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Relativistic Bose–Einstein condensation with disorder

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Abstract

We investigate the thermodynamics of a self-interacting relativistic charged scalar field in the presence of weak disorder. We consider quenched disorder which couples linearly to the mass of the scalar field. After performing noise averages over the free energy of the system, we find that disorder increases the mean-field critical temperature for Bose–Einstein condensation at finite density in the ultrarelativistic limit. In turn, preliminary non-relativistic calculations indicate that the presence of randomness affects the Bose gas in the opposite way in such a limit, i.e. disorder reduces the mean-field condensation temperature. The effect of disorder on the temperature dependence of the chemical potential for a fixed charge density is investigated. Significant differences from the mean-field temperature dependence of the chemical potential are observed as the strength of the noise intensity increases. Finally, the temperature dependence of the chemical potential with fixed total charge and entropy is investigated. It is found that there is no Bose–Einstein condensation for a fixed charge to entropy ratio in the presence of weak disorder. The possible relevance of the findings in the present paper in different areas is discussed.

Keywords: Bose–Einstein condensation, phase transitions, wave propagation in random media, finite temperature field theory

(Some figures may appear in colour only in the online journal)

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1. Introduction and motivation

Disorder plays an important role in the critical behavior of second order phase transitions [1]. The relevance of disorder in the criticality can be assessed qualitatively using the critical exponent α of the specific heat for the disorder-free system [2, 3]; namely, when $\alpha > 0$ (the specific heat diverges at the critical point), the critical behavior of the disordered system is changed, when $\alpha < 0$ (the specific heat is finite), disorder has no effect on the critical behavior. On the other hand, at low temperatures quantum fluctuations may compete with the random fluctuations; an example is the destruction of the ordered ground state of a spin-glass—a disorder strongly correlated system—by quantum fluctuations [4]. Conversely, quantum fluctuations can stabilize a glass phase in a disordered environment; Carleo *et al* [5] demonstrated that repulsively interacting bosons can feature a novel quantum phase displaying both Bose–Einstein condensation and spin-glass behavior due to frustration. In the present paper we investigate the interplay between quantum and random fluctuations in a self-interacting relativistic charged scalar field theory with a finite chemical potential. Disorder in relativistic Bose–Einstein condensation (RBEC) has not been considered in the literature, contrary to the case of non-RBEC, where it has been under intensive study since early seminal works [6–8].

Disorder has a decisive influence on the zero-temperature phase diagram of non-relativistic Bose systems. As emphasized by the literature, there is a quantum phase transition for such systems from a Mott insulating phase to a conducting phase. Since no pure Bose system can be a normal conducting fluid at zero temperature, the conducting-insulator transition must correspond to the onset of superfluidity. As shown in [8], this scenario is changed dramatically in the presence of a random potential. For the case of a Gaussian colored noise, a Bose glass phase also arises and the transition to superfluidity only occurs from this third phase, never directly from the Mott insulator. The introduction of a random potential in such systems may also imply the destruction of the superfluidity phase, as discussed in [7, 9–12]. In particular, a recent study by Lopatin and Vinokur [13] employing the replica method found a negative shift in the condensation temperature of a dilute Bose gas due to disorder—see also [14, 15].

There is an extensive literature on RBEC following the pioneering works of [16–23], which discussed RBEC in flat space–times, and [24–27], which discussed RBEC in curved space–times. While relativistic Bose–Einstein condensates are not yet realizable in controllable experiments like their non-relativistic counterparts, they do relate to observable and experimentally accessible phenomena. One example, of immense current interest, concerns the condensation dynamics in relativistic quantum field theories where creation and annihilation of particles play crucial role, like in far-from-equilibrium stages of the early Universe and in experiments with relativistic heavy-ion collisions [28]. There is also the possibility of Bose–Einstein condensation of pions and kaons [29–32] in neutron stars. The condensation of these mesons will affect the equation of state of matter in the interior of the star, which has direct consequences on the observable mass–radius relation of the star, and will also impact the early evolution of the neutron star. With this respect, we quote [33] in which the authors discuss an interesting model with the purpose of studying in detail the Bose–Einstein condensation within neutron stars due to the formation of Cooper pairs. In turn, in dark-matter models where scalar particles constitute a natural ingredient, relativistic Bose–Einstein condensates assume an important place in the study of the effects of scalar dark-matter background on the equilibrium of degenerate stars [34]. In this case there is particular interest in the charge density and the associated chemical potential.

In real physical situations, the presence of some sort of disorder in the system is unavoidable. The disorder can be due to uncontrollable disturbances external to the system; for instance in a cosmological context such perturbations can originate from standard inflationary fluctuations, required to generate large-scale structures. On the other hand, random fluctuations can also be the result of an incomplete treatment of degrees of freedom associated with fields that couple to the field of interest. Furthermore, researches on the stochastic Gross–Pitaevskii equation for Bose–Einstein condensates indicate that random dynamics is presumed to be a general result of the interaction between long and short frequency modes [35]. As with non-relativistic Bose–Einstein condensates of condensed matter physics, one expects that disorder will impact the critical behavior of RBEC. The present study is a first step toward a systematic study of disorder in relativistic quantum field theory models, in that we focus on a weakly interacting charged scalar field at finite temperature in the presence of nonstatic randomness (the precise meaning for *nonstatic noise* will be defined shortly). Our model is a kind of generalization of the scalar Landau–Ginzburg theory, where the quenched disorder is described by random fluctuations of the effective transition temperature [1].

The organization of this paper is as follows. In section 2 we present our model. The disorder field couples to the charged scalar field via the mass term of the scalar field, just as in the random-temperature Landau–Ginzburg model. We consider weak disorder and implement a perturbative expansion for the free energy as power series expansion in the strength of the disorder field. In section 3 we study the thermodynamics properties of the self-interacting relativistic Bose gas at finite density with randomness. The self-interactions of the scalar field are treated in a mean-field approximation. We calculate the noise average of the free energy. In section 4 we obtain the critical temperature in the presence of random fluctuations. In section 5 we discuss the net total charge associated with the condensate and also the modifications in temperature evolution of the chemical potential due to disorder. Conclusions and Perspectives are presented in section 6. The paper includes appendices containing details of lengthy derivations. Throughout the paper we employ units with $\hbar = c = k_B = 1$.

2. Scalar field thermodynamics and disorder

We are interested in studying the effects of randomness on a charged scalar field φ of mass m in equilibrium with a thermal reservoir at temperature T. We employ the imaginary time formalism of Matsubara [36] to write the partition function of the model in the grand canonical ensemble as [18, 37]

$$Z = \left[N(\beta)\right]^2 \int [D\varphi]\left[D\varphi^*\right] \mathrm{e}^{S\left[\varphi,\varphi^*\right]}, \tag{1}$$

where the action $S[\varphi, \varphi^*]$ reads

$$\begin{aligned} S\left[\varphi, \varphi^*\right] = \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\mathbf{x} \Big[&\left(\partial_t + \mathrm{i}\mu\right) \varphi^* \left(\partial_t - \mathrm{i}\mu\right) \varphi \\ &- \nabla\varphi^*\nabla\varphi - m^2\varphi^*\varphi - \lambda\left(\varphi^*\varphi\right)^2 \Big], \end{aligned} \tag{2}$$

where V is the volume of the system, $\beta = 1/T$, μ the chemical potential associated with the conserved charge, and $\partial_t = \mathrm{i}\,\partial_\tau$. The field φ satisfies the Kubo–Martin–Schwinger [38, 39] boundary condition $\varphi(\tau, \mathbf{x}) = \varphi(\tau + \beta, \mathbf{x})$. $N(\beta)$ is a β-dependent but μ-independent constant that comes from the integration over the canonical momentum conjugated to the field φ [37]. Incidentally, we remark that the existence of a conserved charge is a consequence of the $U(1)$ global symmetry displayed by the theory—by Noether's theorem, there must be a conserved current associated with each continuous symmetry of the Lagrangian.

Next we consider the coupling of a random noise source to the quantum matter field in a similar fashion to the random-temperature Landau–Ginzburg model, but generalized to a τ-dependent noise. That is, we perform the replacement $m^2 \to m^2(1 + \nu)$, where $\nu = \nu(\tau, \mathbf{x})$ is dimensionless. The partition function given in equation (1) becomes replaced by

$$Z[\nu] = \left[N(\beta)\right]^2 \int [D\varphi]\left[D\varphi^*\right] \mathrm{e}^{S_T\left[\nu, \varphi, \varphi^*\right]}, \tag{3}$$

where

$$S_T\left[\nu, \varphi, \varphi^*\right] = S\left[\varphi, \varphi^*\right] + S_I\left[\nu, \varphi, \varphi^*\right], \tag{4}$$

with $S[\varphi, \varphi^*]$ given by equation (2) and $S_I[\nu, \varphi, \varphi^*]$ contains the coupling of the scalar field with the noise field:

$$S_I\left[\nu, \varphi, \varphi^*\right] = -m^2 \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\mathbf{x}\, \nu(\tau, \mathbf{x})\varphi^*(\tau, \mathbf{x})\varphi(\tau, \mathbf{x}). \tag{5}$$

The physical picture is that the random fluctuations describe average effects of external disturbances on the system or of degrees of freedom of unobserved fields. Although similar to a real-time dependence, the τ dependence in $\nu(\tau, \mathbf{x})$ should be understood as being of similar nature of the one that arises naturally in a self-energy for the field φ when integrating out fields in favor of effective interactions of φ. It is important to note that in general, when integrating over unobserved degrees of freedom one obtains also effective vertices, in addition to self-energies. Thereof we stress that there is no implicit assumption here that equation (5) is an exact replacement for all effects of integrating out unobserved fields, but solely that the dependence on τ of the noise field is very natural for non-isolated systems. Hereafter we mean by *static noise* the noise fields that are τ independent and *nonstatic noise* those fields that depend upon τ. Reference [40] presents another situation in which the noise is nonstatic. In addition, the form of the coupling between the noise and the quantum field indicates that $U(1)$ global symmetry is still present in the full theory with randomness, which implies that one must introduce a chemical potential associated with a conserved charge as considered above.

Here we consider the random function $\nu(\tau, \mathbf{x})$ as a Gaussian distribution given by

$$P[\nu] = p_0\, \mathrm{e}^{-1/2\sigma^2 \int \mathrm{d}^d x\, [\nu(x)]^2}, \tag{6}$$

where $x = (\tau, \mathbf{x})$ and p_0 is the normalization constant of the distribution. The quantity σ^2 is a parameter associated with the intensity of the disorder. We will denote the mean value over the random variable as $\overline{(\cdots)}$, defined by

$$\overline{A[\nu]} = \int [D\nu]\, P[\nu]\, A[\nu], \tag{7}$$

with $A[\nu]$ being any functional of ν. From equation (6), we have a white noise with two-point correlation function given by

$$\overline{\nu(\tau, \mathbf{x})\nu\left(\tau', \mathbf{x}'\right)} = \sigma^2\, \delta\left(\tau - \tau'\right)\delta^3\left(\mathbf{x} - \mathbf{x}'\right). \tag{8}$$

As well known, it follows from the Gaussian distribution that

$$\overline{\nu(x_1) \cdots \nu(x_{2n+1})} = 0, \tag{9}$$

$$\overline{\nu(x_1)\cdots\nu(x_{2n})} = \sum_{\text{pair comb.}} \prod_{\text{pairs}} \overline{\nu(x_j)\nu(x_k)}, \tag{10}$$

where n is an integer.

