



Integration over families of Lagrangian submanifolds in BV formalism

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Abstract

Gauge fixing is interpreted in BV formalism as a choice of Lagrangian submanifold in an odd symplectic manifold (the BV phase space). A natural construction defines an integration procedure on families of Lagrangian submanifolds. In string perturbation theory, the moduli space integrals of higher genus amplitudes can be interpreted in this way. We discuss the role of gauge symmetries in this construction. We derive the conditions which should be imposed on gauge symmetries for the consistency of our integration procedure. We explain how these conditions behave under the deformations of the worldsheet theory. In particular, we show that integrated vertex operator is actually an inhomogeneous differential form on the space of Lagrangian submanifolds.

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1. Introduction

Many physical theories have BRST-like structure. This means that there is a nilpotent fermionic symmetry Q , and the space of physical states is the cohomology of Q . Such theories come with an equivalence relation. The theory with the action S is equivalent to the theory with the action $S + Q\Psi$ for any operator Ψ . Physically meaningful quantities should be invariant,

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i.e. should be the same for any two equivalent theories. An important question is, how can we obtain such invariants? One way to obtain an invariant is to take the path integral:

$$\int_{\text{path integral}} [d\phi] \mathcal{O}_1 \dots \mathcal{O}_n e^{\frac{1}{\hbar} S[\phi]} \tag{1}$$

where \mathcal{O}_j are some BRST-closed operators. But actually there are other possibilities [1,2], which we will now describe.

Consider the whole equivalence class of theories. It is infinite-dimensional, because $S \simeq S + Q\Psi$ where Ψ could be more or less arbitrary functional.

Let us choose a basis $\{\Psi_a\}$; the space of BRST trivial deformations is parametrized by the coordinates x^a :

$$S(x) = S + \sum_a x^a Q\Psi_a \tag{2}$$

Let us call this equivalence class \mathcal{M} . It turns out that the following pseudo-differential form on \mathcal{M} is closed:

$$\Omega = \int_{\text{path integral}} [d\phi] \exp\left(S[\phi] + \sum_a \Psi_a[\phi] dx^a\right) \tag{3}$$

We want to obtain BRST invariants by integrating Ω over some closed cycles [1].

| This construction (and related) was also used in [3,4], and on manifolds with a boundary in [4,5].

This procedure was used in [1] to define string amplitudes.² We have to integrate over the moduli space of metrics on a genus g Riemann surface. When we vary the metric, the variation of the worldsheet action is Q -exact:

$$T_{\alpha\beta} = Qb_{\alpha\beta} \tag{4}$$

In this case $\Psi_a dx^a = b_{\alpha\beta} dg^{\alpha\beta}$ and Eq. (3) gives the standard string measure:

$$\Omega = \int [d\phi] \exp\left(S[\phi] + \int_{\text{worldsheet}} b_{\alpha\beta} dg^{\alpha\beta}\right) \tag{5}$$

where ϕ are the fields of the worldsheet sigma-model. In this approach, the choice of a metric $g^{\alpha\beta}$ should be understood as a choice of a representative in the class of physically equivalent theories (a choice of gauge fixing).

We would like to integrate over the *moduli space* of metrics modulo diffeomorphisms. However, the form Ω given by Eq. (5) is *not automatically* base. Although it is true that it is diffeomorphism-invariant, it is generally speaking not horizontal and therefore not a base form. Indeed, changing $dg^{\alpha\beta}$ to $dg^{\alpha\beta} + 2\nabla^{(\alpha} \xi^{\beta)}$ results in the same expression for Ω only when:

$$\nabla^\alpha b_{\alpha\beta} = 0 \tag{6}$$

² BV formalism was applied to string worldsheet theory in [6].

Even in the case of standard bosonic string, this is only true on-shell, leading to restrictions on allowed operator insertions. For a general string theory sigma-model (such as pure spinor formalism in a curved background [7]) we would like to relax the condition (6) by allowing $\nabla^\alpha b_{\alpha\beta}$ to be Q of something. Moreover, for the worldsheet theories of the topological type, such as the pure spinor formalism [7], a deformation of the metric is not necessarily better than any other Q -exact deformation of the action. We want to extend the definition of Ω to the space of *all* BRST-trivial deformations. But with all these generalizations, we still want Ω to be a *base form* with respect to the worldsheet diffeomorphisms.

A construction of base Ω was suggested in [2]. The purpose of this paper is to further develop this construction, and to fill in some technical details.

It appears that the right interpretation of Ω is a pseudo-differential form on the space of Lagrangian submanifolds in BV phase space, which we denote LAG. We will give the definition in Section 3. The space LAG does not have topologically nontrivial closed integration cycles. One can integrate over a non-compact cycle, extending to infinity, if Ω is rapidly decreasing. But this is not the case in string theory. Therefore we have to consider the factorspace of LAG over the action of the group of worldsheet diffeomorphisms. In our approach, the action of the group of diffeomorphisms on the BV phase space should come as part of the definition of the string worldsheet theory. In more general applications (beyond string theory) we may use, in place of diffeomorphisms, some other symmetry group. We develop the formalism for general symmetry group. For our construction of base Ω to work, the action of the symmetry group on the BV phase space should satisfy certain properties, which we describe in Section 5. One special case is when BV formalism comes from BRST formalism; we discuss this in Section 6. Another special case is topologically twisted $N = 2$ superconformal theory (Section 9).

In string theory, one considers the worldsheet theory along with its nontrivial deformations. These deformations are called “integrated vertex operators”.³ In Section 7 we study the corresponding deformation of the string measure. We show that this deformation will generally speaking involve mixing between the components of Ω of different degrees. A closely related concept is unintegrated vertex operators; we will consider them in Section 11. In Section 10 we discuss the case of worldsheet with boundary. This could be useful for the off-shell formulation of string theory [8].

2. Brief review of the BV formalism

The “BV phase space” is a supermanifold M with a non-degenerate odd 2-form ω satisfying $d\omega = 0$. It is **not true** that ω^{-1} defines an antisymmetric bivector. (The symmetry properties of ω^{-1} are not of a bivector, see Appendix A.2.) Instead, it defines a second order odd differential operator Δ on the so-called “half-densities” [9]. A half-density is a geometrical object which transforms under the change of variables as a square root of the volume element. A half-density defines a measure on every Lagrangian submanifold $L \subset M$.

2.1. Canonical transformations

Let M denote the BV phase space. We say that a vector field is Hamiltonian if it preserves the odd symplectic form. Let us assume that M is simply-connected. Then Hamiltonian vector

³ BRST-trivial deformations do not change the physical worldsheet theory. BRST-nontrivial deformations, on the other hand, do change the worldsheet theory, but do not change “the string theory”. They just change the background.

fields are of the form $\{H, _ \}$. Let \mathfrak{g} denote the Lie algebra of Hamiltonian vector fields. The commutator is defined as the standard commutator of vector fields:

$$[\{H_1, _ \}, \{H_2, _ \}] = \{\{H_1, H_2\}, _ \} \tag{7}$$

Let G denote the Lie supergroup of canonical transformations of M . Its Lie superalgebra is the Lie superalgebra of Hamiltonian vector fields:

$$\mathfrak{g} = \text{Lie}(G) \tag{8}$$

Let $C^\infty M$ denote the space of smooth functions on M . Notice that $\Pi C^\infty M$ is a Lie superalgebra under the Poisson bracket. It is a central extension of \mathfrak{g} :

$$\widehat{\mathfrak{g}} = \Pi C^\infty M \tag{9}$$

Let \widehat{G} denote the corresponding central extension of G . It can be realized as the group of automorphisms of some contact manifold — see Section 3.6.1.

2.2. Canonical operator

2.2.1. Definition of Δ_{can}

Any half-density $\rho_{\frac{1}{2}}$ defines a measure on a Lagrangian submanifold L [10,11], which we will denote $\rho_{\frac{1}{2}}|_L$, or sometimes just $\rho_{\frac{1}{2}}$. Given a smooth function H , let us consider the variation of $\int_L \rho_{\frac{1}{2}}$ under the variation of L specified by the Hamiltonian vector field ξ_H corresponding to H . It can only depend on the restriction of H on L . Therefore it should be of the form:

$$\delta_{\{H, _ \}} \int_L \rho_{\frac{1}{2}}|_L = \int_L H \mu_L[\rho_{\frac{1}{2}}] \tag{10}$$

where $\mu_L[\rho_{\frac{1}{2}}]$ is some integral form on L (which of course depends on $\rho_{\frac{1}{2}}$). There exists some half-density on M , which we will denote $\Delta_{\text{can}}\rho_{\frac{1}{2}}$, such that:

$$\mu_L[\rho_{\frac{1}{2}}] = - \left(\Delta_{\text{can}}\rho_{\frac{1}{2}} \right) \Big|_L \tag{11}$$

In other words:

Theorem 1. *Given a half-density $\rho_{\frac{1}{2}}$, there exists another half-density $\Delta_{\text{can}}\rho_{\frac{1}{2}}$, such that for any $H \in \text{Fun}(M)$ and any Lagrangian $L \subset M$:*

$$\delta_{\{H, _ \}} \int_L \rho_{\frac{1}{2}}|_L = - \int_L H \left(\Delta_{\text{can}}\rho_{\frac{1}{2}} \right) \Big|_L \tag{12}$$

Eq. (12) is the definition of Δ_{can} . We will give a proof in [Appendix B](#).

2.2.2. Relation between Δ_{can} and Lie derivative

Let us fix two functions $F \in \text{Fun}(M)$ and $H \in \text{Fun}(M)$. We will assume them to be even. In this case:

$$\{F, H\} = \{H, F\} \tag{13}$$

For any Lagrangian submanifold $L \subset M$, let us consider:

$$\int_L \left(H \mathcal{L}_{\{F, \cdot\}} \rho_{\frac{1}{2}} - F \mathcal{L}_{\{H, \cdot\}} \rho_{\frac{1}{2}} \right) = \int_L \left(\mathcal{L}_{\{F, \cdot\}}(H \rho_{\frac{1}{2}}) - \mathcal{L}_{\{H, \cdot\}}(F \rho_{\frac{1}{2}}) \right) = \tag{14}$$

$$= \delta_{\{F, \cdot\}} \int_L H \rho_{\frac{1}{2}} - \delta_{\{H, \cdot\}} \int_L F \rho_{\frac{1}{2}} = - \int_L F \Delta_{\text{can}}(H \rho_{\frac{1}{2}}) + \int_L H \Delta_{\text{can}}(F \rho_{\frac{1}{2}}) \tag{15}$$

Suppose that H is zero at some point $m \in M$: $H(m) = 0$. Pick a Lagrangian submanifold L passing through m , such that the restriction of H on L is zero. Eq. (15) holds for arbitrary F , therefore $\mathcal{L}_{\{H, \cdot\}} \rho_{\frac{1}{2}} = \Delta_{\text{can}}(H \rho_{\frac{1}{2}})$. We can always achieve $H(m) = 0$ for any point $m \in M$, by adding an appropriate constant to H , i.e. replacing H with $H - H(m)$. Therefore:

$$\mathcal{L}_{\{H, \cdot\}} \rho_{\frac{1}{2}} = (-)^{\bar{H}} \Delta_{\text{can}} \left(H \rho_{\frac{1}{2}} \right) - H \Delta_{\text{can}} \rho_{\frac{1}{2}} \tag{16}$$

Some properties of the Lie derivative of a half-density. We will now discuss some properties of Lie derivative which are independent of its relation to Δ_{can} .

Consider a vector field v on M , and the corresponding 1-parameter group of diffeomorphisms g^t . Let us think of a half-density $\rho_{\frac{1}{2}}$ as a function of x and \mathbf{E} , where x is a point of M and \mathbf{E} a basis in $T_x M$, depending on \mathbf{E} in the following way:

$$\rho_{\frac{1}{2}}(x, A\mathbf{E}) = (\text{SDet}A)^{1/2} \rho_{\frac{1}{2}}(x, \mathbf{E}) \tag{17}$$

By definition, the Lie derivative of $\rho_{\frac{1}{2}}$ along $v \in \text{Vect}(M)$ is:

$$\left(\mathcal{L}_v \rho_{\frac{1}{2}} \right) (x, \mathbf{E}) = \left. \frac{d}{dt} \right|_{t=0} \rho(g^t x, g_*^t \mathbf{E}) \tag{18}$$

Let us multiply v by a function $f \in \text{Fun}(M)$ such that $f(x) = 0$. The flux of fv preserves the point x , and we have:

$$\left(\mathcal{L}_{fv} \rho_{\frac{1}{2}} \right) (x, \mathbf{E}) = (-)^{\bar{f}\bar{v}} \left. \frac{d}{dt} \right|_{t=0} \rho(x, \exp(tv \otimes df)\mathbf{E}) = \frac{(-)^{\bar{f}\bar{v}}}{2} (\mathcal{L}_v f) \rho_{\frac{1}{2}}(x, \mathbf{E})$$

This implies that for any $f \in \text{Fun}(M)$ and $v \in \text{Vect}(M)$:

$$\mathcal{L}_{fv} \rho_{\frac{1}{2}} = f \mathcal{L}_v \rho_{\frac{1}{2}} + (-)^{\bar{f}\bar{v}} \frac{1}{2} (\mathcal{L}_v f) \rho_{\frac{1}{2}} \tag{19}$$

In particular:

$$\begin{aligned} \mathcal{L}_{\{FH, \cdot\}} \rho_{\frac{1}{2}} &= \mathcal{L}_{F\{H, \cdot\}} \rho_{\frac{1}{2}} + (-)^{\bar{F}\bar{H}} \mathcal{L}_{H\{F, \cdot\}} \rho_{\frac{1}{2}} = \\ &= F \mathcal{L}_{\{H, \cdot\}} \rho_{\frac{1}{2}} + (-)^{\bar{F}\bar{H}} H \mathcal{L}_{\{F, \cdot\}} \rho_{\frac{1}{2}} + (-)^{(\bar{H}+1)\bar{F}} \{H, F\} \rho_{\frac{1}{2}} \end{aligned} \tag{20}$$

2.2.3. The canonical operator is nilpotent

Indeed, since the definition of Δ_{can} is geometrically natural, it automatically commutes with canonical transformations and therefore for any $H \in \text{Fun}(M)$:

$$[\Delta_{\text{can}}, \mathcal{L}_{\{H, \cdot\}}] \rho_{\frac{1}{2}} = 0 \tag{21}$$

Comparing this with Eq. (16) we derive:

$$\Delta_{\text{can}}^2 = 0 \tag{22}$$

2.2.4. *Odd Laplace operator on functions*

Given a half-density $\rho_{1/2}$ we can define the odd Laplace operator on functions as follows:

$$(\Delta_{\rho_{1/2}} F)\rho_{1/2} = \Delta_{\text{can}}(F\rho_{1/2}) - (-)^{\bar{F}} F \Delta_{\text{can}}\rho_{\frac{1}{2}} = (-)^{\bar{F}} \mathcal{L}_{\{F, \cdot\}}\rho_{\frac{1}{2}} \tag{23}$$

We will often abbreviate:

$$\Delta F = \Delta_{\rho_{1/2}} F \tag{24}$$

when there is some obvious implicit choice of half-density. Eqs. (16) and (20) imply:

$$\Delta_{\rho_{1/2}}(XY) = (\Delta_{\rho_{1/2}} X)Y + (-)^{\bar{X}} X \Delta_{\rho_{1/2}} Y + (-)^{\bar{X}} \{X, Y\} \tag{25}$$

Notice that the odd Poisson bracket measures the deviation of $\Delta_{\rho_{1/2}}$ from being a differentiation of $\text{Fun}(M)$.

2.3. *Master Equation*

We will always assume that $\rho_{\frac{1}{2}}$ satisfies the Master Equation:

$$\Delta_{\text{can}}\rho_{\frac{1}{2}} = 0 \tag{26}$$

Under this assumption, the operator $\Delta_{\rho_{1/2}}$ is nilpotent:

$$\Delta_{\rho_{1/2}}^2 = 0 \tag{27}$$

2.4. *Moment map*

Let G denote the group of canonical transformations of M . Let us denote G_L and G_R the group of left and right shifts on G (both G_L and G_R are naturally isomorphic to G). Both left and right shifts naturally lift to ΠTG . Let \mathfrak{g} be the Lie algebra corresponding to G .

We consider the right-invariant differential form on \widehat{G} with values in $\widehat{\mathfrak{g}}$, which is denoted $d\widehat{g}\widehat{g}^{-1}$. Because of (9), we can consider it as a differential 1-form with values in $\Pi C^\infty M$. There is some invariance under the left and right shifts, which can be summarized as follows:

$$d\widehat{g}\widehat{g}^{-1} \in C^\infty((\Pi T\widehat{G}) \times_{\widehat{G}_L \times \widehat{G}_R} M) \tag{28}$$

$$d\widehat{g}\widehat{g}^{-1} = (d\widehat{g}\widehat{g}^{-1})^A \mathcal{H}_A \tag{29}$$

$$d_{(g)} F(gx) = \{d\widehat{g}\widehat{g}^{-1}, F\}(gx) \tag{30}$$

where the action of $\widehat{G}_L \times \widehat{G}_R$ on $\Pi T\widehat{G}$ is induced from the following action on \widehat{G} :

$$(g_L, g_R) g = g_L g g_R^{-1} \tag{31}$$

The action of $\widehat{G}_L \times \widehat{G}_R$ on M is:

$$(g_L, g_R) m = g_L m \tag{32}$$

The \widehat{G}_R -invariance of $d\widehat{g}\widehat{g}^{-1}$ follows immediately from the definitions. The \widehat{G}_L -invariance, at the infinitesimal level, follows from the following formula, where it is enough to assume that $\{\xi^B \mathcal{H}_B, \cdot\}$ is an even vector field:

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} (d\widehat{g}g^{-1})^A \mathcal{H}_A \left(e^{-t\{\xi^B \mathcal{H}_B, \cdot\}} x \right) &= - (d\widehat{g}g^{-1})^A \xi^B \{ \mathcal{H}_B, \mathcal{H}_A \} (x) = \\ &= - \left[\xi, d\widehat{g}g^{-1} \right]^A \mathcal{H}_A(x) \end{aligned} \tag{33}$$

The moment map satisfies the Maurer–Cartan equation:

$$d(d\widehat{g}g^{-1}) = -\frac{1}{2} \{ d\widehat{g}g^{-1}, d\widehat{g}g^{-1} \} \tag{34}$$

2.5. Canonical operator is ill-defined in field-theoretic context

The considerations of Section 2 are valid when the BV phase space M is a finite-dimensional supermanifold. But in the field theory context, Δ_{can} is usually ill-defined. We use the BV formalism for the worldsheet sigma-model, *i.e.* in the field-theoretic context.

We hope that it is possible to regularize Δ_{can} , maybe as in [12]. In many cases the string worldsheet theory is essentially free (quadratic Lagrangian) and the path integral can be computed exactly, allowing the explicit verification. In other cases, there is a natural small parameter and the computations can be verified *order by order in perturbation theory*.

In this paper, we just use finite-dimensional formulas with Δ_{can} ; in the infinite-dimensional case the results should be justified by perturbation theory.

3. Form Ω

3.1. Summary

We want to define some pseudo-differential form Ω which can serve as *string measure*. There are several closely related definitions:

1. **PDF on the group of canonical transformations.** In Section 3.2 we will start by constructing Ω as a PDF on \widehat{G} . The definition actually depends on the choice of a fixed Lagrangian submanifold $L \subset M$. Strictly speaking, we can characterize Ω as a map of the following type:

$$\Omega : \text{LAG} \rightarrow \text{Fun}(\Pi T \widehat{G}) \tag{35}$$

(which associates a PDF on \widehat{G} to every Lagrangian submanifold).

2. **PDF on the space of Lagrangian submanifolds.** There is a natural map:

$$\widehat{G} \times \text{LAG} \rightarrow \text{LAG} \tag{36}$$

coming from the action of G on M . A natural question is, does Ω descend to a PDF on LAG? The answer is essentially “yes”, although some minor modifications are needed. In Section 3.5 we will define Ω as a PDF on the space of Lagrangian submanifolds.

3. **PDF on the space of Legendrian submanifolds.** Suppose that we can construct an $\mathbf{R}^{0|1}$ -bundle over M , which we will call \widehat{M} , with a connection whose curvature is ω . Lagrangian submanifolds on M lift to Legendrian submanifolds in \widehat{M} . We can define Ω as a closed PDF on the space LEG of Legendrian submanifolds in \widehat{M} .
4. **PDF on an equivalence class of actions.** In BV formalism the choice of a Lagrangian submanifold $L \subset M$ is closely related to the choice of a quantization scheme. In other words,

it is essentially the choice of a representative in a class of physically equivalent theories. Given S_{BV} and $L \in \text{LAG}$, the restriction $(S_{BV})|_L$ gives a physical action functional which we use in the path integral. A different choice of L gives a BRST equivalent action functional. Therefore it would be natural to try to interpret Ω as a PDF on such an equivalence class. This, however, is not straightforward. The space of Lagrangian submanifolds is actually larger than the space of action functionals; to descend to the space of action functionals one has to take the factorspace over the symmetries of S_{BV} . Generally speaking, Ω does not descend to this factorspace. We will study these issues in Section 5.

