,\bar{\theta}))\Phi(\tilde{y},\theta)\Big)\end{aligned} \quad (3.1.4.9)$$

And we have the well known form in $N=1$ superspace

$$\mathcal{L} = \frac{1}{4\pi}\mathbb{I}m\mathrm{Tr}\tau\Big[\int d\theta^2 W^{\alpha}W_{\alpha}+2\int d^2\theta d^2\bar{\theta}e^{2gV}\Phi^{\dagger}e^{-2gV}\Phi\Big]. \quad (3.1.4.10)$$

Notice that the lagrangian (3.1.4.6) is expressed only by the superfield Ψ and $\Psi^{\dagger}$ does not appear, i.e. the lagrangian is a holomorphic function and we can replaced it by an holomorphic function of Ψ, $\mathcal{F}(\Psi)$. So the general lagrangian in $N=2$ pure SYM is:

$$\mathcal{L} = \frac{1}{4\pi}\mathbb{I}m\mathrm{Tr}\int d^2\theta d^2\bar{\theta}\frac{1}{2}\tau\mathcal{F}(\Psi). \quad (3.1.4.11)$$

Again, we can expand it in components of the $N=1$ superspace and we obtain:

$$\mathcal{L} = \frac{1}{8\pi}\mathbb{I}m\Big(\int d^2\theta\mathcal{F}_{ab}(\Phi)W^{\alpha a}W^{b}_{\alpha}+2\int d^2\theta d^2\bar{\theta}(e^{2gV}\Phi^{\dagger}e^{-2gV})^{a}\mathcal{F}_{a}(\Phi)\Big), \quad (3.1.4.12)$$

where $\mathcal{F}_{ab}(\Phi)=\partial^2\mathcal{F}/\partial\Phi^a\partial\Phi^b$ and $\mathcal{F}_a(\Phi)=\partial\mathcal{F}/\partial\Phi^a$. The general action for a gauge multiplet in $N=2$ SYM theory is:

$$\frac{1}{16\pi}\mathbb{I}m\int dx^4 d^2\theta d^2\bar{\theta}\mathcal{F}(\Psi). \quad (3.1.4.13)$$

The prepotential has a non-trivial form for a $N=2$ SYM lagrangian. The simplest case is

$$\mathcal{F}(\Psi)=\mathcal{F}_{class}(\Psi)=\frac{1}{2}\mathrm{Tr}(\tau\Psi^2), \quad (3.1.4.14)$$

the quadratic form of $\mathcal{F}$ on the superfield Ψ is fixed by renormalization, i.e. Ψ^2 contents at most terms like ϕ^4, $\bar{\psi}\psi$, etc. In their field components, that corresponds to well known renormalizable terms in QFT.

The action for the lagrangian in (3.1.4.12) can be expanded at low energies, we have:

$$\mathcal{F}_a(\Phi) = \mathcal{F}_{ab}(\Phi)\Phi^b + \cdots$$

So, for the interaction term with chiral and vector superfields

$$\int d^2\theta d^2\bar{\theta}\mathbb{I}m(e^{2gV}\Phi^\dagger e^{-2gV})^a \mathcal{F}_{ab}(\Phi)\Phi^b = \int d^2\theta d^2\bar{\theta}\mathbb{I}m\mathcal{F}_{ab}(\Phi)(e^{2gV}\Phi^\dagger e^{-2gV})^a\Phi^b. \tag{3.1.4.15}$$

Here $\mathcal{F}_{ab}(\Phi)$ corresponds to the metric on the space of fields

$$g_{ab} = \mathbb{I}m(\partial_a\partial_b\mathcal{F}(\Phi)). \tag{3.1.4.16}$$

If we want to write a low energy effective action, renormalizability is no a criterion and $\mathcal{F}$ has a more complicated form than $\mathcal{F}_{class}$.

3.1.5 Global symmetries in $N = 2$ theory

In four dimensions, supersymmetry gauge theories with an extended supersymmetric algebra have a global symmetry which corresponds to an unitary transformation among their supercharges. For an extended SUSY, the unitary global symmetry is $U(N)$ and we call it R-symmetry. For instance, in $N = 1$ SUSY, $U(1)$ is the global symmetry group and it acts on the supercharges as $Q \to e^{-i\alpha}Q$ and this transformation extends to the superspace with grassmannian coordinates $\theta \to e^{i\alpha}\theta$ and $\bar{\theta} \to e^{-i\alpha}\bar{\theta}$.

For $N = 2$ SUSY, the global symmetry is $U(2)$ and can be decomposed in $SU(2) \times U(1)$. So we have the $U(1)$ symmetry which acts on the superspace coordinates of $N = 2$, θ^p and $\bar{\theta}_q$ similarly to the $N = 1$ SUSY. To avoid latter confusions we denote to this symmetry sector as $U(1)_R$. The $SU(2)$ sector of the symmetry allows us to rotate the indices of the supercharges Q^i. In order to preserve the invariance of the supersymmetric lagrangian under R-transformation we need to assign an appropriated transformation for its respective superfields. With this information we can easily obtain the behavior of the component fields.
Below we describe how the multiplets of $N = 2$ behave under R-symmetry transformation.

Action on $N = 2$ vector multiplet

We have already described the field and multiplet content in the language of $N = 1$, see appendix. For the $N = 2$ vector multiplet, we have a vector multiplet and a chiral multiplet in $N = 1$, with the field content $V(A_\mu, \lambda)$ and $\Phi(\phi, \psi)$. We can arrange the field content as follow

$$\begin{array}{ccc} & A_\mu & \\ \lambda^\alpha & & \psi^\alpha \\ & \phi & \end{array} \tag{3.1.5.1}$$

, where the $SU(2)$ transformation acts on the rows. It allows us to exchange the Weyl spinors λ and ψ and keep the lagrangian invariant. In the language of $N = 1$ only one part of the $SU(2)$ symmetry is manifest. It does not mix up the fields λ and ψ. We will denote it as $U(1)_J$ a subgroup of $SU(2)$ and it is generates by the transformation $(\lambda, \psi) \to (e^{i\alpha}\lambda, e^{-i\alpha}\psi)$.

We can summarize the transformations as:

$$U(1)_R \;:\; Q \to Q(e^{-i\alpha}\theta), \qquad \tilde{Q} \to \tilde{Q}(e^{-i\alpha}\tilde{\theta}) \tag{3.1.5.2}$$

$$U(1)_J \;:\; Q \to e^{i\alpha}Q(e^{-i\alpha}\theta), \qquad \tilde{Q} \to e^{i\alpha}\tilde{Q}(e^{-i\alpha}\tilde{\theta}) \tag{3.1.5.3}$$

Action on the $N = 2$ hypermultiplet

Matter fields are described in the hypermultiplet for $N = 2$ SYM and its field content is: two complex scalar fields $(q, \tilde{q}^\dagger)$ and two Weyl spinors $(\psi, \tilde{\psi}^\dagger)$. All field components transforms under the fundamental representation of the gauge group. We can arrange the fields as:

$$\begin{array}{ccc} & \psi_q & \\ q & & \tilde{q}^\dagger. \\ & \tilde{\psi}_q^\dagger & \end{array} \tag{3.1.5.4}$$

The $SU(2)$ symmetry acts on the rows and it allows to rotate the complex scalar fields q and $\tilde{q}^\dagger$. In the language of $N = 1$, we have two chiral multiplets $Q(q, \psi_q)$ and $\tilde{Q}(\tilde{q}, \tilde{\psi}_q)$ and the reminding part of $SU(2)$ is $U(1)_J$. The unitary transformations for Q and $\tilde{Q}$ are:

$$U(1)_R \;:\; Q \to Q(e^{-i\alpha}\theta), \qquad \tilde{Q} \to \tilde{Q}(e^{-i\alpha}\tilde{\theta}). \tag{3.1.5.5}$$

$$U(1)_J \;:\; Q \to e^{i\alpha}Q(e^{-i\alpha}\theta), \qquad \tilde{Q} \to e^{i\alpha}\tilde{Q}(e^{-i\alpha}\tilde{\theta}) \tag{3.1.5.6}$$

For future convenience, we write all component fields transformation under $U(1)_R$ and $U(1)_J$

$$\begin{array}{rrcl} U(1)_R : & \phi & \to & e^{2i\alpha}\phi, \\ & (\psi, \lambda) & \to & e^{i\alpha}(\psi, \lambda), \\ & (\psi_q, \tilde{\psi}_q) & \to & e^{-i\alpha}(\psi_q, \tilde{\psi}_q), \\ & (A_\mu, q, \tilde{q}) & \to & (A_\mu, q, \tilde{q}), \end{array} \tag{3.1.5.7}$$

$$\begin{array}{rrcl} U(1)_J & : (\lambda, q) & \to & e^{i\alpha}(\lambda, q), \\ & (\psi, \tilde{q}^\dagger) & \to & e^{-i\alpha}(\psi, \tilde{q}^\dagger), \\ & (A_\mu, \phi, \psi_q, \tilde{\psi}_q^\dagger) & \to & (A_\mu, \phi, \psi_q, \tilde{\psi}_q^\dagger). \end{array} \tag{3.1.5.8}$$

We can combine the two spinorial components from the two chiral multiplets in a single one. $\psi_D = (\psi_q, \tilde{\psi}_q)$. It transforms as $\psi_D \to e^{i\alpha}\psi_D$ under $U(1)_J$ and $\psi_D \to e^{i\alpha\gamma_5}\psi_D$ under $U(1)_R$. It is the four component Dirac spinor. A Dirac spinor can be constructed from λ and $\bar{\psi}$ and transforms similarly to the ψ_D under $U(1)_j$ and $U(1)_R$.

3.1.6 Breaking R-symmetry

Classically, the theory has a global $SU(2) \times U(1)_R$ symmetry group. However, at the quantum level the $U(1)_R$ subgroup is broken because of anomalies. The remaining subgroup of $U(1)_R$, which is still a global symmetry determined by instanton contribution (we will see this in more detail on the next section). To compute the anomaly for the gauge group $SU(N_c)$ is though the index theorem, each left moving fermions on the fundamental or anti-fundamental representations contributes with 1 zero mode and each fermions in the adjoint representation contributes with $2N_c$ zero modes. The correlation function involves integration in the fermionic collective coordinates of their corresponding zero modes. For a non-zero correlation function we should have enough fermion insertions, for N_f flavors, fermions in the fundamental representation. We have the non-vanishing correlation function.

$$G = \langle \lambda(x_1) \cdots \lambda(x_{2N_c}) \psi(y_1) \cdots \psi(y_{2N_c}) \psi_q(z_1) \cdots \psi_q(z_{N_f}) \tilde{\psi}_q(u_1) \cdots \tilde{\psi}_q(u_{N_f}) \quad (3.1.6.1)$$

and G transforms under $U(1)_R$ as:

$$G \to e^{i\alpha(4N_c - 2N_f)} G. \quad (3.1.6.2)$$

So $U(1)_R$ is broken to the discrete group $\mathbb{Z}_{4N_c - 2N_f}$. We will restrict our analysis to a pure gauge theory. So $N_f = 0$, and the discrete group is $\mathbb{Z}_{4N_c}$. The global symmetry would be $SU(2) \times \mathbb{Z}_{4N_c}$. However, $SU(2)$ has an internal symmetry under the exchange $\phi^2 \to -\phi^2$, this corresponds to a $\mathbb{Z}_2$ also it rotates the fermionic fields in π as $(\lambda, \psi) \to e^{i\pi}(\lambda, \psi)$. Finally, we have $SU(2) \times \mathbb{Z}_{4N_c}/\mathbb{Z}_2$.

