Radii of starlikeness of some special functions

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RADII OF STARLIKENESS OF SOME SPECIAL FUNCTIONS

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Abstract. Geometric properties of the classical Lommel and Struve functions, both of the first kind, are studied. For each of them, there different normalizations are applied in such a way that the resulting functions are analytic in the unit disc of the complex plane. For each of the six functions we determine the radius of starlikeness precisely.

1. Introduction and statement of the main results

Let $\mathbb{D}_r$ be the open disk $\{z \in \mathbb{C} : |z| < r\}$, where $r > 0$, and set $\mathbb{D} = \mathbb{D}_1$. By $\mathcal{A}$ we mean the class of analytic functions $f : \mathbb{D}_r \to \mathbb{C}$ which satisfy the usual normalization conditions $f(0) = f'(0) - 1 = 0$. Denote by $\mathcal{S}$ the class of functions belonging to $\mathcal{A}$ which are univalent in $\mathbb{D}_r$ and let $\mathcal{S}^*(\alpha)$ be the subclass of $\mathcal{S}$ consisting of functions which are starlike of order α in $\mathbb{D}_r$, where $0 \le \alpha < 1$. The analytic characterization of this class of functions is

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{S} \ : \ \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ \text{ for all } \ z \in \mathbb{D}_r \right\},$$

and we adopt the convention $\mathcal{S}^* = \mathcal{S}^*(0)$. The real number

$$r^*_\alpha(f) = \sup\left\{ r > 0 \ : \ \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ \text{ for all } \ z \in \mathbb{D}_r \right\},$$

is called the radius of starlikeness of order α of the function f. Note that $r^*(f) = r^*_0(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^*(f)})$ is a starlike domain with respect to the origin.

We consider two classical special functions, the Lommel function of the first kind $s_{\mu,\nu}$ and the Struve function of the first kind $\mathbf{H}_\nu$. They are explicitly defined in terms of the hypergeometric function ${}_1F_2$ by

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right), \quad \frac{1}{2}(-\mu \pm \nu - 3) \notin \mathbb{N}, \tag{1.1}$$

and

$$\mathbf{H}_\nu(z) = \frac{\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\frac{\pi}{4}}\,\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_1F_2\left(1; \frac{3}{2}, \nu+\frac{3}{2}; -\frac{z^2}{4}\right), \quad -\nu-\frac{3}{2} \notin \mathbb{N}. \tag{1.2}$$

A common feature of these functions is that they are solutions of inhomogeneous Bessel differential equations [**Wa**]. Indeed, the Lommel function of the first kind $s_{\mu,\nu}$ is a solution of

$$z^2w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = z^{\mu+1}$$

while the Struve function $\mathbf{H}_\nu$ obeys

$$z^2w''(z) + zw'(z) + (z^2 - \nu^2)w(z) = \frac{4\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}.$$

We refer to Watson's treatise [**Wa**] for comprehensive information about these functions and recall some more recent contributions. In 1972 Steinig [**St**] examined the sign of $s_{\mu,\nu}(z)$ for real μ, ν and positive z. He showed, among other things, that for $\mu < \frac{1}{2}$ the function $s_{\mu,\nu}$ has infinitely many changes of sign on $(0, \infty)$. In 2012 Koumandos and Lamprecht [**KL**] obtained sharp estimates for the location of the zeros of $s_{\mu-\frac{1}{2},\frac{1}{2}}$ when $\mu \in (0, 1)$. The Turán type inequalities for $s_{\mu-\frac{1}{2},\frac{1}{2}}$ were established in [**BK**] while those for the Struve function were proved in [**BPS**].

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Geometric properties of $s_{\mu-\frac{1}{2},\frac{1}{2}}$ and of the Struve function were obtained in [**BS**] and in [**OY**, **YO**], respectively. Motivated by those results we study the problem of starlikeness of certain analytic functions related to the classical special functions under discussion. Since neither $s_{\mu,\nu}$, nor $\mathbf{H}_\nu$ belongs to $\mathcal{A}$, first we perform some natural normalizations. We define three functions originating from $s_{\mu,\nu}$:

$$f_{\mu,\nu}(z) = \left((\mu-\nu+1)(\mu+\nu+1)s_{\mu,\nu}(z)\right)^{\frac{1}{\mu+1}},$$
$$g_{\mu,\nu}(z) = (\mu-\nu+1)(\mu+\nu+1)z^{-\mu}s_{\mu,\nu}(z)$$

and

$$h_{\mu,\nu}(z) = (\mu-\nu+1)(\mu+\nu+1)z^{\frac{1-\mu}{2}}s_{\mu,\nu}(\sqrt{z}).$$

Similarly, we associate with $\mathbf{H}_\nu$ the functions

$$u_\nu(z) = \left(\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)\right)^{\frac{1}{\nu+1}},$$
$$v_\nu(z) = \sqrt{\pi}2^\nu z^{-\nu}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z)$$

and

$$w_\nu(z) = \sqrt{\pi}2^\nu z^{\frac{1-\nu}{2}}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(\sqrt{z}).$$

Clearly the functions $f_{\mu,\nu}$, $g_{\mu,\nu}$, $h_{\mu,\nu}$, u_ν, v_ν and w_ν belong to the class $\mathcal{A}$. The main results in the present note concern the exact values of the radii of starlikeness for these six function, for some ranges of the parameters.

