**THE EUROPEAN
PHYSICAL JOURNAL C**

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On the bilinear covariants associated to mass dimension one spinors

J. M. Hoff da Silva[1,a], **C. H. Coronado Villalobos**[1,b], **R. J. Bueno Rogerio**[1,c], **E. Scatena**[2,d]

[1] UNESP, DFQ, Campus de Guaratinguetá, Avenida Dr. Ariberto Pereira da Cunha, 333, CEP 12516-410 Guaratinguetá, SP, Brazil
[2] Universidade Federal de Santa Catarina-CEE, Rua Pomerode, 710, CEP 89065-300 Blumenau, SC, Brazil

Abstract In this paper we approach the issue of Clifford algebra basis deformation, allowing for bilinear covariants associated to Elko spinors which satisfy the Fierz–Pauli–Kofink identities. We present a complete analysis of covariance, taking into account the involved dual structure associated to Elko spinors. Moreover, the possible generalizations to the recently presented new dual structure are performed.

1 Introduction

The so-called Elko spinors are a new set of spinors with a complex and interesting structure on their own. Historically, they were proposed by Ahluwalia and Grumiller when studying the properties of the Majorana spinor. Similar to them, Elko spinors are eigenspinors of the charge conjugation operator, C, but they have dual helicity and can take positive (self-conjugate) and negative (anti-self-conjugate) eigenvalues of C, while the Majorana ones take only the positive value and carry single helicity.

From the physical point of view, Elko spinors are constructed to be invisible to the other particles (i.e., they do not couple with the fields of the Standard Model, except for the Higgs boson), becoming a natural candidate to dark matter [1]. Mathematically, the dual-helicity peculiarity forces one to a redefinition of the dual spinor structure, as can be found in [1]. This idiosyncrasy reflects itself when constructing the spin sums for the Elko spinors, which break Lorentz symmetry. However, the spin sums are invariant under transformations of the very special relativity (VSR) [2–4]. There are several areas in which Elko spinors have been studied, from accelerator physics [5–8] to cosmology [9–19]. In particular, the appreciation of new dual structures brings about interesting possibilities within the algebraic scope [20,21].

As is well known, much of the physics associated to spinor fields is unveiled from its bilinear covariants for the simple reason that single fermions are not directly experienced. In this context it is indeed important to pay special attention to the subtleties of Clifford algebra when associating real numbers to the bilinear covariants [22]. It may sound as a secondary issue, but in fact the opposite is true. In two outstanding papers in the 1990s [23,24], Crawford worked out several important formalizations concerning the bispinor algebra. Among these, a rigorous procedure to obtain real bilinear covariants was developed. The general aim of this paper is to make use of this procedure to envisage what (if any) bilinear covariants are real when dealing with mass dimension one spinors. It is our hope that the results to be shown here may shed some light on the observables associated to these spinors. With suitable, but important, changes we take advantage of the formalism developed in [23] in order to study the bilinear covariants associated to the Elko spinor case. After a complete analysis, including the right observance of the Fierz–Pauli–Kofink (FPK) [25–27] relations, we arrive at the subset of real bilinear covariants.

Quite recently, new possibilities concerning the field adjoint possibilities were thoroughly investigated in [28,29]. These formalizations may lead to a local and full Lorentz spin 1/2 field also endowed with mass dimension one, evading, thus, the Weinberg no-go theorem [30]. We also have investigated the bilinears to this case, and to some extent the aforementioned program may be applied, leading to similar conclusions.

This paper is organized as follows: in Sect. 2 we introduce the bilinears covariant and proceed with the Clifford algebra basis deformation; then, in Sect. 3, we analyze its covariant structure. Both these sections are related to Elko spinors as objects whose spin sums break Lorentz symmetry. The natural background symmetry is then encoded

[a] e-mail: hoff@feg.unesp.br
[b] e-mail: ccoronado@feg.unesp.br
[c] e-mail: rodolforogerio@feg.unesp.br
[d] e-mail: e.scatena@ufsc.br

into the orthochronous proper Lorentz subgroup, leaving for Appendix B an analogous investigation taking into account VSR Elko spinors. In Sect. 4, we conclude with some remarks about the results we found.