The standard procedure to study Bose–Einstein condensation is to separate from φ the constant zero mode $\langle\varphi\rangle \equiv \xi$:

$$\varphi = \xi + \chi, \tag{11}$$

where χ is a complex field with no zero mode. The χ field is written in terms of real and imaginary parts as

$$\chi = \frac{1}{\sqrt{2}}\left(\chi_1 + \mathrm{i}\chi_2\right), \tag{12}$$

so that the action in equation (4) can be written as

$$\begin{aligned} S_T\left[\nu, \chi_1, \chi_2, \xi\right] = &- \beta V\, U(\xi) + S_0\left[\xi, \chi_1, \chi_2\right] \\ &+ S_{\text{int}}\left[\xi, \chi_1, \chi_2\right] + S_I\left[\nu, \chi_1, \chi_2, \xi\right], \end{aligned} \tag{13}$$

with the following potential

$$U(\xi) = \left(m^2 - \mu^2\right)\xi^2 + \lambda\xi^4, \tag{14}$$

the quadratic part

$$\begin{aligned} S_0\left[\xi, \chi_1, \chi_2\right] = -\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\mathbf{x}\, \Big[&\partial_\tau\chi_1\partial_\tau\chi_1 + \nabla\chi_1\nabla\chi_1 \\ &+ \left(6\lambda\xi^2 + m^2 - \mu^2\right)\chi_1^2 + \partial_\tau\chi_2\partial_\tau\chi_2 \\ &+ \nabla\chi_2\nabla\chi_2 + \left(2\lambda\xi^2 + m^2 - \mu^2\right)\chi_2^2 \\ &- 2\mathrm{i}\mu\left(\chi_2\partial_\tau\chi_1 - \chi_1\partial_\tau\chi_2\right)\Big], \end{aligned} \tag{15}$$

and the self-interacting part

$$\begin{aligned} S_{\text{int}}\left[\xi, \chi_1, \chi_2\right] = -\int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\mathbf{x}\, \Big[&2^{1/2}\lambda\xi\chi_1\left(\chi_1^2 + \chi_2^2\right) \\ &+ \frac{\lambda}{4}\left(\chi_1^2 + \chi_2^2\right)^2\Big]. \end{aligned} \tag{16}$$

In equations (15) and (16) we neglected the linear terms in the field χ, because their contributions will be proportional to terms like $\chi(\mathbf{p} = 0) = 0$. In turn, the random contribution is given by

$$\begin{aligned} S_I\left[\nu, \chi_1, \chi_2, \xi\right] = -m^2 \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\mathbf{x}\, \Big\{&\xi^2\, \nu(\tau, \mathbf{x}) \\ &- \frac{1}{2}\,\nu(\tau, \mathbf{x})\left[\chi_1^2(\tau, \mathbf{x}) + \chi_2^2(\tau, \mathbf{x})\right] \\ &- \sqrt{2}\,\xi\,\nu(\tau, \mathbf{x})\chi_1(\tau, \mathbf{x})\Big\}. \end{aligned} \tag{17}$$

We are interested in studying the thermodynamics of the above system in the presence of disorder. We follow closely the path used for the noiseless case [18–20, 41], in that the transition temperature is determined by analyzing the minimum of the free energy as a function of the variational parameter ξ. Noise average is taken into account using equation (7), with A being the Helmholtz free energy $\Omega(\beta, V, \mu, \xi)$. Specifically, for a uniform infinite volume system we have the relation $\beta\Omega(\beta, V, \mu, \xi) = -\ln Z_R$, where $\ln Z_R$ is the renormalized logarithm of the partition function, and thence:

$$\begin{aligned}\overline{\Omega}(\beta, V, \mu, \xi) &= -\frac{1}{\beta}\int [D\nu]\, P[\nu]\, \ln Z_R[\nu] \\ &= -\frac{1}{\beta}\,\overline{\ln Z_R[\nu]}.\end{aligned} \tag{18}$$

Note that we are considering a situation where one has to deal with two kinds of averages, namely thermal averages and noise averages, which are not treated on the same footing. This can be justified when the characteristic time scale of the change in disorder is much larger then the time of observation of phenomena of interest. This means that in order to calculate random averages of thermodynamic observables, one performs such averages *over the logarithm of the partition function* and not over the partition function itself. The noise average over the partition function is trivial, as one can integrate very easily over $\nu(x)$ using the probability distribution of equation (6). In other words, one calculates the free energy for a given configuration of the noise $\nu(x)$ and then carry out the random average.

Equation (18) requires a method to evaluate the average over noise realizations of the free energy. For static noise and arbitrary noise intensities the replica-trick is widely used [1]. Here we consider the weak-noise limit and use a perturbative approach [42, 43], in that one expands the partition function in a power series in the noise ν. This will be discussed in the next section.

3. Noise average of the free energy

It is known that random mass models generate effective interactions that mimic a negative coupling constant. Because of this, we will consider the mean field approximation for the disorder-free part of the partition function; i.e. one calculates the noiseless free energy neglecting $S_{\text{int}}[\xi, \chi_1, \chi_2]$. As discussed in [41], one might expect this to be a good approximation if both λ and $\lambda\xi$ are small. We note that there is no assumption here that a mean field approximation captures the full richness of the critical behavior of the relativistic interacting Bose gas; the approximation is used because it provides the system with a ground state and a starting point for assessing the role played by disorder in the relativistic model. Therefore, the model only makes sense when the noise-induced interactions are weaker than the self-interactions $\lambda\,(\varphi^*\varphi)^2$ in equation (2). In our treatment we ensure this by treating the noise as a weak interaction on the top of the mean-field generated by the self-interactions $\lambda\;(\varphi^*\varphi)^2$. In other words, the noise is weakly coupled to the scalar field in such a way that the random fluctuations do not destabilize the mean field solution and still allows for the existence of a ground state.

In the weak-disorder limit, the partition function in equation (3) can be expanded in a power series in S_I:

$$Z[\nu] = (N(\beta))^2 \int [\mathrm{d}\chi_1]\,[\mathrm{d}\chi_2]\,\mathrm{e}^{-\beta V\,U(\xi)+S_0} \sum_{n=0}^{\infty} \frac{S_I^n}{n!}, \tag{19}$$

where $S_I = S_I[\nu, \chi_1, \chi_2, \xi]$ and $S_0 = S_0[\xi, \chi_1, \chi_2]$. Taking the logarithm of both sides and then taking the random average leads us to

$$\overline{\ln Z[\nu]} = \ln Z_{\text{MF}} + \overline{\ln Z_I[\nu]}, \tag{20}$$

where the mean-field contribution $\ln Z_{\rm MF}$ is given by $\ln Z_{\rm MF} = -\beta V\, U(\xi) + \ln Z_0$, with

$$\ln Z_0 = \ln\left[(N(\beta))^2 \int [D\chi_1][D\chi_2]\, \mathrm{e}^{S_0} \right]. \tag{21}$$

The quantity $\ln Z_0$ is calculated explicitly in appendix A and the result is

$$\ln Z_0 = -\sum_{\mathbf{p}} \left[\frac{\beta\Omega_+}{2} + \ln\left(1 - \mathrm{e}^{-\beta\Omega_+}\right) + \frac{\beta\Omega_-}{2} + \ln\left(1 - \mathrm{e}^{-\beta\Omega_-}\right) \right], \tag{22}$$

where the quantities $\Omega_\pm$ are properly defined in the appendix A. Therefore, one gets the following mean-field partition function:

$$\ln Z_{\rm MF} = -\beta V\, U(\xi) - \sum_{\mathbf{p}} \left[\frac{\beta\Omega_+}{2} + \ln\left(1 - \mathrm{e}^{-\beta\Omega_+}\right) + \frac{\beta\Omega_-}{2} + \ln\left(1 - \mathrm{e}^{-\beta\Omega_-}\right) \right]. \tag{23}$$

Now let us focus on the corrections to the mean-field solution due to disorder which are given by

$$\overline{\ln Z_I[\nu]} = \overline{\ln\left(1 + \sum_{n=1}^{\infty} \frac{\langle S_I^n \rangle}{n!} \right)}. \tag{24}$$

Here the averages $\langle(\cdots)\rangle$ are defined using the mean-field ensemble represented by the action S_0:

$$\langle(\cdots)\rangle = \frac{\int [D\chi_1][D\chi_2](\cdots)\, \mathrm{e}^{S_0}}{\int [D\chi_1][D\chi_2]\, \mathrm{e}^{S_0}}. \tag{25}$$

Expanding equation (24) up to second order in the noise field, one obtains

$$\overline{\ln Z_I[\nu]} = \overline{\langle S_I \rangle} + \frac{1}{2}\left(\overline{\langle S_I^2 \rangle} - \overline{\langle S_I \rangle^2} \right), \tag{26}$$

where S_I is given by equation (17). From equation (9), we have that $\overline{\langle S_I \rangle} = 0$. The other terms are obtained using equations (17) and (8):

$$\overline{\ln Z_I[\nu]} = m^4\sigma^2 \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}\mathbf{x} \left[\xi^2 \left\langle \chi_1^2 \right\rangle + \frac{1}{8} \left(\left\langle \chi_1^4 \right\rangle + \left\langle \chi_2^4 \right\rangle - \left\langle \chi_1^2 \right\rangle^2 - \left\langle \chi_2^2 \right\rangle^2 + 2 \left\langle \chi_1^2\, \chi_2^2 \right\rangle - 2 \left\langle \chi_1^2 \right\rangle \left\langle \chi_2^2 \right\rangle \right) \right]. \tag{27}$$