Let us set

$$f_\mu(z) = f_{\mu-\frac{1}{2},\frac{1}{2}}(z),\quad g_\mu(z) = g_{\mu-\frac{1}{2},\frac{1}{2}}(z)\quad\text{and}\quad h_\mu(z) = h_{\mu-\frac{1}{2},\frac{1}{2}}(z).$$

The first principal result we establish reads as follows:

Theorem 1. *Let $\mu \in (-1,1)$, $\mu \neq 0$. The following statements hold:*

a) *If $0 \le \alpha < 1$ and $\mu \in \left(-\frac{1}{2},0\right)$, then $r^*_\alpha(f_\mu) = x_{\mu,\alpha}$, where $x_{\mu,\alpha}$ is the smallest positive root of the equation*

$$z\,s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - \alpha\left(\mu+\frac{1}{2}\right)s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = 0.$$

Moreover, if $0 \le \alpha < 1$ and $\mu \in \left(-1,-\frac{1}{2}\right)$, then $r^*_\alpha(f_\mu) = q_{\mu,\alpha}$, where $q_{\mu,\alpha}$ is the unique positive root of the equation

$$izs'_{\mu-\frac{1}{2},\frac{1}{2}}(iz) - \alpha\left(\mu+\frac{1}{2}\right)s_{\mu-\frac{1}{2},\frac{1}{2}}(iz) = 0.$$

b) *If $0 \le \alpha < 1$, then $r^*_\alpha(g_\mu) = y_{\mu,\alpha}$, where $y_{\mu,\alpha}$ is the smallest positive root of the equation*

$$z\,s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - \left(\mu+\alpha-\frac{1}{2}\right)s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = 0.$$

c) *If $0 \le \alpha < 1$, then $r^*_\alpha(h_\mu) = t_{\mu,\alpha}$, where $t_{\mu,\alpha}$ is the smallest positive root of the equation*

$$zs'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - \left(\mu+2\alpha-\frac{3}{2}\right)s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = 0.$$

The corresponding result about the radii of starlikeness of the functions, related to Struve's one, is:

Theorem 2. *Let $|\nu| < \frac{1}{2}$. The following assertions are true:*

a) *If $0 \le \alpha < 1$, then $r^*_\alpha(u_\nu) = \delta_{\nu,\alpha}$, where $\delta_{\nu,\alpha}$ is the smallest positive root of the equation*

$$z\mathbf{H}'_\nu(z) - \alpha(\nu+1)\mathbf{H}_\nu(z) = 0.$$

b) *If $0 \le \alpha < 1$, then $r^*(v_\nu) = \rho_{\nu,\alpha}$, where $\rho_{\nu,\alpha}$ is the smallest positive root of the equation*

$$z\mathbf{H}'_\nu(z) - (\alpha+\nu)\mathbf{H}_\nu(z) = 0.$$

c) *If $0 \le \alpha < 1$, then $r^*_\alpha(w_\nu) = \sigma_{\nu,\alpha}$, where $\sigma_{\nu,\alpha}$ is the smallest positive root of the equation*

$$z\mathbf{H}'_\nu(z) - (2\alpha+\nu-1)\mathbf{H}_\nu(z) = 0.$$

It is worth mentioning that the starlikeness of h_μ, when $\mu \in (-1,1)$, $\mu \neq 0$, as well as of w_ν, under the restriction $|\nu| \le \frac{1}{2}$, were established in [**BS**], and it was proved there that all the derivatives of these functions are close-to-convex in $\mathbb{D}$.

2. Preliminaries

2.1. **The Hadamard's factorization.** The following preliminary result is the content of Lemmas 1 and 2 in [**BK**].

Lemma 1. *Let*

$$\varphi_k(z) = {}_1F_2\left(1; \frac{\mu-k+2}{2}, \frac{\mu-k+3}{2}; -\frac{z^2}{4}\right)$$

where $z \in \mathbb{C}$, $\mu \in \mathbb{R}$ and $k \in \{0,1,\dots\}$ such that $\mu - k$ is not in $\{0,-1,\dots\}$. Then, φ_k is an entire function of order $\rho = 1$ and of exponential type $\tau = 1$. Consequently, the Hadamard's factorization of φ_k is of the form

$$\varphi_k(z) = \prod_{n\geq 1}\left(1 - \frac{z^2}{z^2_{\mu,k,n}}\right), \tag{2.1}$$

where $\pm z_{\mu,k,1}$, $\pm z_{\mu,k,2}, \dots$ are all zeros of the function φ_k and the infinite product is absolutely convergent. Moreover, for z, μ and k as above, we have

$$(\mu-k+1)\varphi_{k+1}(z) = (\mu-k+1)\varphi_k(z) + z\varphi_k'(z),$$

$$\sqrt{z}s_{\mu-k-\frac{1}{2},\frac{1}{2}}(z) = \frac{z^{\mu-k+1}}{(\mu-k)(\mu-k+1)}\varphi_k(z).$$

2.2. **Quotients of power series.** We will also need the following result (see [**BK**, **PV**]):

Lemma 2. *Consider the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$, where $a_n \in \mathbb{R}$ and $b_n > 0$ for all $n \geq 0$. Suppose that both series converge on $(-r,r)$, for some $r > 0$. If the sequence $\{a_n/b_n\}_{n\geq 0}$ is increasing (decreasing), then the function $x \mapsto f(x)/g(x)$ is increasing (decreasing) too on $(0,r)$. The result remains true for the power series*

$$f(x) = \sum_{n\geq 0} a_n x^{2n} \quad \text{and} \quad g(x) = \sum_{n\geq 0} b_n x^{2n}.$$

2.3. **Zeros of polynomials and entire functions and the Laguerre-Pólya class.** In this subsection we provide the necessary information about polynomials and entire functions with real zeros. An algebraic polynomial is called hyperbolic if all its zeros are real.