3.2. Definition of Ω as a PDF on \widehat{G}

For any function⁴ $E : \mathbf{C} \rightarrow \mathbf{C}$, consider the following pseudo-differential form on \widehat{G} :

$$\Omega^{\{E\}} \in \text{Fun}((\Pi T \widehat{G}) \times \text{LAG}) \tag{37}$$

$$\Omega^{\{E\}} = \int_{\widehat{g}L} E(d\widehat{g}g^{-1}) \rho_{\frac{1}{2}} \tag{38}$$

where $d\widehat{g}g^{-1}$ is the moment map of Section 2.4; its main property is that for any function $F \in \text{Fun}(M)$:

$$d_{(g)}(F \circ g) = \{d\widehat{g}g^{-1}, F\} \circ g \tag{39}$$

3.3. Ω is closed

Under the assumption that $\Delta_{\text{can}}\rho_{\frac{1}{2}} = 0$ the form Ω is closed.

Preparation for the proof of closedness. Notice that for any even $\mathcal{H} \in \text{Fun}(M)$:

$$\mathcal{H}\Delta_{\text{can}}(E(\mathcal{H})\rho_{\frac{1}{2}}) - \Delta_{\text{can}}(\mathcal{H}E(\mathcal{H})\rho_{\frac{1}{2}}) + \frac{1}{2}\{\mathcal{H}, \mathcal{H}\}E'(\mathcal{H})\rho_{\frac{1}{2}} + \tag{40}$$

$$+ \Delta_{\text{can}}\left(\left(\int d\mathcal{H} E(\mathcal{H})\right)\rho_{\frac{1}{2}}\right) - \left(\int d\mathcal{H} E(\mathcal{H})\right)\Delta_{\text{can}}\rho_{\frac{1}{2}} = 0 \tag{41}$$

We can interpret this formula using the notion of Lie derivative of half-density as follows:

$$\mathcal{L}_{\{f(\mathcal{H}), \cdot\}}\rho_{\frac{1}{2}} = \mathcal{L}_{\{\mathcal{H}, \cdot\}}\left(f'(\mathcal{H})\rho_{\frac{1}{2}}\right) - \frac{1}{2}\{\mathcal{H}, \mathcal{H}\}f''(\mathcal{H})\rho_{\frac{1}{2}} \tag{42}$$

where $f(\mathcal{H}) = \int d\mathcal{H}E(\mathcal{H})$ (but we prefer to work with $E(\mathcal{H})$ instead of $f(\mathcal{H})$, because it is $E(\mathcal{H})$ that enters in Eq. (38)). When \mathcal{H} is the moment map $\mathcal{H} = d\widehat{g}g^{-1}$, this can be combined with the Maurer–Cartan Eq. (34):

$$(d + \mathcal{L}_{\{d\widehat{g}g^{-1}, \cdot\}})\left(f'(d\widehat{g}g^{-1})\rho_{\frac{1}{2}}\right) = \mathcal{L}_{\{f(d\widehat{g}g^{-1}), \cdot\}}\rho_{\frac{1}{2}} \tag{43}$$

(where d only acts on \widehat{g}).

⁴ We will actually use just $E(z) = e^z$.

Proof of closedness. Taking \mathcal{H} to be the moment map $\mathcal{H} = d\widehat{g}\widehat{g}^{-1}$, we get:

$$d \int_{gL} E(d\widehat{g}\widehat{g}^{-1}) \rho_{\frac{1}{2}} = \tag{44}$$

$$= \int_{gL} d(E(d\widehat{g}\widehat{g}^{-1})) \rho_{\frac{1}{2}} - d\widehat{g}\widehat{g}^{-1} \Delta_{\text{can}} \left(E(d\widehat{g}\widehat{g}^{-1}) \rho_{\frac{1}{2}} \right) + \Delta_{\text{can}} \left(d\widehat{g}\widehat{g}^{-1} E(d\widehat{g}\widehat{g}^{-1}) \rho_{\frac{1}{2}} \right) = \tag{45}$$

$$= \int_{gL} d(d\widehat{g}\widehat{g}^{-1}) E'(d\widehat{g}\widehat{g}^{-1}) \rho_{\frac{1}{2}} + \frac{1}{2} \{d\widehat{g}\widehat{g}^{-1}, d\widehat{g}\widehat{g}^{-1}\} E'(d\widehat{g}\widehat{g}^{-1}) \rho_{\frac{1}{2}} + \tag{46}$$

$$+ \Delta_{\text{can}} \left(\int d\mathcal{H} E(\mathcal{H}) \rho_{\frac{1}{2}} \right) - \int d\mathcal{H} E(\mathcal{H}) \Delta_{\text{can}} \rho_{\frac{1}{2}}$$

$$= 0 \tag{47}$$

where we have taken into account the definition of the canonical odd Laplace operator and the fact that $\int_{gL} \Delta_{\text{can}}(\dots) = 0$. We also used the Maurer–Cartan Eq. (34). In particular, let us take $E(\mathcal{H}) = \exp(\mathcal{H})$. Let us denote:

$$\Omega\langle F \rangle = \int_{\widehat{g}L} \exp\left(d\widehat{g}\widehat{g}^{-1}\right) \rho_{\frac{1}{2}} F \tag{48}$$

$$\Omega = \Omega\langle 1 \rangle \tag{49}$$

Then Eq. (47) implies:

$$d\Omega\langle F \rangle = -\Omega\left\langle \rho_{\frac{1}{2}}^{-1} \Delta_{\text{can}}(\rho_{\frac{1}{2}} F) \right\rangle = -\Omega\langle \Delta F \rangle \tag{50}$$

Also notice the following equation:

$$\iota_{\{H, \cdot\}} \Omega\langle F \rangle = \Omega\langle HF \rangle \tag{51}$$

3.4. Algebraic interpretation

3.4.1. Cone Lie superalgebra: general definition

Let G be a Lie group. It is possible to introduce the structure of a Lie group on ΠTG . In fact:

$$\Pi TG = \text{Map}(\mathbf{R}^{0|1}, G) \tag{52}$$

and the structure of the group is introduced by pointwise multiplication. Consider the corresponding Lie superalgebra:

$$\widetilde{\mathfrak{g}} = \text{Lie}(\Pi TG) \tag{53}$$

We can interpret $\widetilde{\mathfrak{g}}$ as the algebra of maps from $\mathbf{R}^{0|1}$ to \mathfrak{g} ; therefore the elements of $\widetilde{\mathfrak{g}}$ are \mathfrak{g} -valued functions $f(\theta)$ of an odd parameter $\theta \in \mathbf{R}^{0|1}$. It is possible to extend $\widetilde{\mathfrak{g}}$ by an extra odd element ∂_θ with the following commutation relations:

$$[\partial_\theta, f(\theta)] = \frac{\partial}{\partial \theta} f(\theta) \tag{54}$$

We will call this extended algebra $\widetilde{\mathfrak{g}}'$.

3.4.2. Cone Lie superalgebra associated to a BV algebra

Let us consider a BV algebra \mathcal{G} with the generator Δ . Let \mathfrak{g} be the Lie superalgebra which is obtained from \mathcal{G} by forgetting the associative algebra structure and flipping parity, and $\widetilde{\mathfrak{g}}$ the corresponding cone Lie superalgebra. We need to flip parity in order to turn $\{_, _\}$ into a Lie superalgebra operation. If the parity of b as an element of \mathcal{G} , is $|b|$, then the parities of the corresponding elements of $\widetilde{\mathfrak{g}}$ are: $|(b, 0)| = |b| + \bar{1}$ and $|(0, b)| = |b|$.

Theorem 2. *The following formulas define the representation of $\widetilde{\mathfrak{g}}$ on \mathcal{G} :*

$$\rho(d)a = -\Delta a \tag{55}$$

$$\rho((0, b))a = ba \tag{56}$$

$$\rho((b, 0))a = [\rho((0, b)), \rho(d)]a = \tag{57}$$

$$= (-)^{|b|} \Delta(ba) - b\Delta a \tag{58}$$

We have to check that:

$$\begin{aligned} [\rho((b, 0)), \rho((0, c))]a &= \rho((b, 0))\rho((0, c))a - (-)^{|c|(|b|+1)}\rho((0, c))\rho((b, 0))a \\ &= \rho((0, \{b, c\}))a \end{aligned} \tag{59}$$

Indeed, we have:

$$\begin{aligned} \rho((b, 0))\rho((0, c))a &= (-)^{|b|} \Delta(bca) - b\Delta(ca) = (-)^{|b|} (\Delta b)ca + \{b, ca\} = \\ &= \{b, c\}a + (-)^{|b|} (\Delta b)ca + (-)^{|c|(|b|+1)} c\{b, a\} \end{aligned} \tag{60}$$

$$\rho((0, c))\rho((b, 0))a = (-)^{|b|} c\Delta(ba) - cb\Delta a = (-)^{|b|+|c|(|b|+1)} (\Delta b)ca + c\{b, a\} \tag{61}$$

3.4.3. Form $\Omega(\dots)$ as an intertwiner

Let us consider the particular case when \mathcal{G} is the algebra of functions on the odd symplectic manifold M (our BV phase space).

In this case, $\widetilde{\mathfrak{g}}$ naturally acts on the differential forms on \widehat{G} . Indeed, every element $\alpha \in \mathfrak{g} = \text{Lie}(G)$ determines the corresponding right-invariant vector field on \widehat{G} . Then $(\alpha, 0)$ would act as a Lie derivative along this vector field, and $(0, \alpha)$ acts as a contraction.

We can consider $\Omega(_)$ as a linear map from \mathcal{G} to the space of differential forms on \widehat{G} ; for each $a \in \mathcal{G}$, this map computes $\Omega\langle a \rangle$ — the corresponding differential form. Eqs. (50) and (51) can be interpreted as saying that $\Omega(_)$ is an intertwiner:

$$x\Omega\langle a \rangle = \Omega\langle \rho(x)a \rangle \tag{62}$$

In the language of half-densities, we have an intertwiner between:

- action of $\widetilde{\mathfrak{g}}$ on half-densities in M :

$$R(d)\rho_{\frac{1}{2}} = -\Delta\rho_{\frac{1}{2}} \tag{63}$$

$$R(t_{\xi_H})\rho_{\frac{1}{2}} = H\rho_{\frac{1}{2}} \tag{64}$$

$$R(\xi_H)\rho_{\frac{1}{2}} = (-)^{\overline{H}} \Delta(H\rho_{\frac{1}{2}}) - H\Delta\rho_{\frac{1}{2}} \tag{65}$$

and

- action of \tilde{g} on PDFs on \widehat{G} :

$$R(d)\alpha = d\alpha \tag{66}$$

$$R(\iota_{\xi_H})\alpha = \iota_{\xi_H}\alpha \tag{67}$$

$$R(\xi_H)\alpha = \mathcal{L}_{\xi_H}\alpha \tag{68}$$

3.5. Descent to LAG

3.5.1. Deviation of Ω from being horizontal

In Section 3.2 we defined a form Ω on $\Pi T\widehat{G} \times \text{LAG}$. The action of \widehat{G} on LAG defines a natural projection:

$$\widehat{\pi} : \Pi T\widehat{G} \times \text{LAG} \rightarrow \Pi T\text{LAG} \tag{69}$$

It is natural to ask if Ω is constant along the fibers of $\widehat{\pi}$. The answer is, strictly speaking, no, but the dependence on the fiber is rather mild. Let us present $\Pi T\text{LAG}$ as follows:

$$\Pi T\text{LAG} = (\Pi T\widehat{G}) \times_{\widehat{G}} \text{LAG} \tag{70}$$

where $\times_{\widehat{G}}$ means factor over the symmetry:

$$(\widehat{g}, L) \mapsto (\widehat{g}\widehat{f}, \widehat{f}^{-1}L) \tag{71}$$

For the descent to work, Ω should be base, i.e. both invariant and horizontal, with respect to (71). Actually, it is invariant and almost horizontal. The invariance is straightforward. Let us study the question of horizontality, and understand why it is “almost horizontal” instead of “horizontal”. Let $\xi \in \widehat{\mathfrak{g}}$ denote a Hamiltonian whose flux preserves the Lagrangian submanifold L . The horizontality would be equivalent to the statement that $\iota_{\text{Ad}(\widehat{g})\xi}\Omega$ is zero. But in fact, Eq. (51) implies:

$$\iota_{\text{Ad}(\widehat{g})\xi}\Omega = \int_{\widehat{g}L} (\text{Ad}(\widehat{g})\xi) e^{d\widehat{g}\widehat{g}^{-1}} \rho_{\frac{1}{2}} \tag{72}$$

Notice that $\xi \in \text{St}(L)$ implies $\text{Ad}(\widehat{g})\xi \in \text{St}(\widehat{g}L)$. Apriori this only implies that $(\text{Ad}(\widehat{g})\xi)|_{\widehat{g}L}$ is a constant (but not necessarily zero).

This is potentially a problem (we certainly do want Ω to descend on LAG). We will now outline some possible ways of resolving this problem.

3.5.2. Use ghost number symmetry

In string theory there is usually an action of $U(1)$ called “ghost number symmetry”. Various objects are either $U(1)$ -invariant or have definite charge.

Let us restrict ourselves to only considering those Lagrangian submanifolds which are invariant under this $U(1)$ (i.e. request that the orbits of the $U(1)$ be tangent to the Lagrangian submanifold) and request that $d\widehat{g}\widehat{g}^{-1}$ have ghost number -1 . This eliminates the possibility of adding a constant to $d\widehat{g}\widehat{g}^{-1}$ and therefore renders $(d\widehat{g}\widehat{g}^{-1})_L$ unambiguously defined from the variation of L . Our form Ω is now horizontal, and descends from G to LAG.

3.5.3. Use transverse Lagrangian submanifold

Suppose that we can find a Lagrangian submanifold $L^\vee \subset M$ which is transverse to all Lagrangian submanifolds from our family:

$$(\widehat{g}L) \cap L^\vee = p(\widehat{g}L) \quad (\text{one point}) \tag{73}$$

where $p(\widehat{g}L)$ is a marked point on every $\widehat{g}L$.

Let $\mathcal{Z}_{L^\vee} \subset \text{Fun}(M)$ denote the subspace of those Hamiltonians which vanish on L^\vee . Since L^\vee is chosen to be Lagrangian, $\Pi\mathcal{Z}_{L^\vee}$ is a Lie subalgebra of $\mathfrak{g} = \Pi\text{Fun}(M)$. (Slightly smaller than the stabilizer of L^\vee .) It is intuitively clear (from counting the “degrees of freedom”) that we can impose the following gauge condition: $\widehat{g} \in \exp(\Pi\mathcal{Z}_{L^\vee}) \subset \text{St}(L^\vee) \subset \widehat{G}$. This implies that $d\widehat{g}\widehat{g}^{-1}$ vanishes at the marked point: $(d\widehat{g}\widehat{g}^{-1})(p(\widehat{g}L)) = 0$.

This eliminates the ambiguity of a constant in $d\widehat{g}\widehat{g}^{-1}$.

What happens if we change L^\vee to another transversal Lagrangian submanifold \widetilde{L}^\vee ? Let us assume that we can choose:

$$\widehat{f} \in \text{St}(L) \subset \widehat{G} \tag{74}$$

$$\text{such that } \widehat{f}\widetilde{L}^\vee = L^\vee \tag{75}$$

Then we just have to change:

$$\widehat{g} \mapsto \widehat{g}\widehat{f} \tag{76}$$

As Ω is invariant, this would not change the result of the integration.

3.5.4. Upgrade LAG to LAG₊

The most elegant solution is to use, instead of the space of Lagrangian submanifolds LAG, the space LAG₊ of Lagrangian submanifolds with marked point [2].

3.6. Space of Legendrian submanifolds in \widehat{M}

3.6.1. Quantomorphisms

Suppose that there exists an $\mathbf{R}^{0|1}$ -bundle over M :

$$\widehat{M} \xrightarrow{\widehat{\pi}} M \tag{77}$$

with a connection such that the curvature is equal to ω . Then we can realize the central extension \widehat{G} as the group of automorphisms of this bundle.

We have the exact sequence:

$$0 \rightarrow \text{Inv}_{\mathbf{R}^{0|1}}(\Gamma(T\widehat{M}/M)) \rightarrow \text{Inv}_{\mathbf{R}^{0|1}}(\text{Vect}(\widehat{M})) \xrightarrow{\widehat{\pi}_*} \text{Vect}(M) \rightarrow 0 \tag{78}$$

where $\text{Inv}_{\mathbf{R}^{0|1}}$ means that we should consider $\mathbf{R}^{0|1}$ -invariant vector fields. The kernel of $\widehat{\pi}_*$ is the 0|1-dimensional space $\text{Inv}_{\mathbf{R}^{0|1}}(\Gamma(T\widehat{M}/M))$. A connection is a split:

$$\text{lift} : \text{Vect}(M) \rightarrow \text{Inv}_{\mathbf{R}^{0|1}}(\text{Vect}(\widehat{M})) \tag{79}$$

Suppose that we can find a “symplectic potential” α such that $\omega = d\alpha$. Then we can use it to construct the connection satisfying:

$$[\text{lift}(v_1), \text{lift}(v_2)] = \text{lift}([v_1, v_2]) + \omega(v_1, v_2)\partial_\theta \tag{80}$$

where ∂_ϑ is the vector field arising from the action of $\mathbf{R}^{0|1}$ on \widehat{M} . (We can think of ϑ as a coordinate in the fiber; it is only defined locally, but ∂_ϑ is globally well-defined.) Explicitly:

$$\text{lift}(v) = v + (\iota_v \alpha) \partial_\vartheta \tag{81}$$

Let us consider the subalgebra $\mathfrak{g} \subset \text{Vect}(M)$ consisting of Hamiltonian vector fields. For every even (we will restrict to even vector fields for simplicity) $\{H, _ \} \in \mathfrak{g}$ consider the following vector field on \widehat{M} :

$$\widehat{\xi}_H = \{H, _ \} + (\iota_{\{H, _ \}} \alpha + H) \partial_\vartheta \tag{82}$$

It is defined to preserve the connection. An explicit calculation shows that the Lie derivative vanishes:

$$\mathcal{L}_{\widehat{\xi}_H} (d\theta - \alpha) = 0 \tag{83}$$

Notice that the vertical component of $\widehat{\xi}_H$ (with respect to the connection defined in Eq. (81)) is $H \partial_\vartheta$. By construction, the space of vector fields of this form is closed under commutator. We can check it directly, using the formula:

$$\begin{aligned} & [\{H, _ \} + (\iota_{\{H, _ \}} \alpha + H) \partial_\vartheta , \{F, _ \} + (\iota_{\{F, _ \}} \alpha + F) \partial_\vartheta] = \\ & = \{ \{H, F\}, _ \} + (\mathcal{L}_{\{H, _ \}} (\iota_{\{F, _ \}} \alpha) - \mathcal{L}_{\{F, _ \}} (\iota_{\{H, _ \}} \alpha) + 2\{H, F\}) \partial_\vartheta = \\ & = \{ \{H, F\}, _ \} + (\iota_{\{ \{H, F\}, _ \}} \alpha + \{H, F\}) \partial_\vartheta \end{aligned} \tag{84}$$

As a Lie algebra this is $\Pi\text{Fun}(M)$. It integrates to the group of automorphisms of the fiber bundle $\widehat{M} \rightarrow M$ which preserve the connection defined in Eq. (81).

3.6.2. Form Ω as a form on the space of Legendrian submanifolds LEG

A Legendrian submanifold in \widehat{M} projects to a Lagrangian submanifold in M :

$$\pi(\widehat{L}) = L \tag{85}$$

This projection is typically *not* one-to-one; it is a cover. We can define Ω as a PDF on the space of Legendrian submanifolds by interpreting Eq. (48) as integration over the projection:

$$\Omega(\widehat{g}, d\widehat{g}) = \int_{\pi(\widehat{L})} \exp(d\widehat{g}\widehat{g}^{-1}) \rho_{\frac{1}{2}} \tag{86}$$

This descends to a closed PDF on the space of Legendrian submanifolds LEG.

3.7. Reduction to integration over single Lagrangian submanifold

Here we will explain that integration over a family of Lagrangian submanifolds can be reduced to integration over a single Lagrangian submanifold in some larger BV phase space.

A family of Lagrangian submanifolds $\{L(\lambda) \subset M \mid \lambda \in \Lambda\}$ defines a single Lagrangian submanifold in $L_\Lambda \subset M \times \Pi T^*(\Pi T \Lambda)$. We will now describe the construction of L_Λ .

