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Bound States in Minkowski Space in 2 + 1 Dimensions

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Abstract The Nakanishi perturbative integral representation of the Bethe–Salpeter amplitude in three-dimensions $(2+1)$ is used to solve the corresponding homogeneous Bethe–Salpeter equation in Minkowski space. The projection of this equation onto the null-plane, as reported here, leads to a bound-state equation for the Nakanishi weight function. The explicit forms of the integral equation for the Nakanishi weight function are shown in the ladder approximation. In addition, the valence light-front wave function is presented. The formal steps of the formalism are illustrated to some extend, with the resulting equation being applied to a bound state system composed by two identical scalar particles of mass m, interacting through the exchange of another massive scalar particle of mass μ. The results reported in this contribution show quite good agreement between our calculations obtained from the Bethe–Salpeter amplitude with the Nakanishi weight function with direct solutions obtained in the Euclidean space.

1 Introduction

The manifestly covariant homogeneous Bethe–Salpeter (BS) formalism in Minkowski space is derived for two space dimensions and time $(2+1)$, and applied to obtain the bound-state solution of two-bosons in a two-dimensional (2D) surface. By considering such reduced 2D space of the Bethe–Salpeter equation (BSE), we are motivated by well-known studies of electronic excitations of graphene [1,2] within a relativistic framework. It is well know that in deformed or dopped graphene it is possible to open an energy band gap [3,4]; and, in this case, an exciton (bound-state of an electron and an electron-hole) may be created, with an energy smaller than the gap. The work we are reporting, applied to two-boson system, is a first step to develop some basic tools for solving the bound-state problem of two relativistic fermions in $2+1$ dimensions. It will be sufficient to access the main difficulties in treating the Minkowski space BSE within a 2D surface.

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The non-perturbative BS formalism in Minkowski space, starting from the corresponding BS amplitude written in terms of the Nakanishi Perturbative Integral Representation (PTIR) [5], was first developed in Refs. [6,7] for two-boson bound states in four dimensions (3+1). The singularities of the kernel of BSE appearing in Minkowski space are absent in the integral equation for the Nakanishi weight function. This is appealing for numerical treatments, as one can dispense with the Wick rotation.

The Karmanov and Carbonell method, introduced in [8] for bound-state problems in 3+1 dimensions, also makes use of the Nakanishi PTIR for the BS amplitude as in [6]. However, in addition they consider the explicitly covariant light-front (LF) framework [10]. Considerable simplifications in the algebraic manipulations for bound and scattering states are found, as detailed by in Ref. [9]. The essential point about the method is to recognise that the effective dynamics of the LF valence wave function is able to describe the complexity of the BSE projected onto the light-front [11–13].

The present work generalises the Nakanishi PTIR to solve the bound-state problem of relativistic two-particles in 2+1 Minkowski space. In our approach, the solutions of the homogeneous BSE in 2+1 dimensions are found by using the PTIR to represent the BS amplitude, with the corresponding projection onto the light-front. We adopt the ladder approximation for the kernel of this equation, considering two identical scalar bosons with masses m, interacting by the exchange of another massive boson with mass μ. For the derivation of the integral equation for the Nakanishi weight function, we used the method proposed in Ref. [8]. In addition, one could use the uniqueness conjecture of the weight function in the non-perturbative domain, following Ref. [9], by deriving an integral equation suitable for numerical bound-state solutions; a task which is left for a future investigation.

Next sections of the present report are organised as follows. In Sect. 2, the notation is defined by presenting the homogeneous BSE in 2+1 dimensions, as well as the Nakanishi integral representation of the BS amplitude. In Sect. 3, we show the main steps in deriving the formalism leading to the equation for the Nakanishi weight function, by relying on the LF projection. The numerical treatment of the integral equation for the Nakanishi weight function, with the corresponding results, are provided in Sect. 4. Our findings and perspectives are summarised in Sect. 5.

2 The Bethe–Salpeter Equation in 2+1 Dimensions and the Nakanishi PTIR

The homogeneous BSE in 2+1 dimensions for the bound state of two scalar bosons of mass m, with interaction mediated by the exchange of a scalar boson with mass μ, in the ladder approximation, with units such that $\hbar = c = 1$, is given by

$$\Phi(k,p) = \frac{\mathrm{i}}{\left(k+\frac{p}{2}\right)^2 - m^2 + \mathrm{i}\epsilon}\,\frac{\mathrm{i}}{\left(k-\frac{p}{2}\right)^2 - m^2 + \mathrm{i}\epsilon}(\mathrm{i}g)^2 \int \frac{d^3k'}{(2\pi)^3}\frac{\mathrm{i}}{(k'-k)^2 - \mu^2 + \mathrm{i}\epsilon}\Phi(k',p), \quad (1)$$