The simple statement that two real polynomials p and q posses real and interlacing zeros if and only if any linear combinations of p and q is a hyperbolic polynomial is sometimes called Obrechkoff's theorem. We formulate the following specific statement that we shall need.

Lemma 3. *Let $p(x) = 1 - a_1x + a_2x^2 - a_3x^3 + \cdots + (-1)^n a_n x^n = (1 - x/x_1)\cdots(1 - x/x_n)$ be a hyperbolic polynomial with positive zeros $0 < x_1 \leq x_2 \leq \cdots \leq x_n$, and normalized by $p(0) = 1$. Then, for any constant C, the polynomial $q(x) = Cp(x) - x\,p'(x)$ is hyperbolic. Moreover, the smallest zero η_1 belongs to the interval $(0, x_1)$ if and only if $C < 0$.*

The proof is straightforward; it suffices to apply Rolle's theorem and then count the sign changes of the linear combination at the zeros of p. We refer to [**BDR**, **DMR**] for further results on monotonicity and asymptotics of zeros of linear combinations of hyperbolic polynomials.

A real entire function ψ belongs to the Laguerre-Pólya class $\mathcal{LP}$ if it can be represented in the form

$$\psi(x) = cx^m e^{-ax^2+\beta x}\prod_{k\geq 1}\left(1 + \frac{x}{x_k}\right)e^{-\frac{x}{x_k}},$$

with $c, \beta, x_k \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N}\cup\{0\}$, $\sum x_k^{-2} < \infty$. Similarly, ϕ is said to be of type I in the Laguerre-Pólya class, written $\varphi \in \mathcal{LPI}$, if $\phi(x)$ or $\phi(-x)$ can be represented as

$$\phi(x) = cx^m e^{\sigma x}\prod_{k\geq 1}\left(1 + \frac{x}{x_k}\right),$$

with $c \in \mathbb{R}$, $\sigma \geq 0$, $m \in \mathbb{N}\cup\{0\}$, $x_k > 0$, $\sum 1/x_k < \infty$. The class $\mathcal{LP}$ is the complement of the space of hyperbolic polynomials in the topology induced by the uniform convergence on the compact sets of the complex plane while $\mathcal{LPI}$ is the complement of the hyperbolic polynomials whose zeros posses a preassigned constant sign. Given an entire function φ with the Maclaurin expansion

$$\varphi(x) = \sum_{k\geq 0} \gamma_k \frac{x^k}{k!},$$

its Jensen polynomials are defined by

$$g_n(\varphi; x) = g_n(x) = \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j.$$

Jensen proved the following relation in [**Jen**]:

Theorem A. *The function φ belongs to $\mathcal{LP}$ ($\mathcal{LPI}$, respectively) if and only if all the polynomials $g_n(\varphi; x)$, $n = 1, 2, \ldots$, are hyperbolic (hyperbolic with zeros of equal sign). Moreover, the sequence $g_n(\varphi; z/n)$ converges locally uniformly to $\varphi(z)$.*

Further information about the Laguerre-Pólya class can be found in [**Obr**, **RS**] while [**DC**] contains references and additional facts about the Jensen polynomials in general and also about those related to the Bessel function.

A special emphasis has been given on the question of characterizing the kernels whose Fourier transform belongs to $\mathcal{LP}$ (see [**DR**]). The following is a typical result of this nature, due to Pólya [**Po**].

Theorem B. *Suppose that the function K is positive, strictly increasing and continuous on $[0, 1)$ and integrable there. Then the entire functions*

$$U(z) = \int_0^1 K(t)\sin(zt)dt \quad \text{and} \quad V(z) = \int_0^1 K(t)\cos(zt)dt$$

have only real and simple zeros and their zeros interlace.

In other words, the latter result states that both the sine and the cosine transforms of a kernel are in the Laguerre-Pólya class provided the kernel is compactly supported and increasing in the support.

Theorem 3. *Let $\mu \in (-1, 1)$, $\mu \neq 0$, and c be a constant such that $c < \mu + \frac{1}{2}$. Then the functions $z \mapsto zs'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - cs_{\mu-\frac{1}{2},\frac{1}{2}}(z)$ can be represented in the form*

$$\mu(\mu+1)\left(z\, s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) - c\, s_{\mu-\frac{1}{2},\frac{1}{2}}(z)\right) = z^{\mu+\frac{1}{2}}\psi_\mu(z), \tag{2.2}$$

where ψ_μ is an even entire function and $\psi_\mu \in \mathcal{LP}$. Moreover, the smallest positive zero of ψ_μ does not exceed the first positive zero of $s_{\mu-\frac{1}{2},\frac{1}{2}}$.