2 Bilinear analysis

Let ψ be a given spinor field belonging to a section of the vector bundle $\mathbf{P}_{\mathrm{Spin}^e_{1,3}}(\mathcal{M}) \times_{\rho} \mathbb{C}^4$ where ρ stands for the entire representation space $D^{(1/2,0)} \oplus D^{(0,1/2)}$, or a given sector of such. The bilinear covariants associated to ψ, as usual, read

$$\sigma = \psi^{\dagger}\gamma_0\psi, \quad \omega = -\psi^{\dagger}\gamma_0\gamma_{0123}\psi, \quad \mathbf{J} = \psi^{\dagger}\gamma_0\gamma_{\mu}\psi\ \gamma^{\mu},$$
$$\mathbf{K} = \psi^{\dagger}\gamma_0 i\gamma_{0123}\gamma_{\mu}\psi\ \gamma^{\mu}, \quad \mathbf{S} = \frac{1}{2}\psi^{\dagger}\gamma_0 i\gamma_{\mu\nu}\psi\gamma^{\mu}\wedge\gamma^{\nu}, \tag{1}$$

where the Dirac matrices are written in the chiral (or Weyl) representation,

$$\gamma_0 = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} \mathbb{O} & \sigma_i \\ -\sigma_i & \mathbb{O} \end{pmatrix}. \tag{2}$$

On general grounds, it is always expected that one can associate (1) to physical observables. For instance, in the usual case, bearing in mind the relativistic description of the electron, σ is the invariant length, $\mathbf{J}$ is associated to the current density, $\mathbf{K}$ is the spin projection in the momentum direction, and $\mathbf{S}$ is the momentum electromagnetic density. The bilinear covariants, as is well known, obey the so-called Fierz–Pauli–Kofink (FPK) identities, given by [31]

$$\mathbf{J}^2 = \sigma^2 + \omega^2, \quad J_{\mu}K_{\nu} - K_{\mu}J_{\nu} = -\omega S_{\mu\nu} - \frac{\sigma}{2}\epsilon_{\mu\nu\alpha\beta}S^{\alpha\beta},$$
$$J_{\mu}K^{\mu} = 0, \quad \mathbf{J}^2 = -\mathbf{K}^2. \tag{3}$$

It can be seen that the physical requirement of reality can always be satisfied for Dirac spinors bilinear covariants [23], by a suitable deformation of the Clifford basis leading to physical appealing quantities. Unfortunately, the same cannot be stated for mass dimension one spinors, as Elko spinors. Actually, a straightforward calculation shows an incompatibility in the usual construction of bilinear covariants. In fact, one of the FPK identities is violated. This is due to the new dual structure associated to these spinors. It is worth to mention that the main difference between the Crawford deformation [23,24] and the one to be accomplished here is that in the former case, the spinors are understood as Dirac spinors, i.e., spinorial objects endowed with single helicity. Therefore, the dual structure is the usual one $\bar{\psi}(\boldsymbol{p}) = \psi^{\dagger}(\boldsymbol{p})\gamma_0$ and the required normalization is also ordinary. On the other hand Elko spinors, due to their own formal structure, need a dual redefinition $\overline{\lambda}_h^{S/A}(\boldsymbol{p}) = [\Xi(\boldsymbol{p})\lambda_h^{S/A}(\boldsymbol{p})]^{\dagger}\gamma_0$. This redef-

inition leads, ultimately, to a new normalization, culminating in a basis deformation satisfying the FPK identities.