As a first step, let us consider a submanifold $L'' \subset M \times \Lambda$ which is defined as follows:

$$L'' = \{ (m, \lambda) \mid \lambda \in \Lambda, m \in L(\lambda) \} \tag{87}$$

This can be promoted to a subspace $L' \subset M \times \Pi T \Lambda$:

$$L' = \{(m, \lambda, [d\lambda]) \mid (\lambda, [d\lambda]) \in \Pi T \Lambda, m \in L(\lambda)\} \tag{88}$$

Finally, we will construct L_Λ as the following section of the vector bundle $M \times \Pi T^*(\Pi T \Lambda) \longrightarrow M \times \Pi T \Lambda$ restricted to $L' \subset M \times \Pi T \Lambda$:

$$s : L' \rightarrow M \times \Pi T^*(\Pi T \Lambda) \tag{89}$$

$$s(m, \lambda, [d\lambda]) = (m, \lambda, [d\lambda], \sigma(m), 0) \tag{90}$$

where $\sigma(m)$ computes for every tangent vector to Λ the value of its corresponding generating function on $m \in L(\lambda)$. This section defines our big Lagrangian submanifold:

$$L_\Lambda = s(L') \tag{91}$$

There is a natural BV Hamiltonian \widehat{d} on $\Pi T^*(\Pi T \Lambda)$. It describes the lift to ΠT^* of the natural nilpotent vector field d on $\Pi T \Lambda$. We have:

$$\int_\Lambda \Omega = \int_{L_\Lambda} \exp(S_{\text{BV}} + \widehat{d}) \tag{92}$$

$$\text{where } \widehat{d} = [d\lambda]\lambda^* \tag{93}$$

In the case of Yang–Mills theory, λ is called \bar{c} and $[d\lambda]$ is called π (see Section 6.7).

4. Picture changing

4.1. Integration in two steps

In our approach we integrate some closed form Ω over a cycle in the moduli space of Lagrangian submanifolds modulo gauge symmetries. The form Ω itself is defined in terms of the integration of $e^{S_{\text{BV}}}$ over the Lagrangian submanifold (path integral). We therefore have a double integral: first the path integral, and then a finite-dimensional integral over a cycle in the moduli space.

However, we suspect that there is actually no fundamental distinction between these two steps. In principle, one can combine them into one integration.

It is often convenient to pull one or more integration out from the integration of Ω into the path integral or vice versa. This is “picture changing”.

In the case of bosonic string we usually choose a Lagrangian submanifold corresponding to a fixed complex structure (or metric, depending on the flavor of the formalism). In this case the path integration contains the integration over the antifield to metric/complex structure, which is identified with the b -ghost. However, in principle we could also include into the path integral some partial integration over the moduli space of complex structures, and then integrate over the rest later.

It turns out that in the BV formalism, this (at least in some cases) corresponds to changing each Lagrangian submanifolds in the family into a different Lagrangian submanifold. This is a topologically nontrivial change, essentially a change in polarization.

4.2. Baranov–Schwarz transform

Any family of Lagrangian submanifolds can locally be considered as an orbit of some Lagrangian submanifold L_0 by an abelian subgroup of G :

$$UL_0 = \{gL_0 \mid g \in U\} \tag{94}$$

where $U \subset G$ is an abelian subgroup. For every $L(u) = uL_0$ $u \in U$, consider a submanifold $K(u) \subset L(u)$ which is the common zero set of all Hamiltonians of elements of T_uU restricted on $L(u)$. Consider the union:

$$\tilde{L} = \bigcup_{u \in U} K(u) \tag{95}$$

It is a Lagrangian submanifold in M . Moreover:

$$\int_{\Pi T(UL_0)} \Omega = \int_{\tilde{L}} \rho_{\frac{1}{2}} \tag{96}$$

Indeed, let us consider the k -form component of Ω . We have to integrate it over some k -dimensional family of Lagrangian submanifolds. We parametrize the family by s^1, \dots, s^k . Let us perform the Baranov–Schwarz transform by integrating over $d[ds]$. This turns Ω into an integral form. We get:

$$\int \prod_I ds^I d[ds^I] \int_{gL} \rho_{\frac{1}{2}} \exp\left(\sum_I ds^I \frac{\partial g}{\partial s^I} g^{-1}\right) = \tag{97}$$

$$= \int \prod_I ds^I \int_{gL} \rho_{\frac{1}{2}} \prod_{I=1}^k \delta\left(\frac{\partial g}{\partial s^I} g^{-1}\right) \tag{98}$$

(Remember that we identify $\frac{\partial g}{\partial s^I} g^{-1}$ with the actual Hamiltonian, *i.e.* for us $\frac{\partial g}{\partial s^I} g^{-1}$ is a function on M , and in particular a function on L .) This means that actually we are integrating not over the whole Lagrangian submanifold L , but over a submanifold $K \subset L$ of the codimension k , which is defined by the system of equations:

$$\frac{\partial g}{\partial s^I} g^{-1} = 0, \quad I \in \{1, \dots, k\} \tag{99}$$

But on the other hand, remember that we have to integrate over the family of the dimension k . In other words, we lost k integrations by inserting the δ -functions (97), but then regained them as integration over the family. This is equivalent to the “90 degree rotation” $L \mapsto \tilde{L}$.

Therefore integration over a family of Lagrangian submanifolds is equivalent to the integration over a single “rotated” Lagrangian submanifold.

However, in the physical context of path integration, it is hard to make use of this equivalence.

5. Equivariant Ω

5.1. Special canonical transformations

A canonical transformation is called *special* if it preserves $\rho_{\frac{1}{2}}$. Let SG denote the group of special canonical transformations, and \mathfrak{sg} its algebra. We will also consider the corresponding

centrally extended \widehat{SG} and $\widehat{\mathfrak{sg}}$. Elements of $\widehat{\mathfrak{sg}}$ are those Hamiltonians which are annihilated by $\Delta_{\rho_{1/2}}$.

Suppose that two Lagrangian submanifolds L_1 and L_2 are connected by a special canonical transformation:

$$L_2 = gL_1 \quad , \quad g \in SG \tag{100}$$

Naively one would want to consider such Lagrangian submanifolds completely equivalent. However, this does not work. The form Ω is not base with respect to the action of SG on LAG. Moreover, at least in the finite-dimensional case the action of SG on LAG is usually transitive; the factor space $SG \backslash \text{LAG}$ would be a discrete space. Therefore it does not make much sense to consider the factorspace over SG .

On the other hand, if we consider just LAG without any identifications, then there are no integration cycles. For our closed form Ω to produce invariants, we need some integration cycles.

The solution is to consider a factorspace of LAG over a subgroup $H \subset SG$. In fact this H can not be an arbitrary subgroup of SG ; it has to be chosen carefully.⁵

5.2. Straightforward descent does not work

Let \widehat{h} be a special canonical transformation, i.e. $\widehat{h} \in \widehat{SG}$. It follows immediately that $\Omega = \int_{gL} e^{d\widehat{g}\widehat{g}^{-1}}$ is invariant under the left shift $\widehat{g} \mapsto \widehat{h}\widehat{g}$. But Ω is not horizontal; for $\xi \in \widehat{\mathfrak{sg}}$ we have:

$$\iota_{\xi}\Omega = \int_{gL} \xi e^{d\widehat{g}\widehat{g}^{-1}} \rho_{\frac{1}{2}} \neq 0 \tag{101}$$

Therefore Ω does not descend from LAG to $SG \backslash \text{LAG}$.

However, we will now identify a class of subalgebras $\mathfrak{h} \subset \mathfrak{sg}$ for which we can construct the base form $\Omega_{\mathfrak{h}}^{\text{base}}$. Being a base form, it descends to $H \backslash \text{LAG}$, where H is the Lie group generated by the flows of elements of \mathfrak{h} .

5.3. Equivariant half-densities

Suppose that we have the following data:

1. A subgroup H of the group of canonical transformations of an odd symplectic manifold M (the BV phase space); the Lie algebra \mathfrak{h} of H consists of vector fields of the form $\{F, _ \}$ where F runs over Hamiltonians of elements of \mathfrak{h} .
2. A map

$$\mathfrak{h} \rightarrow \text{half-densities on } M \tag{102}$$

$$\xi \mapsto \rho_{\frac{1}{2}}^C(\xi) \tag{103}$$

such that:

⁵ Gauge symmetries were discussed in the context of BV formalism in [13,14]; the requirements for gauge symmetries which we need for our procedure appear to be more restrictive.

$$\mathcal{L}_\xi \rho_{\frac{1}{2}}^C(\eta) = \left. \frac{d}{dt} \right|_{t=0} \rho_{\frac{1}{2}}^C(e^{t \text{ad}_\xi} \eta) \tag{104}$$

$$\Delta_{\text{can}} \left(\rho_{\frac{1}{2}}^C(\{F, _ \}) \right) = F \rho_{\frac{1}{2}}^C(\{F, _ \}) \tag{105}$$

Then the following equation is a cocycle of the H -equivariant Cartan complex on LAG:

$$\Omega(L, dL, \xi) = \int_L e^{d\widehat{g}\widehat{g}^{-1}} \rho_{\frac{1}{2}}^C(\xi) \tag{106}$$

This is proven using Section 3.4.3.

Notice the compatibility of Eqs. (104) and (105):

$$\begin{aligned} \mathcal{L}_{\{F, _ \}} \rho_{\frac{1}{2}}^C(\{F, _ \}) &= -\Delta_{\text{can}} \left(F \rho_{\frac{1}{2}}^C(\{F, _ \}) \right) - F \Delta_{\text{can}} \rho_{\frac{1}{2}}^C(\{F, _ \}) = \\ &= -\Delta_{\text{can}}^2 \left(\rho_{\frac{1}{2}}^C(\{F, _ \}) \right) - F^2 \rho_{\frac{1}{2}}^C(\{F, _ \}) = 0 \end{aligned} \tag{107}$$

Here we used Eq. (16); also, since $\{F, _ \}$ is even, F is odd and therefore $F^2 = 0$.

It seems that the main ingredient in this approach is the choice of a group H . (In the case of bosonic string this is the group of diffeomorphisms.) The rest of the formalism is built around H . We suspect that $\rho_{\frac{1}{2}}^C(\xi)$ is more or less unambiguously determined by the choice of H ; even $\rho_{\frac{1}{2}}^C(0)$ is already unambiguously determined. Indeed, we will now see that the constraints arising from Eqs. (104) and (105) are very tight.

5.4. Expansion of $\Omega(L, dL, \xi)$ in powers of ξ

5.4.1. Derivation of a special subspace $\mathcal{F} \subset \text{Fun}(M)$

Let us expand $\rho_{\frac{1}{2}}^C(\xi)$ in powers of ξ :

$$\rho_{\frac{1}{2}}^C(\xi) = \rho_{\frac{1}{2}}^{(0)} (1 + \Phi\langle \xi \rangle + \dots) \tag{108}$$

[we use angular brackets to emphasize linear dependence: $\Phi\langle \xi \rangle$ is some linear function of $\xi \in \mathfrak{h}$

Eq. (105) implies that $\rho_{\frac{1}{2}}^{(0)}$ satisfies the Master Equation. In this Section we will use $\rho_{\frac{1}{2}}^{(0)}$ to define the odd Laplace operator on functions:

$$\Delta = \Delta_{\rho_{\frac{1}{2}}^{(0)}} \tag{109}$$

Eq. (104) implies that $\rho_{\frac{1}{2}}^{(0)}$ is H -invariant; using Eq. (16) we derive:

$$\mathcal{L}_{\{F, _ \}} \rho_{\frac{1}{2}}^{(0)} = (-)^{\overline{F}} \Delta_{\text{can}} \left(F \rho_{\frac{1}{2}}^{(0)} \right) = 0 \tag{110}$$

Therefore the Hamiltonians generating \mathfrak{h} should be all Δ -closed. Moreover, Eq. (105) implies:

$$\Delta \Phi\langle \xi \rangle = \xi \tag{111}$$

so they are actually all Δ -exact. Eq. (104) implies:

$$\{\Delta \Phi\langle \xi \rangle, \Phi\langle \eta \rangle\} = \Phi\langle [\xi, \eta] \rangle \tag{112}$$

The image of Φ , as a linear statistics-reversing map $\mathfrak{h} \rightarrow \text{Fun}(M)$, will be called \mathcal{F} . Notice that the inverse map to $\Phi : \mathfrak{h} \rightarrow \mathcal{F}$ is $\Delta : \mathcal{F} \rightarrow \mathfrak{h}$. To summarize:

$$\mathfrak{h} = \Pi\Delta\mathcal{F} \tag{113}$$

5.4.2. *Properties of subspace \mathcal{F}*

Therefore the existence of equivariant analogue of Ω implies the existence of a subspace $\mathcal{F} \subset \text{Fun}(M)$ satisfying some special properties, which we will now study. Let us define the bracket:

$$[x, y] = \{x, \Delta y\} \tag{114}$$

which satisfies $\Delta[x, y] = \{\Delta x, \Delta y\}$. Eq. (112) implies that:

$$[x, y] \in \mathcal{F} \tag{115}$$

$$\text{and } [x, y] = (-)^{\bar{x}\bar{y}+1}[y, x] \tag{116}$$

In other words:

- \mathcal{F} is closed under $[_, _]$ and
- $[_, _]$ is antisymmetric

Moreover:

- the Jacobi identity for $\{_, _ \}$, implies the Jacobi identity for $[_, _]$; in other words $\Pi\mathcal{F}$ is a Lie superalgebra (isomorphic to \mathfrak{h})

It is enough to consider the case when \mathcal{F} is even (all Φ are even). Then Eq. (112) implies:

$$\Delta\{\Phi, \Phi\} = 0 \tag{117}$$

where $\Phi = \Phi(\xi)$ for some odd ξ . Moreover, Eq. (105) at the second order of the ξ -expansion implies that $\{\Phi, \Phi\}$ is Δ -exact. In other words, exists a map

$$q : \mathcal{F} \otimes \mathcal{F} \rightarrow \text{Fun}(M) \tag{118}$$

such that:

$$\{x, y\} = \Delta q(x, y) \tag{119}$$

5.4.3. *Higher orders of ξ -expansion*

We can always write, without any further assumptions:

$$\rho_{\frac{1}{2}}^C(\xi) = e^{a(\xi)} \rho_{\frac{1}{2}}^0 \tag{120}$$

So defined $a(\xi)$ should satisfy (we abbreviate $\Phi(\xi)$ to just Φ):

$$\Delta a + \frac{1}{2}\{a, a\} = \Delta\Phi \tag{121}$$

$$\{\Delta\Phi(\eta), a(\xi)\} = \left. \frac{d}{dt} \right|_{t=0} a(e^{t \text{ad}_\eta} \xi) \tag{122}$$

The first two terms in the ξ -expansion (which is the same as Φ -expansion) are:

$$a = \Phi - \frac{1}{2}q(\Phi, \Phi) + \dots \tag{123}$$

The consistency of Eq. (121) requires that $\Delta\{a, a\} = 0$ at every order in Φ . This automatically implies:

$$\Delta\{a, a\} = 2\{\Delta a, a\} = -\{\{a, a\}, a\} + 2\{\Delta\Phi, a\} \tag{124}$$

The first term $-\{\{a, a\}, a\}$ is automatically zero because of the Jacobi identity. The vanishing of $2\{\Delta\Phi, a\}$ follows from Eq. (122):

$$\{\Delta\Phi, a(\Phi)\} = a(\{\Delta\Phi, \Phi\}) = a(0) = 0 \tag{125}$$

Therefore Δ -closedness of $\{a, a\}$ is automatically satisfied order by order. But its Δ -exactness is a nontrivial **additional assumption** which has to be verified on case-by-case basis. Moreover, $\{a, a\}$ should be Δ -exact in an equivariant manner, *i.e.* satisfying Eq. (122). To the first order in Φ :

$$a = \Phi - \frac{1}{2}\Delta^{-1}\{\Phi - \frac{1}{2}\Delta^{-1}\{ \quad \Phi - \frac{1}{2}\Delta^{-1}\{\Phi, \Phi\}, \\ \Phi - \frac{1}{2}\Delta^{-1}\{\Phi, \Phi\} \quad \}, \\ \Phi - \frac{1}{2}\Delta^{-1}\{ \quad \Phi - \frac{1}{2}\Delta^{-1}\{\Phi, \Phi\}, \\ \Phi - \frac{1}{2}\Delta^{-1}\{\Phi, \Phi\} \quad \} + o(\Phi^4) \tag{126}$$

The terms of the order Φ^3 are:

$$\frac{1}{2}\Delta^{-1}\{\Phi, \Delta^{-1}\{\Phi, \Phi\}\} = \frac{1}{2}\Delta^{-1}\{\Phi, q(\Phi, \Phi)\} \tag{127}$$

The terms of the order Φ^4 are:

$$-\frac{1}{2}\Delta^{-1}\left(\{\Phi, \Delta^{-1}\{\Phi, \Delta^{-1}\{\Phi, \Phi\}\}\} + \frac{1}{4}\{\Delta^{-1}\{\Phi, \Phi\}, \Delta^{-1}\{\Phi, \Phi\}\}\right) \tag{128}$$

Notice that:

$$\Delta\left(\{\Phi, \Delta^{-1}\{\Phi, \Delta^{-1}\{\Phi, \Phi\}\}\} + \frac{1}{4}\{\Delta^{-1}\{\Phi, \Phi\}, \Delta^{-1}\{\Phi, \Phi\}\}\right) = \tag{129}$$

$$= \{\Delta\Phi, \Delta^{-1}\{\Phi, \Delta^{-1}\{\Phi, \Phi\}\}\} \tag{130}$$

This is zero by our assumption that the function $\Phi \mapsto \Delta^{-1}\{\Phi, \Delta^{-1}\{\Phi, \Phi\}\}$ is **h**-invariant. This means that in Eq. (127) we are taking Δ^{-1} of a Δ -closed expression. We must assume that this expression is also Δ -exact, so we could take Δ^{-1} of it. This assumption has to be verified order by order.

5.5. Summary: constraints on gauge symmetry

The symmetry group H with Lie algebra \mathfrak{h} acts on the odd symplectic manifold M (the BV phase space), in such a special way that the following requirements are satisfied.

We require to exist the solution of Eqs. (104) and (105) of the form of Eq. (120). We require that $a(\xi)$ be smooth at $\xi = 0$. This implies that $\rho_{\frac{1}{2}}^{(0)}$ defined in Eq. (108) satisfies the Quantum Master Equation: $\Delta_{\text{can}}\rho_{\frac{1}{2}}^{(0)} = 0$ and moreover that $\rho_{\frac{1}{2}}^{(0)}$ is \mathfrak{h} -invariant. This means that every generator of $\xi \in \mathfrak{h}$ is Δ -closed, where $\Delta = \Delta_{\rho_{\frac{1}{2}}^{(0)}}$ is the odd Laplace operator on functions constructed from $\rho_{\frac{1}{2}}^{(0)}$. Moreover, we assume that every element of $\xi \in \mathfrak{h}$ is actually Δ -exact, i.e. exists $\Phi(\xi)$ such that:

$$\xi = \Delta\Phi(\xi) \tag{131}$$

The linear subspace \mathcal{F} of $\text{Fun}(M)$ generated by $\Phi(\xi)$, $\xi \in \mathfrak{h}$, should satisfy the properties described in Section 5.4.2. (In simplest cases all elements of \mathcal{F} are in involution with each other.) Moreover, we must require the existence of a solution a to Eqs. (121) and (122).

5.6. Deformations of equivariant half-density

What can we say about the moduli space of solutions of Eqs. (104), (105)? Suppose that we can find another solution $\tilde{\rho}_{\frac{1}{2}}^{\text{C}}(\xi)$:

$$\tilde{\rho}_{\frac{1}{2}}^{\text{C}}(\xi) = R(\xi)\rho_{\frac{1}{2}}^{\text{C}}(\xi) \tag{132}$$

$$R \in \text{Fun}(\mathfrak{h} \times M) \tag{133}$$

Eqs. (104), (105) imply that R should satisfy:

$$\mathcal{L}_{\xi} R(\eta) = \left. \frac{d}{dt} \right|_{t=0} R(e^{t \text{ad}_{\xi}} \eta) \tag{134}$$

$$\Delta_{\rho_{\frac{1}{2}}^{\text{C}}(\xi)} R(\xi) = 0 \tag{135}$$

This means that infinitesimal deformations of $\rho_{\frac{1}{2}}^{\text{C}}$ are determined by the cohomology of $\Delta_{\rho_{\frac{1}{2}}^{\text{C}}(\xi)}$ in the space of \mathfrak{h} -invariant functions $\mathfrak{h} \rightarrow \text{Fun}(M)$.

Let us assume that $R(\xi)$ is a smooth function of ξ and can be expanded in series in ξ . Suppose that the leading term is of the order n in ξ (in other words, R has a zero of the order n at $\xi = 0$):

$$R(\xi) = R_n(\xi^{\otimes n}) + R_{n+1}(\xi^{\otimes n+1}) + \dots \tag{136}$$

Eq. (135) implies:

$$\Delta R_n = 0 \tag{137}$$

where $\Delta = \Delta_{\rho_{\frac{1}{2}}^{(0)}} = \Delta^{(0)} + \{S_{\text{BV}}, _ \}$. Therefore we can think of R_n as a map:

$$S^n \mathfrak{h} \xrightarrow{R_n} [\text{integrated vertex operators of ghost number } -2n] \tag{138}$$

commuting with the action of \mathfrak{h} . This means that the space of deformations can be computed by the spectral sequence whose initial page is the cohomology of Δ in the space of invariant polynomials on \mathfrak{h} with values in functions on M ; the ghost number should correlate with the degree of polynomial as in Eq. (138).

