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Symplectic Field Theories: Scalar and Spinor Representations

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Abstract. Using elements of symmetry, as gauge invariance, aspects of field theories represented in symplectic space are introduced and analyzed under physical bases. The states of a system are described by symplectic wave functions, which are associated with the Wigner function. Such wave functions are vectors in a Hilbert space introduced from the cotangent-bundle of the Minkowski space. The symplectic Klein–Gordon and the Dirac equations are derived, and a minimum coupling is considered in order to analyze the Landau problem in phase space.

Keywords. Moyal product, Phase space, Field theory.

1. Introduction

This work renders tribute, in memory, to Professor Waldyr Alves Rodrigues Jr., who has inspired generations of physicists in Brazil, mainly working in mathematical physics with Clifford algebras [41]. Here, following his footprints, we study representations of the Poincaré Lie algebra taking, as a representation space, a Hilbert space defined from a symplectic manifold. By emphasizing Clifford algebras and spinor structures, we study symmetry representations of relativistic fields (in particular gauge fields). This analysis of symmetry provides a meaning for symplectic (phase-space) wave functions, which are associated with the Wigner function.

The notion of phase space in quantum mechanics was introduced by Wigner [52], where each operator, A, defined in the Hilbert space, $\mathcal{H}$, is associated with a function, $a_W(q,p)$, in phase space, Γ. This is given by a mapping $\Omega_W : A \rightarrow a_W(q,p)$ such that the associative algebra of operators defined in $\mathcal{H}$ turns out to be an associative algebra in Γ, given by $\Omega_W : AB \rightarrow a_W \star b_W$, where the star (or Moyal)-product $\star$ is defined by

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$$a_W \star b_W = a_W(q,p) \exp\left[\frac{i}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right] b_W(q,p)$$

(we use natural units: $\hbar = c = 1$). The non-commutative nature of linear mappings in $\mathcal{H}$ is associated to a non-commutative structure in Γ, that has been explored in different ways [2,7,9,10,12,14–16,20–26,28,33,34,36, 40,42,45,51,53]. This non-commutative structure provides the consistency of the Wigner representation with the standard Hilbert space representation of quantum mechanics. In other words, the non-commutative product of operators, as in the Heisenberg relations, is mapped by Ω_W in a non-commutative product of functions in phase space, through the Moyal product.

Applications of the Wigner approach take place in multiple direction, as in experiments of reconstruction of quantum states, in stochastic processes, in quantum tomography and in direct measurement of the Wigner function [29–32,35,37,38,44]. From a theoretical point of view, the Wigner formalism is associated with quantum groups and non-commutative geometries [8,11,13,27,46]. In addition, the mapping Ω_W induces the introduction of operators like $\widehat{A} = a_W\star$ acting on functions b_W, such that $\widehat{A}(b_W) = a_W \star b_W$; then it can be used to study symmetry groups. We have explored this fact by analyzing unitary representations of Lie groups generated by operators like $\widehat{A}$ [39]. In the case of the Galilei group such unitary representation gives rise to a unitary evolution which can be written in a form of a symplectic Schrödinger equation. From the Poincaré group, the Klein–Gordon and Dirac equations are derived in phase space. The connection with Wigner functions, $f_W(q,p)$ (that are quasi-probability distributions) can then be derived, providing a physical interpretation for those representations. Considering bosons, for instance, a complex wave function, $\psi_W(q,p)$, is associated with the Wigner function by $f_W(q,p) = \psi_W(q,p) \star \psi_W^*(q,p)$, where $\psi_W^*(q,p)$ is the complex conjugate of $\psi_W(q,p)$ [3–5]. The wave function is, therefore, physically interpreted and is called a quasi-amplitude of probability. As a consequence, this symplectic representation provides a method to consider the Wigner-function formalism on bases of symmetry groups. This procedure includes gauge invariance, which is a intricate task to be accomplished with the standard Wigner approach, since a Wigner function is a real function. For the case of $U(1)$ gauge, the spin 1 representation [6], corresponding to writing the Maxwell equation in phase space, is a realization of the Seiberg-Witten [43] non-commutative field theory, where the field tensor is given by $F^{\mu\nu} = \partial_\nu A^\mu - \partial_\mu A^\nu - i\{A^\mu, A^\nu\}_M$, with $\{f(q,p), g(q,p)\}_M = f(q,p)\star g(q,p) - g(q,p)\star f(q,p)$ being the Moyal-Poisson bracket.

There are many other representations of symmetry groups in phase space, but their physical meaning is no-longer evident [33]. An example is the formalism proposed by Torres-Vega and Frederick [17–19,49], motivated by the Husimi distribution function. In such a work, a frame in the Hilbert space, $|\tau\rangle = |q,p\rangle$, is introduced in phase space, Γ, such that the position and momentum operators are given, respectively, by $\widehat{Q} = q/2 + i\hbar\partial/\partial p$ and $\widehat{P} = p/2 - i\hbar\partial/\partial q$. The Schrödinger equation for bosons is then derived by taking the wave function $|\psi(t)\rangle$ in Γ, i.e. $\psi(q,p;t) = \langle\tau|\psi(t)\rangle$. This formalism has been applied, for example, in the harmonic analysis. Some physical aspects, nevertheless, remain to be clarified [47,48]. For instance, although $\psi(q,p;t)$ is claimed to be associated with the Husimi distribution, which is the case for some particular situations of harmonic oscillators, so far there is no general proof for that. Here we address this problem, deriving such a formalism from symmetry group representations.

We start by analyzing the vector fields on the symplectic manifold Γ, such that the corresponding operators $\widehat{Q}$ and $\widehat{P}$, acting on the Hilbert space of functions on Γ, are particular examples of Van Hove flows in phase space [1, 50]. Doing so, we obtain a generalization of the Torres-Vega and Frederick formalism to relativistic field theories. For spin 1/2 particles, we obtain a symplectic Dirac equation which is applied to analyze an electron under an intensive magnetic field (the Landau problem). This analysis is completed by providing a physical interpretation for spinors of type $\psi(q,p;t)$. In this approach to the representations of symmetries, the spinor states $\psi(q,p;t)$ are obtained from the quasi-amplitude of probability $\psi(q,p;t)_W$. We show how to map one function to each other; and as such, there is an association of $\psi(q,p;t)$ with $f(q,p;t)_W$. This is a way to assure a physical interpretation for the wave functions of type $\psi(q,p;t)$.