where p and k are the total and relative momenta. Within the LF formalism in 2+1 dimensions, we define $p^{\pm} \equiv p^0 \pm p^2$ and $p_{\perp} \equiv p^1$, such that the spatial momentum variables are p^1 and p^2. Within the Nakanishi PTIR, the $s-$wave BS amplitude is given by

$$\Phi(k,p) = -\mathrm{i}\int_{-1}^{1} dz \int_0^{\infty} d\gamma \, \frac{g_B^{(n)}(\gamma, z; \kappa^2)}{\left(\gamma + m^2 - \frac{p^2}{4} - k^2 - p\cdot k\, z - \mathrm{i}\epsilon\right)^{n+2}}, \quad (2)$$

where $g_B^{(n)} \equiv g_B^{(n)}(\gamma, z; \kappa^2)$ is the Nakanishi weight function, for the variables γ and z $(0 < \gamma < \infty, |z| < 1)$, for a fixed positive integer n. In (2), κ^2 is defined by the bound-state mass $M = \sqrt{p^2}$ through

$$\kappa^2 = m^2 - p^2/4 \;\; (>0, \;\; \text{for a bound-state system}). \quad (3)$$

The Euclidean BS amplitude within the Nakanishi PTIR is obtained directly from the Eq. (2) by taking $k^0 \rightarrow \mathrm{i}k_E^0$. This is necessary if one aims to perform a detailed comparison with results obtained from the solution of the BSE in the Euclidean space using the standard method of the Wick rotation. It is important to remind that the Nakanishi weight function must be even in z, otherwise the BS amplitude would not have positive norm (see Refs. [5,6,14]).

In the next section, the final form of the integral equation for $g_B^{(n)}$ is derived by substituting Eq. (2) on both sides of Eq. (1), followed by the $k^-{-}$integration to perform the LF projection.

3 The PTIR and the LF Projection

The projection of the BS amplitude onto the light-front gives the valence wave function as

$$\phi_v(\gamma, z) = \mathrm{i} \int_{-1}^{1} dz' \int_0^{\infty} d\gamma' g_B^{(n)}(\gamma', z'; \kappa^2) \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \frac{1}{\left(\gamma' + \kappa^2 - k^2 - p \cdot k\, z' - \mathrm{i}\epsilon\right)^{n+2}}, \tag{4}$$

where we have introduced $\gamma \equiv k_\perp^2$ (here, $k_\perp \equiv k^1$) and $z = -2k^+/M$. We adopt the rest frame of the bound system. The valence LF wave function, obtained by using the same procedure as in [8], is

$$\phi_v(\gamma, z) = \frac{2}{M(n+1)} \int_0^{\infty} d\gamma' \frac{g_B^{(n)}\left(\gamma', z, \kappa^2\right)}{\left[\gamma' + \gamma + z^2 m^2 + \left(1 - z^2\right)\kappa^2\right]^{n+1}}. \tag{5}$$

The next step is to project onto the LF the right-hand side of the BSE, Eq. (1), according to

$$\begin{aligned}\phi_v(\gamma, z) = (-1)^n (\mathrm{i}g)^2 \int_{-1}^{1} dz' \int_0^{\infty} d\gamma' g_B^{(n)}(\gamma', z'; \kappa^2) \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \frac{1}{\left(k + \frac{p}{2}\right)^2 - m^2 + \mathrm{i}\epsilon} \frac{1}{\left(k - \frac{p}{2}\right)^2 - m^2 + \mathrm{i}\epsilon} \\ \times \int \frac{d^3k'}{(2\pi)^3} \frac{1}{(k - k')^2 - \mu^2 + \mathrm{i}\epsilon} \frac{1}{\left(k'^2 + p \cdot k'\, z' - \gamma' - \kappa^2 + \mathrm{i}\epsilon\right)^{n+2}},\end{aligned} \tag{6}$$

and then perform the k^- integration. Finally, the equation for the Nakanishi weight function for $n = 1$, with $g_B\left(\gamma, z, \kappa^2\right) \equiv g_B^{(1)}\left(\gamma, z, \kappa^2\right)$, can be written as

$$\int_0^{\infty} d\gamma' \frac{g_B\left(\gamma', z, \kappa^2\right)}{\left[\gamma' + \gamma + z^2 m^2 + \left(1 - z^2\right)\kappa^2\right]^2} = \int_{-1}^{1} dz' \int_0^{\infty} d\gamma' V(z, z', \gamma, \gamma') g_B(\gamma', z', \kappa^2), \tag{7}$$

