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Review of Geometric quantization and WKB method

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Resumo

A quantização geométrica é um procedimento para construir uma teoria quântica a partir de elementos geométricos de um sistema clássico considerado como uma variedade simplética. Ele fornece uma abordagem matemática para uma teoria quântica com uma ampla gama de aplicações que vai de sistemas com partículas a teorias de campo quântico, para as quais a variedade simplética é o espao cotangente do espaço de campos (elementos do espaço cotangente são variações infinitesimais). Por outro lado, o método WKB fornece uma maneira de construir uma solução aproximada para a equação de Schrödinger na mecânica quântica a partir de elementos geométricos no espaço de fase de soluções de um sistema clássico. Estas notas são uma revisão de alguns artigos nessas duas abordagens da mecânica quântica.

Palavras Chaves: Geometric quantization, Symplectic geometry, Poisson algebra, WKB, Quantum mechanics

Áreas do conhecimento: Mathematical Physics

Abstract

Geometric quantization is a procedure to construct a quantum theory from geometric elements of a classical system regarded as a symplectic manifold. It provides a mathematical approach to a quantum theory with a wide range of applications that go from systems with particles to quantum field theories, for which the symplectic manifold is the cotangent space of the space of fields (elements of the cotangent space are infinitesimal variations). On the other side, WKB method provides a way to construct an approximate solution to the Schrödinger equation in quantum mechanics from geometric elements on the phase space of solutions of a classical system. These notes are a review of some papers on those two approaches to quantum mechanics.

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Capítulo 1

Introduction

To construct a quantum theory from an associate classical system one needs to follow a procedure called quantization. Geometric quantization (GQ) follows Dirac's quantization conditions up to the point where it leads to a contradiction and then modifying it to achieve its goal of constructing the 'correct' quantum theory (compared to the standart Schrödinger quantization in quantum mechanics).

This method proceeds in 3 stages: Pre-quantization, polarization and metaplectic correction. Pre-quantization starts from a symplectic manifold (M, ω) (classical system) and produces an injective representation of the Poisson algebra into the space of vector fields on M, together with a natural Hilbert space only if the 2-form ω holds certain condition. It is at this point where all the Dirac's quantization conditions can not be fullfilled simultaneously and GQ relax one of them; this stage, called polarization, consist of reducing the Hilbert space selecting a Poisson-commuting set of n variables on the $2n$-dimensional phase space M and consider functions that depend only on these n variables. There is a final stage, called metaplectic correction, which is necessary in the case of real polarizations; it modifies a line bundle constructed in the prequantization stage, as a result it produces the correct energy levels for the quantum harmonic oscillator.

Geometric quantization for anticommuting variables is of particular interest since it can be applied to BRST quantization, we will show some results for this case in chapters 3 and 4, but for a deeper treatment see [1], [2], [3].

WKB method is a procedure with lots of similarities to geometric quantization but a different goal, construct an approximate solution to the Schrödinger equation from a classical solution and geometric structures in it. In these notes we will start from the simplest case and generalize step-by-step until the case of cotangent bundles is treated, we will not treat the case of a general symplectic manifold (in the interest of pedagogy over generality) since it becomes too technical and requires more abstract language.

Capítulo 2

Pre-quantization: even case

Prequantization is the first step in geometric quantization, given a symplectic manifold (M, ω), it allows us to construct a faithful representation of the Poisson algebra $\mathcal{P}$. In order for this construction to work, we need a principal fibre bundle $\pi : Y \to M$ with structure group $\mathbb{R}/D$, D a discrete subgroup of $\mathbb{R}$ ($\mathbb{R}$ with $+$ as the group product), together with a connection 1-form α such that $d\alpha = \pi^*\omega$.

This motivates 2 questions, which are the main point of study in this chapter:

- Under what conditions does there exist such principal fibre bundle (Y, α)?
- If it exists, what are the different possibilities for (Y, α)?

To answer these questions we need some constructions over the 2-form ω and the manifold M.

2.1 Group of periods $Per(\omega)$

We start by choosing an open cover $\mathcal{U} = \{U_i \mid i \in I\}$ of a connected symplectic manifold (M, ω) such that all finite intersections $U_{i_1} \cap \cdots \cap U_{i_k}$ are either contractible or empty (this is always possible since every manifold has a good cover). Applying the Poincaré lemma, there exist on each U_i a 1-form θ_i such that $d\theta_i = \omega$. On each non-empty intersection $U_i \cap U_j$ we have $d(\theta_i - \theta_j) = \omega - \omega = 0$ and because $U_i \cap U_j$ is again contractible there exist smooth functions f_{ij} on $U_i \cap U_j$ such that

$\theta_i - \theta_j = df_{ij}$. On a triple intersection $U_i \cap U_j \cap U_k$ we have $d(f_{ij} + f_{jk} + f_{ki}) = \theta_i - \theta_j + \theta_j - \theta_k + \theta_k - \theta_i = 0$ and hence there exist constant functions a_{ijk} such that $a_{ijk} = f_{ij} + f_{jk} + f_{ki}$. Then we have:

$$\begin{cases} d\theta_i = \omega & \text{on } U_i \\ \theta_i - \theta_j = df_{ij} & \text{on } U_i \cap U_j \\ f_{ij} + f_{jk} + f_{ki} = a_{ijk} & \text{on } U_i \cap U_j \cap U_k \end{cases} \tag{2.1}$$

Definition 2.1.1. The **nerve** of the cover $\mathcal{U}$ is defined as

$$\mathcal{N}(\mathcal{U}) = \{(i_0, \ldots, i_k) \in I^{k+1} \mid k \in \mathbb{N},\ U_{i_0} \cap \cdots \cap U_{i_k} \neq \emptyset\}$$

and an element $(i_0, \cdots, i_k)$ is called a k-simplex.

The abelian group $C_k(\mathcal{U})$ of k-chains consists of all finite formal sums $\sum c_{i_0,\ldots,i_k}(i_0, \ldots, i_k)$ with $(i_0, \ldots, i_k) \in \mathcal{N}(\mathcal{U})$ and $c_{i_0,\ldots,i_k} \in \mathbb{Z}$.
The boundary operator $\partial_k : C_k(\mathcal{U}) \to C_{k-1}(\mathcal{U})$ is the homomorphism defined on the basis of $C_k(\mathcal{U})$ by

$$\partial_k(i_0, \ldots, i_k) = \sum_{j=0}^{k} (-1)^j (i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k) \tag{2.2}$$

Notice that the fact that it is a homomorphism makes ∂_k a 'linear' map and allows us to define it only on the basis of $C_k(\mathcal{U})$. As it is usual for a boundary operator, it can be checked that $\partial_{k-1} \circ \partial_k = 0$.
A homomorphism $h : C_k(\mathcal{U}) \to A$, with A an abelian group, is called a k-cochain if it is totally skew-symmetric on the basis of $C_k(\mathcal{U})$, i.e., h changes sign when one interchanges two entries i_p and i_q in $(i_0, \ldots, i_k)$. The set of all k-cochains with values in the abelian group A is denoted by $C^k(\mathcal{U}, A)$; equipped with pointwise addition of functions this is an abelian group. We can define coboundary operators $\delta_k : C^k(\mathcal{U}, A) \to C^{k+1}(\mathcal{U}, A)$ by:

$$(\delta_k h)(c) = h(\partial_{k+1} c) \qquad \text{for } c \in C_{k+1}(\mathcal{U}) \tag{2.3}$$

From this definition $\partial_{k-1} \circ \partial_k = 0$ implies $\delta_k \circ \delta_{k-1} = 0$.

From (2.1) we see that a_{ijk} form a 2-cochain with values in $A = \mathbb{R}$ (because it is skew-symmetric on the basis); moreover, if we apply δ_2 to $a_{ijk} = f_{ij} + f_{jk} + f_{ki}$ we obtain:

$$\begin{aligned}
(\delta_2 a)(ijkl) &= a(\partial_3(ijkl)) = a((jkl) - (ikl) + (ijl) - (ijk)) \\
&= f_{jk} + f_{kl} + f_{lj} - f_{ik} - f_{kl} - f_{li} + f_{ij} + f_{jl} + f_{li} - f_{ij} - f_{jk} - f_{ki} \\
&= 0
\end{aligned}$$

The 2-cochains with this property will be called 2-cocycles, then a_{ijk} is a 2-cocycle (we will denote the 2-cocycle a).

Definition 2.1.2. $Per(\omega) = im(a : ker(\partial_2) \to \mathbb{R})$, where a is the 2-cocycle defined above, is called the group of periods of the closed 2-form ω (note that $Per(\omega)$ is a subgroup of $(\mathbb{R}, +)$).

Proposition 2.1.1. *$Per(\omega)$ is independent of the choices in the construction of the 2-cocycle a.*

Proof. Looking at the equation (2.1), we see that θ_i can be replaced by $\theta_i + d\phi_i$ (with ϕ_i a function on U_i) without affecting ω; but that will produce the change $f_{ij} \to f_{ij} + \phi_i - \phi_j$, replacing this in the definition of a_{ijk} we see that it remains unchanged. The other possibility is to make $f_{ij} \to f_{ij} + c_{ij}$ (with c_{ij} a constant function on $U_i \cap U_j$), that will produce the change $a_{ijk} \to a_{ijk} + c_{ij} + c_{jk} + c_{ki} = a_{ijk} + (\delta_1 c)_{ijk}$; for any $\alpha \in ker(\partial_2)$ we see that $(\delta_1 c)\alpha = c(\partial_2 \alpha) = 0$, so $Per(\omega)$ does not depend on this choice either. Since these changes exhaust all possible choices in the construction of a, the proposition is proved. □

Remark. If we replace ω by $\omega + d\theta$ then θ_i is replaced by $\theta_i + \theta$ and f_{ij} is not changed; it follows that $Per(\omega)$ depends only upon the cohomology class of ω in the de Rham cohomology group $H^2_{\mathrm{dR}}(M, \mathbb{R})$.

Proposition 2.1.2. *The 2-cocycle 'a' can be chosen such that for all $(i, j, k) \in \mathcal{N}(\mathcal{U})$: $a_{ijk} \in Per(\omega)$*

Proof. This is a purely algebraic statement, independent of the topological properties of $Per(\omega)$. We define the homomorphism $b : C_1(\mathcal{U}) \to \mathbb{R}/Per(\omega)$ as follows. On the subspace $im(\partial_2) \subset C_1(\mathcal{U})$ it is defined by $b = \pi \circ a \circ (\partial_2)^{-1}$, where π denotes the projection $\mathbb{R} \to \mathbb{R}/Per(\omega)$; this map is independent of the choice in $(\partial_2)^{-1}$, to check that lets evaluate $b(\partial_2\alpha)$ where $\alpha \in ker(\partial_2)$ (here α represents the freedom of the choice). This gives us $b(\partial_2\alpha) = \pi(a(\alpha)) = 0$, because $a(\alpha) \in Per(\omega)$ and it is mapped to 0 by π, so it is independent of the choice in $(\partial_2)^{-1}$. Now $\mathbb{R}/Per(\omega)$ is a divisible $\mathbb{Z}$-module (i.e. for all $m \in \mathbb{Z}$, m defines by multiplication a surjective map from $\mathbb{R}/Per(\omega)$ to itself), there exists an extension b to the whole of $C_1(\mathcal{U})$ (because every divisible $\mathbb{Z}$-module is injective, see [4]). Since $C_1(\mathcal{U})$ is a free $\mathbb{Z}$-module, then it is a projective module, and by the lifting property of projective modules this implies that there exists a homomorphism $b' : C_1(\mathcal{U}) \to \mathbb{R}$ satisfying $\pi \circ b' = b$. Finally, because of the freedom of the choice in the definition of f_{ij} we can replace it by $f_{ij} - b'_{ij}$, which changes the cocycle a_{ijk} into $a_{ijk} - (\delta_1 b')_{ijk}$; and by construction of b' it follows that:

$$\begin{aligned}
\pi(a_{ijk} - (\delta_1 b')_{ijk}) &= \pi a - \pi b' \partial_2 \\
&= \pi a - b\partial_2 \\
&= \pi a - (\pi a(\partial_2)^{-1})\partial_2 \\
&= 0
\end{aligned}$$

showing that this modified cocycle has values in $Per(\omega)$ as claimed. □

Proposition 2.1.3. *$Per(\omega) = \{0\} \Leftrightarrow \omega$ is exact.*

Proof. ($\Leftarrow$) By construction $a_{ijk} = 0$.
($\Rightarrow$) Suppose $Per(\omega) = \{0\}$, according to the last proposition we may assume that the constants $a_{ijk} = f_{ij} + f_{jk} + f_{ki}$ are zero (because they can take values in $Per(\omega)$). Let ρ_i be a partition of unity subordinated to the cover $\mathcal{U}$. Then the local 1-forms $\hat{\theta}_i = \theta_i + d(\sum_k \rho_k f_{ki})$ are well-defined on U_i and they coincide on the intersection $U_i \cap U_j : \hat{\theta}_i - \hat{\theta}_j = df_{ij} + d(\sum_k \rho_k(f_{ki} - f_{kj})) = df_{ij} - d((\sum_k \rho_k) f_{ij}) = 0$. These local 1-forms thus define a global 1-form $\hat{\theta}$, which satisfies $d\hat{\theta} = \omega$. □

