Rev Mat Complut (2016) 29:439–454
DOI 10.1007/s13163-015-0187-5

Geometry and equisingularity of finitely determined map germs from $\mathbb{C}^n$ to $\mathbb{C}^3$, $n > 2$

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Received: 8 September 2014 / Accepted: 29 December 2015 / Published online: 11 January 2016

Abstract In this article we describe the geometry and the Whitney equisingularity of finitely determined map germs $f:(\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$ with $n \geq 3$. In the study of the geometry, we first investigate the critical locus $\Sigma(f)$ of the germ, which is in the source. Then the discriminant $\Delta(f)$, the image of the critical locus by the germ f, is studied. Last, but not least we investigate the set $X(f)$, which is the inverse image by f of the discriminant. If the critical locus is not empty, the set $X(f)$ is an hypersurface in the source that has nonisolated singularity at the origin. Concerning the Whitney equisingularity of families, we use some of the properties of the strata to prove that the Whitney equisingularity of an unfolding F is equivalent to the constancy of the Lê numbers of the hypersurfaces $\Delta(f)$ and $X(f)$. From this study we describe some relationship among the invariants needed to describe the Whitney equisingularity of families in these dimensions, we reduce the number of invariants needed

This work was partially supported by CAPES, CNPq, FAPESP and FAPEMIG.

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to a total of $2n + 2$, which improves substantially the number required by Gaffney's theorem.

Keywords Geometry of map germs · Whitney equisingularity · Numerical invariants · Lâ numbers

Mathematics Subject Classification Primary 32S15 · 14B05

1 Introduction

The study of the geometry of the singularities of map germs is one of the main questions in singularity theory, a key tool to better understanding of the geometry is the description of all strata which appear in the critical locus $\Sigma(f)$, in the discriminant $\Delta(f)$, and in the hypersurface $X(f)$. Moreover, in these sets we can study the numerical invariants that control triviality conditions in families of map germs. In this article, first we investigate the geometry of finitely determined map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$ with $n \geq 3$, in the second section we give an explicit description of all strata in these dimensions and, with the aid of a computer system, we show in an explicity way how to compute them in several examples.

Concerning the Whitney equisingularity, Gaffney describes in [1] the following problem: "Given a 1-parameter family of map germs $F: (\mathbb{C} \times \mathbb{C}^n, (0, 0)) \to (\mathbb{C} \times \mathbb{C}^p, (0))$, find analytic invariants whose constancy in the family implies the family is Whitney equisingular." He shows that for the class of finitely determined map germs of discrete stable type, the Whitney equisingularity of such a family is guaranteed by the invariance of the zero stable types and the polar multiplicities associated to all stable types.

A natural question is to find a minimal set of invariants that guarantee the Whitney equisingularity of the family. Gaffney and Vohra in [2] studied map germs from n-space to the plane, with $n \geq 3$. In this case they showed that the top dimensional stratum of the inverse image of the discriminant carries all information necessary to determine if the family of map germs is Whitney equisingular, in fact they showed that Lê numbers of this stratum control all other invariants needed for the Whitney equisingularity.

In [3] the finitely determined map germs in $\mathcal{O}(n, 3)$ with $n \geq 3$ are investigated and $2n + 2$ invariants are obtained under the restriction that the map germ has corank one, it is also shown in [3] that the Lê numbers of $X(f)$ control the Whitney equisingularity of this set. Here we continue the study of such map germs, the main difference on one hand, we show that the Whitney equisingularity of $X(f)$ also implies the Whitney equisingularity of the strata in $\Sigma(f)$, and on the other hand, we use the Lê numbers also for the discriminant $\Delta(f)$, instead of the polar multiplicities as done in [3], moreover we show that the corank one condition is not needed.

We use the Surfex program [4], to produce the pictures in this work.

2 Geometry of finitely determined map germs $f \in \mathcal{O}(n, 3)$

2.1 Stable types of finitely determined map germs

We follow Gaffney in [1] and denote by $\mathcal{O}(n, p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$, $\mathcal{O}_e(n, p)$ denotes the set of germs at the origin but not necessarily origin preserving.

For a germ $f \in \mathcal{O}_e(n, p)$, call $m = \min\{n, p\}$ and denote the ideal generated by the set of $m \times m$ minors of the derivative of f by $J(f)$, the critical set of f, denoted $\Sigma(f)$, consists of the set of points $x \in \mathbb{C}^n$ such that the differential is not an epimorphism. The discriminant $\Delta(f)$ of f is the image of $\Sigma(f)$ by f.

When $n \geq p$ there appear naturally in the inverse image of $\Delta(f)$ other points than the points in the critical set, moreover these points form a set in $\mathbb{C}^n$ with the same codimension than $\Delta(f)$ in $\mathbb{C}^p$. In this case we denote by $X(f)$ the hypersurface $\overline{\left(f^{-1}(\Delta(f)) - \Sigma(f)\right)}$ with reduced structure.

As usual, we denote by $\mathcal{R}$ the Mather's group of local diffeomorphisms on the source $(\mathbb{C}^n, 0)$, by $\mathcal{L}$, the Mather's group of local diffeomorphisms on the target $(\mathbb{C}^p, 0)$, $\mathcal{A}$ denotes the group the product $\mathcal{L} \times \mathcal{R}$ and $\mathcal{K}$ denotes the group of contact.

A given map-germ $f \in \mathcal{O}(n, p)$ is said to be **k-$\mathcal{G}$-determined** ($\mathcal{G} = \mathcal{A}$ or $\mathcal{K}$) if whenever $g \in \mathcal{O}(n, p)$ and $j^k f(0) = j^k g(0)$, then g is $\mathcal{G}$-equivalent to f; f is **finitely $\mathcal{G}$-determined** if it is k-$\mathcal{G}$-determined for some $k < \infty$. In order to lighten notation, and in accordance with standard practice, whenever we say "finitely determined (without specifying whether $\mathcal{A}$ or $\mathcal{K}$)", we shall mean "finitely $\mathcal{A}$-determined".

Map germs which are finitely $\mathcal{K}$-determined are called map germs of **finite singularity type** [5, p. 201]. Using the symmetry between the theory of $\mathcal{A}$-equivalence and the theory of $\mathcal{K}$-equivalence, we know that, in particular if $f \in \mathcal{O}(n, p)$ with $n \geq p$ is finitely $\mathcal{A}$-determined, then f has finite singularity type. In this situation, the restriction $f|_{\Sigma(f)} : \Sigma(f) \to \Delta f$ is a **finite map** [any point in the target has at most a finite number of pre-images], and hence $\dim \Sigma(f) = \dim \Delta f$.