The remaining global symmetry is broken by the Higgs vacuum, then we see that ϕ^2 has four charges under $\mathbb{Z}_{4N_c}$

$$\phi^2 \to e^{2\pi i n/N_c} \phi^2, \quad (3.1.6.3)$$

invariant for the following values of n: $n = N_c, 2N_c, 3N_c, 4N_c$.

Since the vacuum is characterized by a non-zero ϕ^2 the discrete symmetry $\mathbb{Z}_{4N_c}$ is reduced to $\mathbb{Z}_4$. This work for $SU(2)$ gauge group which has $N_c = 2$. ϕ^2 is the only invariant. The final global symmetry group for a pure gauge $N = 2$ SYM lagrangian is

$$SU(2) \times \mathbb{Z}_4/\mathbb{Z}_2. \quad (3.1.6.4)$$

3.2 Low-energy effective action for $N = 2$ gauge theories

We are interested in analyze the properties of the theory $N = 2$ SYM $SU(2)$ pure gauge at low energies. Since we mentioned that this theory turns out to be solved exactly in a low energy regime.

So the behavior of the theory for low energies, lower that a certain cut off Λ which is lower than the characteristic energy scale. For instance, the mass of the lightest particle. We can use the Wilsonian approach to find the effective action with the lower cut off at low energy.

In principle, the theory at low energies can be obtained integrating over all massive and massless states about a cut off at low energy. Even when the procedure seems to be direct, this is extremal complicated and the procedure can not be applied explicitly [?]. At sufficiently low energies, none of the massive states appear as physical states and an effective description of the theory obtained from the massless states of the theory.

3.2.1 Semiclassical description of the moduli space in $SU(2)$

We need to determine the Wilsonian effective action by starting from a microscopic theory with the gauge group $SU(2)$ in $N = 2$ SYM.

The scalar potential of the theory has the form:

$$V = -\frac{1}{2g^2}\mathrm{Tr}([\phi^\dagger, \phi]^2). \tag{3.2.1.1}$$

In order to keep supersymmetry unbroken we require $V(\phi) = 0$ in the vacuum, but this still leaves the possibility to have a scalar expectation value different from zero, $\langle\phi\rangle \neq 0$ as well as the scalar field $\phi^\dagger$ and ϕ commute.

As we are interested in the inequivalent gauge vacuum states the general scalar field in the adjoint representation of $SU(2)$ has the form:

$$\phi(x) = \phi(x)^a\sigma^a = \frac{1}{2}\sum_{a=1}^{3}(a^a(x) + ib^a(x))\sigma^a. \tag{3.2.1.2}$$

It is a triplet in the adjoint representation of $SU(2)$. By a gauge transformation in $SU(2)$ we have $a_1(x) = a_2(x) = 0$ and because of $[\phi^\dagger, \phi] = 0$ then $b_1(x) = b_j(x) = 0$. We can define $a = a_3 + ib_3$ and obtain $\phi = \frac{1}{2}a\sigma_3$ and in the vacuum configuration a should be constant.

Under a gauge transformation we can rotate the axes 1 and 2 of $SU(2)$ in such a way that $a \to -a$ and they are gauge equivalent to $\frac{1}{2}a^2$ or $\mathrm{Tr}\phi^2$. This is true at semiclassical level. However, when we consider quantum fluctuation, we can not assure the gauge equivalence.

We define

$$u = \langle \mathrm{Tr}\phi^2 \rangle, \qquad \langle \phi \rangle = \frac{1}{2}a^2 \tag{3.2.1.3}$$

The complex parameter u labels the inequivalent vacuum. The manifold constructed by these inequivalent vacuums is called moduli space $\mathcal{M}$ of the theory, where u is a coordinate in the manifold $\mathcal{M}$. In other words $\mathcal{M}$ is the complex u-plane. We will see that there exist some singularities and the knowledge of the behavior of the theory near to those singularities allows to determine the effective action S_W.

For a expectation value of the scalar field different to zero the $SU(2)$ gauge symmetry is spontaneously broken by Higgs mechanism. From the kinetic term of ϕ , $|\nabla_\mu \phi|^2$, generates the mass for the gauge bosons. $A^a_\mu, a = 1, 2$ become massive. $\frac{1}{2}m^2 = \frac{1}{g^2}|ga|^2$, then $m = \sqrt{2}a$. Also, ψ^a, λ^a interaction terms, for $a = 1, 2$ become massive, while ϕ describes quantum fluctuations in the σ_3 directions stay massless. The massless modes are described by low energy effective action with $N = 2$ SUSY invariant, even when the gauge group is broken $N = 2$ SUSY remains.

Now, the gauge group is $U(1)$ and V is in the adjoint representation. We can expand $e^{-2gV} \cong 1 - 2gV + \cdots$, here only 1 contributes, then for the abelian case there is no self-coupling of the gauge bosons and for any of the fields in the $N = 2$ SYM vector multiplet. For this reason, they do not carry electric charge.

For the $U(1)$ gauge theory the lagrangian is simpler

$$\mathcal{L}^{U(1)}_{eff} = \frac{1}{4\pi}\mathbb{I}m\Big(\int d^4\theta \frac{\partial \mathcal{F}}{\partial \Phi}\Phi^\dagger + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial \Phi \partial \Phi} W_\alpha W^\alpha\Big) \tag{3.2.1.4}$$

3.2.2 Metric on the moduli space

The effective action of $N = 2$ SYM for $SU(2)$ gauge group.

$$S^{U(1)}_{eff} = \frac{1}{16\pi}\mathbb{I}m \int d^4x \big[\int d^2\theta \mathcal{F}''(\Phi) W^\alpha W_\alpha + \int d^2\theta d^2\bar{\theta} \Phi^\dagger \mathcal{F}'(\Phi)\big] \tag{3.2.2.1}$$

The second term in (3.2.2.1) can be expanded, the chiral superfield and $\mathcal{F}'$ in their field components and by using Taylor expansion respectively. So we have:

$$\Phi^\dagger \mathcal{F}'(\Phi)|_{\theta^2\bar{\theta}^2} = \mathcal{F}''(\phi)|\partial_\mu \phi| - i\mathcal{F}''(\phi)\psi\sigma^\mu \partial_\mu \bar{\psi} + \cdots \tag{3.2.2.2}$$

where the three points corresponds to non-derivative terms. We replace it to the effective action eq(3.2.2.1).

$$\frac{1}{4\pi}\mathbb{I}m \int d^4x \Big[\mathcal{F}''(\phi)|\partial_\mu \phi| - i\mathcal{F}''(\phi)\psi\sigma^\mu \partial_\mu \bar{\psi} + \cdots\Big] \tag{3.2.2.3}$$

It is enough to analyze the massless contribution for the fields, kinetic terms of the scalar and fermion fields.

For the first term in eq(3.2.2.1) we have by expanding in field components the term for the supersymmetric field strength tensor.

$$W^\alpha W_\alpha|_{\theta^2} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \cdots \tag{3.2.2.4}$$

So at the end we have:

$$\frac{1}{4\pi}\mathbf{Im}\int d^4x\Big[-\frac{1}{4}F_{\mu\nu}(F^{\mu\nu} - \tilde{F}^{\mu\nu}) - i\lambda\sigma^\mu\partial_\mu\bar{\lambda}\Big]. \tag{3.2.2.5}$$

Looking at the symmetric terms as part of a σ-model, $\mathcal{F}''(\Phi)$ appears on all of them. It will play the role of a sort of metric for the space of fields. Also, we are defining a metric in the space of inequivalent vacuum configuration i.e. the metric of the moduli space.

As a definition, the metric of the moduli space ($\bar{a}$ denotes the conjugated of a) we have:

$$ds^2 = \mathbf{Im}\mathcal{F}''(a)da d\bar{a}. \tag{3.2.2.6}$$

At classical level, we have the metric determined by $\mathbb{I}m(\tau)\mathrm{Tr}\Phi^2$. So $\mathbb{I}m\mathcal{F}(a) \sim \mathbb{I}m\tau$. And $\tau(a) = \mathcal{F}''(a)$ is the complexified coupling constant, i.e. the metric for the σ-model $g_{\phi^\dagger\phi} \sim \mathcal{F}''(\phi)$. We have replaced the metric for the moduli space $\mathcal{M}$ by the expectation value of ϕ in a point of the metric.

A suitable description of the effective action in terms of its fields Φ ($\Phi^\dagger$) and W^α, for all the values of u in the moduli space. In the case that the kinetic terms are obtained from the eq(3.2.2.3) and eq(3.2.2.5) the field component are defined positive in all $\mathcal{M}$. We can write it as $\mathbb{I}m\tau(a) > 0$. However, from the complex analysis i is shown that this can not be the case. Because $\mathcal{F}(a)$ is an holomorphic function, then $\mathcal{F}''(a)$ is also a holomorphic function and a harmonic function. Harmonic functions can not take a minimum value. And hence, in a compactified complex plane, it can not obey $\mathbb{I}m\tau(a) > 0$ everywhere, unless we are describing it classically, where τ takes a constant value. On the other hand, we need the positive condition for the metric. So we will allow $\mathbb{I}m\tau(a) > 0$ for local description the coordinate a, $\bar{a}$ and the function $\mathcal{F}(a)$ are appropriated only in certain region of $\mathcal{M}$.

When $\mathbb{I}m\tau(a)$ approaches to zero, we need to use a new set of coordinates $\bar{a}$ and $\mathbb{I}m\tau(a)$ are still non-singular. We can achieve this by providing singularities to the metric eq(3.2.2.6). It is only a coordinate singularity. Classically, the prepotential at low energies is given by $\mathcal{F}_{class} = \frac{1}{2}\tau_{class}\Phi^2$. The form of the full perturbative part of it was found by Seiberg (1988)[24]. For this is necessary to study the quantum corrections of the complex coupling constant $\tau(a)$. At this point, we make use of the beta function $\beta(g)$ [22]. We can use a general result from QTF of the beta function at one loop correction when we have a field content of scalar, fermions and gauge fields. It is given by:

$$\beta(g) = \frac{g^3}{16\pi^2}\Big[-\frac{11}{3}C_G + \frac{1}{6}N_S C_S + \frac{4}{3}N_F C_F\Big], \tag{3.2.2.7}$$

where N_s and N_F are the number of complex scalars and Dirac fermions respectively and the constant c depends on the representation of the fields.

$$\mathrm{Tr}[T^a, T^b] = c\delta^{ab}. \tag{3.2.2.8}$$

For the case of out interest, $N = 2$ SYM, because of the non-renormalization theorems 1-loop correction is enough to describe the whole perturbative quantum corrections. Also, we insert $c = N_c$ in the vector multiplet, that has two complex scalar fields and one Dirac spinor.