Similarly, if $|\nu| < \frac{1}{2}$ and d is a constant satisfying $d < \nu + 1$, then

$$\frac{\sqrt{\pi}}{2}\Gamma\left(\nu + \frac{3}{2}\right)\ (\, z\mathbf{H}'_\nu(z) - d\mathbf{H}_\nu(z)\,) = \left(\frac{z}{2}\right)^{\nu+1} \phi_\nu(z), \tag{2.3}$$

where ϕ_ν is an entire function in the Laguerre-Pólya class and the smallest positive zero of ϕ_ν does not exceed the first positive zero of $\mathbf{H}_\nu$.

Proof. First suppose that $\mu \in (0, 1)$. Since, by (1.1),

$$\mu(\mu+1)s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \sum_{k\geq 0} \frac{(-1)^k z^{2k+\mu+\frac{1}{2}}}{2^{2k}\left(\frac{\mu+2}{2}\right)_k\left(\frac{\mu+3}{2}\right)_k},$$

then

$$\mu(\mu+1)z\, s'_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \sum_{k\geq 0} \frac{(-1)^k\left(2k+\mu+\frac{1}{2}\right) z^{2k+\mu+\frac{1}{2}}}{2^{2k}\left(\frac{\mu+2}{2}\right)_k\left(\frac{\mu+3}{2}\right)_k}.$$

Therefore, (2.2) holds with

$$\psi_\mu(z) = \sum_{k\geq 0} \frac{2k+\mu+\frac{1}{2}-c}{\left(\frac{\mu+2}{2}\right)_k\left(\frac{\mu+3}{2}\right)_k}\left(-\frac{z^2}{4}\right)^k.$$

On the other hand, by Lemma 1,

$$\mu(\mu+1)s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = z^{\mu+\frac{1}{2}}\varphi_0(z),$$

and, by [**BK**, Lemma 3], we have

$$z\varphi_0(z) = \mu(\mu+1)\int_0^1 (1-t)^{\mu-1}\sin(zt)dt, \quad \text{for } \mu > 0. \tag{2.4}$$

Therefore φ_0 has the Maclaurin expansion

$$\varphi_0(z) = \sum_{k\geq 0} \frac{1}{\left(\frac{\mu+2}{2}\right)_k \left(\frac{\mu+3}{2}\right)_k} \left(-\frac{z^2}{4}\right)^k.$$

Moreover, (2.4) and Theorem B imply that $\varphi_0 \in \mathcal{LP}$ for $\mu \in (0,1)$, so that the function $\tilde{\varphi}_0(z) := \varphi_0(2\sqrt{z})$,

$$\tilde{\varphi}_0(\zeta) = \sum_{k\geq 0} \frac{1}{\left(\frac{\mu+2}{2}\right)_k \left(\frac{\mu+3}{2}\right)_k} (-\zeta)^k,$$

belongs to $\mathcal{LPI}$. Then it follows form Theorem A that its Jensen polynomials

$$g_n(\tilde{\varphi}_0;\zeta) = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{\left(\frac{\mu+2}{2}\right)_k \left(\frac{\mu+3}{2}\right)_k} (-\zeta)^k$$

are all hyperbolic. However, observe that the Jensen polynomials of $\tilde{\psi}_\mu(z) := \psi_\mu(2\sqrt{z})$ are simply

$$-\frac{1}{2}g_n(\tilde{\psi}_\mu;\zeta) = -\frac{1}{2}\left(\mu + \frac{1}{2} - c\right) g_n(\tilde{\varphi}_0;\zeta) - \zeta\, g_n'(\tilde{\varphi}_0;\zeta).$$

Lemma 3 implies that all zeros of $g_n(\tilde{\psi}_\mu;\zeta)$ are real and positive and that the smallest one precedes the first zero of $g_n(\tilde{\varphi}_0;\zeta)$. In view of Theorem A, the latter conclusion immediately yields that $\tilde{\psi}_\mu \in \mathcal{LPI}$ and that its first zero precedes the one of $\tilde{\varphi}_0$. Finally, the first statement of the theorem for $\mu \in (0,1)$ follows after we go back from $\tilde{\psi}_\mu$ and $\tilde{\varphi}_0$ to ψ_μ and φ_0 by setting $\zeta = -\frac{z^2}{4}$.

Now we prove (2.2) for the case when $\mu \in (-1,0)$. Observe that for $\mu \in (0,1)$ the function [**BK**, Lemma 3]

$$\varphi_1(z) = \sum_{k\geq 0} \frac{1}{\left(\frac{\mu+1}{2}\right)_k \left(\frac{\mu+2}{2}\right)_k} \left(-\frac{z^2}{4}\right)^k = \mu\int_0^1 (1-t)^{\mu-1}\cos(zt)dt$$

belongs also to Laguerre-Pólya class $\mathcal{LP}$, and hence the Jensen polynomials of $\tilde{\varphi}_1(z) := \varphi_1(2\sqrt{z})$ are hyperbolic. Straightforward calculations show that the Jensen polynomials of $\tilde{\psi}_{\mu-1}(z) := \psi_{\mu-1}(2\sqrt{z})$ are

$$-\frac{1}{2}g_n(\tilde{\psi}_{\mu-1};\zeta) = -\frac{1}{2}\left(\mu - \frac{1}{2} - c\right) g_n(\tilde{\varphi}_1;\zeta) - \zeta\, g_n'(\tilde{\varphi}_1;\zeta).$$