The work is presented as follows. In Sect. 2, the Hilbert space is introduced from the cotangent-bundle of the Minkowski space. In Sect. 3, representations of the Poincarè group are analyzed, taking the symplectic Hilbert space as the representation space. The symplectic Klein–Gordon theory, describing spin zero particles, is studied in Sect. 4, while in Sect. 5, we consider the symplectic Dirac equation, including an analysis of gauge transformations and the Landau problem in phase space. In Sect. 6 we analyze the physical content of the symplectic wave fucntions; and in Sect. 7, final concluding remarks are presented.

2. Relativistic Sympletic Hilbert Space

Let $\mathbb{M}$ be a Minkowski manifold where each point is specified by coordinates q^μ, with $\mu = 0,1,2,3$, and metric such that $diag(g)= (+---)$. Let $T^*\mathbb{M}$ be the corresponding cotangent-bundle, where each point is specified by the coordinates (q^μ, p^μ). The space $T^*\mathbb{M}$ can be then equipped with a symplectic structure by defining a 2-form $\omega = dq^\mu \wedge dp_\mu$, called the symplectic form. Let us define the operator on $C^\infty(T^*M)$,

$$\Lambda = \frac{\overleftarrow{\partial}}{\partial q^\mu}\frac{\overrightarrow{\partial}}{\partial p_\mu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\overrightarrow{\partial}}{\partial q_\mu}, \tag{1}$$

such that for C^∞ functions, $f(q,p)$ and $g(q,p)$, we have,

$$\omega(f\Lambda, g\Lambda) = f\Lambda g = \{f,g\}, \tag{2}$$

where $\{f,g\} = \frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial q_\mu}$ is the Poisson bracket. The space $T^*\mathbb{M}$ endowed with this symplectic structure is called the symplectic (or phase) space, and will be denoted by Γ, such that a vector is specified by $\omega =$

$(\omega^1, \omega^2, \ldots, \omega^8)$, with $\omega^1 = q^0, \omega^2 = q^1, \omega^3 = q^2, \omega^4 = q^3$, $\omega^5 = p^0, \omega^6 = p^1, \omega^7 = p^2, \omega^8 = p^3$ and $q = (q^0, \mathbf{q})$ and $p = (p^0, \mathbf{p})$ being vectors in $\mathbb{M}$. The symplectic metric matrix in Γ is given by (η_{ab}), $a, b = 1, \ldots, 8$,

$$\eta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

so that, given two functions $f(\omega)$ and $g(\omega)$ of class C^∞, the scalar product is given by the Poisson bracket written as,

$$\omega(f\Lambda, g\Lambda) = \eta^{ab} \frac{\partial f}{\partial \omega^a} \frac{\partial g}{\partial \omega^b} = \eta^{ab} \partial_a f \partial_b g.$$

The known application of this structure is the phase space in the relativistic classical mechanics. In this case, the components q^μ are the generalized coordinates and the components p^μ are the canonically conjugate momenta. From these general definitions, we can use Γ to introduce a Hilbert space over the phase space Γ, say $\mathcal{H}(\Gamma)$, which can be taken as the representation space for symmetry group. This is our main goal here.

Consider then a vector space $\mathcal{H}(\Gamma) = \{\phi(q,p), \psi(q,p), \ldots\}$ defined by complex functions in Γ of Lebesgue type. We define a scalar product in $\mathcal{H}(\Gamma)$ by,

$$\langle \phi | \psi \rangle = \int \phi^*(q,p) \psi(q,p) d^4q d^4p.$$

For $\phi(q,p) = \psi(q,p)$, we take,

$$\begin{aligned} \langle \psi | \psi \rangle &= \int \psi^*(q,p) \psi(q,p) d^4q d^4p \\ &= \int |\psi(q,p)|^2 d^4q d^4p = 1. \end{aligned}$$

Therefore, $\mathcal{H}(\Gamma)$ is a Hilbert space.

Using as a basis the following functions,

$$f_{k,n}(q,p) = \frac{1}{2\pi} e^{-i(kq+np)},$$

we have $\psi(q,p) = \sum_{k,n} \psi_{k,n} f_{k,n}(q,p)$. The functions $f_{k,n}(q,p)$ can be written in terms of the scalar product as $f_{k,n}(q,p) = \langle q,p|k,n\rangle = \langle q|k\rangle \langle p|n\rangle$. Here, $\langle k,n|$ is the dual of $|k,n\rangle$, such that $\langle k,n|k',n'\rangle = \langle n|n'\rangle\langle k|k'\rangle = \delta_{nn'}\delta_{kk'}$. This leads to,

$$\psi(q,p) = \langle q,p|\psi\rangle = \sum_{k,n} \psi_{k,n} \langle q,p|k,n\rangle$$

and $|\psi\rangle = \sum_{k,n} \psi_{k,n} |k,n\rangle$. The dual is $\langle \psi| = \sum_{k,n} \psi^*_{n,k} \langle k,n|$, such that, consistently, we have $\langle \psi|\psi\rangle = \sum_{k,n} \psi^*_{n,k} \psi_{k,n} = \sum_{k,n} |\psi_{k,n}|^2$.

We analyze now linear mappings in the Hilbert space $\mathcal{H}(\Gamma)$ in order to construct representation of symmetries.

Let $\mathcal{V}$ be the set of linear operators acting on $\mathcal{H}(\Gamma)$. The set $\mathcal{V}$ is equipped with a vector space structure from the following definition of the sum and product by scalars of operators,

$$(\Omega + \alpha W)A = \Omega(A) + \alpha W(A).$$

Throughout this work two classes of operators are used: the Poisson operators, constructed from the Poisson bracket, and the van Hove operators, constructed from the van Hove vector fields [1, 50].