where the interaction kernel is given by

$$V(z, z', \gamma, \gamma') = g^2 \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \frac{P(p, z', k, \gamma', \kappa^2)}{\left[\left(\frac{p}{2} + k\right)^2 - m^2 + \mathrm{i}\epsilon\right]\left[\left(\frac{p}{2} - k\right)^2 - m^2 + \mathrm{i}\epsilon\right]}, \tag{8}$$

with

$$P(p, z, k, \gamma, \kappa^2) \equiv \int \frac{d^3k'}{(2\pi)^3} \frac{1}{(k - k')^2 - \mu^2 + \mathrm{i}\epsilon} \frac{1}{\left(k'^2 + p \cdot k'\, z - \gamma - \kappa^2 + \mathrm{i}\epsilon\right)^3}. \tag{9}$$

We observe that the manipulations of the integrand, as well as the integrations involved in deriving the interaction term (8), given in the following expression, follow the same steps as in [8]. The function $P(p, z, k, \gamma, \kappa^2)$ is obtained as an integral over the Feynman parameters. After that, the integration over k^- is performed analytically, leading to

$$\begin{aligned}V(z, z', \gamma, \gamma') \;=\; \frac{3\, g^2}{64\,\pi} \frac{1}{[\gamma + (1 - z^2)\kappa^2 + z^2 m^2]} \Bigg[\left(\frac{1+z}{1+z'}\right)^{\frac{5}{2}} \theta(z' - z) F(z, z', \gamma, \gamma') + \\ + \left(\frac{1-z}{1-z'}\right)^{\frac{5}{2}} \theta(z - z') F(-z, -z', \gamma, \gamma')\Bigg],\end{aligned} \tag{10}$$

where

$$F(z, z', \gamma, \gamma') \equiv \frac{2\left[8b_1^2 b_2 + 4b_1\left(3b_2^2 + 3b_2 b_3 + 2b_3^2\right) + b_2^2(3b_2 + 2b_3)\right]}{3(b_1 + b_2 + b_3)^{3/2}\left(b_2^2 - 4b_1 b_3\right)^2} - \frac{16\sqrt{b_1}\, b_2}{3\left(b_2^2 - 4b_1 b_3\right)^2}, \tag{11}$$

with b_1, b_2, b_3 defined as

$$b_1 \equiv \frac{1+z}{1+z'}\mu^2; \qquad b_2 \equiv \gamma + m^2 z^2 + \kappa^2(1-z^2) + \frac{1+z}{1+z'}[\gamma' - \mu^2];$$
$$b_3 \equiv \left(z'^2 \frac{M^2}{4} + \kappa^2\right) - \left[\gamma + m^2 z^2 + \kappa^2(1-z^2)\right]. \tag{12}$$

Before closing this section, we would like to add that, in Ref. [6], after introducing the Nakanishi PTIR in the formalism for the BS amplitude (1), the resulting kernel was manipulated in order to factorize a denominator identical to the one appearing in Eq. (2). Then, from the conjecture of uniqueness, an equation for the weight function was derived. Differently, the approach developed in Ref. [9] based on Ref. [8], used the projection onto the LF of the BSE, as already discussed above. Furthermore, in [9], the bound-state equation for the 3+1 problem, analogous to (7), was manipulated in order to apply the uniqueness conjecture, which allowed to simplify the form of the integral equation for the weight function.

4 Numerical Treatment and Results

By following the approach performed in Ref. [14] for the 3+1 bound-state problem, in the present 2+1 model we write Eq. (7) in a basis set expansion, and solve a generalised eigenvalue equation for the function $g_B\left(\gamma, z, \kappa^2\right)$. Therefore, we have expanded the Nakanishi weight function as follows:

$$g_B(\gamma, z; \kappa^2) = \sum_{\ell=0}^{N_z} \sum_{j=0}^{N_g} A_{\ell j}(\kappa^2)\, G_\ell(z)\, \mathcal{L}_j(\gamma), \tag{13}$$

where the functions $G_\ell(z)$ are given in terms of even Gegenbauer polynomials, $C_{2\ell}^{(5/2)}(z)$, as

$$G_\ell(z) = 4(1-z^2)\Gamma(5/2)\sqrt{\frac{(2\ell+5/2)(2\ell)!}{\pi\,\Gamma(2\ell+5)}}C_{2\ell}^{(5/2)}(z), \tag{14}$$

and the functions $\mathcal{L}_j(\gamma)$ are expressed in terms of Laguerre polynomials, $L_j(a\gamma)$,