Lemma 3 implies that for $\mu \in (0,1)$ all zeros of $g_n(\tilde{\psi}_{\mu-1};\zeta)$ are real and positive and that the smallest one precedes the first zero of $g_n(\tilde{\varphi}_1;\zeta)$. This fact, together with Theorem A, yields that $\tilde{\psi}_{\mu-1} \in \mathcal{LPI}$ and that its first zero precedes the one of $\tilde{\varphi}_1$. Consequently, the first statement of the theorem for $\mu \in (-1,0)$ follows after we go back from $\tilde{\psi}_{\mu-1}$ and $\tilde{\varphi}_1$ to $\psi_{\mu-1}$ and φ_1 by setting $\zeta = -\frac{z^2}{4}$ and substituting μ by $\mu+1$.

In order to prove the corresponding statement for (2.3), we recall first that the hypergeometric representation (1.2) of the Struve function is equivalent to

$$\frac{\sqrt{\pi}}{2}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z) = \sum_{k\geq 0} \frac{(-1)^k}{\left(\frac{3}{2}\right)_k \left(\nu+\frac{3}{2}\right)_k} \left(\frac{z}{2}\right)^{2k+\nu+1},$$

which immediately yields

$$\phi_\nu(z) = \sum_{k\geq 0} \frac{2k+\nu+1-d}{\left(\frac{3}{2}\right)_k \left(\nu+\frac{3}{2}\right)_k} \left(-\frac{z^2}{4}\right)^k.$$

On the other hand, the integral representation

$$\mathbf{H}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\sin(zt)dt,$$

which holds for $\nu > -\frac{1}{2}$, and Theorem B imply that the even entire function

$$\mathcal{H}_\nu(z) = \sum_{k\geq 0} \frac{1}{\left(\frac{3}{2}\right)_k \left(\nu+\frac{3}{2}\right)_k} \left(-\frac{z^2}{4}\right)^k$$

belongs to the Laguerre-Pólya class when $|\nu| < \frac{1}{2}$. Then the functions $\tilde{\mathcal{H}}_\nu(z) := \mathcal{H}_\nu(2\sqrt{z})$,

$$\tilde{\mathcal{H}}_\nu(\zeta) = \sum_{k\geq 0} \frac{1}{\left(\frac{3}{2}\right)_k \left(\nu+\frac{3}{2}\right)_k} (-\zeta)^k,$$

is in $\mathcal{LPI}$. Therefore, its Jensen polynomials

$$g_n(\tilde{\mathcal{H}}_\nu;\zeta) = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{\left(\frac{3}{2}\right)_k \left(\nu+\frac{3}{2}\right)_k} (-\zeta)^k$$

are hyperbolic, with positive zeros. Then, by Lemma 3, the polynomial $-\frac{1}{2}(\nu+1-d)\, g_n(\tilde{\mathcal{H}}_\nu;\zeta) - \zeta\, g_n'(\tilde{\mathcal{H}}_\nu;\zeta)$ possesses only real positive zeros. Obviously the latter polynomial coincides with the nth Jensen polynomials of $\tilde{\phi}_\nu(z) = \phi_\nu(2\sqrt{z})$, that is

$$-\frac{1}{2} g_n(\tilde{\phi}_\nu;\zeta) = -\frac{1}{2}(\nu+1-d)\, g_n(\tilde{\mathcal{H}}_\nu;\zeta) - \zeta\, g_n'(\tilde{\mathcal{H}}_\nu;\zeta).$$

Moreover, the smallest zero of $g_n(\tilde{\phi}_\nu;\zeta)$ precedes the first positive zero of $g_n(\tilde{\mathcal{H}}_\nu;\zeta)$. This implies that $\phi_\nu \in \mathcal{LP}$ and that its first positive zero is smaller that the one of $\mathcal{H}_\nu$. □

3. Proofs of the main results

Proof of Theorem 1. We need to show that for the corresponding values of μ and α the inequalities

$$\Re\left(\frac{zf_\mu'(z)}{f_\mu(z)}\right) > \alpha, \quad \Re\left(\frac{zg_\mu'(z)}{g_\mu(z)}\right) > \alpha \quad \text{and} \quad \Re\left(\frac{zh_\mu'(z)}{h_\mu(z)}\right) > \alpha \tag{3.1}$$

are valid for $z \in \mathbb{D}_{r_\alpha^*(f_\mu)}$, $z \in \mathbb{D}_{r_\alpha^*(g_\mu)}$ and $z \in \mathbb{D}_{r_\alpha^*(h_\mu)}$ respectively, and each of the above inequalities does not hold in larger disks. It follows from (2.1) that

$$f_\mu(z) = f_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \left(\mu(\mu+1)s_{\mu-\frac{1}{2},\frac{1}{2}}(z)\right)^{\frac{1}{\mu+\frac{1}{2}}} = z\left(\varphi_0(z)\right)^{\frac{1}{\mu+\frac{1}{2}}},$$

$$g_\mu(z) = g_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \mu(\mu+1)z^{-\mu+\frac{1}{2}} s_{\mu-\frac{1}{2},\frac{1}{2}}(z) = z\varphi_0(z),$$