The derivation of the ensemble averages in equation (27) can be performed in the usual way (see for instance [41]). The result is

$$\overline{\ln Z_I[\nu]} = m^4\sigma^2\left[\xi^2\sum_{n,\mathbf{p}}\mathcal{D}^0_{11}(\omega_n, \mathbf{p}) + \frac{1}{4\,\beta V}\left(\sum_{n,\mathbf{p}}\mathcal{D}^0_{11}(\omega_n, \mathbf{p})\right)^2 + \frac{1}{4\,\beta V}\left(\sum_{n,\mathbf{p}}\mathcal{D}^0_{22}(\omega_n, \mathbf{p})\right)^2\right], \tag{28}$$

where $\mathcal{D}^0_{ij}(\omega_n, \mathbf{p})$, $i, j = 1, 2$ are the zero-order propagators of the fields χ_j. Since the propagators have divergent vacuum contributions, equation (28) must be carefully regularized. The renormalization of the propagators is discussed in appendix B. After carrying out such a procedure we get

$$\overline{\ln Z_I[\nu]} = (\beta V)\frac{m^4\sigma^2}{2}\left[\Pi_m\left(2\xi^2 + \Pi_m\right) - \Pi_v^2\right], \tag{29}$$

where the quantities Π_v and $\Pi_m(\beta, \xi)$ are obtained in appendix B; they are given by

$$\Pi_v = \frac{1}{4}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{W_+(\mathbf{p}, \xi)} + \frac{1}{4}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{W_-(\mathbf{p}, \xi)}, \tag{30}$$

and

$$\Pi_m = \Pi_m(\beta, \mu, \xi) = \frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{W_+(\mathbf{p}, \xi)}\frac{1}{e^{\beta\Omega_+} - 1} + \frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{W_-(\mathbf{p}, \xi)}\frac{1}{e^{\beta\Omega_-} - 1}. \tag{31}$$

The quantities $W_\pm(\mathbf{p}, \xi)$ are properly defined in appendix B. Note that in the above equations we have considered the large-volume limit. Finally, inserting equations (14), (23) and (29) in equation (20) and neglecting for the moment the divergent vacuum contributions, one gets the following expression for the renormalized $\ln Z$ up to second order in noise intensity:

$$\frac{1}{\beta V}\overline{\ln Z_R[\nu]} = \left[\mu^2 - m^2 - \lambda\xi^2 + m^4\sigma^2\,\Pi_m\right]\xi^2 - \frac{1}{\beta}\int\frac{d\mathbf{p}}{(2\pi)^3}\left[\ln\left(1 - e^{-\beta\Omega_+}\right) + \ln\left(1 - e^{-\beta\Omega_-}\right)\right] + \frac{m^4\sigma^2}{2}\,\Pi_m^2. \tag{32}$$

In the next section we discuss the determination of the critical temperature.

4. The critical temperature

The total Helmholtz free energy is obtained by inserting equation (32) in equation (18):

$$\frac{\overline{\Omega}(\beta, V, \mu, \xi)}{V} = \left(m^2 - \mu^2 + \lambda\xi^2 - m^4\sigma^2\,\Pi_m\right)\xi^2 + \frac{1}{\beta}\int\frac{d\mathbf{p}}{(2\pi)^3}\left[\ln\left(1 - e^{-\beta\Omega_+}\right) + \ln\left(1 - e^{-\beta\Omega_-}\right)\right] - \frac{m^4\sigma^2}{2}\,\Pi_m^2. \tag{33}$$

Since we are working in the mean-field approximation, $\lambda\xi \ll 1$, we neglect contributions coming from terms proportional to $\lambda^2\xi^4$ in the definition of $\Omega_\pm(\mathbf{p}, \xi)$ in appendix A. This leads to

$$\Omega_\pm(\mathbf{p}, \xi) \approx \omega(\mathbf{p}) \pm \mu, \; \frac{1}{W_\pm(\mathbf{p}, \xi)} \approx \frac{1}{\omega(\mathbf{p})}, \tag{34}$$

with $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2}$ and $M^2 = m^2 + 4\lambda\xi^2$. Hence

$$\begin{aligned}\frac{\overline{\Omega}(\beta, V, \mu, \xi)}{V} = {} & \left(m^2 - \mu^2 + \lambda\xi^2 - m^4\sigma^2\,\Pi_m\right)\xi^2 \\ & + \frac{1}{\beta}\int\frac{d\mathbf{p}}{(2\pi)^3}\left[\ln\left(1 - e^{-\beta(\omega(\mathbf{p})-\mu)}\right) + \ln\left(1 - e^{-\beta(\omega(\mathbf{p})+\mu)}\right)\right] \\ & - \frac{m^4\sigma^2}{2}\,\Pi_m^2\end{aligned} \tag{35}$$

with

$$\begin{aligned}\Pi_m(\beta, \mu, \xi) = {} & \frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\omega(\mathbf{p})}\frac{1}{e^{\beta(\omega(\mathbf{p})-\mu)} - 1} \\ & + \frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\omega(\mathbf{p})}\frac{1}{e^{\beta(\omega(\mathbf{p})+\mu)} - 1}.\end{aligned} \tag{36}$$

If the above momentum integrals are to be convergent, one must admit μ to be constrained such that $|\mu| \leqslant M$. On the other hand, as remarked in [20], in the absence of randomness the correct interpretation of the emergence of two terms related to each other by $\mu \to -\mu$ is the following: One must identify one term corresponding to particles (with charge $+1$) and the other term corresponding to antiparticles (with charge -1). We claim that this assertion still holds in the full theory with random noise.

From equation (35), one may read off the classical energy density, i.e. the Helmholtz free energy density at zero temperature:

$$\frac{\Omega_{\rm cl}(V, \mu, \xi)}{V} = \left(m^2 - \mu_0^2 + \lambda\xi^2\right)\xi^2, \tag{37}$$

where $\mu_0 = \mu(T = 0)$ is the chemical potential at zero temperature. To such a quantity one should add the contributions coming from the zero-point energy of the fields as well as the divergent vacuum term. This leads us to a divergent vacuum energy density. Its regularization and renormalization are discussed at length in appendix C and the final result is that the renormalized vacuum energy density equals the classical contribution, $\Omega_{\rm cl}/V$.

As discussed in [41], the parameter ξ is not determined *a priori* and it should be treated as a variational parameter, related to the charge carried by the condensed particles. At fixed β and μ, the free energy is an extremum with respect to variations of ξ. The derivative of equation (35) with respect to ξ implies that $\xi = 0$ unless

$$\begin{aligned}\xi_0^2 = {} & \frac{1}{2\lambda\left(1 + \sigma^2 m^4\,\Xi\left(\beta, \mu, \xi_0\right)\right)} \\ & \times\left[\mu^2 - m^2 - 4\lambda_{\rm eff}(\sigma)\,\Pi_m\left(\beta, \mu, \xi_0\right) - 2\lambda\sigma^2 m^4\Pi_m\left(\beta, \mu, \xi_0\right)\Xi\left(\beta, \mu, \xi_0\right)\right],\end{aligned} \tag{38}$$

where

$$\lambda_{\text{eff}}(\sigma) = \lambda - \frac{\sigma^2 m^4}{4}, \tag{39}$$

and ($\omega = \omega(\mathbf{p})$)

$$\Xi(\beta, \mu, \xi) = \frac{1}{2}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} \frac{\mathrm{e}^{\beta(\omega+\mu)}(\beta\omega + 1) - 1}{\omega^3\left(\mathrm{e}^{\beta(\omega+\mu)} - 1\right)^2} + \frac{1}{2}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3} \frac{\mathrm{e}^{\beta(\omega-\mu)}(\beta\omega + 1) - 1}{\omega^3\left(\mathrm{e}^{\beta(\omega-\mu)} - 1\right)^2}. \tag{40}$$

Here we have used equation (34). Since ξ is related to the charge carried by the condensate, at the transition we have $\xi_0 = 0$ and then

$$\mu_c^2 - m^2 - 4\lambda_{\text{eff}}(\sigma)\,\Pi_m\left(\beta_c, \mu_c, 0\right) - 2\lambda\sigma^2 m^4 \Pi_m\left(\beta_c, \mu_c, 0\right)\Xi\left(\beta_c, \mu_c, 0\right) = 0. \tag{41}$$

This equation gives the critical temperature $T_c = \beta_c^{-1}$ in terms of the critical chemical potential $\mu_c = \mu(T_c)$ as function of the parameters of the model: m, λ and σ^2. To clarify the influence of disorder on T_c, let us consider the behavior of the critical temperature in the ultrarelativistic limit of equation (41). Since this is akin to performing a high-temperature expansion, we follow the technique developed in [19, 20] to obtain an analytical expression for the critical temperature. The relevant formulae are collected in appendix D.

Inserting in equation (41) the expression for the ultrarelativistic limit (i.e., $\beta\, m \ll 1$) of $\Pi_m(\beta, \mu, \xi_0)$, equation (D.12), and noting that in this limit the contributions coming from the last term of equation (41) can be neglected one gets for the critical temperature

$$T_c^2 = \left[1 + \frac{\sigma^2 m^4}{4\lambda_{\text{eff}}(\sigma)}\right] T_0^2, \tag{42}$$

where T_0 is the mean field critical temperature in the absence of disorder [18]:

$$T_0^2 = \frac{3}{\lambda}\left(\mu_c^2 - m^2\right). \tag{43}$$

Clearly, disorder implies in an increase of the condensation temperature in the ultrarelativistic limit. Also, there is a critical value for σ^2 for which Bose–Einstein condensation occurs only if $\mu_c = m$—namely, $\sigma_c^2 = 4\lambda/m^4$, which implies in $\lambda_{\text{eff}}(\sigma_c) = 0$. The condition $\mu_c = m$ is precisely the one for condensation of the free relativistic Bose gas [18]. While one should keep in mind that there might be important nonperturbative corrections to the precise value of critical value of the noise intensity, σ_c, it is clear that noise has induced an effective negative self-coupling for the scalar field that competes with the original repulsive coupling $\lambda > 0$.

The T–μ phase diagram $\lambda = 0.1$ is shown in figure 1. We use rescaled quantities T/m and μ/m. The noiseless mean field result is indicated by the (black) solid curve, which is the standard result [18]. The vertical (green) dashed–dotted line is for $\sigma = \sigma_c$, for which the effective coupling $\lambda_{\text{eff}}(\sigma)$ vanishes and, as said above, condensation occurs for $\mu_c = m$.

The chemical potential is a temperature-dependent parameter related to the total charge. This is discussed in the next section.

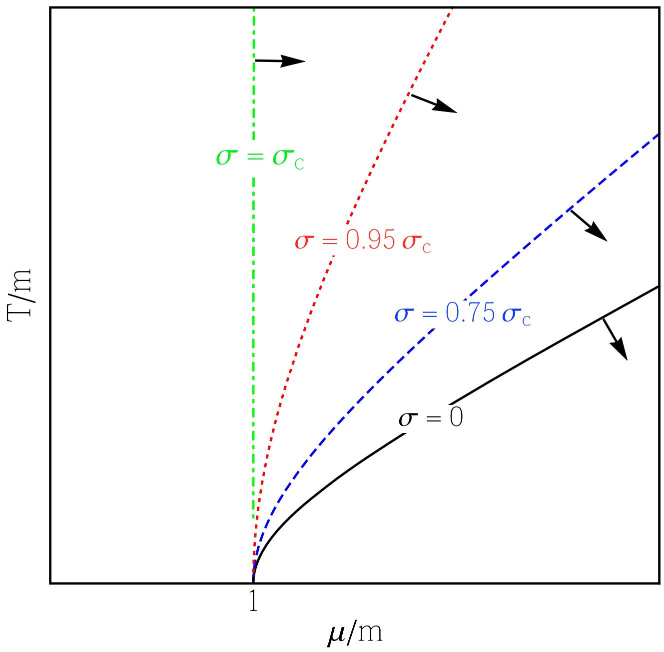

Figure 1. Temperature versus chemical potential phase-diagram for $\lambda = 0.1$ and four different values of the noise intensity σ. The arrows indicate the region of condensation, where $\xi_0 \neq 0$.