We obtain:

$$\beta_{1-loop}(g) = -\frac{2N_c g^3}{16\pi^2} \tag{3.2.2.9}$$

Since the beta function is defined by $\beta(g) = -\mu\frac{\partial g}{\partial\mu}$, where μ is the scale of energy.

In the other hand $\beta(g) = \mu\frac{\partial g}{\partial\mu}$. So we obtain:

$$\frac{4\pi}{e^2(M)} - \frac{4\pi}{g^2(Q)} = \frac{N_c}{\pi}\log\frac{Q}{M}. \tag{3.2.2.10}$$

On the limit of energies near to the cut off $Q \to \Lambda$, we have $g^2(Q) \to \infty$. Finally, from eq (3.2.2.10) we have:

$$\frac{4\pi}{e^2(M)} = \frac{N_c}{\pi}\ln\frac{\Lambda}{M} \Rightarrow \Lambda = Me^{-4\pi^2/N_c g^2(M)}. \tag{3.2.2.11}$$

From eq(3.1.4.14) the prepotential for $N = 2$ SYM, classically is giving by:

$$Im\mathcal{F}(\Psi) = \frac{1}{2}Im\tau_{class}\Psi^2 \tag{3.2.2.12}$$

Now, we have the imaginary part of the complex coupling constant τ_{class}

$$Im\tau_{class} = \frac{N_c}{2\pi}\ln\frac{\Phi^2}{\Lambda^2}. \tag{3.2.2.13}$$

And we can replace the metric of the moduli space given in eq(3.2.2.6) and the coupling constant as the second derivative of the prepotential. Also we have given an analysis at low energies following the description for low energy effective theory.

There, only the scalar fields contributes in the vacuum. So the dependence of $\mathcal{F}$ is only respect to ϕ in the vacuum. Now, we write the metric as:

$$ds^2 = \frac{\partial^2 K}{\partial a_i \partial \bar{a}_j} da_i d\bar{a}_j = \frac{\partial^2}{\partial a_i \partial \bar{a}_j}\big(Im\frac{\partial \mathcal{F}}{\partial a_k} d\bar{a}_k\big) da_i d\bar{a}_j = Im\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} da_i d\bar{a}_j \quad (3.2.2.14)$$

After the spontaneous symmetry breaking of the gauge group $SU(2) \rightarrow U(1)$, there is only one generator left, then we have:

$$ds^2 = Im\frac{\partial^2 \mathcal{F}}{\partial a^2} da d\bar{a}. \quad (3.2.2.15)$$

The complex coupling constant take the form $\tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2}$. So:

$$ds^2 = Im\tau(a) da d\bar{a} \quad (3.2.2.16)$$

We can analyze the behavior of the theory at large limit of a, it means, $a \gg \Lambda$. In a semi-classical description this limit corresponds to the weak coupling limit, the so called Asymptotic freedom. So the 1-loop correction if the preportential is given by:

$$\mathcal{F}_{1-loop} = \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2}. \quad (3.2.2.17)$$

We can also write the prepotential up to perturbative quantum corrections in terms of the superfield in $N = 2$, Ψ. So the combining tree level and the 1-loop results we have:

$$\mathcal{F}_{pert}(\Psi) = \frac{i}{2\pi} \Psi^2 \log \frac{\Psi^2}{\Lambda^2}. \quad (3.2.2.18)$$

This is a full perturbative result due to the non-renormalisable theorems. However, there are non-perturbative corrections which we are going to determine by Instanton contribution.

And we can obtain the expression for the coupling constant τ from eq3.2.2.17.

$$\begin{aligned}
\frac{\partial^2 \mathcal{F}}{\partial a^2} &= \frac{\partial}{\partial a}\big(\frac{i}{2\pi}(2a \ln a - 2a \ln \Lambda + a)\big), \\
&= \frac{1}{\pi}(2 \ln a - 2 \ln \Lambda + 3), \\
&= \frac{i}{\pi}(2 \ln |\frac{a}{\Lambda}| + 3) \quad (3.2.2.19)
\end{aligned}$$

replacing it in the metric, we have:

$$ds^2 = Im\tau(a) da d\bar{a} = \frac{1}{\pi}(2 \ln |\frac{a}{\Lambda}| + 3) da d\bar{a}. \quad (3.2.2.20)$$

However, the metric must be positive. In fact, the metric is positive for large $|a|$, which assure the logarithm be positive, but, for small values of $|a|$, it is $|a| \sim \frac{1}{5}\Lambda$, then the imaginary part of τ is less than zero. i.e. the metric given in eq(3.2.2.20) can not be the correct description for a moduli space for all the space of scalar fields ϕ.

3.2.3 Physical interpretation of singularities

All the information about the $N = 2$ SYM theory at low energies is content in the structure of the moduli space metric. It is important to analyze the behavior of the theory on the singular points of the moduli space. In these singular points the fields of the theory become massless. The structure of this singularity is called monodromy and this depends on the properties of the multiplet which becomes massless.

The theory without spontaneous symmetry breaking as well as the theory when the gauge group is broken share the elements of the moduli space denoting by u

$$u = \frac{1}{2} < Tr\phi^2 >\sim u_{cl} = \frac{1}{2}a^2, \qquad (3.2.3.1)$$

for $u \neq 0$, there is an additional gauge field which become massive. The coordinate u of the moduli space present quantum corrections in a perturbative or non-perturbative treatment. In a quantum correction, u is shifted from zero due to the fact that the quantum fluctuations of the theory can not allow to have $u = 0$ as a solution. Also, because of the Z_2 symmetry of the u-coordinate, the singularities appears in pairs around zero.

So the singularities should be caused by multiplet with spin less than 1/2. More precisely, they must be massive multiplets which becomes massless at those particular points of the moduli space. In our case, we are working on the $N = 2$ SUSY, where our options are strongly restricted. The only one multiplet in the extended supersymmetry $N = 2$ with spin lower than 1/2 is the Hypermultiplet. However, along our description we had not included them. We just consider the Vector multiplet. Then there are not elementary particles in the theory that satisfy the condition of the spin. We have to look for composite objects and it turns out that those objects do exist.

We consider $N = 2$ theory with Hypermultiplets, heavy solitonic objects, which carries electric and magnetic charge called dyons. These are heavy at the weak coupling limit and do not play a role in the low energy effective theory. However, Seiberg and Witten showed that the singularities that appears in $N = 2$ theory due to these dyons become massless on specific points of the moduli space.

3.2.4 Duality

Let us consider the gauge field terms in the full $N = 2$ supersymmetry action including the θ-term, then we have the lagrangian:

$$\mathcal{L}_{A_\mu} = Im\frac{\tau}{32\pi}(F + i\tilde{F})^2, \tag{3.2.4.1}$$

by expanding the complex coupling constant in terms of the gauge coupling constant and the theta angle. $\tau = 4\pi i/g^2 + \theta/2\pi$, then the lagrangian expands as:

$$-Im\frac{\tau}{32\pi}(F + i\tilde{F})^2 = -\frac{1}{16\pi}Im\tau(F^2 + iF\tilde{F}) = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{3.2.4.2}$$

These are the gauge field kinetic term and the theta term. This time, we will not use A_μ as the independent variable as usual, but $F_{\mu\nu}$ as our independent variable. Then, we impose the Bianchi identity $d^*F = 0$ as a constraint of the action. which leads us to introduce a Lagrange multiplier to the action. The Lagrange multiplier we introduce is the abelian vector B_μ with the field strength tensor $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The lagrange multiplier is constructed by coupling the vector field to a magnetic monopole. It satisfy the constraint $\epsilon^{0\rho\mu\nu}\partial_\rho F_{\mu\nu} = 8\pi\delta^{(3)}$ and the Lagrange multiplier term is written:

$$\begin{aligned}\frac{1}{8\pi}\int B_\mu\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} &= -\frac{1}{8\pi}\int \epsilon^{\mu\nu\rho\sigma}\partial_\nu B_\mu F_{\rho\sigma} + \text{total derivative},\\ &= \frac{1}{16\pi}\int \epsilon^{\mu\nu\rho\sigma}(\partial_\mu B_\nu - \partial_\nu B_\mu)F_{\rho\sigma},\\ &= \frac{1}{16\pi}\int \epsilon^{\mu\nu\rho\sigma}G_{\mu\nu}F_{\rho\sigma} = \frac{1}{8\pi}\int \tilde{G}F,\end{aligned} \tag{3.2.4.3}$$

To go from the first to the second line we have used the antisymmetric property of $\epsilon^{\mu\nu\rho\sigma}$ on the μ and ν indices. Also the form of the lagrangian in eq(3.2.4.2) can be expresed in terms of the lagrangian multiplier as

$$\frac{1}{8\pi}\int \tilde{G}F = \frac{1}{16\pi}\int (G + i\tilde{G})(F + i\tilde{F}). \tag{3.2.4.4}$$

If we add the Lagrange multiplier to the action eq(3.2.4.2) and completing squares, we obtain:

$$\begin{aligned}&\tfrac{1}{32\pi}\; Im\int \big(-\tau(F + i\tilde{F})^2 + 2(G + i\tilde{G})(F + i\tilde{F})\big)\\ &= -\frac{1}{32\pi}Im\int \big((\sqrt{\tau}(F + i\tilde{F}) - \frac{1}{\sqrt{\tau}}(G + i\tilde{G}))^2 - \frac{1}{\tau}(G + i\tilde{G})\big)\\ &= \frac{1}{32\pi}Im\frac{1}{\tau}\int (G + i\tilde{G})^2.\end{aligned} \tag{3.2.4.5}$$

The last line is obtained by take the Gaussian path integral over F. As a result, the quadratic term is canceled. Thus we have found a dual theory completely in terms of the Lagrange multiplier gauge field and it has the same functional form of eq(3.2.4.2), but with a coupling constant $-\frac{1}{\tau}$.

Consider the effective action in eq(3.2.2.1). We define the dual to Φ as:

$$\Phi_D = \mathcal{F}' \tag{3.2.4.6}$$

and the function $\mathcal{F}_D(\Phi_D)$ dual to $\mathcal{F}(\Phi)$ by

$$\mathcal{F}'_D(\Phi_D) = -\Phi, \tag{3.2.4.7}$$

where $\mathcal{F}'_D(\Phi_D)$ means $d\mathcal{F}_D/d\Phi_D$. This duality transformation constitutes a Lagrange transformation, with the constraint $\mathcal{F}_D(\Phi_D) = \mathcal{F}(\Phi) - \Phi\Phi_D$ from eq(3.2.4.6) and eq(3.2.4.7).