$$h_\mu(z) = h_{\mu-\frac{1}{2},\frac{1}{2}}(z) = \mu(\mu+1)z^{\frac{3-2\mu}{4}} s_{\mu-\frac{1}{2},\frac{1}{2}}(\sqrt{z}) = z\varphi_0(\sqrt{z}),$$

which in turn imply that

$$\frac{zf_\mu'(z)}{f_\mu(z)} = 1 + \frac{z\varphi_0'(z)}{(\mu+\frac{1}{2})\varphi_0(z)} = 1 - \frac{1}{\mu+\frac{1}{2}} \sum_{n\geq 1} \frac{2z^2}{z_{\mu,0,n}^2 - z^2},$$

$$\frac{zg_\mu'(z)}{g_\mu(z)} = 1 + \frac{z\varphi_0'(z)}{\varphi_0(z)} = 1 - \sum_{n\geq 1} \frac{2z^2}{z_{\mu,0,n}^2 - z^2},$$

$$\frac{zh_\mu'(z)}{h_\mu(z)} = 1 + \frac{1}{2}\frac{\sqrt{z}\varphi_0'(\sqrt{z})}{\varphi_0(\sqrt{z})} = 1 - \sum_{n\geq 1} \frac{z}{z_{\mu,0,n}^2 - z},$$

respectively. We note that for $\mu \in (0,1)$ the function φ_0 has only real and simple zeros (see [**BK**]). For $\mu \in (0,1)$, and $n \in \{1,2,\dots\}$ let $\xi_{\mu,n} = z_{\mu,0,n}$ be the nth positive zero of φ_0. We know that (see [**KL**, Lemma 2.1]) $\xi_{\mu,n} \in (n\pi, (n+1)\pi)$ for all $\mu \in (0,1)$ and $n \in \{1,2,\dots\}$, which implies that $\xi_{\mu,n} > \xi_{\mu,1} > \pi > 1$ for all $\mu \in (0,1)$ and $n \geq 2$. On the other hand, it is known that [**BKS**] if $z \in \mathbb{C}$ and $\beta \in \mathbb{R}$ are such that $\beta > |z|$, then

$$\frac{|z|}{\beta - |z|} \geq \Re\left(\frac{z}{\beta - z}\right). \tag{3.2}$$

Then the inequality

$$\frac{|z|^2}{\xi_{\mu,n}^2 - |z|^2} \geq \Re\left(\frac{z^2}{\xi_{\mu,n}^2 - z^2}\right),$$

holds get for every $\mu \in (0,1)$, $n \in \mathbb{N}$ and $|z| < \xi_{\mu,1}$. Therefore,

$$\Re\left(\frac{zf_\mu'(z)}{f_\mu(z)}\right) = 1 - \frac{1}{\mu+\frac{1}{2}} \Re\left(\sum_{n\geq 1} \frac{2z^2}{\xi_{\mu,n}^2 - z^2}\right) \geq 1 - \frac{1}{\mu+\frac{1}{2}} \sum_{n\geq 1} \frac{2|z|^2}{\xi_{\mu,n}^2 - |z|^2} = \frac{|z|\, f_\mu'(|z|)}{f_\mu(|z|)},$$

$$\Re\left(\frac{zg_\mu'(z)}{g_\mu(z)}\right)=1-\Re\left(\sum_{n\geq1}\frac{2z^2}{\xi_{\mu,n}^2-z^2}\right)\geq1-\sum_{n\geq1}\frac{2\left|z\right|^2}{\xi_{\mu,n}^2-\left|z\right|^2}=\frac{\left|z\right|g_\mu'(\left|z\right|)}{g_\mu(\left|z\right|)}$$

and

$$\Re\left(\frac{zh_\mu'(z)}{h_\mu(z)}\right)=1-\Re\left(\sum_{n\geq1}\frac{z}{\xi_{\mu,n}^2-z}\right)\geq1-\sum_{n\geq1}\frac{\left|z\right|}{\xi_{\mu,n}^2-\left|z\right|}=\frac{\left|z\right|h_\mu'(\left|z\right|)}{h_\mu(\left|z\right|)},$$

where equalities are attained only when $z=|z|=r$. The latter inequalities and the minimum principle for harmonic functions imply that the corresponding inequalities in (3.1) hold if and only if $|z|<x_{\mu,\alpha}$, $|z|<y_{\mu,\alpha}$ and $|z|<t_{\mu,\alpha}$, respectively, where $x_{\mu,\alpha}$, $y_{\mu,\alpha}$ and $t_{\mu,\alpha}$ are the smallest positive roots of the equations

$$rf_\mu'(r)/f_\mu(r)=\alpha,\quad rg_\mu'(r)/g_\mu(r)=\alpha,\ rh_\mu'(r)/h_\mu(r)=\alpha.$$

Since their solutions coincide with the zeros of the functions

$$r\mapsto rs_{\mu-\frac12,\frac12}'(r)-\alpha\left(\mu+\frac12\right)s_{\mu-\frac12,\frac12}(r),\ r\mapsto rs_{\mu-\frac12,\frac12}'(r)-\left(\mu+\alpha-\frac12\right)s_{\mu-\frac12,\frac12}(r),$$

$$r\mapsto rs_{\mu-\frac12,\frac12}'(r)-\left(\mu+2\alpha-\frac32\right)s_{\mu-\frac12,\frac12}(r),$$

the result we need follows from Theorem 3. In other words, Theorem 3 show that all the zeros of the above three functions are real and their first positive zeros do not exceed the first positive zero $\xi_{\mu,1}$. This guarantees that the above inequalities hold. This completes the proof our theorem when $\mu\in(0,1)$.