We say f is a **stable germ** if every "nearby germ is $\mathcal{A}$-equivalent to f". A **stable type** is an $\mathcal{A}$-equivalence class of stable germs. A stable type $\mathcal{Q}$ appears in a versal unfolding F of f, if for any representative $F = (id, f_u(x))$ of F, there exists a point $(u, y) \in \mathbb{C}^s \times \mathbb{C}^p$ such that the germ $f_u : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y)$ is a stable germ of type $\mathcal{Q}$, $S = f^{-1}(y) \cap \Sigma(f_u)$. The points (u, y) and (u, x) with $x \in S$ are called points of stable type $\mathcal{Q}$ in the target and in the source, respectively.

A finitely determined germ f has **discrete stable type** if there exists a versal unfolding of f in which only a finite number of stable types occur. In particular, if the numbers (n, p) are in Mather's "nice dimensions" [which is true for $(n, 3)$, $n \geq 3$, our focus here] or on the boundary thereof, then every finitely determined germ $f \in \mathcal{O}(n, p)$ has discrete stable type.

The Mather–Gaffney **geometric criterion** of finite determinacy states that: *A germ f is finitely determined if, and only if, there exist open neighborhoods U of 0 in the source and V of 0 in the target such that $f^{-1}(0) \cap U \cap \Sigma(f) = 0$ and for each $y \in V$, $y \neq 0$, the germ $f : (\mathbb{C}^n, \mathbf{S}) \to (\mathbb{C}^p, y)$ is a stable germ, where* $\mathbf{S} = f^{-1}(y) \cap U \cap \Sigma(f)$. See [1, Proposition 1.3], or [6].

2.2 Geometry of the critical set and the discriminant

The critical set $\Sigma(f)$ of f is described by its Thom–Boardman stratification, which we recover next.

For any Boardman symbol $i = (i_1, \ldots, i_r)$ with $1 \leq r \leq 3$, denote by $\Sigma^i(f)$ the set of points in $\Sigma(f)$ of type i. The sets $\Sigma^i(f)$ are obtained as the zero sets of the iterated Jacobian ideals, notion due to Morin in [7]. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map-germ and $I \subset \mathcal{O}_n$ an ideal generated by the system $g_1, \ldots, g_r$. For each $\ell \in \{1, \ldots, n\}$ we define the **jacobian extension of rank** ℓ for the pair (f, I), by $\Delta_\ell(f, I) := I + I_\ell(d(f_1, \ldots, f_p, g_1, \ldots, g_r))$, where $I_\ell(d(f_1, \ldots, f_p, g_1, \ldots, g_r))$ denotes the ideal generated by minors of size $\ell \times \ell$ from the jacobian matrix of the $f_1, \ldots, f_p, g_1, \ldots, g_r$ and $(d(f_1, \ldots, f_p, g_1, \ldots, g_r))$.

If $i = (i_1, \ldots, i_k)$ is a Boardman number, we inductively define the **Iterated Jacobian Ideal** for i, $J_i(f)$ in the following manner: $J_i(f) = \begin{cases} \Delta_{n-i_1+1}(f, \{0\}) & \text{if } k = 1 \\ \Delta_{n-i_k+1}(f, J_{i_1,\ldots,i_k}(f)) & \text{if } k > 1. \end{cases}$

Therefore for all $n \geq 3$ in any finitely determined map germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^3, 0)$, $\Sigma(f) = \Sigma^{n-2}(f)$ is the zero set of the Jacobian ideal $J_{n-2}(f)$, $\Sigma^{n-2,1}(f)$ is the zero set of the first iterated Jacobian ideal $J_{n-2,1}(f)$ and $\Sigma^{n-2,1,1}(f)$ is the zero set of the ideal $J_{n-2,1,1}(f)$.

Lemma 2.1 *For any finitely determined map germ from* $(\mathbb{C}^n, 0)$ *to* $(\mathbb{C}^3, 0)$ *with* ≥ 3,

(i) $\Sigma(f)$, *the critical set of* f, *is either empty or* 2*-dimensional*;
(ii) *The set* $\Sigma^{n-2,1}(f)$ *is either empty or* 1*-dimensional*;
(iii) *The set* $\Sigma^{n-2,1,1}(f)$ *is either empty or* 0*-dimensional.*

Proof These results follow by the Mather–Gaffney geometric criterion of finite determinacy of the germ f, this implies that f has isolated instability at the origin, therefore the sets above have the expected dimension. □

Lemma 2.2 *For any finitely determined map germ from* $(\mathbb{C}^n, 0)$ *to* $(\mathbb{C}^3, 0)$ *with* ≥ 3, *the sets* $f^{-1}(0)$, $\Sigma^{n-2}(f)$, $\Sigma^{n-2,1}(f)$ *and* $\Sigma^{n-2,1,1}(f)$ *have isolated singularity.*

Proof Since f is finitely determined, it follows from the Geometric criterion of Mather-Gaffney (see [6]), that there exist neighborhoods U and V of 0 in $\mathbb{C}^n$ such that $f^{-1}(0) \cap U \cap \Sigma(f) = 0$ and for each $y \in V$, $y \neq 0$, $x \neq 0$ with $x \in f^{-1}(\Delta)$ and $x \in \Sigma(f)$, the germ $f : (\mathbb{C}^n, x) \to (\mathbb{C}^3, y)$ is stable, hence it is written as $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, g(x_1, x_2, x_3) + \sum_{j=4}^n x_j^2)$ and the germ $(x_1, x_2, g(x_1, x_2, x_3))$ is a stable germ from $\mathbb{C}^3$ to $\mathbb{C}^3$, therefore the result follows from [8, Section 3]. We remember that such germ f is called a suspension of the germ $(x_1, x_2, g(x_1, x_2, x_3))$. □

To describe the discriminant $\Delta(f) = f(\Sigma^{n-2}(f))$, we remember that for all $n \geq 3$, the stable singularities which appear in the target are precisely the same type than map germs from $\mathbb{C}^3$ to $\mathbb{C}^3$, namely the ordinary double point curve, the cuspidal edge curve, the ordinary triple points, swallowtails, the transversal crossings of a cuspidal edge with a plane and the regular part, denoted $A_1(f)$. According to Arnold's notation we denote these sets by $A_{1,1}(f)$, $A_2(f)$, $A_{1,1,1}(f)$, $A_3(f)$ and $A_{2,1}(f)$ respectively, and call $A(f)$ the set $A(f) = A_{1,1,1}(f) \cup A_{2,1}(f) \cup A_3(f) \cup A_2(f) \cup A_{1,1}(f)$, hence $\Delta(f) = A_1(f) \cup A(f)$. The curves $A_{1,1}(f)$ and $A_2(f)$ are the 1-stable singularities of f and the sets $A_{1,1,1}(f)$, $A_3(f)$ and $A_{2,1}(f)$ are the 0-stable singularities of f.