Let us make these assertions more clear by explicitly showing the mentioned problem. In order to guarantee the sequential readability of the paper we leave for Appendix A a brief, but self contained, overview on the spinorial formal structure of Lorentz breaking Elko fields. Taking advantage of what was there defined, we use as an example the spinor $\lambda_{\{-,+\}}^S(\boldsymbol{p})$ and its dual given by, respectively,

$$\lambda_{\{-,+\}}^S(\boldsymbol{p}) = \Upsilon_- \begin{pmatrix} -i\sin(\theta/2)e^{-i\phi/2} \\ i\cos(\theta/2)e^{i\phi/2} \\ \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} \tag{4}$$

and[1]

$$\overline{\lambda}_{\{-,+\}}^S(\boldsymbol{p}) = \Upsilon_+$$
$$\times (-i\sin(\theta/2)e^{i\phi/2}\ i\cos(\theta/2)e^{-i\phi/2}\ -\cos(\theta/2)e^{i\phi/2}\ -\sin(\theta/2)e^{-i\phi/2}). \tag{5}$$

We stress once again that the dual structure associated to Elko spinors is obtained in a very judicious fashion in [29], leaving no space for modifications, an exception being made for the generalizations found in Ref. [1,29]. Using (4) and (5), as a direct calculation shows, Eq. (1) gives

$$\sigma = -2m, \tag{6}$$
$$\omega = 0, \tag{7}$$
$$J_0 = 0,$$
$$J_1 = 2im\cos\theta\cos\phi,$$
$$J_2 = 2im\cos\theta\sin\phi,$$
$$J_3 = -2im\sin\theta, \tag{8}$$
$$K_0 = 0,$$
$$K_1 = -2m\sin\phi,$$
$$K_2 = 2m\cos\phi,$$
$$K_3 = 0, \tag{9}$$

and

$$S_{01} = -2im\sin\theta\cos\phi,$$
$$S_{02} = -2im\sin\theta\sin\phi,$$
$$S_{03} = -2im\cos\theta,$$
$$S_{12} = S_{13} = S_{23} = 0. \tag{10}$$

As can be verified, the above bilinear covariants do not obey the FPK equations. More specifically, the relation containing $S_{\mu\nu}$ is not satisfied. In view of this problem, we revisited the formulation performed in [23] in order to find an appropriate Clifford basis upon which the bilinear covariants can be constructed, leading to the right verification of the FPK

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[1] The explicit form of the *boost factor*, $\Upsilon_{\pm}$, is defined in (A7).

relations. The price to be paid is that only a subset of bilinear covariants comprises real quantities.

2.1 Deformation of the Clifford algebra basis

As is well known, the very constitutive relation of the Clifford algebra is given by

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}, \quad \mu, \nu = 0, 1, 2, \ldots, N-1, \tag{11}$$

where $g_{\mu\nu}$ is an $N = 2n$ even-dimensional space-time metric, which in Cartesian coordinates has the form diag$(1, -1, \ldots, -1)$. The generators of the Clifford algebra are then the identity I and the vectors γ_μ, usually represented as square matrices. The standard approach dictates the complementation of the Clifford algebra basis, in order to guarantee real bilinear covariants. This complement is performed by the composition of the vector basis, used as building blocks [23],

$$\tilde{\gamma}_{\mu_1\mu_2\cdots\mu_{N-M}} \equiv \frac{1}{M!}\epsilon_{\mu_1\mu_2\cdots\mu_N}\gamma^{\mu_{N-M+1}}\gamma^{\mu_{N-M+2}\cdots}\gamma^{\mu_N}. \tag{12}$$

As is easy to see, the lowest M value is 2 (the smallest combination), nevertheless, it runs in the range $M = 2, 3, \ldots, N$. In this respect, the elements that form the (real) Clifford algebra basis are

$$\{\Gamma_i\} \equiv \{\mathrm{I}, \gamma_\mu, \tilde{\gamma}_{\mu_1\mu_2\ldots\mu_{N-2}}, \ldots, \tilde{\gamma}_\mu, \tilde{\gamma}\}, \tag{13}$$

where $\tilde{\gamma} \equiv \tilde{\gamma}_{\mu_1\mu_2\cdots\mu_{N-M}}$.