Consider $f \in \mathcal{H}(\Gamma)$; then we define the complex-Poisson vector field $\overline{f} \in \mathcal{V}$ by [1],

$$\overline{f} = -iX_f = -i\left(\frac{\partial f}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial}{\partial p_\mu}\right).$$

Some particular examples are in order here. For $f \equiv q_\nu$, we have,

$$\overline{q}_\nu = -i\left(\frac{\partial q_\nu}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial q_\nu}{\partial q^\mu}\frac{\partial}{\partial p_\mu}\right) = i\frac{\partial}{\partial p^\nu}.$$

Now consider $f \equiv p_\nu$. Then,

$$\overline{p}_\nu = -i\left(\frac{\partial p_\nu}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial p_\nu}{\partial q^\mu}\frac{\partial}{\partial p_\mu}\right) = -i\frac{\partial}{\partial q^\nu}.$$

Given $f \in \mathcal{H}(\Gamma)$, we define $\hat{F} \in \mathcal{V}$ as a generalization of the van Hove vector field so that,

$$\hat{F} = f - \frac{1}{2}\left(q^\mu\frac{\partial f}{\partial q^\mu} + p^\mu\frac{\partial f}{\partial p^\mu}\right) + \overline{f}.$$

For $f = q_\nu$, we obtain,

$$\begin{aligned}\widehat{Q}_\nu &= q_\nu - \frac{1}{2}\left(q^\mu\frac{\partial q_\nu}{\partial q^\mu} + p^\mu\frac{\partial q_\nu}{\partial p^\mu}\right) + i\frac{\partial}{\partial p^\nu}\\ &= \frac{1}{2}q_\nu + i\frac{\partial}{\partial p^\nu}.\end{aligned}$$

For $f \equiv p_\nu$, $\hat{P}_\nu$ reads,

$$\begin{aligned}\widehat{P}_\nu &= p_\nu - \frac{1}{2}\left(q^\mu\frac{\partial p_\nu}{\partial q^\mu} + p^\mu\frac{\partial p_\nu}{\partial p^\mu}\right) - i\frac{\partial}{\partial q^\nu}\\ &= \frac{1}{2}p_\nu - i\frac{\partial}{\partial q^\nu}.\end{aligned}$$

From the definitions above, it follows that, for $f(q,p) = g(q,p) + ah(q,p)$, $a \in \mathbb{C}$, we have linearity, i.e.,

$$\widehat{g + ah}(q,p) = \widehat{g}(q,p) + a\widehat{h}(q,p).$$

For $f(q,p) = g(q,p)h(q,p)$, we have a modified derivation, i.e.,

$$\widehat{gh}(q,p) = g(q,p)\widehat{h}(q,p) + h(q,p)\widehat{g}(q,p) - g(q,p)h(q,p)$$

Note that the vector fields $\overline{f}$ and $\hat{f}$ are Hermitian operators acting on $\mathcal{H}(\Gamma)$.

Proposition 1. *The operator $\hat{P}_\mu$ is the generator of a unitary space translation group with elements given by $U(a) = \exp[-ia_\mu\hat{P}^\mu]$ such that $U(a)\hat{Q}_\mu U(a)^\dagger = \hat{Q}_\mu + a_\mu$ and $U(a)\psi(q,p) = e^{ia_\mu p^\mu}\psi(q^\mu + a^\mu, p)$, with $[\hat{Q}_\mu, \hat{P}_\nu] = ig_{\mu\nu}$ and $[\hat{P}_\mu, \hat{P}_\nu] = 0$.*

With these results, $\hat{P}_\mu$ can be interpreted physically as a momentum, while $\hat{Q}_\mu$, transforming under the translation $U(a)$ as a position, can be interpreted as a position observable. It is important to notice that we have similar results, by taking $\hat{Q}_\mu$ as a c-number. This will be useful when we consider gauge fields in the last section. We are in position to obtain representations for the Poincaré symmetries.

3. Symplectic Poincaré Lie Algebra

Introducing

$$\hat{M}_{\alpha\beta} = \hat{Q}_\alpha \hat{P}_\beta - \hat{Q}_\beta \hat{P}_\alpha, \tag{3}$$

we have

$$\begin{aligned}\hat{M}_{\alpha\beta} = \frac{1}{4}(q_\alpha p_\beta - q_\beta p_\alpha) - \frac{i}{2}\left(q_\alpha \frac{\partial}{\partial q^\beta} - q_\beta \frac{\partial}{\partial q^\alpha}\right) \\ +\frac{i}{2}\left(p_\beta \frac{\partial}{\partial p^\alpha} - p_\alpha \frac{\partial}{\partial p^\beta}\right) + \frac{\partial^2}{\partial p^\alpha q^\beta} - \frac{\partial^2}{\partial p^\beta q^\alpha}.\end{aligned}$$

These operators satisfy the commutation relations of the Poincaré Lie algebra, i.e.

$$[\hat{M}_{\mu\nu}, \hat{P}_\sigma] = i(g_{\nu\sigma}\hat{P}_\mu - g_{\sigma\mu}\hat{P}_\nu), \tag{4}$$

and

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] = -i(g_{\mu\sigma}\hat{M}_{\nu\rho} + g_{\nu\rho}\hat{M}_{\mu\sigma} - g_{\mu\rho}\hat{M}_{\nu\sigma} - g_{\nu\sigma}\hat{M}_{\mu\rho}). \tag{5}$$

This is proved by direct calculation, i.e.