Definition 2.3 We call *source double point curve*, denoted by $D(f)$ the curve given by the intersection $f^{-1}(A_{1,1}(f))$ with $\Sigma^{n-2}(f)$.

To obtain this curve we consider its associated ideal $I(D(f)$ as the reduction of the corresponding ideals, or $I(D(f)) := \sqrt{\langle I(f^{-1}(A_{1,1}(f))), J_{n-2}(f)\rangle}$.

We denote by $g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, the defining equation of the discriminant with reduced structure or $g^{-1}(0) = \Delta(f)$.

Example 2.4 Let $f(x, y, z, w) = (x, y, z^6 + xz + yz^2 + w^2)$. This germ from $\mathbb{C}^4$ to $\mathbb{C}^3$ is a suspension of the germ $f_0(x, y, z) = (x, y, z^6 + xz + yz^2)$ and in this case it is easier to compute the stables types which appear in $\Sigma(f)$ and in $\Delta(f)$ from the suspension.

First we compute the critical set $\Sigma(f_0) = \Sigma^1(f_0)$ and the curve $\Sigma^{1,1}(f_0)$ using the iterated Jacobian ideals $J_1(f_0) = I(\Sigma(f_0)) := \langle x + 2\,yz + 6\,z^5\rangle$ and $J_{1,1}(f_0) = I(\Sigma^{1,1}(f_0)) := \langle x - 24z^5, y + 15z^4\rangle$.

For the germ f we have the iterated Jacobian ideals $J_2(f) = I(\Sigma(f)) := \langle x + 2\,yz + 6\,z^5, w\rangle$ and $J_{2,1}(f) = I(\Sigma^{2,1}(f)) := \langle x - 24z^5, y + 15z^4, w\rangle$.

To obtain the source double point curve of the germ f_0 we apply the method described in [8] which uses the Vandermonde matrix associated to the set $D^2(f_0) \subset \mathbb{C}^4$ consisting of the points (x, y, z_1, z_2) with $z_1 \neq z_2$ in $\mathbb{C}^4$ such that $f_0(x, y, z_1) = f_0(x, y, z_2)$. The associated ideal $I(D^2(f_0))$ is:

$$I(D^2(f_0)) := \langle x + 12z_1^2z_2^3 + 12z_1^3z_2^2 + 6z_1z_2^4 + 6z_1^4z_2, y - 3z_1^4 - 3z_2^4 - 12z_1z_2^3 - 12z_2z_1^3 - 15z_1^2z_2^2, 4z_1^3 + 6z_1^2z_2 + 6z_2^2z_1 + 4z_2^3\rangle.$$

The projection of $D^2(f_0)$ to $\mathbb{C}^3$ gives the double point set $D(f_0)$ and

$$I(D(f_0)) := \langle x + 2yz + 6z^5, 16y^3 + 153y^2z^4 + 540yz^8 + 675z^{12}\rangle.$$

For the germ f we can not apply the results shown in [8] since the results there are for map germs from $\mathbb{C}^n$ to $\mathbb{C}^n$, then in general we compute $D(f)$ using the definition of its associated ideal $I(D(f))$, however, as this germ is a suspension of f_0 we know that $I(D(f)) := \langle I(D(f_0)), w\rangle$ (Fig. 1).

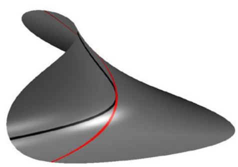

Fig. 1 The real part of the set $\Sigma(f_0)$ and the real parts of the curves $\Sigma^{1,1}(f_0)$ and $D(f_0)$

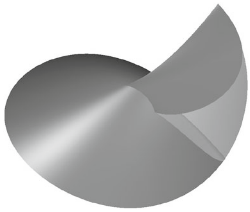

Fig. 2 The real part of the discriminant $\Delta(f) = \Delta(f_0)$ in the target

Fixing (X, Y, Z) as the target variables, we obtain the defining equation of the **discriminant** $\Delta(f)$:

$$g(X, Y, Z) = 46656Z^5 + 43200Z^2Y^2X^2 + 13824Y^3Z^3 + 1024ZY^6 + 22500ZYX^4 + 256Y^5X^2 + 3125X^6.$$

The cuspidal curve $A_2(f)$ satisfies the equations $25X^2 + 96YZ = 0$ and $135Z^2 + 4Y^3 = 0$ and the ideal of the double point curve in the target is

$$I(A_{1,1}(f)) := \langle 25X^3 + 36XYZ, 675X^2Z + 432YZ^2 + 64Y^4, 135XZ^2 - 16XY^3, 27Z^3 + 5X^2Y^2 + 4Y^3Z \rangle.$$

or

$$I(A_{1,1}(f)) := \langle 27Z^2 + 4Y^3, X \rangle \cap \langle -135Z^2 + 16Y^3, 25X^2 + 36YZ \rangle.$$

To obtain these equations in the target we use the Fitting ideals of the discriminant, concept presented by Mond and Pellikaan in [9, sections 1. and 2.], for the computation of the Fitting ideals we apply the results in [10,11] (Fig. 2).

The Milnor numbers $\mu(\Sigma(f))$, $\mu(\Sigma^{2,1}(f))$ and $\mu(D(f))$ are shown in the table below.

Germ f	$\mu(\Sigma(f)) = \mu(\Sigma(f_0))$	$\mu(\Sigma^{2,1}(f)) = \mu(\Sigma^{1,1}(f_0))$	$\mu(D(f)) = \mu(D(f_0))$
$(x, y, z^6 + xz + yz^2 + w^2)$	0	0	22

Next we show the numbers of 0-stable singularities and also the Milnor numbers $\mu(A_2(f))$ and $\mu(A_{1,1}(f))$.

Germ f	$\sharp A_3(f)$	$\sharp A_{1,1,1}(f)$	$\sharp A_{2,1}(f)$	$\mu(A_2(f))$	$\mu(A_{1,1}(f))$
$(x, y, z^6 + xz + yz^2 + w^2)$	3	1	6	8	17

We remember that the numbers $\sharp A_3(f)$, $\sharp A_{2,1}(f)$ and $\sharp A_{1,1,1}(f)$ mean the number of corresponding points in a stable perturbation of the germ f. In general, the space curves $A_{1,1}(f)$ and $A_2(f)$ are not a complete intersection and the Milnor number refers to the Buchweitz–Greuel definition of the Milnor number for such curves, see [12]. To compute these numbers we use the software Singular [13].

2.3 The geometry of $f^{-1}(\Delta)$

For map germs from $\mathbb{C}^n$ to $\mathbb{C}^p$ with $n \geq p$ there appear naturally in the inverse image of $\Delta(f)$ other points than the points in the critical set, moreover these points form a set in $\mathbb{C}^n$ with the same codimension than $\Delta(f)$ in $\mathbb{C}^p$.