5. Temperature dependence of the chemical potential

5.1. Fixed charged density

Here we investigate the T dependence of the chemical potential for a fixed total charge density $\rho = Q/V$, with $Q > 0$, i.e. particles outnumber antiparticles. As usual, the charge density is calculated by differentiating with respect to μ the Helmholtz free energy at its minimum ($\xi = \xi_0$):

$$\rho = -\frac{1}{V}\left(\frac{\partial\overline{\Omega}}{\partial\mu}\right)_{\beta,V,\xi}, \tag{44}$$

where it should be understood that $\overline{\Omega} = \overline{\Omega}(\beta, V, \mu, \xi)$. For temperatures above the critical temperature, one has $\xi_0 = 0$; below the critical temperature ξ_0 is a solution of equation (38). Inserting equation (35) in the above equation (44) and employing equation (34) one obtains for ρ:

$$\rho = \left(2\mu + m^4\sigma^2\frac{\partial\Pi_m}{\partial\mu}\right)\xi_0^2 + \rho^*\left(\beta, \mu, \xi_0\right) + \rho_I\left(\beta, \mu, \xi_0\right), \tag{45}$$

where ρ^* is the mean-field thermal contribution:

$$\rho^*(\beta, \mu, \xi) = \int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\left[\frac{1}{\mathrm{e}^{\beta(\omega(\mathbf{p})-\mu)} - 1} - \frac{1}{\mathrm{e}^{\beta(\omega(\mathbf{p})+\mu)} - 1}\right], \tag{46}$$

and ρ_I is the contribution due to disorder

$$\rho_I(\beta,\ \mu,\ \xi)\ = \sigma^2 m^4\ \Pi_m \frac{\partial \Pi_m}{\partial \mu}. \tag{47}$$

From the discussions aforementioned, equation (46) illustrates the fact that each term in such an expression may be interpreted as the number density for particles and antiparticles, respectively. In turn, the above constrain $|\mu| \leqslant M$ can be envisaged as a requisite that the number densities be non-negative. We require that such statements are retained in the full model with randomness, at least in the mean-field approximation.

As above, we are interested in analyzing the ultrarelativistic limit of our results. For a fixed ρ, equation (45) can be formally inverted to give the chemical potential as a function of the temperature. Using the expressions derived in appendix D, one obtains for ρ^*:

$$\rho^*(\beta,\ \mu,\ \xi) \approx \frac{\mu}{3\beta^2} + \frac{M^2\mu}{4\pi^2} - \frac{\mu^3}{6\pi^2}, \tag{48}$$

and for ρ_I:

$$\rho_I(\beta,\ \mu,\ \xi) \approx \frac{\sigma^2 m^4 \mu}{16\pi^2}\left(\frac{\mu}{2\pi^2} - \frac{1}{3\beta^2}\right). \tag{49}$$

Inserting these results in equation (45), one obtains

$$\rho \approx 2\mu\left[1 + \frac{1}{2\pi^2}\lambda_{\rm eff}(\sigma)\right]\xi_0^2 + \frac{\mu}{3\beta^2}\left(1 - \frac{\sigma^2 m^4}{16\pi^2}\right). \tag{50}$$

The first term is the charge density associated with the condensate (zero-momentum mode)

$$\rho_c \approx 2\mu\left[1 + \frac{1}{2\pi^2}\lambda_{\rm eff}(\sigma)\right]\xi_0^2, \tag{51}$$

and the second is the charge density associated with the thermal particle excitations (finite-momentum modes)

$$\rho_{\rm th} \approx \frac{\mu}{3\beta^2}\left(1 - \frac{\sigma^2 m^4}{16\pi^2}\right). \tag{52}$$

Using equations (38), (41), and (D.12), one obtains for the condensate ξ_0:

$$\xi_0^2 \approx \frac{1}{2\lambda}\left[\mu^2 - \mu_c^2 - \frac{\lambda_{\rm eff}(\sigma)}{3}\left(\frac{1}{\beta^2} - \frac{1}{\beta_c^2}\right)\right]. \tag{53}$$

At the critical temperature $\xi_0 = 0$ and

$$\rho \approx \frac{\mu_c}{3\beta_c^2}\left(1 - \frac{\sigma^2 m^4}{16\pi^2}\right). \tag{54}$$

Finally, one can obtain the expression of the chemical potential in function of the the temperature. Inserting equation (53) in equation (50), one obtains a cubic equation for $\mu = \mu_\sigma(T)$ in terms of the total charge density ρ. For temperatures just below T_c the approximate solution is given by

$$\mu_\sigma(T) \approx \mu_c + \frac{\lambda\left(T_c^2 - T^2\right)}{6\tilde{\mu}_c^2 + \lambda T_c^2}\ \mu_c\left[\left(1 + \frac{3\lambda}{4\pi^2}\right)\frac{\sigma^2 m^4}{4\lambda} - \frac{\lambda}{2\pi^2}\right], \tag{55}$$

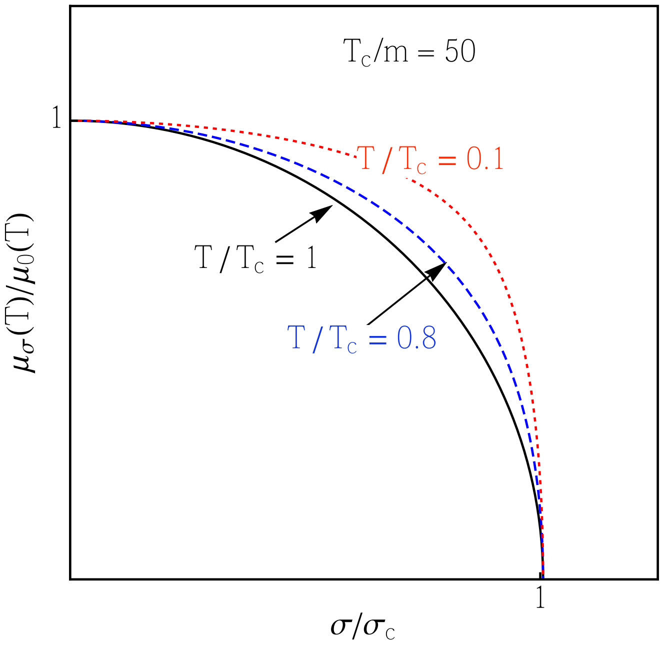

Figure 2. Chemical potential as a function of σ for three different temperatures $T \leqslant T_c$, for $\lambda = 0.1$ and $T_c/m = 50$.

where

$$\tilde{\mu}_c^2 = \left[1 + \frac{1}{2\pi^2}\lambda_{\text{eff}}(\sigma)\right]\mu_c^2. \tag{56}$$

A close inspection of equation (55) reveals that for $\sigma = 0$ one has the mean-field solution. As the temperature is reduced beyond T_c, the mean-field $\mu(T)$ continues to decrease, even though for sufficiently low temperatures such an expression ceases to be a good approximation. This is in agreement with the usual results of [18–20]. This scenario is modified for $\sigma \neq 0$. Neglecting the term $3\lambda/4\pi^2$, in order to keep the same behavior one must require that $m^4\sigma^2 < 2\lambda^2/\pi^2$. This situation respects the stability assumption: $m^4\sigma^2 \ll \lambda$. However the case in which $m^4\sigma^2 > 2\lambda^2/\pi^2$ is also possible provided that the stability condition remains valid. Actually, for the special case $m^4\sigma^2 \approx 2\lambda^2/\pi^2$, $\mu(T) \approx \mu_c$, even though the system is not at the critical point. Within the scenario in which $m^4\sigma^2 > 2\lambda^2/\pi^2$, $\mu(T)$ increases as the temperature is reduced. We interpret this as an energetically non-favorable situation and we conjecture that disorder may destabilize the condensate. In order to confirm such a conjecture, one should consider field self-interactions beyond the mean-field approximation employed here, which is outside the scope of the present work.

Let us analyze the behavior of $\mu_\sigma(T)$, equation (55), as a function of the noise intensity σ and for a fixed temperature $T \leqslant T_c$, depicted in figure 2. For small values of σ, the chemical potential is approximately constant and thereafter it starts to decrease. Close to the critical value of σ, the chemical potential is close to zero: this is the region where the induced interactions balance the field self-interactions; for $\sigma > \sigma_c$, the mean-field solution is destabilized.

Substituting equation (55) in (51) and using (54) one gets, for $0 \ll T \lesssim T_c$ and λ, $m^4\sigma^2 \ll \beta m \ll 1$

$$\rho_c \approx \rho\left[1 - \left(\frac{\beta_c}{\beta}\right)^2\right]. \tag{57}$$

This is the same behavior as found for the ground-state charge density of the ideal gas, see e.g. [19, 20]. This corresponds to a temperature-dependent ξ_0 given by

$$\xi_0^2 \approx \frac{1}{6}\left(T_c^2 - T^2\right)\left[\frac{1 - \sigma^2 m^4/16\pi^2}{1 + \lambda_{\rm eff}(\sigma)/2\pi^2}\right]. \tag{58}$$

For completeness, let us present the critical temperature as a function of the fixed charge density ρ:

$$\rho \approx \frac{m}{3\beta_c^2}\left[1 + \frac{\lambda}{6(\beta_c m)^2} - \frac{\sigma^2 m^4}{24(\beta_c m)^2}\right]. \tag{59}$$

Hence, the ultrarelativistic critical temperature in the weak-disorder limit is given by:

$$\beta_c^{-1} \approx \left(\frac{3\rho}{m}\right)^{1/2}\left[1 - \frac{\lambda}{12(\beta_c m)^2} + \frac{\sigma^2 m^4}{48(\beta_c m)^2}\right]. \tag{60}$$

Solving this equation by iteration one arrives at a power series expansion of β_c^{-1} in the effective coupling $m^4\sigma^2$. At first order, one has

$$\beta_c^{-1} \approx \beta_u^{-1}\left[1 - \frac{\lambda_{\rm eff}(\sigma)}{12(\beta_u m)^2}\right], \tag{61}$$

where β_u^{-1} is the ultrarelativistic critical temperature for the free gas

$$\beta_u^{-1} = \left(\frac{3\rho}{m}\right)^{1/2}. \tag{62}$$

As above, we find a positive shift in the critical temperature due to random fluctuations. Again, for $\lambda_{\rm eff}(\sigma_c) = 0$ the transition temperature is the same as the free case, i.e. the system behaves effectively as a free Bose gas.