2.2 Condition for the existence of the $\mathbb{R}/D$ principal fibre bundle (Y, α)

First, let's give a definition of a principal fibre bundle in terms of transition functions:

Definition 2.2.1. Given a manifold M, a cover $\{U_i\}$ of M and smooth functions $g_{ij} : U_i \cap U_j \to G$, where G is a Lie group, which obey the following cocycle conditions:

$$g_{ij}(m) \cdot g_{ji}(m) = e \qquad \forall m \in U_i \cap U_j \tag{2.4}$$

$$g_{ij}(m) \cdot g_{jk}(m) \cdot g_{ki}(m) = e \qquad \forall m \in U_i \cap U_j \cap U_k \tag{2.5}$$

A principal G fibre bundle over M is the space $P = \bigcup_i U_i \times G/\sim$, where $(m, g) \sim (m, g_{ij}(m)g)$, together with the projection map $\pi : (m, g) \mapsto m$ and the action of G on P given by $(m, g') \cdot g \mapsto (m, g'g)$

Using the same notation as the section before, we have the following theorem:

Theorem 2.2.1. *A principal $\mathbb{R}/D$ fibre bundle Y over M with a compatible connection form α exists if and only if $Per(\omega) \subseteq D$.*

Proof. ($\Leftarrow$) Suppose that $Per(\omega) \subseteq D$, then as a consequence of proposition 2.1.2, we can assume that the functions f_{ij} have been chosen such that the cocycle a_{ijk} takes its values in $Per(\omega) \subseteq D$. It follows that the functions $g_{ij} : U_i \cap U_j \to \mathbb{R}/D$ defined by $g_{ij}(m) = \pi'(f_{ij}(m))$ satisfy the cocycle condition (here $\pi' : \mathbb{R} \to \mathbb{R}/D$):

$$g_{ij} + g_{jk} + g_{ki} = \pi(f_{ij} + f_{jk} + f_{ki}) = \pi(a_{ijk}) = 0$$

Having this condition we proceed to the construction of a principal fibre bundle $\pi : Y \to M$ with structure group $\mathbb{R}/D$; on a local chart $U_i \times \mathbb{R}/D$, with projection $\pi(m, x_i) = m$, we define the transition between charts as:

$$(m, x_i) \longmapsto (m, x_j) \equiv (m, x_i + g_{ij}(m))$$

Notice that this transition function is well-defined, because on $U_i \cap U_j \cap U_k$ we have the transitions $x_i + g_{ij} = x_j$ and $x_j + g_{jk} = x_k$ which together with the cocycle condition on g_{ij} implies $x_i + g_{ij} + g_{jk} = x_k \Rightarrow x_i = x_k + g_{ki}$.
The action $r \in \mathbb{R}/D$ on Y is given on a local chart $U_i \times \mathbb{R}/D$ by $(m, x_i) \cdot r = (m, x_i + r)$. On Y we can define a global 1-form α as follows, on the local chart $U_i \times \mathbb{R}/D$ it is given by

$$\alpha = \pi^* \theta_i + dx_i \tag{2.6}$$

where x_i is a local coordinate on $\mathbb{R}/D$ (but dx_i is globally defined 1-form). To check that it is a global 1-form we evaluate the following difference on $U_i \cap U_j$:

$$\begin{aligned}(\pi^*\theta_i + dx_i) - (\pi^*\theta_j + dx_j) &= \pi^*(\theta_i - \theta_j) + d(x_i - x_j) \\ &= \pi^*(df_{ij}) + d(-g_{ij}) \\ &= d(\pi(f_{ij})) - d(g_{ij}) \\ &= 0\end{aligned}$$

Now, from the definition of α it is easy to see that $d\alpha = \pi^*\omega$; and that it is an Ehresmann connection (i.e., for any element $x_i\partial_{x_i}$ of the Lie algebra of $\mathbb{R}/D$ the following 2 conditions holds $\iota(x_i\partial_{x_i})\alpha = x_i$ and $\iota(x_i\partial_{x_i})d\alpha = 0$).
($\Rightarrow$) Suppose (Y, α) exists, then there exist transition functions $g_{ij} : U_i \cap U_j \to \mathbb{R}/D$. From the 2 conditions which holds for an Ehresmann connection and the compatibility condition $d\alpha = \pi^*\omega$ we obtain the local expression $\alpha = \pi^*\theta_i + dx_i$ with $d\theta_i = \omega$. However, since α is defined globally, we must impose the following constraint $\pi^*\theta_i + dx_i = \pi^*\theta_j + dx_j$ which using the transition functions transforms into $\pi^*(\theta_i - \theta_j) = dg_{ij}$. Now, by contractibility of $U_i \cap U_j$, there exist smooth functions f_{ij} such that $\theta_i - \theta_j = df_{ij}$, replacing this in the last equation we see that we can choose $\pi(f_{ij}) = g_{ij}$. Finally, from $g_{ij} = x_j - x_i$ we get the cocycle condition $g_{ij} + g_{jk} + g_{ki} = 0$ which at the same time implies $\pi(f_{ij} + f_{jk} + f_{ki}) = \pi(a_{ijk}) = 0 \Rightarrow a_{ijk} \in D$, showing that $Per(\omega) \subseteq D$. $\Box$

Remark. There exists a classification of inequivalent $\mathbb{R}/D$ principal fibre bundles (prequantum bundles), where two prequantum bundles (Y, α) and (Y', α') are regarded as equivalent if there is a principal fibre bundle isomorphism $\psi : Y \to Y'$ (i.e. ψ commutes with $\pi : Y \to M$ and ψ is equivariant with respect to the action of the structure group $\mathbb{R}/D$) such that $\psi^*\alpha' = \alpha$, it follows that $\omega = \omega'$, hence their group of periods are the same. It turns out that inequivalent prequantum bundles are classified by $H^1(M, \mathbb{R}/D)$ (we will use the fact that $H^1(M, \mathbb{R}/D) = Hom(\pi_1(M), \mathbb{R}/D)$ in some examples later, this last equality is a result of the universal coefficient theorem (see for instance theorem 15.14 in reference [5]). A proof of this classification can be found in [6].

2.3 Lifting infinitesimal symmetries of (M, ω)

Definition 2.3.1. A vector field X on M is called an infinitesimal symmetry of the pair (M, ω) if $\mathcal{L}_X \omega = 0$.

Assuming we have the bundle (Y, α) over (M, ω), we want to know if there exist a vector field X' on Y such that $\pi_* X' = X$ (it is lifted from X) and $\mathcal{L}_{X'} \alpha = 0$ (it is a symmetry of (Y, α)). We have the following result:

Proposition 2.3.1. *Such X' exists if and only if there exist a function f on M such that $\iota(X)\omega + df = 0$ (i.e. only Hamiltonian vector fields can be lifted).*

Proof. $(\Rightarrow)$ Suppose X' exists, then we have the equations $\pi_* X' = X$ and $\mathcal{L}_{X'} \alpha = 0$ which imply:

$$0 = \iota(X')d\alpha + d(\iota(X')\alpha) = \iota(X')\pi^*\omega + d(\alpha(X')) = \pi^*(\iota(X)\omega) + d(\alpha(X'))$$

On the right side of the last equation the first term is a function that depends only on M and not on the fibers, this implies that $\alpha(X')$ must also be a function on M; a general function on M can be written as $f \circ \pi = \pi^* f \equiv \alpha(X')$. Now, the last equation transforms into $\pi^*(\iota(X)\omega + df) = 0$ which implies $\iota(X)\omega + df = 0$ because π^* is injective.
$(\Leftarrow)$ Suppose $\iota(X)\omega + df = 0$, we can use the function f to fix the vertical component of the vector field X', i.e. $\alpha(X') = \pi^* f$. This implies:

$$\mathcal{L}_{X'}\alpha = \iota(X')d\alpha + d(\iota(X')\alpha) = \pi^*(\iota(X)\omega) + d(\pi^* f) = \pi^*(\iota(X)\omega + df) = 0$$

Which concludes the proof. □

Remark. Notice that X' is uniquely determined by the global equations $\pi_* X' = X$ (horizontal component) and $\alpha(X') = \pi^* f$ (vertical component); on the local chart $U_i \times \mathbb{R}/D$ the equation $\pi_* X' = X$ imply:

$$X' = X + X'_V \frac{\partial}{\partial x_i}$$

Where X'_V remains to be determined; using the local expression $\alpha = \pi^*\theta_i + dx_i$, the equation $\alpha(X') = \pi^* f$ gives

$$X' = X + \pi^*(f - \theta_i(X)) \frac{\partial}{\partial x_i} \equiv \xi_f \tag{2.7}$$

From now on, we will denote the Hamiltonian vector fields X as X_f and the corresponding lifted symmetry X' as ξ_f.

Proposition 2.3.2. *The vector fields ξ_f with the usual commutator of vector fields form a faithful representation of the Poisson algebra $\mathcal{P}$.*

Proof. We showed that ξ_f is uniquely determined given an $f \in C^\infty(M)$, we only need to check that the map $f \mapsto \xi_f$ is a morphism of Lie algebras:

$$\begin{aligned}
[\xi_f, \xi_g] &= \left[X_f + \pi^*(f - \theta_i(X_f))\tfrac{\partial}{\partial x_i},\ \ X_g + \pi^*(g - \theta_i(X_g))\tfrac{\partial}{\partial x_i}\right] \\
&= [X_f, X_g] + \pi^*\left(\{f, g\} - \mathcal{L}_{X_f}(\iota(X_g)\theta_i) - \{g, f\} + \mathcal{L}_{X_g}(\iota(X_f)\theta_i)\right)\tfrac{\partial}{\partial x_i} \\
&= X_{\{f,g\}} + \pi^*\left(2\{f, g\} - \iota_{[X_f,X_g]}\theta_i - \iota(X_g)(\mathcal{L}_{X_f}\theta_i) + \mathcal{L}_{X_g}(\iota(X_f)\theta_i)\right)\tfrac{\partial}{\partial x_i} \\
&= X_{\{f,g\}} + \pi^*\Big(2\{f, g\} - \iota_{X_{\{f,g\}}}\theta_i - \iota(X_g)(\iota(X_f)d\theta_i)\Big)\tfrac{\partial}{\partial x_i} \\
&= X_{\{f,g\}} + \pi^*\Big(2\{f, g\} - \iota_{X_{\{f,g\}}}\theta_i - \omega(X_f, X_g)\Big)\tfrac{\partial}{\partial x_i} \\
&= X_{\{f,g\}} + \pi^*\Big(\{f, g\} - \iota_{X_{\{f,g\}}}\theta_i\Big)\tfrac{\partial}{\partial x_i} \\
&= \xi_{\{f,g\}}
\end{aligned}$$

Which shows $f \mapsto \xi_f$ is an injective morphism of Lie algebras, i.e. a faithful representation. □

2.4 Quantization conditions and construction of the quantum Hilbert space

In physics, when one refers to quantization one is referring to an assignment

$$\mathcal{Q} : f \mapsto \mathcal{Q}(f) \tag{2.8}$$

from classical observables $f \in C^\infty(M)$ to operators $\mathcal{Q}(f)$ on some Hilbert space $\mathcal{H}$. The best situation according to Dirac (see [7] chapter IV) would be when this map satisfies the following requirements:

- **Q1**: $\mathbb{R}$-linearity

$$\mathcal{Q}(af + g) = a\mathcal{Q}(f) + \mathcal{Q}(g) \qquad \forall a \in \mathbb{R}, f, g \in C^\infty(M) \tag{2.9}$$

- **Q2**: Lie algebra homomorphism of the Poisson algebra into certain algebra structure on the set of operators on $\mathcal{H}$

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar\mathcal{Q}(\{f, g\}) \tag{2.10}$$

This condition is motivated by the Ehrenfest theorem in quantum mechanics, where observables that commute with the Hamiltonian (and also does not depend explicitly in time) are conserved in mean (i.e. its mean values are constant in time). Also, a result from classical mechanics shows that if a classical observable f satisfies the equation $\{f, H\} = 0$, where H is the Hamiltonian, then f is conserved in time; so, the last homomorphism assure us that the classically conserved quantities will be conserved quantum mechanically in mean. Also notice that when the $\hbar$ constant tends to 0 we recover the usual commutation of functions for the operators $\mathcal{Q}(f)$ and $\mathcal{Q}(g)$; meaning that when one works with macroscopic quantities, where the quantity $\hbar$ can be neglected, quantum effects can be neglected.