$$\begin{aligned}[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] &= [\hat{Q}_\mu\hat{P}_\nu - \hat{Q}_\nu\hat{P}_\mu, \hat{Q}_\rho\hat{P}_\sigma - \hat{Q}_\sigma\hat{P}_\rho] \\ &= [\hat{Q}_\mu\hat{P}_\nu, \hat{Q}_\rho\hat{P}_\sigma] - [\hat{Q}_\mu\hat{P}_\nu, \hat{Q}_\sigma\hat{P}_\rho] \\ &\quad -[\hat{Q}_\nu\hat{P}_\mu, \hat{Q}_\rho\hat{P}_\sigma] + [\hat{Q}_\nu\hat{P}_\mu, \hat{Q}_\sigma\hat{P}_\rho] \\ &= [\hat{Q}_\mu, \hat{Q}_\rho\hat{P}_\sigma]\hat{P}_\nu + \hat{Q}_\mu[\hat{P}_\nu, \hat{Q}_\rho\hat{P}_\sigma] \\ &\quad -[\hat{Q}_\mu, \hat{Q}_\sigma\hat{P}_\rho]\hat{P}_\nu - \hat{Q}_\mu[\hat{P}_\nu, \hat{Q}_\sigma\hat{P}_\rho] \\ &\quad -[\hat{Q}_\nu, \hat{Q}_\rho\hat{P}_\sigma]\hat{P}_\mu - \hat{Q}_\nu[\hat{P}_\mu, \hat{Q}_\rho\hat{P}_\sigma] \\ &\quad +[\hat{Q}_\nu, \hat{Q}_\sigma\hat{P}_\rho]\hat{P}_\mu + \hat{Q}_\nu[\hat{P}_\mu, \hat{Q}_\sigma\hat{P}_\rho].\end{aligned}$$

Using the derivation of the commutator we have

$$\begin{aligned}[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] &= [\hat{Q}_\mu, \hat{Q}_\rho]\hat{P}_\sigma\hat{P}_\nu + \hat{Q}_\rho[\hat{Q}_\mu, \hat{P}_\sigma]\hat{P}_\nu \\ &\quad +\hat{Q}_\mu\hat{Q}_\rho[\hat{P}_\nu, \hat{P}_\sigma] + \hat{Q}_\mu[\hat{P}_\nu, \hat{Q}_\rho]\hat{P}_\sigma \\ &\quad -[\hat{Q}_\mu, \hat{Q}_\sigma]\hat{P}_\rho\hat{P}_\nu - \hat{Q}_\sigma[\hat{Q}_\mu, \hat{P}_\rho]\hat{P}_\nu \\ &\quad -\hat{Q}_\mu\hat{Q}_\sigma[\hat{P}_\nu, \hat{P}_\rho] - \hat{Q}_\mu[\hat{P}_\nu, \hat{Q}_\sigma]\hat{P}_\rho \\ &\quad -[\hat{Q}_\nu, \hat{Q}_\rho]\hat{P}_\sigma\hat{P}_\mu - \hat{Q}_\rho[\hat{Q}_\nu, \hat{P}_\sigma]\hat{P}_\mu \\ &\quad -\hat{Q}_\nu\hat{Q}_\rho[\hat{P}_\mu, \hat{P}_\sigma] - \hat{Q}_\nu[\hat{P}_\mu, \hat{Q}_\rho]\hat{P}_\sigma \\ &\quad +[\hat{Q}_\nu, \hat{Q}_\sigma]\hat{P}_\rho\hat{P}_\mu + \hat{Q}_\sigma[\hat{Q}_\nu, \hat{P}_\rho]\hat{P}_\mu \\ &\quad +\hat{Q}_\nu\hat{Q}_\sigma[\hat{P}_\mu, \hat{P}_\rho] + \hat{Q}_\nu[\hat{P}_\mu, \hat{Q}_\sigma]\hat{P}_\rho.\end{aligned}$$

This leads to Eq. (5). With the Lie algebra of the Poincaré group being satisfied, we proceed with the construction of the relativistic fields in phase space. The Casimir invariants of the Poincar é algebra is given by $C_1 = \hat{P}^2 = \hat{P}_\mu \hat{P}^\mu = m^2$ and $C_2 = \hat{W}_\mu \hat{W}^\mu = -m^2 s(s+1)$, where $\hat{W}_\mu$ is the Pauli-Lubanski pseudovector $\hat{W} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{M}^{\nu\rho}\hat{P}^\sigma$. The first invariant of the Poincaré algebra is related to the mass shell condition.

If we consider the generators of linear transformation $\hat{M}_{\mu\rho}$, which can be defined from the definition $M_{\alpha\beta} = q_\alpha p_\beta - p_\alpha q_\beta$, by using the van Hove vector field operators, we have

$$\hat{M}^{v-H}_{\alpha\beta} = M_{\alpha\beta} - \frac{1}{2}\left(q^\mu \frac{\partial M_{\alpha\beta}}{\partial q^\mu} + p^\mu \frac{\partial M_{\alpha\beta}}{\partial p^\mu}\right) \\ -i\left(\frac{\partial M_{\alpha\beta}}{\partial p_\mu}\frac{\partial}{\partial q^\mu} - \frac{\partial M_{\alpha\beta}}{\partial q^\mu}\frac{\partial}{\partial p_\mu}\right).$$

Using Proposition 1, we obtain

$$\hat{M}^{v-H}_{\alpha\beta} = ip_\beta \frac{\partial}{\partial p^\alpha} - ip_\alpha \frac{\partial}{\partial p^\beta} + iq_\beta \frac{\partial}{\partial q^\alpha} - iq_\alpha \frac{\partial}{\partial q^\beta}.$$

Such operator is another representation for the Poincaré Lie algebra. Here we explore Eq. (3) since $\hat{Q}_\alpha$ and $\hat{P}_\alpha$ are identified as observables.