Even though the following analysis falls out of the scope of the present article, it is instructive to briefly discuss the critical temperature as a function of the fixed charge density ρ in the non-relativistic limit. From the results derived in appendix D one gets

$$\rho^*(\beta, \mu, \xi) = \left(\frac{M}{2\pi\beta}\right)^{3/2} {\rm Li}_{3/2}\left(U_{\rm NR}\right), \tag{63}$$

and

$$\rho_I(\beta, \mu, \xi) = \sigma^2 m^4 \frac{M}{32\pi^3\beta^2}\,{\rm Li}_{3/2}\left(U_{\rm NR}\right){\rm Li}_{1/2}\left(U_{\rm NR}\right), \tag{64}$$

where ${\rm Li}_s(z)$ is the polylogarithm function whose expression is given in appendix D and $U_{\rm NR} = U_{\rm NR}(\beta, \mu_{\rm NR}) = {\rm e}^{\beta\mu_{\rm NR}}, \mu_{\rm NR} = \mu - M$ being the non-relativistic chemical potential. At the critical temperature, one gets

$$\rho = \left(\frac{m}{2\pi\beta_c}\right)^{3/2} {\rm Li}_{3/2}\left(U_{{\rm NR}c}\right)\left[1 + \frac{\sigma^2 m^4}{4\,(2\pi)^{3/2}}\,\frac{{\rm Li}_{1/2}\left(U_{{\rm NR}c}\right)}{\sqrt{\beta_c m}}\right], \tag{65}$$

where $U_{{\rm NR}c} = U_{\rm NR}(\beta_c, \mu_{{\rm NR}c})$. As above, one may solve this equation by iteration to obtain a power series expansion of β_c^{-1} in the effective coupling $m^4\sigma^2$. At first order, one gets the non-relativistic critical temperature in the weak-disorder limit:

$$\beta_c^{-1} \approx \beta_{\rm NR}^{-1}\left[1 - \frac{\sigma^2 m^4}{6\,(2\pi)^{3/2}}\,\frac{{\rm Li}_{1/2}\left(U_{{\rm NR}c}\right)}{\sqrt{\beta_{\rm NR}\, m}}\right], \tag{66}$$

where $\beta_{\rm NR}^{-1}$ is given by

$$\beta_{\rm NR}^{-1} = \frac{2\pi}{m}\left(\frac{\rho}{{\rm Li}_{3/2}\left(U_{{\rm NR}c}\right)}\right)^{2/3}. \tag{67}$$

In order to understand in detail the shift in the critical temperature due to random fluctuations one must insert in the above expression the chemical potential given by equation (41) in the non-relativistic limit. From the calculations presented in appendix D, one easily notes that the most important contributions to μ_c come from the function $\Xi(\beta_c, \mu_c, \xi_0)$. In addition, such contributions have negative values which implies that $\mu_c < m \therefore \mu_{{\rm NR}c} < 0$ ($\mu_{{\rm NR}c}$ is also close to zero in the weak-disorder limit). This is the expected behavior, since in order for ρ in this limit to be real one should have a negative chemical potential. The most important consequence of this is that in the leading order ${\rm Li}_{1/2}({\rm e}^{-z}) \approx \sqrt{\pi}/\sqrt{z}$ for $z \ll 1$. Incidentally, note also that in the leading order ${\rm Li}_{3/2}({\rm e}^{-z}) \approx \zeta(3/2)$ and thus equation (67) describes the non-relativistic critical temperature for the free Bose gas. Such considerations suggest that the effect of random fluctuations in the non-relativistic regime is to reduce the condensation temperature, in sharp contrast with the result uncovered in the ultrarelativistic limit. Such features find a close parallel with the results of [13].

5.2. Charge and entropy fixed

As discussed in the Introduction, RBEC has important cosmological implications. In such a context, in most cases the volume V changes with temperature but the net total charge Q and the entropy $\mathcal{S}$ remain constant. Therefore it is crucial to study the temperature evolution of the chemical potential with Q and $\mathcal{S}$ fixed (as in the early Universe). The entropy is given by

$$\mathcal{S} = \beta^2\left(\frac{\partial\overline{\Omega}}{\partial\beta}\right)_{V,\mu,\xi}, \tag{68}$$

where again it is to be understood that one must set $\xi = \xi_0$ after taking the above derivative. Inserting equation (35) in the above expression and employing equation (34), one gets

$$\frac{\mathcal{S}}{V} = \frac{\mathcal{S}_{\rm MF}}{V} + \frac{\mathcal{S}_I}{V}, \tag{69}$$

where the mean-field entropy is given by

$$\frac{\mathcal{S}_{\rm MF}}{V} = -I\left(\beta, \mu, \xi_0\right) + \beta\,\frac{\partial I\left(\beta, \mu, \xi_0\right)}{\partial\beta}, \tag{70}$$

with

$$I(\beta, \mu, \xi) = \int\frac{{\rm d}\mathbf{p}}{(2\pi)^3}\left[\ln\left(1 - {\rm e}^{-\beta(\omega(\mathbf{p})+\mu)}\right) + \ln\left(1 - {\rm e}^{-\beta(\omega(\mathbf{p})-\mu)}\right)\right], \tag{71}$$

whereas the corrections due to random fluctuations are given by

$$\frac{\mathcal{S}_I}{V} = -\sigma^2 m^4\,\beta^2\left(\xi_0^2 + \Pi_m\right)\frac{\partial\Pi_m}{\partial\beta}, \tag{72}$$

with $\Pi_m = \Pi_m(\beta, \mu, \xi_0)$ and $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2 + 4\lambda\xi_0^2}$. Using the expressions derived in appendix D, one obtains, in the ultrarelativistic limit

$$\frac{\mathcal{S}}{V} \approx \frac{4\pi^2}{45\beta^3} + \left(\xi_0^2 + \frac{1}{12\beta^2}\right)\frac{\sigma^2 m^4}{6\beta}. \tag{73}$$

Taking the ratios of the associated charge densities ρ_c and $\rho_{\rm th}$, given in equations (51) and (52), respectively, with the expression above leads to

$$\frac{Q_c}{\mathcal{S}} \approx \frac{45\mu\beta^3}{2\pi^2}\left(1 - \frac{5\sigma^2 m^4}{32\pi^2}\right)\tilde{\xi}_0^2 \tag{74}$$

and

$$\frac{Q_t}{\mathcal{S}} \approx \frac{15\mu\beta}{4\pi^2}\left(1 - \frac{7\sigma^2 m^4}{32\pi^2}\right), \tag{75}$$

where terms proportional to $\beta^2\sigma^2 m^4$ were dropped and

$$\tilde{\xi}_0^2 = \xi_0^2\left[1 + \frac{1}{2\pi^2}\lambda_{\rm eff}(\sigma)\right], \tag{76}$$

with ξ_0^2 given by equation (53). The sum of equations (74) and (75) produces a term independent of T. In the high-temperature region, the total net charge Q is given by equation (75). Lowering the temperature, there would be a point such that

$$\frac{Q}{\mathcal{S}} \approx \frac{15\hat{\mu}_c\hat{\beta}_c}{4\pi^2}\left(1 - \frac{7\sigma^2 m^4}{32\pi^2}\right), \tag{77}$$

where $1/\hat{\beta}_c = \hat{T}_c$ is the critical temperature for a fixed $Q/\mathcal{S}$ and $\hat{\mu}_c$ is the associated critical chemical potential. After a little algebra one can show that

$$\hat{\beta}_c^2 \approx \frac{1}{m^2}\left[\frac{\sigma^2 m^4}{12} - \frac{\lambda}{3} + \frac{16\pi^4}{225}\left(1 + \frac{7\sigma^2 m^4}{16\pi^2}\right)\left(\frac{Q}{\mathcal{S}}\right)^2\right]. \tag{78}$$

Remembering the stability assumption mentioned earlier and assuming that $Q/\mathcal{S}$ is sufficiently small, the above expression shows that $\hat{\beta}_c^2 < 0$ and thus in this case adiabatic cooling will not lead to symmetry breaking. Thence all the net charge is correctly given by equation (75) and there will be no Bose–Einstein condensation. This result is similar to the one in the absence of random fluctuations [19, 20].

Thus, there appears that assuming that $Q/\mathcal{S}$ is large enough so that $\hat{\beta}_c^2 > 0$, one could expect Bose–Einstein condensation to take place. However, this expectation falls apart when one soon realizes that there are other inconsistencies plaguing this particular case. Summing equations (74) and (75) and inserting equation (53) in the result yields a cubic equation for $\mu(T)$ in terms of the temperature and $Q/\mathcal{S}$. If $\hat{\mu}_c$ is the value of the chemical potential at the critical temperature, then just below $\hat{T}_c$ the approximate solution is

$$\mu \approx \hat{\mu}_c + \frac{2\lambda\left(\hat{T}_c^2 - T^2\right)}{6\tilde{\hat{\mu}}_c^2 + \lambda\hat{T}_c^2}\,\hat{\mu}_c\left[\frac{\sigma^2 m^4}{8\lambda}\left(1 + \frac{33\lambda}{16\pi^2}\right) - \frac{3}{4} - \frac{\lambda}{4\pi^2}\right], \tag{79}$$

where again terms proportional to $\lambda\,\sigma^2 m^4$ and $\sigma^4 m^8$ were dropped and

$$\tilde{\hat{\mu}}_c^2 = \hat{\mu}_c^2\left(1 + \frac{\lambda}{2\pi^2} - \frac{9\sigma^2 m^4}{32\pi^2}\right). \tag{80}$$

Inserting equation (79) in equation (74) one gets

$$\frac{Q_c}{\mathcal{S}} \approx -\frac{Q}{2\mathcal{S}}\left[1 - \left(\frac{\hat{\beta}_c}{\beta}\right)^2\right], \tag{81}$$

for $0 \ll T \lesssim T_c$ and $\lambda,\ \sigma^2 m^4 \ll \beta m \ll 1$, where $Q/\mathcal{S}$ is given by equation (77). In the present context, this corresponds to the following temperature-dependent ξ_0

$$\xi_0^2 \approx -\frac{1}{12}\left(\hat{T}_c^2 - T^2\right)\left[1 - \frac{\lambda}{2\pi^2} + \frac{\sigma^2 m^4}{16\pi^2}\right]. \tag{82}$$

Note that $\xi_0^2 < 0$, which is clearly unphysical; on the other hand $Q_c/\mathcal{S}$ is also negative which contradicts our initial assumption that particles outnumber antiparticles. Thence for a fixed $Q/\mathcal{S}$ there will be no Bose–Einstein condensation, a result already expected within the mean-field theory in the absence of disorder. As emphasized in [19, 20] this is due to the fact that $m^2 > 0$. We conclude that a nonstatic weak disorder does not change such a behavior.

6. Conclusions and perspectives

In this paper we investigated the effect of weak disorder on a weakly interacting relativistic charged scalar field in thermal equilibrium with a reservoir. We studied the effect of coupling of a random field to the scalar field in the situation where Bose–Einstein condensation takes place. We considered a quenched disorder which couples linearly to the mass of the scalar field, just as in the random-temperature Landau–Ginzburg model. After performing noise averages of the free energy, we obtained the corrections to the mean field critical temperature for the interacting Bose gas at finite density.