- **Q3**: The constant function 1 is mapped to the identity operator

$$\mathcal{Q}(1) = 1 \tag{2.11}$$

- **Q4**: Quantum operators coming from classical observables must be self-adjoint

$$\mathcal{Q}(f) = \mathcal{Q}(f)^\dagger \tag{2.12}$$

Observables in quantum theory are given by the spectrum of the corresponding operators, so this condition will assure that the observables are real numbers.

- **Q5**: To state this condition precisely, we say that the set of functions $\{f_1, f_2, \cdots, f_n\}$ is a complete set of classical observables if any function which has vanishing Poisson brackets with all of them is a constant, the same for a complete set of quantum observables but with the commutator instead of the Poisson brackets. Then the condition is:

If $\{f_1, f_2, \cdots, f_n\}$ *is a complete set of classical observables,*
$\{\mathcal{Q}(f_1), \mathcal{Q}(f_2), \cdots, \mathcal{Q}(f_n)\}$ *is a complete set of quantum observables.*

Although it would be ideal from the point of view of Dirac to impose all these conditions, we have a no-go theorem (usually called Groenewold-Van Hove theorem, see for example [8]) which tell us even in the simplest case of $\mathbb{R}^{2n}$ with its canonical symplectic form, the map $\mathcal{Q}$ can not exist; so we are forced to relax or replace some condition. The way geometric quantization proceeds is by replacing the quantization condition $Q5$, but before going further in this direction, we will construct what is called the prequantum Hilbert space from the quantization conditions $Q1$, $Q2$, $Q3$ and $Q4$.

We start with the symplectic manifold (M, ω) which carries a natural volume form $\varepsilon \equiv \omega^n$ where $dim(M) = 2n$. With this volume form, one would be tempted to define in a canonical way the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(M, \varepsilon)$ of square integrable complex valued functions on M and the quantum operators $\mathcal{Q}(f) = -i\hbar X_f$ (Hamiltonian vector field) associated to the classical observable $f \in C^\infty(M)$. But this definition does not satisfy $Q3$ because $Q(1) = 0$, this is the reason of why we needed the construction of the faithful representation of the Poisson algebra made in the previous sections.

The natural way to proceed is to construct the Hilbert space $\mathcal{H}$ out of functions on Y (where Y comes from the $\mathbb{R}/D$ principal fibre bundle (Y, α)), (Y, α) also have a canonical volume form $\varepsilon \equiv \alpha \wedge (d\alpha)^n$, which is always non-zero as can be checked from the local expression of α in (2.6) (these kind of manifolds (Y, α) where $\alpha \wedge (d\alpha)^n \neq 0$ are called contact manifolds which are a straightforward generalization of symplectic manifolds for the case of odd dimension $dim(Y) = 2n+1$, and α is called contact structure). Again with this form we define the Hilbert space $\mathcal{H} = L^2_{\mathbb{C}}(Y, \varepsilon)$ and the quantum operators $\mathcal{Q}(f) = -i\hbar\xi_f$ (see equation (2.7)), the extra $\hbar$ factor is set so that it coincides with the usual operators in quantum mechanics as we will see later. We still have the problem that $\mathcal{Q}(1) = -i\hbar\partial_{x_i}$ on the whole of $L^2_{\mathbb{C}}(Y, \varepsilon)$, so we define $\mathcal{H}$ to be the subspace of $L^2_{\mathbb{C}}(Y, \varepsilon)$ on which $\mathcal{Q}(1) = 1$. On a local chart $U_i \times \mathbb{R}/D$, for $\psi \in L^2_{\mathbb{C}}(Y, \varepsilon)$:

$$\mathcal{Q}(1)\psi = \psi \;\;\Rightarrow\;\; -i\hbar\frac{\partial\psi}{\partial x_i}(m, x) = \psi(m, x) \;\;\Rightarrow\;\; \psi(m, x) = \psi(m) \cdot exp(ix/\hbar) \quad (2.13)$$

Here we have the problem that when $D = \{0\}$, the integration of $|\psi|^2$ over the fibre $\mathbb{R}$ diverges except when $\psi(m) = 0$; the other possibility is $D = d\mathbb{Z}$ with $d \in \mathbb{R}/\{0\}$, in this case $exp(ix/\hbar)$ should be well defined on $\mathbb{R}/d\mathbb{Z}$ which implies $exp(id/\hbar) = 1 \Rightarrow d \in 2\pi\hbar\mathbb{Z}$. Then we can describe the Hilbert space $\mathcal{H}$ in terms of the action of $\mathbb{R}/2\pi\hbar\mathbb{Z}$ on Y:

$$\mathcal{H} \equiv \mathcal{PQH} = \{\psi \in L^2_{\mathbb{C}}(Y, \varepsilon) \mid \forall a \in \mathbb{R}/2\pi\hbar\mathbb{Z} : \psi(m+a) = \psi(m) \cdot exp(ia/\hbar)\} \tag{2.14}$$

This Hilbert space is called the prequantum Hilbert space, notice that now we have the condition $Per(\omega) \subset 2\pi\hbar\mathbb{Z}$.

Remark. The self-adjointness of the operator $\mathcal{Q}(f) = -i\hbar\eta_f$ comes from the fact that ξ_f is an infinitesimal symmetry of (Y, α) (i.e. $\mathcal{L}_{\xi_f}\alpha = 0$), it follows that the Lie derivative of the volume form $\varepsilon = \alpha \wedge d\alpha$ on Y in the direction of ξ_f is zero which shows that $-i\hbar\xi_f$ is formally a symmetric operator on $L^2_{\mathbb{C}}(Y, \varepsilon)$. Now, if we assume ξ_f is a complete vector field, then its flow is a 1-parameter group of diffeomorphisms of Y which leave ε invariant, on elements of $L^2_{\mathbb{C}}(Y, \varepsilon)$ they define a one-parameter group of unitary transformations $U_t = exp(t\xi_f)$ since ξ_f leaves the inner product (integral) invariant. Stone's theorem on one parameter unitary groups tell us that U_t can be written as $U_t = exp(itA)$ where A is a self-adjoint operator, comparing the 2 expression for U_t we see that $\mathcal{Q}(f) = -i\hbar\xi_f$ is self-adjoint.

Now we see how the prequantum Hilbert space $\mathcal{PQH}$ satisfies the quantization conditions $Q1$, $Q2$, $Q3$ and $Q4$; and we return to the problem of replacing the quantization condition $Q5$, let's analyze what happens in the case $M = T^*\mathbb{R}$ with coordinates (q, p) and standard symplectic form $\omega = dp \wedge dq = d(pdq) = d\alpha$:

$$\mathcal{Q}(q) = i\hbar\frac{\partial}{\partial p} + q(-i\hbar\frac{\partial}{\partial x_i}) = i\hbar\frac{\partial}{\partial p} + q \tag{2.15}$$

$$\mathcal{Q}(p) = -i\hbar\frac{\partial}{\partial q} \tag{2.16}$$

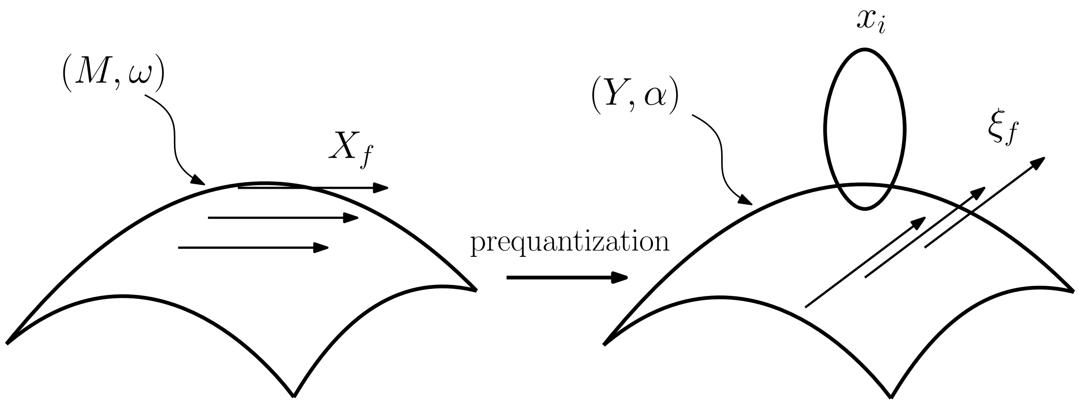
both of them commute with $\frac{\partial}{\partial p}$ and therefore they fail to be a complete set, besides the term $\mathcal{Q}(q)$ has an extra term $i\hbar\frac{\partial}{\partial p}$ compared to the usual q operator in quantum mechanics, they coincide if the elements of $\mathcal{PQH}$ depend only on q (as it is usual for the wave function in quantum mechanics). This observation suggests us that we should reduce the prequantum Hilbert space $\mathcal{PQH}$ to elements only depending on "q" coordinates, the way to do that for a general symplectic manifold is by choosing a maximally commuting set of observables $\{F_k\}$ (like the set $\{q^1, q^2, \cdots, q^n\}$). Then the quantization condition $Q5$ is replaced by the condition of finding a maximally commuting set of observables $\{F_k\}$ and reducing the prequantum Hilbert space $\mathcal{PQH}$ to elements only depending on $\{F_k\}$, the resulting Hilbert space is called quantum Hilbert space and denoted by $\mathcal{QH}$.

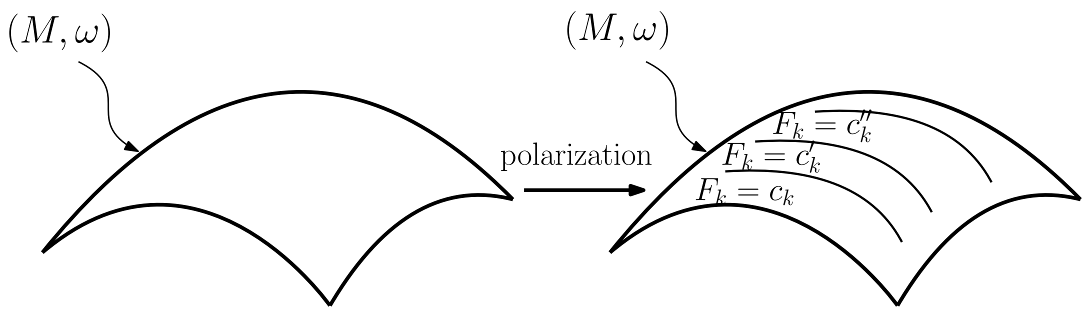
The procedure of choosing a maximally commuting set of observables $\{F_k\}$ is called polarization, the condition

$$\{F_i, F_j\} = \omega(X_{F_i}, X_{F_j}) = -dF_i(X_{F_j}) = 0 \qquad \forall i, j \tag{2.17}$$

tell us that the Hamiltonian vector fields X_{F_i} are tangent to $N = \bigcap_i \{F_i = c_i\}$ for some constants c_i so that they locally span TN. Also from the last equation we see that $\omega|_{TN} = 0$ and by the fact that it a is maximally set $dim(N) = \frac{1}{2}dim(M) = n$, these kind of submanifolds are called a Lagrangian. By varying the constants c_k in the equations $F_k = c_k$ one obtains a foliation, this also comes from the polarization condition $[X_{F_i}, X_{F_j}] = X_{\{F_i, F_j\}} = 0$ (integrability condition from Frobenius theorem).