4. Sympletic Klein Gordon Equation

Let us consider the representation of spin zero, $s = 0$, in the Pauli-Lubanski invariant; and so we write for the other Casimir invariant $\hat{P}^2 = \hat{P}_\mu \hat{P}^\mu = m^2 I$ (where we have used the Schur's lemma). Then we obtain the wave equation for the phase space scalar field $\phi(q, p)$,

$$\hat{P}^2\phi(q,p) = \hat{P}^\mu \hat{P}_\mu \phi(q,p) = m^2\phi(q,p),$$

or

$$-\frac{\partial^2\phi}{\partial q^\mu \partial q_\mu} - ip^\mu \frac{\partial\phi}{\partial q^\mu} + [p^\mu p_\mu/4 - m^2]\phi = 0,$$

which is a Klein–Gordon equation in phase space. A general solution is given by

$$\phi(q,p) = f(p^\mu)e^{\frac{i}{2}p^\mu q_\mu}.$$

This equation, and its complex conjugate, can also be obtained by the Lagrangian density in phase space (we use $\partial_\mu = \partial/\partial q^\mu$)

$$\mathcal{L}_0 = \partial^\mu\phi(q,p)\partial_\mu\phi^*(q,p) + \frac{i}{2}p^\mu[\phi(q,p)\partial_\mu\phi^*(q,p) \\ -\phi^*(q,p)\partial_\mu\phi(q,p)] + [p^\mu p_\mu/4 - m^2]\phi(q,p)\phi^*(q,p). \tag{6}$$

From the Noether theorem, the energy-momentum tensor is given by

$$\theta^{\mu\nu} = \frac{\partial\mathcal{L}_0}{\partial(\frac{\partial\phi(q,p)}{\partial\eta^\mu})}\frac{\partial\phi(q,p)}{\partial\eta_\nu} \tag{7}$$

$$+\frac{\partial\mathcal{L}_0}{\partial(\frac{\partial\phi^*(q,p)}{\partial\eta^\mu})}\frac{\partial\phi^*(q,p)}{\partial\eta_\nu} - g^{\mu\nu}\mathcal{L}_0. \tag{8}$$

where $\eta^\mu = (q^\mu, p^\mu)$. Using the Lagrangian density, $\theta^{\mu\nu}$ reads

$$\theta^{\mu\nu} = \partial^\mu \phi(q,p)\partial^\nu \phi^*(q,p) + [p^\mu p^\nu/4 - m^2 g^{\mu\nu}]\phi(q,p)\phi^*(q,p).$$

We can include interaction by defining the Lagrangian to be $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$, with $\mathcal{L}_I = \frac{\lambda\phi^4}{4!}$. Using the Euler Lagrange equations, $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0$, we obtain

$$-\frac{\partial^2 \phi(q,p)}{\partial q^\mu \partial q_\mu} - ip^\mu \frac{\partial \phi(q,p)}{\partial q^\mu} + [p^\mu p_\mu/4 - m^2]\phi(q,p) = \frac{\lambda\phi^3}{3!}.$$

In the sequence we derive the symplectic Dirac equation.

5. Symplectic Dirac Equation

In this section we consider spin 1/2 representation and the minimum gauge coupling to a spinor field. We follow as much as possible standard procedure, and then we apply the formalism to an electron in a external field—corresponding to the Landau-problem in phase space.

5.1. Spin 1/2 Symplectic Representaion

In order to study a representation for spin $\frac{1}{2}$ particles, we introduce the operator $\gamma^\mu \hat{P}_\mu$, where $\hat{P}_\mu = \frac{1}{2}p_\mu - i\partial_\mu$, such that, acting on a 4-component Dirac spinor in phase space $\Psi(q,p)$, we obtain

$$\gamma^\mu \hat{P}_\mu \Psi(q,p) = m\Psi(q,p),$$

or

$$\gamma^\mu \left(\frac{1}{2}p_\mu - i\partial_\mu\right)\Psi(q,p) = m\Psi(q,p), \tag{9}$$

which is the Dirac equation in phase space. Consistency with the mass-shell condition is obtained by following usual steps. First we write

$$(\gamma^\mu \hat{P}_\mu)(\gamma^\nu \hat{P}_\nu)\Psi(q,p) = m^2\Psi(q,p),$$

such that,

$$\gamma^\mu \left(\frac{1}{2}p_\mu - i\partial_\mu\right)\gamma^\nu \left(\frac{1}{2}p_\nu - i\partial_\nu\right)\Psi(q,p) = m^2\Psi(q,p).$$

Therefore,

$$(\gamma^\mu \hat{P}_\mu)(\gamma^\nu \hat{P}_\nu) = m^2 = \hat{P}_\mu \hat{P}^\mu = \hat{P}^2$$

Since $\hat{P}_\mu \hat{P}_\nu = \hat{P}_\nu \hat{P}_\mu$, we have

$$\gamma^\mu \gamma^\nu \hat{P}_\mu \hat{P}_\nu = \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\mu \gamma^\nu)\hat{P}_\mu \hat{P}_\nu,$$

such that

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g_{\mu\nu}.$$

Equation (9) is derived from the Lagrangian for spin $\frac{1}{2}$ particles in phase space, which is given by

$$\mathcal{L} = -\frac{i}{2}(\partial_\mu \bar{\Psi}\gamma^\mu \Psi - \partial_\mu \Psi \gamma^\mu \bar{\Psi}) - \left(\frac{1}{2}\not{p} - m\right)\Psi\bar{\Psi},$$

where $\bar{\Psi} = \gamma^0 \Psi^\dagger$ and $\not{p} = \gamma^\mu p_\mu$. This can be used to obtain the energy-momentum tensor in phase space for the Dirac field,

$$\begin{aligned}\theta^{\mu\nu} &= \frac{i}{2}\gamma^\mu \bar{\Psi}\partial^\nu \Psi - \frac{i}{2}\gamma^\mu \Psi \partial^\nu \bar{\Psi} \\ &\quad -\frac{i}{2}\gamma^\mu(\bar{\Psi}\partial^\nu \Psi - \Psi\partial^\nu \bar{\Psi}) - g^{\mu\nu}\mathcal{L}.\end{aligned}$$