Following this construction of dual fields, the second term of the action eq(3.2.2.1) can be written as

$$\begin{aligned} Im\int d^4xd^2\theta d^2\bar{\theta}\Phi^\dagger\mathcal{F}'(\Phi) &= Im\int d^4xd^2\theta d^2\bar{\theta}(-\mathcal{F}'_D(\Phi_D))^\dagger\Phi_D, \\ &= Im\int d^4xd^2\theta d^2\bar{\theta}\Phi^\dagger_D\mathcal{F}'_D(\Phi_D). \end{aligned} \tag{3.2.4.8}$$

We have probed that the functional form of the second term in the action eq(3.2.2.1) is invariant under duality transformation defined on eq(3.2.4.6) and eq(3.2.4.7).

Next, we consider the first term in the action q(3.2.2.1). The duality transformation is local on Φ. However, it is not the case for the transformation of W_α. Recall that W_α behaves as the supersymmetric field strength tensor and it contains the $U(1)$ which is $F_{\mu\nu}$. We can translate this as the Bianchi identity $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = \partial_\nu\tilde{F}^{\mu\nu} = 0$. It corresponds to the constraint on superfields as $Im(D_\alpha W^\alpha) = 0$, where D_α is a derivative in the superspace.

In the functional integral we can integrate over V only, or over W^α by imposing the constraint $Im(D_\alpha W^\alpha) = 0$ with a real Lagrange multiplier called V_D

$$\begin{aligned} &\int \ DV\exp\big[\frac{i}{16\pi}Im\int d^4xd^2\theta d^2\bar{\theta}\mathcal{F}''(\Phi)W^\alpha W_\alpha\big], \\ \cong &\int DWDV_D\exp\big[\frac{i}{16\pi}Im\int d^4x\big(d^2\theta\mathcal{F}''(\Phi)W^\alpha W_\alpha + \frac{1}{2}\int d^2\theta d^2\bar{\theta}V_D D_\alpha W^\alpha\big)\big] \end{aligned} \tag{3.2.4.9}$$

The last term can be modify as

$$\begin{aligned} \int d^2\theta d^2\bar{\theta}V_D D_\alpha W^\alpha &= -\int d^2\theta d^2\bar{\theta}D_\alpha V_D W^\alpha = +\int d^2\theta\bar{D}^2(D_\alpha V_D W^\alpha), \\ &= \int d^2\theta(\bar{D}^2 D_\alpha V_D)W^\alpha = -4\int d^2\theta(W_D)_\alpha W^\alpha \end{aligned} \tag{3.2.4.10}$$

where we have used $\bar{D}_{\dot{\beta}}W^{\alpha} = 0$ and the dual W_D is defined from V_D in analogy to the abelian version of W in terms of V as $W_D = -\frac{1}{4}\bar{D}^2 D_\alpha V_D$. After doing functional integral over W, we obtain:

$$\int DV_D \exp\big[\frac{i}{16\pi} Im \int d^4x d^2\theta\big(-\frac{1}{\mathcal{F}''(\Phi)} W_D^\alpha W_{D\alpha}\big)\big]. \qquad (3.2.4.11)$$

We have re-expressed the $N = 1$ supersymmetric Yang-Mills action in terms of the dual Yang-Mills action with effective coupling $\tau(a) = \mathcal{F}''(a)$ replaced by $-\frac{1}{\tau(a)}$. But $\tau(a) = \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g^2(a)}$, so $\tau \to -\frac{1}{\tau}$. It is the generalization of the coupling constant. Also, we can see that W_D describes the $F_{\mu\nu}$ dual, $\tilde{F}_{\mu\nu}$. All this manipulation on eq(3.2.4.11) constitutes a dual transformation that generalizes the old electromagnetic duality discussed by Montonen and Olive. By expressing $-\frac{1}{\mathcal{F}''(\Phi)}$ in terms of Φ_D one can see that $\mathcal{F}_D''(\Phi_D) = -\frac{d\Phi}{d\Phi_D} = -\frac{1}{\mathcal{F}''}$. then we have:

$$-\frac{1}{\tau(a)} = \tau_D(a_D). \qquad (3.2.4.12)$$

The whole action eq(3.2.2.1) can be written as

$$\frac{1}{16\pi}\mathbb{I}m \int d^4x \big[\int d^2\theta \mathcal{F}_D''(\Phi_D) W_D^\alpha W_{D\alpha} + \int d^2\theta d^2\bar{\theta} \Phi_D^\dagger \mathcal{F}_D'(\Phi_D)\big] \qquad (3.2.4.13)$$

Chapter 4

Conclusions

We were able to obtain a classical description of the electric-magnetic duality. However, as it was pointed by Montonen and Olive, there are still some problems when we want to extend this duality to a quantum treatment.

To make a good quantum treatment of the theory which preserves the electric-magnetic duality we introduce supersymmetry. It turns out that in $N = 2$ susy we can obtain a non-trivial situation which is possible to compute the solution for the low energy limit. This solution was found by Seiberg and Witten in 1994.

Looking for the correct structure of the moduli space in a quantum treatment of the $N = 2$ vector multiplet. Because of the holomorphism of the Lagrangian, the properties of the moduli space is strongly restricted.

We also showed in section3.1.3 that, the central charges of the $N = 2$ supersymmetry respect the Bogomolny bound which we described on section2.4.

The next step following the analysis of Seiberg and Witten, is the description and study of the behavior of the moduli space around the singularities, we can obtain it by the use of a powerful mathematical tool, the monodromies. With these information, we obtain diferential equations that describe the prepotential at low energies and in order to solve the equations we introduce the elliptic curves solving the theory in an exact way.

The progress on the understanding of the non-perturbative properties of the supersymmetric field and string theories was developed by Seiberg and Witten for $N = 2$ SYM theory. Similarly, Hull and Townsend worked on a duality description for the Heterotic and Type II string theories and they found a string equivalence. Aslo we can work on other extentions of the original SW duality developed in 1994. Argyres and Douglas 1995 study the case for $SU(2)$ as gauge group. In 2000 Davide Gaiotto generalized it to the $SU(N)$ case.

Appendix A

A.1 Conventions

We are going to work with flat metric $\eta = diag(-1,1,1,1)$ and the spinorial representation of the Lorentz group is $SL(2,\mathbb{Z}) \sim SU(2)_L \times SU(2)_R$, for each $SU(2)$ we are going to denote undotted elements for the left(L) and dotted elements for the right(R). They transform in the following way.

$$\psi'_\alpha = M_\alpha^\beta \psi_\alpha, \qquad \bar{\psi}'_{\dot{\alpha}} = M^{*\dot{\beta}}_{\ \dot{\alpha}} \bar{\psi}_{\dot{\beta}}, \tag{A.1.0.1}$$

those spinor indices are raised and lowered by using the following antisymmetric tensor $\epsilon_{\alpha\beta}$

$$\epsilon^{\dot{\alpha}\dot{\beta}} = \epsilon^{\alpha\beta} = \imath\sigma_2. \tag{A.1.0.2}$$

And this tensor is invariant under $SL(2,\mathbb{Z})$ transformation: $\epsilon^{\alpha\beta} = M_\gamma^\alpha \epsilon^{\gamma\delta} M_\delta^\beta$. This can be written in a more compact way by: $M^T\sigma_2 M = \sigma_2$. It also implies $\sigma_2 M = (M^T)^{-1}\sigma_2$. Using this we can write a transformation of the spinor with raised indices as

$$\psi'^\alpha = \psi^\beta (M^{-1})_\beta^\alpha, \qquad \bar{\psi}'^{\dot{\alpha}} = \bar{\psi}^{\dot{\beta}} (M^*)^{-1\dot{\alpha}}_{\dot{\beta}}. \tag{A.1.0.3}$$

Now, let us define:

$$(\sigma^\mu)_{\alpha\dot{\alpha}} \equiv (1, \vec{\sigma}), \tag{A.1.0.4}$$

where $\sigma^\imath$ are the Pauli matrices for $i = 1,2,3$. When we multiplied this Pauli matrices to the four-momentum P_μ we have

$$\sigma^\mu P_\mu = \begin{pmatrix} P_0 - P_3 & P_1 - \imath P_2 \\ P_1 + \imath P_2 & P_0 + P_3 \end{pmatrix}. \tag{A.1.0.5}$$

We see that the determinant $det(\sigma^\mu P_\mu) = P_\mu P^\mu$ which is $P^2 = m^2$. Also, we can raise the indices of σ^μ with the ϵ-tensor as

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = -(\alpha^\mu)^{\alpha\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}}. \tag{A.1.0.6}$$

Some extra properties for the sigma matrices can be obtained by defining

$$(\sigma^{\mu\nu})_\alpha^\beta = \frac{1}{4}[\sigma^\mu_{\alpha\dot\beta}\bar\sigma^{\nu\dot\beta\beta} - (\mu \leftrightarrow \nu)], \tag{A.1.0.7}$$

$$(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\dot\beta} = \frac{1}{4}[\bar\sigma^{\mu\dot\alpha\beta}\sigma^\nu_{\beta\dot\beta} - (\mu \leftrightarrow \nu)]. \tag{A.1.0.8}$$

Another useful property is the following

$$\sigma^i\sigma^j = \delta^{ij} + \imath\epsilon^{ijk}\sigma^k. \tag{A.1.0.9}$$

The scalar product for spinors, we use the following conventions,

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi, \tag{A.1.0.10}$$

$$\bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = \bar\chi\bar\psi, \tag{A.1.0.11}$$

$$(\psi\chi)^\dagger\bar\chi_{\dot\alpha}\bar\psi^\alpha = \bar\chi\bar\psi = \bar\psi\bar\chi. \tag{A.1.0.12}$$

We list here some more spinor properties

$$\chi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\chi, \tag{A.1.0.13}$$

$$(\chi\sigma^\mu\bar\psi)^\dagger = \psi\sigma^\mu\bar\chi, \tag{A.1.0.14}$$

$$\chi\sigma^\mu\bar\sigma^\nu\psi = \psi\sigma^\nu\bar\sigma^\mu\chi, \tag{A.1.0.15}$$

$$(\chi\sigma^\mu\bar\sigma^\nu\psi)^\dagger\bar\psi\bar\sigma^\nu\sigma^\mu\bar\chi. \tag{A.1.0.16}$$

In the above basis, the Dirac matrices and Dirac and Majorana spinors are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \qquad \psi_D = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}, \qquad \psi_M = \begin{pmatrix} \psi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} \tag{A.1.0.17}$$

As usual, one defines

$$\gamma_5 \sim \imath\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.1.0.18}$$

Consider a massless fermion "moving" in z-direction. So we can define a direction for the four-momentum $P^\mu = E(1,0,0,-1)$. From the Dirac equation we have $(\gamma^0 - \gamma^3)\psi = 0$. The helicity is determined by the γ^5 with eigenvalues ± 1.

