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Dynamical Localization for Discrete Anderson Dirac Operators

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Abstract We establish dynamical localization for random Dirac operators on the d-dimensional lattice, with $d \in \{1, 2, 3\}$, in the three usual regimes: large disorder, band edge and 1D. These operators are discrete versions of the continuous Dirac operators and consist in the sum of a discrete free Dirac operator with a random potential. The potential is a diagonal matrix formed by different scalar potentials, which are sequences of independent and identically distributed random variables according to an absolutely continuous probability measure with bounded density and of compact support. We prove the exponential decay of fractional moments of the Green function for such models in each of the above regimes, i.e., (j) throughout the spectrum at larger disorder, (jj) for energies near the band edges at arbitrary disorder and (jjj) in dimension one, for all energies in the spectrum and arbitrary disorder. Dynamical localization in theses regimes follows from the fractional moments method. The result in the one-dimensional regime contrast with one that was previously obtained for 1D Dirac model with Bernoulli potential.

Keywords Anderson Dirac operators · Dynamical localization · Fractional moments method

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1 Introduction

The study of localization properties for random operators, in special the phenomenon of dynamical localization, has become an important field of research. Several models of random operators have been considered in the literature, such as the Anderson Schrödinger model, Anderson-type random hamiltonians, the unitary Anderson model, random block operators, among others (see references [1–4,8,10–12,14,17,19,20,26–28]). The prototypical model in the study of localization properties of quantum states of single electrons in disordered solids is the Anderson Schrödinger model, for which the property of dynamical localization is known to hold in each of the following regimes (see [27] and references therein): (j) in any spacial dimension and for all energies in the spectrum at larger disorder, (jj) near band edges of the spectrum in any spacial dimension and for arbitrary disorder, and (jjj) for all energies in the spectrum and arbitrary disorder, in dimension one.

In this paper we study dynamical localization in regimes (j)–(jjj) above for a class of random Dirac operators, which we call discrete Anderson Dirac (DAD) model, defined by

$$H_\omega(m,c) := H_0(m,c) + V_\omega, \tag{1}$$

acting on the Hilbert space $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ with $\nu = \begin{cases} 2 & \text{if } \ d \in \{1,2\} \\ 4 & \text{if } \ d = 3 \end{cases}$, where $\omega \in \Omega = \mathbb{R}^{\mathbb{Z}^d}$, $m \geq 0$ is the mass of a particle in the lattice $\mathbb{Z}^d$ and $c > 0$ represents the speed of light. The free Dirac operator $H_0(m,c)$ and the random potential V_ω are defined as follows:

(i) $H_0(m,c) := c\mathcal{D}_d + mc^2\mathcal{B}_d$ with

$$\mathcal{D}_1 = \begin{pmatrix} 0 & d_1^* \\ d_1 & 0 \end{pmatrix}, \ \mathcal{D}_2 = \begin{pmatrix} 0 & d_1^* - id_2^* \\ d_1 + id_2 & 0 \end{pmatrix}, \ \mathcal{D}_3 = \begin{pmatrix} O_2 & \sigma \cdot D^* \\ \sigma \cdot D & O_2 \end{pmatrix},$$

$$\mathcal{B}_1 = \mathcal{B}_2 = \sigma_3, \quad \mathcal{B}_3 = \begin{pmatrix} I_2 & O_2 \\ O_2 & -I_2 \end{pmatrix}, \tag{2}$$

where I_2 is the 2×2 identity matrix, O_2 is the 2×2 null matrix, $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the triple of usual Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

$D = (d_1, d_2, d_3)$ is the triple of finite difference operators $d_j = \tau_j - Id$, $j \in \{1,2,3\}$, each acting on $l^2(\mathbb{Z}^d, \mathbb{C})$, where τ_j is the right shift only in the coordinate j, and $D^* = (d_1^*, d_2^*, d_3^*)$ is the triple of adjoints of d_1, d_2 and d_3, respectively. We have

$$\sigma \cdot D = \sum_{j=1}^{3} \sigma_j d_j = \begin{pmatrix} d_3 & d_1 - id_2 \\ d_1 + id_2 & -d_3 \end{pmatrix},$$

$$\sigma \cdot D^* = \sum_{j=1}^{3} \sigma_j d_j^* = \begin{pmatrix} d_3^* & d_1^* - id_2^* \\ d_1^* + id_2^* & -d_3^* \end{pmatrix}.$$

The operator $H_0(m,c)$ is bounded and self-adjoint on $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$.

(ii) V_ω is a diagonal matrix formed by different random potentials $V_\omega^{(\alpha)} : \mathbb{Z}^d \to \mathbb{R}$ given by $V_\omega^{(\alpha)}(j) = \omega_{j\alpha}$ with $j \in \mathbb{Z}^d$ and $\alpha \in \mathcal{A} = \{1, \ldots, \nu\}$ ($\nu = 2$ or $\nu = 4$). We write

$$V_\omega = \operatorname{diag}\left(V_\omega^{(1)}, \ldots, V_\omega^{(\nu)}\right). \tag{4}$$

We assume that $\omega = \{\omega_{j\alpha}\}_{j\in\mathbb{Z}^d,\alpha\in\mathcal{A}}$ is a 2-parameter family of independent, identically distributed (i.i.d.) random variables with common Borel probability measure μ which is absolutely continuous with bounded density $\rho \in \mathrm{L}^\infty(\mathbb{R})$ and of compact support. We denote by

$$\mathrm{supp}\ \mu := \{x \in \mathbb{R} : \mu(x-\epsilon, x+\epsilon) > 0,\ \forall\epsilon > 0\} = [\omega_{\min}, \omega_{\max}]$$

the support of μ, where $\omega_{\min} = \inf(\mathrm{supp}\ \mu)$ and $\omega_{\max} = \sup(\mathrm{supp}\ \mu)$. Let $\mathbb{P}$ be the product measure generated by the pre-measure induced by μ on the Borel cylinder sets in $\Omega = \mathbb{R}^{\mathbb{Z}^d}$. Under the above assumptions, the potential matrix V_ω is a bounded multiplication operator and therefore each $H_\omega(m, c)$, defined in (1), is a bounded self-adjoint operator on $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$. These operators $H_\omega(m, c)$ are discrete versions of the continuous Dirac operators in quantum mechanics [29]. Dynamical and spectral properties of the model (1) in dimension $d = 1$ have already been studied by the authors with other potentials [7,10,23]; the explicitly versions of this model in $d = 2$ and $d = 3$ are being considered here for the first time (to the best of our knowledge).

In [10], two of the present authors have studied dynamical localization (uniform boundedness in time of each moment of the position operator, i.e., relation (6) in Sect. 3) for the model (1) in dimension $d = 1$ with potentials $V_\omega(n)$, $n \in \mathbb{Z}$, being i.i.d. Bernoulli random variables taking values $\pm V$ with $V > 0$. It was shown in [10] that the massive case (when $m > 0$) has dynamical localization and the zero mass case presents dynamical delocalization for specific values of the energy. The method that was used to obtain dynamical localization was the multiscale analysis, a technique initially developed by Fröhlich and Spencer in 1983 [13].

In the present work, we establish dynamical localization in the strong exponential sense of Definition 3.1 (see Sect. 3), for the DAD model (1) in the regimes (j)–(jj) mentioned above, for dimensions $d \in \{1, 2, 3\}$, and for dimension $d = 1$ in the regime (jjj). This concept of dynamical localization implies that $H_\omega(m, c)$ has pure point spectrum for almost every ω (see Sect. 3) and is a way of asserting that solutions of the Dirac equation $H_\omega(m, c)\Psi(t) = i\partial_t\Psi(t)$ keep strongly localized in space, uniformly in time, so that all moments of the position operator are bounded in time (see relation (6) in Sect. 3, which is interpreted as absence of quantum transport). The method used here to obtain dynamical localization is the fractional moments method (FMM), described in Theorem 3.1, which was introduced by Aizenman and Molchanov in 1993 [2]; it was further developed in [3,16] and recently it was extended for random operators acting in spaces l^2 on general graphs [4] (see also the review paper [27]). This method gives dynamical localization under the condition that the fractional moments of the Green function decay exponentially. Although it requires that the distribution of potential values is absolutely continuous with bounded density and of compact support, the FMM is technically simpler than the multiscale analysis, and in general it also gives stronger results on dynamical localization. The applications of the FMM to DAD model are described in Theorems 3.2, 3.3 and 3.4 (see Sect. 3.2 for details), which shows the exponential decay of fractional moments of the Green function in the three usual regimes: large disorder, band edge and 1D; these are the main results of this paper.

The main motivation for studying dynamical localization for DAD using FMM comes from the d-dimensional discrete Anderson Schrödinger model. Although we have gotten final statements similar to the Schrödinger case, the application of the method to the Dirac setting is not immediate, and each step needs to be verified. Now we mention some points of contrast or connection with respect to the Schrödinger case:

(i) In contrast to the Schrödinger case, model (1) have different actions depending on the spatial dimension $d \in \{1, 2, 3\}$; hence, we have some specific calculations and details in each dimension, as in the proof of Theorem 3.2.

(ii) In model (1), the potential V_ω is a diagonal matrix (4) formed by different potentials $V_\omega^{(\alpha)}(j) = \omega_{j\alpha}$ which are i.i.d. random variables with the same probability measure. In addition, we assume the independence of the variables $\omega_{j\alpha}$ and $\omega_{k\beta}$ not only with respect to positions j, k, but also with respect to the components α, β (two distinct sequences are independent). These facts are essential in the proofs of Proposition 4.1 and Theorems 3.2, 3.3 and 3.4; we underline that the analysis here does not cover the case of potential which does not depend on the component variables. In the one-dimensional Schrödinger case the potentials are sequences of scalars, but since the potential here is assumed to depend on the component variables, our model really can be reformulated as a Schrödinger model on a product graph (e.g., $\mathbb{Z}^d \times \{1, 2, 3, 4\}$); so, from some perspective, this is an interesting case of random Schrödinger models with particular potential physical relevance.

(iii) The definition of Green function is different from the Schrödinger case because the matrix elements $G_\omega^{\alpha\beta}(j, k; z)$ depend not only on the positions $j, k \in \mathbb{Z}^d$, but also on components α, β. Thus, we obtain recursive relations (30)–(40) between the fractional moments of the components of Green function, in dimensions $d \in \{1, 2, 3\}$ (see proof of Theorem 3.2), which implies, by using Proposition 4.1, in exponential decay of those moments in each component.

(iv) The spectrum of the free Dirac operator $H_0(m, c)$ consists of two disjoint intervals (except for $m = 0$) with the famous *negative energies* (see Theorem 2.1), whereas just one interval for the Schrödinger model on $\mathbb{Z}^d$; this implies in four extremes energy intervals $I_\delta^\pm(m, c)$ (for $m > 0$ and $0 < \omega_{\max} < 2mc^2$) for which $H_\omega(m, c)|_{\mathcal{H}_\pm}$ exhibits dynamical localization (see Theorem 3.3).

(v) In contrast with the 1D Bernoulli potentials mentioned above, for DAD with null mass, i.e., $m = 0$, it is concluded in Theorem 3.4 that dynamical localization holds true for all energies in the spectrum (under arbitrary disorder), in the case where the distribution of the potential is absolutely continuous with bounded density of compact support. There is, therefore, a drastic difference in dynamical behavior of the DAD between a continuous and a extremely singular distribution (i.e., Bernoulli).

(vi) In the proof of our 1D localization result (Theorem 3.4), we work with the FMM in infinite volume and with complex energies, while in the localization result obtained for one-dimensional Anderson Schrödinger model (Theorem 4.1 in [18]), the authors have established the exponential decay of fractional moments of the Green function in finite volume and for real energies.

The results are divided into three parts. In the first one, which refers to large disorder regime, we establish the boundedness of the fractional moments of the Green function of the operator $H_{\omega,\lambda}(m, c)$, for every $\lambda > 0$ (Proposition 4.1). Using this result together with the decoupling Lemma 5.1, we obtain the exponential decay of fractional moments of the Green function, for dimensions $d \in \{1, 2, 3\}$, for large disorder λ and for all energies in the spectrum (see Theorem 3.2(i)). Dynamical localization for $H_{\omega,\lambda}(m, c)$ at large disorder and for all energies in the spectrum (see Theorem 3.2(ii)) then follows from fractional moments method in the Dirac context (Theorem 3.1). In the second part, which refers to band edge regime, we establish the exponential decay of fractional moments of the Green function of the operator $H_\omega(m, c)$ (restricted to subspaces of negative and positive energies), for dimensions $d \in \{1, 2, 3\}$ and $m > 0$, at energies near the bottom of the spectrum (see

Theorem 3.3(i) for precise statements). The general idea in this case is the following: we control the expectation value of fractional moments of the infinite volume Green function in terms of the corresponding expectation values of finite volume restrictions of the operator (see Proposition 8.1 and the decoupling estimate (61) in Sect. 10). Dynamical localization for $H_\omega(m, c)$ at energies near the bottom of the spectrum (see Theorem 3.3(ii)) follows, again, by Theorem 3.1. In the third part, which refers to 1D localization regime, we establish the exponential decay of fractional moments of the Green function of $H_\omega(m, c)$ at all energies of the spectrum (see Theorem 3.4(i)), which implies dynamical localization in this regime (Theorem 3.4(ii)). In the one-dimensional case, we use the formalism of transfer matrices (see Sect. 12), which allows us to obtain the exponential decay in Lemma 12.1, as well as a special representation (65) of the Green function in terms of generalized eigenfunctions.

In the case of null mass $m = 0$ or the massive case $m > 0$ with $\omega_{\max} \geq 2mc^2$, there is no gap in the spectrum of $H_\omega(m, c)$ (see Sect. 2) and the proof of Lemma 7.1 (initial estimate of Lifshitz tails) does not hold; consequently, we can not guarantee dynamical localization for $H_\omega(m, c)$ in these cases (except if $\omega_{\max}$ is sufficiently large so that the large disorder regime is reached).

1.1 Connection with Other Works

At first glance, the results presented here may be seem as particular cases of those in [12,28], where localization in the large disorder regime for Schrödinger operators with nonmonotone random potentials and for Anderson models on locally finite graphs have been studied, respectively, or even [11,14], where weakly disordered regime for Bogoliubov-de Gennes operators and band edge regime for random block operators have been considered, respectively. However, there are some differences that are worth mentioning.

First, by (ii) above, differently from [11] (where the potential V is defined in their relation (19) with the assumptions required for the operator be self-adjoint and with an uniformly α-Hölder continuous distribution of i.i.d. random variables) and [12] (where the nonmonotone potentials sequences of i.i.d. random variables has an α-regular distribution with a finite q-moment), the potential considered in DAD model is a monotone diagonal matrix (4) with i.i.d. random variables in each entry, which distribute according to an absolutely continuous probability measure with bounded density and of compact support, with each entry completely independent of the others; this is a fundamental technical assumption in our proofs. In [11], localization is proven in the weak disorder regime for the spectrum in the central gap, with quite different techniques from this paper. Our strategy in this work is to use the FMM in infinite volume (Theorem 3.1), whereas in [12] the authors base their arguments on eigenfunction correlators in finite volumes and a Wegner estimate.

In [28], the localization in large disorder regime is studied specifically for Anderson Schrödinger models on locally finite graphs, a class of operator different from DAD. The work [14] uses the technique of multiscale analysis to establish dynamical localization for random block operators in a neighborhood of the internal band edges and, once again, their model does not include DAD as a particular case.

Finally, the Dirac model (1) can be seen as a particular case of models studied in [2], but our localization results (Theorems 3.2 and 3.3) are technically different from the obtained in [2] (in particular, they depend on some specific constructions of DAD; see comments in Sect. 3.2).

1.2 Some Physical Remarks

The introduction of the discrete Dirac model in 1D [9,10] was motivated by the desire to present a corresponding version of the usual tight-binding Schrödinger case, with the possibility of a zero mass m parameter, where its transfer matrices are quite similar to the dimerized Schrödinger transfer matrices (which results in specific physical properties in case of $m = 0$, as delocalization for some Bernoulli potentials). We have then considered this model with sparse potentials [7,23] and obtained some dynamical bounds for specific models [22]. The 1D Dirac model has also been used in related but different contexts [5,15, 21,24,25,30].

It is natural to consider the discrete Dirac model in two and three dimensions, particularly in the context of Anderson potentials and localization, both spectral and dynamical. This is the subject of this work, which discuss the three usual regimes of dynamical localization for the DAD model, using the FMM in infinite volume.

1.3 Organization

The organization of the paper is as follows. In Sect. 2 we describe explicitly the spectrum of the Dirac operators (1). Sect. 3 is divided into two Sects. 3.1 and 3.2: in the first, we present the FMM adapted to the DAD model (Theorem 3.1); in the second, we present our results on dynamical localization for DAD in the regimes: large disorder (Theorem 3.2), band edge (Theorem 3.3) and 1D (Theorem 3.4). In Sect. 4 we prove the boundedness of fractional moments. Section 5 is dedicated to the proof of Theorem 3.2. In Sect. 6 we define finite volume restrictions of $H_\omega(m, c)$ with Neumann and simple boundary conditions. Section 7 is devoted to establishing probabilistic estimates on the eigenvalues of finite volume restrictions next to bottom of spectrum of $H_\omega(m, c)$. In Sect. 8 we obtain the exponential decay of the fractional moments of the Green function of finite volume operators given by Proposition 8.1, which is a fundamental result for the proof of Theorem 3.3. The geometric resolvent equation is described in Sect. 9. In Sect. 10 we establish relations of decoupling of fractional moments that allow us to write the fractional moments of the infinite volume Green function in terms of the corresponding finite volume Green function. Section 11 is dedicated to the proof of Theorem 3.3. In Sect. 12 we present properties of the 1D Dirac model. Finally, Sect. 13 is dedicated to the proof of Theorem 3.4.

2 Spectrum of the Anderson Dirac Operators

The goal of this section is to characterize explicitly the spectrum of the Dirac operators $H_\omega(m, c)$, defined by (1). We denote it by $\sigma(H_\omega(m, c))$. First, we find the spectrum of the free operator $H_0(m, c)$, which in the 1D case has already been established in Proposition 2.8 in [7] through the behavior of the m-function. Here we obtain the spectrum, for dimensions $d \in \{1, 2, 3\}$, via Fourier transform.