Now we prove that the inequalities in (3.1) also hold for $\mu\in(-1,0)$, except the first one, which is valid for $\mu\in\left(-\frac12,0\right)$. In order to do this, suppose that $\mu\in(0,1)$ and adapt the above proof, substituting μ by $\mu-1$, φ_0 by the function φ_1 and taking into account that the nth positive zero of φ_1, denoted by $\zeta_{\mu,n}=z_{\mu,1,n}$, satisfies (see [**BS**]) $\zeta_{\mu,n}>\zeta_{\mu,1}>\frac{\pi}{2}>1$ for all $\mu\in(0,1)$ and $n\geq2$. It is worth mentiontioning that

$$\Re\left(\frac{zf_{\mu-1}'(z)}{f_{\mu-1}(z)}\right)=1-\frac{1}{\mu-\frac12}\Re\left(\sum_{n\geq1}\frac{2z^2}{\zeta_{\mu,n}^2-z^2}\right)\geq1-\frac{1}{\mu-\frac12}\sum_{n\geq1}\frac{2\left|z\right|^2}{\zeta_{\mu,n}^2-\left|z\right|^2}=\frac{\left|z\right|f_{\mu-1}'(\left|z\right|)}{f_{\mu-1}(\left|z\right|)},$$

remains true for $\mu\in\left(\frac12,1\right)$. In this case we use the minimum principle for harmonic functions to ensure that (3.1) is valid for $\mu-1$ instead of μ. Thus, using again Theorem 3 and replacing μ by $\mu+1$, we obtain the statement of the first part for $\mu\in\left(-\frac12,0\right)$. For $\mu\in(-1,0)$ the proof of the second and third inequalities in (3.1) go along similar lines.

To prove the statement for part **a** when $\mu\in\left(-1,-\frac12\right)$ we observe that the counterpart of (3.2) is

$$\mathrm{Re}\left(\frac{z}{\beta-z}\right)\geq\frac{-|z|}{\beta+|z|},\tag{3.3}$$

and it holds for all $z\in\mathbb{C}$ and $\beta\in\mathbb{R}$ such that $\beta>|z|$ (see [**BKS**]). From (3.3), we obtain the inequality

$$\Re\left(\frac{z^2}{\zeta_{\mu,n}^2-z^2}\right)\geq\frac{-\left|z\right|^2}{\zeta_{\mu,n}^2+\left|z\right|^2},$$

which holds for all $\mu\in\left(0,\frac12\right)$, $n\in\mathbb{N}$ and $|z|<\zeta_{\mu,1}$ and it implies that

$$\Re\left(\frac{zf_{\mu-1}'(z)}{f_{\mu-1}(z)}\right)=1-\frac{1}{\mu-\frac12}\Re\left(\sum_{n\geq1}\frac{2z^2}{\zeta_{\mu,n}^2-z^2}\right)\geq1+\frac{1}{\mu-\frac12}\sum_{n\geq1}\frac{2\left|z\right|^2}{\zeta_{\mu,n}^2+\left|z\right|^2}=\frac{i\left|z\right|f_{\mu-1}'(i\left|z\right|)}{f_{\mu-1}(i\left|z\right|)}.$$

In this case equality is attained if $z=i\left|z\right|=ir$. Moreover, the latter inequality implies that

$$\Re\left(\frac{zf_{\mu-1}'(z)}{f_{\mu-1}(z)}\right)>\alpha$$

if and only if $|z|<q_{\mu,\alpha}$, where $q_{\mu,\alpha}$ denotes the smallest positive root of the equation $irf_{\mu-1}'(\mathrm{ir})/f_{\mu-1}(\mathrm{ir})=\alpha$, which is equivalent to

$$irs_{\mu-\frac32,\frac12}'(ir)-\alpha\left(\mu-\frac12\right)s_{\mu-\frac32,\frac12}(ir)=0,\ \text{for}\ \mu\in\left(0,\frac12\right).$$

Substituting μ by $\mu+1$, we obtain

$$irs'_{\mu-\frac{1}{2},\frac{1}{2}}(ir)-\alpha\left(\mu+\frac{1}{2}\right)s_{\mu-\frac{1}{2},\frac{1}{2}}(ir)=0,\ \text{for}\ \mu\in\left(-1,-\frac{1}{2}\right).$$

It follows from Theorem 3 that the first positive zero of $z\mapsto izs'_{\mu-\frac{1}{2},\frac{1}{2}}(iz)-\alpha\left(\mu+\frac{1}{2}\right)s_{\mu-\frac{1}{2},\frac{1}{2}}(iz)$ does not exceed $\zeta_{\mu,1}$ which guarantees that the above inequalities are valid. All we need to prove is that the above function has actually only one zero in $(0,\infty)$. Observe that, according to Lemma 2, the function

$$r\mapsto\frac{irs'_{\mu-\frac{1}{2},\frac{1}{2}}(ir)}{s_{\mu-\frac{1}{2},\frac{1}{2}}(ir)}$$

is increasing on $(0,\infty)$ as a quotient of two power series whose positive coefficients form the increasing "quotient sequence" $\left\{2k+\mu+\frac{1}{2}\right\}_{k\geq0}$. On the other hand, the above function tends to $\mu+\frac{1}{2}$ when $r\to0$, so that its graph can intersect the horizontal line $y=\alpha\left(\mu+\frac{1}{2}\right)>\mu+\frac{1}{2}$ only once. This completes the proof of part **a** of the theorem when $\mu\in(-1,0)$. □