Other useful properties for grassmanian variables are:

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta, \qquad \theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta, \tag{A.1.0.19}$$

$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = \frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta, \qquad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta, \tag{A.1.0.20}$$

$$\theta^{\alpha}\bar{\theta}^{\dot{\beta}}\frac{1}{2}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\theta\sigma_{\mu}\bar{\theta}, \tag{A.1.0.21}$$

$$\theta\sigma^{\mu}\bar{\theta}\theta\sigma^{\nu}\bar{\theta} = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}g^{\mu\nu}, \tag{A.1.0.22}$$

$$\theta\psi\theta\chi = -\frac{1}{2}\psi\chi\theta\theta, \qquad \bar{\theta}\bar{\psi}\bar{\theta}\bar{\chi} = -\frac{1}{2}\bar{\theta}\bar{\theta}\bar{\psi}\bar{\chi}, \tag{A.1.0.23}$$

$$\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^{\beta}} = -\frac{\partial}{\partial\theta_{\alpha}}, \tag{A.1.0.24}$$

$$Tr(\sigma^{\mu}\bar{\sigma}^{\nu}) = 2g^{\mu\nu}, \tag{A.1.0.25}$$

$$Tr(\sigma^{\mu}\bar{\sigma}^{\nu}\sigma^{\rho}\bar{\sigma}^{\sigma}) = 2(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho} + \imath\epsilon^{\mu\nu\rho\sigma}). \tag{A.1.0.26}$$

A.2 Supersymmetric algebra

Besides the commutators from the Poincare algebra:

$$\begin{aligned}
[P_{\mu}, P_{\nu}] &= 0, && \text{(A.2.0.27)}\\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma}, && \text{(A.2.0.28)}\\
[M_{\mu\nu}, P_{\rho}] &= -i\eta_{\rho\mu}P_{\nu} + i\eta_{\rho\nu}P_{\mu}. && \text{(A.2.0.29)}
\end{aligned}$$

We also have the following relations for the generators of the supersymmetry algebra:

$$\begin{aligned}
[P_{\mu}, Q^{I}_{\alpha}] &= 0, && \text{(A.2.0.30)}\\
[P_{\mu}, \bar{Q}_{\dot{\alpha}I}] &= 0, && \text{(A.2.0.31)}\\
[M_{\mu\nu}, Q^{I}_{\alpha}] &= i(\sigma_{\mu\nu})^{\beta}_{\alpha}, && \text{(A.2.0.32)}\\
[M_{\mu\nu}, \bar{Q}_{\dot{\alpha}I}] &= i(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}}, && \text{(A.2.0.33)}\\
\{Q^{I}_{\alpha}, \bar{Q}_{\dot{\alpha}J}\} &= 2\sigma^{\mu}_{\alpha\dot{\beta}}P_{\mu}\delta^{I}_{J}, && \text{(A.2.0.34)}\\
\{Q^{I}_{\alpha}, Q^{J}_{\beta}\} &= 2\sqrt{2}\epsilon_{\alpha\beta}Z^{IJ}, \qquad Z^{IJ} = -Z^{JI}, && \text{(A.2.0.35)}\\
\{\bar{Q}_{\dot{\alpha}I}, \bar{Q}_{\dot{\beta}J}\} &= 2\sqrt{2}\epsilon_{\dot{\alpha}\dot{\beta}}(Z_{IJ})^{*}. && \text{(A.2.0.36)}
\end{aligned}$$

Where P_{μ} and $M_{\mu\nu}$ are the generators of the Poincare algebra. Four-momentum and Lorentz generators respectively. Also $Q's$ and $\bar{Q}'s$ are the supersymmetry generators. The indices I and J correspond to the number of supersymmetry $I, J = 1, \cdots, N$. And $\alpha = 1, 2$ and $\dot{\alpha} = \dot{1}, \dot{2}$ correspond to the spinorial indices. $\sigma^{\mu}_{\alpha\dot{\beta}}$ are defined as $(\mathbf{1}, \vec{\sigma})$. The operators Z^{IJ} and Z^{*}_{IJ} are the central charges.

We will see that it is enough to work with the supersymmetric algebra without central charges for the massless representations, while for the massive representations which we are going to define later as extended supersymmetry the concept of central charges will be useful.

A.3 Representations of the supersymmetry algebra

We need a short review of the representations of the Poincare algebra before to work out on the supersymmetry algebra properly. The Poincare algebra has 10 generators: four generators for translations P_μ and six generators from the Lorentz algebra $M_{\mu\nu}$ (three generators correspond to rotations and the other three correspond to the boosts).

The Poincare algebra has two Casimirs, $P^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$, P_μ the four-momentum and $W_\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$, W^μ is the Pauli-Lubanski vector. P^2 and W^2 commute with all the generators of the algebra (i.e. they are the Casimirs of the Poincare algebra).

The Casimir operators are useful to classify irreducible representations of a group. For the case of the Poincare algebra, these representations are nothing but the often called "particles". The first distinction of representation are the massive and massless representations of the Poincare algebra.

Let us consider first a massive representation. In the rest frame the four momentum $P_\mu = (m, 0, 0, 0)$, m is the mass. Also, $P^2 = m^2$ and for the Pauli-Lubanski vector W^μ. The only possibility for $\nu = 0$, with this $\epsilon^{\mu 0\rho\sigma} \to \epsilon^{0ijk} = \epsilon^{ijk}$, for $i, j, k = 1, 2, 3$. So $W^i = \frac{m}{2}\epsilon^{ijk} M_{jk}$

$$\begin{aligned} W^i = -\frac{1}{2}\epsilon^{0ijk} m M_{jk} \quad &= \quad -\frac{m}{2}\epsilon^{ijk} M_{jk}. \\ \Rightarrow \quad & W^2 = W_i W^i = \frac{m^2}{4}\epsilon^{ijk}\epsilon_{ilm} M_{jk} M^{lm}, \\ & W^2 = \frac{m}{2}(\delta^j_l \delta^k_m - \delta^j_m \delta^k_l) M_{jk} M^{lm}, \\ & = \frac{m^2}{2} M_{lk} M^{lk}. \end{aligned} \tag{A.3.0.37}$$

Also, $M_{ij} = \epsilon_{ijk} J_k$. Then $W^2 = m^2 J_k J^k = m^2 j(j+1)$, where j is the eingenvalue of $\hat{J}$ and corresponds to the spin.

Also, we notice that W_μ is orthogonal to P_μ.

$$P_\mu W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu P_\mu M_{\rho\sigma} \tag{A.3.0.38}$$

but $P_\mu P_\nu$ is symmetric in their indices and ϵ is antisymmetric in all its indices. So the product PW is zero. In the rest frame, $W_0 = 0$ to guarantee its orthogonality with P_μ, $W_i = (0, -\frac{m}{2}\epsilon_{ijk} M^{jk})$. Now, conclude that massive particles can be distinguishable by its mass m and its spin j.

Let us consider the massless representation of the Poincare algebra. We have $P^2 = 0$ and $W^2 = 0$. In the rest frame, we have the four momentum as $P_\mu = (E, 0, 0, E)$, where E is the energy of the massless particle. It implies $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$, $P_\nu = 0$ for $\nu = 1, 2$. In order to obtain $W^2 = 0$ we need the time component of W^μ be different to zero. So we have $W^0 = \frac{1}{2}\epsilon^{03ij}P_3 M_{ij}$, here $i.j = 1, 2$. Finally, we have $W^\mu = M_{12}P^\mu$, i.e. for massless particles P^μ and W^μ are proportional and the proportional constant is M_{12} which corresponds to the helicity j. $M_{12} = \pm j$.

The states of a massless particle are distinguishable by its energy and its helicity. Also the spin is fixed (i.e. The photon is massless with spin ± 1 and helicity ± 1).

Now, as we have called particles to the irreducible representations of the Poincare algebra, we are going to call superparticles to the irreducible representations of the supersymmetry algebra. Since the Poincare algebra is a subalgebra of the supersymmetry algebra. We say that any irreducible representation of the supersymmetry algebra is a representation of the Poincare algebra, which in general are reducible representations of it. This leads us to the following conclusion that a superparticle corresponds to a collection of particles and they are related by the action of the supersymmetry generators on these particles (Q^I_α and $\bar{Q}^I_{\dot{\alpha}}$ supersymmetry generators). The superparticle is often called as supermultiplet and the spin of its particles diferes by units of one-half.

It is important to remark some details of the representation of the supersymmetry algebra. We will see that these properties have important physical implications.

1. While the Casimirs in the Poincare algebra are P^2 and W^2 for the case of the supersymmetry algebra only P^2 is still a Casimir. W^2 it is not a Casimir anymore because of its commutator with the supersymmetry algebra generators. i.e. the particles that belong to the same multiplete share the same value of mass but their spins are different.

Since the spins of the particles of the multiplet differ by factor of 1/2, the degeneracy of mass for bosons and fermions is something that we have not seen in any known particle spectrum. This implies that supersymmetry must break in nature.

2. A supermultiplet has the same number of d.o.f (degrees of freedom) for bosons and fermions, $n_F = n_B$. The fermion number operator is defined by:

$$(-1)^{N_F} = \begin{cases} -1 & \text{, fermionic state} \\ +1 & \text{, bosonic state} \end{cases} \tag{A.3.0.39}$$

N_F can be taken as twice the spin $N_F = 2s$.

So we have:

$$(-1)^{N_F}|B\rangle = |B\rangle, \qquad (-1)^{N_F}|F\rangle = -|F\rangle \tag{A.3.0.40}$$

We can show that $Tr(-1)^{N_F} = 0$. Taken the trace on a finite dimensional representation of the supersymmetry algebra.

$$\{Q^I_\alpha, (-1)^{N_F}\} = 0 \longrightarrow Q^I_\alpha(-1)^{N_F} = -(-1)^{N_F} Q^I_\alpha \tag{A.3.0.41}$$

multiply this by $\bar{Q}_{\dot\beta J}$ in the right hand side and taken the trace.

$$0 = Tr(Q^I_\alpha(-1)^{N_F}\bar{Q}_{\dot\beta J} + (-1)^{N_F} Q^I_\alpha \bar{Q}_{\dot\beta J}), \tag{A.3.0.42}$$

by the cyclic property of the trace.

$$0 = Tr((-1)^{N_F}\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\}) = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^I_J Tr(-1^{N_F}). \tag{A.3.0.43}$$

We can choose a frame where $P_\mu \neq 0$, necessarily $Tr(-1)^{N_F}$.

Since P^2 is still a Casimir of the supersymmetry algebra we can classify massive and massless supermultiplets as representations in the supersymmetry algebra.

A.3.1 Massless representation

Let us assume that there is not central charge, i.e. $Z^{IJ} = 0$. For the case of massless representation this is the only relevant case. Let us construct now the irreducible representations.