Theorem 2.1 *The spectrum of $H_0(m, c)$ is given by*

$$\sigma(H_0(m, c)) = \left[-\sqrt{m^2c^4 + a_d c^2}, -mc^2\right] \cup \left[mc^2, \sqrt{m^2c^4 + a_d c^2}\right]$$

with the constant

$$a_d = \begin{cases} 4 & \text{if} \quad d = 1, \\ 6 + 4\sqrt{2} \ \approx 11.66 & \text{if} \quad d = 2, \\ \max\limits_{x \in [0,2\pi]^3} f_+(x) \ \approx 17.23 & \text{if} \quad d = 3, \end{cases}$$

where

$$f_+(x) = \ 2\sum_{j=1}^{3}(1 - \cos(x_j)) + \sqrt{8 \sum_{(u,v)\in\mathcal{C}} (1 - \cos u)(1 - \cos v)(1 - \cos(u - v))},$$

for $x = (x_1, x_2, x_3) \in [0, 2\pi]^3$ *and* $\mathcal{C} = \{(x_1, x_2), (x_1, x_3), (x_2, x_3)\}$. *Furthermore, the spectrum of* $H_0(m, c)$ *is purely absolutely continuous.*

Proof As usual, we will show that the free Dirac operator $H_0(m, c)$ is unitarily equivalent, via Fourier transform, to a diagonal matrix operator formed by multiplication operators. In fact, consider the Fourier transform $F : \mathrm{L}^2([0, 2\pi]^d, \mathbb{C}^\nu) \longrightarrow l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ and its inverse F^{-1}, in dimensions $d \in \{1, 2, 3\}$, given by

$$(Fg)(j) = \frac{1}{(2\pi)^{d/2}} \int_{[0,2\pi]^d} e^{-ij\cdot x} g(x)\, dx \quad \text{and} \quad (F^{-1}u)(x) = \frac{1}{(2\pi)^{d/2}} \sum_{j\in\mathbb{Z}^d} u(j)e^{ij\cdot x},$$

where $j \cdot x = \sum_{n=1}^{d} j_n x_n$. A calculation shows that

$$\left[\left(F^{-1} H_0(m, c) F\right) g\right](x) = M_d \ g(x),$$

for all $g \in \mathrm{L}^2([0, 2\pi]^d, \mathbb{C}^\nu)$, $x = (x_1, \ldots, x_d) \in [0, 2\pi]^d$, with

$$M_d \ = \ c \sum_{n=1}^{d} S_n^{(d)} + mc^2 \mathcal{B}_d, \quad d \in \{1, 2, 3\},$$

where $\mathcal{B}_d$ are the diagonal matrices defined in (2) and

$$S_1^{(1)} = S_1^{(2)} = \sigma_1 U_+(x_1), \quad S_2^{(2)} = \sigma_2 U_+(x_2), \quad S_n^{(3)} = \begin{pmatrix} O_2 & (\sigma_n U_-(x_n))^* \\ \sigma_n U_-(x_n) & O_2 \end{pmatrix},$$

with $\sigma_n, \ n \in \{1, 2, 3\}$, being the Pauli Matrices defined in (3) and

$$U_\pm(x_n) = \begin{pmatrix} e^{-ix_n} - 1 & 0 \\ 0 & e^{\pm ix_n} - 1 \end{pmatrix}.$$

After diagonalizing the matrices M_d (for x fixed), we obtain the diagonal matrix

$$D_d = \mathrm{diag}\,(\lambda_1(x), \lambda_2(x)) \quad \text{for} \quad d \in \{1, 2\},$$

where $\lambda_1(x) = -\lambda_2(x) = \sqrt{m^2c^4 + c^2 f_+(x)}$ and

$$f_+(x) = \begin{cases} 2(1 - \cos x) & \text{if} \quad d = 1, \\ 2(1 - \cos x_1)(1 - \sin x_2) + 2(1 - \cos x_2)(1 + \sin x_1) & \text{if} \quad d = 2. \end{cases}$$

In dimension $d = 3$,

$$D_d = \mathrm{diag}\,(\lambda_1(x), \lambda_2(x), \lambda_3(x), \lambda_4(x))$$

where $\lambda_1(x) = -\lambda_3(x) = \sqrt{m^2c^4 + c^2 f_+(x)}\,$, $\lambda_2(x) = -\lambda_4(x) = \sqrt{m^2c^4 + c^2 f_-(x)}$ and

$$f_\pm(x) = 2\sum_{j=1}^{3}(1 - \cos(x_j)) \pm \sqrt{8 \sum_{(u,v)\in\mathcal{C}} (1 - \cos u)(1 - \cos v)(1 - \cos(u - v))},$$

for $x = (x_1, x_2, x_3) \in [0, 2\pi]^3$ and $\mathcal{C} = \{(x_1, x_2), (x_1, x_3), (x_2, x_3)\}$.

It follows that $H_0(m, c)$ is unitarily equivalent to the diagonal matrix operator D_d, which is formed by multiplication operators $\mathcal{M}_{\lambda_i(x)}$, $i \in \mathcal{A}$. Thus

$$\begin{aligned}\sigma(H_0(m, c)) = \sigma(D_d) &= \bigcup_{i\in\mathcal{A}} \sigma(\mathcal{M}_{\lambda_i(x)}) \\ &= \left[-\max_{x\in[0,2\pi]^d} \lambda_1(x), -\min_{x\in[0,2\pi]^d} \lambda_1(x)\right] \bigcup \left[\min_{x\in[0,2\pi]^d} \lambda_1(x), \max_{x\in[0,2\pi]^d} \lambda_1(x)\right]\end{aligned}$$

with

$$\min_{x\in[0,2\pi]^d} \lambda_1(x) = \sqrt{m^2c^4 + c^2 \min_{x\in[0,2\pi]^d} f_+(x)} = \sqrt{m^2c^4 + c^2 \cdot 0} = mc^2$$

and

$$\max_{x\in[0,2\pi]^d} \lambda_1(x) = \sqrt{m^2c^4 + c^2 \max_{x\in[0,2\pi]^d} f_+(x)},$$

where $\max_{x\in[0,2\pi]^d} f_+(x) = a_d$ is given in the statement of the theorem. Moreover, the function $\lambda_i : [0, 2\pi]^d \longrightarrow \mathbb{R}$ is continuous and inverse images of sets of zero Lebesgue measure in $\mathbb{R}$ under the function λ_i are sets of zero Lebesgue measure in $\mathbb{R}^d$. Therefore, the spectrum $\sigma(H_0(m, c)) = \bigcup_{i\in\mathcal{A}} \sigma(\mathcal{M}_{\lambda_i(x)})$ is purely absolutely continuous. □

The next step is to determine the spectrum of $H_\omega(m, c)$. It follows, as a consequence of the general theory of ergodic operators [19], for which the Anderson Dirac operator is a special case, that the spectrum of $H_\omega(m, c)$ is almost surely deterministic, i.e., there exists a closed subset Σ of $\mathbb{R}$ such that

$$\sigma(H_\omega(m, c)) = \Sigma, \qquad \mathbb{P}\text{-a.s.}$$

The following result explicitly describes the spectrum of the Dirac operators $H_\omega(m, c)$. The proof will be omitted since it is analogous the proof of Theorem 3.9 in [19] or Theorem 2 in [27].

Theorem 2.2 *The spectrum of $H_\omega(m, c)$ is almost surely given by*

$$\sigma(H_\omega(m, c)) = \sigma(H_0(m, c)) + \operatorname{supp} \mu\,.$$

By hypothesis, $\operatorname{supp} \mu = [\omega_{\min}, \omega_{\max}]$. Without loss of generality we will assume that $\omega_{\min} = 0$, since otherwise just consider the operator $H_\omega(m, c) - \omega_{\min} I_2$. Thus $V_\omega^{(\alpha)}(j) = \omega_{j\alpha} \geq 0$ for all $j \in \mathbb{Z}^d$ and $\alpha \in \mathcal{A}$. By Theorems 2.2 and 2.1, the spectrum of $H_\omega(m, c)$ is almost surely

$$\sigma(H_\omega(m, c)) = \Sigma = \left[-\sqrt{m^2c^4 + a_dc^2}, -mc^2 + \omega_{\max}\right] \cup \left[mc^2, \sqrt{m^2c^4 + a_dc^2} + \omega_{\max}\right].$$

Furthermore, if one assumes that $0 < \omega_{\max} < 2mc^2$ for $m > 0$, which guarantees the existence of the gap $(-mc^2 + \omega_{\max}, mc^2)$ in the spectrum of $H_\omega(m, c)$ for $m > 0$, then the values

$$-\sqrt{m^2c^4 + a_dc^2}, \quad -mc^2 + \omega_{\max}, \quad mc^2 \quad \text{and} \quad \sqrt{m^2c^4 + a_dc^2} + \omega_{\max}$$

are the band edges of $\sigma(H_\omega(m, c))$. In the case $m = 0$, the spectrum of $H_\omega(m, c)$ does not have a gap and $-\sqrt{m^2c^4 + a_dc^2}$ and $\sqrt{m^2c^4 + a_dc^2} + \omega_{\max}$ are the band edges of $\sigma(H_\omega(m, c))$.

3 Results on Localization

In this section we discuss the concept of dynamical localization (Definition 3.1 below) and some of its consequences. We further include two Sects. 3.1 and 3.2: in the former, we present the criteria of the fractional moments adapted to the DAD model (Theorem 3.1), which is used to obtain dynamical localization; in the latter, we present the main results of this work (Theorems 3.2, 3.3 and 3.4), i.e., dynamical localization for the Anderson Dirac operators (1).

Denote by $\{e_{j\alpha}\}_{j\in\mathbb{Z}^d,\alpha\in\mathcal{A}} = \{e_j \otimes \delta_\alpha\}_{j\in\mathbb{Z}^d,\alpha\in\mathcal{A}}$ the canonical orthonormal basis of $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$, whose elements $e_j \otimes \delta_\alpha$ are tensor products of elements of the canonical basis $\{e_j\}_{j\in\mathbb{Z}^d}$ of $l^2(\mathbb{Z}^d, \mathbb{C})$ with elements of canonical basis $\{\delta_\alpha\}_{\alpha\in\mathcal{A}}$ of $\mathbb{C}^\nu$, where $\mathcal{A} = \{1, 2\}$ if $d \in \{1, 2\}$ and $\mathcal{A} = \{1, 2, 3, 4\}$ if $d = 3$. Throughout this paper we will use the norm $|j| := |j_1| + \cdots + |j_d|$ of $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$.

Definition 3.1 The operator $H_\omega(m, c)$ exhibits dynamical localization in an bounded set $I \subset \mathbb{R}$ if there exist finite numbers $C > 0$ and $\eta > 0$ such that, for each $j, k \in \mathbb{Z}^d$ and each $\alpha, \beta \in \mathcal{A}$,

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left|\left\langle e_{j\alpha}, e^{-itH_\omega(m,c)}\chi_I(H_\omega(m, c))e_{k\beta}\right\rangle\right|\right) \le Ce^{-\eta|j-k|}. \quad (5)$$

Here, $\mathbb{E}(X) = \int_\Omega X\, d\mathbb{P}$ denotes the expectation with respect to the probability measure $\mathbb{P}$ for random variables X on Ω and $\chi_I(H_\omega(m, c))$ is the spectral projection of $H_\omega(m, c)$ associated with I.

Dynamical localization in the form (5) implies that all moments of the position operator are bounded in time (see [27]; it is similar for the Dirac model), i.e., for all $p > 0$ and for any $\Psi \in l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ of compact support,

$$\sup_{t\in\mathbb{R}} \left\| |X|^p\, e^{-itH_\omega(m,c)}\chi_I(H_\omega(m, c))\Psi \right\| < \infty \quad \mathbb{P}\text{-a.s.}, \quad (6)$$

where the "position" operator $|X|$ is defined by $(|X|\,\Phi)\,(n) = |n|\,\Phi(n)$. Thus, the solutions of the time-dependent Dirac equation $H_\omega(m, c)\Psi(t) = i\partial_t\Psi(t)$ stay localized in space, uniformly for all times, which is interpreted as absence of quantum transport.

It also true that dynamical localization implies that $H_\omega(m, c)$ has pure point spectrum, for almost every ω, via the RAGE Theorem (see Proposition 5.3 of [27]; a similar result is valid for the Dirac model). The proof that the spectrum is pure point does not imply exponential decay of the corresponding eigenfunctions. Nonetheless, it is shown in [2,4] how this follows directly from exponential decay of the fractional moments of the Green function (estimate (8) of the Theorem 3.1 below) using the Simon–Wolff's criterion.

3.1 Fractional Moments Criteria for Localization

The Green function $\mathcal{G}_\omega(m, c, z)$ of $H_\omega(m, c)$ is the matrix representation of the resolvent $(H_\omega(m, c) - zI)^{-1}$, whose elements are given by

$$G_\omega^{\alpha\beta}(j,k;z) := \left\langle e_{j\alpha}, (H_\omega(m,c) - zI)^{-1}\, e_{k\beta} \right\rangle, \tag{7}$$

with $j, k \in \mathbb{Z}^d$, $\alpha, \beta \in \mathcal{A}$ and $z \in \mathbb{C} \setminus \mathbb{R}$.

The following result, which is the fractional moments method adapted to DAD, is used as a criterion to establish dynamical localization for $H_\omega(m,c)$. This result is a consequence of Theorem 7.7 in [4] (where the result is formulated in the context of general graphs) and Definition 3.1 above; it can also be obtained from a version that was developed for the discrete Anderson Schrödinger model (see [2,27]). Nevertheless, there is also a version of this method in finite volume (Theorem 7.11 in [4]), which applies to Dirac Hamiltonians $H_\omega(m,c)$ restricted to finite subsets of $\mathbb{Z}^d$ (see also [3,27])). In order to obtain dynamical localization, we just need to prove that the fractional moments $\mathbb{E}\left(|G_\omega^{\alpha\beta}(j,k;z)|^s\right)$, for $s \in (0,1)$, decay exponentially with $|j-k|$.

Theorem 3.1 *(FMM) Let $I \subset \mathbb{R}$ be a bounded set. If there exist $s \in (0,1)$, $0 < C < \infty$ and $\eta > 0$ such that, for each $j, k \in \mathbb{Z}^d$, $\alpha, \beta \in \mathcal{A}$, all $E \in I$ and $\epsilon > 0$,*

$$\mathbb{E}\left(|G_\omega^{\alpha\beta}(j,k;E+i\epsilon)|^s\right) \le C e^{-\eta|j-k|}, \tag{8}$$

then $H_\omega(m,c)$ exhibits dynamical localization in the form (5) on the bounded set I.

3.2 Localization Regimes

We will show that for each of the regimes (j)–(jjj) described in the Introduction, we have the exponential decay of fractional moments of the Green function (Theorems 3.2, 3.3 and 3.4 below). Thus dynamical localization in each of these regimes follows by Theorem 3.1. In the proofs of our results we work with $H_\omega(m,c)$ in infinite volume, that is, with DAD acting on the space $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ (in the especial case of Theorem 3.3, we relate $H_\omega(m,c)$ with their finite volume restrictions). However, these localization proofs can be adapted for the context of finite volume, using ideas from [3,18].

In order to discuss the large disorder regime, we introduce a disorder parameter $\lambda > 0$ in the DAD model and consider

$$H_{\omega,\lambda}(m,c) := H_0(m,c) + \lambda V_\omega \tag{9}$$

with $H_0(m,c)$ and V_ω defined by **(i)-(ii)** in the Introduction. We denote by $\mathcal{G}_{\omega,\lambda}(m,c,z)$ the Green function of $H_{\omega,\lambda}(m,c)$ with matrix elements $G_{\omega,\lambda}^{\alpha\beta}(j,k;z)$ defined as in (7).

The first main goal is to check the fractional moment condition (8) for $H_{\omega,\lambda}(m,c)$ with large enough disorder λ. More precisely, we prove in the Sect. 5 the following result:

Theorem 3.2 (Localization at large disorder) *Consider the operator $H_{\omega,\lambda}(m,c)$ defined by* (9).

(i) *Let $0 < s < 1$. There exists $\lambda_0 > 0$ such that for $\lambda \ge \lambda_0$, there exist finite numbers $C = C(\lambda_0) > 0$ and $\eta = \eta(\lambda_0) > 0$ such that*

$$\mathbb{E}\left(|G_{\omega,\lambda}^{\alpha\beta}(j,k;z)|^s\right) \le C e^{-\eta|j-k|},$$

for all $j, k \in \mathbb{Z}^d$, $\alpha, \beta \in \mathcal{A}$ and all $z \in \mathbb{C} \setminus \mathbb{R}$.

(ii) *For every $\lambda \ge \lambda_0$, the operator $H_{\omega,\lambda}(m,c)$ exhibits dynamical localization in the form (5) on its spectrum.*

Theorem 3.2 can be obtained as an application of the Theorem 3.1 and Lemma 3.2 in [2], since due to example (c) of the Sect. 4 in [2], the DAD model can be considered as a particular case of the operators $H = T + U_0(x) + \lambda V_x$ with diagonal disorder. However, we would like to stress the following:

- In [2], the distribution of the potential is conditionally absolutely continuous and conditionally τ-regular, with some $0 < \tau \leq 1$. In DAD, the distribution of the potential is absolutely continuous with bounded density and of compact support.
- Our proof uses Proposition 4.1 (boundedness of the fractional moments of the Green function), which is also used in the proof of our localization results in band edge and 1D regimes (Theorems 3.3 and 3.4 below).
- The proofs of Proposition 4.1 and Theorem 3.2 explicitly need the independence of the potentials $V_\omega^{(\alpha)}(j)$ with respect to positions j and components α.
- The proof of our result does not use the fact that $H_\omega(m, c)$ has almost surely a complete set of orthonormal eigenfunctions that decay exponentially (as in Theorem 3.1 in [2]), but so recursive relations (30)–(40) between the fractional moments of the components of Green function, which are specific of DAD. Such relations result, by using Proposition 4.1, in exponential decay of those moments in each component.

Let $\mathcal{H}_+$ and $\mathcal{H}_-$ be the subspaces of $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ of negative and positive energies, respectively. We denote by $\mathcal{P}_\pm$ the orthogonal projection operators onto $\mathcal{H}_\pm$ (for more details see [29]). We can write $l^2(\mathbb{Z}^d, \mathbb{C}^\nu) = \mathcal{H}_+ \oplus \mathcal{H}_-$ and every state $\Psi \in l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ can be uniquely written as $\Psi = \Psi_+ + \Psi_-$, where $\Psi_+ = \mathcal{P}_+\Psi$ and $\Psi_- = \mathcal{P}_-\Psi$. Moreover, for nonzero vectors, $\langle \Psi_+, H_0(m, c)\Psi_+\rangle > 0$ and $\langle \Psi_-, H_0(m, c)\Psi_-\rangle < 0$. We denote by

$$H_\omega(m, c)|_{\mathcal{H}_\pm} = H_0(m, c)|_{\mathcal{H}_\pm} + V_\omega|_{\mathcal{H}_\pm} \tag{10}$$

the restriction of $H_\omega(m, c)$ to the subspaces $\mathcal{H}_\pm$. Let $\mathcal{G}_\omega^{\mathcal{H}_\pm}(m, c, z)$ be the Green function of $H_\omega(m, c)|_{\mathcal{H}_\pm}$, with matrix elements $G_\omega^{\alpha\beta,\mathcal{H}_\pm}(j, k; z)$ defined as in (7).

The following theorem is the main technical result of this paper on band edge localization, which we prove in Sect. 11. This result is different from Theorem 3.2 in [2] and its proof uses the initial estimates of Lifshitz tails given in Lemma 7.1, Proposition 8.1 and the decoupling estimate (61).