Schematically:





Before going into examples, let us check the relation of geometric quantization and path integrals for the case $M = \mathbb{R}^{2n}$. The Hamiltonian vector field X_f generates a flow

$$\Phi_t^f : m \mapsto \Phi_t^f(m) \tag{2.18}$$

of canonical transformations of M. For an element $\psi \in \mathcal{PQH}$ the action of the lifted Hamiltonian vector field ξ_f is given by the pull-back:

$$\hat{\Phi}_t^f : \psi \mapsto exp(t\xi_f)^*(\psi) \tag{2.19}$$

Taking a partition $0 < t_1 < \cdots < t_p < t$ and introducing the term $\mathcal{L}_f = \theta_i(X_f) - f$:

$$\begin{aligned}
(\hat{\Phi}_t^f\psi)(m) &\approx \cdots exp[(t_2-t_1)X_f - \tfrac{\hbar}{i}(t_2-t_1)\mathcal{L}_f(i\hbar\partial_{x_i})]^* exp[t_1X_f - (t_1-0)\mathcal{L}_f(i\hbar\partial_{x_i})]^*\psi(m) \\
&\approx \cdots exp[(t_2-t_1)X_f - \tfrac{\hbar}{i}(t_2-t_1)\mathcal{L}_f]^* exp[t_1X_f - (t_1-0)\mathcal{L}_f]^*\psi(m) \\
&\approx \cdots exp[-i(t_2-t_1)\hbar X_f - (t_2-t_1)\mathcal{L}_f(\Phi_{t_1}^f(m)) - (t_1-0)\mathcal{L}_f(m)]^*\psi(\Phi_{t_1}^f(m))
\end{aligned} \tag{2.20}$$

In the limit $p \to \infty$:

$$(\hat{\Phi}_t^f\psi)(m) = \psi(\Phi_t^f(m)) \cdot exp\left(-\frac{i}{\hbar}\int_0^t \mathcal{L}_f(\Phi_{t'}^f(m))dt'\right) \tag{2.21}$$

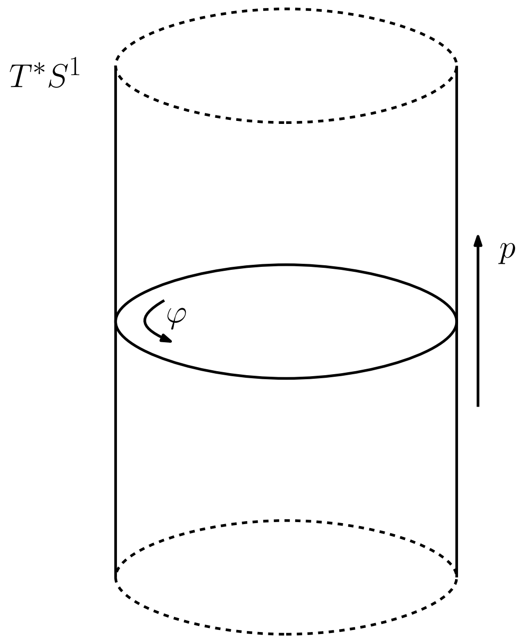
Thus evolution in "time" is given by the exponential of the classical action. Notice that in coordinates:

$$\mathcal{L}_f = \theta_i(X_f) - f = p_k\frac{\partial f}{\partial p_k} - f \tag{2.22}$$

So in fact $\mathcal{L}_f$ is the lagrangian from classical mechanics when f is the Hamiltonian.

There is one extra step in geometric quantization called Half-form correction (or metaplectic correction) but we are not going a treat it here because it is too technical and requires much more abstract language (for reference, see [13]). Now, let's look at some examples:

Example 2.4.1. $M = T^*S^1$



In this case we have $\omega = dp \wedge d\varphi = d(pd\varphi)$, since ω is exact, then the period group is $\{0\}$ and the prequantum bundle (Y, α) is trivial with connection:

$$\alpha = pd\varphi + dx_i \tag{2.23}$$

We mentioned that inequivalent prequantum bundles are classified by $H^1(M, \mathbb{R}/D) = Hom(\pi_1(M), \mathbb{R}/D)$, from $\pi_1(T^*S^1) = \mathbb{Z}$ we obtain that

$$H^1(T^*S^1, U(1)) = Hom(\mathbb{Z}, U(1)) = U(1) \tag{2.24}$$

by looking at the homomorphism $f : \mathbb{Z} \to U(1)$ where $f(n) = g^n$ for a fixed $g \in U(1)$. So we expect $U(1)$ inequivalent prequantizations, which are generated by the fact that $d\varphi$ is closed but not exact (if it were exact $\int_{S^1} d\varphi = 0$ from Stokes theorem). Thus we can modify the connection to:

$$\alpha_\lambda = (p + \lambda)d\varphi + dx_i \ , \qquad \lambda \in [0, 1) \simeq U(1) \tag{2.25}$$

note that we still have the equation $\omega = d\alpha$. The modified prequantum operator for the momentum is

$$\mathcal{Q}^\lambda(p) = -i\hbar\frac{\partial}{\partial\varphi} + \lambda, \tag{2.26}$$

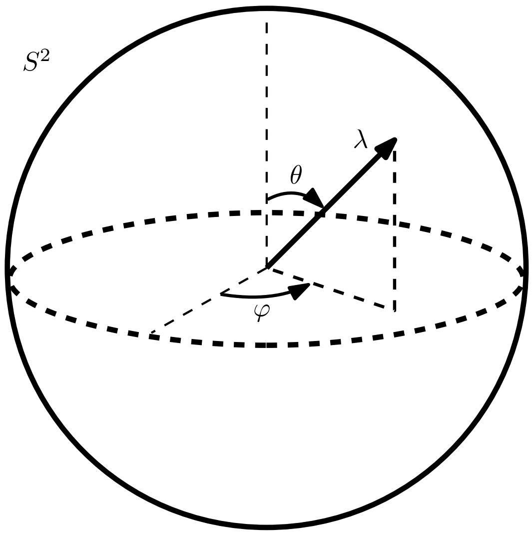
to calculate its spectrum notice that a general element $\psi \in L^2(S^1)$ can be written as $\psi(\varphi) = \sum_{n \neq 0} \psi_n e^{in\varphi}$, so in the end we get:

$$Spec(\mathcal{Q}^\lambda(p)) = \{(n+\lambda)\hbar,\ n \in \mathbb{Z}\} \tag{2.27}$$

This shows that inequivalent quantum theories are obtained from inequivalent prequantizations (Y, α_λ) parametrized by $\lambda \in [0,1) \simeq U(1)$.

The parameter λ contributes to the holonomy picked up by a state upon parallel transport around S^1. It can thus be regarded as a toy-model of the Aharonov-Bohm effect, where $\lambda d\varphi$ represents the magnetic field running through the interior of S^1. Note that connections differing by exact 1-forms df are related by gauge transformations $x_i \to x_i + f$ on the fibre bundle $\mathbb{R}/(2\pi\mathbb{Z})$, in our case we have connections differing by a non-exact 1-form $\lambda d\varphi$ which are related by 'large' gauge transformations (those not homotopic to the identity) which lead to inequivalent quantum theories.

Example 2.4.2. $M = S^2$

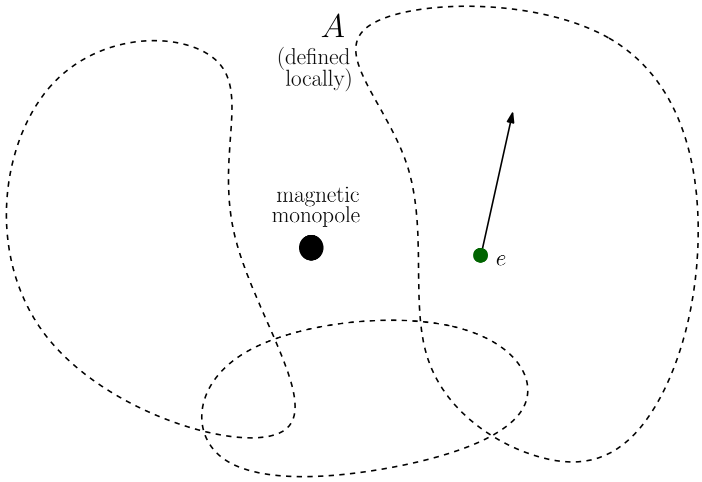


Here we have the canonical symplectic form $\omega = \lambda\ sin\theta\ d\theta \wedge d\varphi$, the group of periods $Per(\omega)$ is obtained integrating ω over S^2 ($\int_{S^2} \omega = 4\pi\lambda$)

$$Per(\omega) = 4\pi\lambda\mathbb{Z} \tag{2.28}$$

It follows that the quantization condition $Per(\omega) \subset 2\pi\hbar\mathbb{Z}$ is equivalent to $\lambda = \frac{1}{2}n\hbar$, $n \in \mathbb{Z}$, giving the well known quantization of spin if we interpret λ as the classical spin and S^2 as its phase space. From the fact that $\pi_1(S^2) = \{0\}$ we obtain $H^1(S^2, U(1)) = Hom(\{0\} \to U(1)) = \{0\}$, which tell us that there is a unique prequantum bundle (Y, α_λ) for each admissible value of λ (topological monopole sector).

Example 2.4.3. Quantization condition on the electric charge



Consider the cotangent bundle T^*Q with its standard symplectic form $\omega = d\theta$ where θ is the Liouville form (also called Tautological form), in local coordinates $(q_1, \cdots, q_n)$ on Q

$$\theta = \sum_j p_j dq_j \tag{2.29}$$

From the standard minimal coupling prescription $p_j \to p_j - eA_j(q^k)$ the standard symplectic from can be written in local coordinates as:

$$\omega_F = \sum_j d(p_j - eA_j(q^k)) \wedge dq_j = \sum_j dp_j \wedge dq_j - e\sum_{j,l} \frac{\partial A_j}{\partial q_l} dq_l \wedge dq_j = \omega + eF \tag{2.30}$$

where $F = d(A_j dq_j)$ (summation convention assumed). Thus the quantization condition of (T^*Q, ω_F) is equivalent to the integrality of the form $\frac{e}{2\pi\hbar}F$. If F represents a non-trivial cohomology class (this is the case for a charged particle moving in the field of a magnetic monopole where A is only defined locally), then the values of e are restricted and the prequantum bundle (Y, α) will be non-trivial as well. As a note aside, the coupling constant quantization conditions in some field theories like WZW model and topological massive gauge theory can be understood in the same way.

In the next chapter we will proceed with a brief review of supermanifolds in order to understand geometric quantization in the context of Grassmann coordinates instead of usual 'bosonic' coordinates.

Capítulo 3

Review of supergeometry

3.1 Supermanifolds

Definition 3.1.1. Let X be a topological space, and let $\mathbf{C}$ be a category. A **presheaf** F on X is a functor with values in $\mathbf{C}$ given by the following data:

- For each open set U of X, there corresponds an object $F(U)$ in $\mathbf{C}$.
- For each inclusion of open sets $V \subseteq U$, there corresponds a morphism $\mathrm{res}_{V,U} \colon F(U) \to F(V)$ in the category C.

The morphisms $\mathrm{res}_{V,U}$ are called restriction morphisms. If $s \in F(U)$, then its restriction $\mathrm{res}_{V,U}(s)$ will be denoted by $s|_V$. The restriction morphisms are required to satisfy two properties:

- For every open set U of X, the restriction morphism $\mathrm{res}_{V,V} \colon F(V) \to F(V)$ is the identity morphism on $F(V)$.
- If we have three open sets $W \subseteq V \subseteq U$, then the composite $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U}$ equals $\mathrm{res}_{W,U}$.

Definition 3.1.2. A **sheaf of supercommutative algebras** is a presheaf with values in the category of supercommutative algebras that satisfies the following two axioms:

- (Locality) If (U_i) is an open covering of an open set U, and if $s, t \in F(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each set U_i of the covering, then $s = t$.

- (Gluing) If (U_i) is an open covering of an open set U, and if for each i a section $s_i \in F(U_i)$ is given such that for each pair U_i, U_j of the covering sets the restrictions of s_i and s_j agree on the overlaps: $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each i.

Definition 3.1.3. An $(n|m)$-supermanifold $\mathcal{M}$ is a sheaf $\mathcal{O}_{\mathcal{M}}$, over a smooth n-manifold M (the body of $\mathcal{M}$), of supercommutative algebras locally isomorphic to algebras of form $C^\infty(U) \otimes \wedge^\bullet V^*$ with $U \subset M$ open and V a fixed m-dimensional vector space, i.e., there is an atlas on M comprised by open subsets $U_\alpha \subset M$ with chart maps $\phi_\alpha : U_\alpha \to W = \mathbb{R}^n$ with isomorphisms of supercommutative algebras $\Phi_\alpha : \mathcal{O}_{\mathcal{M}}(U_\alpha) \to C^\infty(\phi_\alpha(U_\alpha)) \wedge^\bullet V^* \equiv \mathcal{A}_\alpha$

Locally a function on $\mathcal{M}$ is an element of $\mathcal{A}_\alpha$, i.e., has local form

$$f|_{U_\alpha} = \sum_k \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq m} f_{i_1 \ldots i_k}(x) \cdot \theta_{i_1} \ldots \theta_{i_k}$$

with $x_1, x_2, \ldots, x_n$ the local even coordinates on M and $\theta_1, \theta_2, \ldots, \theta_n$ the odd coordinates on V.