We have shown that the effect of the randomness is to increase the critical temperature for fixed charge density $\rho = Q/V$ in the ultrarelativistic limit. By contrast, a preliminary non-relativistic calculation as performed above seems to indicate that disorder tends to lower the critical temperature for fixed charge density. We observed significant differences from the mean-field temperature dependence of the chemical potential as the strength of the noise intensity increases. In particular, we found that for a critical noise intensity, the model behaves as a free field theory. In addition, having in mind application in the physics of the early Universe, we have investigated the temperature dependence of the chemical potential with fixed total charge and entropy. We found that there is no Bose–Einstein condensation for a fixed charge to entropy ratio in the presence of weak disorder. Within this context, we expect our model to be suitable to investigate realistic Higgs potentials in the presence of disorder in a straightforward way. In fact, a natural extension of this work is to consider the problem of examining more complicated models with several Higgs multiplets.

Naturally, one should keep in mind that these are results valid for weak disorder and obtained in the framework of a perturbative expansion in the noise intensity. It remains to be seen if the same results are attainable with a nonperturbative calculation, e.g. using a replica trick. For τ-independent noise, application of the replica-trick consists in the following [1]: using the fact that one can write

$$\ln Z = \lim_{n \to 0} \left(Z^n - 1 \right) / n,$$

one has that $\overline{\ln Z[\nu]} = \lim_{n \to 0}(Z_n - 1)/n$, where $Z_n = \overline{Z^n[\nu]}$; the Z_n's are interpreted as the partition functions of new systems, formed from n statistically independent copies of the original system. The quenched free energy functional is defined as $F_q(h) \equiv -\lim_{n \to 0}(Z_n - 1)/n$, showing that the quenched free energy functional can be calculated from a zero-component field theory. On the other hand, still at mean-field level one could also address the same problem considered in this work within the formalism developed by Cooper and collaborators which amounts to express the Lagrangian for a dilute Bose gas in terms of auxiliary fields associated with the normal and anomalous condensate densities [44, 45].

We remark on an important point with respect to the fact that random mass models generate effective interactions that mimic a negative coupling constant. Many authors claim that non-relativistic bosons only make sense in a random potential when they present repulsive interactions [8]. Nevertheless, there are many examples that even for a free theory one can define the theory in a controllable fashion. For instance, relativistic scalar field models with negative coupling constant were investigated in the literature and meaningful results were obtained—see for example [46–51]. Based on the results obtained in [52], where it has been shown that the theory with a negative coupling constant develops a condensate, Arias *et al* [53] discussed the thermodynamics of a asymptotically free Euclidean self-interacting scalar field defined in a compact spatial region without boundaries.

To conclude, we mention that effects of randomness over quantum fields have been discussed in different physical scenarios. In particular, on the basis of the results of [54–56], it was proposed in a condensed-matter-physics setting an analog model for fluctuations of the light cone [42]. Also, a free massive scalar field in inhomogeneous random media was studied in [43]. After performing the averages over the random functions, the two- and four-point causal Green's function of the model were presented up to one-loop approximation. Likewise, [57] and [58] investigated the influence of fluctuations of the event horizon on the transition rate of a two-level system which interact with a quantum field. More recently studies of effects of light-cone fluctuations over the renormalized vacuum expectation value of the stress–energy tensor of a real massless scalar field were carried out in [59]. In this case the field was defined in a flat space–time with non-trivial topology. In [60] the influence of such random fluctuations upon the zero-point energy associated with a free massless scalar in the presence of boundaries was investigated. Nonperturbative extensions of such works are under investigation by the authors.

Acknowledgments

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Appendix A. Calculation of the partition function

In this appendix we calculate the logarithm of the free partition function given by equation (21). We start by introducing Fourier series to the fields χ_1 and χ_2:

$$\chi_i(\tau,\,\mathbf{x}) = \left(\frac{\beta}{V}\right)^{1/2}\sum_n\sum_{\mathbf{p}} \mathrm{e}^{\mathrm{i}(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\chi_{i;n}(\mathbf{p}), \tag{A.1}$$

with $i = 1,\,2$ and $\beta\omega_n = 2\pi n$ due to the constraint of periodicity $\chi_i(0,\,\mathbf{x}) = \chi_i(\beta,\,\mathbf{x})$ for all $\mathbf{x}$. Inserting this last result in the free field action given by equation (15) we obtain, after performing an integration by parts:

$$S_0 = -\frac{1}{2}\sum_{n,\mathbf{p}}\begin{pmatrix}\chi_{1;-n}(-\mathbf{p}) & \chi_{2;-n}(-\mathbf{p})\end{pmatrix}\Theta\begin{pmatrix}\chi_{1;n}(\mathbf{p})\\ \chi_{2;n}(\mathbf{p})\end{pmatrix}, \tag{A.2}$$

where we discarded a total derivative term and we defined the matrix $\Theta = \Theta_n(\mathbf{p})$ as

$$\Theta_n(\mathbf{p}) = \beta^2\begin{pmatrix}\omega_n^2 + \omega_1^2(\mathbf{p}) - \mu^2 & -2\mu\omega_n\\ 2\mu\omega_n & \omega_n^2 + \omega_2^2(\mathbf{p}) - \mu^2\end{pmatrix}, \tag{A.3}$$

with $\omega_i(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_i^2}$, $i = 1,\,2$, $m_1^2 = 6\lambda\xi^2 + m^2$ and $m_2^2 = 2\lambda\xi^2 + m^2$. Thus, using equation (A.2) the logarithm of the free partition function now becomes

$$\ln Z_0 = \ln(N(\beta))^2 + \ln J(\beta,\,\mu), \tag{A.4}$$

where

$$\begin{aligned} J(\beta,\,\mu) = &\prod_n\prod_{\mathbf{p}}\int \mathrm{d}\chi_{2;n}(\mathbf{p})\exp\left\{-\frac{1}{2}[\Theta_n(\mathbf{p})]_{22}\chi_{2;n}(\mathbf{p})\chi_{2;-n}(-\mathbf{p})\right\}\\ &\times\int \mathrm{d}\chi_{1;n}(\mathbf{p})\exp\left\{-\frac{1}{2}[\Theta_n(\mathbf{p})]_{11}\chi_{1;n}(\mathbf{p})\chi_{1;-n}(-\mathbf{p})\right\}\\ &\times\exp\left\{-\frac{1}{2}[\Theta_n(\mathbf{p})]_{12}\chi_{2;n}(\mathbf{p})\chi_{1;-n}(-\mathbf{p}) - \frac{1}{2}[\Theta_n(\mathbf{p})]_{21}\chi_{1;n}(\mathbf{p})\chi_{2;-n}(-\mathbf{p})\right\}. \end{aligned} \tag{A.5}$$

Noting that $\chi_{i;-n}(-\mathbf{p}) = \chi_i^*(\mathbf{p})$, $i = 1,\,2$, as required by the reality of the fields $\chi_i(\tau,\,\mathbf{x})$, the above integrals are just generic Gaussian integrals. Therefore

$$J(\beta,\,\mu) = \prod_n\prod_{\mathbf{p}}\left[\det\Theta_n(\mathbf{p})\right]^{-1/2}, \tag{A.6}$$

where we have discarded an overall constant multiplicative factor. Inserting this last expression in equation (A.4), we get:

$$\ln Z_0 = \ln(N(\beta))^2 - \frac{1}{2}\ln\left[\prod_n\prod_{\mathbf{p}}\det\Theta_n(\mathbf{p})\right], \tag{A.7}$$

where

$$\begin{aligned} &\ln\left[\prod_n\prod_{\mathbf{p}}\det\Theta_n(\mathbf{p})\right]\\ &= \ln\left\{\prod_n\prod_{\mathbf{p}}\beta^4\left[\left(\omega_n^2 + \omega_1^2(\mathbf{p}) - \mu^2\right)\left(\omega_n^2 + \omega_2^2(\mathbf{p}) - \mu^2\right) + 4\mu^2\omega_n^2\right]\right\}, \end{aligned} \tag{A.8}$$

and, according to [37], in the large-volume limit

$$\ln(N(\beta)) = -V\,\ln\beta\sum_n\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}. \tag{A.9}$$

It is possible to factorize the quantity inside the square brackets in equation (A.8) by defining the 'effective mass'

$$M^2 = \frac{m_1^2 + m_2^2}{2} + \frac{\lambda^2 \xi^4}{\mu^2}. \tag{A.10}$$

One gets

$$\begin{aligned}&\left[\omega_n^2 + \omega_1^2(\mathbf{p}) - \mu^2\right]\left[\omega_n^2 + \omega_2^2(\mathbf{p}) - \mu^2\right] + 4\mu^2\omega_n^2\\ &\quad = \left[\omega_n^2 + \Omega_+^2(\mathbf{p}, \xi)\right]\left[\omega_n^2 + \Omega_-^2(\mathbf{p}, \xi)\right],\end{aligned} \tag{A.11}$$

where

$$\Omega_\pm^2(\mathbf{p}, \xi) = (\omega(\mathbf{p}) \pm \mu)^2 - \frac{\lambda^2\xi^4}{\mu^2}, \tag{A.12}$$

with $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2}$. Hence

$$\begin{aligned}\ln Z_0 = &-\frac{1}{2}\sum_{n,\mathbf{p}} \ln\left\{\beta^2\left[\left(\omega_n^2 + \Omega_+^2\right)\right]\right\} - \frac{1}{2}\sum_{n,\mathbf{p}} \ln\left\{\beta^2\left[\left(\omega_n^2 + \Omega_-^2\right)\right]\right\}\\ &- 2V\ln\beta\sum_n\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3},\end{aligned} \tag{A.13}$$

where $\Omega_\pm = \Omega_\pm(\mathbf{p}, \xi)$. The frequency sums can be performed using standard procedures [61] and the result is given by equation (22).