5.2. Electron in an External Field

Let us examine the gauge symmetries in phase space demanding the invariance of the Lagrangian by a local gauge transformation given by $e^{\Lambda(q,p)}\Psi$. This leads to the minimum coupling,

$$\hat{P}^\mu \Psi \to (\hat{P}^\mu - eA^\mu)\Psi = \left(\frac{1}{2}p^\mu - i\partial^\mu - eA^\mu\right)\Psi,$$

where $A^\mu = A^\mu(q)$ is the four-potential. This describes an electron in an external field, with Dirac equation given by

$$\left[\gamma_\mu \left(\frac{1}{2}p^\mu - i\partial^\mu - eA^\mu\right) - m\right]\Psi = 0. \tag{10}$$

In order to illustrate such results, let us consider an electron in a external field given by $A = (A^0, \mathbf{A})$, with $A^0 = 0, A^i = \frac{1}{2}\epsilon_{ijk}B^j q^k$ and $\mathbf{B} = (0, 0, B)$. We are using for the coordinates $q = (q^0, \mathbf{q}); \mathbf{q} = (q^1, q^2, q^3)$. From the definition of the field, we have

$$A^1 = -\frac{1}{2}Bq^2; \quad A^2 = \frac{1}{2}Bq^1; \quad A^3 = 0, \tag{11}$$

such that the particle is free along the direction of the magnetic field. We consider the following representation for the Dirac matrix, γ^μ:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix},$$

where σ^i are the Pauli matrices. Multiplying Eq. (10) by $[\gamma_\mu(\frac{1}{2}p^\mu - i\partial^\mu - eA^\mu) + m]$, we obtain

$$\left[\left(\frac{1}{2}p^\mu - i\partial^\mu - eA^\mu\right)^2 - m^2\right]\Psi(q,p) = 0.$$

In terms of components, this equation reads

$$\begin{aligned}(m^2 - eB\Sigma^{12})\Psi = \Bigg[&\left(\frac{1}{2}p^0 - i\partial^0\right)^2 + \left(\frac{1}{2}p_1 - i\partial_1 - eA_1\right)^2 \\ &+ \left(\frac{1}{2}p_2 - i\partial_2 - eA_2\right)^2\Bigg]\Psi,\end{aligned}$$

where $\Sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/2 = (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/2$. Writing

$$\Psi(q,p) = \begin{pmatrix} \varphi(q^0, p^0)\phi(\mathbf{q},\mathbf{p}) \\ -\varphi(q^0, p^0)\phi(\mathbf{q},\mathbf{p}) \end{pmatrix},$$

where $\mathbf{q} = (q^1, q^2)$ and $\mathbf{p} = (p^1, p^2)$ in $\phi(\mathbf{q},\mathbf{p})$, we obtain

$$\left(\frac{1}{2}p^0 - i\partial^0)^2\varphi(q^0, p^0\right) = E^2\varphi(q^0, p^0)$$

and

$$\lambda^2\phi(\mathbf{q},\mathbf{p}) = \left[\left(\frac{1}{2}p_1 - i\partial_1 - \frac{1}{2}\omega q_2\right)^2 + \left(\frac{1}{2}p_2 - i\partial_2 - \frac{1}{2}\omega q_1\right)^2 - \omega\sigma^3\right]\phi(\mathbf{q},\mathbf{p}), \qquad (12)$$

where $\lambda^2 = E^2 - m^2$ and $\omega = eB$. We have used the field components A_1 and A_2, given in Eq. (11).

Let us define the creation and annihilation operators,

$$a_1 = \frac{\sqrt{\omega}}{2}\left(q_1 + \frac{i}{\omega}p_1 + \frac{2}{\omega}\partial_1\right),$$

$$a_1^+ = \frac{\sqrt{\omega}}{2}\left(q_1 - \frac{i}{\omega}p_1 - \frac{2}{\omega}\partial_1\right),$$

$$a_2 = \frac{\sqrt{\omega}}{2}\left(q_2 + \frac{i}{\omega}p_2 + \frac{2}{\omega}\partial_2\right),$$

$$a_2^+ = \frac{\sqrt{\omega}}{2}\left(q_2 - \frac{i}{\omega}p_2 - \frac{2}{\omega}\partial_2\right),$$

such that $[a_i, a_j^+] = \delta_{ij}$. Then Eq. (12) is written in a standard way

$$[a_1a_1^+ + a_2a_2^+ + 1 + i(a_1a_2^+ - a_1^+a_2) - \sigma^3]\phi = \frac{\lambda^2}{\omega}\phi.$$

This equation is rewritten by writing $a_+ = (a_1 + ia_2)$ and $a_- = (a_1 - ia_2)$ and the number operators $N_+ = a_+^+a_+$ and $N_- = a_-^+a_-$, with eigenvalues and eigen-states giving by $N_\pm|n_+, n_-\rangle = n_\pm|n_+, n_-\rangle$. This leads to

$$E = \pm[eB\,(2n_- + 1 - s) + m^2]^{1/2},$$

with $s = \pm 1$ being the eigenvalue of σ^3 in number basis: energy E is the Landau energy levels. This result shows the physical consistency of the representation with a non-trivial application.

Next section we investigate the physical meaning of the wave functions in phase space.

6. Symplectic Wave Functions and Wigner Funcion

As mentioned in the Introduction, in a previous work, we had studied symplectic representations of symmetries of space time (the Galilei and the

Poincaré groups) by using operators in the form $\widehat{A} = a_W\star$. In such cases the position and momentum observables, respectively, are given by

$$\widehat{Q} = q\star = q + \frac{i}{2}\frac{\partial}{\partial p},$$
$$\widehat{P} = p\star = p - \frac{i}{2}\frac{\partial}{\partial q}.$$

For example, representations of the Poincaré Lie algebra were studied by initially introducing $\hat{M}_{\alpha\beta} = \widehat{Q}_\alpha \widehat{P}_\beta - \widehat{Q}_\beta \hat{P}_\alpha$. Next we wrote the Klein–Gordon and the Dirac equation in phase space variables. Accordingly, the quasi-amplitude of probabilities $\psi_W(q,p)$ could be associated to the Wigner function $f_W(q,p)$ by $f_W(q,p) = \psi_W(q,p) \star \psi_W(q,p)^*$. In this section, our goal is to show that there is a mapping from the quasi-amplitude of probabilities $\psi_W(q,p)$ to the states derived from the van Hove operators, given in the previous section. We will denote the wave functions of those states by $\psi_{vH}(q,p)$.