In view of the new elements appearing in the definition of the Elko dual, it is necessary to adapt the Clifford algebra basis complementation. As shown previously, it is absolutely necessary for the right appreciation of the FPK relations. We shall stress that for the Dirac spinorial case, the set (13) is suitably deformed (by a slightly different normalization) in order to provide real bilinear covariants. We shall pursue something similar here, and we will be successful in correcting the problem related to the FPK relations. This notwithstanding, only a subset of bilinear covariants ends up as real in the Elko spinorial case.

The first two bilinears arising from the Clifford algebra basis are

$$\sigma \equiv \overline{\lambda}_h(\boldsymbol{p})\mathrm{I}\lambda_h(\boldsymbol{p}), \tag{14}$$

$$J_\mu \equiv \overline{\lambda}_h(\boldsymbol{p})\gamma_\mu\lambda_h(\boldsymbol{p}), \tag{15}$$

where $\overline{\lambda}_h(\boldsymbol{p}) = [\Xi(\boldsymbol{p})\lambda_h(\boldsymbol{p})]^\dagger\gamma_0$ is identified as the Elko dual spinor.[2] The operator $\Xi(\boldsymbol{p})$ is responsible to change the spinor "helicity" (or rather the type of Elko spinor), here labeled by h. All the details concerning such an operator can

be found in Ref. [29]. The requirement $\sigma = \sigma^\dagger$ leads automatically to $\gamma_0 = \Xi^\dagger(\boldsymbol{p})\gamma_0^\dagger\Xi(\boldsymbol{p})$ since $\Xi^2(\boldsymbol{p}) = 1$. This constraint is readily satisfied, in such a way that (14) is real.[3] By the same token, one should require $J_\mu = J_\mu^\dagger$. This imposition leads to the constraint $\gamma_0\gamma_\mu = \Xi^\dagger(\boldsymbol{p})\gamma_\mu^\dagger\gamma_0^\dagger\Xi(\boldsymbol{p})$ which, however, cannot be fulfilled. This is an important point. In fact, the counterpart associated to Dirac spinors is simply $\gamma_0^{-1}\gamma_\mu^\dagger\gamma_0 = \gamma_\mu$, a constraint naturally achieved. The new dual structure, therefore, has forced a new interpretation of the bilinear covariants. First of all, as a matter of fact, for mass dimension one spinors, J^μ cannot be associated to the conserved current. Obviously, in order to have $\partial_\mu J^\mu = 0$ it is necessary to use the Dirac equation. This truism has lead to interesting algebraic possibilities [20], but the point to be emphasized here is that there is no problem in having a complex quantity related to the bilinear J^μ. The additional important consequence of an imaginary J^μ is that in order to satisfy the FPK equations K^μ or $S^{\mu\nu}$ must also be imaginary.

Notice that, instead of the usual Crawford deformation, here we do not arrive at an entire real bilinear set. In fact, in trying to implement the full reality condition it is mandatory to change the building block of the Clifford basis γ_μ. It would inevitably lead, however, to a change in the constitutive algebraic relation of the Clifford algebra (11). Therefore, this change must be excluded. It is important to emphasize, moreover, that even if we were willing to accept a modification of (11), the resulting constraint needed to get a real set of bilinears cannot be fulfilled.

Having said that, we may proceed deforming the usual basis in order to redefine bilinear covariants which satisfy the FPK identities. Making use of Eq. (12) and considering that the norm for the Elko spinor is real we have

$$[\overline{\lambda}_h(\boldsymbol{p})\tilde{\gamma}_{\mu_1\mu_2\cdots\mu_{N-M}}\lambda_h(\boldsymbol{p})]^\dagger$$
$$= (-1)^{M(M-1)/2}\overline{\lambda}_h(\boldsymbol{p})\Xi(\boldsymbol{p})\tilde{\gamma}_{\mu_1\mu_2\cdots\mu_{N-M}}\Xi(\boldsymbol{p})\lambda_h(\boldsymbol{p}). \tag{16}$$