In particular, since for a given n and $\mathbf{p}$ the propagators can be expressed as functional derivatives of the partition function [41]

$$\mathcal{D}_{ij}(\omega_n, \mathbf{p}) = 2\left(\mathcal{D}_{ij}^0\right)^2\frac{\delta\ln Z[\nu]}{\delta\mathcal{D}_{ij}^0}. \tag{A.14}$$

$i, j = 1, 2$, one notes that the zero-order propagators $\mathcal{D}_{ij}^0(\omega_n, \mathbf{p})$ are given by

$$\mathcal{D}_{ij}^0(\omega_n, \mathbf{p}) = -2\beta^2\frac{\delta\ln Z_0}{\delta\left[\Theta_n(\mathbf{p})\right]_{ij}}. \tag{A.15}$$

Appendix B. Renormalization of propagators

Here we examine the renormalization of the finite-temperature propagators $\mathcal{D}_{11}^0(\omega_n, \mathbf{p})$ and $\mathcal{D}_{22}^0(\omega_n, \mathbf{p})$. Following [41] we define the self-energy $\Pi_1 = \Pi_1(\omega_n, \mathbf{p})$ with respect to the averaged propagator $\overline{\mathcal{D}_{11}}$ as

$$\overline{\mathcal{D}_{11}(\omega_n, \mathbf{p})} = \left(1 + \mathcal{D}_{11}^0\Pi_1\right)^{-1}\mathcal{D}_{11}^0. \tag{B.1}$$

A similar expression holds for $\Pi_2 = \Pi_2(\omega_n, \mathbf{p})$ which is the self-energy with respect to the averaged propagator $\overline{\mathcal{D}_{22}}$. Hence, recalling equations (A.14) and (A.15), we get, up to second order in ν:

$$\left(1 + \mathcal{D}_{11}^0\Pi_1\right)^{-1} = 1 + 2\mathcal{D}_{11}^0\frac{\delta\overline{\ln Z_I[\nu]}}{\delta\mathcal{D}_{11}^0}, \tag{B.2}$$

and

$$\left(1+\mathcal{D}_{22}^{0}\,\Pi_{2}\right)^{-1}=1+2\mathcal{D}_{22}^{0}\,\frac{\delta\,\overline{\ln Z_{I}[\nu]}}{\delta\mathcal{D}_{22}^{0}}. \tag{B.3}$$

Therefore, inserting equation (28) in the above equations and expanding their left-hand sides to first order yields the following expressions for the self-energies

$$\Pi_{1}(\omega_{n},\,\mathbf{p})=-2m^{4}\sigma^{2}\xi^{2}-\frac{m^{4}\sigma^{2}}{\beta V}\sum_{n,\mathbf{p}}\mathcal{D}_{11}^{0}(\omega_{n},\,\mathbf{p}), \tag{B.4}$$

and

$$\Pi_{2}(\omega_{n},\,\mathbf{p})=-\frac{m^{4}\sigma^{2}}{\beta V}\sum_{n,\mathbf{p}}\mathcal{D}_{22}^{0}(\omega_{n},\,\mathbf{p}). \tag{B.5}$$

On the other hand, remembering equation (A.15) one gets

$$\begin{aligned}\frac{1}{\beta V}\sum_{n,\mathbf{p}}\mathcal{D}_{ii}^{0}(\omega_{n},\,\mathbf{p})=&\frac{1}{4}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^{3}}\frac{1}{W_{-}(\mathbf{p},\,\xi)}\left[1+\frac{2}{\mathrm{e}^{\beta\Omega_{-}}-1}\right]\\&+\frac{1}{4}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^{3}}\frac{1}{W_{+}(\mathbf{p},\,\xi)}\left[1+\frac{2}{\mathrm{e}^{\beta\Omega_{+}}-1}\right],\end{aligned} \tag{B.6}$$

for $i=1,\,2$ and

$$\frac{1}{W_{\pm}(\mathbf{p},\,\xi)}=\frac{1}{\Omega_{\pm}(\mathbf{p},\,\xi)}\pm\frac{\mu}{\omega\Omega_{\pm}(\mathbf{p},\,\xi)}.$$

In equation (B.6) we consider V to be large compared to all other physical lengths so we can replace the sum over $\mathbf{p}$ with an integral. In this way we have

$$\Pi_{1}(\omega_{n},\,\mathbf{p})=-m^{4}\sigma^{2}\left[2\xi^{2}+\Pi_{v+}+\Pi_{v-}+\,\Pi_{m+}(\beta,\,\xi)+\Pi_{m-}(\beta,\,\xi)\right], \tag{B.7}$$

and

$$\Pi_{2}(\omega_{n},\,\mathbf{p})=-m^{4}\sigma^{2}\left[\Pi_{v+}+\Pi_{v-}+\,\Pi_{m+}(\beta,\,\xi)+\Pi_{m-}(\beta,\,\xi)\right], \tag{B.8}$$

where we have defined

$$\Pi_{v\pm}=\frac{1}{4}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^{3}}\frac{1}{W_{\pm}(\mathbf{p},\,\xi)}, \tag{B.9}$$

and

$$\Pi_{m\pm}(\beta,\,\xi)=\frac{1}{2}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^{3}}\frac{1}{W_{\pm}(\mathbf{p},\,\xi)}\frac{1}{\mathrm{e}^{\beta\Omega_{\pm}}-1}. \tag{B.10}$$

Since Π_v is a divergent quantity, in order to avoid physically meaningless results the following counterterm must be added to the original action:

$$\begin{aligned}\delta S=&-\,\delta\sigma^{2}\int_{0}^{\beta}\mathrm{d}\tau\int_{V}\mathrm{d}\mathbf{x}\;\varphi\varphi^{*}\\=&-\,\frac{\delta\sigma^{2}}{2}\int_{0}^{\beta}\mathrm{d}\tau\int_{V}\mathrm{d}\mathbf{x}\left(2\xi^{2}+\chi_{1}^{2}+\chi_{2}^{2}\right),\end{aligned} \tag{B.11}$$

where we have droped terms linear in χ_1 and χ_2. Treating this as an additional interaction, we see from equation (24) that to lowest order this counterterm contributes to $\ln Z_I$ as

$$-(\beta V)\delta\sigma^2\xi^2 - \frac{\delta\sigma^2}{2}\int_0^\beta \mathrm{d}\tau\int_V \mathrm{d}\mathbf{x}\left(\left\langle\chi_1^2\right\rangle + \left\langle\chi_2^2\right\rangle\right). \tag{B.12}$$

The counterterm should be chosen so that

$$\delta\sigma^2 - m^4\sigma^2\left(\Pi_{\nu-} + \Pi_{\nu+}\right) = 0. \tag{B.13}$$

In this way we get a finite result for the propagators. Whence, collecting the above results, the contribution to $\ln Z$ up to second order in the noise will be

$$\overline{\ln Z_I[\nu]} = (\beta V)\frac{m^4\sigma^2}{2}\left[\Pi_m\left(2\xi^2 + \Pi_m\right) - \Pi_\nu^2\right], \tag{B.14}$$

where one has that $\Pi_m = \Pi_{m+}(\beta, \xi) + \Pi_{m-}(\beta, \mu, \xi)$ and also $\Pi_\nu = \Pi_{\nu+} + \Pi_{\nu-}$.

Appendix C. Renormalization of the vacuum energy density

In this appendix we discuss the renormalization of the classical energy density, equation (37). In the expression (35) we have neglected the shift in $\Omega_{\text{cl}}(\xi)$ coming from the zero-point energy density of the vacuum as well as the divergent contribution Π_ν which results from the renormalization of the propagators considered in detail in the previous appendix (e.g., see equation (B.14)). Since we are in the mean field approximation, $\lambda\xi \ll 1$, we take into account the same approximation mentioned in section 4. Namely, we neglect the contributions coming from the terms proportional to $\lambda^2\xi^4$ in the definition of $\Omega_\pm(\mathbf{p}, \xi)$ in equation (A.12). This means that, in this approximation the zero-point energy density is given by:

$$E_0 = \int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\left(\frac{\Omega_+}{2} + \frac{\Omega_-}{2}\right) = \int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\omega(\mathbf{p}),$$

whereas Π_ν becomes

$$\Pi_\nu = \Pi_{\nu+} + \Pi_{\nu-} = \frac{1}{2}\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\frac{1}{\omega(\mathbf{p})}.$$

As a regularization procedure we simply choose to place a high-momentum cutoff Λ_c on the integration over $|\mathbf{p}|$. In this way, we get

$$E_0 = \frac{1}{64\pi^2}\left[4M^2\Lambda_c^2 - 2M^4\ln\left(\frac{\Lambda_c^2}{M^2}\right) - M^4\right], \tag{C.1}$$

and

$$\Pi_\nu = \frac{1}{32\pi^2}\left[\Lambda_c^2 - M^2\ln\left(\frac{\Lambda_c^2}{M^2}\right)\right], \tag{C.2}$$

where, due to the aproximation earlier observed, $M^2 = m^2 + 4\lambda\xi^2$. Also, in equations (C.1) and (C.2) we have dropped constants and terms which vanish as $\Lambda_c \to \infty$. In order to renormalize the vacuum energy density, we demand that the final result should be independent of Λ_c. Also, we require its minimum to be at the same location as the classical energy density, i.e., at $\xi_c^2 = (\mu_0^2 - m^2)/2\lambda$. This is achieved by adding to the original action counterterms which depend on the bare parameters m^2 and λ as well as on Λ_c. In addition, one should specify a suitable set of normalization conditions. Here we choose

$$\frac{d^2\overline{\Omega(0, \mu_0, \xi)}}{d\xi^2}|_{\xi=\xi_c} = 4(\mu_0^2 - m^2)$$
$$\frac{d^4\overline{\Omega(0, \mu_0, \xi)}}{d\xi^4}|_{\xi=\xi_c} = 24\lambda, \tag{C.3}$$

where $\overline{\Omega(0, \mu_0, \xi)} = \Omega_{\rm cl}(\xi)$ plus divergent vacuum terms. These are reminiscent of the usual normalization conditions employed in the effective potential approach of quantum field theories.

In both expressions for E_0 and Π_v we have terms proportional to $\ln(1 + 4\lambda\xi^2/m^2)$ which could render the renormalization procedure somewhat cumbersome. Since $\lambda\,\xi \ll 1$ by assumption, for simplicity we may Taylor expand this logarithmic function and keep terms up to $\lambda^2\xi^4$. Using this technique for E_0 and Π_v and adding the resulting divergent term $E_0 + m^4\sigma^2\Pi_v^2/2$ to the classical energy density (37) results in the following vacuum energy density:

$$\begin{aligned}\overline{\Omega(0, \mu_0, \xi)} = &\frac{1}{1024\pi^4}\Bigg\{64\pi^2m^2\Lambda_c^2 - 16\pi^2m^4\\
&- 32\pi^2m^4\ln\left(\frac{\Lambda_c^2}{m^2}\right) + m^4\left[\ln\left(\frac{\Lambda_c^2}{m^2}\right)\right]^2\\
&+ \frac{m^4\sigma^2\Lambda_c^2}{2}\left[I(m, \Lambda_c) - m^2\ln\left(\frac{\Lambda_c^2}{m^2}\right)\right]\Bigg\}\\
&+ \xi^2\Bigg\{m^2 - \mu_0^2 + \delta m^2 + \frac{\lambda}{4\pi^2}I(m, \Lambda_c) - \frac{\lambda m^2}{128\pi^4}\ln\left(\frac{\Lambda_c^2}{m^2}\right)\\
&+ \frac{m^4\sigma^2}{2}\left[g(\lambda, m, \Lambda_c) - \frac{\lambda\Lambda_c^2}{128\pi^4}\ln\left(\frac{\Lambda_c^2}{m^2}\right)\right]\Bigg\}\\
&+ \xi^4\Bigg\{\lambda + \delta\lambda + f_+(\lambda) + \ln\left(\frac{\Lambda_c^2}{m^2}\right)f_-(\lambda)\\
&+ m^4\sigma^2\left[\frac{\lambda}{m^2}g(\lambda, m, \Lambda_c) - \frac{\lambda^2}{32\pi^4}\ln\left(\frac{\Lambda_c^2}{m^2}\right)\right]\Bigg\},\end{aligned} \tag{C.4}$$

where $\delta\, m^2\xi^2$ and $\delta\lambda\xi^4$ are the above mentioned counterterms and

$$f_\pm(\lambda) = \frac{\lambda^2}{2\pi^2}\left(\frac{1}{32\pi^2} \pm 1\right),$$
$$g(\lambda, m, \Lambda_c) = \frac{\lambda\Lambda_c^2}{128\pi^4} + \frac{\lambda m^2}{128\pi^4}\left[\ln\left(\frac{\Lambda_c^2}{m^2}\right)\right]^2,$$
$$I(m, \Lambda_c) = \Lambda_c^2 - m^2\ln\left(\frac{\Lambda_c^2}{m^2}\right). \tag{C.5}$$