Theorem 3.3 (Band edge localization) *Consider the operators $H_\omega(m, c)|_{\mathcal{H}_\pm}$ defined by (10) with $m > 0$ and let $0 < \omega_{\max} < 2mc^2$.*

(i) *For every $s \in (0, 1)$, there exist finite numbers $\delta > 0$, $C > 0$ and $\eta > 0$ such that*

$$\mathbb{E}\left(|G_\omega^{\alpha\beta,\mathcal{H}_\pm}(j, k; E + i\epsilon)|^s\right) \leq Ce^{-\eta|j-k|}, \tag{11}$$

for all $j, k \in \mathbb{Z}^d$, $\alpha, \beta \in \mathcal{A}$, $E \in I_\delta^\pm(m, c)$ and $\epsilon > 0$, where $I_\delta^+(m, c)$ is any of the positive energy intervals

$$[mc^2, mc^2 + \delta] \quad or \quad \left[\sqrt{m^2c^4 + a_d c^2} + \omega_{\max} - \delta, \sqrt{m^2c^4 + a_d c^2} + \omega_{\max}\right], \tag{12}$$

and $I_\delta^-(m, c)$ is any of the negative energy intervals

$$[-mc^2 + \omega_{\max} - \delta, -mc^2 + \omega_{\max}] \quad or \quad \left[-\sqrt{m^2c^4 + a_d c^2}, -\sqrt{m^2c^4 + a_d c^2} + \delta\right]. \tag{13}$$

(ii) *The operator $H_\omega(m, c)|_{\mathcal{H}_\pm}$ exhibits dynamical localization in the form (5) in $I_\delta^\pm(m, c)$.*

The strategy of the proof of item (i) of Theorem 3.3 is the usual in the FMM, namely, we control the expectation value of fractional moments of the infinite volume Green function, $\mathbb{E}\big(|G_{\omega}^{\alpha\beta,\mathcal{H}_{\pm}}(j,k;E+i\epsilon)|^{s}\big)$ for $s\in(0,1)$, in terms of the expectation value of fractional moments of the Green function of a finite volume restriction of the operator [see Proposition 8.1 and the decoupling estimate (61)].

For the one-dimensional DAD, dynamical localization holds throughout the spectrum, independent of the disorder strength. More precisely,

Theorem 3.4 (1D localization) *Consider the operator $H_{\omega}(m,c)$ defined by (1) on $l^{2}(\mathbb{Z},\mathbb{C}^{2})$.*

(i) *There exists a number $s_0\in(0,1)$ such that, for each $0<s\leq s_0$, there exist finite numbers $C>0$ and $\eta>0$ such that*

$$\mathbb{E}\left(|G_{\omega}^{\alpha\beta}(j,k;E+i\epsilon)|^{s}\right)\leq Ce^{-\eta|j-k|}\,, \tag{14}$$

for all $j,k\in\mathbb{Z}$, $\alpha,\beta\in\{1,2\}$, $E\in\sigma(H_{\omega}(m,c))$ and $\epsilon>0$.

(ii) *The operator $H_{\omega}(m,c)$ exhibits dynamical localization in the form (5) on its spectrum.*

Theorem 3.4 is proven in Sect. 13. In the one-dimensional case, the proof of localization is different from the established in large disorder and band edge regimes. In this case, one has the formalism of the transfer matrices (see Sect. 12), which allows one to obtain the exponential decay described in Lemma 12.1 and the special representation (65) of the Green function in terms of generalized eigenfunctions. In Sect. 13, we show how this leads to exponential decay in (14).

4 Boundedness of Fractional Moments

The proof of Theorem 3.2 follows general ideas from [2,27] used in the context of the Anderson Schrödinger model, but with specific constructions of DAD as have been mentioned in Introduction and Sect. 3.2. The first important point in this proof, which is the goal of this section, is to establish the boundedness of the fractional moments of the Green function (for any parameter $\lambda>0$); this is the content of Proposition 4.1 below (a-priori bound, adapted from [27]). The following two lemmas, whose proofs can be found in [16,17,28], will be used in the proof of Proposition 4.1.

Lemma 4.1 *Let $g:\mathbb{R}\to\mathbb{R}$ be a nonnegative function with $g\in\mathrm{L}^{1}(\mathbb{R})\cap\mathrm{L}^{\infty}(\mathbb{R})$ and $s\in(0,1)$. Then, for $C(s,g)=\|g\|_{\infty}^{s}\,\|g\|_{1}^{1-s}\,\dfrac{2^{s}s^{-s}}{1-s}<\infty$ one has, for each $\theta\in\mathbb{C}$,*

$$\int_{\mathrm{supp}\ g}\frac{1}{|v+\theta|^{s}}\,g(v)dv\ \leq\ C(s,g)\,.$$

Lemma 4.2 *For every $s\in(0,1)$ and $a>0$, there exists a constant $C(a,s)<\infty$ such that*

$$\int_{-a}^{a}\|(M-uI)^{-1}\|^{s}\,du\ \leq\ C(a,s)\,,$$

for each matrix $M\in\mathbb{M}_{2\times2}(\mathbb{C})$ such that either $\mathrm{Im}\,M=\dfrac{1}{2i}(M-M^{*})\geq0$ *or* $\mathrm{Im}\,M\leq0$.

Proposition 4.1 *For each $s \in (0,1)$, there exists a constant $C_1 = C_1(s,\rho) < \infty$ such that, for each $j,k \in \mathbb{Z}^d$, $\alpha,\beta \in \mathcal{A}$, $z \in \mathbb{C}\setminus\mathbb{R}$ and $\lambda > 0$,*

$$\mathbb{E}_{j,k}^{\alpha\beta}\left(|G_{\omega,\lambda}^{\alpha\beta}(j,k;z)|^s\right) \le C_1\lambda^{-s}\,, \tag{15}$$

where

$$\mathbb{E}_{j,k}^{\alpha\beta}(\cdots) := \int\int \cdots \rho(\omega_{j\alpha})d\omega_{j\alpha}\ \rho(\omega_{k\beta})d\omega_{k\beta}$$

is the conditional expectation with $(\omega_{l\gamma})_{l\in\mathbb{Z}^d\setminus\{j,k\}\ or\ \gamma\neq\alpha,\beta}$ fixed. Consequently,

$$\mathbb{E}\left(|G_{\omega,\lambda}^{\alpha\beta}(j,k;z)|^s\right) \le C_1\lambda^{-s}. \tag{16}$$

Proof Let us fix the parameters m, c and denote $H_{\omega,\lambda} = H_{\omega,\lambda}(m,c)$. The proof of (15) is split into two cases:

(1) $(j,\alpha) = (k,\beta)$. In this case, for $j \in \mathbb{Z}^d$ and $\alpha \in \mathcal{A}$ fixed, write $\omega = (\hat{\omega}, \omega_{j\alpha})$ where $\hat{\omega} = (\omega_{l\gamma})_{l\in\mathbb{Z}^d\setminus\{j\}\ or\ \gamma\neq\alpha}$. Let $P_{e_{j\alpha}} := \langle e_{j\alpha},\ \cdot\rangle e_{j\alpha}$ be the orthogonal projection onto the subspace spanned by $e_{j\alpha}$. We can separate the $\omega_{j\alpha}$ and $\hat{\omega}$ dependence of $H_{\omega,\lambda}$ as

$$H_{\omega,\lambda} = H_{\hat{\omega},\lambda} + \lambda\omega_{j\alpha}P_{e_{j\alpha}}.$$

Observe that the operators $H_{\omega,\lambda}$ and $H_{\hat{\omega},\lambda}$ are bounded self-adjoints on $l^2(\mathbb{Z}^d,\mathbb{C}^\nu)$ and $z \notin \sigma(H_{\omega,\lambda}) \cup \sigma(H_{\hat{\omega},\lambda})$. Thus, by the second resolvent identity we have

$$(H_{\omega,\lambda} - zI)^{-1} = (H_{\hat{\omega},\lambda} - zI)^{-1} - \lambda\omega_{j\alpha}(H_{\hat{\omega},\lambda} - zI)^{-1}P_{e_{j\alpha}}(H_{\omega,\lambda} - zI)^{-1}.$$

It follows from this relation that the corresponding diagonal elements of the Green functions satisfy

$$G_{\omega,\lambda}^{\alpha\alpha}(j,j;z) = G_{\hat{\omega},\lambda}^{\alpha\alpha}(j,j;z) - \lambda\omega_{j\alpha}G_{\hat{\omega},\lambda}^{\alpha\alpha}(j,j;z)G_{\omega,\lambda}^{\alpha\alpha}(j,j;z). \tag{17}$$

Since the Green function satisfies $\dfrac{\operatorname{Im} G_{\hat{\omega},\lambda}^{\alpha\alpha}(j,j;z)}{\operatorname{Im} z} > 0$ for $\operatorname{Im} z \neq 0$, then $G_{\hat{\omega},\lambda}^{\alpha\alpha}(j,j;z) \neq 0$ and it follows from (17) that

$$G_{\omega,\lambda}^{\alpha\alpha}(j,j;z) = \frac{1}{\xi + \lambda\omega_{j\alpha}} \quad \text{with} \quad \xi = \frac{1}{G_{\hat{\omega},\lambda}^{\alpha\alpha}(j,j;z)}. \tag{18}$$

Note that ξ not depend on $\omega_{j\alpha}$. Thus, writing $\mathbb{E}_j^\alpha(\cdots) := \int \cdots \rho(\omega_{j\alpha})d\omega_{j\alpha}$ and using (18), we find that

$$\mathbb{E}_j^\alpha\left(|G_{\omega,\lambda}^{\alpha\alpha}(j,j;z)|^s\right) = \frac{1}{\lambda^s}\int_{\text{supp}\ \rho} \frac{1}{\left|\omega_{j\alpha} + \frac{\xi}{\lambda}\right|^s}\,\rho(\omega_{j\alpha})d\omega_{j\alpha} \ \le\ \frac{1}{\lambda^s}\,C(s,\rho)\,,$$

with the constant $C(s,\rho)$ given by Lemma 4.1 (applied for $\theta = \xi/\lambda$ and $g = \rho$), which is independent of λ and ξ, and thus independent of $\hat{\omega}$, z, j and α.

(2) $(j,\alpha) \neq (k,\beta)$. We prove this case replacing the rank-one perturbation argument above by a rank-two perturbation argument. For $j,k \in \mathbb{Z}^d$ and $\alpha,\beta \in \mathcal{A}$ fixed, we write $\omega = (\hat{\omega}, \omega_{j\alpha}, \omega_{k\beta})$ where $\hat{\omega} = (\omega_{l\gamma})_{l\in\mathbb{Z}^d\setminus\{j,k\}\ or\ \gamma\neq\alpha,\beta}$. Let $P = P_{e_{j\alpha}} + P_{e_{k\beta}}$ and write

$$H_{\omega,\lambda} = H_{\hat{\omega},\lambda} + \lambda\omega_{j\alpha}P_{e_{j\alpha}} + \lambda\omega_{k\beta}P_{e_{k\beta}}.$$

Using the second resolvent identity we obtain

$$P(H_{\omega,\lambda} - zI)^{-1}P = P(H_{\hat{\omega},\lambda} - zI)^{-1} \times P - \lambda P(H_{\hat{\omega},\lambda} - zI)^{-1}P(\omega_{j\alpha}P_{e_{j\alpha}} + \omega_{k\beta}P_{e_{k\beta}}) \times P(H_{\omega,\lambda} - zI)^{-1}P. \tag{19}$$

Now the operators $P(H_{\omega,\lambda} - zI)^{-1}P$ and $P(H_{\hat{\omega},\lambda} - zI)^{-1}P$ can be represented by the matrices (in the space $\mathbb{M}_{2\times 2}(\mathbb{C})$)

$$Q = \begin{pmatrix} G^{\alpha\alpha}_{\omega,\lambda}(j, j; z) & G^{\alpha\beta}_{\omega,\lambda}(j, k; z) \\ G^{\beta\alpha}_{\omega,\lambda}(k, j; z) & G^{\beta\beta}_{\omega,\lambda}(k, k; z) \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} G^{\alpha\alpha}_{\hat{\omega},\lambda}(j, j; z) & G^{\alpha\beta}_{\hat{\omega},\lambda}(j, k; z) \\ G^{\beta\alpha}_{\hat{\omega},\lambda}(k, j; z) & G^{\beta\beta}_{\hat{\omega},\lambda}(k, k; z) \end{pmatrix},$$

respectively, since $G^{\alpha\beta}_{\omega,\lambda}(j, k; z) = \langle e_{j\alpha}, P(H_{\omega,\lambda} - zI)^{-1}Pe_{k\beta}\rangle$. Moreover, the operator $\omega_{j\alpha}P_{e_{j\alpha}} + \omega_{k\beta}P_{e_{k\beta}}$ is represented by the matrix $\operatorname{diag}(\omega_{j\alpha}, \omega_{k\beta})$. Replacing these matrix representations in the relation (19) and developing it, we get

$$Q = \left[\hat{Q}^{-1} + \lambda \begin{pmatrix} \omega_{j\alpha} & 0 \\ 0 & \omega_{k\beta} \end{pmatrix}\right]^{-1}, \tag{20}$$

which is a special case of the Krein formula (Lemma 2.2 of [2]).
Using (20) and the fact that $G^{\alpha\beta}_{\omega,\lambda}(j, k; z)$ is one of the matrix-elements of Q, we find

$$\begin{aligned} \mathbb{E}^{\alpha\beta}_{j,k}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, k; z)|^s\right) &\le \frac{1}{\lambda^s}\,\mathbb{E}^{\alpha\beta}_{j,k}\left(\left\|\left[-\frac{1}{\lambda}\hat{Q}^{-1} - \begin{pmatrix} \omega_{j\alpha} & 0 \\ 0 & \omega_{k\beta} \end{pmatrix}\right]^{-1}\right\|^s\right) \\ &\le \frac{\|\rho\|^2_\infty}{\lambda^s}\int_{-a}^{a}\int_{-a}^{a}\left\|\left[-\frac{1}{\lambda}\hat{Q}^{-1} - \begin{pmatrix} \omega_{j\alpha} & 0 \\ 0 & \omega_{k\beta} \end{pmatrix}\right]^{-1}\right\|^s d\omega_{j\alpha}d\omega_{k\beta}, \end{aligned}$$

with $a > 0$ large so that $[-a, a] \supset \operatorname{supp} \rho$. In the double integral we use the change of variables $u_\pm = (\omega_{j\alpha} \pm \omega_{k\beta})/2$, which gives a Jacobian factor $\left|\frac{\partial(\omega_{j\alpha},\omega_{k\beta})}{\partial(u_+,u_-)}\right| = 2$. As $\omega_{j\alpha}, \omega_{k\beta} \in [-a, a]$ implies $u_\pm \in [-a, a]$, we obtain the bound

$$\begin{aligned} \mathbb{E}^{\alpha\beta}_{j,k}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, k; z)|^s\right) &\le \frac{2\,\|\rho\|^2_\infty}{\lambda^s}\int_{-a}^{a}\int_{-a}^{a}\left\|\left[-\frac{1}{\lambda}\hat{Q}^{-1} + \begin{pmatrix} -u_- & 0 \\ 0 & u_- \end{pmatrix} - u_+ I\right]^{-1}\right\|^s du_+du_- \\ &\le \frac{4a\,\|\rho\|^2_\infty}{\lambda^s}\,C(a, s) = \frac{C(s,\rho)}{\lambda^s}, \end{aligned}$$

with the constant $C(a, s)$ given by Lemma 4.2, which was applied to the matrix

$$M = -\frac{1}{\lambda}\hat{Q}^{-1} + \begin{pmatrix} -u_- & 0 \\ 0 & u_- \end{pmatrix}$$

since $\operatorname{Im} M = \frac{1}{2i}(M - M^*) > 0$ if $\operatorname{Im} z > 0$ and $\operatorname{Im} M < 0$ if $\operatorname{Im} z < 0$. Indeed,

$$\frac{\operatorname{Im} M}{\operatorname{Im} z} = \frac{1}{\lambda}\,(\hat{Q}^{-1})^* P\,\frac{\operatorname{Im}\,(H_{\hat{\omega},\lambda} - zI)^{-1}}{\operatorname{Im} z}\,P\hat{Q}^{-1} > 0\,.$$

Thus, we have concluded the proof of estimate (15). Taking the expectation of both sides of (15) we get (16); indeed,

$$\mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, k; z)|^s\right) = \mathbb{E}\left(\mathbb{E}^{\alpha\beta}_{j,k}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, k; z)|^s\right)\right) \le \mathbb{E}(C_1\lambda^{-s}) = C_1\lambda^{-s}.$$

□

5 Proof of Localization at Large Disorder

This section is dedicated to the proof of Theorem 3.2. Besides Proposition 4.1, we also need the following result (decoupling Lemma 4.2 in [27]):

Lemma 5.1 *Let* $0 < s < 1$. *For each function* $g \in \mathrm{L}^\infty(\mathbb{R})$ *of compact support, there exists a constant* $C_2 < \infty$ *such that, for each* $\xi, \theta \in \mathbb{C}$,

$$\int \frac{1}{|v-\theta|^s}\, g(v)dv \;\le\; C_2 \int \frac{|v-\xi|^s}{|v-\theta|^s}\, g(v)dv\,.$$

Proof of Theorem 3.2 (i) Let $j, k \in \mathbb{Z}^d, \alpha, \beta \in \mathcal{A}$ and $z \in \mathbb{C}\backslash\mathbb{R}$. If $j = k$, the result follows directly by Proposition 4.1. Suppose $j \neq k$. Then, for dimension $d = 1$, we have

$$\begin{aligned} 0 &= \langle e_{j\alpha}, e_{k1}\rangle = \langle e_{j\alpha}, (H_{\omega,\lambda} - zI)^{-1}(H_{\omega,\lambda} - zI)e_{k1}\rangle \\ &= \langle e_{j\alpha}, (H_{\omega,\lambda} - zI)^{-1}\left[\left(mc^2 + \lambda\omega_{k1} - z\right)e_{k1} + c\left(e_{(k+1)2} - e_{k2}\right)\right]\rangle, \end{aligned}$$

which implies

$$\left[\lambda\omega_{k1} - \left(z - mc^2\right)\right] G^{\alpha 1}_{\omega,\lambda}(j, k; z) = c\left[G^{\alpha 2}_{\omega,\lambda}(j, k; z) - G^{\alpha 2}_{\omega,\lambda}(j, k+1; z)\right]. \quad (21)$$

Similarly for the other component in $d = 1$, we have

$$\left[\lambda\omega_{k2} - \left(z + mc^2\right)\right] G^{\alpha 2}_{\omega,\lambda}(j, k; z) = c\left[G^{\alpha 1}_{\omega,\lambda}(j, k; z) - G^{\alpha 1}_{\omega,\lambda}(j, k-1; z)\right]. \quad (22)$$

For dimension $d = 2$, the analogous relations to (21) and (22) are, with $j = (j_1, j_2), k = (k_1, k_2) \in \mathbb{Z}^2$, respectively,

$$\begin{aligned} &\left[\lambda\omega_{k1} - \left(z - mc^2\right)\right] G^{\alpha 1}_{\omega,\lambda}(j, k; z) \\ &\quad = c\left[(1+i)G^{\alpha 2}_{\omega,\lambda}(j, k; z) - G^{\alpha 2}_{\omega,\lambda}(j, (k_1+1, k_2); z) - iG^{\alpha 2}_{\omega,\lambda}(j, (k_1, k_2+1); z)\right] \end{aligned} \quad (23)$$

and

$$\begin{aligned} &\left[\lambda\omega_{k2} - \left(z + mc^2\right)\right] G^{\alpha 2}_{\omega,\lambda}(j, k; z) \\ &\quad = c\left[(1-i)G^{\alpha 1}_{\omega,\lambda}(j, k; z) - G^{\alpha 1}_{\omega,\lambda}(j, (k_1-1, k_2); z) + iG^{\alpha 1}_{\omega,\lambda}(j, (k_1, k_2-1); z)\right]. \end{aligned} \quad (24)$$