When $p = 3$ in $f^{-1}(\Delta(f))$ there appear the inverse images by f of the sets $A_{1,1}(f)$, $A_2(f)$, $A_{1,1,1}(f)$, $A_3(f)$, $A_{2,1}(f)$ which are also codimension preserving, or in other words, $f^{-1}(A_{1,1}(f))$ and $f^{-1}(A_2(f))$ are $(n-2)$-dimensional, since $A_{1,1}(f)$ and $A_2(f)$ are curves in $\mathbb{C}^3$, $f^{-1}(A_{1,1,1}(f))$, $f^{-1}(A_3(f))$ and $f^{-1}(A_{2,1}(f))$ are $(n-3)$-dimensional, since $A_{1,1,1}(f)$, $A_3(f)$ and $A_{2,1}(f)$ are isolated points in $\mathbb{C}^3$. We remark also that $f^{-1}(0)$ is $(n-3)$-dimensional.

In general the critical sets of $f^{-1}(A(f))$, $f^{-1}(A_{1,1}(f))$, $f^{-1}(A_2(f))$, $f^{-1}(A_{1,1,1}(f))$, $f^{-1}(A_3(f))$ and $f^{-1}(A_{2,1}(f))$ possible have nonisolated singularity at the origin.

Remark 2.5 When $n = p = 3$ the hypersurface $f^{-1}(\Delta)$ in $\mathbb{C}^n$ has two components, the surfaces $X(f) := \overline{(f^{-1}(\Delta) - \Sigma(f))}$ and $\Sigma(f)$. The intersection of $X(f) \cap \Sigma(f)$ is formed by the curves $\Sigma_{1,1}(f)$ and $D(f)$ together with the origin. In $\Sigma_{1,1}(f)$ this intersection is tangential while in $D(f)$ this intersection is transversal.

On the other hand, for $n > 3$ the critical set $\Sigma(f)$ is two dimensional while $X(f)$ is $(n-1)$-dimensional, in this case $\Sigma(f) \subset X(f)$. In fact, in these dimensions one has $X(f) = f^{-1}(\Delta)$.

Now we study the geometry of f in the source from the point of view of the defining equation of the discriminant, denoted $g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$. We denote by $h : (\mathbb{C}^n, 0) \to \mathbb{C}$ the composition $h := g \circ f$.

Lemma 2.6 1. *If* $\Sigma(f) \neq \emptyset$, *then* $\Sigma(f) \cap f^{-1}(0) = \{0\}$.
2. *g is a submersion at each point of* $\Delta(f) \backslash A(f)$.
3. $\Sigma(f) \subseteq \Sigma(h) \subseteq \Sigma(f) \cup f^{-1}(A(f))$.
4. *When* $n > 3$, $X := X(f) = \mathcal{V}(h)$, *and* $\mathcal{I}(X) = (h)$.

Proof (1) If $\Sigma(f)$ is nonempty, it is 2-dimensional, and it is clear that both $\Sigma(f)$ and $f^{-1}(0)$ contain the point $\{0\}$ in the source. The desired result now holds from the finite determinacy of the germ f as this implies that $f|_{\Sigma(f)}$ is a finite map.

(2) For all points $\overline{z}$ in $\Delta(f)\backslash A(f)$ sufficiently close to 0, we know that $\Delta(f)$ is smooth at $\overline{z}$; since g defines $\Delta(f)$ with reduced structure, we conclude that $Dg(\overline{z}) \neq 0$.

(3) We first show that $\Sigma(f) \subseteq \Sigma(g \circ f)$. It clear that this inclusion holds if $\Sigma(f) = \emptyset$; consider the case of $\Sigma(f) \neq \emptyset$, then for any $z \in \Sigma(f)$, with $z \neq 0$ and z close to zero, we know that z is in the set $\Sigma^{n-2}(f)$ (fold) or in the set $\Sigma^{n-2,1}(f)$ (cusp), and in both cases we see that all minors 3×3 of the Jacobian matrix at z are equal to 0, this also holds for the point $z = 0$ since it is part of $\Sigma(f)$, then it follows that locally the inclusion $\Sigma(f) \subseteq \Sigma(g \circ f)$ holds since the gradient matrix of $g \circ f$ is also zero at these points.

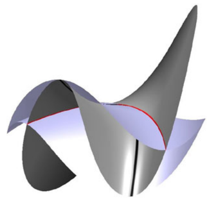

Fig. 3 The real part of the hypersurface $f_0^{-1}(\Delta(f_0))$ in $\mathbb{C}^3$

In regard the second inclusion $\Sigma(g \circ f) \subseteq \Sigma(f) \cup f^{-1}(A(f))$, consider a point $z \in \Sigma(g \circ f) \backslash \Sigma(f)$; it suffices to show that z belongs to $f^{-1}(A(f))$. At such point z, we see that f is a submersion, since $z \notin \Sigma(f)$ and as $D(g \circ f)(z) = 0$ it follows from the chain rule that we then must have $Dg(f(z)) = 0$. Since we also know that g is a submersion at each point of $\Delta f \backslash A(f)$, we conclude that $z \in f^{-1}(A(f))$.

(4) This proof is done by using Serre's criterion $R0$ and $S1$, see [14] or [15]. Since h defines a hypersurface in $\mathbb{C}^n$ it is Cohen–Macaulay and we have $S1$. Condition $R0$ means that the hypersurface defined by h is smooth in codimension ≥ 1, but this follows from that fact that $\Sigma(h) = \Sigma(g \circ f)$ has dimension $\leq n-2$ by item 2. Hence, h is reduced by Serre's criterion. □

Remark 2.7 For $n > 3$ we see from the item 4 of this lemma that one of the following inequalities $\Sigma(f) \subseteq \Sigma(g \circ f) \subseteq \Sigma(f) \cup f^{-1}(A)$ of the item 3 is always an equality. But as the discriminant set is given by the union $\Delta(f) = A_1(f) \cup A(f)$ the equality $\Sigma(f) = \Sigma(g \circ f)$ occurs if, and only if, $A(f)$ is empty, or in other words, f is a fold and this case is simple.

Therefore we consider from now on that f is not a fold and that the equality $\Sigma(g \circ f) = \Sigma(f) \cup f^{-1}(A(f))$ always holds.

We remember also that for all $n \geq 3$, $f^{-1}(A_2) \cap f^{-1}(A_{1,1}) = f^{-1}(0)$ as the germ f is finitely determined.