1. In the rest reference frame $P_\mu = (E, 0, 0, E)$. the supersymmetry algebra obeys.

$$\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^I_J. \tag{A.3.1.1}$$

but

$$\sigma^\mu P_\mu = \begin{pmatrix} -P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & -P_0 - P_3 \end{pmatrix} = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}. \tag{A.3.1.2}$$

So:

$$\begin{pmatrix} \{Q^I_1, \bar{Q}_{\dot 1 J}\} & 0 \\ 0 & \{Q^I_2, \bar{Q}_{\dot 2 J}\} \end{pmatrix} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix} \delta^I_J. \tag{A.3.1.3}$$

And because of positivity of the Hilbert space

$$\langle\psi|\{Q^I_2, \bar{Q}_{\dot 2 J}\}|\psi\rangle = \|Q^I_2|\psi\rangle\|^2 + \|\bar{Q}_{\dot 2 J}|\psi\rangle\|^2 = 0 \tag{A.3.1.4}$$

then Q^I_2 and $\bar{Q}_{\dot 2 J}$ are found trivialy equal to zero.

2. For $\{Q^I_1, \bar{Q}_{\dot 1 J}\} = 4E\delta^I_J$ non-trivial, we normalize generators as:

$$a_I = \frac{1}{\sqrt{4E}} Q_{1I}, \qquad a^\dagger_I = \frac{1}{\sqrt{4E}} \bar{Q}^\dagger_{\dot 1 I}, \tag{A.3.1.5}$$

where the a_I and $a^\dagger_I$ operators satisfy the anticommutation relations:

$$\{a_I, a_J\} = 0, \qquad \{a^\dagger_I, a^\dagger_J\} = 0, \qquad \{a_I, a^\dagger_J\}. \tag{A.3.1.6}$$

As we said before when the operators a and $a^\dagger$ (constructed from the non-trivial generators of the supersymmetry algebra) act on the states low and rise the helicity in units of 1/2 respectively .

$$\begin{aligned}[M_{12}, Q_1^I] &= i(\sigma_{12})_1^1 Q_1^I = -\frac{1}{2}Q_1^I,\\ [M_{12}, \bar{Q}_{\dot{1}I}] &= \frac{1}{2}\bar{Q}_{\dot{1}I}.\end{aligned} \tag{A.3.1.7}$$

3. In order to construct a representation, we start by choosing a state which is annihilated by all generators $a_I's$. This state has $m = 0$ and helicity λ_0 and is written as $|E, \lambda_0\rangle$. We write it shortly as $|\lambda_0\rangle$

$$a_I|\lambda_0\rangle = 0. \tag{A.3.1.8}$$

The state $|\lambda_0\rangle$ is called Clifford vacuum.

This state can be bosonic or fermionic and it is not the vacuum of the theory (as state of minimal energy).

4. The complete representation is obtained by the action of creation operators $a_I^\dagger$ on $|\lambda_0\rangle$.

$$\begin{aligned}&|\lambda_0\rangle, \qquad a_I^\dagger|\lambda_0\rangle = |\lambda_0 + 1/2\rangle_I, \qquad a_I^\dagger a_J^\dagger|\lambda_0\rangle = |\lambda_0 + 1\rangle_{IJ},\\ &\qquad\qquad \cdots a_1^\dagger a_2^\dagger \cdots a_N^\dagger|\lambda_0\rangle = |\lambda_0 + N/2\rangle.\end{aligned} \tag{A.3.1.9}$$

The state with larger helicity in the massless representation is $\lambda = \lambda_0 = N/2$. Because of the antisymmetry of $I \leftrightarrow J$. The number of states with helicity $\lambda_0 + k/2$ is counted by the combinatory C_k^N with $k = 0, 1, 2 \cdots N$.

5. The total number of states of the representation is the sum of all the combinatories for different values of k:

$$\sum_{k=0}^{N} C_k^N = 2^N = (2^{N-1})_B + (2^{N-1})_F, \tag{A.3.1.10}$$

where we have separate the number of bosonic and fermionic states.

6. A CPT transformation flips the sign of the helicity unless, the helicity is symmetrically distributed around 0, but it is not the most general case. The supermultiplet is not CPT invariant. If we want a CPT invariant theory, we must double the supersymmetry by adding its CPT conjugate. Tis procedure is not needed for a CPT self-conjugate. Only if $\lambda_0 = -N/4$ then the helicity is symmetrically distributed.

Now we are going to follow the procedure to construct different irreducible representations of the supersymmetry algebra. We can distinguish the different cases for its number of supersymmetry.

N=1 Supersymmetry

We have shown how to construct the particle spectrum in a supermultiplet from the Clifford vacuum state and how the creation operators $a_I^\dagger$ act on the vacuum state

$$|\lambda_0\rangle, \quad a_I^\dagger|\lambda_0\rangle = |\lambda_0 + 1/2\rangle_I. \tag{A.3.1.11}$$

Then we have two possible states for the helicity λ_0 (0 and 1/2). For :
$\lambda_0 = 0$: Chiral multiplet (Matter multiplet),
$\lambda_0 = 1/2$: Vector multiplet (gauge multiplet).

∗ Matter multiplet (Chiral multiplet).

$$\lambda_0 = 0 \Rightarrow (0, \frac{1}{2}) \oplus (-\frac{1}{2}, 0)$$

.

Here we have added the CPT conjugate to complete the representation.

The number of d.o.f. are one Weyl spinor and a complex scalar. This is the representation to describe matter for $N = 1$ supersymmetry.

∗ Gauge multiplet (Vector multiplet)

$$\lambda_0 = 1/2 \Rightarrow (\frac{1}{2}, 1) \oplus (-1, \frac{1}{2})$$

.

The content of d.o.f. is one Weyl spinor and one vector. The gauge fields are written in this representation.

The gauge transformation for the Weyl spinor should have the same transformation than the vector (i.e. the adjoint representation) because they belong to the same supermultiplet. Also, this representation has an internal symmetry (R-symmetry) which we will describe latter.

Although we are going to focus in a rigid supersymmetric theory, i.e. without consider gravity. Just for completeness, we are going to mention that there are other representations with higher helicity, but they are not of our interest in our description.

Gravitino multiplet

$$\lambda_0 = 1 \Rightarrow (1, \frac{3}{2}) \oplus (-\frac{3}{2}, -1)$$

, Graviton multiplet

$$\lambda_0 = 3/2 \Rightarrow (\frac{3}{2}, 2) \oplus (-2, -\frac{3}{2})$$

.

N=2 Supersymmetry

$*$ Matter multiplet (Hypermultiplet)

$$\lambda_0 = -1/2 \Rightarrow (-\frac{1}{2}, 0, 0, +\frac{1}{2}) \oplus (-\frac{1}{2}, 0, 0, +\frac{1}{2})$$

.

The d.o.f. are two Weyl spinors and two complex scalars. Matter is described in this representation for $N = 2$ supersymmetry. The field content in the language of the $N = 1$ supersymmetry are two Chiral multiplets with oppositive chirality.

Under $SU(2)$ R-symmetry, the scalar states with 0 helicity behaves as doublets and the fermions as singlets. In principle, the representations has the CPT self-conjugate condition $\lambda_0 = -N/2$, but this multiplet can not be CPT invariant. If it were the case, the scalar fields would be real and would not form a doublet in $SU(2)$. Because of the 2-dimensional representation of $SU(2)$ is pseudoreal. Therefore, the doublet of scalars should be complex.

$*$ Gauge multiplet (vector multiplet)

$$\lambda_0 = 0 \Rightarrow (0, \frac{1}{2}, \frac{1}{2}, 1) \oplus (-1, -\frac{1}{2}, -\frac{1}{2}, 0)$$

The d.o.f. are one vector, two Weyl fermions and a complex scalar. In the language of $N = 1$, we have a Chiral multiplet and a Gauge multiplet. It is important to note that both are in the adjoint representation of the gauge group.

Also, we have the other representations of higher spin: gravitino and graviton multiplet.

N=4 Supersymmetry

$$\lambda_0 = -1 \Rightarrow (-1, 4 \times -\frac{1}{2}, 6 \times 0, 4 \times \frac{1}{2}, +1)$$

The d.o.f. are one vector, four Weyl spinors and three complex scalars. In the $N = 1$ language this multiplet corresponds to one vector multiplet and three matter multiplets. All the fields transform in the adjoint representation.
-Vector is a singlet in $SU(4)$. (R-symmetry)
-Fermions transform in the fundamental representation.
-Scalar fields transform as twice the antisymmetric representation of $SU(4)$

A.3.2 Massive representation

Let us consider a state with mass m in the rest frame. The four momentum is $P_\mu = (m, 0, 0, 0)$. It is easy to see the difference from the massless case, the number of non-trivial generators does not dismiss.

$$\begin{pmatrix} \{Q^I_1, \bar{Q}_{\dot{1}J}\} & 0 \\ 0 & \{Q^I_2, \bar{Q}_{\dot{2}J}\} \end{pmatrix} = \begin{pmatrix} 2m & 0 \\ 0 & 2m \end{pmatrix} \delta^I_J. \tag{A.3.2.1}$$

We can write it as:

$$\{Q^I_\alpha, \bar{Q}_{\dot{\beta}J}\} = 2m\delta_{\alpha\dot{\beta}}\delta^I_J. \tag{A.3.2.2}$$

It means that the massive representation is larger than the massless representation. Now, it is convinient to write spin rather than helicity for j and $j(j+1)$ are the eingenvalues of $\hat{J}^2$. Also the Clifford vacuum has a degeneracy $2j+1$, since j_3 takes values from $-j$ to j.

N=1 Supersymmetry

Now we have both creation and annihilation operators.

$$a_{1,2} = \frac{1}{\sqrt{2m}} Q_{1,2}, \qquad a^\dagger_{1,2} = \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{1},\dot{2}}. \tag{A.3.2.3}$$

The supersymmetry algebra is reduced to

$$\{a^I_\alpha, a^\dagger_{\dot{\beta}J}\}\delta_{\alpha\dot{\beta}}\delta^{IJ}. \tag{A.3.2.4}$$

All the other anticommutators vanish.

The Clifford vacuum is defined by (A.3.1.8) and the representation is constructed when we act $a^{I\dagger}_\alpha$ on $|\lambda_0\rangle$. For this case, we have C^{2N}_k for a level $|\lambda_0 + k/2\rangle$ after apply creation operator $a^\dagger$.

The total dimension of the representation is given by

$$\sum_{k=0}^{2N} C^{2N}_k = 2^{2N}. \tag{A.3.2.5}$$

The highest spin value we can obtain is $N/2$ instead of N, because

$$(a^I_1)^\dagger (a^J_2)^\dagger = \frac{1}{2}\epsilon^{\alpha\beta}(a^I_\alpha)^\dagger (a^J_\beta)^\dagger$$

, which is a scalar. Then for $k = 2N$, the state has spin zero.

$*$ Matter multiplet

We have the states:

$$|j_0\rangle \quad , \qquad a^\dagger_\alpha |j_0\rangle, \qquad \frac{1}{2}\epsilon^{\alpha\beta} a^\dagger_\alpha a^\dagger_\beta |j_0\rangle. \tag{A.3.2.6}$$

In the language of $N = 1$, there are two matter multiplets with opposive chirality. In terms of fields there are one Majorana spinor and one complex scalar.