For dimension $d = 3$, relations (21) and (22) are replaced by the following four relations (with $j = (j_1, j_2, j_3), k = (k_1, k_2, k_3) \in \mathbb{Z}^3$):

$$\begin{aligned} \left[\lambda\omega_{k1} - \left(z - mc^2\right)\right] G^{\alpha 1}_{\omega,\lambda}(j, k; z) \;&= c[G^{\alpha 3}_{\omega,\lambda}(j, k; z) - G^{\alpha 3}_{\omega,\lambda}(j, (k_1, k_2, k_3+1); z) \\ &\quad + (1+i)G^{\alpha 4}_{\omega,\lambda}(j, k; z) \\ &\quad - G^{\alpha 4}_{\omega,\lambda}(j, (k_1+1, k_2, k_3); z) \\ &\quad - iG^{\alpha 4}_{\omega,\lambda}(j, (k_1, k_2+1, k_3); z)], \end{aligned} \quad (25)$$

$$\begin{aligned} \left[\lambda\omega_{k2} - \left(z - mc^2\right)\right] G^{\alpha 2}_{\omega,\lambda}(j, k; z) \;&= c[(1-i)G^{\alpha 3}_{\omega,\lambda}(j, k; z) \\ &\quad - G^{\alpha 3}_{\omega,\lambda}(j, (k_1+1, k_2, k_3); z) \\ &\quad + iG^{\alpha 3}_{\omega,\lambda}(j, (k_1, k_2+1, k_3); z) \\ &\quad + G^{\alpha 4}_{\omega,\lambda}(j, (k_1, k_2, k_3+1); z) - G^{\alpha 4}_{\omega,\lambda}(j, k; z)], \end{aligned} \quad (26)$$

$$\begin{aligned}\left[\lambda\omega_{k3}-\left(z+mc^2\right)\right]G^{\alpha 3}_{\omega,\lambda}(j,k;z) &= c[G^{\alpha 1}_{\omega,\lambda}(j,k;z)-G^{\alpha 1}_{\omega,\lambda}(j,(k_1,k_2,k_3-1);z)\\ &\quad +(1+i)G^{\alpha 2}_{\omega,\lambda}(j,k;z)\\ &\quad -G^{\alpha 2}_{\omega,\lambda}(j,(k_1-1,k_2,k_3);z)\\ &\quad -iG^{\alpha 2}_{\omega,\lambda}(j,(k_1,k_2-1,k_3);z)]\end{aligned} \tag{27}$$

and

$$\begin{aligned}\left[\lambda\omega_{k4}-\left(z+mc^2\right)\right]G^{\alpha 4}_{\omega,\lambda}(j,k;z) &= c[(1-i)G^{\alpha 1}_{\omega,\lambda}(j,k;z)\\ &\quad -G^{\alpha 1}_{\omega,\lambda}(j,(k_1-1,k_2,k_3);z)\\ &\quad +iG^{\alpha 1}_{\omega,\lambda}(j,(k_1,k_2-1,k_3);z)\\ &\quad +G^{\alpha 2}_{\omega,\lambda}(j,(k_1,k_2,k_3-1);z)-G^{\alpha 2}_{\omega,\lambda}(j,k;z)].\end{aligned} \tag{28}$$

Note that $G^{\alpha\beta}_{\omega,\lambda}(j,k;z)$ is the upper right entry of the matrix Q which appears on the left-hand side of relation (20). Explicitly, by expanding the right-hand side of (20), one obtains

$$G^{\alpha\beta}_{\omega,\lambda}(j,k;z) \;=\; \frac{\tau}{\lambda\omega_{k\beta}-\zeta} \tag{29}$$

with

$$\tau=\frac{G_2}{G_4+\lambda\omega_{j\alpha}(G_1G_4-G_2G_3)} \quad\text{and}\quad \zeta=\frac{-1-\lambda\omega_{j\alpha}G_1}{G_4+\lambda\omega_{j\alpha}(G_1G_4-G_2G_3)},$$

where $G_1=G^{\alpha\alpha}_{\hat{\omega},\lambda}(j,j;z)$, $G_2=G^{\alpha\beta}_{\hat{\omega},\lambda}(j,k;z)$, $G_3=G^{\beta\alpha}_{\hat{\omega},\lambda}(k,j;z)$ and $G_4=G^{\beta\beta}_{\hat{\omega},\lambda}(k,k;z)$ are the entries of the matrix $\hat{Q}$. Observe that τ and ζ are independent of $\omega_{k\beta}$.

Using (29), Lemma 5.1 with $g=\rho$ and $\xi=(z\mp mc^2)/\lambda$, relations (21)–(22) and the inequality $\left|\sum_{n=1}^k y_n\right|^s\le\sum_{n=1}^k|y_n|^s$, we find for $d=1$,

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j,k;z)|^s\right) &= \frac{1}{\lambda^s}\,\mathbb{E}\left(\frac{|\tau|^s}{\left|\omega_{k1}-\frac{\zeta}{\lambda}\right|^s}\right) \le \frac{C_2}{\lambda^s}\,\mathbb{E}\left(\frac{|\tau|^s\left|\omega_{k1}-\frac{z-mc^2}{\lambda}\right|^s}{\left|\omega_{k1}-\frac{\zeta}{\lambda}\right|^s}\right)\\ &= \frac{C_2}{\lambda^s}\,\mathbb{E}\left(\left|\lambda\omega_{k1}-(z-mc^2)\right|^s|G^{\alpha 1}_{\omega,\lambda}(j,k;z)|^s\right)\\ &\le \frac{C_2\,c^s}{\lambda^s}\left[\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j,k+1;z)|^s\right)\right]\end{aligned} \tag{30}$$

and

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j,k;z)|^s\right) &\le \frac{C_2}{\lambda^s}\,\mathbb{E}\left(\left|\lambda\omega_{k2}-(z+mc^2)\right|^s|G^{\alpha 2}_{\omega,\lambda}(j,k;z)|^s\right)\\ &\le \frac{C_2\,c^s}{\lambda^s}\left[\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j,k-1;z)|^s\right)\right].\end{aligned} \tag{31}$$

Substituting (31) into (30) and similarly (30) into (31), we obtain

$$\mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j,k;z)|^s\right) \;\le\; \Delta^{(1)}_\lambda\left[\mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j,k-1;z)|^s\right)+\mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j,k+1;z)|^s\right)\right],$$

for $\alpha, \beta \in \{1, 2\}$, where $\Delta_\lambda^{(1)} := \dfrac{a_\lambda^2}{1 - 2a_\lambda^2}$ with $a_\lambda = C_2 c^s \lambda^{-s}$. For given j and k one can iterate $|j-k|$ times, in each step picking up a factor $2\Delta_\lambda^{(1)}$. This results in the bound

$$\mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, k; z)|^s\right) \le \left(2\Delta_\lambda^{(1)}\right)^{|j-k|} \sup_{l\in\mathbb{Z}} \mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, l; z)|^s\right). \tag{32}$$

For dimension $d = 2$, repeating the calculations in (30)–(31) and, in the last passage, using relations (23)–(24) we get

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j, k; z)|^s\right) &\le \frac{C_2}{\lambda^s}\, \mathbb{E}\left(\left|\lambda\omega_{k1} - (z - mc^2)\right|^s |G^{\alpha 1}_{\omega,\lambda}(j, k; z)|^s\right)\\ &\le \frac{C_2\, c^s(\sqrt{2})^s}{\lambda^s}\, [\, \mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j, k; z)|^s\right)\\ &\quad +\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j, (k_1+1, k_2); z)|^s\right) +\\ &\quad + \mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j, (k_1, k_2+1); z)|^s\right)\,]\end{aligned} \tag{33}$$

and

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j, k; z)|^s\right) &\le \frac{C_2}{\lambda^s}\, \mathbb{E}\left(\left|\lambda\omega_{k2} - (z + mc^2)\right|^s |G^{\alpha 2}_{\omega,\lambda}(j, k; z)|^s\right)\\ &\le \frac{C_2\, c^s(\sqrt{2})^s}{\lambda^s}\, [\, \mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j, k; z)|^s\right)\\ &\quad +\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j, (k_1-1, k_2); z)|^s\right) +\\ &\quad + \mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j, (k_1, k_2-1); z)|^s\right)\,].\end{aligned} \tag{34}$$

Substituting relation (34) into (33) and similarly (33) into (34), one finds

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, k; z)|^s\right) \le \Delta_\lambda^{(2)}\Big[\, &\mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, (k_1-1, k_2); z)|^s\right)\\ &+ \mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, (k_1, k_2-1); z)|^s\right)\\ &+ \mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, (k_1+1, k_2); z)|^s\right)\\ &+ \mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, (k_1+1, k_2-1); z)|^s\right)\\ &+ \mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, (k_1, k_2+1); z)|^s\right)\\ &+ \mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, (k_1-1, k_2+1); z)|^s\right)\Big]\end{aligned}$$

for $\alpha, \beta \in \{1, 2\}$, where $\Delta_\lambda^{(2)} := \dfrac{b_\lambda^2}{1 - 3b_\lambda^2}$ with $b_\lambda = C_2 c^s(\sqrt{2})^s\lambda^{-s} = (\sqrt{2})^s a_\lambda$. Analogously to the case $d = 1$, one can iterate $|j-k|$ times, in each step picking up a factor $6\Delta_\lambda^{(2)}$. This results in

$$\mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, k; z)|^s\right) \le \left(6\Delta_\lambda^{(2)}\right)^{|j-k|} \sup_{l\in\mathbb{Z}^2} \mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j, l; z)|^s\right). \tag{35}$$

For dimension $d = 3$, analogous calculations to (30)–(31), together with relations (25)–(28), lead to

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j,k;z)|^s\right) &\leq \frac{C_2}{\lambda^s}\,\mathbb{E}\left(\left|\lambda\omega_{k1}-(z-mc^2)\right|^s|G^{\alpha 1}_{\omega,\lambda}(j,k;z)|^s\right)\\ &\leq b_\lambda\Big[\,\mathbb{E}\left(|G^{\alpha 3}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha 3}_{\omega,\lambda}(j,(k_1,k_2,k_3+1);z)|^s\right)\\ &\quad+\mathbb{E}\left(|G^{\alpha 4}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha 4}_{\omega,\lambda}(j,(k_1+1,k_2,k_3);z)|^s\right)\\ &\quad+\mathbb{E}\left(|G^{\alpha 4}_{\omega,\lambda}(j,(k_1,k_2+1,k_3);z)|^s\right)\Big],\end{aligned}\tag{36}$$

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j,k;z)|^s\right) &\leq \frac{C_2}{\lambda^s}\,\mathbb{E}\left(\left|\lambda\omega_{k2}-(z-mc^2)\right|^s|G^{\alpha 2}_{\omega,\lambda}(j,k;z)|^s\right)\\ &\leq b_\lambda\Big[\,\mathbb{E}\left(|G^{\alpha 3}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha 3}_{\omega,\lambda}(j,(k_1+1,k_2,k_3);z)|^s\right)\\ &\quad+\mathbb{E}\left(|G^{\alpha 3}_{\omega,\lambda}(j,(k_1,k_2+1,k_3);z)|^s\right)+\mathbb{E}\left(|G^{\alpha 4}_{\omega,\lambda}(j,k;z)|^s\right)\\ &\quad+\mathbb{E}\left(|G^{\alpha 4}_{\omega,\lambda}(j,(k_1,k_2,k_3+1);z)|^s\right)\Big],\end{aligned}\tag{37}$$

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha 3}_{\omega,\lambda}(j,k;z)|^s\right) &\leq \frac{C_2}{\lambda^s}\,\mathbb{E}\left(\left|\lambda\omega_{k3}-(z+mc^2)\right|^s|G^{\alpha 3}_{\omega,\lambda}(j,k;z)|^s\right)\\ &\leq b_\lambda\Big[\,\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j,(k_1,k_2,k_3-1);z)|^s\right)\\ &\quad+\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j,(k_1-1,k_2,k_3);z)|^s\right)\\ &\quad+\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j,(k_1,k_2-1,k_3);z)|^s\right)\Big]\end{aligned}\tag{38}$$

and

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha 4}_{\omega,\lambda}(j,k;z)|^s\right) &\leq \frac{C_2}{\lambda^s}\,\mathbb{E}\left(\left|\lambda\omega_{k4}-(z+mc^2)\right|^s|G^{\alpha 4}_{\omega,\lambda}(j,k;z)|^s\right)\\ &\leq b_\lambda\Big[\,\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j,(k_1-1,k_2,k_3);z)|^s\right)\\ &\quad+\mathbb{E}\left(|G^{\alpha 1}_{\omega,\lambda}(j,(k_1,k_2-1,k_3);z)|^s\right)+\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j,k;z)|^s\right)\\ &\quad+\mathbb{E}\left(|G^{\alpha 2}_{\omega,\lambda}(j,(k_1,k_2,k_3-1);z)|^s\right)\Big].\end{aligned}\tag{39}$$

Substituting (38)–(39) into (36)–(37) and similarly (36)–(37) into (38)–(39), in each case adding the two remaining inequalities, isolating the sum

$$\mathbb{E}\left(|G^{\alpha\beta_1}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha\beta_2}_{\omega,\lambda}(j,k;z)|^s\right),\quad \text{for } (\beta_1,\beta_2)\in\{(1,2),(3,4)\},$$

on the left-hand side of the inequality and iterating $|j-k|$ times, we obtain

$$\begin{aligned}&\mathbb{E}\left(|G^{\alpha\beta_1}_{\omega,\lambda}(j,k;z)|^s\right)+\mathbb{E}\left(|G^{\alpha\beta_2}_{\omega,\lambda}(j,k;z)|^s\right)\\ &\quad\leq\left(24\Delta^{(3)}_\lambda\right)^{|j-k|}\sup_{l\in\mathbb{Z}^3}\left[\mathbb{E}\left(|G^{\alpha\beta_1}_{\omega,\lambda}(j,l;z)|^s\right)+\mathbb{E}\left(|G^{\alpha\beta_2}_{\omega,\lambda}(j,l;z)|^s\right)\right],\end{aligned}\tag{40}$$

where $\Delta^{(3)}_\lambda := \dfrac{b^2_\lambda}{1-7b^2_\lambda}$. Now choose $\lambda_0 > 0$ sufficiently large such that $b^2_{\lambda_0} < \dfrac{1}{31}$, which implies

$$2\Delta^{(1)}_{\lambda_0} < 6\Delta^{(2)}_{\lambda_0} < 24\Delta^{(3)}_{\lambda_0} < 1.$$

Writing $\eta = -\ln\left(24\Delta^{(3)}_{\lambda_0}\right) > 0$, it follows by the estimates (32), (35), (40) and Proposition 4.1 that for $d \in \{1, 2, 3\}$, $\alpha, \beta \in \mathcal{A}$ and $\lambda \geq \lambda_0$,

$$\mathbb{E}\left(|G^{\alpha\beta}_{\omega,\lambda}(j,k;z)|^s\right) \leq e^{-\eta|j-k|}C_1\lambda^{-s} \leq Ce^{-\eta|j-k|},$$

with $C = C_1\lambda_0^{-s}$. This completes the proof.

(ii) It follows directly from item (i) of this theorem and Theorem 3.1. This completes the proof of the theorem.

□

6 Finite Volume Operators and Boundary Conditions

Consider the finite subset $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$, with $L \in \mathbb{N}$. In this section we define finite volume operators $H^{\Lambda_L}_\omega(m, c)$ and $(H^{\Lambda_L}_\omega(m, c))^N$, respectively the restrictions of $H_\omega(m, c)$ to $l^2(\Lambda_L, \mathbb{C}^\nu)$ with Neumann and simple boundary conditions on Λ_L (see Definitions 6.1 and 6.2). These operators will be used in the next sections, especially in Lemma 7.1 and Proposition 8.1, which are important preparation results for the proof of Theorem 3.3.

Let $\left\{e_{j\alpha}\right\}_{j\in\mathbb{Z}^d,\alpha\in\mathcal{A}}$ be the canonical basis of $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$, defined in Sect. 3. The matrix elements of an operator $\mathcal{O}$ on $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ are the entries of the $\nu \times \nu$ matrix defined, for each pair $j, k \in \mathbb{Z}^d$, by

$$\mathcal{O}_{jk} = \left(\langle e_{j\alpha}, \mathcal{O}e_{k\beta}\rangle\right)_{\alpha,\beta\in\mathcal{A}},$$

where α varies over the rows and β over the columns of $\mathcal{O}_{jk}$. For $\Psi \in l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$, we have

$$(\mathcal{O}\Psi)(j) = \sum_{k\in\mathbb{Z}^d} \mathcal{O}_{jk}\Psi(k)$$

and the elements $\mathcal{O}_{jk}$ define the operator $\mathcal{O}$ uniquely.

For convenience, we introduce matrices $M^{(d)}_{n,\pm}$ and $M^{(d)}_n := M^{(d)}_{n,+} + M^{(d)}_{n,-}$ defined, for $d \in \{1, 2, 3\}$ and $1 \leq n \leq d$ with $n \in \mathbb{N}$, by

$$M^{(1)}_{1,+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M^{(1)}_{1,-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$M^{(2)}_{1,\pm} = M^{(1)}_{1,\pm}, \quad M^{(2)}_{2,+} = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \quad M^{(2)}_{2,-} = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix},$$

$$M^{(3)}_{n,+} = \begin{pmatrix} O_2 & \sigma_n \\ O_2 & O_2 \end{pmatrix}, \quad M^{(3)}_{n,-} = \begin{pmatrix} O_2 & O_2 \\ \sigma_n & O_2 \end{pmatrix},$$

where σ_n, $n \in \{1, 2, 3\}$, are the Pauli matrices as in (3) and O_2 is the 2×2 null matrix. Note that $M^{(1)}_1 = \sigma_1$, $M^{(2)}_2 = \sigma_2$ and $M^{(3)}_n = \begin{pmatrix} O_2 & \sigma_n \\ \sigma_n & O_2 \end{pmatrix}$.

For $d \in \{1, 2, 3\}$, let $\left\{v^{(d)}_n : 1 \leq n \leq d,\ n \in \mathbb{N}\right\}$ be the canonical basis of $\mathbb{R}^d$, formed by the vectors $v^{(1)}_1 = 1$, $v^{(2)}_1 = (1, 0)$, $v^{(2)}_2 = (0, 1)$, $v^{(3)}_1 = (1, 0, 0)$, $v^{(3)}_2 = (0, 1, 0)$ and

$v_3^{(3)} = (0, 0, 1)$. The matrix elements of the free Dirac operator $H_0(m, c)$ are

$$(H_0(m,c))^{(d)}_{jk} = \begin{cases} -c\sum_{n=1}^{d} M_n^{(d)} + mc^2\mathcal{B}_d & \text{if } \ j=k, \\ cM^{(d)}_{n,\pm} & \text{if } j = k \pm v_n^{(d)},\ 1 \le n \le d, \\ O^{(d)} & \text{if } |j-k| \ge 2, \end{cases}$$

where $\mathcal{B}_d$ are the diagonal matrices defined in (2) and $O^{(d)} = \begin{cases} O_2 & \text{if} \quad d \in \{1,2\} \\ O_4 & \text{if} \quad d = 3, \end{cases}$ with O_n being the $n \times n$ null matrix. Thus, the matrix elements of the discrete Anderson Dirac operator $H_\omega(m, c)$ are given by

$$(H_\omega(m,c))^{(d)}_{jk} = (H_0(m,c))^{(d)}_{jk} + (V_\omega)^{(d)}_{jk}\ , \text{ with } j,k \in \mathbb{Z}^d,$$

where the potential V_ω acts as a diagonal multiplication operator.