5.7. Base form

Let $\Omega_{\mathfrak{h}}^C$ denote the equivariant form, given by Eq. (120). We will denote the corresponding base form $\Omega_{\mathfrak{h}}^{\text{base}}$:

$$\begin{array}{ccc} & \text{LAG} & \\ & \downarrow & \\ \Omega_{\mathfrak{h}}^{\text{base}} \in \Omega^\bullet(\text{LAG}) & \text{base for} & \downarrow \end{array} \tag{139}$$

$$H \backslash \text{LAG}$$

We will work under the assumption that the action of \mathfrak{h} on LAG does not have fixed points; $\text{LAG} \rightarrow H \backslash \text{LAG}$ can be considered a principal H -bundle. In order to construct the base form $\Omega_{\mathfrak{h}}^{\text{base}}$ from the Cartan’s $\Omega_{\mathfrak{h}}^C$, we first choose on this principal bundle some connection \mathcal{A} . (We understand the connection as a \mathcal{F} -valued 1-form on LAG computing the “vertical component” of a vector.) Then we apply the horizontal projection i.e. replace

$$d\widehat{g}g^{-1} \mapsto d\widehat{g}g^{-1} - \mathcal{A} \tag{140}$$

$$\text{in } \Omega_{\mathfrak{h}}^C = \int_{gL} e^{d\widehat{g}g^{-1}} \Gamma(\Phi) \rho_{\frac{1}{2}} \tag{141}$$

Finally, we replace Φ with the curvature $d\mathcal{A} + \mathcal{A}^2$ of the connection \mathcal{A} ; we get:

$$\Omega_{\mathfrak{h}}^{\text{base}} = \int_{gL} e^{d\widehat{g}g^{-1} - \mathcal{A}} \Gamma(d\mathcal{A} + \mathcal{A}^2) \rho_{\frac{1}{2}} \tag{142}$$

5.8. Example: Duistermaat–Heckman integration

Consider the case when our odd symplectic submanifold is the odd cotangent bundle of some symplectic supermanifold X :

$$M = \Pi T^* X \tag{143}$$

Let $p : M \rightarrow X$ be the projection. Let π denote the Poisson bivector of X . It defines the following solution of the Quantum Master Equation:

$$\rho_{\frac{1}{2}}(x, x^*, \mathbf{B}) = \exp(\pi^{ij}(x)x_i^*x_j^*) \tag{144}$$

$$\text{where } \mathbf{B} = \{e_1, \dots, e_n, f^1, \dots, f^n, \epsilon^{\vee 1}, \dots, \epsilon^{\vee n}, \eta_{\vee 1}, \dots, \eta_{\vee n}\} \tag{145}$$

$$\text{where } \{e_1, \dots, e_n, f^1, \dots, f^n\} \text{ is a Darboux basis in } T_x X \tag{146}$$

$$\text{and } \{\epsilon^{\vee 1}, \dots, \epsilon^{\vee n}, \eta_{\vee 1}, \dots, \eta_{\vee n}\} \text{ its dual} \tag{147}$$

In this formula we define a half-density at $m = (x, x^*) \in M$ as a function on the space of bases (“tetrads”) in $T_m M \simeq T_x X \oplus \Pi T_x^* X$, such that the value on two different bases differs by a multiplication by the super-determinant.

Let H be a subgroup of the group of canonical transformations of X . A canonical transformation h of X can be lifted to a BV-canonical transformation ΠT^*h of ΠT^*X as follows:

$$(\Pi T^*h)(\xi, x) = ((h^{-1}(x)_*)^*\xi, h(x)) \tag{148}$$

An infinitesimal canonical transformation of X generated by a Hamiltonian F lifts to an infinitesimal BV-canonical transformation of ΠT^*X generated by the BV-Hamiltonian $\Delta_{\rho_{\frac{1}{2}}}(p^*F)$. Every point $x \in X$ defines a Lagrangian submanifold in ΠT^*X : the fiber ΠT_x^*X . Therefore, we can think of X as a family of Lagrangian submanifolds in ΠT^*X . In this case \mathcal{F} is the p^* of the subspace of Hamiltonians generating H . The equivariant form Ω becomes:

$$\Omega(x, [dx], F) = \int_{\Pi T_x^*X} \rho_{\frac{1}{2}} \exp\left([dx]^i x_i^* + F(x)\right) = \tag{149}$$

$$= \exp\left(\varpi_{ij}(x)[dx]^i [dx]^j + F(x)\right) \tag{150}$$

where $\varpi = \pi^{-1}$.

5.9. Example: an interpretation of Cartan complex

Let us consider the case:

$$M = \Pi T^*(\Pi T X) \tag{151}$$

The odd tangent bundle $\Pi T X$ has a natural volume form. This induces a half-density on M . Let $\rho_{\frac{1}{2}}^{(0)}$ be the half-density on M induced from the volume form on $\Pi T X$. It satisfies the Quantum Master Equation. Now consider a different half-density, which also satisfies the Quantum Master Equation⁶:

$$\rho_{\frac{1}{2}} = e^{\widehat{d}} \rho_{\frac{1}{2}}^{(0)} \tag{152}$$

where \widehat{d} is the BV Hamiltonian generating the lift to $\Pi T^*(\Pi T X)$ of the canonical odd vector field d on $\Pi T X$.

Suppose that a Lie group H acts on X . This action can be lifted to $\Pi T X$ and $\Pi T^*(\Pi T X)$. An infinitesimal action of $\xi \in \mathfrak{h} = \text{Lie}(H)$ on M is generated by the BV Hamiltonian:

$$\widehat{\mathcal{L}}_{\xi} = \{\widehat{d}, \widehat{t}_{\xi}\} \tag{153}$$

Therefore, in this case:

$$a(\xi) = \Phi\langle \xi \rangle = \widehat{t}_{\xi} \tag{154}$$

$$\rho_{\frac{1}{2}}^c(\xi) = e^{\widehat{d} + \widehat{t}_{\xi}} \rho_{\frac{1}{2}}^{(0)} \tag{155}$$

It satisfies Eq. (105):

$$\Delta \left(e^{\widehat{d} + \widehat{t}_{\xi}} \rho_{\frac{1}{2}}^{(0)} \right) = \widehat{\mathcal{L}}_{\xi} e^{\widehat{d} + \widehat{t}_{\xi}} \rho_{\frac{1}{2}}^{(0)} \tag{156}$$

⁶ We can replace $e^{\widehat{d}}$ with an arbitrary function of \widehat{d} . This will still satisfy the Quantum Master Equation.

This formula can be generalized as follows. Suppose that α is a function on $\mathfrak{h} \times \Pi T X$. For any $\xi \in \mathfrak{h}$, we think of $\alpha(\xi)$ as a function on $\Pi T X$, i.e. a pseudo-differential form on X . With a slight abuse of notations, $\alpha(\xi)$ will also denote the pullback of $\alpha(\xi)$ along the projection, from $\Pi T X$ to $\Pi T^*(\Pi T X)$. Then we have:

$$\Delta \left(\alpha(\xi) e^{\widehat{d} + \widehat{\iota}_\xi} \rho_{\frac{1}{2}}^{(0)} \right) = \widehat{\mathcal{L}}_\xi \alpha(\xi) e^{\widehat{d} + \widehat{\iota}_\xi} \rho_{\frac{1}{2}}^{(0)} + \tag{157}$$

$$+ ((d + \iota_\xi) \alpha(\xi)) e^{\widehat{d} + \widehat{\iota}_\xi} \rho_{\frac{1}{2}}^{(0)} \tag{158}$$

Therefore $\alpha \mapsto \alpha e^{\widehat{d} + \widehat{\iota}}$ is an intertwiner from the Cartan complex to the complex of equivariant half-densities on $\Pi T^*(\Pi T X)$.

5.10. Equivariant effective action

When some fields are integrated out, we get an “effective action” for the remaining fields.

Mini-review of BV effective action. Let (M, ω) be an odd symplectic supermanifold (the BV phase space), and $\rho_{\frac{1}{2}}$ be a half-density on M . Suppose that we are given a submanifold $E \subset M$ which comes with the structure of a fiber bundle with fibers isotropic submanifolds over some base B . (This can be thought of as a family of isotropic submanifolds.) For every $b \in B$ we have the corresponding fiber, an isotropic submanifold which we denote $I(b)$. Moreover, we require:

$$\ker \omega|_E = TI \tag{159}$$

(i.e. the degenerate subspace of the restriction of ω to E is the tangent space of the fiber). Since $d\omega = 0$, this condition implies that the Lie derivative of $\omega|_E$ along the fiber vanishes, and therefore ω defines an odd symplectic form on the base which we will denote ω_B .

Finally, we require certain maximality property:

$$\begin{aligned} E \text{ cannot be embedded in any larger} \\ \text{submanifold of } M \text{ where } TI \text{ would still be isotropic} \end{aligned} \tag{160}$$

Let $L \subset B$ be a Lagrangian submanifold of B . Under the above conditions, it can be lifted to a Lagrangian submanifold in M as $\pi^{-1}L$, where π is a natural projection:

$$\pi : E \rightarrow B \tag{161}$$

We then define a half-density $\pi_* \rho_{\frac{1}{2}}$ on B so that for every Lagrangian submanifold $L \subset B$ and every function $f \in \text{Fun}(B)$:

$$\int_L f \pi_* \rho_{\frac{1}{2}} = \int_{\pi^{-1}(L)} (f \circ \pi) \rho_{\frac{1}{2}} \tag{162}$$

The proof of existence of such $\pi_* \rho_{\frac{1}{2}}$ is similar to the proof of [Theorem 1](#). We define a measure $\mu_L[\rho_{\frac{1}{2}}]$ on L as push-forward of $\rho_{\frac{1}{2}}|_{\pi^{-1}L}$ to L . We observe that for any canonical transformation g bringing fibers of E into fibers: $g^* \mu_{gL}[\rho_{\frac{1}{2}}] = \mu_L[g^* \rho_{\frac{1}{2}}]$. Indeed:

$$\begin{aligned} \int_L (f \circ g) g^* \mu_{gL}[\rho_{\frac{1}{2}}] &= \int_{gL} f \mu_{gL}[\rho_{\frac{1}{2}}] = \int_{\pi^{-1}(gL)} (f \circ \pi) \rho_{\frac{1}{2}} = \\ &= \int_L (f \circ g \circ \pi) g^* \rho_{\frac{1}{2}} \end{aligned} \tag{163}$$

Then we define a half-density $\sigma_{\frac{1}{2}}[L, \rho_{\frac{1}{2}}]$ on B so that $\mu_L[\rho_{\frac{1}{2}}]$ is its restriction to L . The same argument as in [Appendix B](#) shows that $\sigma_{\frac{1}{2}}[L, \rho_{\frac{1}{2}}]$ does not depend on L . We define $\pi_* \rho_{\frac{1}{2}} = \sigma_{\frac{1}{2}}[L, \rho_{\frac{1}{2}}]$.

Suppose that we are given a function Ψ on M whose restriction on E is constant along I . Then it defines a function on B which we denote $\pi_* \Psi$:

$$(\pi_* \Psi) \circ \pi = \Psi|_E \tag{164}$$

Notice that in this case the flux $\{\Psi, _ \}$ is tangent to E .

It is enough to prove that for any $\phi \in \text{Fun}(M)$ constant on E : $\{\Psi, \phi\} = 0$. From the maximality of B , as defined in [Eq. \(160\)](#), follows that $\{\phi, _ \}$ is tangent to I . Since Ψ is constant along I , it follows that indeed $\{\Psi, \phi\} = 0$.

We have:

$$\forall m \in E : \pi(m)_* (\{\Psi, _ \}(m)) = \{\pi_* \Psi, _ \}(\pi(m)) \tag{165}$$

Equivalently, for two functions Ψ_1 and Ψ_2 on M both constant along I :

$$\{\Psi_1, \Psi_2\} = \{\pi_* \Psi_1, \pi_* \Psi_2\} \circ \pi \tag{166}$$

In order to prove [Eq. \(165\)](#), notice that any vector field tangent to I can be written as $\sum_a f_a \{\phi_a, _ \}$ where f_a are some functions on M and ϕ_a are some functions on M constant on E . The commutator with $\{\Psi, _ \}$ of such a vector field is $\sum_a \{\Psi, f_a\} \{\phi_a, _ \}$ — again tangent to I . This means that the flow of Ψ brings fibers to fibers, which is equivalent to [Eq. \(165\)](#).

We will now prove that for any $\Psi \in \text{Fun}(M)$ whose restriction on E is constant along I :

$$\pi_* \left(\mathcal{L}_{\{\Psi, _ \}} \rho_{\frac{1}{2}} \right) = \mathcal{L}_{\{\pi_* \Psi, _ \}} \pi_* \rho_{\frac{1}{2}} \tag{167}$$

Indeed, for any “test function” $f \in \text{Fun}(B)$:

$$\delta_{\{\pi_* \Psi, _ \}} \int_L f \pi_* \rho_{\frac{1}{2}} = \int_L \left(\{\pi_* \Psi, f\} \pi_* \rho_{\frac{1}{2}} + (-)^{(\bar{\Psi}+1)\bar{f}} f \mathcal{L}_{\{\pi_* \Psi, _ \}} \pi_* \rho_{\frac{1}{2}} \right) = \tag{168}$$

$$= \delta_{\{\Psi, _ \}} \int_{\pi^{-1}L} (f \circ \pi) \rho_{\frac{1}{2}} = \int_{\pi^{-1}L} \left(\{\Psi, f \circ \pi\} \rho_{\frac{1}{2}} + (-)^{(\bar{\Psi}+1)\bar{f}} f \mathcal{L}_{\{\Psi, _ \}} \rho_{\frac{1}{2}} \right) = \tag{169}$$

$$= \int_L \left(\{\pi_* \Psi, f\} \pi_* \rho_{\frac{1}{2}} + (-)^{(\bar{\Psi}+1)\bar{f}} f \pi_* \left(\mathcal{L}_{\{\Psi, _ \}} \rho_{\frac{1}{2}} \right) \right) \tag{170}$$

Equality of [Lines \(168\)](#) and [\(170\)](#) implies [Eq. \(167\)](#). In particular: if $\rho_{\frac{1}{2}}$ satisfies the Quantum Master Equation on M , then $\pi_* \rho_{\frac{1}{2}}$ satisfies the Quantum Master Equation on B . Indeed, $\rho_{\frac{1}{2}}$

satisfying the Quantum Master Equation is equivalent to the statement that for any functions Ψ and F :

$$\delta_{\{\Psi, \cdot\}} \int_L F \rho_{\frac{1}{2}} = (-)^{\bar{F}+1} \int_L \Psi \mathcal{L}_{\{F, \cdot\}} \rho_{\frac{1}{2}} \tag{171}$$

When $\rho_{\frac{1}{2}}$ satisfies the QME on M , considering Eq. (171) with both Ψ and F constant along the fiber of $E \rightarrow B$ and using Eq. (167) proves that $\pi_* \rho_{\frac{1}{2}}$ also satisfies the QME.

Equivariant case. Now we will consider partial integration when the half-density satisfies the equivariant Master Equation. We show that the resulting effective theory still satisfies the equivariant Master Equation.

Suppose that a Lie group H acts on M by canonical transformations, and moreover that every Hamiltonian $\xi \in \mathfrak{h}$ is constant on the fibers of $E \xrightarrow{\pi} B$:

$$\forall \xi \in \mathfrak{h} \exists \pi_* \xi \in \text{Fun}(B) : \xi|_E = (\pi_* \xi) \circ \pi \tag{172}$$

Suppose that we are given an equivariant half-density $\xi \mapsto \rho_{\frac{1}{2}}^C(\xi)$ on M , i.e.:

$$\mathcal{L}_\eta \rho_{\frac{1}{2}}^C(\xi) = \left. \frac{d}{dt} \right|_{t=0} \rho_{\frac{1}{2}}^C(e^{t\{\eta, \cdot\}} \xi) \tag{173}$$

$$\Delta \rho_{\frac{1}{2}}^C(\xi) = \xi \rho_{\frac{1}{2}}^C(\xi) \tag{174}$$

Then the result of the partial integration is an equivariant half-density on B :

$$\mathcal{L}_\eta \pi_* \rho_{\frac{1}{2}}^C(\xi) = \left. \frac{d}{dt} \right|_{t=0} \pi_* \rho_{\frac{1}{2}}^C(e^{t\{\eta, \cdot\}} \xi) \tag{175}$$

$$\Delta \pi_* \rho_{\frac{1}{2}}^C(\xi) = (\pi_* \xi) \pi_* \rho_{\frac{1}{2}}^C(\xi) \tag{176}$$

Indeed, Eq. (174) is equivalent to the statement that for any two functions Ψ and F :

$$\delta_{\{\Psi, \cdot\}} \int_L F \rho_{\frac{1}{2}}^C(\xi) = (-)^{\bar{F}+1} \int_L \left(\Psi \mathcal{L}_{\{F, \cdot\}} \rho_{\frac{1}{2}}^C(\xi) + \Psi F \xi \rho_{\frac{1}{2}}^C(\xi) \right) \tag{177}$$

Eq. (176) follows from Eqs. (167) and (177) considering the case when both Ψ and F are constant along the fiber of $E \rightarrow B$.

In this sense, the property of solving the equivariant Master Equation (104), (105) survives passing to effective action.

6. BRST formalism

We do not have a complete description of symmetry groups H satisfying the properties summarized in Section 5.5. But we do understand the special case, when BV formalism comes from BRST formalism.

6.1. Brief review of BRST formalism

One starts with the “classical action” S_{cl} which is invariant under some gauge symmetry. Let X be the “classical” space of fields, e.g. for the Yang–Mills theory the fields are $A_\mu(x)$.

Suppose the gauge symmetry is H , with the Lie algebra $\mathfrak{h} = \text{Lie}(H)$. We introduce ghost fields, geometrically:

$$\text{fields and ghosts} \in \frac{\Pi T H \times X}{H} \tag{178}$$

where the action of $h_0 \in H$ is via right shift on H :

$$h_0(dh, h, x) = (dh h_0^{-1}, h h_0^{-1}, h_0 \cdot x) \tag{179}$$

We use the coordinates dh, h on $\Pi T H$ (denoting “ dh ” the coordinate on the fiber $\Pi T_h H$) and x on X . Notice that this commutes with $Q_{\text{BRST}} = dh \frac{\partial}{\partial h}$. We can always find a representative with $h = 1$ (i.e. choose $h_0 = h^{-1}$). Then, with the standard notation $c = dh$:

$$Q_{\text{BRST}} = \frac{1}{2} f_{AB}{}^C c^A c^B \frac{\partial}{\partial c^C} + c^A v_A^i \frac{\partial}{\partial x^i} \tag{180}$$

Functions on $\frac{\Pi T H \times X}{H}$ satisfy:

$$f(dh h_0^{-1}, h h_0^{-1}, h_0 \cdot x) = f(dh, h, x) \tag{181}$$

$$(Q_{\text{BRST}} f)(dh, 1, x) = \frac{\partial}{\partial \epsilon} f(dh, 1 + \epsilon dh, x) = \frac{\partial}{\partial \epsilon} f(dh + \epsilon dh dh, 1, (1 + \epsilon dh)x) \tag{182}$$

6.2. Integration measure

We assume that X/H comes with some integration measure:

$$\mu = e^{S_{\text{cl}}(x)} \tag{183}$$

This μ be understood as an integration measure, i.e. a density of weight 1, rather than a function of x . The product of this measure with the canonical measure on $\Pi T H$ gives us a measure on $\frac{\Pi T H \times X}{H}$ which we will call μ_{BRST} . Notice that Q_{BRST} preserves this measure. This can be proven as follows. For any function $f \in \text{Fun}(\frac{\Pi T H \times X}{H})$:

$$\int_{\frac{\Pi T H \times X}{H}} \mu_{\text{BRST}} Q_{\text{BRST}} f = 0 \tag{184}$$

because Q_{BRST} comes from the canonical odd vector field on $\Pi T H$.

6.3. Lift of symmetries to BRST configuration space

Original gauge symmetries can be lifted to the BRST field space as *left* shifts on H :

$$h_0 \cdot (dh, h, x) = (h_0 dh, h_0 h, x) \tag{185}$$

These left shifts commute with the right shifts used in Eq. (179), therefore they act consistently on the factorspace. They also commute with Q_{BRST} . Moreover, infinitesimal left shifts L_ξ are actually BRST-exact:

$$\frac{\partial}{\partial t} \Big|_{t=0} (e^{t\xi} dh, e^{t\xi} h, x) = \left[(\xi h) \frac{\partial}{\partial dh}, dh \frac{\partial}{\partial h} \right] (dh, h, x) \tag{186}$$

$$\text{therefore } L_\xi = \left[(\xi h) \frac{\partial}{\partial dh}, Q_{\text{BRST}} \right] \tag{187}$$

The measure on $\frac{\Pi T H \times X}{H}$ is invariant under left shifts, because it is constructed from the canonical measure on $\Pi T H$ which is invariant.