It can be readily verified that the following redefinition of $\tilde{\gamma}_{\mu_1\mu_2\cdots\mu_{N-M}}$ is appropriate to ensure K^μ to be a real quantity:

$$\tilde{\gamma}_{\mu_1\mu_2\cdots\mu_{N-M}}$$
$$= (i^{M(M-1)/2}/M!)\Xi(\boldsymbol{p})\epsilon_{\mu_1\cdots\mu_N}\gamma^{\mu_{N-M+1}\cdots}\gamma^{\mu_N}\Xi(\boldsymbol{p}). \tag{17}$$

With the redefinition above, one is able to define the bispinor Clifford algebra basis as in (13), but with the gammas given by (17). As an example, consider the four-dimensional space-time. In this case the basis is given, accordingly, by

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[2] We once again refer the reader to Appendix A for the definition of the dual, as well as its necessity.

[3] Notice that it must be proved that the bilinears composed with $\Xi(\boldsymbol{p})$'s are covariant. We shall investigate this issue in the next section.

$$M = 4 \quad \Rightarrow \quad \tilde{\gamma} = -i\,\Xi(p)\gamma_5\Xi(p), \tag{18}$$

$$M = 3 \quad \Rightarrow \quad \tilde{\gamma}_\mu = -\Xi(p)\gamma_5\gamma_\mu\Xi(p), \tag{19}$$

$$M = 2 \quad \Rightarrow \quad \tilde{\gamma}_{\mu\nu} = \frac{i}{2}\Xi(p)\gamma_\mu\gamma_\nu\Xi(p), \tag{20}$$

where $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$. Now, with the real Clifford algebra basis at hand, it is possible to construct the bilinear forms, given by

$$
\begin{aligned}
\mathrm{I} &\Rightarrow \sigma_E = \overline{\lambda}_h(p)\mathrm{I}\lambda_h(p), \\
\gamma_\mu &\Rightarrow J_{\mu_E} = \overline{\lambda}_h(p)\gamma_\mu\lambda_h(p), \\
\tilde{\gamma} &\Rightarrow \omega_E = -i\overline{\lambda}_h(p)\gamma_5\lambda_h(p), \\
\tilde{\gamma}_\mu &\Rightarrow K_{\mu_E} = -\overline{\lambda}_h(p)\Xi(p)\gamma_5\gamma_\mu\Xi(p)\lambda_h(p), \\
\tilde{\gamma}_{\mu\nu} &\Rightarrow S_{\mu\nu_E} = i\overline{\lambda}_h(p)\Xi(p)\gamma_\mu\gamma_\nu\Xi(p)\lambda_h(p).
\end{aligned}
\tag{21}
$$

From the above construction, after a simple but tedious checking process, one ensures that the slight modifications of the bilinear covariants are enough to guarantee that the FPK identities (3) are satisfied. After all, we arrive at σ and K^μ as real non-null quantities. As remarked in the Introduction, it is our hope that these quantities shall be considered in the determination of possible experimental outcomes of the Elko construction.

3 Covariant structure

So far we have worked out quantities defined as (21) claiming that they must be faced as bilinear covariants. While they are bilinear quantities, their covariant structure must be evinced. The whole issue is related to the (necessary) presence of the $\Xi(p)$ operator.