Definition 6.1 The free Dirac operator $H_0^{\Lambda_L}(m, c)$ restricted to $l^2(\Lambda_L, \mathbb{C}^\nu)$ with simple boundary condition on Λ_L, in dimensions $d \in \{1, 2, 3\}$, is defined by

$$(H_0^{\Lambda_L}(m,c))^{(d)}_{jk} = (H_0(m,c))^{(d)}_{jk}\ , \text{ with } j,k \in \Lambda_L.$$

We also define the Anderson Dirac operator restricted to $l^2(\Lambda_L, \mathbb{C}^\nu)$ by

$$H_\omega^{\Lambda_L}(m,c) = H_0^{\Lambda_L}(m,c) + V_\omega^{\Lambda_L},$$

where $V_\omega^{\Lambda_L}$ is the restriction of V_ω to Λ_L.

Let us define the boundary of Λ_L as the set

$$\partial\Lambda_L = \Big\{(j,k) \in \mathbb{Z}^d \times \mathbb{Z}^d : |j-k| = 1 \text{ and either } j \in \Lambda_L, k \notin \Lambda_L \text{ or } j \notin \Lambda_L, k \in \Lambda_L\Big\}.$$

It consists of the edges connecting points in Λ_L with points outside Λ_L. We also consider the inner boundary of Λ_L by

$$\partial^{\mathrm{i}}\Lambda_L = \{j \in \mathbb{Z}^d :\ j \in \Lambda_L,\ \exists\, k \notin \Lambda_L \text{ with } (j,k) \in \partial\Lambda_L\} = \{j \in \mathbb{Z}^d : \|j\|_\infty = L\}$$

and the outer boundary by

$$\partial^{\mathrm{o}}\Lambda_L = \{k \in \mathbb{Z}^d :\ k \notin \Lambda_L,\ \exists\, j \in \Lambda_L \text{ with } (j,k) \in \partial\Lambda_L\} = \{k \in \mathbb{Z}^d : \|k\|_\infty = L+1\},$$

where $\|n\|_\infty := \max\{|n_1|, \dots, |n_d|\}$ denote the maximum norm on $\mathbb{Z}^d$.

For any subset $\Lambda_L \subset \mathbb{Z}^d$ with $d \in \{1, 2, 3\}$, we define the boundary operator $\mathcal{F}_{\Lambda_L}$ by its matrix elements:

$$\left(\mathcal{F}_{\Lambda_L}\right)^{(d)}_{jk} = \begin{cases} cM^{(d)}_{n,\pm} & \text{if} \quad j = k \pm v_n^{(d)},\ 1 \le n \le d,\ (j,k) \in \partial\Lambda_L, \\ O^{(d)} & \text{otherwise.} \end{cases}$$

Thus, for the Hamiltonian $H_\omega(m, c) = H_0(m, c) + V_\omega$ we have the important relation

$$H_\omega(m,c) = H_\omega^{\Lambda_L}(m,c) \oplus H_\omega^{(\Lambda_L)^c}(m,c) + \mathcal{F}_{\Lambda_L} \tag{41}$$

with spacial decomposition $l^2(\mathbb{Z}^d, \mathbb{C}^\nu) = l^2(\Lambda_L, \mathbb{C}^\nu) \oplus l^2((\Lambda_L)^c, \mathbb{C}^\nu)$, where $(\Lambda_L)^c = \mathbb{Z} \setminus \Lambda_L$.

Now, the operator $H_0^{\Lambda_L}(m, c)$ on $l^2(\Lambda_L, \mathbb{C}^\nu)$ can be written as

$$H_0^{\Lambda_L}(m, c) = -c \sum_{n=1}^{d} M_n^{(d)} + mc^2 \mathcal{B}_d \; + A_{\Lambda_L} \, ,$$

where A_{Λ_L} is the adjacency matrix on Λ_L given by the matrix elements

$$\left(A_{\Lambda_L}\right)_{jk}^{(d)} = \begin{cases} cM_{n,\pm}^{(d)} & \text{if} \quad j, k \in \Lambda_L \text{ and } j = k \pm v_n^{(d)}, \; 1 \le n \le d, \\ O^{(d)} & \text{otherwise.} \end{cases}$$

Definition 6.2 We define the Neumann free Dirac operator $(H_0^{\Lambda_L}(m, c))^N$ on $l^2(\Lambda_L, \mathbb{C}^\nu)$, in dimensions $d \in \{1, 2, 3\}$ and with boundary conditions on Λ_L, by

$$(H_0^{\Lambda_L}(m, c))^N := \mathcal{N}_{\Lambda_L} + mc^2 \mathcal{B}_d \; + A_{\Lambda_L} \, ,$$

where $\mathcal{N}_{\Lambda_L}$ is the multiplication operator by the matrix

$$\mathcal{N}_{\Lambda_L}(k) := -c \sum_{n=1}^{d} M_n^{(d)} g_L(k_n)$$

with $k = (k_1, \ldots, k_d) \in \Lambda_L$ e the function $g_L(k_n) := \begin{cases} 1 & \text{if} \quad k_n \neq L \\ 0 & \text{if} \quad k_n = L \end{cases}, \; 1 \le n \le d,$ indicates that for $\psi \in l^2(\Lambda_L, \mathbb{C})$ one has $(d_n\psi)(k) \neq 0$ if $k_n \neq L$ and $(d_n\psi)(k) = 0$ if $k_n = L$. Here we consider Neumann boundary condition in sites $k \in \partial^{\mathrm{i}}\Lambda_L$ with $k_n = L$ and simple boundary condition in sites $k \in \partial^{\mathrm{i}}\Lambda_L$ with $k_n = -L$. We also define the Neumann Anderson Dirac operator restricted to $l^2(\Lambda_L, \mathbb{C}^\nu)$ by

$$(H_\omega^{\Lambda_L}(m, c))^N := (H_0^{\Lambda_L}(m, c))^N + V_\omega^{\Lambda_L},$$

where $V_\omega^{\Lambda_L}$ is the restriction of V_ω to Λ_L.

Note that the Anderson Dirac operators $H_\omega^{\Lambda_L}(m, c)$ and $(H_\omega^{\Lambda_L}(m, c))^N$ defined above are bounded and self-adjoint on $l^2(\Lambda_L, \mathbb{C}^\nu)$. We denote by

$$H_\omega^{\Lambda_L}(m, c)|_{\mathcal{H}_\pm} = H_0^{\Lambda_L}(m, c)|_{\mathcal{H}_\pm} \; + \; V_\omega^{\Lambda_L}|_{\mathcal{H}_\pm} \tag{42}$$

and

$$\left(H_\omega^{\Lambda_L}(m, c)\right)_{\mathcal{H}_\pm}^N = \left(H_0^{\Lambda_L}(m, c)\right)_{\mathcal{H}_\pm}^N \; + \; V_\omega^{\Lambda_L}|_{\mathcal{H}_\pm} \tag{43}$$

the restrictions of $H_\omega^{\Lambda_L}(m, c)$ and $(H_\omega^{\Lambda_L}(m, c))^N$ to the subspaces $\mathcal{H}_\pm$ of $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ of negative and positive energies.

Denote by $|B|$ the number of elements of a set $B \subset \mathbb{Z}^d$. Since $|\Lambda_L| = (2L + 1)^d$, the operators $H_0^{\Lambda_L}(m, c)|_{\mathcal{H}_\pm}$ act on spaces of finite dimension, i.e., they are finite matrices. Thus, their spectra $\sigma\left(H_0^{\Lambda_L}(m, c)|_{\mathcal{H}_\pm}\right)$ consist of eigenvalues $E_n^\pm\left(H_0^{\Lambda_L}(m, c)\right)$ which can be enumerate in crescent order,

$$\cdots \le E_2^-\left(H_0^{\Lambda_L}(m, c)\right) \le E_1^-\left(H_0^{\Lambda_L}(m, c)\right) \le E_1^+\left(H_0^{\Lambda_L}(m, c)\right) \le E_2^+\left(H_0^{\Lambda_L}(m, c)\right) \le \cdots ,$$

with $E_1^+\left(H_0^{\Lambda_L}(m,c)\right) = \inf\sigma\left(H_0^{\Lambda_L}(m,c)|_{\mathcal{H}_+}\right)$, $E_1^-\left(H_0^{\Lambda_L}(m,c)\right) = \sup\sigma\left(H_0^{\Lambda_L}(m,c)|_{\mathcal{H}_-}\right)$ and those eigenvalues are symmetric with respect to origin:

$$E_n^-\left(H_0^{\Lambda_L}(m,c)\right) = -E_n^+\left(H_0^{\Lambda_L}(m,c)\right), \quad \text{for all } n. \tag{44}$$

A similar description is valid for the eigenvalues $E_n^{\pm}\left((H_0^{\Lambda_L}(m,c))^N\right)$ of the operators $\left(H_0^{\Lambda_L}(m,c)\right)^N_{\mathcal{H}_\pm}$. Considering the matrix $\left[\left(H_0^{\Lambda_L}(m,c)\right)^N_{\mathcal{H}_+}\right]^2$, which corresponds to a Schrödinger Hamiltonian with Neumann boundary conditions on the sites $k \in \partial^{\mathrm{i}}\Lambda_L$ with $k_j = \pm L$, and the matrix $\left[H_0^{\Lambda_L}(m,c)|_{\mathcal{H}_+}\right]^2$, which corresponds to a Schrödinger Hamiltonian with Neumann boundary condition in the sites $k \in \partial^{\mathrm{i}}\Lambda_L$ with $k_j = -L$ and simple boundary condition in the sites $k \in \partial^{\mathrm{i}}\Lambda_L$ with $k_j = L$, one verifies that their eigenvalues satisfy $\tilde{E}_n^+\left(\left[\left(H_0^{\Lambda_L}(m,c)\right)^N\right]^2\right) \le \tilde{E}_n^+\left(\left[H_0^{\Lambda_L}(m,c)\right]^2\right)$, for all n. Since $\left(H_0^{\Lambda_L}(m,c)\right)^N_{\mathcal{H}_+} > 0$ and $H_0^{\Lambda_L}(m,c)|_{\mathcal{H}_+} > 0$, we obtain

$$E_n^+\left((H_0^{\Lambda_L}(m,c))^N\right) \le E_n^+\left(H_0^{\Lambda_L}(m,c)\right), \quad \text{for all } n, \tag{45}$$

and therefore $\left(H_0^{\Lambda_L}(m,c)\right)^N_{\mathcal{H}_+} \le H_0^{\Lambda_L}(m,c)|_{\mathcal{H}_+}$.

7 Initial Estimate of Lifshitz Tails

The proof of Theorem 3.3 follows general ideas from [17,27], developed in the context of the discrete Anderson Schrödinger model and for the unitary Anderson model. The first important point in this proof is to estimate the fractional moments of the Green function in finite volume (see Proposition 8.1). For this, Lemma 7.1 will be fundamental and its proof will make use of the following inequality (Proposition 4.15 in [4] or Lemma 6.3 in [19]):

Proposition 7.1 (Temple's inequality) *Let H be a self-adjoint operator. Suppose that H has an isolated nondegenerate eigenvalue $E_1 = \inf\sigma(H)$, and let $E_2 = \inf(\sigma(H)\setminus\{E_1\})$. If $\psi \in dom(H)$ with $\|\psi\| = 1$ satisfies $\langle\psi, H\psi\rangle < E_2$, then the following inequality holds:*

$$E_1 \ \ge\ \langle\psi, H\psi\rangle - \frac{\langle\psi, H^2\psi\rangle - \langle\psi, H\psi\rangle^2}{E_2 - \langle\psi, H\psi\rangle}.$$

Lemma 7.1 *Consider the operators $H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_\pm}$ given by (42) with $m > 0$ and let $0 < \omega_{\max} < 2mc^2$. For each $r \in (0,1)$, there are finite constants $\hat{\eta} > 0$ and $\hat{C} > 0$ such that the following estimates hold true for all $L \in \mathbb{N}$ sufficiently large:*

(i) $\mathbb{P}\left\{\omega : \inf\left(\sigma\left(H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_+}\right) - mc^2\right) \le L^{-r}\right\} \ \le\ \hat{C}L^d e^{-\hat{\eta}L^{rd/2}}$,

(ii) $\mathbb{P}\left\{\omega : \left(-mc^2 + \omega_{\max} - \sup\sigma\left(H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_-}\right)\right) \le L^{-r}\right\} \ \le\ \hat{C}L^d e^{-\hat{\eta}L^{rd/2}}$,

(iii) $\mathbb{P}\left\{\omega : \left(\inf\sigma\left(H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_-}\right) + \sqrt{m^2c^4 + a_d c^2}\right) \le L^{-r}\right\} \ \le\ \hat{C}L^d e^{-\hat{\eta}L^{rd/2}}$,

(iv) $\mathbb{P}\left\{\omega : \left(\sqrt{m^2c^4 + a_d c^2} + \omega_{\max} - \sup\sigma\left(H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_+}\right)\right) \le L^{-r}\right\} \ \le\ \hat{C}L^d e^{-\hat{\eta}L^{rd/2}}$.

Proof (i) The proof is based on ideas from [4,19] used in the context of random Schrödinger operators. For the operators $H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_+}$ and $\left(H_\omega^{\Lambda_L}(m,c)\right)_{\mathcal{H}_+}^N$ given by (42) and (43), due to relation (45) and supp $\mu = [0, \omega_{\max}]$, we have

$$\inf \sigma \left(\left(H_\omega^{\Lambda_L}(m,c)\right)_{\mathcal{H}_+}^N \right) \leq \inf \sigma \left(H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_+} \right).$$

Thus, it is sufficient to check the estimate in (i) for the operator $\left(H_\omega^{\Lambda_L}(m,c)\right)_{\mathcal{H}_+}^N$. In order to use Proposition 7.1, consider the constant eigenfunction Ψ, with $\|\Psi\| = 1$, of the operator $\left(H_0^{\Lambda_L}(m,c)\right)_{\mathcal{H}_+}^N$ in $l^2(\Lambda_L, \mathbb{C}^\nu)$, defined by

$$\Psi(n) = \begin{pmatrix} \Psi_1(n) \\ \vdots \\ \Psi_\nu(n) \end{pmatrix} \quad \text{with} \quad \Psi_\alpha(n) = \begin{cases} \left(2^{p-1}|\Lambda_L|\right)^{-1/2} & \text{if } 1 \leq \alpha \leq p \\ 0 & \text{if } p < \alpha \leq \nu \end{cases},$$

for $n \in \Lambda_L$ and $p = \nu/2 = \begin{cases} 1 & \text{if} \quad d \in \{1,2\} \\ 2 & \text{if} \quad d = 3 \end{cases}$, associated with the first (smallest) positive eigenvalue $E_1^+\left((H_0^{\Lambda_L}(m,c))^N\right) = mc^2$. For dimensions $d \in \{1,2,3\}$, we have

$$\begin{aligned} \left\langle \Psi, \left[\left(H_\omega^{\Lambda_L}(m,c)\right)_{\mathcal{H}_+}^N - mc^2 I \right] \Psi \right\rangle &= \left\langle \Psi, \left[\left(H_0^{\Lambda_L}(m,c)\right)_{\mathcal{H}_+}^N - mc^2 I \right] \Psi \right\rangle \\ &\quad + \left\langle \Psi, V_\omega^{\Lambda_L}|_{\mathcal{H}_+} \Psi \right\rangle \\ &= \frac{1}{2^{p-1}|\Lambda_L|} \sum_{n \in \Lambda_L} \sum_{i=1}^{p} V_\omega^{(i)}(n). \end{aligned}$$

Note that these sums ($p \in \{1,2\}$) are arithmetic means of i.i.d. random variables and

$$\frac{1}{|\Lambda_L|} \sum_{n \in \Lambda_L} V_\omega^{(i)}(n) \longrightarrow \mathbb{E}(V_\omega^{(i)}(0)) > 0 \quad \mathbb{P}\text{-a.s.}, \tag{46}$$

for $i \in \{1,2\}$, as $L \to \infty$. To apply Proposition 7.1, we need

$$\frac{1}{2^{p-1}|\Lambda_L|} \sum_{n \in \Lambda_L} \sum_{i=1}^{p} V_\omega^{(i)}(n) < E_2^+\left((H_\omega^{\Lambda_L}(m,c))^N\right) - mc^2, \tag{47}$$

where $E_2^+\left((H_\omega^{\Lambda_L}(m,c))^N\right)$ is the second positive eigenvalue of $\left(H_\omega^{\Lambda_L}(m,c)\right)_{\mathcal{H}_+}^N$. However, $E_2^+\left((H_\omega^{\Lambda_L}(m,c))^N\right) - mc^2 \longrightarrow 0$ as $L \to \infty$, which contradicts relations (46) and (47). The rate at which $E_2^+\left((H_\omega^{\Lambda_L}(m,c))^N\right) - mc^2$ vanishes is given by the estimate

$$E_2^+\left((H_\omega^{\Lambda_L}(m,c))^N\right) - mc^2 \geq E_2^+\left((H_0^{\Lambda_L}(m,c))^N\right) - mc^2 \geq K(m,c)L^{-2}, \tag{48}$$

for a suitable constant $K(m,c) > 0$. In order to obtain relation (47), we introduce a new potential $W_\omega^{(L)}|_{\mathcal{H}_+}$, which is a diagonal matrix formed by random potentials (in each site)

$$W_\omega^{(L)(\alpha)}(n) = \min\left\{V_\omega^{(\alpha)}(n),\ \frac{1}{3}K(m,c)L^{-2}\right\},\quad \text{with } \alpha \in \mathcal{A},\ n \in \Lambda_L.$$

For fixed L, the random variables $W_\omega^{(L)(\alpha)}(n)$ are still i.i.d.. Defining the operator

$$H_\omega^{(L)}(m,c) := \left(H_0^{\Lambda_L}(m,c)\right)^N_{\mathcal{H}_+} + W_\omega^{(L)}|_{\mathcal{H}_+}$$

on $l^2(\Lambda_L, \mathbb{C}^\nu)$, we have that $E_1^+\left(\left(H_\omega^{\Lambda_L}(m,c)\right)^N\right) \geq E_1^+\left(H_\omega^{(L)}(m,c)\right)$. It follows from the definition of $W_\omega^{(L)}|_{\mathcal{H}_+}$ and relation (48) that

$$\begin{aligned}\left\langle\Psi, \left(H_\omega^{(L)}(m,c) - mc^2 I\right)\Psi\right\rangle &= \frac{1}{2^{p-1}|\Lambda_L|}\sum_{n\in\Lambda_L}\sum_{i=1}^{p} W_\omega^{(L)(i)}(n)\\ &\leq \frac{1}{3}K(m,c)L^{-2}\\ &< E_2^+\left((H_\omega^{(L)}(m,c))\right) - mc^2.\end{aligned}$$

By applying Proposition 7.1 to $H_\omega^{(L)}(m,c) - mc^2 I$ and Ψ, we have for $d \in \{1,2,3\}$,

$$\begin{aligned}&E_1^+\left(\left(H_\omega^{\Lambda_L}(m,c)\right)^N\right) - mc^2 \geq E_1^+\left(H_\omega^{(L)}(m,c)\right) - mc^2\\ &\geq \left\langle\Psi, \left(H_\omega^{(L)}(m,c) - mc^2 I\right)\Psi\right\rangle - \frac{\left\langle\Psi, \left(H_\omega^{(L)}(m,c) - mc^2 I\right)^2\Psi\right\rangle}{K(m,c)L^{-2} - \left\langle\Psi, \left(H_\omega^{(L)}(m,c) - mc^2 I\right)\Psi\right\rangle}\\ &\geq \frac{1}{2^{p-1}|\Lambda_L|}\sum_{n\in\Lambda_L}\sum_{i=1}^{p} W_\omega^{(L)(i)}(n) - \frac{\frac{1}{2^{p-1}|\Lambda_L|}\sum_{n\in\Lambda_L}\sum_{i=1}^{p}\left(W_\omega^{(L)(i)}(n)\right)^2}{\left(K(m,c) - \frac{1}{3}K(m,c)\right)L^{-2}}\\ &\geq \frac{1}{2^p|\Lambda_L|}\sum_{n\in\Lambda_L}\sum_{i=1}^{p} W_\omega^{(L)(i)}(n)\ .\end{aligned}$$