In equation (C.4) we again have retained terms up to $\lambda^2\xi^4$. Employing the normalization conditions (C.3) we find that

$$\delta m^2 = \frac{\lambda m^2}{128\pi^4} \ln\left(\frac{\Lambda_c^2}{m^2}\right) - \frac{\lambda}{4\pi^2} I\left(m,\, \Lambda_c\right) + \frac{m^4\sigma^2}{2}\left[\frac{\lambda\Lambda_c^2}{128\pi^4}\ln\left(\frac{\Lambda_c^2}{m^2}\right) - g\left(\lambda,\, m,\, \Lambda_c\right)\right], \tag{C.6}$$

and

$$\delta\lambda = -f_+(\lambda) - \ln\left(\frac{\Lambda_c^2}{m^2}\right) f_-(\lambda) - m^4\sigma^2\left[\frac{\lambda}{m^2} g\left(\lambda,\, m,\, \Lambda_c\right) - \frac{\lambda^2}{32\pi^4}\ln\left(\frac{\Lambda_c^2}{m^2}\right)\right]. \tag{C.7}$$

In this way, after a straightforward calculation one gets

$$\overline{\Omega\left(0,\, \mu_0,\, \xi\right)} = \Omega_{\rm cl}(\xi) + K\left(\lambda,\, m,\, \Lambda_c\right), \tag{C.8}$$

where

$$K\left(\lambda,\, m,\, \Lambda_c\right) = -\frac{m^4}{64\pi^2} + \frac{m^2}{32\pi^2}\left\{\Lambda_c^2 + I\left(m,\, \Lambda_c\right) + \frac{m^2}{32\pi^2}\left[\ln\left(\frac{\Lambda_c^2}{m^2}\right)\right]^2\right\} + \frac{m^4\sigma^2\Lambda_c^2}{2048\pi^4}\left[I\left(m,\, \Lambda_c\right) - m^2\ln\left(\frac{\Lambda_c^2}{m^2}\right)\right]. \tag{C.9}$$

The (infinite) constant term $K(\lambda,\, m,\, \Lambda_c)$ can be set to zero by shifting the vacuum energy density by a constant amount. This can always be done since in non-gravitational physics only energy differences are measurable. In this way, we finally get that the renormalized vacuum energy is just the classical energy density, $\Omega_{\rm cl}(\xi) = \overline{\Omega(0,\, \mu_0,\, \xi)}$.

Appendix D. Ultrarelativistic limit and non-relativistic limit of $\Pi_m(\beta, \mu, \xi)$

Employing spherical coordinates, one can express $\Pi_m(\beta,\, \mu,\, \xi)$ as

$$\Pi_m(\beta,\, \mu,\, \xi) = \frac{1}{4\pi^2}\int_0^\infty {\rm d}p\, \frac{p^2}{\omega}\left[\frac{1}{{\rm e}^{\beta(\omega-\mu)} - 1} + \frac{1}{{\rm e}^{\beta(\omega+\mu)} - 1}\right], \tag{D.1}$$

where a partial integration was made and $\omega = \omega(\mathbf{p})$. Let us define

$$g_l(y,\, r) = \frac{1}{\Gamma(l)}\int_0^\infty {\rm d}x\, \frac{x^{l-1}}{\exp\left[\left(x^2 + y^2\right)^{1/2} - ry\right] - 1}, \tag{D.2}$$

$$h_l(y,\, r) = \frac{1}{\Gamma(l)}\int_0^\infty \frac{{\rm d}x}{\left(x^2 + y^2\right)^{1/2}}\, \frac{x^{l-1}}{\exp\left[\left(x^2 + y^2\right)^{1/2} - ry\right] - 1}. \tag{D.3}$$

The functions of interest here are

$$G_l(y,\, r) = g_l(y,\, r) - g_l(y,\, -r), \tag{D.4}$$

$$H_l(y,\, r) = h_l(y,\, r) + h_l(y,\, -r). \tag{D.5}$$

Therefore, with $y = \beta M$, $r = \mu/M$ and after a simple change of variables we get

$$\Pi_m(y, r) = \frac{1}{2\pi^2\beta^2} H_3(y, r). \tag{D.6}$$

The calculation of the functions G_l and H_l is discussed at length in [20]. Here we simply quote the quantities which are relevant for our computations. The following recursion relations ought to be employed:

$$\frac{dG_{l+1}}{dy} = lrH_{l+1} - \frac{y}{l} G_{l-1} + \frac{y^2 r}{l} H_{l-1}, \tag{D.7}$$

$$\frac{dH_{l+1}}{dy} = \frac{r}{l} G_{l-1} - \frac{y}{l} H_{l-1}, \tag{D.8}$$

with the initial conditions $G_l(0, 0) = 0$, $l > 0$, and $H_l(0, 0) = 2\zeta(l-1)/(l-1)$, $l > 2$, $\zeta(s)$ being the usual Riemann zeta function. Consequently, knowledge of G_1 and H_1 will yield G_l and H_l for all positive odd l.

The small y expansions of the functions G_1 and H_1 are given by, respectively:

$$G_1(y, r) = \frac{\pi r}{\left(1 - r^2\right)^{1/2}} - ry + 2\pi r \times \sum_{k=1}^{\infty} (-1)^{k+1} a_k\, \zeta(2k+1) \left(\frac{y}{2\pi}\right)^{2k+1}, \tag{D.9}$$

and

$$H_1(y, r) = \frac{\pi}{y\left(1 - r^2\right)^{1/2}} + \ln\left(\frac{y}{4\pi}\right) + \gamma + \sum_{k=1}^{\infty} (-1)^k b_k\, \zeta(2k+1) \left(\frac{y}{2\pi}\right)^{2k}, \tag{D.10}$$

with $\gamma = 0.5772...$ being the Euler's constant. The quantities a_k and b_k are simple polynomials in r. For $k = 1$ one has $a_1 = 1$ and $b_1 = r^2 + 1/2$. We refer the reader to [20] for all important details concerning the derivations of the above relations.

The $y \ll 1$ limit allows retain just the first term of the summations in G_1 and H_1. Employing equations (D.7)–(D.10) with the aforementioned initial conditions one obtains, after a straightforward calculation:

$$H_3(y, r) = \frac{\pi^2}{6} - \frac{y}{2} \pi \left(1 - r^2\right)^{1/2} + \frac{y^2}{8}\left[\ln\left(\frac{16\pi^2}{y^2}\right) - 2\gamma + 1 - 2r^2\right] + \frac{y^4}{64\pi^2}\left(1 + 4r^2\right)\zeta(3), \tag{D.11}$$

where we used the fact that $\zeta(2) = \pi^2/6$. Hence inserting the above expressions in equation (D.6) we get

$$\Pi_m(\beta, \mu, \xi) \approx \frac{1}{12\beta^2} - \frac{\mu^2}{8\pi^2} + \frac{M^2}{16\pi^2} B_1(\beta), \tag{D.12}$$

where

$$B_1(\beta) = \ln\left(\frac{16\pi^2}{\beta^2 M^2}\mathrm{e}^{-2\gamma+1}\right). \tag{D.13}$$

Terms proportional to β^2 or higher powers of β were dropped. On the other hand, the large y expansion of the functions $g_l(y, r)$, $h_l(y, r)$ are given by, respectively (for details, see [20]):

$$g_l(y, r) = \frac{\Gamma(l/2)}{\Gamma(l)} \sum_{n=0}^{\infty} \frac{1}{\Gamma(l/2 - n)\, n!}\left(\frac{1}{2y}\right)^{n+1-l/2} \times \left[y\, \Gamma(l/2 + n)\, \mathrm{Li}_{n+l/2}\left(\mathrm{e}^{(r-1)y}\right) + \Gamma(l/2 + n + 1)\, \mathrm{Li}_{n+l/2+1}\left(\mathrm{e}^{(r-1)y}\right)\right] \tag{D.14}$$

and

$$h_l(y, r) = \frac{\Gamma(l/2)}{\Gamma(l)} \sum_{n=0}^{\infty} \frac{\Gamma(l/2 + n)}{\Gamma(l/2 - n)\, n!}\left(\frac{1}{2y}\right)^{n+1-l/2} \mathrm{Li}_{n+l/2}\left(\mathrm{e}^{(r-1)y}\right), \tag{D.15}$$

where $\Gamma(z)$ is the gamma function and $\mathrm{Li}_s(z)$ is the polylogarithm or Jonquière's function:

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

In order to present an expression for $\Pi_m(\beta, \mu, \xi)$ in the non-relativistic limit, one must consider the functions $h_3(y, \pm r)$. In turn, note that, in this limit:

$$h_3(y, -r) \approx \left(\frac{\pi y}{8}\right)^{1/2} \mathrm{Li}_{3/2}\left(U_{\mathrm{NR}}^{-1}\, \mathrm{e}^{-2y}\right) \approx \left(\frac{\pi y}{8}\right)^{1/2} \frac{\mathrm{e}^{-2y}}{U_{\mathrm{NR}}}, \tag{D.16}$$

where we have kept only the leading terms and we have defined $U_{\mathrm{NR}} = U_{\mathrm{NR}}(\beta, \mu_{\mathrm{NR}}) = \mathrm{e}^{(r-1)y} = \mathrm{e}^{\beta\mu_{\mathrm{NR}}}$, $\mu_{\mathrm{NR}} = \mu - m$ being the non-relativistic chemical potential (in the non-relativistic limit, $-\infty < \mu_{\mathrm{NR}} < 0$). The above equation implies that the contribution of antiparticles is exponentially small in the non-relativistic limit. Hence, one may keep only the function $h_3(y, r)$ in order to reach a final expression for the function $\Pi_m(\beta, \mu, \xi)$ in the non-relativistic limit. The leading term in such a limit reads

$$\Pi_m(\beta, \mu, \xi) = \left(\frac{M}{32\pi^3\beta^3}\right)^{1/2} \mathrm{Li}_{3/2}\left(U_{\mathrm{NR}}\right). \tag{D.17}$$

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