$*$ Gauge multiplet

$$j_0 = 1/2 \Rightarrow (-1, 2 \times -\frac{1}{2}, 2 \times 0, 2 \times \frac{1}{2}, 1)$$

.

There is a massive vector, a real massive scalar and a Dirac massive spinor. In the language of $N = 1$, it corresponds to the field content of one matter multiplet and one vector multiplet.

A.4 Local Representations of N=1 Supersymmetry

We are going to describe the action of supersymmetry on the local fields for a quantum field theory. All objects apart of the elements of the Poincare Group transforms as components of tensors or spinors defined on the space-time manifold. In a similar way, supersymmetric transformations act on an extension of the space-time, called "superspace". Now, quantum fields transform as components of the "superfield" defined in the superspace. We have seen that superfield multiplets are built in terms of fields. We shall describe the construction of the superspace as well as we will introduce the Chiral and Vector superfields.

A.4.1 Superspace

The superspace is obtained by adding four spinors degrees of freedom θ^α, $\bar{\theta}_{\dot{\alpha}}$ as coordinates in this new and bigger space and the space-time coordinates x^μ.

Under supersymmetric transformations we have for the space-time coordinates and the spinorial new coordinates the following transformations (For N=1 and the transformation parameters ξ and $\bar{\xi}$):

$$x^\mu \to x'^\mu = x^\mu + \imath\theta\sigma^\mu\bar{\xi} - \imath\xi\sigma^\mu\bar{\theta}, \tag{A.4.1.1}$$

$$\theta \to \theta' = \theta + \xi, \tag{A.4.1.2}$$

$$\bar{\theta} \to \bar{\theta}' = \bar{\theta} + \bar{\xi}. \tag{A.4.1.3}$$

A supersymmetric differential transformation for any object $F(x, \theta, \bar{\theta})$, with space-time coordinates or spinorial coordinates for the "superspace". It can be represented by

$$\delta_\xi F(x, \theta, \bar{\theta}) = (\xi Q + \bar{\xi}\bar{Q})F(x, \theta, \bar{\theta}), \tag{A.4.1.4}$$

where:

$$
\begin{aligned}
Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} - \imath\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, & Q^\alpha &= -\frac{\partial}{\partial\theta_\alpha} + \imath\bar\theta_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\alpha}\partial_\mu, \\
\bar Q^{\dot\alpha} &= \frac{\partial}{\partial\bar\theta_{\dot\alpha}} - \imath(\bar\sigma^\mu)^{\dot\alpha\alpha}\partial_\mu, & \bar Q_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \imath\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu. \quad \text{(A.4.1.5)}
\end{aligned}
$$

They satisfy he following algebra:

$$\left\{Q_\alpha, \bar Q_{\dot\alpha}\right\} = 2\imath\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \quad \text{(A.4.1.6)}$$

$$\{Q_\alpha, Q_\beta\} = \left\{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\right\} = 0. \quad \text{(A.4.1.7)}$$

As superfields are functions of the space-time variables and the two Weyl spinors θ_α and $\bar\theta_{\dot\alpha}$. They hold for the supersymmetric transformation in (A.4.1.4)

It is useful to define the supersymmetric covariant derivative:

$$
\begin{aligned}
D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + \imath\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, & D^\alpha &= -\frac{\partial}{\partial\theta_\alpha} + \imath\bar\theta_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\alpha}\partial_\mu, \\
\bar D^{\dot\alpha} &= \frac{\partial}{\partial\bar\theta_{\dot\alpha}} + \imath(\bar\sigma^\mu)^{\dot\alpha\alpha}\partial_\mu, & \bar D_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \imath\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \text{(A.4.1.8)}
\end{aligned}
$$

They have the following anticommutation rules

$$
\begin{aligned}
\left\{D_\alpha, \bar D_{\dot\alpha}\right\} &= -2\imath\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \\
\{D_\alpha, D_\beta\} = \left\{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\right\} &= 0. \quad \text{(A.4.1.9)}
\end{aligned}
$$

An arbitrarily superfield can be expanded in terms of normal fields as follows:

$$
\begin{aligned}
F(x,\theta,\bar\theta) &= f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) \\
&+ \theta\theta m(x) + \bar\theta\bar\theta n(x) + \theta\sigma^\mu\bar\theta v_\mu \\
&+ \theta\theta\bar\theta\bar\lambda(x) + \bar\theta\bar\theta\theta\psi(x) + \theta\theta\bar\theta\bar\theta d(x). \quad \text{(A.4.1.10)}
\end{aligned}
$$

A.4.2 Chiral Superfields

The N=1 scalar supermultiplet is represented by a superfield with the constraint.

$$\bar D_{\dot\alpha}\Phi = 0. \quad \text{(A.4.2.1)}$$

this is refered to as chiral fields. Note that for $y^\mu = x^\mu + \imath\theta\sigma^\mu\bar\theta$, we have:

$$\bar D_{\dot\alpha} y^\mu = 0, \qquad \bar D_{\dot\alpha}\theta^\beta = 0 \quad \text{(A.4.2.2)}$$

Any function of these variable satisfy,

$$\begin{aligned}\Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \\ &= \phi(x) + \imath\theta\sigma^{\mu}\bar{\sigma}\partial_{\mu}\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\phi(x) \\ &+ \sqrt{2}\theta\psi(x) - \frac{\imath}{\sqrt{2}}\theta\theta\partial_{\mu}\psi(x)\sigma^{\mu}\bar{\theta} + \theta\theta F(x),\end{aligned} \tag{A.4.2.3}$$

where ϕ and ψ are the scalar and fermionic fields respectively and F is the auxiliar field required for the on-shell closure of the algebra. Similarly, the antichiral superfield is defined and respects the following constraint $D_{\alpha}\Phi^{\dagger} = 0$ and can be expanded as:

$$\begin{aligned}\Phi^{\dagger} &= \phi^{\dagger}(\bar{y}) + \sqrt{2}\bar{\theta}\bar{\psi}(\bar{y}) + \bar{\theta}\bar{\theta}\bar{F}(\bar{y}) \\ &= \phi^{\dagger}(x) - \imath\theta\sigma^{\mu}\bar{\sigma}\partial_{\mu}\bar{\phi}(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\bar{\phi}(x) \\ &+ \sqrt{2}\bar{\theta}\bar{\psi}(x) + \frac{\imath}{\sqrt{2}}\bar{\theta}\bar{\theta}\partial_{\mu}\psi(x)\sigma^{\mu}\theta + \bar{\theta}\bar{\theta}\bar{F}(x),\end{aligned} \tag{A.4.2.4}$$

the product of chiral superfields is a chiral superfield again. In general, any functions of only chiral superfield is a chiral superfield.

$$\mathcal{W}(\Phi_i) = \mathcal{W}(\phi_i + \sqrt{2}\theta\psi_i\theta\theta F_i) = \mathbb{W}(\phi_i) + \frac{\partial\mathcal{W}}{\partial\phi_i}\sqrt{2}\theta\psi_i + \theta\theta(\frac{\partial\mathcal{W}}{\partial\phi_i}F_i - \frac{1}{2}\frac{\partial^2\mathcal{W}}{\partial\phi_i\phi_j}\psi_i\psi_j). \tag{A.4.2.5}$$

It useful to give the transformation laws for the component fields which belong to the chiral and antichiral superfields. (δ corresponds to the supersymmetric transformation)

$$\delta\phi = \sqrt{2}\xi\psi, \tag{A.4.2.6}$$

$$\delta\psi_{\alpha} = \sqrt{2}\xi_{\alpha}F + \imath\sqrt{2}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_{\mu}\phi, \tag{A.4.2.7}$$

$$\delta F = \imath\sqrt{2}\bar{\xi}\bar{\sigma}^{\mu}\partial_{\mu}\psi. \tag{A.4.2.8}$$

Similarly, for the antichiral fields

$$\delta\bar{\phi} = \sqrt{2}\bar{\xi}\bar{\psi}, \tag{A.4.2.9}$$

$$\delta\bar{\psi}^{\dot{\alpha}} = \sqrt{2}\bar{\xi}^{\dot{\alpha}}\bar{F} + \imath\sqrt{2}(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha}\xi_{\alpha}\partial_{\mu}\bar{\phi}, \tag{A.4.2.10}$$

$$\delta\bar{F} = \imath\sqrt{2}\xi\sigma^{\mu}\partial_{\mu}\bar{\psi}. \tag{A.4.2.11}$$

We are interested in to write invariant supersymetric terms for a lagrangian (Kinetic and potential terms) in a superfield notation. It is useful to have product of superfields in terms of component fields

$$\begin{aligned}
\bar{\Phi}_i\Phi_j &= \bar{\phi}_i\phi_j + \sqrt{2}\theta\psi_j\bar{\phi}_i + \sqrt{2}\bar{\theta}\bar{\psi}_i\phi_j + \theta^2\bar{F}_i\phi_j \\
&+ \theta\sigma^\mu\bar{\theta}[\imath(\bar{\phi}_i\partial_\mu\phi_j - \phi_j\partial_\mu\bar{\phi}_i) - \bar{\psi}_i\bar{\sigma}_\mu\psi_j] + \sqrt{2}\bar{\theta}^2\theta\psi_j\bar{F}_i + \sqrt{2}\theta^2\bar{\theta}\bar{\psi}_i F_j \\
&+ \frac{1}{\sqrt{2}}\theta^2[\bar{\theta}\bar{\sigma}^\mu\partial_\mu\psi_i\bar{\phi}_j - \bar{\theta}\sigma^\mu\psi_j\partial_\mu\bar{\phi}_i] + \frac{i}{\sqrt{2}}\bar{\theta}^2[\theta\sigma^\mu\partial_\mu\bar{\psi}_j\phi_i - \theta\sigma^\mu\psi_i\partial_\mu\phi_j] \\
&+ \theta^2\bar{\theta}^2[\bar{F}_iF_j - \frac{1}{4}\bar{\phi}_i\partial_\mu\partial^\mu\phi_j - \frac{1}{4}\partial_\mu\partial^\mu\bar{\phi}_i\phi_j + \frac{1}{2}\partial_\mu\bar{\phi}_i\partial^\mu\phi_j \\
&+ \frac{i}{2}\partial_\mu\bar{\psi}_i\bar{\sigma}^\mu\psi_j - \frac{i}{2}\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_j]. \qquad (A.4.2.12)
\end{aligned}$$

We can also obtain the following formulas for quadratic terms in the fields

$$\Phi_i(x,\theta)\Phi_j(x,\theta) = \cdot + \theta^2[\phi_iF_j + F_i\phi_j - \psi_i\psi_j], \qquad (A.4.2.13)$$

and for the cubic term for chiral superfields

$$\begin{aligned}
\Phi_i(x,\theta)\Phi_j(x,\theta)\Phi_k(x,\theta) = \cdot = \theta^2[&F_i\phi_j\phi_k + \phi_iF_j\phi_k + \phi_i\phi_jF_k \qquad (A.4.2.14)\\
&-\psi_i\psi_j\phi_k - \phi_i\psi_j\psi_k - \psi_i\phi_j\psi_k]. \qquad (A.4.2.15)
\end{aligned}$$

Now, we write the most general supersymmetric action. It has scalar and spinor fields and it is given by the sum of a kinetic and potential term:

$$S = \int d^4x\{\int d^2\theta d^2\bar{\theta}\bar{\Phi}_i\Phi_i + [(\int d^2\theta\mathcal{W}(\Phi)) + h.c.]\}. \qquad (A.4.2.16)$$

Expanding the superpotential $\mathcal{W}$ around the scalar field, we can see that the terms of superfields with $\theta^2\bar{\theta}^2$ or θ^2 components are the important at the moment to integrate on the fermionic degrees of freedom.