$$\mathcal{L}_j(\gamma) = \sqrt{a}\; L_j(a\gamma)\; e^{-a\gamma/2}. \tag{15}$$

The above, Gegenbauer and Laguerre polynomials fulfils the following orthonormality conditions:

$$\int_{-1}^{1} dz\; G_\ell(z)\; G_n(z) = \delta_{\ell n}, \quad \int_0^\infty d\gamma \mathcal{L}_j(\gamma)\mathcal{L}_\ell(\gamma) = a\int_0^\infty d\gamma\; e^{-a\gamma} L_j(a\gamma) L_\ell(a\gamma) = \delta_{j\ell}. \tag{16}$$

For the cases with $\mu = 0.1$ and $\mu = 0.5$, we have considered $a = 24$ and $a = 2$, respectively, for the parameter a. The variable γ has been rescaled according to $\gamma \to 2\gamma/a_0$ with $a_0 = 12$, for numerical convenience. The parameters a and a_0 have been used to control the numerical convergence. After introducing the basis set expansion of the Nakanishi weight function, the schematic form for the generalised eigenvalue equation is given by

$$g^{-2} \sum_j \left(D_{ij}(M) + \epsilon\, \delta_{ij}\right)\; g_j(\kappa^2) = \sum_j A_{ij}(M)\; g_j(\kappa^2), \tag{17}$$

where g_j are associated with $A_{\ell j}$, written in terms of only one subindex. We introduce a small parameter ϵ to provide stability to the matrix inversion in the left-hand side of Eq. (17), as discussed in [8].

For both, z and γ integrals, we have used Gaussian–Legendre or Gaussian–Laguerre quadratures with 36 points and extended the basis up to $N_z = 18$ and $N_g = 32$ for all values of μ/m and B/m. The numerical stability pattern found in the solution of the corresponding 3+1 problem [14] is also repeated for the 2+1 case; namely, the eigenvalues have faster convergence than the corresponding eigenvectors. The Euclidean BS

Table 1 Results for g^2/m^3 for different binding energies, B, and masses of the exchanged boson, μ, from the numerical solution of Eq. (7)

B/m	$\mu = 0.1$		$\mu = 0.5$	
	Eq. (7)	Eucl.	Eq. (7)	Eucl.
0.01	0.82	0.79	5.33	5.31
0.1	4.26	4.26 | 4.268[a]	14.88	14.88
0.2	8.07	8.06	22.67	22.67
0.5	19.50	19.51	42.33	42.33
1	36.05	36.03 | 36.052[a]	67.38	67.39

[a] Results from Euclidean space calculations and the corresponding accurate results from Ref. [15] are given in the table

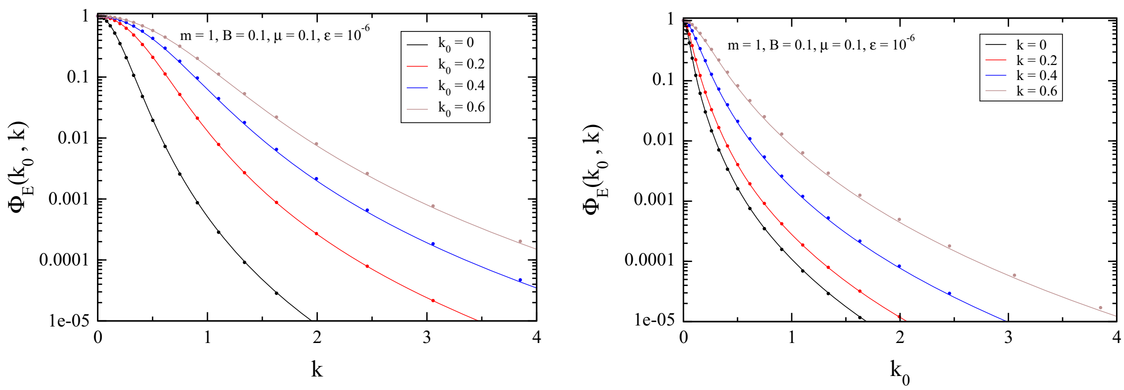

Fig. 1 (Color online) The Euclidean Bethe–Salpeter amplitude with arbitrary normalisation for $\mu/m = 0.1$ and $B/m = 0.1$ in 2+1 dimensions. The BS amplitude in the *left panel* is computed for $k_0 \equiv k_E$ values of 0, 0.2, 0.4 and 0.6 from bottom to top. In the *right panel* the results are obtained for k values of 0, 0.2, 0.4 and 0.6 from bottom to top. The *solid lines* are Euclidean BS amplitude calculated with the Nakanishi weight function and the *full circles* are the solutions of the standard Euclidean BSE. Units are such that $m = 1$