Proof of Theorem 2. As in the proof of Theorem 1 we need show that, for the corresponding values of ν and α, the inequalities

$$\Re\left(\frac{zu'_\nu(z)}{u_\nu(z)}\right)>\alpha,\ \ \Re\left(\frac{zv'_\nu(z)}{v_\nu(z)}\right)>\alpha\ \text{and}\ \Re\left(\frac{zw'_\nu(z)}{w_\nu(z)}\right)>\alpha \tag{3.4}$$

are valid for $z\in\mathbb{D}_{r^*_\alpha(u_\nu)}$, $z\in\mathbb{D}_{r^*_\alpha(v_\nu)}$ and $z\in\mathbb{D}_{r^*_\alpha(w_\nu)}$ respectively, and each of the above inequalities does not hold in any larger disk.

If $|\nu|\leq\frac{1}{2}$, then (see [**BPS**, Lemma 1]) the Hadamard factorization of the transcendental entire function $\mathcal{H}_\nu$, defined by

$$\mathcal{H}_\nu(z)=\sqrt{\pi}2^\nu z^{-\nu-1}\Gamma\left(\nu+\frac{3}{2}\right)\mathbf{H}_\nu(z),$$

reads as follows

$$\mathcal{H}_\nu(z)=\prod_{n\geq1}\left(1-\frac{z^2}{h^2_{\nu,n}}\right),$$

which implies that

$$\mathbf{H}_\nu(z)=\frac{z^{\nu+1}}{\sqrt{\pi}2^\nu\Gamma\left(\nu+\frac{3}{2}\right)}\prod_{n\geq1}\left(1-\frac{z^2}{h^2_{\nu,n}}\right),$$

where $h_{\nu,n}$ stands for the nth positive zero of the Struve function $\mathbf{H}_\nu$.

We know that (see [**BS**, Theorem 2]) $h_{\nu,n}>h_{\nu,1}>1$ for all $|\nu|\leq\frac{1}{2}$ and $n\in\mathbb{N}$. If $|\nu|\leq\frac{1}{2}$ and $|z|<h_{\nu,1}$, then (3.2) imples

$$\Re\left(\frac{zu'_\nu(z)}{u_\nu(z)}\right)=1-\frac{1}{\nu+1}\Re\left(\sum_{n\geq1}\frac{2z^2}{h^2_{\nu,n}-z^2}\right)\geq1-\frac{1}{\nu+1}\sum_{n\geq1}\frac{2\left|z\right|^2}{h^2_{\nu,n}-\left|z\right|^2}=\frac{|z|\,u'_\nu(|z|)}{u_\nu(|z|)},$$

$$\Re\left(\frac{zv'_\nu(z)}{v_\nu(z)}\right)=1-\Re\left(\sum_{n\geq1}\frac{2z^2}{h^2_{\nu,n}-z^2}\right)\geq1-\sum_{n\geq1}\frac{2\left|z\right|^2}{h^2_{\nu,n}-\left|z\right|^2}=\frac{|z|\,v'_\nu(|z|)}{v_\nu(|z|)}$$

and

$$\Re\left(\frac{zw'_\nu(z)}{w_\nu(z)}\right)=1-\Re\left(\sum_{n\geq1}\frac{z}{h^2_{\nu,n}-z}\right)\geq1-\sum_{n\geq1}\frac{|z|}{h^2_{\nu,n}-|z|}=\frac{|z|\,w'_\nu(|z|)}{w_\nu(|z|)},$$

where equalities are attained when $z=|z|=r$. Then minimum principle for harmonic functions implies that the corresponding inequalities in (3.4) hold if and only if $|z|<\delta_{\nu,\alpha}$, $|z|<\rho_{\nu,\alpha}$ and $|z|<\sigma_{\nu,\alpha}$, respectively, where $\delta_{\nu,\alpha}$, $\rho_{\nu,\alpha}$ and $\sigma_{\nu,\alpha}$ are the smallest positive roots of the equations

$$ru'_\nu(r)/u_\nu(r)=\alpha,\ \ rv'_\nu(r)/v_\nu(r)=\alpha,\ rw'_\nu(r)/w_\nu(r)=\alpha.$$

The solutions of these equations are the zeros of the functions

$$r\mapsto r\mathbf{H}'_\nu(r)-\alpha(\nu+1)\mathbf{H}_\nu(r),\ r\mapsto r\mathbf{H}'_\nu(r)-(\alpha+\nu)\mathbf{H}_\nu(r),\ r\mapsto r\mathbf{H}'_\nu(r)-(2\alpha+\nu-1)\mathbf{H}_\nu(r),$$

which, in view of Theorem 3, have only real zeros and the smallest positive zero of each of them does not exceed the first positive zeros of $\mathbf{H}_\nu$. □

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