We start with some standard facts on multilinear algebra. Let A be an operator on a Hilbert space H_1 and B an operator on another Hilbert space H_2. These operators induce a transformation $A \otimes B$ of the $H_1 \otimes H_2$ determined by

$$(A \otimes B)(|\psi_1\rangle \otimes |\psi_2\rangle) = A|\psi_1\rangle \otimes B|\psi_2\rangle, \tag{13}$$

where $|\psi_1\rangle \in H_1$ and $|\psi_2\rangle \in H_2$. If we set $A = e^{tM}$ and $B = e^{tN}$ then it follows that

$$(e^{tM} \otimes e^{tN})(|\psi_1\rangle \otimes |\psi_2\rangle) = e^{tM}|\psi_1\rangle \otimes e^{tN}|\psi_2\rangle, \tag{14}$$

The tensor product of the two exponentials can be expressed as one single exponential. In order to see this we differentiate Eq. (14) with respect to the parameter t and compute it at $t = 0$. One obtains,

$$\frac{d}{dt}[(e^{tM} \otimes e^{tN})(|\psi_1\rangle \otimes |\psi_2\rangle)]_{t=0} = (M \otimes 1^{(2)} + 1^{(1)} \otimes N) \times(|\psi_1\rangle \otimes |\psi_2\rangle), \tag{15}$$

where $1^{(2)}$ and $1^{(1)}$ are the identity operators on the Hilbert spaces H_2 and H_1 respectively. This shows that $(e^{tM} \otimes e^{tN})$ can be written as e^{tO}, where O is the operator $M \otimes 1^{(2)} + 1^{(1)} \otimes N$.

Consider the operators $X \otimes 1^{(2)}$ and $1^{(1)} \otimes Y$ on the space $H_1 \otimes H_2$, where X is an operator on H_1 and Y is an operator on H_2. Of course, $[X \otimes 1^{(2)}, 1^{(1)} \otimes Y] = 0$. It is easily seen that,

$$Xe^{tM}|\psi_1\rangle \otimes e^{tN}|\psi_2\rangle = (X \otimes 1^{(2)})(e^{tM} \otimes e^{tN}) (|\psi_1\rangle \otimes |\psi_2\rangle),$$

and similarly,

$$e^{tM}|\psi_1\rangle \otimes Ye^{tN}|\psi_2\rangle = (1^{(1)} \otimes Y)(e^{tM} \otimes e^{tN}) \times(|\psi_1\rangle \otimes |\psi_2\rangle). \tag{16}$$

Once we can find transformations on the Hilbert spaces H_1 and H_2 such that,

$$Xe^{tM} = e^{tM}X^{'}, \tag{17}$$

for X' operator on H_1, and

$$Ye^{tN} = e^{tN}Y^{'}, \tag{18}$$

for Y' operator on H_2, it follows that,

$$\begin{aligned}(X \otimes 1^{(2)})(e^{tM} \otimes e^{tN})(|\psi_1\rangle \otimes |\psi_2\rangle) = &(e^{tM} \otimes e^{tN}) \\ &\times(X' \otimes 1^{(2)}) \\ &\times(|\psi_1\rangle \otimes |\psi_2\rangle).\end{aligned}$$

Analogously,

$$\begin{aligned}(1^{(1)} \otimes Y)(e^{tM} \otimes e^{tN})(|\psi_1\rangle \otimes |\psi_2\rangle) = &(e^{tM} \otimes e^{tN}) \\ &\times(1^{(1)} \otimes Y') \\ &\times(|\psi_1\rangle \otimes |\psi_2\rangle).\end{aligned}$$

We shall now apply these algebraic ideas to the symplectic representations. Consider the pair of operators $\overline{Q} = q1$ and $\overline{P} = p1$, satisfying $[\overline{Q}, \overline{P}] = 0$, which means that they share common eigenvectors, despite that such operators transform as position and momentum. Let us be more specific about such operators and make the following identifications,

$$\begin{aligned}\overline{Q} &= q1 = q(1^{(1)} \otimes 1^{(2)}) \\ &= (q1^{(1)} \otimes 1^{(2)}) = Q^{(1)} \otimes 1^{(2)},\end{aligned}$$

where $Q^{(1)} = q1^{(1)}$. Similarly,

$$\begin{aligned}\overline{P} &= p1 = p(1^{(1)} \otimes 1^{(2)}) \\ &= (1^{(1)} \otimes p1^{(2)}) = 1^{(1)} \otimes P^{(2)},\end{aligned}$$

where $P^{(2)} = p1^{(2)}$. Next we identify $\{|q\rangle\}$ and $\{|p\rangle\}$ as two independent basis of the two Hilbert spaces H_1 and H_2 respectively. Thus we reproduce the previous results,

$$\overline{Q}|q, p\rangle = q|q, p\rangle, \tag{19}$$

$$\overline{P}|q, p\rangle = p|q, p\rangle, \tag{20}$$

where $|q, p\rangle = |q\rangle \otimes |p\rangle$. It is important to observe that, by knowing how the operators act on a basis of the product space, we know how they act on any vector of that space.