Example 3.1.1. (Split supermanifolds) Let $E \to M$ be a rank m vector bundle over an n-manifold M. We can define a split $(n|m)$-supermanifold ΠE with body M and sheaf $\mathcal{O}_{\Pi E} \equiv \Gamma(M, \wedge^\bullet E^*)$, here $\Gamma(M, -)$ denotes smooth sections over M and note that $\wedge^\bullet E^*$ can be seen as a supercommutative algebra.
Applying the construction above to the tangent and cotangent bundle of an n-manifold M, we obtain two distinguished $(n|n)$-supermanifolds ΠTM and ΠT^*M.

Remark. The augmentation map $\wedge^\bullet V^* \to \mathbb{R}$ (i.e. homomorphism which maps an element of the graded algebra to its homogeneous component of degree 0) induces a globally well-defined augmentation map $\mathcal{O}_{\mathcal{M}} \to C^\infty(M)$

Definition 3.1.4. A morphism of supermanifolds $\phi : \mathcal{M} \to \mathcal{N}$ consists of the data of:

- A smooth map between the bodies $f : M \to N$,

- An extension of f to a morphism of sheaves of supercommutative algebras $\phi^* : \mathcal{O}_\mathcal{N} \to \mathcal{O}_\mathcal{M}$. In particular, for an open set $U \subset N$, we have a morphism $\phi^*_U : \mathcal{O}_\mathcal{N}(U) \to \mathcal{O}_\mathcal{M}(f^{-1}(U))$ commuting with the augmentation maps:

$$\begin{array}{ccc} \mathcal{O}_\mathcal{N}(U) & \xrightarrow{\phi^*} & \mathcal{O}_\mathcal{M}(f^{-1}(U)) \\ \downarrow & & \downarrow \\ C^\infty(U) & \xrightarrow{f^*} & C^\infty(f^{-1}(U)) \end{array}$$

Theorem 3.1.1. *(Batchelor's theorem) Every smooth supermanifold with body M is non-canonically isomorphic to a split-supermanifold ΠE for some vector bundle $E \to M$.*

Definition 3.1.5. A vector field $v \in \mathfrak{X}(\mathcal{M})$ of parity $|v| \in [0,1]$ (with the convention $0 =$even, $1 =$odd) is a derivation of $\mathcal{O}_\mathcal{M}$ of parity $|v|$, i.e., an $\mathbb{R}$-linear map $v : \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}$ satisfying

$$\begin{aligned} & v(f \cdot g) = v(f) \cdot g + (-1)^{|v||f|} f \cdot v(g) \\ & |v(f)| = |v| + |f| \;\; mod\, 2 \end{aligned}$$

Vector fields on $\mathcal{M}$ form a Lie superalgebra with Lie bracket

$$[v, w] \equiv v \circ w - (-1)^{|v||w|} w \circ v$$

3.2 Odd-symplectic manifolds

The generalization of symplectic manifolds for the odd case is straightforward:

Definition 3.2.1. Let $\mathcal{M}$ be a supermanifold, an odd-symplectic structure on $\mathcal{M}$ is a closed 2-form ω which is:

- closed, $d\omega = 0$
- odd form, i.e. in local coordinates x_i, θ_α on $\mathcal{M}$, with x^i even and θ^α odd, $\omega = \sum_{i,\alpha} \omega_{i,\alpha}(x,\theta)\, dx^i \wedge d\theta^\alpha$ where $\omega_{i,\alpha}(x,\theta)$ is a matrix of local functions on $\mathcal{M}$.
- non-degenerate, the matrix $\omega_{i,\alpha}(x,\theta)$ is invertible.

We call $(\mathcal{M}, \omega)$ an odd-symplectic manifold if $\mathcal{M}$ is a supermanifold endowed with an odd-symplectic structure.

Properties of odd-symplectic manifolds are summarized in the following theorem (for details of the proof see [9]):

Theorem 3.2.1. *Let $(\mathcal{M}, \omega)$ be an odd symplectic manifold with body M ($dim(M) = n$).*

- *The dimension of $\mathcal{M}$ is $(n|n)$. (This comes from the non-degeneracy of the odd-symplectic structure)*
- *In the neighborhood of any point of M one can find local coordinates (x^i, θ_i) on $\mathcal{M}$ such that $\omega = \sum_i dx^i \wedge d\theta_i$ (Darboux coordinates)*
- *There exists a global symplectomorphism $\psi : (\mathcal{M}, \omega) \to (\Pi T^* M, \omega_{stand})$ where $\omega_{stand} = \sum_i dx^i \wedge d\theta_i$ (the standard odd-symplectic structure on $\Pi T^* M$)*

The first 2 results of this theorem are natural generalizations from usual symplectic geometry, the first one related to the even dimension of symplectic manifolds and the second one related to the Darboux coordinates. The third result (very much unlike the case of ordinary simplectic geometry) tell us that up to symplectomorphism all odd-symplectic manifolds are odd-cotangent bundles. This last result is of particular interest for prequantization in the odd case, as we will see in the next chapter.

Capítulo 4

Pre-quantization: odd case

Now, following the same steps as we did for the non-super case, we will study the pre-quantization procedure for the case of an odd-symplectic manifold $(\mathcal{M}, \omega)$. Recall that we needed a connection 1-form α defined on a principal fibre bundle $\pi : \mathcal{Y} \to \mathcal{M}$ such that $d\alpha = \pi^*\omega$; this implies (since ω is odd) that our connection α must be odd and that our structure group is $\mathbb{R}^{0|1}$ (for the odd case we can not have the discrete subgroup D because there is no constant Grassmann variables on $\mathcal{M}$).

4.1 Construction of the $\mathbb{R}^{0|1}$ bundle over $(\mathcal{M}, \omega)$

We start by choosing an open cover $\mathcal{U} = \{U_i \mid i \in I\}$ of an odd-symplectic manifold $(\mathcal{M}, \omega)$ such that all finite intersections $U_{i_1} \cap \cdots \cap U_{i_k}$ are either contractible or empty. Applying the super Poincaré lemma (repeating the same steps as in the non-super case) we have:

$$\begin{cases} d\theta_i = \omega & \text{on } U_i \\ \theta_i - \theta_j = df_{ij} & \text{on } U_i \cap U_j \\ f_{ij} + f_{jk} + f_{ki} = a_{ijk} & \text{on } U_i \cap U_j \cap U_k \end{cases} \tag{4.1}$$

But the difference with the non-super case is that now we have odd functions f_{ij} and odd 1-forms θ_i, thus the condition $d(a_{ijk}) = d(f_{ij} + f_{jk} + f_{ki}) = 0$ implies that a_{ijk} is even and real which is only possible if $a_{ijk} = 0$. Now, let ρ_i be a partition of unity subordinated to $\mathcal{U}$, we can redefine the local odd 1-forms θ_i to $\hat{\theta}_i = \theta_i + d(\sum_k \rho_k f_{ki})$

defined on U_i. On the intersection $U_i \cap U_j$:

$$\hat{\theta}_i - \hat{\theta}_j = \theta_i - \theta_j + d(\sum_k (\rho_k(f_{ki} - f_{kj}))) = d(f_{ij}) + d((\sum_k \rho_k)(-f_{ij})) = 0$$

Them $\hat{\theta}_i$ define a global 1-form $\hat{\theta}$ which satisfies $d\hat{\theta} = \omega$. Our last result can be stated as follows:

Proposition 4.1.1. *Any closed odd 2-form is exact.*

This tell us that the group of periods is trivial and we don't need a quantization condition. So in the end we can summarize the prequantization procedure in the following theorem:

Theorem 4.1.2. *Let $(\mathcal{M}, \omega)$ be and odd-symplectic manifold with body M.*

- *There exists a $\mathbb{R}^{0|1}$-bundle $\mathcal{Y}$ with connection $\alpha = \theta + d\eta_i$ where θ is a global 1-form satisfying $d\theta = \omega$ and η_i is a global coordinate on $\mathbb{R}^{0|1}$*
- *For each function f on $\mathcal{M}$ there exists a unique vector field $\xi_f \in Symm(\mathcal{Y}, \alpha)$ given in coordinates by*

$$\xi_f = X_f + (f - \iota(X_f)\theta)\frac{\partial}{\partial \eta_i} \tag{4.2}$$

with the following properties

$$\pi_* \xi_f = X_f, \qquad \mathcal{L}_{\xi_f}\alpha = 0, \qquad \alpha(\xi_f) = \pi^* f \tag{4.3}$$

- *The map $f \to \xi_f$ is an isomorphism of Lie algebras. (Faithful representation of the Poisson algebra)*

Capítulo 5

WKB method

In the context of quantum mechanics, the WKB method is a way to obtain approximate solutions to the Schrödinger equation (to first order in $\hbar$) which will be called semiclassical states. These solutions can be formalized as being Lagrangian submanifolds of the phase space of the classical system equipped with a half-density. The aim of these sections will be to explain this method and see its relation to geometric quantization.

5.1 Lagrangian submanifolds as classical states

Let's consider a 1-dimensional system with hamiltonian

$$H(q,p) = \frac{p^2}{2m} + V(q) \tag{5.1}$$

The corresponding time-dependent Schrödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{5.2}$$

The stationary states are obtained using the ansatz $\psi(x,t) = \varphi(x)e^{-i\omega t}$, which give us the time-independent Schrödinger equation

$$\hat{H}\varphi = E\varphi \qquad\qquad \text{where } E = \hbar\omega \tag{5.3}$$

Now we use the WKB ansatz $\varphi(x) = e^{\frac{iS(x)}{\hbar}}$ and we obtain the following equation:

$$(\hat{H}-E)\varphi = \left[\frac{(S'(x))^2}{2m} + (V-E) - \frac{i\hbar}{2m}S''(x)\right]e^{\frac{iS(x)}{\hbar}} = 0 \tag{5.4}$$

To order $\mathcal{O}(\hbar^0)$ we get the Hamilton-Jacobi equation

$$H(x, S'(x)) = \frac{(S'(x))^2}{2m} + V(x) = E \tag{5.5}$$

Solving for $S'(x)$ (note that $S'(x)$ is the canonical momentum)

$$S'(x) = \pm\sqrt{2m(E - V(x))} = p \tag{5.6}$$

The differential $dS = S'(x)dx$ can be viewed as a mapping $dS : \mathbb{R} \to T^*\mathbb{R}$ defined by $x \mapsto S'(x)dx$. Then S is a solution of the Hamilton-Jacobi equation if the image of dS lies in the level manifold $H^{-1}(E)$ (this notation comes from $H(q,p) = E$)

For the case of a n-dimensional system with hamiltonian

$$H(q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n) = \sum_j \frac{p_j^2}{2m} + V(q_1, q_2, \cdots, q_n) \tag{5.7}$$

We get the Hamilton-Jacobi equation

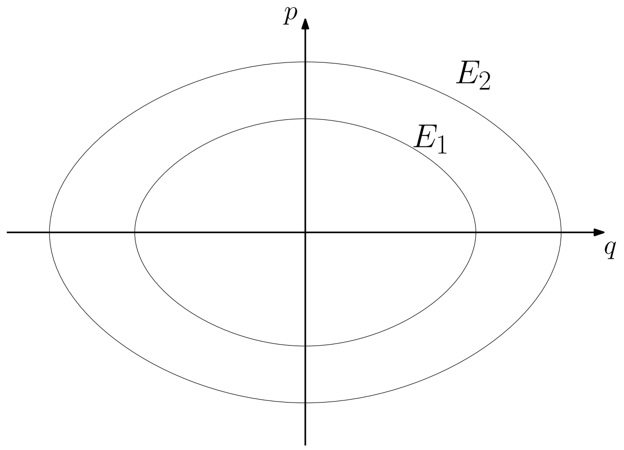
$$\frac{||\nabla S||^2}{2m} + V = E \tag{5.8}$$

If there exists a solution $S : \mathbb{R}^n \to \mathbb{R}$ to the Hamilton-Jacobi equation then $L \equiv im(dS)$ is characterized by 3 geometric properties:

1. L is a submanifold of $H^{-1}(E)$ (because S is a solution to the Hamilton-Jacobi equation)
2. The pullback to L of the form $\alpha = \sum_j p_j dq_j$ on $\mathbb{R}^{2n}$ is exact (because $p_j = \frac{\partial S}{\partial q_j}$)
3. The restriction of the canonical projection $\pi : T^*\mathbb{R} \to \mathbb{R}^n$ to L induces a diffeomorphism $L \simeq \mathbb{R}^n$ (because S is the inverse of this restriction)

Applications, like the harmonic oscillator, show us that the only geometric property that we must preserve is a weakened version of condition (2), in which 'exact' is replaced by 'closed'. Such submanifolds are called lagrangian.