6.4. BV from BRST

The BV phase space is:

$$M = \Pi T^* \left(\frac{\Pi T H \times X}{H} \right) \tag{188}$$

The zero section $\frac{\Pi T H \times X}{H}$ is a Lagrangian submanifold. It comes with the integration measure, which lifts to a half-density on M of the form $e^{S_{\text{BV}}}$, where:

$$S_{\text{BV}} = S_{\text{cl}}(x) + (Q_{\text{BRST}} c^A) c_A^* + (Q_{\text{BRST}} x^i) x_i^* \tag{189}$$

where Q_{BRST} is as defined in Eq. (180).

We can imagine a more general situation when we have a functional S_{cl} with an odd symmetry Q_{BRST} nilpotent off-shell. But, just to describe the “standard BRST formalism”, we explicitly break the fields into c^A and x^i .

We have realized the gauge algebra \mathfrak{h} as symmetries of the BRST configuration space as left shifts on $\Pi T H$. Since the BV phase space is the odd cotangent bundle, we can further lift them to the BV phase space. The symmetry corresponding to the infinitesimal left shift (185) is generated by the BV Hamiltonian:

$$H(\xi) = \Delta(\xi^\alpha c_\alpha^*) \tag{190}$$

In other words:

$$\mathcal{F} \text{ is generated by } c_\alpha^* \tag{191}$$

6.5. Form Ω in BRST formalism

Although the “BRST formalism in the proper sense of this word” requires splitting ϕ into “physical fields” φ and “ghosts” c , in many cases such a split is not required and does not play any role. Let us forget about the split for a moment; just require that BRST operator is nilpotent off-shell:

$$Q_{\text{BRST}}^2 = 0 \text{ off-shell} \tag{192}$$

But the split into φ and c will come back very soon in Section 6.6.

Form Ω in BRST formalism is given by the following expression:

$$\Omega = \int [d\phi] \exp (S_{\text{cl}} + Q_{\text{BRST}} \Psi + d\Psi) \tag{193}$$

$$d\Omega = 0 \tag{194}$$

The family Λ is a family of gauge fermions $\Psi(\phi)$. To prove $d\Omega = 0$ we use that $\int [d\phi] Q_{\text{BRST}}(\dots) = 0$ (no BRST anomaly). Notice that we deform the action $S_{\text{cl}} \rightarrow S_{\text{cl}} + Q_{\text{BRST}}\Psi$ but do not deform Q_{BRST} .

Generally speaking, we would like to treat as “gauge symmetries” those Ψ which are BRST exact plus equations of motion (here “equations of motion” are those derived from $S_{\text{cl}} + Q_{\text{BRST}}\Psi$). In other words, for any vector field ζ on the field space and any functional F the following $\delta\Psi$ should correspond to a symmetry:

$$\delta_{F,\zeta}\Psi = Q_{\text{BRST}}F + \mathcal{L}_\zeta S_{\text{cl}} + [\mathcal{L}_\zeta, Q_{\text{BRST}}]\Psi \tag{195}$$

This is just a generic field redefinition plus adding a BRST-exact term⁷

No horizontality. The form Ω given by Eq. (193) is not horizontal:

$$\begin{aligned} \iota_{\delta_{F,\zeta}}\Omega &= \int [d\phi] (Q_{\text{BRST}}F + \mathcal{L}_\zeta S_{\text{cl}} + [Q_{\text{BRST}}, \mathcal{L}_\zeta]\Psi) \\ &\quad \times \exp(S_{\text{cl}}(\phi) + Q_{\text{BRST}}\Psi + d\Psi) \end{aligned} \tag{196}$$

Cartan form. In order to construct equivariant (and then base) form, we will need to restrict ζ and F to belong to some linear subspaces:

$$\zeta \in \mathcal{F}_{\text{Vect}} \subset \text{Vect}(\text{space of fields } \phi) \tag{197}$$

$$F \in \mathcal{F}_{\text{Fun}} \subset \text{Fun}(\text{space of fields } \phi) \tag{198}$$

We will need to require that these subspaces are such that for any $\zeta \in \mathcal{F}_{\text{Vect}}$ and $F \in \mathcal{F}_{\text{Fun}}$:

- the following conditions (analogous to $\{\Phi, \Phi\} = 0$) hold:

$$\mathcal{L}_\zeta^2 = 0 \tag{199}$$

$$\mathcal{L}_\zeta F = 0 \tag{200}$$

- transformations $\delta_{F,\zeta}$ form a closed Lie superalgebra
- the map $\zeta \mapsto [\mathcal{L}_\zeta, Q_{\text{BRST}}]$ is injective (*i.e.* nothing goes to zero)

Then the following expression:

$$\Omega^c(\Psi, d\Psi, F, \zeta) = \int [d\phi] \exp(S_{\text{cl}} + (Q_{\text{BRST}} + d + \mathcal{L}_\zeta)\Psi + F) \tag{201}$$

is annihilated by the Cartan differential.

Proof. Eqs. (199) and (200) imply:

$$(Q_{\text{BRST}} + d + \mathcal{L}_\zeta)^2 = [Q_{\text{BRST}}, \mathcal{L}_\zeta] \tag{202}$$

Therefore:

⁷ The second term on the RHS, $\mathcal{L}_\zeta S_{\text{cl}} + [\mathcal{L}_\zeta, Q_{\text{BRST}}]\Psi$, can be interpreted as the term $\mathcal{L}_\zeta(S_{\text{cl}} + Q_{\text{BRST}}\Psi)$ which vanishes on-shell, plus $Q_{\text{BRST}}(\mathcal{L}_\zeta\Psi)$ which could in principle be absorbed by a shift of F .

$$d \int [d\phi] \exp (S_{cl}(\varphi) + (Q_{BRST} + d + \mathcal{L}_\zeta)\Psi + F) = \tag{203}$$

$$= \int [d\phi] (Q_{BRST} + d + \mathcal{L}_\zeta) \exp (S_{cl}(\varphi) + (Q_{BRST} + d + \mathcal{L}_\zeta)\Psi + F) = \tag{204}$$

$$= \int [d\phi] (Q_{BRST}F + \mathcal{L}_\zeta S_{cl} + [Q_{BRST}, \mathcal{L}_\zeta]\Psi) \times \exp (S_{cl}(\varphi) + (Q_{BRST} + d + \mathcal{L}_\zeta)\Psi + F) \tag{205}$$

In passing from (203) to (204) we assumed that both Q_{BRST} and ζ preserve the measure of integration $[d\phi]$, in other words $\int [d\phi] (Q_{BRST} + \mathcal{L}_\zeta)(\dots) = 0$. To complete the proof, we notice that the last line coincides with Eq. (196).

6.6. Lifting the gauge symmetry to BRST formalism

As we have just explained, our ζ and F are restricted to belong to some subspace $\mathcal{F}_{\text{Vect}} \oplus \mathcal{F}_{\text{Fun}}$, which should satisfy certain conditions. A geometrically natural solution to these conditions can be found in the case of “traditional” BRST formalism where the fields ϕ are split into physical fields φ and ghosts c . It corresponds to the following choice of ζ and F :

$$F = 0 \tag{206}$$

$$\zeta^A(\phi) \frac{\partial}{\partial \phi^A} = \xi^A \frac{\partial}{\partial c^A} \tag{207}$$

— constant (= field-independent) shifts of ghosts. Eq. (201) becomes:

$$\Omega^C(\Psi, d\Psi, \xi) = \int [d\phi] \exp \left(S_{cl} + Q_{BRST}\Psi + d\Psi + \xi^A \frac{\partial \Psi}{\partial c^A} \right) \tag{208}$$

6.7. Faddeev–Popov integration procedure

The naive (or “standard”) Lagrangian submanifold is:

$$\varphi^* = c^* = 0 \tag{209}$$

But the restriction of S_{BV} to this Lagrangian submanifold coincides with S_{cl} , and therefore is degenerate; we cannot integrate. In order to resolve the degeneracy, we have to deform to another Lagrangian submanifold. However, we face a complication. It is desirable to keep the ghost number symmetry. But if we restrict ourselves to only those Lagrangian submanifolds which are invariant under the ghost number symmetry, then the standard one of Eq. (209) is *rigid*. It does not admit deformations. Indeed, the deforming gauge fermion should have ghost number -1 , but there are no fields with negative ghost number. Therefore, there are no deformations.

One solution to this problem is to introduce non-minimal fields (the BRST quartet). Here we will describe another solution (giving the same answer), based on the consideration of families of Lagrangian submanifolds. It turns out that exist ghost number preserving *families* of Lagrangian submanifolds deforming the standard one. We can then use our form Ω to integrate over these families, thus obtaining a regularized theory with ghost number symmetry. Let us outline the construction of such families. Let X be the space of fields φ . Suppose that we have a linear space V and a map $F : X \rightarrow V$ such that the orbits of the symmetry are transversal to the level set

$Y = F^{-1}(0)$. This F is called “gauge fixed condition”. For each $\bar{c} \in \Pi V^*$ we can consider the following section of ΠT^*X :

$$s(x) = (F_*(x))^* \bar{c} \tag{210}$$

where $F_*(x) : T_x X \rightarrow V$ is the derivative of F at the point x . The section (210) defines a Lagrangian submanifold of ΠT^*X . (When $\bar{c} = 0$ this is $X \subset \Pi T^*X$.) Therefore, we have a family of Lagrangian submanifolds parametrized by elements of ΠV^* . Let us integrate our PDF Ω , which in this case is equal to:

$$\Omega(\bar{c}, d\bar{c}) = \int_{L(\bar{c})} \exp(S_{\text{BV}} + \langle d\bar{c}, F \rangle) \tag{211}$$

over this family. The corresponding integral form (the density on ΠV^*) is:

$$I(\bar{c}) = \int_{L(\bar{c})} \delta(F) e^{S_{\text{BV}}} \tag{212}$$

This means that we have to integrate:

$$\int_{\Pi V^*} [d\bar{c}] \int_{L(\bar{c})} \delta(F) e^{S_{\text{BV}}} \tag{213}$$

$$\text{where } S_{\text{BV}}|_{L(\bar{c})} = S_{\text{cl}}(x) + \langle Q_{\text{BRST}}x, (F_*(x))^* \bar{c} \rangle = \tag{214}$$

$$= S_{\text{cl}} + c^A T_A^i \partial_i F^a \bar{c}_a \tag{215}$$

This is the standard Faddeev–Popov integral (see e.g. Chapter 8 of [15]). The integration is convergent for an appropriate choice of a contour.

Yang–Mills theory.

$$S_{\text{cl}} = \int d^4x \text{tr} (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^2 \tag{216}$$

$$Q_{\text{BRST}} A_\mu = D_\mu c \tag{217}$$

$$Q_{\text{BRST}} c = \frac{1}{2} [c, c] \tag{218}$$

The Landau gauge corresponds to the following function $F : X \rightarrow V$, where V is the space of functions on the four-dimensional spacetime:

$$F(A) = \partial^\mu A_\mu \tag{219}$$

Notice that V (and therefore our family of Lagrangian submanifolds) is now infinite-dimensional. In this case $(F_*)^* \bar{c}$ is $A^{*\mu} = \partial^\mu \bar{c}$ and:

$$S_{\text{BV}}|_{L(\bar{c})} = S_{\text{cl}}(x) + D_\mu c \partial^\mu \bar{c} \tag{220}$$

The integration contour is such that \bar{c} is complex conjugate to c . The insertion of $\delta(F)$ is usually done by means of a Lagrange multiplier.

6.8. Conormal bundle to the constraint surface

Here we will describe Lagrangian submanifolds of the type used in the bosonic string worldsheet theory. They provide another solution to the problem discussed in the beginning of Section 6.7.

Conormal bundle. Let \mathcal{Y} be some family of submanifolds $Y \subset X$ closed under the action of the gauge symmetry⁸ For each $Y \in \mathcal{Y}$, the *odd conormal bundle* of Y (denoted $\Pi(TY)^\perp$) is a subbundle of the odd cotangent bundle $\Pi T^*X|_Y$ which consists of those covectors which evaluate to zero on vectors tangent to Y . For each $Y \subset X$, the corresponding odd conormal bundle is a Lagrangian submanifold. Given such a family \mathcal{Y} , let us define a family of Lagrangian submanifolds in the BV phase space in the following way: for every Y , the corresponding Lagrangian submanifold is the odd conormal bundle of Y , times the space of c -ghosts:

$$L(Y) = \Pi(TY)^\perp \times [c\text{-ghosts}] = \Pi(TY)^\perp \times \Pi\mathbf{h} \tag{221}$$

Irreducibility and completeness. Let us ask the following question: under what conditions the restriction of S_{BV} to each $L(Y)$ is non-degenerate? Or, in case if it is degenerate, how can we characterize the degeneracy? We have:

$$S_{\text{BV}}|_{L(Y)} = S_{\text{cl}}(\varphi)|_{\varphi \in Y} + T_A^i c^A \varphi_i^* |_{\varphi^* \in \Pi(TY)^\perp} \tag{222}$$

The second term $T_A^i c^A \varphi_i^*$ is the evaluation of the covector φ^* on the tangent vector $Q\varphi \in TX$.

Let us assume that the restriction of S_{BV} to any $L(Y)$ has a critical point, and study the quadratic terms in the expansion of $S_{\text{BV}}|_L$ around that critical point. To define the perturbation theory, we need already *the quadratic terms* to be non-degenerate. Assuming that the critical point is at $\varphi = 0$:

$$S_{\text{cl}}(\varphi) = k_{ij} \varphi^i \varphi^j + o(\varphi^2) \tag{223}$$

Suppose that all degeneracies of S_{cl} and of k are due to symmetries (the *completeness*). In other words:

$$\ker k = \text{im } \tau \tag{224}$$

where $\tau : \mathbf{h} \rightarrow TX$ (225)

$$\tau^i(\xi) = T_A^i \xi^A \tag{226}$$

Let s be the quadratic part of $S_{\text{BV}}|_L$:

$$s = k_{ij} \varphi^i \varphi^j \Big|_{\varphi \in TY} + \tau^i(c) \varphi_i^* \Big|_{\varphi^* \in (TY)^\perp} \tag{227}$$

The degeneracy is characterized by the isotropic subspace of s which we denote $\ker s$:

$$\ker s = (\text{im } \tau \cap TY) \oplus \Pi \left((\text{im } \tau \cap TY) \oplus \ker \tau \oplus (\text{im } \tau + TY)^\perp \right) \tag{228}$$

Let us make the following assumptions:

⁸ This means that if $Y_1 \in \mathcal{Y}$ then for any gauge transformation h , $hY_1 \in \mathcal{Y}$.

1. The space $\text{im } \tau \cap TY$ is zero, in other words Y is transverse to the orbits of H . This is a constraint on the choice of Y .
2. The next term, $\ker \tau$, is also zero. This kernel being nonzero corresponds to “reducible” gauge symmetries.

But the last term $\Pi(\text{im } \tau + TY)^\perp$ is essentially nonzero. It can be identified with the cotangent space to our family:

$$(\text{im } \tau + TY)^\perp = T_Y^*(\mathcal{Y}/H) \tag{229}$$

where \mathcal{Y} is the moduli space of submanifolds $Y \subset X$. Therefore the quadratic part of S_{BV} is degenerate. However this degeneration is removed by the factor $e^{d\widehat{g}g^{-1}}$. Indeed, in this case:

$$d\widehat{g}g^{-1} = \varphi_i^* dy^i \tag{230}$$

When we integrate over \mathcal{Y}/H , the differentials dy^i span the complement of $\text{im } \tau$ in TX/TY . Since we require that the family \mathcal{Y} be H -closed, τ defines a map $\mathbf{h} \rightarrow TX/TY$ which we denote $[\tau]$. With these notations:

$$(\text{im } \tau + TY)^\perp = (\text{coker } [\tau])^* \tag{231}$$

In the case of bosonic string (Section 8.2) $\dim \text{coker } [\tau] = 3g - 3$ and $e^{d\widehat{g}g^{-1}}$ contributes $\prod_{i=1}^{3g-3} b^{\alpha\beta} \delta g_{\alpha\beta}^{(0)}$.

7. Integrated vertex operators

7.1. Deformations of BV action

Suppose that we infinitesimally deformed the solution of the Master Equation:

$$S_{\text{BV}}(\varepsilon) = S_{\text{BV}} + \varepsilon U \tag{232}$$

Such functions U generalize the notion of *integrated vertex operator* in string theory. In order for this to satisfy the Master Equation to the first order in ε we require:

$$\Delta U = \rho_{\frac{1}{2}}^{-1} \Delta_{\text{can}}(\rho_{\frac{1}{2}} U) = \Delta^{(0)} U + \{S_{\text{BV}}, U\} = 0 \tag{233}$$

We will also postulate that U is diffeomorphism invariant, which means in our formalism that:

$$\{\Delta\Phi, U\} = 0 \tag{234}$$

As we deform S_{BV} , what happens to \mathcal{F} ? The answer is simple if U satisfies the Siegel gauge condition:

$$\forall \Phi \in \mathcal{F} : \{\Phi, U\} = 0 \tag{235}$$

In this case \mathcal{F} remains undeformed. But if Eq. (235) is not satisfied, then \mathcal{F} should also get deformed.

7.2. Deformations of \mathcal{F}

Eqs. (233) and (234) imply that $\{U, \Phi\}$ is Δ -closed:

$$\Delta\{U, \Phi\} = 0 \tag{236}$$

Let us also require that it is Δ -exact:

$$\exists a_U : \mathcal{F} \rightarrow \text{Fun}(M) : \tag{237}$$

$$\{U, \Phi\} = -\Delta a_U(\Phi) \tag{238}$$

$$\text{and } [\Delta\Phi\langle\xi\rangle, a_U(\Phi\langle\eta\rangle)] = a_U(\Phi\langle[\xi, \eta]\rangle) \text{ (h-invariance of } a_U) \tag{239}$$

Under these assumptions we can deform:

$$\Phi \mapsto \Phi + \varepsilon a_U(\Phi) \tag{240}$$

$$\text{so that } (\Delta + \varepsilon\{U, _})(\Phi + \varepsilon a_U(\Phi)) = \Delta\Phi \tag{241}$$

In other words:

- the space \mathcal{F} does deform, according to Eq. (240), but the action of diffeomorphisms remains the same.

In particular, the BV Hamiltonian of diffeomorphisms $\Delta\Phi$ stays undeformed (Eq. (241)).

Is it true that the deformed Φ remain in involution modulo Δ -exact? Notice that the deformed $\{\Phi, \Phi\}$ is automatically $\Delta + \varepsilon\{U, _}$ -closed under already taken assumptions:

$$(\Delta + \varepsilon\{U, _})\{\Phi + \varepsilon a_U(\Phi), \Phi + \varepsilon a_U(\Phi)\} = 2\{\Delta\Phi, \Phi + \varepsilon a_U(\Phi)\} = 0 \tag{242}$$

Opening the parentheses, we derive that the expression $2\{\Phi, a_U(\Phi)\} - \{U, q(\Phi)\}$ is Δ -closed:

$$\Delta\left(2\{\Phi, a_U(\Phi)\} - \{U, q(\Phi)\}\right) = 0 \tag{243}$$

Let us assume that it is also Δ -exact:

$$2\{\Phi, a_U(\Phi)\} - \{U, q(\Phi)\} = \Delta q'(\Phi) \tag{244}$$

(the validity of this assumption depends on the cohomology of Δ). Therefore, to the first order in the bosonic infinitesimal parameter ε :

$$\{\Phi + \varepsilon a(\Phi), \Phi + \varepsilon a(\Phi)\} = (\Delta + \varepsilon\{U, _})(q(\Phi) + \varepsilon q'(\Phi)) \tag{245}$$

This equation can also be derived immediately from Eq. (242) under the assumption that the ghost number 3 cohomology of $\Delta + \varepsilon\{U, _}$ vanishes

Therefore the condition of being in involution persists, but with deformed q :

$$q \mapsto q + \varepsilon q' \tag{246}$$

Remember that the construction of equivariant form Ω^C requires solving the equation:

$$\Delta_{\rho_{1/2}} a(\xi) + \frac{1}{2}\{a(\xi), a(\xi)\} = \xi \quad (= \Delta\Phi\langle\xi\rangle) \tag{247}$$

The solutions deforms:

$$a(\xi) \mapsto a(\xi) + \underbrace{\varepsilon a'_U(\xi)}_{\text{linear in } \xi} - \underbrace{\varepsilon q'_U(\xi)}_{\text{quadratic in } \xi} + \dots \tag{248}$$

And the string measure deforms:

$$\Omega^C(L, \Psi, \xi) = \int_L \exp \left(\underbrace{S_{\text{BV}}}_{\text{becomes } S_{\text{BV}} + \varepsilon U} + \Psi + \underbrace{a(\xi)}_{\text{becomes } a + \varepsilon a'_U - \varepsilon q'_U + \dots} \right) \tag{249}$$

In the base form Ω^{base} we substitute for ξ the curvature of the connection — the 2-form. In other words, $a(\xi)$ becomes a *two-form* on LAG. Therefore a vertex operator is actually an *inhomogeneous PDF* on LAG (and not just a function). If the theory has ghost number, then U has ghost number zero and $a(\xi)$ has ghost number -2 ; the sum of the ghost number and the degree of the form is zero. Lower ghost number components of vertex operators were recently used in the context of pure spinor formalism in [16].