The above estimate implies that

$$\mathbb{P}\left\{\omega : E_1^+\left(\left(H_\omega^{\Lambda_L}(m,c)\right)^N - mc^2\right) \leq E\right\} \leq \mathbb{P}\left\{\omega : \frac{1}{|\Lambda_L|}\sum_{n\in\Lambda_L}\sum_{i=1}^{p} W_\omega^{(L)(i)}(n) \leq 2^p E\right\}.$$

By Lemma 6.4 in [19] (see also [4,26]), for $L = \lfloor \delta(2^p E)^{-1/2}\rfloor$ (where $\lfloor x\rfloor$ denotes the largest integer $\leq x$) with $\delta > 0$ and E small enough,

$$\mathbb{P}\left\{\omega : \frac{1}{|\Lambda_L|}\sum_{n\in\Lambda_L}\sum_{i=1}^{p} W_\omega^{(L)(i)}(n) \leq 2^p E\right\} \leq e^{-\gamma_0|\Lambda_L|}$$

for some $\gamma_0 > 0$, where $p \in \{1, 2\}$. It follows by these estimates that, for $d \in \{1, 2, 3\}$, there exist $l \in \mathbb{N}$ large enough, $b > 0$ and $\hat{\gamma} > 0$ such that

$$\mathbb{P}\left\{\omega : E_1^+\left(\left(H_\omega^{\Lambda_l}(m, c)\right)^N - mc^2\right) \le bl^{-2}\right\} \le e^{-\hat{\gamma} l^d}. \tag{49}$$

Now, we build a big cube Λ_L by grouping disjoint copies of the cube Λ_l; more precisely,

$$\Lambda_L = \bigcup_{j \in \mathcal{R}} \Lambda_l(j) \tag{50}$$

with $\Lambda_l(j) = [-l + j, l + j]^d \cap \mathbb{Z}^d$. Indeed, for any odd integer n we take $L = nl + \frac{n-1}{2}$. The set $\mathcal{R}$ indicated in (50) contains n^d points. For every $r \in (0, 1)$, by choosing $n \approx b^{-1/r} l^{2/r-1}$, it follows that $L \approx b^{-1/r} l^{2/r}$. Hence $L^{-r} \approx bl^{-2}$ and $l \approx b^{1/2} L^{r/2}$. Similarly to the Schrödinger case [4,19,26], we have

$$\left(H_\omega^{\Lambda_L}(m, c)\right)^N_{\mathcal{H}_+} \ge \bigoplus_{j \in \mathcal{R}} \left(H_\omega^{\Lambda_l(j)}(m, c)\right)^N_{\mathcal{H}_+}.$$

Therefore, the first positive eigenvalue can be estimated by

$$E_1^+\left(\left(H_\omega^{\Lambda_L}(m, c)\right)^N\right) \ge \min_{j \in \mathcal{R}} E_1^+\left((H_\omega^{\Lambda_l(j)}(m, c))^N\right).$$

Thus, by applying relation (49), we obtain for $d \in \{1, 2, 3\}$,

$$\begin{aligned}
&\mathbb{P}\left\{\omega : \inf \sigma\left(H_\omega^{\Lambda_L}(m, c)|_{\mathcal{H}_+} - mc^2\right) \le L^{-r}\right\} \\
&\le \mathbb{P}\left\{\omega : E_1^+\left(\left(H_\omega^{\Lambda_L}(m, c)\right)^N - mc^2\right) \le L^{-r}\right\} \\
&\le \mathbb{P}\left\{\omega : E_1^+\left((H_\omega^{\Lambda_l(j)}(m, c))^N - mc^2\right) \le bl^{-2} \text{ for some } j \in \mathcal{R}\right\} \\
&\le n^d\, \mathbb{P}\left\{\omega : E_1^+\left((H_\omega^{\Lambda_l}(m, c))^N - mc^2\right) \le bl^{-2}\right\} \\
&\le n^d e^{-\hat{\gamma} l^d} \approx (Ll^{-1})^d e^{-\hat{\gamma} l^d} \approx b^{-d/2} L^{d(1-r/2)} e^{-\hat{\gamma}\, b^{d/2} L^{rd/2}} \\
&\le \hat{C} L^d e^{-\hat{\eta} L^{rd/2}},
\end{aligned}$$

where $\hat{C} = b^{-d/2} > 0$ and $\hat{\eta} = \hat{\gamma}\, b^{d/2} > 0$. This completes the proof of item (**i**).

(ii) By Theorem 2.2 and relation (44) one has, $\mathbb{P}$-a.s.,

$$\begin{aligned}
\inf \sigma\left(H_\omega^{\Lambda_L}(m, c)|_{\mathcal{H}_+}\right) &= \inf \sigma\left(H_0^{\Lambda_L}(m, c)|_{\mathcal{H}_+}\right) + \inf(\operatorname{supp} \mu) \\
&= -\sup \sigma\left(H_0^{\Lambda_L}(m, c)|_{\mathcal{H}_-}\right) \\
&= -\sup \sigma\left(H_\omega^{\Lambda_L}(m, c)|_{\mathcal{H}_-}\right) + \omega_{\max}.
\end{aligned}$$

Therefore (ii) follows from (i).

(iii) We can change the boundary condition on Λ_L by defining a new operator $\hat{\mathcal{N}}_{\Lambda_L}$ for each dimension $d \in \{1, 2, 3\}$, so that the corresponding free Neumann Dirac operator

$$(H_0^{\Lambda_L}(m, c))^{\hat{N}} := \hat{\mathcal{N}}_{\Lambda_L} + mc^2 \mathcal{B}_d + A_{\Lambda_L},$$

restricted to the subspace $\mathcal{H}_-$, has the lower extreme energy $-\sqrt{m^2c^4 + a_d c^2}$ of its spectrum as the lowest negative eigenvalue associated with an eigenfunction Φ in Λ_L whose entries have constant absolute values. Thus, the proof of (**iii**) follows analogously the one of (**i**), but with the operator $\left(H_0^{\Lambda_L}(m,c)\right)_{\mathcal{H}_+}^N$ and the energy mc^2 replaced by $\left(H_0^{\Lambda_L}(m,c)\right)_{\mathcal{H}_-}^{\hat{N}}$ and $-\sqrt{m^2c^4+a_dc^2}$, respectively.

(iv) Similarly to (**ii**), one shows that, $\mathbb{P}$-a.s.,

$$\inf\sigma\left(H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_-}\right) = -\sup\sigma\left(H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_+}\right) + \omega_{\max}\ .$$

Therefore (**iv**) follows from (**iii**).

□

8 Estimates of Fractional Moments in Finite Volume

Let $\mathcal{G}_{\omega,\Lambda_L}(m,c,z)$ and $\mathcal{G}_{\omega,\Lambda_L}^{\mathcal{H}_\pm}(m,c,z)$ be the Green functions, respectively, of $H_\omega^{\Lambda_L}(m,c)$ and $H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_\pm}$, whose matrix elements are given by

$$G_{\omega,\Lambda_L}^{\alpha\beta}(j,k;z) := \left\langle e_{j\alpha}, \left(H_\omega^{\Lambda_L}(m,c) - zI\right)^{-1} e_{k\beta}\right\rangle$$

and

$$G_{\omega,\Lambda_L}^{\alpha\beta,\mathcal{H}_\pm}(j,k;z) := \left\langle e_{j\alpha}, \left(H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_\pm} - zI\right)^{-1} e_{k\beta}\right\rangle ,$$

where $j,k\in\Lambda_L, \alpha,\beta\in\mathcal{A}$ and $z\in\mathbb{C}\backslash\mathbb{R}$. The goal of this section is to estimate the fractional moments of the Green function in finite volume, i.e., $\mathbb{E}(|G_{\omega,\Lambda_L}^{\alpha\beta,\mathcal{H}_\pm}(j,k;z)|^s)$ for $s\in(0,1)$ (Proposition 8.1 below). This result will be used in Sect. 11, together with the decoupling estimate (61), to obtain Theorem 3.3. In order to estimate $\mathbb{E}(|G_{\omega,\Lambda_L}^{\alpha\beta,\mathcal{H}_\pm}(j,k;z)|^s)$, we will use Lemma 7.1, Proposition 4.1 (valid also for the Green function in finite volume) and Lemma 8.1 below (the Combes-Thomas argument for discrete Dirac operators with bounded potential, as DAD). Lemma 8.1 was obtained in Proposition 1 in [23] for the one-dimensional Dirac model, with direct adaptations to $d\in\{2,3\}$.

Lemma 8.1 (Combes–Thomas) *Let $H(m,c) = H_0(m,c)+V$ be a Dirac operator on $l^2(\mathbb{Z}^d,\mathbb{C}^\nu)$ with bounded potential V. If $z\notin\sigma(H(m,c))$, let $\Delta := dist\{z,\sigma(H(m,c))\} > 0$. Then, there exists $a>0$ such that for every $j,k\in\mathbb{Z}^d$, $\alpha,\beta\in\mathcal{A}$,*

$$|G^{\alpha\beta}(j,k;z)| := \left|\langle e_{j\alpha}, (H(m,c)-zI)^{-1} e_{k\beta}\rangle\right| \ \le\ \frac{2}{\Delta}\, e^{-a\Delta|j-k|}.$$

We are now ready to discuss the main result of this section, namely, the exponential decay of the fractional moments of the Green function in finite volume.

Proposition 8.1 *For every $s\in(0,1)$, there exist finite constants $\tilde{C}>0$ and $\tilde{\eta}>0$ such that*

$$\mathbb{E}(|G_{\omega,\Lambda_L}^{\alpha\beta,\mathcal{H}_\pm}(j,k;E+i\epsilon)|^s) \ \le\ \tilde{C}L^d e^{-\tilde{\eta}L^{d/(d+2)}} \tag{51}$$

for $L\in\mathbb{N}$ sufficiently large, for all $\alpha,\beta\in\mathcal{A}$, $j,k\in\Lambda_L$ with $|j-k|\ge L/2$, $\epsilon>0$ and $E\in I_{\delta_L}^{\pm}(m,c)$, where $I_{\delta_L}^{\pm}(m,c)$ are the intervals of negative and positive energies described in (12)–(13) with $\delta_L = \frac{1}{2}L^{-2/(d+2)}$.

Proof Let $I_{\delta_L}^{+}(m,c) = [mc^2, mc^2+\delta_L]$ and $r\in(0,1)$. Due to Lemma 7.1(i), consider the set

$$\Omega_G := \left\{\omega : \inf\sigma\left(H_\omega^{\Lambda_L}(m,c)|_{\mathcal{H}_+} - mc^2\right) > L^{-r}\right\}$$

and its complement $\Omega_B := \Omega \setminus \Omega_G$. Then, $\chi_\Omega = \chi_{\Omega_G} + \chi_{\Omega_B}$ and for every $s \in (0,1)$ we have

$$\begin{aligned}
&\mathbb{E}(|G^{\alpha\beta,\mathcal{H}_+}_{\omega,\Lambda_L}(j,k;E+i\epsilon)|^s) \\
&\quad = \mathbb{E}(|G^{\alpha\beta,\mathcal{H}_+}_{\omega,\Lambda_L}(j,k;E+i\epsilon)|^s \chi_{\Omega_G}) + \mathbb{E}(|G^{\alpha\beta,\mathcal{H}_+}_{\omega,\Lambda_L}(j,k;E+i\epsilon)|^s \chi_{\Omega_B}). \qquad (52)
\end{aligned}$$

Pick $p > 1$ sufficiently small such that $s \cdot p < 1$ and let $q > 1$ be such that $1/p + 1/q = 1$. By applying Hölder's inequality in the second term on the right hand side of (52), together with Proposition 4.1 and Lemma 7.1(i), we get

$$\begin{aligned}
\mathbb{E}(|G^{\alpha\beta,\mathcal{H}_+}_{\omega,\Lambda_L}(j,k;E+i\epsilon)|^s \chi_{\Omega_B}) &\leq \left[\mathbb{E}(|G^{\alpha\beta,\mathcal{H}_+}_{\omega,\Lambda_L}(j,k;E+i\epsilon)|^{sp})\right]^{1/p} [\mathbb{P}(\Omega_B)]^{1/q} \\
&\leq C_0 L^{d/q} e^{-\frac{\hat{\eta}}{q} L^{rd/2}}, \qquad (53)
\end{aligned}$$

with $C_0 = C_1^{1/p} \hat{C}^{1/q}$ and $\hat{\eta} > 0$.

For $\omega \in \Omega_G$, one has $\mathrm{dist}\{E, \sigma(H^{\Lambda_L}_\omega(m,c)|_{\mathcal{H}_+})\} \geq \frac{1}{2} L^{-r}$. In this case, by Lemma 8.1,

$$\mathbb{E}(|G^{\alpha\beta,\mathcal{H}_+}_{\omega,\Lambda_L}(j,k;E+i\epsilon)|^s \chi_{\Omega_G}) \;\leq\; 4^s L^{rs} e^{-\frac{a}{2} s L^{-r}|j-k|}\, \mathbb{P}(\Omega_G) \;\leq\; 4^s L^{rs} e^{-\frac{a}{4} s L^{1-r}}. \qquad (54)$$

Choosing $r = \frac{2}{d+2}$ and inserting relations (53)–(54) into (52), we conclude that

$$\mathbb{E}(|G^{\alpha\beta,\mathcal{H}_+}_{\omega,\Lambda_L}(j,k;E+i\epsilon)|^s) \;\leq\; 4^s L^{\frac{2}{d+2}s} e^{-\frac{a}{4} s L^{d/(d+2)}} + C_0 L^{d/q} e^{-\frac{\hat{\eta}}{q} L^{d/(d+2)}} \;\leq\; \tilde{C} L^d e^{-\tilde{\eta} L^{d/(d+2)}},$$

with $\tilde{C} = 2\max\{4^s, C_0\} < \infty$ and $\tilde{\eta} = \min\left\{\frac{a}{4}s, \frac{\hat{\eta}}{q}\right\} > 0$.

Similarly, one proves estimate (51) for the other three energies intervals $I^{\pm}_{\delta_L}(m,c)$ using the corresponding estimates obtained in Lemma 7.1(ii)–(iv). □

9 Geometric Resolvent Equation

The idea of the proof of Theorem 3.3 is to obtain the exponential decay (11) from the exponential decay (51) obtained in Proposition 8.1. For this, we will use the so-called geometric decoupling method, which is well known in the context of random Schrödinger operators [3,27].

Due to relation (41), we can write the Hamiltonian (1) in the form

$$H_\omega(m,c) = H_{\omega,L}(m,c) + \mathcal{F}_{\Lambda_L}, \qquad (55)$$

where $H_{\omega,L}(m,c) = H^{\Lambda_L}_\omega(m,c) \oplus H^{(\Lambda_L)^c}_\omega(m,c)$ and $\mathcal{F}_{\Lambda_L}$ is the boundary operator defined in Sect. 6. Given $z \in \mathbb{C} \setminus \mathbb{R}$, we write $G_\omega = (H_\omega(m,c) - zI)^{-1}$ and $G_{\omega,L} = (H_{\omega,L}(m,c) - zI)^{-1}$. We perform a double decoupling, once on Λ_L and then on Λ_{L+1}. By applying the second resolvent identity on (55) twice, we get

$$\begin{aligned}
G_\omega &= G_{\omega,L} - G_{\omega,L}\mathcal{F}_{\Lambda_L} G_\omega \\
&= G_{\omega,L} - G_{\omega,L}\mathcal{F}_{\Lambda_L} G_{\omega,L+1} + G_{\omega,L}\mathcal{F}_{\Lambda_L} G_\omega \mathcal{F}_{\Lambda_{L+1}} G_{\omega,L+1}\,. \qquad (56)
\end{aligned}$$

By translation invariance we have $\mathbb{E}\left(|G^{\alpha\beta}_\omega(j,k;z)|^s\right) = \mathbb{E}\left(|G^{\alpha\beta}_\omega(0,k-j;z)|^s\right)$. Thus, it is sufficient to prove (11) for $j = 0$. Taking matrix-elements in (56),

$$\begin{aligned}
&G^{\alpha\beta}_\omega(0,k;z) = \\
&\quad = G^{\alpha\beta}_{\omega,L}(0,k;z) - \langle e_{0\alpha}, G_{\omega,L}\mathcal{F}_{\Lambda_L} G_{\omega,L+1} e_{k\beta}\rangle + \langle e_{0\alpha}, G_{\omega,L}\mathcal{F}_{\Lambda_L} G_\omega \mathcal{F}_{\Lambda_{L+1}} G_{\omega,L+1} e_{k\beta}\rangle.
\end{aligned}$$

For $k \in \mathbb{Z}^d$ with $|k| \geq L+2$, we have $G^{\alpha\beta}_{\omega,L}(0,k;z) = 0$ and $\langle e_{0\alpha}, G_{\omega,L}\mathcal{F}_{\Lambda_L}G_{\omega,L+1}e_{k\beta}\rangle = 0$. Thus,

$$G^{\alpha\beta}_{\omega}(0,k;z) = \langle e_{0\alpha}, G_{\omega,L}\mathcal{F}_{\Lambda_L}G_{\omega}\mathcal{F}_{\Lambda_{L+1}}G_{\omega,L+1}e_{k\beta}\rangle. \tag{57}$$

For convenient notation, we introduce the sets $X^{(d)}_{n,\pm} \subset \mathcal{A} \times \mathcal{A}$ defined, for $d \in \{1,2,3\}$ and $1 \leq n \leq d$ with $n \in \mathbb{N}$, by

$$X^{(1)}_{1,+} = \{(1,2)\}, \quad X^{(1)}_{1,-} = \{(2,1)\},$$
$$X^{(2)}_{1,+} = X^{(2)}_{2,+} = \{(1,2)\}, \quad X^{(2)}_{1,-} = X^{(2)}_{2,-} = \{(2,1)\},$$
$$X^{(3)}_{1,+} = X^{(3)}_{2,+} = \{x_+ = (1,4), y_+ = (2,3)\}, \quad X^{(3)}_{3,+} = \{z_+ = (1,3), w_+ = (2,4)\},$$
$$X^{(3)}_{1,-} = X^{(3)}_{2,-} = \{x_- = (4,1), y_- = (3,2)\}, \quad X^{(3)}_{3,-} = \{z_- = (3,1), w_- = (4,2)\}.$$

Note that $X^{(d)}_{n,-}$ is obtained from $X^{(d)}_{n,+}$ by permutation of coordinates.