Example 2.8 Let $f(x, y, z, w) = (x, y, z^6 + xz + yz^2 + w^2)$. First we obtain the defining equation of the hypersurfaces $f_0^{-1}(\Delta(f_0))$ in $\mathbb{C}^3$ and $f^{-1}(\Delta(f))$ in $\mathbb{C}^4$ for both germs, f_0 and its suspension f. Then we have $f_0^{-1}(\Delta(f_0)) := (7424\, y^3xz^3 + 13200\, y^2x^2z^2 + 10000\, x^3yz + 5616\, z^{16}y + 9072\, z^{12}y^2 + 6672\, z^8y^3 + 9156\, x^3z^5 + 10908\, x^2z^{10} + 6048\, xz^{15} + 24156\, x^2z^6y + 21312\, xz^7y^2 + 19872\, xz^{11}y + 1296\, z^{20} + 2176\, y^4z^4 + 3125\, x^4 + 256\, y^5)(x + 2\, yz + 6\, z^5)^2$.

Here $f_0^{-1}(\Delta(f_0))$ splits in two components $\Sigma(f_0)$ and $X(f_0)$, as $\Sigma(f_0) := x + 2\, yz + 6\, z^5$, we obtain $X(f_0) := 7424\, y^3xz^3 + 13200\, y^2x^2z^2 + 10000\, x^3yz + 5616\, z^{16}y + 9072\, z^{12}y^2 + 6672\, z^8y^3 + 9156\, x^3z^5 + 10908\, x^2z^{10} + 6048\, xz^{15} + 24156\, x^2z^6y + 21312\, xz^7y^2 + 19872\, xz^{11}y + 1296\, z^{20} + 2176\, y^4z^4 + 3125\, x^4 + 256\, y^5$.

We remark that since $n = 3$ one has $\Sigma(f_0) \cap X(f_0) = D(f_0) \cup \Sigma^{1,1}(f_0)$ (Fig. 3).

On the other hand, $X(f) := f^{-1}(\Delta(f)) = 1024\, y^7z^2 + 1024\, y^6w^2 + 2799360\, z^{15}xyw^2 + 2799360\, z^9xyw^4 + 2799360\, z^{11}xy^2w^2 + 2799360\, z^{10}x^2yw^2 + 1399680\, x^2z^4yw^4 + 1486080\, x^2z^6y^2w^2 + 933120\, x^3z^5yw^2 + 1399680\, xz^5y^2w^4 + 1016064\, xz^7y^3w^2 + 933120\, xz^3yw^6 + 86400\, y^2x^3zw^2 + 127872\, y^3x^2z^2w^2 + 41472\, y^3xzw^4 + 82944\, y^4xz^3w^2 + 933120\, z^{19}xw^2 + 1399680\, z^{14}yw^4 + 933120\, z^{20}yw^2 + 1399680\, z^{17}xy^2 + 1399680\, z^{16}y^2w^2 + 1399680\, z^{14}x^2w^2 + 933120\, z^{11}x^3y + 1399680\, z^{13}xw^4 + 1399680\, z^{16}x^2y + 933120\, z^{21}xy + 1399680\, z^8x^2w^4 + 933120$

$z^9x^3w^2 + 1442880\,z^{12}x^2y^2 + 974592\,z^{13}xy^3 + 933120\,z^7xw^6 + 1399680\,z^{10}y^2w^4 + 974592\,z^{12}y^3w^2 + 933120\,z^8yw^6 + 255780\,x^4z^6y + 466560\,x^3z^3w^4 + 233280\,x^4z^4w^2 + 552960\,x^3z^7y^2 + 594432\,x^2z^8y^3 + 466560\,x^2z^2w^6 + 316224\,xz^9y^4 + 233280\,xzw^8 + 508032\,y^3z^6w^4 + 316224\,y^4z^8w^2 + 466560\,y^2z^4w^6 + 233280\,yz^2w^8 + 65700\,y^2x^4z^2 + 100224\,y^3x^3z^3 + 84672\,y^4x^2z^4 + 43200\,y^2x^2w^4 + 41472\,y^4z^2w^4 + 41472\,y^5z^4w^2 + 41472\,y^5xz^5 + 1024\,y^6xz + 22500\,x^5yz + 22500\,x^4yw^2 + 256\,x^2y^5 + 233280\,z^{25}x + 233280\,z^{26}y + 233280\,z^{24}w^2 + 466560\,z^{20}x^2 + 466560\,z^{22}y^2 + 466560\,z^{18}w^4 + 466560\,z^{15}x^3 + 480384\,z^{18}y^3 + 466560\,z^{12}w^6 + 233280\,z^{10}x^4 + 274752\,z^{14}y^4 + 233280\,z^6w^8 + 46656\,x^5z^5 + 88128\,y^5z^{10} + 14848\,y^6z^6 + 13824\,y^3w^6 + 46656\,w^{10} + 46656\,z^{30} + 3125\,x^6.$

In $X(f)$ there exists the singular surface obtained as the inverse image of the double point curve with associated ideal

$$\begin{aligned} &I(f^{-1}(A_{1,1}(f)))\\ &\quad := \langle x, 4y^3 + 27w^4 + 54yz^2w^2 + 27y^2z^4 + 54z^6w^2 + 54yz^8 + 27z^{12}\rangle\\ &\quad \cap \langle -3125x^4 + 768y^5, 25x^2 + 36xyz + 36yw^2 + 36y^2z^2 + 36yz^6, 64y^4\\ &\qquad + 375x^3z + 375x^2w^2 + 375x^2yz^2 + 375x^2z^6, -32y^3 - 105x^2z^2 + 540xzw^2\\ &\qquad + 270w^4 - 270y^2z^4 + 540xz^7 + 540z^6w^2 + 270z^{12}\rangle. \end{aligned}$$

There exists also the singular surface obtained as the inverse image of the cuspidal curve and its associated ideal is

$$\begin{aligned} I(f^{-1}(A_2(f))) := \langle &25x^2 + 96xyz + 96yw^2 + 96y^2z^2 + 96yz^6, 4y^3 + 135x^2z^2\\ &+ 270xzw^2 + 135w^4 + 270xyz^3 + 270yz^2w^2 + 135y^2z^4\\ &+ 270xz^7 + 270z^6w^2 + 270yz^8 + 135z^{12}\rangle. \end{aligned}$$

If we consider $w = 0$ in the ideals above one has the corresponding ideals of these curves for the germ $f_0 : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$:

$I(f_0^{-1}(A_{1,1}(f_0))) := \langle 25x^3 + 36x^2yz + 36xy^2z^2 + 36xyz^6, 200x^2y^2 + 416xy^3z - 160y^4z^2 - 2955x^3z^3 - 4455x^2yz^4 - 1080xy^2z^5 - 920y^3z^6 - 4455x^2z^8 - 1080y^2z^{10} + 1080xz^{13} + 1080yz^{14} + 1080z^{18}, 64xy^3 - 165x^3z^2 - 540x^2yz^3 - 1080x^2z^7 - 540xyz^8 - 540xz^{12}, 64y^4 + 375x^3z + 675x^2yz^2 + 432xy^2z^3 + 432y^3z^4 + 675x^2z^6 + 432xyz^7 + 864y^2z^8 + 432yz^{12}\rangle$.