7.3. Exact vertex operators

Let us consider the case when U is Δ -exact:

$$U = \Delta W \tag{250}$$

Let us also assume that W is H -invariant:

$$\{\Delta \Phi, W\} = 0 \tag{251}$$

In this case, we can take:

$$a_U(\Phi) = -\{W, \Phi\} \tag{252}$$

We observe that the resulting deformation of the equivariant form is the Lie derivative along $\{W, _ \}$:

$$\mathcal{L}_{-\{W, _ \}} \Omega^C = \Omega^C \langle \Delta W - \{W, \Phi\} \rangle = \Omega^C \langle U + a_U(\Phi) \rangle \tag{253}$$

This means that:

- deforming with an exact integrated vertex is equivalent to an infinitesimal change of the integration cycle.

▮ To construct the base form from the Cartan form, we need to choose a connection. When deforming with an exact vertex, we need to also adjust the connection by carrying it along with $\{W, _ \}$

7.4. Relaxing Siegel gauge

Let us deform S_{BV} by adding an integrated vertex operator not in the Siegel gauge. In this case we have to use the full base form Ω^{base} from Section 5.7. Suppose that there exists W such that $U + \Delta W$ is in Siegel gauge:

$$\{\Phi, U + \Delta W\} = 0 \tag{254}$$

In practice W could be complicated. But if it exists, then this implies that we can satisfy the Siegel gauge by slightly deforming the integration cycle.

Usually in string theory models, having the Siegel gauge satisfied on a family of Lagrangian submanifolds implies that Φ vanishes on Lagrangian submanifolds in this family; then our form Ω^{base} reduces to the standard string theory measure. (In the case of bosonic string this is discussed in Section 8.3.) Combined with the observation that the Siegel gauge can be satisfied by deforming the contour, this implies that the amplitudes obtained by integrating our Ω^{base} give the same result as the standard prescription for string amplitudes.

8. Bosonic string

Now we will explain how this formalism can be applied to the bosonic string worldsheet theory.

8.1. Solution of Master Equation

The fundamental fields of the worldsheet theory are: matter fields x^m , complex structure I_β^α and the diffeomorphism ghosts c^α . The matter part of the action depends on matter fields and complex structure:

$$S_{\text{mat}}[I, x] = \frac{1}{2} \int \partial_+ x^m \wedge \partial_- x^m \tag{255}$$

Following the general scheme, we find the solution of the Master Equation:

$$S_{\text{BV}} = S_{\text{mat}} + \int \langle \mathcal{L}_c I, I^\star \rangle + \langle \mathcal{L}_c x, x^\star \rangle + \frac{1}{2} \langle [c, c], c^\star \rangle \tag{256}$$

There is a constraint: $I^2 = -1$, or in components: $I_\beta^\alpha I_\gamma^\beta = -\delta_\gamma^\alpha$. The antifield I^\star can be identified with a symmetric tensor $I_{\alpha\beta}^\star = I_{\beta\alpha}^\star$ with only nonzero components I_{++}^\star and I_{--}^\star (which are also denoted I_{zz}^\star and $I_{\bar{z}\bar{z}}^\star$.) The coupling to δI is defined as follows:

$$\int \langle I^\star, \delta I \rangle = \int dz^\alpha \wedge dz^\beta I_{\alpha\gamma}^\star \delta I_\beta^\gamma \tag{257}$$

8.2. Family of Lagrangian submanifolds

Motivation for changing polarization. Notice that the dependence of S_{BV} on the antifields (letters with \star) is at most linear. Indeed, this S_{BV} corresponds to “just the usual BRST operator” of the form $c^A t_A + \frac{1}{2} f_{BC}^A c^B c^C \frac{\partial}{\partial c^A}$.

It would seem to be natural to choose the Lagrangian submanifold setting all the antifields to zero. However, the restriction of S_{BV} to this Lagrangian submanifold (i.e. S_{cl}) turns out to be problematic from the point of view of quantization (the Nambu–Goto string). The standard approach in bosonic string is to switch to a different Lagrangian submanifold so that the restriction of S_{BV} to this new Lagrangian submanifold is quadratic. However there is some price to pay: BRST operator is only nilpotent on-shell.

New polarization. Let us choose some reference complex structure $I^{(0)}$ and parametrize the nearby complex structures by their corresponding Dolbeault cocycles, which we denote b^\star . Locally it is possible to choose $I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. To summarize:

$$I = I(b^*) \tag{258}$$

$$b^* \in H_{\mathbb{Q}}^1(T^{1,0}) \tag{259}$$

$$I_{\bar{z}}^z(b^*) = (b^*)_{\bar{z}}^z + o(b^*) \tag{260}$$

$$I_{\bar{z}}^z(b^*) = i + o(b^*) \tag{261}$$

The other components are:

$$I_{\bar{z}}^{\bar{z}} = \overline{I_{\bar{z}}^z}, \quad I_{\bar{z}}^{\bar{z}} = \overline{I_{\bar{z}}^z} \tag{262}$$

And we rename I^* as b :

$$b_{\alpha\beta} = I_{\alpha\beta}^* \tag{263}$$

We have just changed the polarization; I^* is now a field (called b) and I an antifield (called b^*). The action can be written in the new coordinates:

$$S_{\text{BV}} = S_{\text{mat}}[I(b^*), x] + \int \langle \mathcal{L}_c(I(b^*)), b \rangle + \langle \mathcal{L}_c x, x^* \rangle + \left\langle \frac{1}{2}[c, c], c^* \right\rangle \tag{264}$$

Family of Lagrangian submanifolds. We now choose the Lagrangian submanifold in the following way:

$$b^* = x^* = c^* = 0 \tag{265}$$

On this Lagrangian submanifold the action is quadratic. In particular:

$$\int \langle b, \mathcal{L}_c(I(b^*)) \rangle \text{ becomes } \int (b_{++}\partial_-c^+ + c.c) \tag{266}$$

BRST structure. The BRST operator Q_{BRST} of the bosonic string can be understood as follows. We expand S_{BV} in powers of the antifields x^*, c^*, b^* and consider only the linear term. The corresponding Hamiltonian vector field preserves the Lagrangian submanifold and is the symmetry of the restriction of S_{BV} on the Lagrangian submanifold. There are also higher order terms, because the dependence on b^* is nonlinear; $\int b_{\alpha}^{\gamma} T_{\gamma\beta} dz^{\alpha} \wedge dz^{\beta}$ is just the linear approximation. In constructing the Q_{BRST} we simply neglect those higher order terms. This leads to Q_{BRST} being nilpotent only on-shell. The explicit formula for Q_{BRST} can be read from Eq. (264):

$$Q_{\text{BRST}} = T_{\alpha\beta} \frac{\partial}{\partial b_{\alpha\beta}} + (c^{\alpha} \partial_{\alpha} x^m) \frac{\partial}{\partial x^m} + \frac{1}{2}[c, c]^{\alpha} \frac{\partial}{\partial c^{\alpha}} \tag{267}$$

Ghost numbers.

x	x^*	I	b	c	c^*
0	-1	0	-1	1	-2

8.3. Integration over the family of Lagrangian submanifolds

Here we will first use our prescription to construct the equivariant analogue of Ω , depending on some motivated choice of the subspace $\mathcal{F} \subset \text{Fun}(M)$. We will then implement the standard procedure to construct a closed form on $H \backslash \text{LAG}$.

Choice of \mathcal{F} . We will here make use of the standard choice of \mathcal{F} always applicable in the BRST case as we explained in Section 6.4. We will choose \mathcal{F} so that $\Pi\Delta\mathcal{F}$ is the algebra of diffeomorphisms of the worldsheet:

$$\Phi = \int c_\alpha^* \xi^\alpha \tag{268}$$

where $\xi^\alpha = \xi^\alpha(z, \bar{z})$ is a vector field on the worldsheet.

Equivariant form $\Omega_{\text{Vect}(\Sigma) \oplus \text{Weyl}}^{\mathcal{C}}$. The resulting equivariant form is:

$$\Omega_{\text{Vect}(\Sigma) \oplus \text{Weyl}}^{\mathcal{C}}(\mathbf{t}) = \int_{gL} \exp \left(S_{\text{BV}} + \int \langle dI, b \rangle + \int c_\alpha^* \mathbf{t}^\alpha \right) \tag{269}$$

where the term $\int \langle dI, b \rangle$ comes from dgg^{-1} and $\int c_\alpha^* \mathbf{t}^\alpha$ from Φ of Eq. (268). The pairing $\langle dI, b \rangle$ is as in Eq. (257).

Recovery of the standard approach. Usually in the literature, the integration cycle is chosen so that $c^* = 0$, and therefore the term $\int c_\alpha^* \mathbf{t}^\alpha$ vanishes. Moreover, we do not even need to do the horizontal projection of dgg^{-1} . This is a consequence of the following general statement. Suppose that $\{\Phi, \Phi\} = 0$ and the integration cycle in the moduli space of Lagrangian submanifolds is such that:

$$\Phi|_{gL} = 0 \tag{270}$$

In this case the construction simplifies:

$$\Omega_{\text{Vect}(\Sigma)} = \Omega \tag{271}$$

We can now use the Baranov–Schwarz transform (Section 4.2) and interpret the integration of Ω as the integration of e^S over some new Lagrangian submanifold, which can be described as follows. Consider any $3g - 3$ -dimensional surface in the space of metrics $g..$ parametrized by (s_1, \dots, s_{3g-3}) :

$$S \subset \text{MET}, \quad \dim S = 3g - 3 \tag{272}$$

(no need to require any holomorphicity). Let us consider a submanifold

$$\mathcal{N}_S \subset \Pi T^* \text{MET} \tag{273}$$

defined as follows: it is the bundle over S whose fiber at a point $s \in S$ consists of the subspace of $\Pi T_s^* \text{MET}$ orthogonal to the tangent space to S at that point. Notice that this submanifold is Lagrangian. We will promote \mathcal{N}_S to a Lagrangian submanifold $\widehat{\mathcal{N}}_S$ in the BV phase space of the bosonic string by adding x and c^{\cdot} (and keeping $x^* = 0$ and $c^* = 0$). We have:

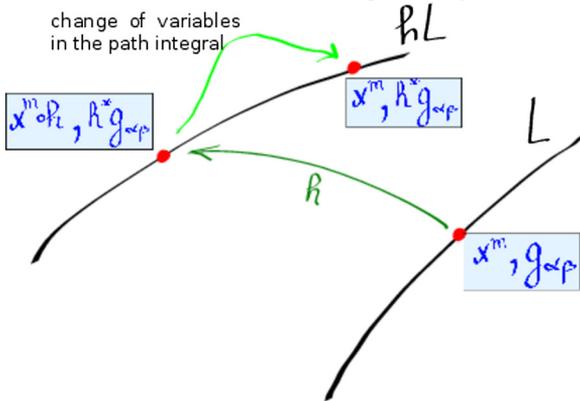
$$\int_S \Omega = \int_{\widehat{\mathcal{N}}_S} e^{S_{\text{BV}}} \tag{274}$$

Notice that we can use Ω instead of $\Omega_{\mathcal{B}}$ because Eq. (270) is satisfied in this case. Indeed, $\Phi(\xi) = \int \xi^\alpha c_\alpha^*$ and we choose the Lagrangian submanifolds so that $c^* = 0$; this proves Eq. (270).

When some part of $\widehat{\mathcal{N}}_S$ contains a gauge-trivial direction $\dot{g}_{\alpha\beta} = 2\nabla_{(\alpha} \xi_{\beta)}$, then the integral of $e^{S_{\text{BV}}}$ over that part is automatically zero. Indeed, in this case all the $b^{\cdot} \in \Pi T_s^* \text{MET}$ orthogonal to the tangent space to S satisfy in particular $b^{\alpha\beta} \nabla_\alpha \xi_\beta = 0$ and therefore the integral of $e^{S_{\text{BV}}}$ over c^{\cdot} will give zero (because of the zero mode $c^\alpha \simeq \xi^\alpha$). In this sense, $e^{S_{\text{BV}}}$ is a “base integral form”.

Standard integration cycle. We will now discuss the “usual” (in the bosonic string theory) integration cycle on the moduli space of Lagrangian submanifolds. Our choice of \mathcal{F} is such that $\mathfrak{h} = \Pi\Delta\mathcal{F}$ is the algebra of diffeomorphisms of the worldsheet. This allows us to construct the base form on the space of Lagrangian submanifolds modulo diffeomorphisms.

On a Lagrangian submanifold from the standard family the metric $g_{\alpha\beta}$ (same thing as $b^{\alpha\beta}$) is fixed, and the path integral goes over $b^{\alpha\beta}, c^\alpha, x^m$. This picture explains why factorization by the action of H results in closed integration cycles:



Here h is a large diffeomorphism; the integration cycle is a $3g - 3$ -dimensional family of metrics $g_{\alpha\beta}$; it only becomes closed after we make an identification of $g_{\alpha\beta}$ and $h^* g_{\alpha\beta}$.

9. Topologically twisted $N = 2$ model

In this Section we will give an example of the construction of \mathcal{F} in the case which is not covered by the scheme described in Section 6.4.

9.1. BV action

Consider the topologically B-twisted $N = 2$ superconformal theory [17]. Let us restrict to the case of flat target space. The fields are:

bosons:	complex scalars x^a and $\bar{x}^{\bar{a}}$
fermions:	one-form ρ^a and scalars ϑ_a and $\bar{\eta}^{\bar{a}}$

The “classical” action is:

$$S_{cl} = \int_{\Sigma} -\frac{1}{2} d\bar{x}^{\bar{a}} \wedge *dx^a - \rho^a \wedge (*d\bar{\eta}^{\bar{a}} + d\vartheta_a) \tag{275}$$

The BV action is:

$$S_{BV} = S_{cl} + \int_{\Sigma} \bar{\eta}^{\bar{a}} \bar{x}^{\bar{a}*} - \frac{1}{2} \langle dx^a, \rho_a^* \rangle \tag{276}$$

9.2. Action of diffeomorphisms

Let us consider the infinitesimal diffeomorphism (= vector field) of the worldsheet:

$$v^L(z^L, z^R) \frac{\partial}{\partial z^L} + v^R(z^L, z^R) \frac{\partial}{\partial z^R} \tag{277}$$

The corresponding BV generation function is $\mathcal{V}\langle v \rangle = \{S_{\text{BV}}, \Phi\langle v \rangle\}$ where:

$$\Phi\langle v \rangle = \int_{\Sigma} (\mathcal{L}_v \overline{x^a}) \overline{\eta^a}^* - (\mathcal{L}_{(v_L - v_R)} \overline{x^a}) \vartheta^{*a} - 2(\iota_v \rho^a) x_a^* + \iota_{\rho_a^*} \iota_v \vartheta^{*a} \tag{278}$$

The last term $\iota_{\rho_a^*} \iota_v \vartheta^{*a}$ requires explanation. As ϑ is a scalar field, its antifield ϑ^* is a 2-form. The expression $\iota_v \vartheta^*$ is the usual contraction of a vector field v with the 2-form ϑ^* . Finally, $\iota_{\rho_a^*} \iota_v \vartheta^*$ should be understood in the following way. We interpret ρ^* as a vector-valued two-form, i.e. a section of $(T \otimes \Omega^2)\Sigma$. We contract its vector index with the single remaining covector index of $\iota_v \vartheta^*$. What remains is a two-form, which is just integrated over Σ .

Explicitly:

$$\begin{aligned} \mathcal{V}\langle v \rangle &= \{S_{\text{BV}}, \Phi\langle v \rangle\} = \\ &= \int_{\Sigma} (\mathcal{L}_v x^a) x_a^* + (\mathcal{L}_v \overline{x^a}) \overline{x_a}^* + \langle (\mathcal{L}_v \rho^a), \rho_a^* \rangle + (\mathcal{L}_v \overline{\eta^a}) \overline{\eta_a}^* + (\mathcal{L}_v \vartheta_a) \vartheta^{*a} + \\ &\quad + d\overline{x^a} \wedge *d(\iota_v \rho^a) - (\mathcal{L}_v \overline{x^a}) d * \rho^a - (\mathcal{L}_{(v_L - v_R)} \overline{x^a}) d\rho^a \\ &= \int_{\Sigma} (\mathcal{L}_v x^a) x_a^* + (\mathcal{L}_v \overline{x^a}) \overline{x_a}^* + \langle (\mathcal{L}_v \rho^a), \rho_a^* \rangle + (\mathcal{L}_v \overline{\eta^a}) \overline{\eta_a}^* + (\mathcal{L}_v \vartheta_a) \vartheta^{*a} - \\ &\quad - *d\overline{x^a} \wedge \mathcal{L}_v \rho^a - (\mathcal{L}_v \overline{x^a}) d * \rho^a \end{aligned} \tag{279}$$

Identities useful in proving this:

$$\begin{aligned} &\left\{ \int_{\Sigma} \langle dx^a, \rho_a^* \rangle, \int_{\Sigma} (\iota_v \rho^b) x_b^* \right\} = \\ &= \int_{\Sigma} (\mathcal{L}_v x^a) x_a^* + \langle (d\iota_v \rho^a), \rho_a^* \rangle \end{aligned} \tag{280}$$

$$\begin{aligned} &\left\{ \int_{\Sigma} \overline{\eta^a} \overline{x_a}^*, \int_{\Sigma} (\mathcal{L}_v \overline{x^a}) \overline{\eta_a}^* - (\mathcal{L}_{(v_L - v_R)} \overline{x^a}) g_{ab} \vartheta^{*b} \right\} = \\ &= \int_{\Sigma} (\mathcal{L}_v \overline{\eta^a}) \overline{\eta_a}^* + (\mathcal{L}_v \overline{x^a}) \overline{x_a}^* - (\mathcal{L}_{(v_L - v_R)} \overline{\eta^a}) g_{ab} \vartheta^{*b} \end{aligned} \tag{281}$$

$$\left\{ S_{\text{cl}}, \int_{\Sigma} \iota_{\rho_a^*} \iota_v \vartheta^{*a} \right\} =$$

$$\begin{aligned}
&= \left\{ \int_{\Sigma} - * d\bar{\eta}^a \wedge \rho^a + \vartheta_a d\rho^a, \int_{\Sigma} \iota_{\rho_a^*} \iota_v \vartheta^{*a} \right\} = \\
&= \int_{\Sigma} \langle (\iota_v d\rho^a), \rho_a^* \rangle + (d\vartheta_a + *d\bar{\eta}^a) \wedge \iota_v \vartheta^{*a} = \\
&= \int_{\Sigma} \langle (\iota_v d\rho^a), \rho_a^* \rangle + (\mathcal{L}_v \vartheta_a) \vartheta^{*a} + (\mathcal{L}_{(v_L - v_R)} \bar{\eta}^a) \vartheta^{*a} \tag{282}
\end{aligned}$$

9.3. Closedness of \mathcal{F} under $[_, _]$

We need to prove:

$$\{\{S_{\text{BV}}, \Phi(w)\}, \Phi(v)\} = \Phi([w, v]) \tag{283}$$

The only nontrivial computation is the bracket of the last term in Eq. (278) with the last line of (279):

$$\left\{ \int_{\Sigma} \iota_{\rho_a^*} \iota_v \vartheta^{*a}, \int_{\Sigma} - * d\bar{x}^a \wedge \mathcal{L}_w \rho^a - (\mathcal{L}_w \bar{x}^a) d * \rho^a \right\} = \tag{284}$$

$$= \int_{\Sigma} - * d\bar{x}^a \wedge \mathcal{L}_w \iota_v \vartheta^{*a} - (\mathcal{L}_w \bar{x}^a) d * \iota_v \vartheta^{*a} = \tag{285}$$

$$= \int_{\Sigma} - * d\bar{x}^a \wedge \mathcal{L}_w \iota_v \vartheta^{*a} - *d(\mathcal{L}_w \bar{x}^a) \iota_v \vartheta^{*a} \tag{286}$$

and the bracket of the term $-(\mathcal{L}_{(v_L - v_R)} \bar{x}^a) \vartheta^{*a}$ in Eq. (278) with $\widehat{\mathcal{L}}_w$:

$$\left\{ \int_{\Sigma} - \iota_v * d\bar{x}^a \vartheta^{*a}, \widehat{\mathcal{L}}_w \right\} = \tag{287}$$

$$= \left\{ \int_{\Sigma} - * d\bar{x}^a \wedge \iota_v \vartheta^{*a}, \widehat{\mathcal{L}}_w \right\} = \tag{288}$$

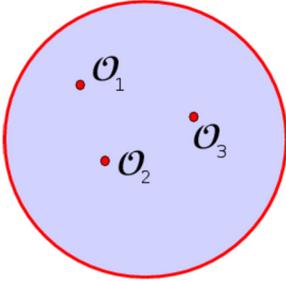
$$= \int_{\Sigma} *d\bar{x}^a \wedge \iota_v \mathcal{L}_w \vartheta^{*a} + *d(\mathcal{L}_w \bar{x}^a) \iota_v \vartheta^{*a} \tag{289}$$

Eqs. (286) and (289) combine into $\int_{\Sigma} *d\bar{x}^a \wedge \iota_{[w, v]} \vartheta^{*a}$ which is the required term $-(\mathcal{L}_{([w, v]_L - [w, v]_R)} \bar{x}^a) \vartheta^{*a}$ in $\Phi([w, v])$.