Let $\left\{v^{(d)}_n : 1 \leq n \leq d,\ n \in \mathbb{N}\right\}$ be the canonical basis of $\mathbb{R}^d$, defined in Sect. 6. Using the fact that $\left\{e_{j\alpha}\right\}_{j\in\mathbb{Z}^d,\alpha\in\mathcal{A}}$ is an orthonormal basis of $l^2(\mathbb{Z}^d, \mathbb{C}^\nu)$ and the definition of the boundary operator $\mathcal{F}_{\Lambda_L}$, we expand the right side of (57) obtaining, for $|k| \geq L+2$,

$$\begin{aligned} G^{\alpha\beta}_{\omega}(0,k;z) = \sum_{1\leq n,l\leq d} C^{(d)}_{n,\pm}C^{(d)}_{l,\pm} \sum_{(u,u')\in\Gamma^{(d)}_{L,n,\pm}} \sum_{(v,v')\in\Gamma^{(d)}_{L+1,l,\pm}} G^{\alpha\beta_u}_{\omega,L}(0,u;z) \\ G^{\alpha_u\beta_v}_{\omega}(u',v;z)G^{\alpha_v\beta}_{\omega,L+1}(v',k;z), \end{aligned} \tag{58}$$

with

$$\Gamma^{(d)}_{L,n,\pm} := \left\{(u,u') \in \partial\Lambda_L : u = u' \pm v^{(d)}_n,\ (\beta_u,\alpha_u) \in X^{(d)}_{n,\pm}\right\}, \quad 1 \leq n \leq d,$$

and constants

$$C^{(1)}_{1,\pm} = c, \quad C^{(2)}_{1,\pm} = c, \quad C^{(2)}_{2,\pm} = \mp ci, \quad C^{(3)}_{1,\pm} = \begin{cases} c & \text{if} \quad (\beta_u,\alpha_u) = x_\pm \\ c & \text{if} \quad (\beta_u,\alpha_u) = y_\pm \end{cases},$$

$$C^{(3)}_{2,\pm} = \begin{cases} \mp ci & \text{if} \quad (\beta_u,\alpha_u) = x_\pm \\ \pm ci & \text{if} \quad (\beta_u,\alpha_u) = y_\pm \end{cases} \quad \text{and} \quad C^{(3)}_{3,\pm} = \begin{cases} c & \text{if} \quad (\beta_u,\alpha_u) = z_\pm \\ -c & \text{if} \quad (\beta_u,\alpha_u) = w_\pm. \end{cases}$$

Relation (58) is called *geometric resolvent equation*.

10 Decoupling of Fractional Moments

In this section we establish relations, of decoupling of fractional moments, that allow us to write the fractional moments of the infinite volume Green function in terms of the corresponding finite volume Green function.

The following result says that the fractional moments $\mathbb{E}(|G^{\alpha\beta}_{\omega}(0,k;z)|^s)$ can be decoupled along the boundary of Λ_L. Recall from Sect. 6 that $u \in \partial^{\mathrm{i}}\Lambda_L$ if and only if $\|u\|_\infty = L$, and $u' \in \partial^{\mathrm{o}}\Lambda_L$ if and only if $\|u'\|_\infty = L+1$.

Lemma 10.1 *For every $s \in (0,1)$ and $d \in \{1,2,3\}$, there exists a constant $C = C(s,\rho,c,d) > 0$ such that*

$$\mathbb{E}(|G_\omega^{\alpha\beta}(0,k;z)|^s) \leq C \sum_{\beta_u,\alpha_v \in \mathcal{A}} \sum_{u \in \partial^{\mathrm{i}}\Lambda_L} \mathbb{E}(|G^{\alpha\beta_u}_{\omega,\Lambda_L}(0,u;z)|^s) \\ \times \sum_{v' \in \partial^{\mathrm{o}}\Lambda_{L+1}} \mathbb{E}(|G^{\alpha_v\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)|^s)$$

for all $z \in \mathbb{C}\setminus\mathbb{R}$, $\alpha,\beta \in \mathcal{A}$, $L \in \mathbb{N}$ and $k \in \mathbb{Z}^d$ with $|k| \geq L+2$.

Proof Let $s \in (0,1)$. Taking fractional moments in (58) we get, for $|k| \geq L+2$ and $d \in \{1,2,3\}$,

$$\mathbb{E}(|G_\omega^{\alpha\beta}(0,k;z)|^s) \\ \leq c^{2s} \sum_{1\leq n,l\leq d} \sum_{(u,u')\in\Gamma^{(d)}_{L,n,\pm}} \sum_{(v,v')\in\Gamma^{(d)}_{L+1,l,\pm}} \mathbb{E}\left(|G^{\alpha\beta_u}_{\omega,\Lambda_L}(0,u;z)G^{\alpha_u\beta_v}_\omega(u',v;z)G^{\alpha_v\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)|^s\right) \tag{59}$$

where $G^{\alpha\beta_u}_{\omega,L}(0,u;z)$ was replaced by $G^{\alpha\beta_u}_{\omega,\Lambda_L}(0,u;z)$ since $0,u \in \Lambda_L$, and $G^{\alpha_v\beta}_{\omega,L+1}(v',k;z)$ was replaced by $G^{\alpha_v\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)$ since $v',k \in (\Lambda_{L+1})^c$. Now for (u,u'), (v,v'), (β_u,α_u) and (β_v,α_v) fixed, consider the term

$$G^{\alpha\beta_u}_{\omega,\Lambda_L}(0,u;z)\; G^{\alpha_u\beta_v}_\omega(u',v;z)\; G^{\alpha_v\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)$$

and note that the first and last of the three factors are independent of $\omega_{u'\alpha_u}$ and $\omega_{v\beta_v}$. Thus, by considering the conditional expectation of the central term we have, by Proposition 4.1,

$$\mathbb{E}^{\alpha_u\beta_v}_{u',v}(|G^{\alpha_u\beta_v}_\omega(u',v;z)|^s) \leq C_1$$

for all $\alpha_u, \beta_v \in \mathcal{A}$ and for a constant $C_1 = C_1(s,\rho)$. Hence,

$$\mathbb{E}\left(|G^{\alpha\beta_u}_{\omega,\Lambda_L}(0,u;z)G^{\alpha_u\beta_v}_\omega(u',v;z)G^{\alpha_v\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)|^s\right) \\ = \mathbb{E}\left(|G^{\alpha\beta_u}_{\omega,\Lambda_L}(0,u;z)|^s\; \mathbb{E}^{\alpha_u\beta_v}_{u',v}(|G^{\alpha_u\beta_v}_\omega(u',v;z)|^s)\; |G^{\alpha_v\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)|^s\right) \\ \leq C_1\, \mathbb{E}\left(|G^{\alpha\beta_u}_{\omega,\Lambda_L}(0,u;z)|^s\right)\, \mathbb{E}\left(|G^{\alpha_v\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)|^s\right) \tag{60}$$

where in the last step we have used the fact that the two terms are stochastically independent. By substituting (60) into (59), it follows that there exists a constant $C = C_1 c^{2s} d^2 > 0$ such that

$$\mathbb{E}(|G_\omega^{\alpha\beta}(0,k;z)|^s) \leq C \sum_{\beta_u,\alpha_v \in \mathcal{A}} \sum_{u \in \partial^{\mathrm{i}}\Lambda_L} \mathbb{E}(|G^{\alpha\beta_u}_{\omega,\Lambda_L}(0,u;z)|^s) \\ \times \sum_{v' \in \partial^{\mathrm{o}}\Lambda_{L+1}} \mathbb{E}(|G^{\alpha_v\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)|^s)$$

for all $z \in \mathbb{C}\setminus\mathbb{R}$, $\alpha,\beta \in \mathcal{A}$ and $k \in \mathbb{Z}^d$ with $|k| \geq L+2$. □

We want to use the estimate in Lemma 10.1 as the first step in an iterative argument. The next step consists of finding a bound for $\mathbb{E}(|G^{\alpha\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)|^s)$ similar to the bound for $\mathbb{E}(|G^{\alpha\beta}_\omega(0,k;z)|^s)$ given by Lemma 10.1, with v' as the new origin. Following the idea of Lemma 2.3 in [3] (which was also used in Lemma 7.3 of [27]), the next result relates the Green function $G_{\omega,(\Lambda_{L+1})^c}$ with the full Green function G_ω.

Lemma 10.2 *For every $s \in (0,1)$ and $d \in \{1,2,3\}$, there exists a constant $\tilde{C} = \tilde{C}(s,\rho,c,d) > 0$ such that*

$$\mathbb{E}(|G^{\alpha\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)|^s) \le \mathbb{E}(|G^{\alpha\beta}_{\omega}(v',k;z)|^s) + \tilde{C} \sum_{u'\in\partial^{\mathrm{o}}\Lambda_{L+1}} \sum_{\gamma\in\mathcal{A}} \mathbb{E}(|G^{\gamma\beta}_{\omega}(u',k;z)|^s)$$

for all $z \in \mathbb{C}\setminus\mathbb{R}$, $\alpha,\beta \in \mathcal{A}$, $L \in \mathbb{N}$, $v',k \in \mathbb{Z}^d$ with $v' \in \partial^{\mathrm{o}}\Lambda_{L+1}$ and $|k| \ge L+2$.

Proof Using the second resolvent identity on (55) gives $G_{\omega,L+1} = G_\omega + G_{\omega,L+1}\mathcal{F}_{\Lambda_{L+1}}G_\omega$. Taking matrix-elements and arguing as done for the geometric resolvent equation (58), we obtain,

$$\begin{aligned} G^{\alpha\beta}_{\omega,L+1}(v',k;z) &= G^{\alpha\beta}_{\omega}(v',k;z) + \langle e_{v'\alpha}, G_{\omega,L+1}\mathcal{F}_{\Lambda_{L+1}}G_\omega e_{k\beta}\rangle \\ &= G^{\alpha\beta}_{\omega}(v',k;z) + \sum_{1\le n,l\le d} C^{(d)}_{n,\pm} \\ &\quad \times \sum_{(u,u')\in\Gamma^{(d)}_{L+1,n,\pm}} G^{\alpha\beta_u}_{\omega,L+1}(v',u;z)G^{\alpha_u\beta}_{\omega}(u',k;z). \end{aligned}$$

Taking fractional moments in this relation and using the following decoupling inequality (analogous to Lemma C.1 of [3]; see also [27]),

$$\mathbb{E}(|G^{\alpha\beta_u}_{\omega,L+1}(v',u;z)|^s|G^{\alpha_u\beta}_{\omega}(u',k;z)|^s) \le C_1\, \mathbb{E}(|G^{\alpha_u\beta}_{\omega}(u',k;z)|^s)$$

with $0 < C_1 = C_1(s,\rho) < \infty$, we get

$$\mathbb{E}(|G^{\alpha\beta}_{\omega,L+1}(v',k;z)|^s) \le \mathbb{E}(|G^{\alpha\beta}_{\omega}(v',k;z)|^s) + \tilde{C} \sum_{u'\in\partial^{\mathrm{o}}\Lambda_{L+1}} \sum_{\gamma\in\mathcal{A}} \mathbb{E}(|G^{\gamma\beta}_{\omega}(u',k;z)|^s)$$

with the constant $\tilde{C} = C_1 c^s d > 0$. Replacing $G^{\alpha\beta}_{\omega,L+1}(v',k;z)$ by $G^{\alpha\beta}_{\omega,(\Lambda_{L+1})^c}(v',k;z)$, since $v',k \in (\Lambda_{L+1})^c$, the result follows. □

In order to obtain the relation that will be used as the starting point of an iteration, we insert the estimate in Lemma 10.2 into the estimate in Lemma 10.1, and use that $|\partial\Lambda_{L+1}| \le C_d L^{d-1}$, together with Proposition 4.1, to obtain the existence of a constant $\tilde{C}_1 = \tilde{C}_1(s,\rho,c,d) > 0$ such that

$$\begin{aligned} &\mathbb{E}(|G^{\alpha\beta}_{\omega}(0,k;z)|^s) \\ &\quad \le \tilde{C}_1 L^{d-1} \sum_{\beta_u\in\mathcal{A}} \sum_{u\in\partial^{\mathrm{i}}\Lambda_L} \mathbb{E}(|G^{\alpha\beta_u}_{\omega,\Lambda_L}(0,u;z)|^s) \sum_{u'\in\partial^{\mathrm{o}}\Lambda_{L+1}} \sum_{\gamma\in\mathcal{A}} \mathbb{E}(|G^{\gamma\beta}_{\omega}(u',k;z)|^s) \end{aligned} \tag{61}$$

for all $z \in \mathbb{C}\setminus\mathbb{R}$, $\alpha,\beta \in \mathcal{A}$, $L \in \mathbb{N}$ and $k \in \mathbb{Z}^d$ with $|k| \ge L+2$.

11 Proof of Band Edge Localization

This section is dedicated to the proof of Theorem 3.3. For this, we will use the decoupling estimate (61) for the Green functions $G^{\mathcal{H}_\pm}_{\omega}$ and $G^{\mathcal{H}_\pm}_{\omega,\Lambda_L}$ restricted to subspaces $\mathcal{H}_\pm$, together with Proposition 8.1.

Proof of Theorem 3.3 (i) Let $z = E + i\epsilon$ with $\epsilon > 0$ and $E \in I^{\pm}_{\delta_L}(m,c)$ for $\delta_L = \frac{1}{2}L^{-2/(d+2)}$. Considering the estimate (61) for the Green functions $G^{\mathcal{H}_\pm}_{\omega}$, $G^{\mathcal{H}_\pm}_{\omega,\Lambda_L}$ and inserting it into the bound obtained in Proposition 8.1 for $\mathbb{E}(|G^{\alpha\beta_u,\mathcal{H}_\pm}_{\omega,\Lambda_L}(0,u;z)|^s)$, we obtain for L sufficiently large,

$$\begin{aligned}
&\mathbb{E}(|G^{\alpha\beta,\mathcal{H}_\pm}_{\omega}(0,k;z)|^s)\\
&\leq \tilde{C}_1 L^{d-1}\,\tilde{C} L^d e^{-\tilde{\eta} L^{d/(d+2)}} |\mathcal{A}|\,|\partial\Lambda_L| \sum_{u'\in\partial^o\Lambda_{L+1}} \sum_{\gamma\in\mathcal{A}} \mathbb{E}(|G^{\gamma\beta,\mathcal{H}_\pm}_{\omega}(u',k;z)|^s)\\
&\leq C_2 L^{3d-2} e^{-\tilde{\eta} L^{d/(d+2)}} \sum_{u'\in\partial^o\Lambda_{L+1}} \sum_{\gamma\in\mathcal{A}} \mathbb{E}(|G^{\gamma\beta,\mathcal{H}_\pm}_{\omega}(u',k;z)|^s),
\end{aligned} \tag{62}$$

for $0 < C_2 = 4\tilde{C}_1\tilde{C}C_d < \infty$, for all $\alpha,\beta \in \mathcal{A}$ and $k \in \mathbb{Z}^d$ with $|k| \geq L+2$. With the constant C_2 from (62), we fix $L = L_0$ sufficiently large such that

$$b := 4C_2 L_0^{3d-2} e^{-\tilde{\eta} L_0^{d/(d+2)}} |\{u' \in \mathbb{Z}^d : \|u'\|_\infty \leq L_0+2\}| \;<\; 1$$

and, by (62),

$$\mathbb{E}(|G^{\alpha\beta,\mathcal{H}_\pm}_{\omega}(0,k;z)|^s) \;\leq\; b \sup_{u':\|u'\|_\infty\leq L_0+2}\; \sup_{\gamma\in\mathcal{A}} \mathbb{E}(|G^{\gamma\beta,\mathcal{H}_\pm}_{\omega}(u',k;z)|^s)\,. \tag{63}$$

Note that $\mathbb{E}(|G^{\gamma\beta,\mathcal{H}_\pm}_{\omega}(u',k;z)|^s) = \mathbb{E}(|G^{\gamma\beta,\mathcal{H}_\pm}_{\omega}(0,k-u';z)|^s)$, which allows us to iterate (63). We choose $\delta := \delta_{L_0} = \frac{1}{2}L_0^{-2/(d+2)}$ and let $E \in I^\pm_\delta(m,c)$. The relation (63) is the first step of the iteration; the second step is

$$\mathbb{E}(|G^{\gamma\beta,\mathcal{H}_\pm}_{\omega}(u',k;z)|^s) \;\leq\; b \sup_{u_2:\|u_2\|_\infty\leq 2(L_0+2)}\; \sup_{\gamma_2\in\mathcal{A}} \mathbb{E}(|G^{\gamma_2\beta,\mathcal{H}_\pm}_{\omega}(u_2,k;z)|^s)\,,$$

and thus successively. If $u', u_2, u_3, \ldots$ is one chain of sites obtained in this way, with a chain of associated components $\gamma, \gamma_2, \gamma_3, \ldots$, then the iteration may be continued as long as $|u_j - k| \geq L_0+2$, i.e., at least $\left\lfloor \frac{|k|}{L_0+2} \right\rfloor - 1$ times. After this number of steps we use Proposition 4.1 to bound the last fractional moment $\mathbb{E}(|G^{\gamma_j\beta,\mathcal{H}_\pm}_{\omega}(u_j,k;z)|^s)$ in the chain by the constant $C(s,\rho) > 0$. In (63) this leads to the bound

$$\mathbb{E}(|G^{\alpha\beta,\mathcal{H}_\pm}_{\omega}(0,k;z)|^s) \;\leq\; C(s,\rho)\, b^{\frac{|k|}{L_0+2}-1} \;=\; \frac{C(s,\rho)}{b}\, e^{\frac{\log b}{L_0+2}|k|}.$$

Thus we have proven Theorem 3.3**(i)** with $C = \frac{C(s,\rho)}{b}$ and $\eta = -\frac{\log b}{L_0+2} > 0$.

(ii) It is direct consequence of item **(i)** of this theorem and Theorem 3.1 applied to the operators $H_\omega(m,c)|_{\mathcal{H}_\pm}$ (see (10)). This completes the proof of the theorem.

□

12 Properties of the 1D Dirac Model

In this section we present properties of the one-dimensional DAD model that we need to prove Theorem 3.4.

The action of $H_\omega(m,c)$ on $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \in l^2(\mathbb{Z},\mathbb{C}^2)$ is given by

$$[H_\omega(m,c)\psi](n) = \begin{pmatrix} (mc^2+\omega_{n1})\psi_1(n) + c\,(\psi_2(n-1)-\psi_2(n)) \\ c(\psi_1(n+1)-\psi_1(n)) + (-mc^2+\omega_{n2})\psi_2(n) \end{pmatrix}. \tag{64}$$

If ψ is a solution of the equation $H_\omega(m,c)\psi = z\psi$ ($z \in \mathbb{C}$), then

$$\begin{pmatrix} \psi_1(n+1) \\ \psi_2(n) \end{pmatrix} = T_m^\omega(n;z) \begin{pmatrix} \psi_1(n) \\ \psi_2(n-1) \end{pmatrix},$$

with

$$T_m^\omega(n;z) = \begin{pmatrix} 1 + \dfrac{\left(mc^2 - z + \omega_{n1}\right)\left(mc^2 + z - \omega_{n2}\right)}{c^2} & \dfrac{mc^2 + z - \omega_{n2}}{c} \\ \dfrac{mc^2 - z + \omega_{n1}}{c} & 1 \end{pmatrix}.$$

The transfer matrix from site k to site n is

$$\Phi_m^\omega(n,k;z) = \begin{cases} T_m^\omega(n;z)\, T_m^\omega(n-1;z) \cdots T_m^\omega(k+1;z) & \text{if} \quad n > k\,, \\ Id_2 & \text{if} \quad n = k\,, \\ \left(T_m^\omega(n+1;z)\right)^{-1} \left(T_m^\omega(n;z)\right)^{-1} \cdots \left(T_m^\omega(k;z)\right)^{-1} & \text{if} \quad n < k\,. \end{cases}$$

Due to the hypothesis that the measure μ is absolutely continuous with bounded density and of compact support, we have the following result, which is a version of Lemma 5.1 in [6] for the one-dimensional DAD (64).