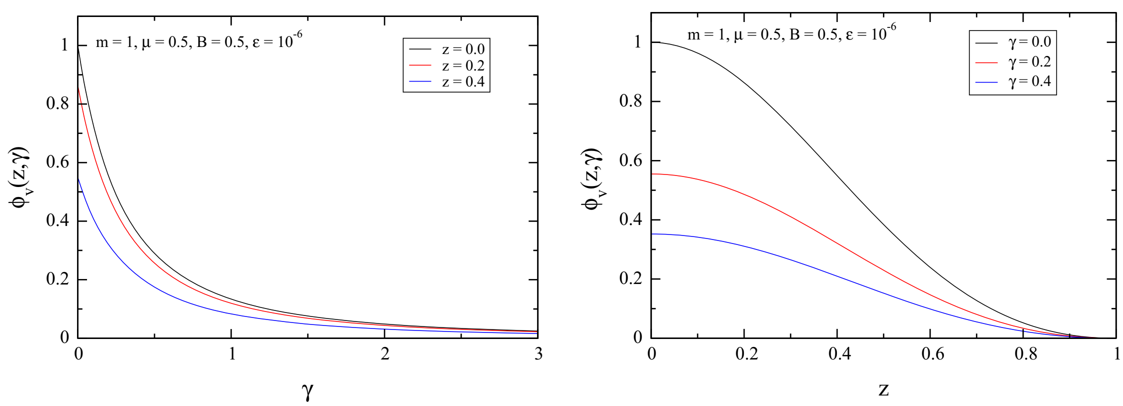

Fig. 2 (Color online) Light-front valence wave function $\phi_v(\gamma, z)$ for $\mu = 0.5$ and $B = 0.5$ with the arbitrary normalisation $\phi_v(0, 0) = 1$. *Left panel*: $\phi_v(\gamma, z)$ as a function of γ for fixed z values of 0, 0.2 and 0.4 from top to bottom. *Right panel*: $\phi_v(\gamma, z)$ as a function of z for fixed γ values of 0, 0.2 and 0.4 from top to bottom. Units are such that $m = 1$

amplitude and the LF wave function, obtained from $g_B(\gamma, z; \kappa)$ by the integrals (2) with $k^0 \to \imath k^0_E$ and (5), respectively, are numerically stable as the eigenvalues.

Our results are shown in Table 1 and in Figs. 1 and 2. A detailed comparison between the eigenvalues obtained from Eq. (7) and the solution of the Euclidean BSE are presented in Table 1. For the sake of comparison with our results, we also present the accurate calculation of Nieuwenhuis and Tjon [15] using the O(4) expansion of the Euclidean ladder BSE.

The results for the BS amplitude obtained from the direct solution in Euclidean space and the ones computed with the analytical extension of Eq. (2) for $k^0 \to \imath k^0_E$ are compared in Fig. 1. It is remarkable, that both solutions, as a function of k and k^0_E agree in ranges where the BS amplitude varies in several order of magnitude. In addition the value of $\epsilon = 10^{-6}$ does not influence the amplitude at large momentum. The LF wave function is presented in Fig. 2 for one example with $\mu = m = 0.5$. The wave function has no zeros, as expected, since it corresponds to the valence component of the two-boson ground state.

5 Summary and Perspectives

The Nakanishi integral representation was successfully extended in the present work, in order to solve the bound-state problem of relativistic two-particles in 2D space in the Minkowski space. In our investigation, we have used the BS equation in 2+1 dimensions in the ladder approximation for two identical bosons interacting by the exchange of another massive boson (Yukawa model). In the present approach, it was reduced to 2+1 dimensions the 2+1 dimensional method proposed in Ref. [8], and extended to the scattering region in [9] relying on the LF projection of the BS equation. It is also important to note that the integral equation derived for the Nakanishi weight function are free from singularities.

Following the present work, we plan to derive the integral equation for the Nakanishi weight function in 2+1 dimensions based on the uniqueness conjecture in the non-perturbative regime and explore the numerical solutions of the integral equation, as well as by extending the approach to the continuum. Another challenge, worthwhile to be pursued, is the extension of the framework to calculate the bound-state properties of two-fermion in 2+1 dimensions. Results of such investigation may be quite relevant in the study of excitons in graphene, in the presence of impurities or defects that open the energy gap, considering as a dynamical framework the interaction Lagrangian with scalar and gauge-fields [16,17].

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