Suppose that the Elko spinor belongs to a linear representation of the symmetry group in question, in such a way that, seen in another frame, the field undergoes a transformation

$$\lambda_h^{\prime S/A}(p') = S(\Lambda)\lambda_h^{S/A}(p). \tag{22}$$

There is a Dirac-like operator that annihilates Elko spinors [29] (not related to the field dynamics) given by

$$(\gamma_\mu p^\mu \Xi(p) \pm m)\lambda_h^{S/A}(p) = 0, \tag{23}$$

from which we shall investigate the covariance. Applying the transformation (22) for the fields in Eq. (23), we find

$$(\gamma_\mu p'^\mu \Xi(p) \pm m)\lambda_h^{\prime S/A}(p') = 0. \tag{24}$$

The momentum can be written as $p^\mu \leftrightarrow i\partial^\mu$ and the partial derivative transforms usually as $\partial'^\mu = \Lambda^\mu{}_\beta \partial^\beta$. Therefore, in order to ensure covariance of Eq. (23) the following behavior of the Dirac matrices and the $\Xi(p)$ operator, respectively, is necessary:

$$\gamma'_\beta = S(\Lambda)\gamma_\mu S^{-1}(\Lambda)\Lambda^\mu{}_\beta, \tag{25}$$

$$\Xi'(p) = S(\Lambda)\Xi(p)S^{-1}(\Lambda). \tag{26}$$

Equation (25) is the usual requirement to be posed to the gamma matrices in order to achieve a covariant Dirac equation. The requirement (26) is the new ingredient of the Elko theory, which must be investigated.

Interestingly enough, from Eq. (A13), along with (25), it is possible to see that [29,32]

$$
\begin{aligned}
\Xi'(p) &= \frac{1}{2m}(\lambda_{\{+-\}}^{\prime S}(p')\bar{\lambda}_{\{+-\}}^{\prime S}(p') + \lambda_{\{-+\}}^{\prime S}(p')\bar{\lambda}_{\{-+\}}^{\prime S}(p') \\
&\quad \times (p') - \lambda_{\{+-\}}^{\prime A}(p')\bar{\lambda}_{\{+-\}}^{\prime A}(p') - \lambda_{\{-+\}}^{\prime A}(p')\bar{\lambda}_{\{-+\}}^{\prime A}(p')), \\
&= \frac{1}{2m}S(\Lambda)(\lambda_{\{+-\}}^{S}(p)\bar{\lambda}_{\{+-\}}^{S}(p) + \lambda_{\{-+\}}^{S}(p) \\
&\quad \times \bar{\lambda}_{\{-+\}}^{S}(p) - \lambda_{\{+-\}}^{A}(p)\bar{\lambda}_{\{+-\}}^{A}(p) \\
&\quad - \lambda_{\{-+\}}^{A}(p)\bar{\lambda}_{\{-+\}}^{A}(p))\gamma_0 S^\dagger(\Lambda)\gamma_0,
\end{aligned}
$$

and therefore

$$\Xi'(p) = S(\Lambda)\Xi(p)S^{-1}(\Lambda), \tag{27}$$

as expected. Once having verified the right transformations, we are able to evince the bilinear quantities. Starting from σ, we have

$$
\begin{aligned}
\sigma'_E &= \overline{\lambda}_h^{\prime S/A}(p')\lambda_h^{\prime S/A}(p') \\
&= \lambda_h^{\dagger S/A}(p)S^\dagger(\Lambda)S^{-1\dagger}(\Lambda)\Xi^\dagger(p)S^\dagger(\Lambda)\gamma_0 S(\Lambda)\lambda_h^{S/A}(p) \\
&= \overline{\lambda}_h^{S/A}(p)\lambda_h^{S/A}(p), \\
&= \sigma_E,
\end{aligned}
$$

implying σ to be a scalar. Repeating the same procedure for the remaining bilinear forms, we obtain

$$J'_{\mu_E} \to \Lambda_\mu^\nu \overline{\lambda}_h(p)\gamma_\nu\lambda_h(p), \quad \text{(Vector)},$$

$$\omega'_E \to \det(\Lambda)i\overline{\lambda}_h(p)\Xi(p)\gamma_5\Xi(p)\lambda_h(p), \quad \text{(Scalar)},$$

$$K'_{\mu_E} \to -\det(\Lambda)\Lambda^\nu{}_\mu i\overline{\lambda}_h(p)\Xi(p)\gamma_5\gamma_\nu\Xi(p)\lambda_h(p),$$
$$\text{(Vector)},$$

$$S'_{\mu\nu_E} \to \frac{i}{2}\Lambda^\alpha{}_\rho \Lambda^\beta{}_\vartheta \overline{\lambda}_h(p)\Xi(p)\gamma_\alpha\gamma_\beta\Xi(p)\lambda_h(p),$$
$$\text{(Bivector)}.$$