Example 5.1.1. 1-dimensional harmonic oscillator



Here $L = im(dS)$ is 1-dimensional, which implies the pullback to L of the 1-form $\alpha = pdq$ is closed. But if we assume α is exact, i.e. $\alpha = df$, using Stoke's theorem

$$\text{Area of the Ellipse} = \int_C dq \wedge dp = -\int_C d\alpha = -\int_{\partial C = H^{-1}(E)} \alpha = -\int_{\partial^2 C} f = 0$$

which only holds for the particular case $E = 0$, then the 1-form α must not be exact. also L is not diffeomorphic to $\mathbb{R}$

This example show us that the classical state of a system should be represented by a lagrangian submanifold L rather than by a phase function S.

5.2 Lagrangian submanifolds equipped with a half-density as semiclassical states

The WKB ansatz $\varphi(x) = e^{\frac{iS(x)}{\hbar}}$ satisfies $|\varphi(x)| = 1$, this is a problem since in quantum mechanics $|\varphi(x)|^2$ represents the probability of the particle being at the position x. We can modify the WKB ansatz φ multiplying it by an 'amplitude function' a and work with the folllowing ansatz $\varphi(x) = a(x)e^{\frac{iS(x)}{\hbar}}$, replacing this in the time-independent Schrödinger equation we obtain:

$$(\hat{H} - E)\varphi = -\frac{\hbar^2}{2m}\sum_j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\left[a(x)e^{\frac{iS(x)}{\hbar}}\right] + (V - E)a(x)e^{\frac{iS(x)}{\hbar}} = 0 \tag{5.9}$$

But $S(x)$ is a solution to the Hamilton-Jacobi equation $||\nabla S||^2 = -2m(V - E)$ which implies

$$(\hat{H} - E)\varphi = -\frac{1}{2m}\left[i\hbar\left(a\Delta S + 2(\nabla a)\cdot(\nabla S)\right) + \hbar^2 \Delta a\right] e^{\frac{iS}{\hbar}} = 0 \tag{5.10}$$

To order $\mathcal{O}(\hbar^1)$ we get the so-called homogeneous transport equation

$$a\Delta S + 2(\nabla a)\cdot(\nabla S) = 0 \tag{5.11}$$

Multiplying by a and factorizing:

$$\nabla \cdot (a^2 \nabla S) = 0 \tag{5.12}$$

Here $a^2\nabla S$ is a vector field on $\mathbb{R}^n$ (the space with coordinates $\{x_j\}$), we can lift this condition to the lagrangian submanifold $L = im(dS)$, i.e. where $p_j = \frac{\partial S}{\partial q_j}$. Notice first that the restriction of X_H to the lagrangian submanifold is (here we set for simplicity $m = 1$ in the hamiltonian):

$$\begin{aligned} X_H|_L &= \sum_j \left(\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} \right)\Bigg|_L \\ &= \sum_j \left(p_j \frac{\partial}{\partial q_j} - \frac{\partial V}{\partial q_j}\frac{\partial}{\partial p_j} \right)\Bigg|_L \\ &= \sum_j \left(\frac{\partial S}{\partial q_j}\frac{\partial}{\partial q_j} - \frac{\partial V}{\partial q_j}\frac{\partial}{\partial p_j} \right) \end{aligned} \tag{5.13}$$

If we denote by $\pi : L \to \mathbb{R}^n$ the projection to $\mathbb{R}^n$ then $\pi_*(X_H|_L) = \sum_j \frac{\partial S}{\partial q_j}\frac{\partial}{\partial q_j} = \nabla S$, this allow us to write the homogeneous transport equation as $\nabla \cdot (a^2\pi_*(X_H|_L)) = 0$. Using the property $\mathcal{L}_X\mu = (\nabla \cdot X)\mu$ with $\mu = |dx| \equiv |dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n|$ on $\mathbb{R}^n$ and $X = a^2\pi_*(X_H|_L)$ we obtain

$$\mathcal{L}_{a^2\pi_*(X_H|_L)}\mu = (\nabla \cdot (a^2\pi_*(X_H|_L)))\mu = 0 \tag{5.14}$$

With the help of the Cartan's magic formula $\mathcal{L}_X = d\iota_X + \iota_X d$ we can transfer the factor a^2 to the canonical density $\mu = |dx|$

$$\mathcal{L}_{\pi_*(X_H|_L)}(a^2|dx|) = 0 \tag{5.15}$$

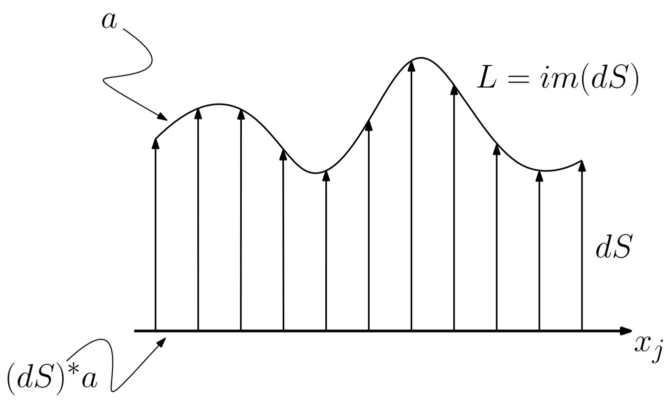
Now, using once again the Cartan's magic formula and the fact that the pullback map commutes with the exterior derivative d, the last equation transforms into

$$\mathcal{L}_{(X_H|_L)}(\pi^* a^2|dx|) = 0 \tag{5.16}$$

Then $\pi^*a^2|dx|$ is a density on L invariant under X_H, which suggest that a solution of the homogeneous transport equation should be represented by a half-density a, invariant under X_H.

As a consequence we have the following result:

> If S is a solution to the Hamilton-Jacobi equation and a is a half-density on $L = im(dS)$ which is invariant under X_H, then $e^{\frac{iS}{\hbar}}(dS)^*a$ is a second-order approximate solution to the time-independent Schrödinger equation.



Up to this point we have (for the case $\mathbb{R}^{2n}$):

Semiclassical state	Lagrangian submanifold L of $\mathbb{R}^{2n}$ and half-density a on L
Stationary semiclassical state	$L \subset H^{-1}(E)$, $\mathcal{L}_{X_H} a = 0$

5.3 WKB method for cotangent bundles

5.3.1 About cotangent bundles

Let's first set some notation about cotangent bundles, the cotangent bundle T^*M of any manifold M is equipped with a natural 1-form, known as the Liouville form (also called Tautological form or symplectic potential), defined by

$$\theta_M((x,y))(v) = y(\pi_* v), \tag{5.17}$$

where $\pi : T^*M \to M$ is the canonical projection and y is a 1-form on M. From the local coordinates $(x_1, \cdots, x_2)$ on M we get the corresponding local coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ on T^*M by the equations

$$q_j(x,y) = x_j \qquad\qquad p_j(x,y) = y\left(\frac{\partial}{\partial x_j}\right) \tag{5.18}$$

which imply

$$\theta_M = \sum_{j=1}^{n} p_j dq_j \tag{5.19}$$

Definition 5.3.1. Given a cotangent bundle T^*M, if L is a smooth manifold of dimension $n = dim(M)$ and $\iota : L \to T^*M$ is an immersion such that $\iota^*\omega_M = 0$ (ω_M is the standard symplectic form on T^*M), we say that the pair (L, ι) is a **lagrangian immersion**.

(L, ι) is called projectable if $\pi_L = \pi \circ \iota$ is a diffeomorphism. We can parametrize projectable lagrangian submanifolds with the help of the Liouville form, in order to do this we will denote ι_η to denote a 1-form η on M when we want to think of it as a map from M to T^*M.

Lemma 5.3.1. *Let η be a 1-form on M, then*

$$\iota_\eta^*\theta_M = \eta \tag{5.20}$$

Proof. ι_η satisfies $\pi \circ \iota_\eta = id$ because it is a section. Using the definition of θ_M, it follows that for each $v \in T_pM$

$$\iota_\eta^*\theta_M(p)(v) = \theta_M(\iota_\eta(p))(\iota_{\eta*}v) = \eta(\pi_*\iota_{\eta*}v) = \eta((\pi \circ \iota_\eta)_*v) = \eta(v) \tag{5.21}$$

what was to be shown. □

Taking exterior derivative to the last equation (and using the fact that it commutes with the pullback map)

$$d\eta = d\iota_\eta^*\theta_M = \iota_\eta^* d\theta_M = -\iota_\eta^*\omega_M \tag{5.22}$$

This equation tell us that the image of ι_η is a lagrangian submanifold of T^*M if and only if η is closed; as a consequence we have the following result:

Corollary. *The relation $\eta \leftrightarrow (M, \iota_\eta)$ defines a bijective correspondence between closed 1-forms on M and the set of projectable lagrangian submanifolds of T^*M.*

Now, from the WKB terminology, we say that $S : M \to \mathbb{R}$ is a phase function for a projectable lagrangian embedding $(L, \iota) \subset T^*M$ (recall that an embedding is an injective immersion) provided that $\iota(L) = dS(M)$. Then, the preceding remarks imply:

Lemma 5.3.2. *If $(L, \iota) \subset T^*M$ is a projectable lagrangian embedding, then $S : M \to \mathbb{R}$ is a phase function if and only if $d(S \circ \pi_L \circ \iota) = \iota^*\theta_M$.*

5.3.2 WKB approximation

Recall that $e^{i\frac{S}{\hbar}}(dS)^*a$ is a second-order approximate solution to the time-independent Schrödinger equation provided that the phase function S and the half-density a satisfy the equations:

$$H \circ \iota_{dS} = E \qquad \text{Hamilton-Jacobi equation} \tag{5.23}$$

$$a\Delta S + 2\mathcal{L}_{\nabla S}a = 0 \qquad \text{Homogeneous transport equation} \tag{5.24}$$

We can formulate this construction for the cotangent bundle T^*M by considering:

- A projectable, exact lagrangian embedding $\iota : L \to T^*M$
- For any primitive $\phi : L \to \mathbb{R}$ of $\iota^*\alpha_M$ (i.e. $d\phi = \iota^*\alpha_M$), the composition $S = \phi \circ \pi_L^{-1}$ is a phase function for (L, ι). (see Lemma 5.3.2)
- If $H : T^*M \to \mathbb{R}$ is any smooth function, then (L, ι) satisfies the Hamilton-Jacobi equation provided that E is a regular value of H and $H \circ \iota = E$
- If a is a half-density on L, then a satisfies the homogeneous transport equation if $\mathcal{L}_{X_H}a = 0$.

Suppose the last 4 conditions holds, if that's the case, then the half-density $e^{i\frac{\phi}{\hbar}}a$ can be pulled-back to yield a second-order approximate solution $(\pi_L^{-1})^*e^{i\frac{\phi}{\hbar}}a$ to the time-independent Schrödinger equation on M. Then, we have for the cotangent bundle case:

Semiclassical state	(L, ι, ϕ, a) where ι is a projectable and exact
Stationary semiclassical state	(L, ι, ϕ, a) such that $H \circ \iota = E$ and $\mathcal{L}_{X_H} a = 0$

In cases like the harmonic oscillator, the lagrangian submanifold L is neither projectable nor exact so we must generalize this procedure.