Next we make explicit how the representation based on the van Hove operators relates to the representation of the quasi-amplitude of probabilities by a tensor product of two gauge transformations. In order to do so we go back to Eq. (14) and set,

$$M = p \cdot Q^{(1)}, \tag{21}$$

where p is a c-number with units of momentum. In addition, we have,

$$N = -2q \cdot P^{(2)}, \tag{22}$$

where q is also a c-number with units of position. We shall point out we are working here in the active point of view for transformations of states. The choices of M and N are taken in order to obtain the desired effects of the transformations on wave functions. Therefore, by setting the parameter t in Eq. (14) equal to $-i$, then we obtain for the exponential operator,

$$(e^{tp\cdot Q^{(1)}} \otimes e^{-2tq\cdot P^{(2)}}) = e^{-i(p\cdot Q^{(1)}\otimes 1^{(2)} - 1^{(1)}\otimes(2q\cdot P^{(2)}))}. \tag{23}$$

Making the gauge transformation on ψ_W, we obtain the wave function ψ,

$$\psi(q,p) = e^{-i(p\cdot Q^{(1)}\otimes 1^{(2)} - 1^{(1)}\otimes(2q\cdot P^{(2)}))}\psi_W(q,p).$$

Under such transformation, the operators on $\psi(q,p)$ are transformed into $\widehat{Q}$ and $\widehat{P}$. Let us write,

$$\widehat{P}_{vH} = 1^{(1)} \otimes \left(-i\frac{\partial}{\partial q}\right)^{(2)}. \tag{24}$$

Now, we compute,

$$\begin{aligned}\widehat{P}_{vH}\psi(q,p) &= 1^{(1)} \otimes \left(-i\frac{\partial}{\partial q}\right)^{(2)} \\ &\quad\times \left(e^{-i(p\cdot Q^{(1)}\otimes 1^{(2)} - 1^{(1)}\otimes(2q\cdot \widehat{P}^{(2)}))}\right)\psi_W(q,p) \\ &= 1^{(1)} \otimes \left(-i\frac{\partial}{\partial q}\right)^{(2)} \\ &\quad\times \left(e^{tp\cdot Q^{(1)}} \otimes e^{-2tq\cdot P^{(2)}}\right)\psi_W(q,p)\end{aligned}$$

Since we label operators with (1) and (2), we know where they act, so we can drop the tensor product sign, to write

$$\begin{aligned}\widehat{P}_{vH}\psi(q,p) &= e^{-ip\cdot Q^{(1)}}\left(-i\frac{\partial}{\partial q}\right)^{(2)}(e^{2iq\cdot P^{(2)}}\psi_W(q,p)) \\ &= e^{-ip\cdot Q^{(1)}}e^{2iq\cdot P^{(2)}}\left[2p - i\frac{\partial}{\partial q}\right]\psi_W(q,p) \\ &= \left(e^{-ip\cdot Q^{(1)}} \otimes e^{2iq\cdot P^{(2)}}\right)(1^{(1)} \otimes 2\widehat{P})\psi_W(q,p).\end{aligned}$$

Writing for another Van Hove operator

$$\widehat{Q}_{vH} = \left(q + i\frac{\partial}{\partial p}\right)^{(1)} \otimes 1^{(2)}, \tag{25}$$

and performing an analogous calculation, we obtain

$$\begin{aligned}\widehat{Q}_{vH}\psi(q,p) &= \left(\left(q+i\frac{\partial}{\partial p}\right)^{(1)} \otimes 1^{(2)}\right) \\ &\quad \times \left(e^{-ip\cdot Q^{(1)}} \otimes e^{2iq\cdot P^{(2)}}\right)\psi_W(q,p) \\ &= e^{-ip\cdot Q^{(1)}} e^{2iq\cdot P^{(2)}}\left[2q+i\frac{\partial}{\partial p}\right]\psi_W(q,p) \\ &= \left(e^{-ip\cdot Q^{(1)}} \otimes e^{2iq\cdot P^{(2)}}\right)(2\widehat{Q}\otimes 1^{(2)})\psi_W(q,p).\end{aligned}$$

Therefore, we have that, under proper gauge transformation, the van Hove operators are transformed into the operators $\widehat{Q}$ and $\widehat{P}$ and vice-versa; in short:

$$\widehat{Q}_{vH}\psi = (2\widehat{Q}\otimes 1^{(2)})\psi_W, \qquad (26)$$

$$\widehat{P}_{vH}\psi = (1^{(1)}\otimes 2\widehat{P})\psi_W. \qquad (27)$$

Using these results, the representations based on the van Hove operators are mapped into the representations based on the quasi-amplitudes of probabilities.

The representation for spin 1/2 based on the quasi-amplitudes is given by,

$$\gamma^{\mu}\left(p_{\mu} - i\frac{1}{2}\partial_{\mu}\right)\Psi_W(q,p) = m\Psi_W(q,p). \qquad (28)$$

It can be shown that the function $f(q,p) = \Psi_W(q,p) \star \overline{\Psi}_W(q,p)$ satisfies all the properties of a Wigner function, including the fact that it is real, normalized to unity and non positive-defined, as well as the equation of motion for fermions, i.e.,

$$\gamma^{\mu}p_{\mu} \star f(q,p) - f(q,p) \star \gamma^{\mu}p_{\mu} = 0 \qquad (29)$$

These results provide a physical interpretation for both representations, since the symplectic wave functions are associated to each other, and in turn are related to the Wigner function.

7. Concluding Remarks

In this work we have studied symplectic representations of the Poincaré space-time symmetry, considering initially van Hove vector fields as linear mappings in a Hilbert space defined on a relativistic symplectic manifold. In this context, we derive the Klein–Gordon and the Dirac equation in phase space. The physical meaning of such representations are considered and for the case of the spin 1/2, we have studied the Landau problem, providing consistent physical results.

Some aspects, which remain to be explored, are important to be mentioned here. Indeed, the formalism can be generalized to non-abelian groups and perturbation methods can be developed in phase space following in parallel to the usual ones. In this representation based on the van Hove operators, the observable describing position can be taken as a c-number operator. This makes the calculations for interacting fields similar to the usual procedures. In particular, for the case of many-body physics and the symplectic Fock space, the interaction is described by c-number operators, with standard rules for the quantization. We have seen this in the application of the Landau problem. In this case the phase space wave function deserves to be closely analyzed. These aspects are left for another work that is in preparation.

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