We remark that this ideal is not radical and $\sqrt{I(f_0^{-1}(A_{1,1}(f_0)))} := I(D(f_0)) \cap H^D(f_0)$, with

$$\begin{aligned} H^D(f_0) := \langle &75625x^2 + 93800xyz + 69696y^2z^2 + 75696xz^5 + 97200yz^6\\ &+ 33696z^10, 18875xy + 7360y^2z + 14487xz^4 + 16656yz^5 + 8352z^9,\\ &\times 20y^2 - 7xz^3 + 19yz^4 + 3z^8\rangle. \end{aligned}$$

or $H^D(f_0) := \langle 125y^2 + 138yz^4 + 45z^8, 25x + 11yz + 15z^5\rangle \cap \langle 4y + 3z^4, x\rangle$.

The branch corresponding to the ideal $H^D(f_0)$ belongs to $X(f_0)$.

The inverse image of the curve $A_2(f_0)$ by f_0 has corresponding ideal:

$$I\left(f_0^{-1}(A_2(f_0))\right) := \langle 25x^2 + 96xyz + 96y^2z^2 + 96yz^6, 128y^3 + 3195x^2z^2 + 4320xyz^3 + 8640xz^7 + 4320yz^8 + 4320z^{12}\rangle.$$

With radical ideal: $\sqrt{I(f_0^{-1}(A_2(f_0)))} := I(\Sigma^{1,1}(f_0)) \cap H^{1,1}(f_0)$, where $H^{1,1}(f_0) := \langle 2000y^3 + 3528y^2z^4 + 2565yz^8 + 675z^{12}, -20y^2 + 42xz^3 + 9yz^4 + 27z^8, 300xy + 352y^2z - 75xz^4 + 360yz^5, 375x^2 + 240xyz + 32y^2z^2 + 300xz^5\rangle$. The branch corresponding to the ideal $H^{1,1}(f_0)$ belongs to $X(f_0)$.

The Buchweitz–Greuel Milnor numbers of the curves $\sqrt{(f_0^{-1}(A_2(f_0)))}$, $\sqrt{(g^{-1}(A_{1,1}(f_0)))}$, $H^{1,1}(f_0)$, $H^D(f_0)$ and $H^D(f_0)$ in the set $X(f_0)$ of the germ f_0 are shown below.

Germ f_0	$\mu\sqrt{(f_0^{-1}(A_2(f_0)))}$	$\mu\sqrt{(g^{-1}(A_{1,1}(f_0)))}$	$\mu(H^{1,1}(f_0))$	$\mu(H^D(f_0))$
$\left(x, y, z^6 + xz + yz^2\right)$	31	67	16	16

Example 2.9 Let $f : \mathbb{C}^4 \to \mathbb{C}^3 : f(x, y, u, v) = (x, y, yu + xv + u^3 + v^3)$. This germ from $\mathbb{C}^4$ to $\mathbb{C}^3$ is not a suspension, then we need to compute all its stable types directly.

The critical set in $\mathbb{C}^4$ is two dimensional with Jacobian ideal $J_2(f) = I(\Sigma^2(f)) := \langle y + 3u^2, x + 3v^2\rangle$.

The ideal for the source critical curve is: $J_{2,1}(f) = I(\Sigma^{2,1}(f)) := \langle x + 3v^2, y + 3u^2, uv\rangle$ and $\mu(J_{2,1}(f)) = 1$.

Now compute the defining equation of the **discriminant** $\Delta(f)$:

$$g(X, Y, Z) := \langle 729Z^4 + 216X^3Z^2 + 216Y^3Z^2 + 16X^6 - 32X^3Y^3 + 16Y^6\rangle.$$

The ideals of the singular curves of the discriminant are: $I(A_2(f)) := \langle XY, 27Z^2 + 4X^3 + 4Y^3\rangle$ with $\mu(A_2(f)) = 7$ and $I(A_{1,1}(f))) := \langle Z, X^3 - Y^3\rangle$ with $\mu(A_{1,1}(f)) = 4$.

In the source $\mathbb{C}^4$ we obtain the ideal of the hypersurface $X(f) = f^{-1}(\Delta(f))$:

$$\begin{aligned} I(X(f)) := \langle & 16x^6 - 32x^3y^3 + 16y^6 + 216x^3y^2u^2 + 216y^5u^2 + 432x^4yuv \\ & + 432xy^4uv + 216x^5v^2 + 216x^2y^3v^2 + 432x^3yu^4 + 1161y^4u^4 \\ & + 432x^4u^3v + 3348xy^3u^3v + 4374x^2y^2u^2v^2 + 3348x^3yuv^3 \\ & + 432y^4uv^3 + 1161x^4v^4 + 432xy^3v^4 + 216x^3u^6 + 3132y^3u^6 \\ & + 8748xy^2u^5v + 8748x^2yu^4v^2 + 3348x^3u^3v^3 + 3348y^3u^3v^3 \\ & + 8748xy^2u^2v^4 + 8748x^2yuv^5 + 3132x^3v^6 + 216y^3v^6 \\ & + 4374y^2u^8 + 8748xyu^7v + 4374x^2u^6v^2 + 8748y^2u^5v^3 \\ & + 17496yu^4v^4 + 8748x^2u^3v^5 + 4374y^2u^2v^6 + 8748xyuv^7 \end{aligned}$$

$$+4374x^2v^8 + 2916yu^{10} + 2916xu^9v + 8748yu^7v^3 + 8748xu^6v^4$$
$$+8748yu^4v^6 + 8748xu^3v^7 + 2916yuv^9 + 2916xv^{10} + 729u^{12}$$
$$+2916u^9v^3 + 4374u^6v^6 + 2916u^3v^9 + 729v^{12}\rangle.$$

We also have the corresponding ideals to the $(n-2)$-dimensional surfaces $f^{-1}(A_2(f))$ and $f^{-1}(A_{1,1}(f))$.

$I(f^{-1}(A_2(f))) := \langle y, 4x^3 + 27x^2v^2 + 54xu^3v + 54xv^4 + 27u^6 + 54u^3v^3 + 27v^2\rangle \cap \langle x, 4y^3 + 27y^2u^2 + 54yv^3u + 54yu^4 + 27u^6 + 54u^3v^3 + 27v^2\rangle$ and $I(f^{-1}(A_{1,1}(f))) := \langle x^3 - y^3, yu + xv + u^3 + v^3\rangle$.

We remark that these two ideals are radical, complete intersections and the origin is not isolated singularity.

The source double point curve $D(f)$ is obtained from the definition of its associated ideal $I(D(f)$.

$$I(D(f)) = \langle f^{-1}(A_{1,1}(f)), J_2(f)\rangle = \langle x + 3v^2, y + 3u^2, u^3 + v^3\rangle.$$

This ideal is radical and the curve is an isolated complete intersection singularity, or ICIS for short, and $\mu(D(f)) = 4$.