5.3.3 Non-exact lagrangian submanifold case

To generalize the last procedure, suppose now that $\iota : L \to T^*M$ is a projectable but not necessarily exact lagrangian embedding. Using the Poincaré lemma, $d\iota^*\theta_M = 0$ on L implies that we can choose a good cover $\{L_i\}$ of L and functions $\phi_i : L_i \to \mathbb{R}$ such that $d\phi_i = \iota^*\theta_M|_{L_i}$. Given a (global) half density on L, we set $a_i = a|_{L_i}$; and define a half density on $\pi_L(L_i)$ by quantizing $(L_i, \iota|_{L_i}, \phi_i, a_i)$ in the sense above

$$I_i = (\pi_{L_i}^{-1})^* e^{\frac{i\phi_i}{\hbar}} a_i \tag{5.25}$$

To make this quantity a globally well-defined half-density we need to impose a condition over ϕ_i, notice that a_i is already globally well-defined. We need that the functions ϕ_i agree where their domains overlap:

$$\phi_i - \phi_j \in 2\pi\hbar \cdot \mathbb{Z} \equiv \mathbb{Z}_\hbar \qquad \text{on } L_i \cap L_j \tag{5.26}$$

From the equivalence of de Rham and Čech cohomology and the equation $d\phi_i = \iota^*\theta_M|_{L_i}$, the last condition is the same as:

$$\int_{L_i} \iota^*\theta_M \ \in \mathbb{Z}_\hbar \tag{5.27}$$

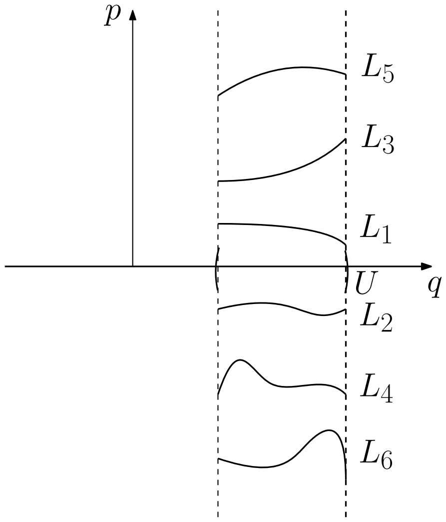
Notice that the functions ϕ_i are defined only up to an additive constant that must be fixed.

For the moment we cannot treat the non-projectable case (since π_L cannot be used to push-forward half-densities from L to M because there is no π_L^{-1}), the most we can do is work with the set of regular points of π_L in order to pass from half-densities on L to half-densities defined near non-caustic points of L.

If $p \in \pi_L(L)$ is non-caustic and π_L is proper (i.e. the preimage under π_L of a compact set is compact), then there is a contractible neighborhood $U \subset M$ of p for which $\pi_L^{-1}(U)$ consist of finitely many disjoint open subsets $L_i \subset L$ such that each $(L_i, \iota|_{L_i})$ is a projectable lagrangian submanifold of T^*U. Choosing a generalized phase function $\phi_i : L_i \to \mathbb{R}$ for each L_i and imposing that the quantization is linear under the sum of half densities we get:

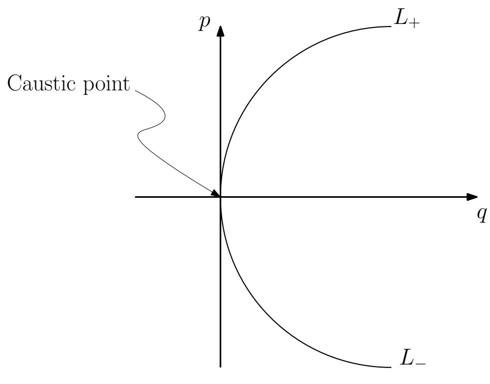
$$I_\hbar(L, \iota, a) = \sum_i (\pi_{L_i}^{-1})^* e^{\frac{i\phi_i}{\hbar}} a_i \tag{5.28}$$

The following picture is an example of this situation:



Since ϕ_i is defined up to a constant this sum is ambiguous. We must decide how to specify the relative phases of $e^{\frac{i\phi_i}{\hbar}}$ if we want to consistently quantize half-densities on L.

Example 5.3.1. For $L = \mathbb{R}$ and $\iota(x) = (x^2, x)$ we have the following picture



A phase function $\phi : L \to \mathbb{R}$ is given by

$$\phi(x) = \frac{2x^2}{3} \tag{5.29}$$

Note that $d\phi = 2x^2dx = xd(x^2) = pdq$.

If $a = B(x)|dx|^{1/2}$ is any half-density on L, then for an arbitrary function f on the q-coordinate:

$$\left((\pi_{L\pm}^{-1})_* \frac{\partial}{\partial q} f \right)\Bigg|_q = \frac{\partial}{\partial q} f(\pi_{L\pm}^{-1}(q)) = \frac{\partial}{\partial q} f(\pm\sqrt{q}) = \pm\frac{1}{2\sqrt{q}}\frac{\partial}{\partial x} f \tag{5.30}$$

which reduces to:

$$(\pi_{L\pm}^{-1})_* \frac{\partial}{\partial q} = \pm\frac{1}{2\sqrt{q}}\frac{\partial}{\partial x} \tag{5.31}$$

applying dx to both sides

$$(\pi_{L\pm}^{-1})^* dx \left(\frac{\partial}{\partial q}\right) = \pm\frac{1}{2\sqrt{q}} \qquad \Longrightarrow \qquad (\pi_{L\pm}^{-1})^* dx = \pm\frac{1}{2\sqrt{q}} dq \tag{5.32}$$

From this equation and the definition of a half-density we obtain:

$$(\pi_{L\pm}^{-1})^* a = \frac{1}{\sqrt{2} q^{1/4}} B(\pm\sqrt{q})|dq|^{1/2} \tag{5.33}$$

Thus, the prequantization of (L, ι, a) is given for $q > 0$ by

$$I_\hbar(L,\iota,a)(q) = \frac{1}{\sqrt{2}q^{1/4}} \left(e^{2iq^{3/2}/3\hbar} B(q^{1/2}) + e^{-2iq^{3/2}/3\hbar} B(-q^{1/2}) \right) |dq|^{1/2} \tag{5.34}$$

The parabola $\iota(L)$ lies in the regular level set $H^{-1}(0)$ of the hamiltonian

$$H(q,p) = \frac{1}{2}(p^2 - q) \tag{5.35}$$

and we can check that the induced vector field X_H on L is $X_H = \frac{1}{2}\frac{\partial}{\partial x}$. Thus the condition of a being invariant under the flow of X_H translates into $B(x) = B$ for B a constant. From the expression above, we obtain

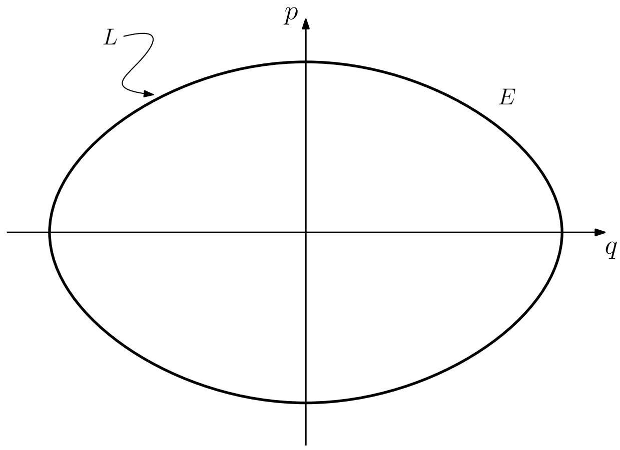
$$I_\hbar(L,\iota,a)(q) = \frac{B}{\sqrt{2}q^{1/4}} \left(e^{2iq^{3/2}/3\hbar} + e^{-2iq^{3/2}/3\hbar} \right) |dq|^{1/2} \tag{5.36}$$

as a semiclassical approximation solution to the Schrödinger equation

$$-\frac{\hbar^2}{2}\frac{\partial^2 \psi}{\partial x^2} - \frac{x}{2}\psi = E\psi$$

a problem with this solution is that it blows at $q = 0$ and is not defined for $q < 0$.

Example 5.3.2. 1-dimensional harmonic oscillator



If we impose the condition $\int_L \iota^*\theta_M \in \mathbb{Z}_\hbar$ we get

$$\text{Area of the Ellipse} = 2\pi E = \int_L \iota^*\theta_M = 2\pi\hbar n \qquad \text{for some } n \in \mathbb{N}$$

which implies

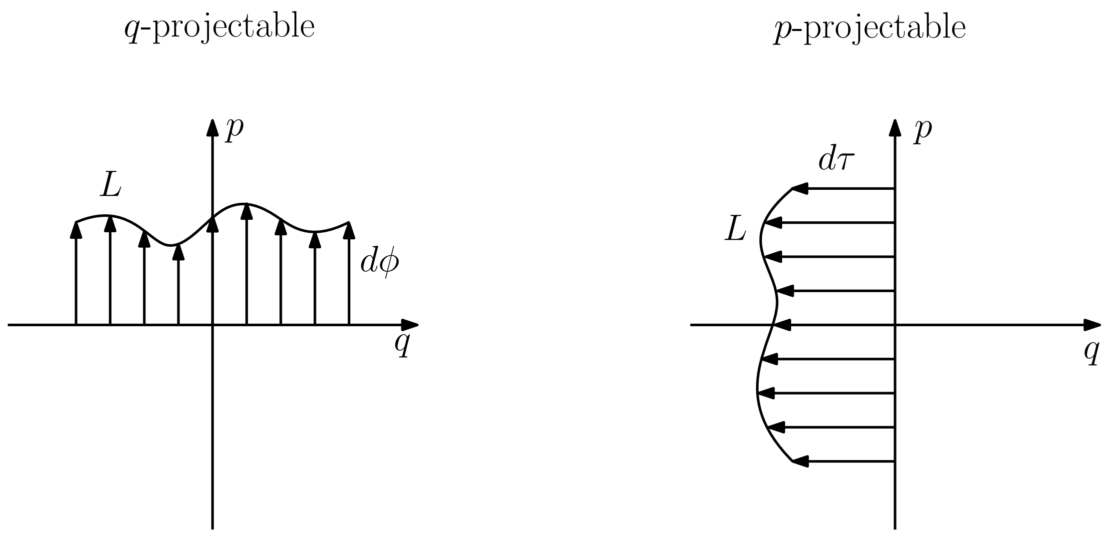
$$E = n\hbar$$

but the real energy eigenvalues for the harmonic oscillator (solving the harmonic oscillator directly from the Schrödinger equation) are $E = (n + \frac{1}{2})\hbar$. The extra $\frac{\hbar}{2}$ factor comes from the non-projectability of the lagrangian submanifold.

5.3.4 Non-projectable lagrangian submanifold case

The quantization procedure of the preceding section is incorrect, since it does not take into consideration the relation of L to the fibers $\pi : M \to T^*M$ (projectability). This factor will lead to a correction in our quantization procedure known as Maslov correction.

Let's analyze the simplest case $M = \mathbb{R}$ with lagrangian immersion $\iota : \mathbb{R} \to T^*\mathbb{R} \simeq \mathbb{R}^2$, we denote by π_p the composition of ι with the projection of $T^*\mathbb{R}$ onto the p-axis. We say (L, ι) is p-projectable if π_p is a diffeomorphism in which case we have a 'rotated' phase function $\tau : L \to \mathbb{R}$ satsifying $d\tau = \iota^*(-qdp)$.