Lemma 12.1 *For each compact subset* $\Lambda \subset \mathbb{C}$*, there exist finite* $\gamma_1 = \gamma_1(\Lambda, m) > 0$*,* $\delta = \delta(\Lambda, m) > 0$*,* $C = C(\Lambda, m) > 0$ *such that*

$$\mathbb{E}\left(\left\|\Phi_m^\omega(n,0;z)u\right\|^{-\delta}\right) \le Ce^{-\gamma_1|n|}$$

for all $z \in \Lambda$*,* $n \in \mathbb{Z}$ *and unit vector* $u \in \mathbb{C}^2$*.*

We omit the proof of this lemma which is similar to the one given for the 1D Anderson Schrödinger operator in [6].

For $j \in \mathbb{Z}$ we write, for notation convenience, $[j,\infty) = \{j, j+1, \ldots\}$. Let $H_\omega^{[j,\infty)}(m,c)$ be the restriction of $H_\omega(m,c)$ to $l^2([j,\infty),\mathbb{C}^2)$ with Dirichlet boundary condition at the endpoint j, and $\mathcal{G}_{\omega,[j,\infty)}(m,c,z)$ the corresponding Green function with matrix elements

$$G_{\omega,[j,\infty)}(n,k;z) = \begin{pmatrix} G^{11}_{\omega,[j,\infty)}(n,k;z) & G^{12}_{\omega,[j,\infty)}(n,k;z) \\ G^{21}_{\omega,[j,\infty)}(n,k;z) & G^{22}_{\omega,[j,\infty)}(n,k;z) \end{pmatrix}$$

where $G^{\alpha\beta}_{\omega,[j,\infty)}(n,k;z) := \left\langle e_{n\alpha}, (H_\omega^{[j,\infty)}(m,c) - zI)^{-1} e_{k\beta} \right\rangle$ for $n,k \in [j,\infty)$, $\alpha, \beta \in \{1,2\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. A important property of the one-dimensional model is that $G_{\omega,[j,\infty)}(n,k;z)$ can be expressed in terms of two generalized eigenfunctions $u_j = u_j(\cdot,z) = \begin{pmatrix} u_{1,j} \\ u_{2,j} \end{pmatrix}$ and $u_\infty = u_\infty(\cdot,z) = \begin{pmatrix} u_{1,\infty} \\ u_{2,\infty} \end{pmatrix}$, where u_j satisfy the boundary condition at the endpoint j and u_∞ is square-summable at $+\infty$. More precisely, we have (see [7])

$$G_{\omega,[j,\infty)}(n,k;z) = \frac{1}{W(u_j,u_\infty)} \begin{cases} \begin{pmatrix} u_{1,j}(n)u_{1,\infty}(k) & u_{1,j}(n)u_{2,\infty}(k) \\ u_{2,j}(n)u_{1,\infty}(k) & u_{2,j}(n)u_{2,\infty}(k) \end{pmatrix} & \text{if} \quad n \le k\,, \\ \begin{pmatrix} u_{1,j}(k)u_{1,\infty}(n) & u_{2,j}(k)u_{1,\infty}(n) \\ u_{1,j}(k)u_{2,\infty}(n) & u_{2,j}(k)u_{2,\infty}(n) \end{pmatrix} & \text{if} \quad n > k\,, \end{cases} \tag{65}$$

where u_j and u_∞ are the solutions of the equation $\left(H_\omega^{[j,\infty)}(m,c)u\right)(n) = zu(n)$ with $u_{1,j}(j) = 1$, $u_{2,j}(j-1) = 0$, and u_∞ is the unique solution (up to a scalar) which is square-summable at $+\infty$ (in particular, there exists a constant $0 < M < \infty$ such that $\left|u_{i,\infty}(k)\right| \leq M$ for $i \in \{1, 2\}$ and for all $k \geq j$). The (constant) Wronskian of u_j and u_∞ is given by

$$W(u_j, u_\infty)(n) = u_{1,j}(n+1)u_{2,\infty}(n) - u_{2,j}(n)u_{1,\infty}(n+1).$$

13 Proof of 1D Localization

This section is dedicated to the proof of Theorem 3.4. The proof of item **(i)** follows general ideas of [18] used in the context of the one-dimensional Anderson Schrödinger model; here we work in infinite volume and with complex energies.

Proof of Theorem 3.4. (i) If $j = k$, then relation (14) holds by Proposition 4.1. We assume that $j < k$; if $j > k$ use that $|G^{\alpha\beta}_\omega(j, k; E + i\epsilon)| = |G^{\alpha\beta}_\omega(k, j; E - i\epsilon)|$. Let $z = E + i\epsilon$ with $E \in \sigma(H_\omega(m, c))$ and $\epsilon > 0$. By the second resolvent identity we have

$$\begin{aligned}(H_\omega(m, c) - zI)^{-1} = {} & (H^{[j,\infty)}_\omega(m, c) - zI)^{-1} + \\ & + (H_\omega(m, c) - zI)^{-1} \left(H^{[j,\infty)}_\omega(m, c) - H_\omega(m, c)\right) \\ & (H^{[j,\infty)}_\omega(m, c) - zI)^{-1}.\end{aligned}$$

Taking matrix-elements in the relation above, we obtain that

$$G^{\alpha\beta}_\omega(j, k; z) = G^{\alpha\beta}_{\omega,[j,\infty)}(j, k; z) + c\, G^{\alpha 2}_\omega(j, j-1; z)\, G^{1\beta}_{\omega,[j,\infty)}(j, k; z). \tag{66}$$

It suffices to prove the exponential decay

$$\mathbb{E}\left(|G^{\alpha\beta}_{\omega,[j,\infty)}(j, k; z)|^s\right) \leq C_0\, e^{-\eta_0|j-k|} \tag{67}$$

for $s \leq s_1 \in (0, 1)$, $0 < C_0 < \infty$, $\eta_0 > 0$ and for all $j, k \in \mathbb{Z}$, $\alpha, \beta \in \{1, 2\}$, $E \in \sigma(H^{[j,\infty)}_\omega(m, c))$ and $\epsilon > 0$. To see this, we take fractional powers of (66) and use Hölder's inequality to obtain

$$\begin{aligned}\mathbb{E}\left(|G^{\alpha\beta}_\omega(j, k; z)|^s\right) \leq {} & \mathbb{E}\left(|G^{\alpha\beta}_{\omega,[j,\infty)}(j, k; z)|^s\right) + c^s \\ & \left[\mathbb{E}\left(|G^{\alpha 2}_\omega(j, j-1; z)|^{2s}\right)\right]^{1/2} \left[\mathbb{E}\left(|G^{1\beta}_{\omega,[j,\infty)}(j, k; z)|^{2s}\right)\right]^{1/2}.\end{aligned}$$

It follows by (67) and Proposition 4.1 that

$$\mathbb{E}\left(|G^{\alpha\beta}_\omega(j, k; z)|^s\right) \;\leq\; C_0\, e^{-\eta_0|j-k|} + c^s C_1^{1/2} C_0^{1/2} e^{-\frac{\eta_0}{2}|j-k|} \;\leq\; C e^{-\eta|j-k|}$$

for $s \leq s_1/2 = s_0 \in (0, 1)$, $C = 2\max\left\{C_0, c^s C_1^{1/2} C_0^{1/2}\right\} > 0$ and $\eta = \eta_0/2 > 0$, which proves the exponential decay in (14). We will then prove (67).
By relation (65) with $n = j < k$, we have

$$G_{\omega,[j,\infty)}(j, k; z) = \frac{1}{W(u_j, u_\infty)(j)} \begin{pmatrix} u_{1,\infty}(k) & u_{2,\infty}(k) \\ u_{2,j}(j)u_{1,\infty}(k) & u_{2,j}(j)u_{2,\infty}(k) \end{pmatrix}$$

where

$$W(u_j, u_\infty)(j) = u_{1,j}(j+1)u_{2,\infty}(j) - u_{2,j}(j)u_{1,\infty}(j+1).$$

Using that u_j is a solution of $\big(H_\omega^{[j,\infty)}(m,c)u\big)(n) = zu(n)$ with $u_{1,j}(j) = 1$, $u_{2,j}(j-1) = 0$, we obtain that

$$u_{2,j}(j) = \frac{-(z - mc^2 - \omega_{j1})}{c} \tag{68}$$

and

$$u_{1,j}(j+1) = \frac{-(z + mc^2 - \omega_{j2})(z - mc^2 - \omega_{j1})}{c^2} + 1. \tag{69}$$

Observe that $u_{2,j}(j) \neq 0$, $u_{1,j}(j+1) \neq 0$ and by hypothesis ω_{j1} and ω_{j2} are independent.

Let $\mathbb{E}_j^2(\cdots) := \int_{\mathrm{supp}(\rho)} \cdots\ \rho(\omega_{j2})d\omega_{j2}$ be the expectation with respect to the random variables $(\omega_{l\gamma})_{l\in\mathbb{Z}^d\setminus\{j\} \text{ or } \gamma\neq 2}$, where $\|\rho\|_\infty < \infty$ and $\mathrm{supp}(\rho) \subset [\omega_{\min}, \omega_{\max}]$. For $s \in (0,1)$ the expectation of the s-moment of $G^{11}_{\omega,[j,\infty)}(j,k;z)$ is given by

$$\begin{aligned}\mathbb{E}\left(|G^{11}_{\omega,[j,\infty)}(j,k;z)|^s\right) &= \mathbb{E}\left[\mathbb{E}_j^2\left(\frac{|u_{1,\infty}(k)|^s}{|u_{1,j}(j+1)u_{2,\infty}(j) - u_{2,j}(j)u_{1,\infty}(j+1)|^s}\right)\right]\\ &= \mathbb{E}\left[\int_{\omega_{\min}}^{\omega_{\max}} \frac{|u_{1,\infty}(k)|^s}{|u_{1,j}(j+1)u_{2,\infty}(j) - u_{2,j}(j)u_{1,\infty}(j+1)|^s}\ \rho(\omega_{j2})d\omega_{j2}\right]\end{aligned}$$

where $u_{2,j}(j)$ and $u_{1,j}(j+1)$ are given by (68) and (69).

For the number $\delta > 0$ obtained in Lemma 12.1, choose $s_1 \in (0,1)$ such that $\delta/s_1 > 1$. Using Lemma 13.1 below, the boundedness $\big|u_{1,\infty}(k)\big| \leq M, \forall k \geq j$, Hölder's inequality and Lemma 12.1, we obtain that for all $s \leq s_1$,

$$\begin{aligned}\mathbb{E}\left(|G^{11}_{\omega,[j,\infty)}(j,k;z)|^s\right) &\leq M^{s_1} C\ \mathbb{E}\left(\left\|\begin{pmatrix} u_{1,\infty}(j+1)\\ u_{2,\infty}(j)\end{pmatrix}\right\|^{-s_1}\right)\\ &\leq M^{s_1} C\ \left[\mathbb{E}\left(\left\|\Phi_m^\omega(j,k;z)\begin{pmatrix} u_{1,\infty}(k+1)\\ u_{2,\infty}(k)\end{pmatrix}\right\|^{-\delta}\right)\right]^{s_1/\delta}\\ &\leq C_0\, e^{-\eta_0|j-k|}\end{aligned}$$

for a constant $C_0 = C_0(m,c,s_1,\rho) > 0$, $\eta_0 = \frac{\gamma_1 s_1}{\delta} > 0$, for all $j,k \in \mathbb{Z}$ and $z = E + i\epsilon$ with $E \in \sigma(H_\omega^{[j,\infty)}(m,c))$ and $\epsilon > 0$. Similarly, one proves the exponential decay of $\mathbb{E}\left(|G^{\alpha\beta}_{\omega,[j,\infty)}(j,k;z)|^s\right)$ with $(\alpha,\beta) \in \{(1,2),(2,1),(2,2)\}$. We have thus established (67), which completes the proof.

(ii) It follows directly from item (i) of this theorem and Theorem 3.1. This completes the proof of the theorem.

□

In the above proof we have used the following result:

Lemma 13.1 *For $z = E + i\epsilon$ with $E \in \sigma(H_\omega^{[j,\infty)}(m,c))$ and $\epsilon > 0$, and for $s \in (0,1)$ there exists $0 < C = C(m,c,s,\rho) < \infty$ such that*

$$\begin{aligned}&\int_{\omega_{\min}}^{\omega_{\max}} \frac{1}{|u_{1,j}(j+1)u_{2,\infty}(j) - u_{2,j}(j)u_{1,\infty}(j+1)|^s}\ \rho(\omega_{j2})d\omega_{j2}\\ &\leq\ C\ \left\|\begin{pmatrix} u_{1,\infty}(j+1)\\ u_{2,\infty}(j)\end{pmatrix}\right\|^{-s}\end{aligned}$$

where $u_{2,j}(j)$ and $u_{1,j}(j+1)$ are given by (68) and (69), respectively.

Proof Consider the set

$$A=\left\{\left|z\pm mc^2-\omega\right| : \omega\in[\omega_{\min},\omega_{\max}],\, z=E+i\epsilon,\;\; E\in\sigma(H_\omega^{[j,\infty)}(m,c)) \text{ and } \epsilon>0 \text{ fixed}\right\}.$$

Let $K_1=\sup(A)<\infty$ and $K_2=\inf(A)>0$. Note that $u_{2,\infty}(j)$, $u_{1,\infty}(j+1)$ and $u_{2,j}(j)$ are independent of ω_{j2}. The proof is divided into two cases.

Case 1: $u_{2,\infty}(j)=0$. In this case, we have

$$\frac{1}{|u_{1,j}(j+1)u_{2,\infty}(j)-u_{2,j}(j)u_{1,\infty}(j+1)|^s} \;\le\; \left(\frac{c}{K_2}\right)^s \left\|\begin{pmatrix} u_{1,\infty}(j+1)\\ u_{2,\infty}(j)\end{pmatrix}\right\|^{-s},$$

which implies

$$\int_{\omega_{\min}}^{\omega_{\max}} \frac{1}{|u_{1,j}(j+1)u_{2,\infty}(j)-u_{2,j}(j)u_{1,\infty}(j+1)|^s}\,\rho(\omega_{j2})d\omega_{j2} \;\le\; C\left\|\begin{pmatrix} u_{1,\infty}(j+1)\\ u_{2,\infty}(j)\end{pmatrix}\right\|^{-s},$$

with $C=\left(\frac{c}{K_2}\right)^s\|\rho\|_\infty\,(\omega_{\max}-\omega_{\min})$.

Case 2: $u_{2,\infty}(j)\neq 0$. In this case, we write

$$\int_{\omega_{\min}}^{\omega_{\max}} \frac{1}{|u_{1,j}(j+1)u_{2,\infty}(j)-u_{2,j}(j)u_{1,\infty}(j+1)|^s}\,\rho(\omega_{j2})d\omega_{j2} = \frac{1}{\left|u_{2,\infty}(j)\right|^s\left|u_{2,j}(j)\right|^s}\int_{\omega_{\min}}^{\omega_{\max}} \frac{1}{\left|\frac{u_{1,\infty}(j+1)}{u_{2,\infty}(j)}-\frac{u_{1,j}(j+1)}{u_{2,j}(j)}\right|^s}\,\rho(\omega_{j2})d\omega_{j2}\,.$$

Note that

$$\left|\frac{u_{1,j}(j+1)}{u_{2,j}(j)}\right| = \left|\frac{z+mc^2-\omega_{j2}}{c}-\frac{c}{z-mc^2-\omega_{j1}}\right| \;\le\; K$$

with $K=\frac{K_1}{c}+\frac{c}{K_2}$. Now we separate into two subcases:

(i) if $\left|\frac{u_{1,\infty}(j+1)}{u_{2,\infty}(j)}\right|>2K$, it follows that

$$\left|\frac{u_{1,\infty}(j+1)}{u_{2,\infty}(j)}-\frac{u_{1,j}(j+1)}{u_{2,j}(j)}\right| \;\ge\; \frac{1}{2}\left|\frac{u_{1,\infty}(j+1)}{u_{2,\infty}(j)}\right|.$$

On the other hand

$$\frac{1}{|u_{1,\infty}(j+1)|^s} \;\le\; \left(1+\frac{1}{4K^2}\right)^{s/2}\left\|\begin{pmatrix} u_{1,\infty}(j+1)\\ u_{2,\infty}(j)\end{pmatrix}\right\|^{-s}.$$

Using these estimates we conclude that

$$\begin{aligned}&\int_{\omega_{\min}}^{\omega_{\max}} \frac{1}{|u_{1,j}(j+1)u_{2,\infty}(j)-u_{2,j}(j)u_{1,\infty}(j+1)|^s}\,\rho(\omega_{j2})d\omega_{j2} \;\le\\ &\le \left(\frac{c}{K_2}\right)^s \frac{1}{\left|u_{2,\infty}(j)\right|^s}\int_{\omega_{\min}}^{\omega_{\max}} \frac{1}{\left(\frac{1}{2}\right)^s\left|\frac{u_{1,\infty}(j+1)}{u_{2,\infty}(j)}\right|^s}\,\rho(\omega_{j2})d\omega_{j2}\\ &\le C\left\|\begin{pmatrix} u_{1,\infty}(j+1)\\ u_{2,\infty}(j)\end{pmatrix}\right\|^{-s},\end{aligned}$$

with $C = \left(\frac{c}{K_2}\right)^s 2^s \, \|\rho\|_\infty \, (\omega_{\max} - \omega_{\min}) \left(1 + \frac{1}{4K^2}\right)^{s/2}$.

(ii) if $\left|\frac{u_{1,\infty}(j+1)}{u_{2,\infty}(j)}\right| \le 2K$, then

$$\frac{1}{|u_{2,\infty}(j)|^s} \le \left(1 + 4K^2\right)^{s/2} \left\| \begin{pmatrix} u_{1,\infty}(j+1) \\ u_{2,\infty}(j) \end{pmatrix} \right\|^{-s}.$$

Using these estimate together with Lemma 4.1, we get

$$\begin{aligned} &\int_{\omega_{\min}}^{\omega_{\max}} \frac{1}{|u_{1,j}(j+1)u_{2,\infty}(j) - u_{2,j}(j)u_{1,\infty}(j+1)|^s} \, \rho(\omega_{j2}) d\omega_{j2} \\ &\le \left(\frac{c}{K_2}\right)^s \frac{1}{\left|u_{2,\infty}(j)\right|^s} \int_{\omega_{\min}}^{\omega_{\max}} \frac{1}{\left|\frac{u_{1,\infty}(j+1)}{u_{2,\infty}(j)} + \frac{z+mc^2-\omega_{j2}}{c} - \frac{c}{z-mc^2-\omega_{j1}}\right|^s} \, \rho(\omega_{j2}) d\omega_{j2} \\ &\le C \left\| \begin{pmatrix} u_{1,\infty}(j+1) \\ u_{2,\infty}(j) \end{pmatrix} \right\|^{-s}, \end{aligned}$$

with $C = \left(\frac{c^2}{K_2}\right)^s C_1(s,\rho) \left(1 + 4K^2\right)^{s/2}$, where in the last step we have used Lemma 4.1 with $g = \rho$ and $\theta = -c\left(\frac{u_{1,\infty}(j+1)}{u_{2,\infty}(j)} - \frac{c}{z-mc^2-\omega_{j1}}\right) - (z + mc^2)$, since $u_{1,\infty}(j+1)$, $u_{2,\infty}(j)$ and ω_{j1} are independent of ω_{j2}.

□

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