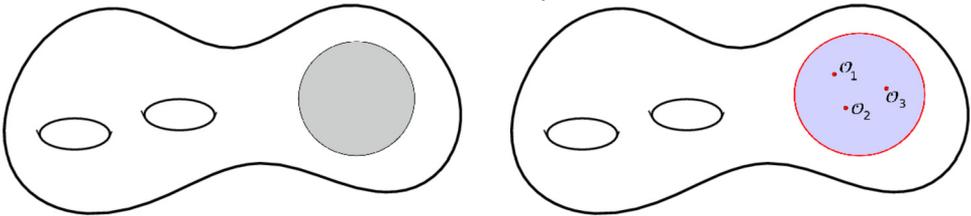
10. Worksheet with boundary

The consideration of this Section is not rigorous for the reason explained in Section 2. We hope that more rigorous treatment could be obtained by an axiomatic approach along the lines of [4].

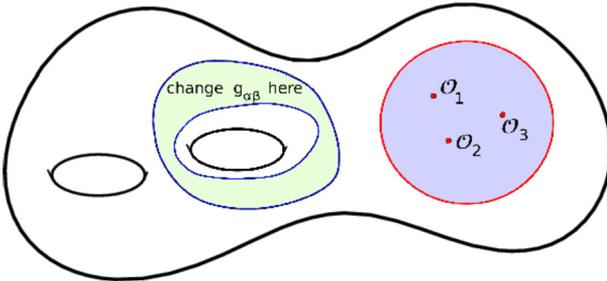
Consider string worldsheet theory on a flat disk D . Insertion of some operators inside the disk creates a state on the boundary:



Let us consider a Riemann surface with a boundary, and insert our flat disk:



Consider the variations of the Lagrangian submanifold (for example, metric), limited inside some compact region on the Riemann surface:



As we vary the metric, the state on the boundary (marked red) remains the same state in the same theory. This is possible because we do not change the theory in the region of insertion.

If we limit the variations of the Lagrangian submanifold in this way, then we can construct the form Ω (and its base analogue) in the same way as we did on the closed Riemann surface. In this case the form Ω is not closed:

$$d\Omega_{\Psi}^{\text{base}} = \Omega_{Q_{\text{BRST}}\Psi}^{\text{base}} \tag{290}$$

We will derive this formula in a moment. If we restrict ourselves with the insertions of only physical operators Ψ , then the form $\Omega_{\Psi}^{\text{base}}$ is closed.

10.1. Variation of the wave function

Let us study the quantum theory on a region $\Sigma \setminus D$ (the complement of the disk on the Riemann surface); its boundary is $-\partial D$. On the picture D is painted blue. Notice that there are no operator insertions inside D .

Let us bring our theory to some first order formalism, so that the Lagrangian is of the form $p\dot{q} - H$. Fix the Lagrangian submanifold L . Fix some polarization in the restriction of the fields

to ∂D , for example the standard polarization where the leaves have constant q . Let us consider the wave function:

$$\Psi(q) = \int_L \rho_{1/2} \tag{291}$$

— the integration is over the field configurations inside D , and q enters through boundary conditions. More generally, let $\Psi_{\mathcal{O}}(q)$ denote the wave function obtained by the path integration with the insertion of some operator \mathcal{O} with compact support, not touching the boundary:

$$\Psi_{\mathcal{O}}(q) = \int_L \mathcal{O} \rho_{1/2} \tag{292}$$

Any functional Ξ on the odd phase space, with compact support on the worldsheet, determines an operator insertion, just by its restriction on the Lagrangian submanifold. We will denote it Ξ_0 :

$$\Xi_0 = \Xi|_L \tag{293}$$

Theorem 3. *Suppose that $\rho_{\frac{1}{2}}$ satisfies the Master Equation. The expansion of S_{BV} around the Lagrangian submanifold in powers of antifields defines some Q_{BRST} . Then:*

$$\Psi_{Q_{\text{BRST}}\Xi_0}(q) = - \int_L \mathcal{L}_{\{\Xi, _ \}} \rho_{\frac{1}{2}} \tag{294}$$

where Ξ_0 is the restriction of Ξ to L and \mathcal{L}_{Ξ} is the Lie derivative of the half-density $\rho_{\frac{1}{2}}$ along the Hamiltonian vector field $\{\Xi, _ \}$ (see Eq. (16)). Notice:

- Eq. (294) is not affected by the presence of a boundary.

Proof. Let us introduce some Darboux coordinates near L , so that L is given by the equation $\phi^* = 0$. Let us expand Ξ in powers of ϕ^* . Let us first assume that only a constant in antifields term is present:

$$\Xi = \Xi_0(\phi) \tag{295}$$

In this case the Lie derivative is equivalent to inserting $Q_{\text{BRST}}\Xi_0$ into the path integral, giving Eq. (294).

If Ξ also depends on antifields, then we have to be careful restricting ourselves to such Ξ that the Lie derivative $\mathcal{L}_{\Xi}\rho_{\frac{1}{2}}$ is well-defined, because otherwise Eq. (294) does not make sense. This assumption must include the vanishing of the integration by parts:

$$\int_L \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi^*} \left(\Xi_1 \phi^* \rho_{\frac{1}{2}} \right) = 0 \tag{296}$$

This is equivalent to the linear term in the antifield expansion of Ξ not contributing to the RHS of Eq. (294).

Theorem 4. *Eq. (294) actually holds even without assuming that $\rho_{\frac{1}{2}}$ satisfies the Master Equation. In this case we define Q_{BRST} using the expansion of $\rho_{\frac{1}{2}}$ in Darboux coordinates:*

$$\rho_{\frac{1}{2}} = e^{S_{\text{cl}} + Q_{\text{BRST}}^a \phi_a^* + \dots}$$

Proof. Nothing in the proof of [Theorem 3](#) requires the use of Master Equation.

Definition. Suppose that $\rho_{\frac{1}{2}}$ satisfies the Master Equation sufficiently close to the boundary, in the sense that $\Delta_{\text{can}}\rho_{\frac{1}{2}}$ has compact support which does not touch the boundary. Then Q_{BRST} defined as in [Theorem 4](#) becomes a symmetry of $S_{\text{BV}}|_L$ sufficiently close to the boundary. Let us define $\widehat{Q}_{\text{BRST}}\Psi_{\mathcal{O}}$ via the insertion of the BRST current near the boundary:

$$\widehat{Q}_{\text{BRST}}\Psi_{\mathcal{O}} \stackrel{\text{def}}{=} \int_{\text{path integral}} [d\phi] \left(\oint_{\text{near the boundary}} j_{\text{BRST}} \right) \mathcal{O} \rho_{\frac{1}{2}}|_L \tag{297}$$

If $\rho_{\frac{1}{2}}$ satisfies the Master Equation, then j_{BRST} is conserved and:

$$\widehat{Q}_{\text{BRST}}\Psi_{\mathcal{O}} = \Psi_{Q_{\text{BRST}}\mathcal{O}} \tag{298}$$

Proof of Eq. (290). Now we are ready to prove [Eq. \(290\)](#). Let us first study the usual “old” (not equivariant) form Ω with the boundary. The derivation parallels the case with no boundary:

$$\begin{aligned} & d \int_{gL} E(d\widehat{g}g^{-1}) \rho_{\frac{1}{2}} = \\ &= \int_{gL} \left(d(E(d\widehat{g}g^{-1})) \rho_{\frac{1}{2}} + \mathcal{L}_{d\widehat{g}g^{-1}} \left(E(d\widehat{g}g^{-1}) \rho_{\frac{1}{2}} \right) \right) = \\ &= \int_{gL} \left(-\frac{1}{2} \{ \mathcal{H}, \mathcal{H} \} E'(\mathcal{H}) \rho_{\frac{1}{2}} + \mathcal{L}_{d\widehat{g}g^{-1}} \left(E(d\widehat{g}g^{-1}) \rho_{\frac{1}{2}} \right) \right) = \\ & \quad \text{here we use Eq. (42) for Lie derivative with } f(\mathcal{H}) = \int E(\mathcal{H}) d\mathcal{H} \\ &= \int_{gL} \text{Lie derivative of } \rho_{\frac{1}{2}} \text{ along the Hamiltonian flux of} \\ & \quad \int d\mathcal{H} E(\mathcal{H}) \Big|_{\mathcal{H}=d\widehat{g}g^{-1}} = \\ &= Q_{\text{BRST}} \left[q \mapsto \int_{gL} \left(\int d\mathcal{H} E(\mathcal{H}) \Big|_{\mathcal{H}=d\widehat{g}g^{-1}} \rho_{\frac{1}{2}} \right) \right] \tag{299} \end{aligned}$$

We should choose $E(\mathcal{H}) = \exp(\mathcal{H})$; in this case $\int d\mathcal{H} E(\mathcal{H}) = E(\mathcal{H}) = e^{\mathcal{H}}$.

10.2. Interpretation of Ω as an intertwiner in the presence of a boundary

The equivariant Ω^c is a particular case of $\Omega(e^a)$ when a is a solution to [Eq. \(121\)](#).

We interpret the path integral in the theory on $\Sigma \setminus D$, with a boundary state, as the path integral over the whole Σ with insertions $\mathcal{O}_1, \mathcal{O}_2, \dots$ (inside D) determining this boundary state. Then

our form Ω is defined by the path integral in the theory on the *whole* compact Riemann surface Σ :

$$\Omega_{\mathcal{O}_1\mathcal{O}_2\dots} = \int_{gL} e^{d\widehat{gg}^{-1}} \mathcal{O}_1\mathcal{O}_2\dots e^a \rho_{\frac{1}{2}} \tag{300}$$

But we only allow the variations of L which do not change the theory inside the disk D . In other words, we restrict to $d\widehat{gg}^{-1}$ of compact support inside $\Sigma \setminus D$. We also assume that a also has compact support localized inside $\Sigma \setminus D$.

Then, as we explained in Section 3.4.3, Ω intertwines d with Δ :

$$d\Omega_{\mathcal{O}_1\mathcal{O}_2\dots} = - \int_{gL} e^{d\widehat{gg}^{-1}} \Delta_{\rho_{\frac{1}{2}}} (\mathcal{O}_1\mathcal{O}_2\dots e^a) \rho_{\frac{1}{2}} \tag{301}$$

Since the support of $\mathcal{O}_1\mathcal{O}_2\dots$ is in D , and the support of a is in $\Sigma \setminus D$, we have:

$$d\Omega_{\mathcal{O}_1\mathcal{O}_2\dots} = - \int_{gL} e^{d\widehat{gg}^{-1}} \Delta_{\rho_{\frac{1}{2}}} (\mathcal{O}_1\mathcal{O}_2\dots) e^a \rho_{\frac{1}{2}} - \tag{302}$$

$$- \int_{gL} e^{d\widehat{gg}^{-1}} \mathcal{O}_1\mathcal{O}_2\dots (\Delta_{\rho_{\frac{1}{2}}} e^a) \rho_{\frac{1}{2}} \tag{303}$$

The first term should be interpreted as a nilpotent operator Q_{BRST} acting on the inserted state:

$$\Delta_{\rho_{\frac{1}{2}}} (\mathcal{O}_1\mathcal{O}_2\dots) = Q_{\text{BRST}}(\mathcal{O}_1\mathcal{O}_2\dots) \tag{304}$$

Therefore we have:

$$d\Omega_{\mathcal{O}_1\mathcal{O}_2\dots} \langle e^a \rangle = -\Omega_{Q_{\text{BRST}}(\mathcal{O}_1\mathcal{O}_2\dots)} \langle e^a \rangle - \Omega_{\mathcal{O}_1\mathcal{O}_2\dots} \langle \Delta_{\rho_{\frac{1}{2}}} e^a \rangle \tag{305}$$

On a compact Riemann surface $(\Delta_{\rho_{\frac{1}{2}}} e^a) \rho_{\frac{1}{2}}$ is the same as $\Delta_{\text{can}}(e^a \rho_{\frac{1}{2}})$ because $\rho_{\frac{1}{2}}$ satisfies the Master Equation (see Section 2.2). But in the presence of a boundary, these two expressions are different, because $\Delta_{\text{can}} \rho_{\frac{1}{2}}$ produces nonzero boundary terms [18,4]. On the other hand, $\Delta_{\rho_{\frac{1}{2}}} \mathcal{O}$ is of compact support if \mathcal{O} is of compact support.

We conclude that the presence of a boundary modifies the intertwiner property of Ω . This is a particular case of the following general construction. Let $\rho : \widetilde{\mathfrak{g}} \rightarrow \text{End}(V)$ be a representation of the cone Lie superalgebra. Then any complex K, d_K defines a new representation of $\widetilde{\mathfrak{g}}$:

$$\rho_1 : \widetilde{\mathfrak{g}} \rightarrow \text{End}(V \otimes K) \tag{306}$$

$$\rho_1(d) = \rho(d) + d_K \tag{307}$$

$$\rho_1|_{\widetilde{\mathfrak{g}}} = \rho|_{\widetilde{\mathfrak{g}}} \tag{308}$$

In our case K is the Hilbert space of states on the boundary, with $d_K = Q_{\text{BRST}}$.

10.3. Base form

In the presence of a boundary our construction of the base form does not give a closed form:

$$d\Omega_{\Psi}^{\text{base}} = \Omega_{Q_{\text{BRST}}\Psi}^{\text{base}} \tag{309}$$

At the same time, Ω^{base} is still horizontal and invariant. Horizontality follows from the construction, as we obtain our base form from the equivariant form by horizontal projection. Invariance follows from the fact that $d\Psi^{\text{base}}$ is also horizontal.

For our constructions presented in this part, it is important that we restrict to only those variations of the complex structure which are zero at the boundary. Otherwise, the variation would change the Hilbert space of states. In such case it would be nontrivial to define $d\Psi$, as we would need a connection on the bundle of Hilbert spaces.

11. Unintegrated vertex operators

Recall that H is the group of diffeomorphisms of the worldsheet and \mathfrak{h} its Lie algebra.

11.1. Definition of unintegrated vertex

Let us try to relax the condition of the \mathfrak{h} -invariance for the vertex, i.e. do not require that $\{\Delta\Phi, U\} = 0$. In this case the deformed action will not be \mathfrak{h} -invariant. It does not seem to be possible to modify the definition of \mathcal{F} so that the deformed action $S + \varepsilon U$ be invariant.⁹ We will deal with this complication in the following way. Let us assume that, instead of being an integral over the worldsheet, U is actually a sum of insertions of some operators into fixed points on the worldsheet. We can then restrict the group of diffeomorphisms to a subgroup which preserves those points. Let \mathfrak{h}_U be the subalgebra of \mathfrak{h} which preserves the points of insertions and $\mathcal{F}_U = \Delta^{-1}\mathfrak{h}$. In other words:

$$\forall \Phi \in \mathcal{F}_U : \{\Delta\Phi, U\} = 0 \tag{310}$$

Remember that we have to integrate Ω over some cycle in the moduli space of Lagrangian submanifolds. We postulate that, in case of unintegrated vertex:

- the integration cycle should include the variations of the positions of the insertion points.

We will now outline the procedure of integration over those insertion points.

11.2. Integration over the location of insertion points

11.2.1. Fixing the Lagrangian submanifold

Therefore we have to study the restriction of Ω to the subgroup $H \subset G$. We will for now assume that elements of \mathcal{F} are in involution (i.e. $q = 0$ in Section 5.4.2). We will make use of the fact that H (as opposed to full G) preserves $\rho_{\frac{1}{2}}$. This implies that the integration measure can be transformed to a fixed Lagrangian submanifold:

$$\int_{hL} \rho_{\frac{1}{2}} \exp(dhh^{-1}) U = \tag{311}$$

$$= \int_L \rho_{\frac{1}{2}} \exp((dhh^{-1}) \circ h) U \circ h \tag{312}$$

⁹ Notice however that it is invariant up to BRST-exact terms, because $\{\Delta\Phi, U\} = \Delta\{\Phi, U\}$.

11.2.2. Modified de Rham complex of H

Definition. We define the “modified de Rham complex” of H as the space of H -invariants:

$$(\text{Fun}(M \times \Pi TH))^H \tag{313}$$

where the action of $h_0 \in H$ is induced by the right shift on H and the action of $H \subset G$ on M ; in particular, any function of the form $h, x \mapsto F(hx)$ is H -invariant. The differential comes from the canonical odd vector field on ΠTH ; we will denote it $d_{(h)}$. The integrand in Eq. (312) belongs to this space:

$$\text{exp}\left((dhh^{-1}) \circ h\right) U \circ h \in (\text{Fun}(M \times \Pi TH))^H \tag{314}$$

We will show that this is the same as the Lie algebra cohomology complex of \mathfrak{h} with coefficients in $\text{Fun}(M)$. This is a version of the well-known theorem saying that Serre–Hochschild complex of the Lie algebra with trivial coefficients is the same as right-invariant differential forms on the Lie group. This is a general statement, true for a Lie group acting on a manifold.

Notations and useful identities.

$$\text{define } \Psi \text{ so that } \Delta\Psi = dhh^{-1} = (dhh^{-1})^A \mathcal{H}_A \text{ (the moment map)} \tag{315}$$

$$\text{and } \tilde{\Psi} = \Psi \circ h \tag{316}$$

$$\text{notice that } \Delta\tilde{\Psi} = \widetilde{\Delta\Psi} = (dhh^{-1}) \circ h \tag{317}$$

In this Section the tilde over a letter will denote the composition with h :

$$\tilde{f} = f \circ h \text{ (a function } x \mapsto f(hx)) \tag{318}$$

Here are some identities that we will need:

$$d_{(h)}\Delta\Psi = -\frac{1}{2}\{\Delta\Psi, \Delta\Psi\} \tag{319}$$

$$d_{(h)}\Psi = \frac{1}{2}\{\Psi, \Delta\Psi\} \tag{320}$$

$$d_{(h)}\tilde{\Psi} = -\frac{1}{2}\{\tilde{\Psi}, \Delta\tilde{\Psi}\} \tag{321}$$

$$d_{(h)}\Delta\tilde{\Psi} = \frac{1}{2}\{\Delta\tilde{\Psi}, \Delta\tilde{\Psi}\} \tag{322}$$

$$d_{(h)}\tilde{f} = \{\Delta\tilde{\Psi}, \tilde{f}\} = \{\widetilde{\Delta\Psi}, \tilde{f}\} = \{\widetilde{\Delta\Psi}, f\} \tag{323}$$

Elements of the space $(\text{Fun}(M \times \Pi TH))^H$ can be obtained from letters $\tilde{\Psi}$ and \tilde{U} by operations of multiplication and computing the odd Poisson bracket, or applying Δ .

11.2.3. Intertwiner between $-d_{(h)} + \Delta$ and Δ

Consider any function $f \in \text{Fun}(M \times \Pi TH)$ (not necessarily H -invariant). We have:

$$(\Delta - d_{(h)})\left(e^{\Delta\Psi \circ h} f \circ h\right) = e^{\Delta\Psi \circ h} ((\Delta - d_{(h)})f) \circ h \tag{324}$$

(Here $e^{\Delta\Psi \circ h} f \circ h$ is just the product of two functions, $e^{\Delta\Psi \circ h}$ and $f \circ h$.) In particular, when $f(x, h)$ only depends on x and does not depend neither on h nor on dh . (i.e. when $f \in \text{Fun}(M)$):

$$(\Delta - d_{(h)})e^{\Delta\tilde{\Psi}} \tilde{f} = e^{\Delta\tilde{\Psi}} \Delta\tilde{f} \tag{325}$$

In other words, the operator of multiplication by $e^{\Delta\tilde{\Psi}}$ intertwines between Δ and $\Delta - d_{(h)}$. After we integrate over the Lagrangian submanifold, $\Delta - d_{(h)}$ becomes just $-d_{(h)}$.

11.2.4. Integration

The one-form component.

$$\int_L \rho_{\frac{1}{2}} \Delta \tilde{\Psi} \tilde{U} = \int_L \rho_{\frac{1}{2}} \{\tilde{\Psi}, \tilde{U}\} = \int_L \rho_{\frac{1}{2}} \{\Psi, U\} \circ h \tag{326}$$

The two-form component.

$$\begin{aligned} & \int_L \rho_{\frac{1}{2}} \Delta \tilde{\Psi} \Delta \tilde{\Psi} \tilde{U} = \\ & = \int_L \rho_{\frac{1}{2}} \{\tilde{\Psi}, \{\tilde{\Psi}, \tilde{U}\}\} - 2d_{(h)} \int_L \rho_{\frac{1}{2}} \tilde{\Psi} \tilde{U} \end{aligned} \tag{327}$$

11.3. Cohomology of Δ vs Lie algebra cohomology

In this Section we will show that the Modified de Rham complex of H is the same as the Lie algebra cohomology complex of \mathfrak{h} with coefficients in $\text{Fun}(M)$.