We can check that this curve is already the source double point curve as follows.

From the generators of $I(D(f))$ write $x = -3v^2$, $y = -3u^2$, $u^3 = -v^3$, then as $u = -v$, one has

$$f_{|D(f)}(x, y, u, v) = (-3v^2, -3u^2, -2u^3 - 2v^3) = (-3v^2, -3u^2, 0).$$

Now, calling X, Y and Z for the variables in the target one has : $X = -3u^2$, $Y = -3v^3$ and $Z = 0$.

Therefore since $u^3 = -v^3$, $u^6 = v^6$ and $X^3 = Y^3$, or $X^3 - Y^3 = 0$, therefore as $Z = 0$ one has $I(f(D(f))) = I(A_{1,1}(f)) = \langle X^3 - Y^3, Z\rangle$.

3 Whitney equisingularity of map germs

Let $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^p, (0,0))$, $F = (t, \overline{f}(t,x))$, be a 1-parameter unfolding of a finitely determined germ f, such that $\overline{f}(t, -)$ preserves the origin for all t. Let $T := \mathbb{C} \times \{0\}$. F is a *good unfolding* of f if there exist neighbourhoods U, W of the origin in $\mathbb{C} \times \mathbb{C}^n$ and $\mathbb{C} \times \mathbb{C}^p$ respectively such that $F^{-1}(W) = U$, F maps $U \cap \Sigma(F) - T$ to $W - T$ and if $(t_0, y_0) \in W - T$, then the germ $f_{t_0} : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y_0)$ is stable, where $S = F^{-1}(t_0, y_0) \cap \Sigma(F)$.

A good unfolding is *excellent* if all the 0-stable invariants are constant in the unfolding and f is of discrete type.

An unfolding F of f is *Whitney equisingular* along the parameter space T if there exists a regular stratification of the source and the target, with T a stratum of the source and the target and these stratifications are Whitney equisingular along T, i.e. satisfy the Whitney conditions **a** and **b** and Thom's A_F condition.

One of the questions of main interest is to show when an excellent unfolding $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^p, (0,0))$ of a finitely determined germ $f \in \mathcal{O}(n, p)$ is Whitney equisingular.

Using the polar invariants, i.e, the polar multiplicities of the polar varieties of the stable types Gaffney showed in the proof of the Theorem 7.1 of [1] the following.

Theorem 3.1 [1, Theorem 7.1] *Suppose that* $F : (\mathbb{C}\times\mathbb{C}^n, (0,0)) \to (\mathbb{C}\times\mathbb{C}^p, (0,0))$ *is an excellent unfolding of a finitely determined germ* $f \in \mathcal{O}(n, p)$. *Also suppose that the polar invariants of all stable types defined in*:

1. *the discriminant* $\Delta(f_t) = f_t(\Sigma(f_t))$,
2. *the singular set* $\Sigma(f_t)$,
3. $X(f_t) := \overline{(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t))}$,

are constant at the origin for all t. *Then the unfolding is Whitney equisingular.*

As a consequence of a convenient version of Thom's isotopy lemma [1, Theorem 6.1], the Theorem 7.1 of Gaffney shows the topological triviality of the family.

The theorem above remains valid if we replace the term "an excellent unfolding" in the hypothesis by "a 1-parameter unfolding" which, when stratified by stable types and by the parameter axis T, has only the parameter axis T as 1-dimensional stratum at the origin [2].

Here we use the geometry of the germ to control the polar invariants needed to ensure the Whitney equisingularity of a family of map germs in $\mathcal{O}(n, 3)$ when $n > 3$.

Now we fix $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^3, (0,0))$ with $n > 3$ be a *good* 1-parameter unfolding of a finitely determined map germ $f \in \mathcal{O}(n, 3)$. For each fixed t we call $f_t(x) = F(t, x)$, then according to the Theorem 3.1 of Gaffney we need to control all polar invariants in the target and also in the source of the map germs f_t. Call $X_t := X(f_t) = \overline{(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t))}$, $X_1(f_t) := \overline{(f_t^{-1}(A_{1,1}(f_t)) - D(f_t))}$ and $X_2(f_t) := \overline{(f_t^{-1}(A_2(f_t)) - \Sigma^{n-2,1}(f_t))}$. We remember that if g_t is the defining equation of the discriminant $\Delta(f_t)$ and $h_t := (g_t \circ f_t) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, then $X_t = X(f_t) = \mathcal{V}(h_t)$.

Consequently, if we suppose that $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^3, (0,0))$, $n > 3$ is an excellent unfolding of a finitely determined germ $f \in \mathcal{O}(n, 3)$, $n > 3$, we need of the constancy of $4n + 10$ polar invariants (at the origin for all t) to guarantee the unfolding is Whitney equisingular.

To control all polar invariants needed to apply the Theorem 3.1 of Gaffney we show some equations involving all these polar invariants and invariants associated to the set X_t.

First we recover basic definitions about the Lê numbers and relative polar multiplicities necessary to better understand the results shown here.

To study the set $X(f)$, as it has possibly non isolated singularity at zero, we need to understand the Lê cycles and relative polar multiplicities associated to it. For this we recover these concepts, given by Massey in [16], for any analytic function $h : (U, 0) \to (\mathbb{C}, 0)$ with U an open subset of $\mathbb{C}^{N+1}$ containing the origin.

We assume that the reader is familiar with the notion of coherent sheaves, gap sheaves, schemes and cycles. In order to fix the notation, for a sheaf α and an analytic subset W in some affine space, we denote by α/W the corresponding gap sheaf and by $V(\alpha)/W$ the scheme associated with the sheaf α/W, where $V(\alpha)$ denotes the analytic space defined by the vanishing of α. We shall at times enclose cycles in square brackets, $[\,\cdot\,]$.

For the definitions in this section let $h : (U, 0) \to (\mathbb{C}, 0)$ be an analytic function, U an open subset of $\mathbb{C}^{N+1}$ containing the origin and $z = (z_0, z_1, \dots, z_N)$ a linear choice of coordinates in $\mathbb{C}^{N+1}$. We fix the dimension of the singular set $\Sigma(h)$ of h is s with $0 \leq s \leq N - 1$.

Definition 3.2 For $0 \leq k \leq N$, the kth (relative) *polar variety*, $\Gamma^k_{h,z}$, of h with respect to z is the scheme $V\left(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_N}\right) / \Sigma(h)$.