A simple example of a p-projectable embedded lagrangian submanifold of $T^*\mathbb{R}$ is the vertical line $\iota(x) = (q_0, x)$ (note that this lagrangian immersion is not q-projectable). The wave function corresponding to a particle at the fixed position q_0 and a constant half-density a on L should correspond to a probability distribution supported at q_0, i.e. a delta function supported at q_0. The idea of Maslov is to proceed pretending that p is the position and q is the momentum and then quantizing to obtain the wave function on p-space. Using the phase function $\tau(x) = -q_0 x$, we obtain:

$$(\pi_p^{-1})^* e^{i\tau/\hbar} |dx|^{1/2} = e^{-iq_0 p/\hbar} |dp|^{1/2} \tag{5.37}$$

Notice that $\mathcal{F}^{-1}(e^{-iq_0 p/\hbar}) = (2\pi\hbar)^{-1/2}\delta(q - q_0)$ (the Dirac delta that we expect to be the wave function for a particle at a fixed position) where $\mathcal{F}$ is the Fourier transform defined (for $\mathbb{R}^n$) as:

$$(\mathcal{F}u)(p) = (2\pi\hbar)^{-n/2} \int_{\mathbb{R}^n} e^{-iq\cdot p/\hbar} u(q)\ d^n q \tag{5.38}$$

with inverse

$$(\mathcal{F}^{-1}v)(q) = (2\pi\hbar)^{-n/2} \int_{(\mathbb{R}^n)^*} e^{iq\cdot p/\hbar} v(p)\ d^n p \tag{5.39}$$

This observation suggest us that we can generalize this case by defining a function B on p-space by the equation

$$B|dp|^{1/2} = (\pi_p^{-1})^* e^{i\tau/\hbar} a \tag{5.40}$$

and then define the Maslov quantization of (L, ι, τ, a) as

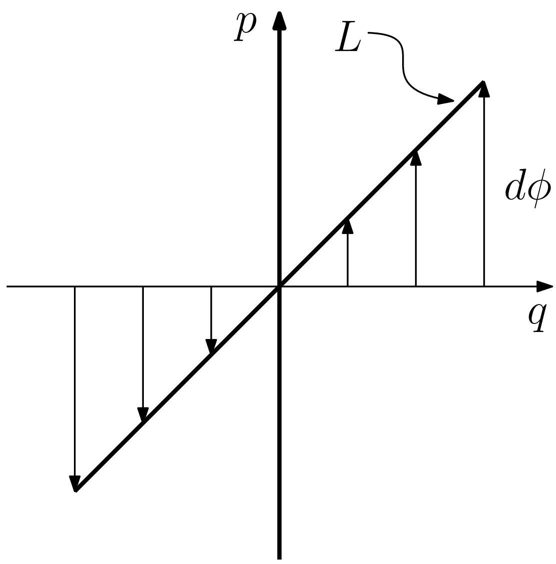
$$J_\hbar(L, \iota, \tau, a) \equiv \mathcal{F}^{-1}(B)|dq|^{1/2} \tag{5.41}$$

Let's look at an example to see how this quantization is related to the previous quantization by pull-back (in a case where the lagrangian submanifold is q and p-projectable)

Example 5.3.3. For real $k \neq 0$, consider the case $L = \mathbb{R}$ and $\iota = (x, kx)$. In this case we have the phase function $\phi : L \to \mathbb{R}$

$$\phi(x) = \frac{kx^2}{2} \tag{5.42}$$

Note that $d\phi = kxdx = pdq$.



If $a = A|dx|^{1/2}$ is a constant half-density in L, then

$$(\pi_L^{-1})^* a = A|dq|^{1/2} \tag{5.43}$$

and quantization by pull-back gives

$$I_\hbar(L, \iota, \tau, a) = e^{ikq^2/2\hbar} A|dq|^{1/2} \tag{5.44}$$

Now, since L is also $p - projectable$, we have the phase function

$$\tau = -kx^2/2 \tag{5.45}$$

which holds $d\tau = -xd(kx) = -qdp$. For the same half-density a we have:

$$(\pi_p^{-1})^* a = A \left| \frac{dx}{dp} \right|^{1/2} |dp|^{1/2} = A|k|^{-1/2}|dp|^{1/2} \tag{5.46}$$

we also need the term

$$\mathcal{F}^{-1}((\pi_p^{-1})^* e^{i\tau/\hbar})(q) = \mathcal{F}^{-1}(e^{-ip^2/2k\hbar}) = |k|^{1/2} e^{-i\pi \cdot sgn(k)/4} e^{ikq^2/2\hbar} \tag{5.47}$$

The last equality comes from the general formula for the Fourier transform of a complex Gaussian

$$\mathcal{F}(e^{-\alpha q^2}) = (2\pi\hbar)^{-1/2} \sqrt{\frac{\pi}{\alpha}} e^{-p^2/4\alpha\hbar^2} \qquad \forall \alpha \in \mathbb{C} : \ Re(\alpha) > 0 \tag{5.48}$$

Thus the Maslov quantization condition gives

$$J_\hbar(L, \iota, \tau, a) = \mathcal{F}^{-1}(A|k|^{-1/2}(\pi_p^{-1})^* e^{i\tau/\hbar})|dq|^{1/2} = e^{-i\pi \cdot sgn(k)/4} \left(e^{ikq^2/2\hbar} A|dq|^{1/2} \right)$$

$$J_\hbar(L, \iota, \tau, a) = e^{-i\pi \cdot sgn(k)/4} I_\hbar(L, \iota, \phi, a) \tag{5.49}$$

In this case quantization by pull-back and Maslov quantization differ by a constant phase shift.

Example 5.3.4. For the lagrangian embedding $\iota(x^2, x)$ of $L = \mathbb{R}$ into $\mathbb{R}^2$ we have the phase function $\tau = -x^3/3$ ($d\tau = -x^2 dx = -qdp$). Thus, the Maslov quantization of the half density $a = B(x)|dx|^{1/2}$ on L is

$$J_\hbar(L, \iota, \tau, a) = \mathcal{F}^{-1}\left(e^{-ip^3/3\hbar} B(p)\right) |dq|^{1/2} = (2\pi\hbar)^{-1/2} \left(\int_{\mathbb{R}^*} e^{\frac{i}{\hbar}(pq - p^3/3)} B(p) dp \right) |dq|^{1/2} \tag{5.50}$$

since $(\pi_p^{-1})^* a = B(p)|dp|^{1/2}$. The last equation can be calculated using the stationary phase approximation for the critical points of the function $S(p) = pq - p^3/3$ which occur precisely when $q = p^2$, i.e. when $(q, p) \in \iota(L)$. Then the integral for the two halves of L ($p = q^{1/2}$ and $p = -q^{1/2}$) is

$$\begin{aligned} J_\hbar(L, \iota, \tau, a) \quad = \quad & \Big(B(-q^{1/2}) e^{\frac{i}{\hbar} S(-q^{1/2})} \sqrt{\frac{2\pi\hbar}{S''(-q^{1/2})}} \, e^{i\pi sgn(S''(-q^{1/2}))/4} + \\ & B(q^{1/2}) e^{\frac{i}{\hbar} S(q^{1/2})} \sqrt{\frac{2\pi\hbar}{S''(q^{1/2})}} \, e^{i\pi sgn(S''(q^{1/2}))/4} \Big) (2\pi\hbar)^{-1/2} |dq|^{1/2} \\ & + \mathcal{O}(\hbar) \end{aligned} \tag{5.51}$$

Evaluating the value of S at $q^{1/2}$ and $-q^{1/2}$:

$$J_\hbar(L,\iota,\tau,a) = \frac{1}{\sqrt{2}q^{1/4}}\left(e^{2iq^{3/2}/3\hbar}B(q^{1/2})e^{-i\pi/4} + e^{-2iq^{3/2}/3\hbar}B(-q^{1/2})e^{i\pi/4}\right)|dq|^{1/2}+\mathcal{O}(\hbar) \tag{5.52}$$

Comparing this result to (5.34) we see that each term in the Maslov's procedure acquires one extra phase factor $e^{\pm i\pi/4}$. However, the advantage of this procedure is that the full expression for $J_\hbar(L,\iota,\tau,a)$ (not the term in the stationary phase approximation) is not singular at the caustic point $q=0$ as long as the half-density a has compact support.

Having these examples in mind, we will proceed to deduce the quantization condition for the Maslov's procedure in terms of an invariant quantity of smooth paths in a Lagrangian submanifolds, known as Maslov index. In order to do that, let's define a p-dependent phase function $T=\tau\circ\pi_p^{-1}$ for $L\subset\mathbb{R}^2$, then the Maslov half-density is :

$$J_\hbar(L,\iota,\tau,a) = (2\pi\hbar)^{-1/2}\int_{\mathbb{R}} e^{i(pq+T(p))/\hbar}B(p)dp\;|dq|^{1/2} \tag{5.53}$$

The critical points of the exponential are precisely at the points where $T'(p)=-q$, we saw from examples (5.3.3) and (5.3.4) that if a change of sign in T'' happens at this point we must multiply by the factor $e^{-i\pi sgn(T'')/4}$ to the usual quantization by pull-back. Assuming that that T' has only non-degenerate critical points, we can assign an index by summing the changes in $sgn(T'')$ (it changes by ±2 in the vicinity of a critical point) while traversing L in a prescribed direction, this index is index is twice an integer known as Maslov index $m_{L,\iota}$.

Given an open interval of non-caustic points U, if Maslov's technique is applied to L_j, then from an application of the stationary phase approximation (just as in the previous example) the Maslov's half- densities I_j on $L_j\cap\pi_L^{-1}(U)$ are given by:

$$\hat{I}_j = e^{-i\pi s_j/4}e^{i\phi_j/\hbar}a \tag{5.54}$$

where s_j are integers depending on each component of $L_j\cap\pi_L^{-1}(U)$ (s_j is zero if L_j is q-projectable and quantized by pull-back, otherwise s_j is $sgn(T'')$ for a suitable

p-dependent phase function), and ϕ_j is a local phase function. If we want $\hat{I}_j$ to be globally well-defined they must coincide in the intersections $L_i \cap L_j$, i.e.

$$e^{-i\pi(s_i - s_j)/4} e^{i(\phi_i - \phi_j)/\hbar} = 1 \tag{5.55}$$

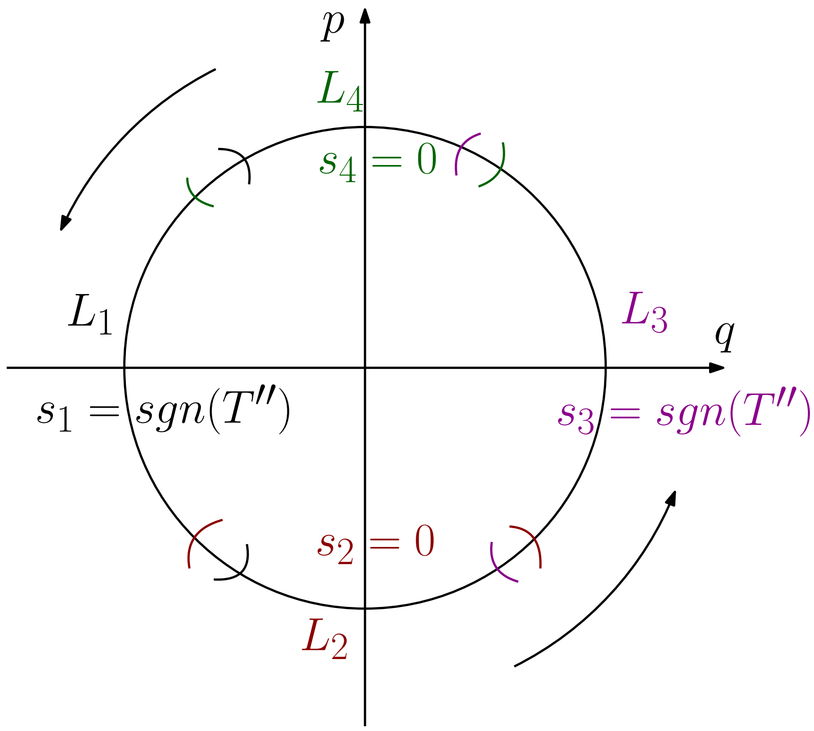
at each point of $L_i \cap L_j$. Since $d(\phi_i - \phi_j) = 0$, we can define the constant quantity

$$\hat{a}_{ij} = (\phi_i - \phi_j) - \pi\hbar\frac{(s_i - s_j)}{4} \tag{5.56}$$

so that the quantization condition becomes:

$$\hat{a}_{ij} \in \mathbb{Z}_\hbar \tag{5.57}$$

This condition is called the Maslov quantization condition. In the case L is a circle



the quantization condition becomes:

$$\int_L \iota^*\alpha - \frac{\pi\hbar}{2} m_{L,\iota} \in \mathbb{Z}_\hbar \tag{5.58}$$

Example 5.3.5. Now we can return to the case of the quantum harmonic oscillator where we have $m_{L,\iota} = 2$, then:

$$\int_L \iota^*\alpha - \frac{\pi\hbar}{2} m_{L,\iota} = 2\pi E - \frac{\pi\hbar}{2} m_{L,\iota} \ \in \mathbb{Z}_\hbar \tag{5.59}$$

Thus the energy levels are:

$$E = \hbar \left(n + \frac{1}{2} \right) \tag{5.60}$$

So we can conclude that the Maslov quantization condition gives the precise energy levels for the quantum harmonic oscillator.

Capítulo 6

Conclusions

These notes are meant as an introduction to geometric quantization and WKB method, they are by no means mathematically rigorous but provides some motivation into the idea that the quantization process should be formulated in geometric terms. We saw through examples that these 2 procedures work for the case of particles in quantum mechanics, so this serves as a starting point to generalize this procedure to the case of quantum field theory (where the phase space is infinite dimensional) and the AKSZ-BV procedure (where the symplectic form is odd) in the interest of research.

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