A.4.3 Vector Multiplet

This multiplet is represented by a real superfield satisfying $V = V^\dagger$. In components, it takes the form:

$$\begin{aligned}
V(x,\theta,\bar{\theta}) &= \phi + \imath\theta\chi - \imath\bar{\theta}\bar{\chi} + \frac{i}{2}\theta^2(M + \imath N) \\
&- \frac{\imath}{2}\bar{\theta}^2(M - \imath N) - \theta\sigma^\mu\bar{\theta}A_\mu \\
&+ \imath\theta^2\bar{\theta}(\bar{\lambda} + \frac{\imath}{2}\bar{\sigma}^\mu\partial_\mu\xi) - \imath\bar{\theta}^2\theta(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}) \\
&+ \frac{1}{2}\theta^2\bar{\theta}^2(D - \frac{1}{2}\Box\phi), \qquad (A.4.3.1)
\end{aligned}$$

Many of these components can be gauge away using the abelian gauge transformation $V \to V + \Lambda + \Lambda^\dagger$, where $\Lambda(\Lambda^\dagger)$ are chiral (antichiral) superfields. In the called Wess-Zumino gauge, we set $\phi = M = N = \chi = 0$. So we have

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + \imath\theta^2\bar{\theta}\bar{\lambda} - \imath\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D. \tag{A.4.3.2}$$

In this gauge, the quadratic and cubic term are reduced to the following compact expressions, $V^2 = \frac{1}{2}A_\mu A^\mu\theta^2\bar{\theta}^2$ and $V^3 = 0$. These result will be helpful for computing later expressions.

A.5 Supersymmetric Gauge Theories in D=4

For this section we are going to construct the N=1 supersymmetric lagrangian for scalar and vector supermultiplets, and these supermultiplets are the building blocks for the N=2 supersymmetric lagrangian which is our point of interest.

1 **N=1 super Yang Mills**

This theory involves only strength W^a_α and a vector superfield V_a and its lagrangian is given by

$$\mathcal{L} = -\frac{i}{16\pi}\int d^2\theta\tau W^\alpha W_\alpha + h.c. = \frac{1}{8\pi}Im[\tau\int d^2\theta W^\alpha W_\alpha]. \tag{A.5.0.3}$$

In component fields we get

$$\mathcal{L} = -\frac{1}{4e^2}F^a_{\mu\nu}F^{\mu\nu}_a - \frac{i}{e^2}\bar{\lambda}^a\bar{\sigma}^\mu(D_\mu\lambda)^a + \frac{1}{2e^2}D^2 + \frac{\theta}{32\pi^2}F^a_{\mu\nu}{}^*F^{\mu\nu}_a \tag{A.5.0.4}$$

Lagrangians for vector multiplets:

Given the vector multiplet expansion and following the Wess-Zumino gauge, reduce the superfield to

$$V = -\theta\sigma^\mu\bar{\theta}A_\mu + \imath\theta^2\bar{\theta}\bar{\lambda} - \imath\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D \tag{A.5.0.5}$$

With these we define the abelian field strength tensor by

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V, \qquad \bar{W}_{\dot{\alpha}} = \frac{1}{4}D^2\bar{D}_{\dot{\alpha}}V. \tag{A.5.0.6}$$

where W_α is a chiral superfield has the form:

$$W_\alpha = -\imath\lambda_\alpha(y) + \theta_\alpha D - \frac{\imath}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} + \theta^2(\sigma^\mu\partial_\mu\bar{\lambda})_\alpha. \tag{A.5.0.7}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the abelian strength tensor.

In the non-abelian case, V belongs to the adjoint representation of the gauge group: $V = V_T^a$, where $T^{a\dagger} = T^a$. Now the gauge transformation is implemented by

$$e^{-2V} \to e^{-\imath\Lambda^\dagger} e^{-2V} e^{\imath\Lambda}, \tag{A.5.0.8}$$

where $\Lambda = \Lambda^a T^a$.

Following the previous gauge transformation, the non-Abelian gauge field strength is defined by

$$W_\alpha = \frac{1}{8}\bar{D}^2 e^{2V} D_\alpha e^{-2V}. \tag{A.5.0.9}$$

and it transforms as

$$W_\alpha \to W'_\alpha = e^{-\imath\Lambda} W_\alpha e^{\imath\Lambda}. \tag{A.5.0.10}$$

In components, it can be obtained after using the following property

$$e^{2V} D_\alpha e^{-2V} = D_\alpha V - \frac{1}{2}[V, D_\alpha V]. \tag{A.5.0.11}$$

After this, we replace (A.5.0.5) on (A.5.0.11) and using this to construct the supersymmetric non-Abelian strength tensor, which has a similar form like the Abelian case.

$$W_\alpha = T^a[-\imath\lambda_\alpha^a + \theta_\alpha D^a - \frac{\imath}{2}(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu}^a + \theta^2\sigma^\mu D_\mu\bar{\lambda}^a], \tag{A.5.0.12}$$

where,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ef^{abc}A_\mu^b A_\nu^c, \qquad (D_\mu\bar{\lambda})^a = \partial_\mu\bar{\Lambda} + ef^{abc}A_\mu^b\bar{\lambda}^c. \tag{A.5.0.13}$$

From the expansion for the non-Abelian gauge field strength tensor. We construct the invariant term for the lagrangian. We compute $W^\alpha W_\alpha|_{\theta\theta}$, it means, we care only on the components with two θ,

$$W^\alpha W_\alpha|_{\theta\theta} = -2\imath\lambda\sigma^\mu\partial_\mu\bar{\lambda} + d^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{\imath}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{A.5.0.14}$$

In the Abelian case the lagrangian does not contain the $F_{\mu\nu}\tilde{F}^{\mu\nu}$ term, but in the non-Abelian it does. Similarly, for the abelian lagrangian we have

$$\mathcal{L} = \frac{1}{4g^2}\left(\int d^2\theta W^\alpha W_\alpha + \int d^2\bar{\theta}\bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\right). \tag{A.5.0.15}$$

We have for the non-Abelian case the same expression, but we need top take trace since the fields are matrices:

$$Tr(W^\alpha W_\alpha|_{\theta\theta}) = -2\imath\lambda^a\sigma^\mu D_\mu\bar{\lambda}^a + D^a D^a - \frac{1}{2}F^{a\mu\nu}F_{\mu\nu}^a + \frac{\imath}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^a F_{\rho\sigma}^a, \tag{A.5.0.16}$$

and, the lagrangian

$$\mathcal{L} = \frac{1}{4g^2} Tr \left(\int d^2\theta W^\alpha W_\alpha + \int d^2\bar{\theta} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right). \tag{A.5.0.17}$$

And we can expand it on field components. Also we are interested in to describe a lagrangian which contains (**??**) to achieve this we introduce as the classical lagrangian the θ parameter and define the complex coupling constant $\tau = \theta/2\pi + 4\pi\imath/g^2$

We obtain

$$\begin{aligned} \mathcal{L} &= \frac{1}{8\pi} Im \left(\tau Tr \int d^2\theta W^\alpha W_\alpha \right) \\ &- \frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} + \frac{1}{g^2} (\frac{1}{2} D^a D^a - \imath \lambda^a \sigma^\mu D_\mu \bar{\lambda}^a) \end{aligned} \tag{A.5.0.18}$$

Note that τ can be seen as a chiral superfield

Interaction terms and the General N=1 Lagrangian: It is important to describe the supersymmetric extension of matter interacting with Yang-Mills theory. The chiral superfields Φ_i belong to a given representation of the gauge group and the generators are the matrices T^a_{ij}.

We have the chiral superfield belong to a certain representation, where the generators are matrices T^a_{ij}. The kinetic term is $\Phi^\dagger_i \Phi_i$ and it is invariant under a global transformation $\Phi' = e^{-\imath\Lambda}\Phi$. For the local case, we make Λ a chiral superfield, to assure the chirality of Φ. We want to describe interacting term which respects local invariance in such a way that we need this term to respect the gauge transformation for non-Abelian gauge fields. In this sense, our kinetic term will be $\Phi^\dagger e^{-2V}\Phi$.

Now, we are able to write the full N=1 supersymmetric lagrangian

$$\mathcal{L} = \frac{1}{8\pi} Im \left(\tau Tr \int d^2\theta W^\alpha W_\alpha + h.c. \right) + \int d^4\theta \Phi^\dagger e^{-2V} \Phi + \int d^2\theta \mathcal{W}(\Phi) + \int d^2\bar{\theta} \bar{\mathcal{W}}(\Phi^\dagger) \tag{A.5.0.19}$$

A.5.1 N=1 Lagrangian scalar multiplet

Now we are able to construct a lagrangian which correspons to the kinetic term for the fields.

$$\mathcal{L} = \Phi^\dagger_i \Phi_i|_{\theta^2\bar{\theta}^2} = \partial_\mu \phi^\dagger_i \partial^\mu \phi_i + F^\dagger_i F_i - i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i. \tag{A.5.1.1}$$

We can extend the form of the kinetic term including a non-trivial metric for the space of fields, and we will have the following kinetic term $g^{ij}\partial_\mu \phi^\dagger_i \partial^\mu \phi_j$. The appropriate modification for the other terms in the lagrangian (A.5.1.1) corresponds to a free lagrangian replaced by the non-linear sigma model. In general, the lagrangian in terms of the superfields has the following form:

$$\mathcal{L} = \int d\theta^2 d\bar{\theta}^2 K(\Phi, \Phi^\dagger) + \int d\theta^2 \mathcal{W}(\Phi) + \int d\bar{\theta}^2 \bar{\mathcal{W}}(\Phi^\dagger). \tag{A.5.1.2}$$

In terms of holomorphic functions $K(\Phi, \Phi^\dagger)$ and the metric in the field space is giving by:

$$g^{ij} = \partial^2 K(\Phi, \Phi^\dagger)/\partial\Phi_i \partial\Phi_j^\dagger. \tag{A.5.1.3}$$

We refer to the function $K(\Phi, \Phi^\dagger)$ as Kahler potential.

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