The kth (relative) *polar multiplicity* , denoted $m_k(h, z)$, is the multiplicity of $\Gamma^k_{h,z}$

On the level of ideals, $\Gamma^k_{h,z}$ consists of the components of $V\left(\frac{\partial h}{\partial z_k}, \dots, \frac{\partial h}{\partial z_N}\right)$ which are not contained in $\Sigma(h)$. We denote by $[\Gamma^k_{h,z}]$ the cycle associated with this scheme. In particular we note that $\Gamma^0_{h,z}$ is empty and we call $\Gamma^{N+1}_{h,z} = U$.

Definition 3.3 For $0 \leq k \leq N$, the kth *Lê cycle*, $\Lambda^k_{h,z}$, of h with respect to z is the cycle

$$\left[\Gamma^{k+1}_{h,z} \cap V\left(\frac{\partial h}{\partial z_k}\right) \right] - [\Gamma^k_{h,z}].$$

In general we shall denote this cycle simply $\Lambda^k_{h,z}$, and not $[\Lambda^k_{h,z}]$, because unlike the polar varieties which are defined as schemes and we have to consider the associated cycle, this definition is given in terms of cycles.

If the intersection of $\Lambda^k_{h,z}$ with the cycle of $V(z_0 - p_0, \dots, z_{k-1} - p_{k-1})$ is purely 0-dimensional at a point $p = (p_0, p_1, \dots, p_N)$, i.e., either p is an isolated point of the intersection or p is not in the intersection, it is possible to define the Lê numbers as follows:

Definition 3.4 For $0 \leq k \leq N$, the kth *Lê number*, $\lambda^k_{h,z}(p)$, of h with respect to z at p, is defined as the intersection number

$$(\Lambda^k_{h,z} \cdot V(z_0 - p_0, \dots, z_{k-1} - p_{k-1}))_p.$$

Definition 3.5 For any linear generic subspace $L^k \subseteq (\mathbb{C}^{N+1}, 0)$, with $k = 0, \dots, N+1$ the **reduced Euler characteristic of the Milnor fiber** of the $h \mid L^k$ at 0, denoted $\chi^{(k)}$, is defined by

$$\chi^{(k)} := m_k(h) + \lambda^{N+1-k}(h) - \lambda^{N+1-(k-1)}(h) + \lambda^{N+1-(k-2)}(h) + \cdots + \lambda^{N+1-s}(h)$$

with $s = \dim V(J(h))$ and $m_k(h)$ denotes the kth (relative) *polar multiplicity* at 0.

The **Euler characteristic**, denoted just $\chi^{(*)}$, is the following sequence:

$$\chi^{(*)} := (\chi^{(N+1)}, \dots, \chi^{(2)}).$$

The number $\chi^{(k)}$ does not depend on the choice of the L^k.

Next we show a condition for the parameter axis $T = \mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}^3$ to be stratum of a Whitney stratification of the discriminant $\Delta(F)$.

Theorem 3.6 [17, p.32] *The pair $(\Delta(F), T)$ is Whitney equisingular if, and only if, the sequence $(m_1(g_t), m_2(g_t), \chi^*_{g_t})$ is independent of t.*

This theorem ensures the Whitney equisingularity of the pair $(\Delta(F), T)$ using invariants defined in the **target**, since each germ $g_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ gives the defining equation of the discriminant $\Delta(f_t)$ in $\mathbb{C}^3$.

Now we denote $X(F) := \overline{(F^{-1}(\Delta(F)) - \Sigma(F))}$ and show how to control the Whitney equisingularity of the pair $(X(F), T)$ in the source.

Theorem 3.7 [3, Theorem 6.3] *Suppose that the stratification by the stable types of F has only the parameter space $T = \mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}^n$ as a locus of instability and the singular set of $X(F)$ is Cohen Macaulay. Then the pair $(X(F), T)$ is Whitney equisingular if, and only if, the sequence $(m_1(h_t), \ldots, m_{n-1}(h_t), \chi^*_{h_t})$ is independent of t.*

From now on we consider that $F : (\mathbb{C} \times \mathbb{C}^n, (0, 0)) \to (\mathbb{C} \times \mathbb{C}^3, (0, 0))$ with $n > 3$ is a *good* 1-parameter unfolding of a finitely determined map germ $f \in \mathcal{O}(n, 3)$. We also suppose that the stratification by the stable types of F has only the parameter space $T = \mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}^n$ as a locus of instability and the singular set of $X(F)$ is Cohen Macaulay. Then we have the following

Proposition 3.8 *The unfolding F is Whitney equisingular if, and only if, the sequences*:

$$(m_1(g_t), m_2(g_t), \chi^*_{g_t}) \quad \textit{and} \quad (m_1(h_t), \ldots, m_{n-1}(h_t), \chi^*_{h_t})$$

are independent of t.

Proof In this setup we obtain from the Remark 2.7 that the singular set of $X(F)$ is equal to $\Sigma(F) \cup F^{-1}(A)$ and from the Theorem 3.7 we obtain that the independence of the sequence

$$(m_1(h_t), \ldots, m_{n-1}(h_t), \chi^*_{h_t})$$

also guarantees the Whitney equisingularity along the parameter space T of all other strata of $\Sigma(F)$ and $F^{-1}(A)$.

Therefore the result appears as a consequence of the Theorems 3.6 and 3.7. □

We also can show the following:

Proposition 3.9 *Let $F : (\mathbb{C} \times \mathbb{C}^n, (0, 0)) \to (\mathbb{C} \times \mathbb{C}^3, (0, 0))$ with $n > 3$ be a good 1-parameter unfolding of a finitely determined map germ $f \in \mathcal{O}(n, 3)$. Suppose that the stratification by the stable types of F has only the parameter space $T = \mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}^n$ as a locus of instability and the singular set of $X(F)$ is Cohen Macaulay. Then the unfolding F is Whitney equisingular if, and only if, the sequences*:

$$(m_1(g_t), m_2(g_t), \lambda^0(g_t), \lambda^1(g_t)) \quad \textit{and}$$
$$(m_1(h_t), \ldots, m_{n-1}(h_t), \lambda^0(h_t), \ldots, \lambda^{n-2}(h_t))$$

are independent of t.

Proof From [18, p.726] we can get the following equivalences:

(i) $(m_1(g_t), m_2(g_t), \chi^*_{g_t})$ is independent of t if, and only if,

$$(m_1(g_t), m_2(g_t), \lambda^0(g_t), \lambda^1(g_t))$$

is independent of t.

(ii) $(m_1(h_t), \ldots, m_{n-1}(h_t), \chi^*_{h_t})$ is independent of t if, and only if,

$$(m_1(h_t), \ldots, m_{n-1}(h_t), \lambda^0(h_t), \ldots, \lambda^{n-2}(h_t))$$

is independent of t and the result follows.

□

Remark 3.10 From the Proposition 3.9 we obtain that there are needed $2n + 2$ invariants, for instance when $n = 4$, we only need 10 invariants, while 26 were required from the Theorem 3.1.

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