11.3.1. Definition of the Lie algebra cohomology $H^\bullet(\mathfrak{h}, \text{Fun}(M))$

The space $\text{Fun}(M)$ is a representation of \mathfrak{h} ; the action of \mathfrak{h} is slightly easier to write down at the level of the corresponding action of the Lie group H ; $h \in H$ acts on $f \in \text{Fun}(M)$ as follows:

$$(h.f)(x) = f(h^{-1}x), \text{ in other words: } h.f = f \circ h^{-1} \tag{328}$$

Therefore:

$$d_{\text{Lie}} f = -\{dhh^{-1}, f\} = -\{\Delta\Psi, f\} \tag{329}$$

We define $d_{\text{Lie}}\Psi$ to coincide with $-d_{(h)}\Psi$ of Eq. (319):

$$d_{\text{Lie}}\Delta\Psi = \frac{1}{2}\{\Delta\Psi, \Delta\Psi\} \tag{330}$$

$$d_{\text{Lie}}\Psi = -\frac{1}{2}\{\Psi, \Delta\Psi\} \tag{331}$$

To follow the Faddeev–Popov notations, we introduce the Faddeev–Popov ghost:

$$\mathbf{c}^A = (dhh^{-1})^A \tag{332}$$

$$\text{then } \Psi = (-)^{\bar{A}+1} \mathbf{c}^A (\Delta^{-1} \mathcal{H}_A) \text{ (cp. Eq. (315))} \tag{333}$$

$$\mathcal{H}_A \in \text{Fun}(M) \tag{334}$$

Beware that Ψ is not just the Faddeev–Popov ghost; it is the product of the Faddeev–Popov ghost \mathbf{c}^A with $\Delta^{-1} \mathcal{H} \in \text{Fun}(M)$.

Eqs. (332) and (333) are equivalent to saying:

$$d_{\text{Lie}} \mathbf{c}^C = \frac{1}{2} (-)^{\bar{A}(\bar{B}+1)} \mathbf{c}^A \mathbf{c}^B f_{AB}{}^C \tag{335}$$

where $f_{AB}{}^C$ is the structure constants of \mathfrak{h} :

$$\{\mathcal{H}_A, \mathcal{H}_B\} = f_{AB}{}^C \mathcal{H}_C \tag{336}$$

$$f_{AB}{}^C = (-)^{\bar{A}\bar{B}+1} f_{BA}{}^C \tag{337}$$

It is straightforward to verify using Eqs. (329) and (331) that:

$$d_{\text{Lie}}^2 f = 0 \tag{338}$$

The subgroup $H \subset G$ preserves $\rho_{\frac{1}{2}}$ and therefore Δ :

$$\Delta(h.f) = h.\Delta f \tag{339}$$

This implies:

$$\Delta d_{\text{Lie}} + d_{\text{Lie}} \Delta = 0 \tag{340}$$

11.3.2. Proof that $d_{(h)}$ is the same as d_{Lie}

This is similar to the statement that for any Lie group G , the de Rham subcomplex of right-invariant forms on G is the same as the Lie cohomology complex of \mathfrak{g} with coefficients in the trivial representation:

$$(\text{Fun}(\Pi TG))^G = C^\bullet(\mathfrak{g}, \mathbf{C}) \tag{341}$$

Our case is a variation on this theme:

$$(\text{Fun}(M \times \Pi TH))^H = C^\bullet(\mathfrak{h}, \text{Fun}(M)) \tag{342}$$

As we explained, $(\text{Fun}(M \times \Pi TH))^H$ consists of functions of $\tilde{\Psi}$ and \tilde{U} . To obtain the corresponding element of $C^\bullet(\mathfrak{h}, \text{Fun}(M))$, we replace $\tilde{\Psi}$ with Ψ with $(dgg^{-1})^A \mapsto \mathbf{c}^A$ (as in Eq. (333)), and \tilde{U} with U . Under this identification $d_{(h)}$ becomes $-d_{\text{Lie}}$.

11.4. Another intertwiner between $d_{\text{Lie}} + \Delta$ and Δ

One intertwiner between $d_{\text{Lie}} + \Delta$ and Δ is already provided by Eq. (325), but it is non-local (because each $\Delta\Psi$ contains one integration). Motivated by the integration procedure of Section 11.2.4, we will now construct another intertwiner, a local one.

Let us assume that elements of subspace \mathcal{F} are all in involution, i.e. $q(x, y) = 0$. In this case:

$$\{\Psi, \{\Psi, \Delta\Psi\}\} = 0 \tag{343}$$

We denote $e^{\{\Psi, \cdot\}}$ the following operation:

$$e^{\{\Psi, \cdot\}} : \text{Fun}(M) \longrightarrow (\text{polynomials of } \mathbf{c}) \otimes \text{Fun}(M) \tag{344}$$

$$U \mapsto U + \{\Psi, U\} + \frac{1}{2}\{\Psi, \{\Psi, U\}\} + \frac{1}{6}\{\Psi, \{\Psi, \{\Psi, U\}\}\} + \dots \tag{345}$$

This operation has the following property:

$$(d_{\text{Lie}} + \Delta)e^{\{\Psi, \cdot\}} = e^{\{\Psi, \cdot\}} \Delta \tag{346}$$

The action of Δ on the left hand side is only on $\text{Fun}(M)$ (it does not touch the \mathbf{c} -ghosts)

11.5. Descent procedure

Here we will show that the intertwining operator of Eq. (346) can be interpreted as the generalization of the string theory descent procedure, which relates unintegrated and integrated vertex operators.

Consider an unintegrated vertex operator U . We interpret it as an element of the cohomology of Δ with some ghost number n :

$$U \in H^n(\Delta) \tag{347}$$

usually $n = 2$ in closed string theory and $n = 1$ in open string theory.

Eq. (346) allows us to construct from U a cohomology class of $\Delta + d_{\text{Lie}}$, where d_{Lie} is the Lie algebra cohomology differential of our Lie algebra \mathfrak{h} with coefficients in $\text{Fun}(M)$, as follows:

$$e^{\{\Psi, \cdot\}} U \in H(\Delta + d_{\text{Lie}}) \tag{348}$$

This expression is “inhomogeneous”, in the sense that different components have different ghost numbers. Each application of $\{\Psi, \cdot\}$ decreases the ghost number by one, but at the same time rises the degree of the Lie algebra cochain. In the context of closed string, the top component coincides with U , then goes $\{\Psi, U\}$, then $\{\Psi, \{\Psi, U\}\}$ and so on. In particular, $\{\Psi, \{\Psi, U\}\}$ is our interpretation of the *integrated* vertex operator.

We could have used $e^{\Delta\Psi}$ instead of $e^{\{\Psi, \cdot\}}$. We prefer to use $e^{\{\Psi, \cdot\}}$ because it leads to the local result. Although Ψ contains integration, the odd Poisson bracket is local (i.e. involves a delta-function) and therefore removes the integral.

In string theory the use of such an inhomogeneous expression is often referred to as “the descent procedure”.

11.5.1. Integrated vertex and Lie algebra cohomology

We have shown that the cohomology of Δ is the same as the cohomology of $\Delta + d_{\text{Lie}}$. The cohomology of $\Delta + d_{\text{Lie}}$ can be computed using the spectral sequence, corresponding to the filtration by the ghost number. Let $F^p \subset \text{Fun}(M)$ consist of the functions with the ghost number $\geq p$. At the first page, we have:

$$E_1^{p,q} = \frac{\ker d_{\text{Lie}} : F^p C^q \rightarrow F^p C^{q+1}}{\text{im } d_{\text{Lie}} : F^p C^{q-1} \rightarrow F^p C^q} \tag{349}$$

Therefore, if $E_1 = E_\infty$, then the cohomology of Δ is equivalent to the cohomology of \mathfrak{h} with values in $\text{Fun}(M)$.

11.5.2. Comparison of $e^{\{\Psi, \cdot\}}$ and $e^{\Delta\Psi}$

We have two operators satisfying the identical intertwining relations:

$$d_{\text{Lie}} e^{\Delta\Psi} + [\Delta, e^{\Delta\Psi}] = 0 \tag{350}$$

$$d_{\text{Lie}} e^{\{\Psi, \cdot\}} + [\Delta, e^{\{\Psi, \cdot\}}] = 0 \tag{351}$$

This suggests the existence of some operator A such that:

$$e^{\Delta\Psi} = e^{\{\Psi, \cdot\}} + d_{\text{Lie}} A + [\Delta, A] \tag{352}$$

This A is an inhomogeneous operator-form:

$$AU = \Psi U + \frac{1}{2} \Psi \Delta \Psi U + \frac{1}{2} \Psi \{\Psi, U\} + \dots \tag{353}$$

11.5.3. Relation between integrated and unintegrated vertices

Consider the special case of flat worldsheet. There is a subalgebra $\mathbf{R}^2 \subset \mathfrak{h}$ consisting of translations $(\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$). Let us restrict h to this subalgebra. This simplifies the computation because: $H^{n>2}(\mathbf{R}^2, \text{Fun}(M)) = 0$. Therefore we have:

$$e^{\{\Psi, \cdot\}} U = U + \{\Psi, U\} + \frac{1}{2} \{\Psi, \{\Psi, U\}\} \tag{354}$$

Going back from the Faddeev–Popov notations to the form notations: $\mathbf{c}^A \mapsto (dhh^{-1})^A$ we obtain:

$$\{\Psi, \{\Psi, U\}\} = dz \wedge d\bar{z} b_{-1} \bar{b}_{-1} U \tag{355}$$

(here dz and $d\bar{z}$ is what remains of dhh^{-1}). This is the usual integrated vertex operator of the bosonic string theory.

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Appendix A. Supermanifolds

A.1. Contraction and Lie derivative

We define ι_V for a vector field V as follows. If V is even, we pick a Grassmann odd parameter ϵ and define:

$$(\iota_V \omega)(Z, dZ) = \frac{\partial}{\partial \epsilon} \omega(Z, dZ + \epsilon V) \tag{356}$$

Remember that dZ parametrizes a point in the fiber of ΠTM over the point Z in M . Then $dZ + \epsilon V$ is a new point in the same fiber, linearly depending on ϵ .

If V is odd, we define ι_V as follows: $\iota_V \omega = \frac{\partial}{\partial \epsilon} \iota_{\epsilon V} \omega$. In coordinates:

$$\iota_V = V^A \frac{\partial}{\partial dZ^A} \tag{357}$$

The relation to Lie derivative:

$$[\iota_V, d] = \mathcal{L}_V \tag{358}$$

A.2. Symplectic structure and Poisson structure

Consider a supermanifold M , with local coordinates Z^A , equipped with an odd Poisson bracket of the form:

$$\{F, G\} = F \frac{\overleftarrow{\partial}}{\partial Z^A} \pi^B \frac{\partial}{\partial Z^B} G \tag{359}$$

The Poisson bivector ${}^A\pi^B$ should be symmetric in the following sense:

$${}^A\pi^B = (-)^{\bar{A}\bar{B}+\bar{A}+\bar{B}} B\pi^A \tag{360}$$

The odd symplectic form ω can be defined from the following equation:

$$dF = (-)^{\bar{F}+1} \iota_{\{F, _ \}} \omega \tag{361}$$

A.3. Darboux coordinates

In Darboux coordinates:

$$\{F, G\} = F \left(\frac{\overleftarrow{\partial}}{\partial\phi_A^*} \frac{\partial}{\partial\phi^A} - \frac{\overleftarrow{\partial}}{\partial\phi^A} \frac{\partial}{\partial\phi_A^*} \right) G \tag{362}$$

$$\omega = (-1)^A d\phi^A d\phi_A^* \tag{363}$$

Appendix B. Proof of the theorem-definition 1

Lemma 1. *Our $\mu_L[\rho_{\frac{1}{2}}]$ (which is a density on L defined, given $\rho_{\frac{1}{2}}$, by Eq. (10)) only depends on $\rho_{\frac{1}{2}}$ through restriction to the first infinitesimal neighborhood of L . In other words, if we replace $\rho_{\frac{1}{2}}$ with $e^f \rho_{\frac{1}{2}}$ where f is a function on M having second order zero on $L \subset M$, then μ_L will not change.*

This is slightly counterintuitive, because Δ_{can} is actually a second order differential operator. It is important that L is Lagrangian.

Proof. The definition of μ is given by Eq. (10); $\rho_{\frac{1}{2}}$ only enters the light hand side of Eq. (10) through the first infinitesimal neighborhood of L .

Lemma 2. *For any fixed positive integer n , any smooth function Ψ on L can be extended to a function $\tilde{\Psi}$ on M such that the derivative of $\rho_{\frac{1}{2}}$ along the flux of $\{\tilde{\Psi}, _ \}$ has zero of the order n on L , i.e.:*

$$\mathcal{L}_{\{\tilde{\Psi}, _ \}} \rho_{\frac{1}{2}} = f \rho_{\frac{1}{2}} \tag{364}$$

where $f \in \text{Fun}(M)$ vanishes

$$\text{in the } n\text{-th infinitesimal neighborhood of } L \subset M \tag{365}$$

Proof. Direct computation in coordinates. Let us choose some Darboux coordinates ϕ^i, ϕ_i^* , so that L is at $\phi^* = 0$. Let us use these coordinates to identify half-densities with functions. Without loss of generality, we can assume that in the vicinity of m :

$$\rho_{\frac{1}{2}} = e^S \tag{366}$$

$$\text{where } S = s(\phi) + Q^i \phi_i^* + \dots \tag{367}$$

where ... stand for terms of the higher order in ϕ^* . Then our problem is to find $\tilde{\Psi}(\phi, \phi^*)$ solving:

$$\frac{\partial}{\partial \phi^i} \frac{\partial}{\partial \phi_i^*} \tilde{\Psi} + \{S, \tilde{\Psi}\} = 0 \tag{368}$$

$$\tilde{\Psi}(\phi, 0) = \Psi(\phi) \tag{369}$$

Solutions can always be found, order by order in ϕ^* , to any order n .

Lemma 3.

$$g^* \mu_{gL}[\rho_{\frac{1}{2}}] = \mu_L[g^* \rho_{\frac{1}{2}}] \tag{370}$$

Proof. For any $H \in \text{Fun}(M)$:

$$\int_L (H \circ g) g^*(\mu_{gL}[\rho_{\frac{1}{2}}]) = \tag{371}$$

$$= \int_{gL} H \mu_{gL}[\rho_{\frac{1}{2}}] = \frac{d}{dt} \Big|_{t=0} \int_{e^{t(H, \cdot)} gL} \rho_{\frac{1}{2}} \Big|_L = \tag{372}$$

$$= \frac{d}{dt} \Big|_{t=0} \int_L g^* (e^{t(H, \cdot)})^* \rho_{\frac{1}{2}}$$

$$= \frac{d}{dt} \Big|_{t=0} \int_L (\exp(t\{H \circ g, \cdot\}))^* g^* \rho_{\frac{1}{2}} = \tag{373}$$

$$= \int_L (H \circ g) \mu_L[g^* \rho_{\frac{1}{2}}] \tag{374}$$

Proof of Theorem. We can in any case define $\sigma_{\frac{1}{2}}[L, \rho_{\frac{1}{2}}]$ by the formula:

$$\sigma_{\frac{1}{2}}[L, \rho_{\frac{1}{2}}] \Big|_L = \mu_L[\rho_{\frac{1}{2}}] \tag{375}$$

What we have to prove is that:

$$\text{so defined } \sigma_{\frac{1}{2}}[L, \rho_{\frac{1}{2}}] \text{ does not depend on } L \tag{376}$$

Consider any $x \in M$ and a Lagrangian submanifold $L \subset M$ such that $x \in L$ and e_1, \dots, f^1, \dots in $T_x M$ such that e_1, \dots are tangent to L and $\omega(f_i, e^j) = \delta_i^j$. Then, Eq. (375) says:

$$\text{by definition } \sigma_{\frac{1}{2}}[L, \rho_{\frac{1}{2}}](x, e_1, \dots, f^1, \dots) = \mu_L[\rho_{\frac{1}{2}}](x)(e_1, \dots) \tag{377}$$

Let us consider Eq. (370) in the special case when $g \in G$ is such that $g(x) = x$. We get:

$$\sigma_{\frac{1}{2}}[gL, \rho_{\frac{1}{2}}](x, g_* e_1, \dots, g_* f^1, \dots) = \sigma_{\frac{1}{2}}[L, g^* \rho_{\frac{1}{2}}](x, e_1, \dots, f^1, \dots) \tag{378}$$

Consider an infinitesimal variation of L specified by some ‘‘gauge fermion’’ $\Psi \in \text{Fun}(L)$. Let us use Lemma 3 to extend it to $\tilde{\Psi}$, and put $g = \exp(t\{\tilde{\Psi}, \cdot\})$. Lemma 2 implies that $\frac{d}{dt} \Big|_{t=0}$ of the RHS of Eq. (378) vanishes. This proves that the variation with respect to L of the LHS of Eq. (378) vanishes, and therefore $\sigma_{\frac{1}{2}}[L, \rho_{\frac{1}{2}}]$ does not depend on L .

References

- [1] A.S. Schwarz, Topological quantum field theories, arXiv:hep-th/0011260.
- [2] A. Mikhailov, A. Schwarz, Families of gauge conditions in BV formalism, *J. High Energy Phys.* 07 (2017) 063, [https://doi.org/10.1007/JHEP07\(2017\)063](https://doi.org/10.1007/JHEP07(2017)063), arXiv:1610.02996.
- [3] F. Bonechi, A.S. Cattaneo, P. Mnev, The Poisson sigma model on closed surfaces, *J. High Energy Phys.* 01 (2012) 099, [https://doi.org/10.1007/JHEP01\(2012\)099](https://doi.org/10.1007/JHEP01(2012)099), arXiv:1110.4850.
- [4] A.S. Cattaneo, P. Mnev, N. Reshetikhin, Perturbative quantum gauge theories on manifolds with boundary, arXiv:1507.01221.
- [5] A.S. Cattaneo, M. Schiavina, On time, arXiv:1607.02412.
- [6] B. Craps, K. Skenderis, Comments on BRST quantization of strings, *J. High Energy Phys.* 0505 (2005) 001, <https://doi.org/10.1088/1126-6708/2005/05/001>, arXiv:hep-th/0503038.
- [7] N. Berkovits, P.S. Howe, Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring, *Nucl. Phys. B* 635 (2002) 75–105, arXiv:hep-th/0112160.
- [8] A. Sen, Off-shell amplitudes in superstring theory, *Fortschr. Phys.* 63 (2015) 149–188, <https://doi.org/10.1002/prop.201500002>, arXiv:1408.0571.
- [9] O.M. Khudaverdian, Delta-operator on semidensities and integral invariants in the Batalin–Vilkovisky geometry, arXiv:hep-ph/9909117.
- [10] A.S. Schwarz, Geometry of Batalin–Vilkovisky quantization, *Commun. Math. Phys.* 155 (1993) 249–260, <https://doi.org/10.1007/BF02097392>, arXiv:hep-th/9205088.
- [11] A.S. Schwarz, Semiclassical approximation in Batalin–Vilkovisky formalism, *Commun. Math. Phys.* 158 (1993) 373–396, <https://doi.org/10.1007/BF02108080>, arXiv:hep-th/9210115.
- [12] K. Costello, *Renormalization and Effective Field Theory*, American Mathematical Soc., 2011.
- [13] A. Sen, B. Zwiebach, A note on gauge transformations in Batalin–Vilkovisky theory, *Phys. Lett. B* 320 (1994) 29–35, [https://doi.org/10.1016/0370-2693\(94\)90819-2](https://doi.org/10.1016/0370-2693(94)90819-2), arXiv:hep-th/9309027.
- [14] M.A. Grigoriev, A.M. Semikhatov, I.Yu. Tipunin, Gauge symmetries of the master action, *J. Math. Phys.* 40 (1999) 1792–1806, <https://doi.org/10.1063/1.532835>, arXiv:hep-th/9804156.
- [15] P. Ramond, Field theory. A modern primer, *Front. Phys.* 51 (1981) 1–397, *Front. Phys.* 74 (1989) 1.
- [16] N. Berkovits, Untwisting the pure spinor formalism to the RNS and twistor string in a flat and $AdS_5 \times S^5$ background, *J. High Energy Phys.* 06 (2016) 127, [https://doi.org/10.1007/JHEP06\(2016\)127](https://doi.org/10.1007/JHEP06(2016)127), arXiv:1604.04617.
- [17] A. Klemm, Preliminary notes: introduction in topological string theory on Calabi–Yau manifolds, <https://www.google.com.br/url?sa=t&rct=j&q=&esrc=s&source=web&cd=4&cad=rja&uact=8&ved=0ahUKEwiP5LPNh-zYAhUJJAKHccpDzEQFghZMAM&url=https%3A%2F%2Fwww.math.tecnico.ulisboa.pt%2F-strings%2FAGTS%2Ftopstrings.pdf&usq=AOvVaw2aPbZHDZZxgAaEivjyEmEf>.
- [18] A. Cattaneo, P. Mnev, N. Reshetikhin, Classical BV theories on manifolds with boundary, *Commun. Math. Phys.* 332 (2014) 535–603, <https://doi.org/10.1007/s00220-014-2145-3>, arXiv:1201.0290.