Therefore, the nomenclature previously adopted is indeed adequate to the present case. We shall conclude pointing out that the investigation of covariance, which concerns the SIM(2), HOM(2) Lorentz subgroups, is taken into account in Appendix B. The analysis is quite analogous, and the physical statements are essentially the same.

4 Final remarks

In this paper we have shown that it is necessary to deform the usual Clifford algebra in order to ascertain the right observance of the Fierz–Pauli–Kofink identities, regarding Elko spinor fields. As a result, only a subset of bilinear covariants are real. After have found the proper deformation, and having observed its dependence on the new dual operator, we study the covariance of the relevant quantities showing explicitly its behavior under a typical transformation of the relativistic group in question.

We would like to conclude this paper by returning to the point raised at the end of the introductory section. As already mentioned, in Refs. [29,30] a subtle way is described to evade Weinberg's no-go theorem, by exploring another possibility of the dual structure, this time constructing the dual with the additional requirement of Lorentz invariant spin sums. Additional mathematical support was given in [28]. In the formulation presented in [29], great care was taken in treating the new dual structure in order not to jeopardize the simplest bilinear σ, i.e., the spinor was required to be an eigenspinor of the new operator, say $\mathcal{O}$, entering into the dual redefinition.[4] By assuming associativity between $\mathcal{O}$ and S, whose meaning is the same of Sect. 3 (except that the full Lorentz group is the relativistic group at hand), and that Eq. (23) is still holding, we have

$$(i\gamma_\mu \Lambda^\mu_\beta \partial'^\beta \Xi(\boldsymbol{p}) \pm m)(S(\Lambda)\mathcal{O})^{-1}\mathcal{O}'\lambda'^{S/A}_\alpha = 0, \tag{28}$$

which can be recast into the form

$$(i\gamma'_\beta \partial'^\beta S(\Lambda)\Xi(\boldsymbol{p})\mathcal{O}^{-1}S^{-1}(\Lambda)\mathcal{O}'$$
$$\pm m S(\Lambda)(S(\Lambda)\mathcal{O})^{-1}\mathcal{O}')\lambda'^{S/A}_\alpha = 0. \tag{29}$$

Therefore the following identifications immediately hold:

$$S(\Lambda)\mathcal{O}^{-1}S^{-1}(\Lambda)\mathcal{O}' = 1, \tag{30}$$

$$\Xi'(\boldsymbol{p}) = S(\Lambda)\Xi(\boldsymbol{p})(S(\Lambda)\mathcal{O})^{-1}\mathcal{O}'. \tag{31}$$

From (30) one sees that the new operator must transform as $\mathcal{O}' = S(\Lambda)\mathcal{O}S^{-1}(\Lambda)$, leading to the same transformation rule of (26) for the operator $\Xi(\boldsymbol{p})$. In this vein the covariance of (23) is ensured. The whole covariant analysis of the corresponding bilinear quantities then follows immediately, leading essentially to the same results.

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4 In this context, we work in an abstract way, denoting the new operators $\mathcal{A}$ and $\mathcal{B}$, defined in [29], by $\mathcal{O}$, aiming to simplify the notation.

Appendix A: A brief overview: the Elko formal structure

Elko spinors are eigenspinors of the charge conjugation operator C

$$C\lambda^{S/A}(\boldsymbol{p}) = \pm\lambda^{S/A}(\boldsymbol{p}), \tag{A1}$$

where

$$C = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K, \tag{A2}$$

and K takes the complex conjugates of any spinor that appears on the right side, while Θ is the Wigner time-reversal operator in the spin $1/2$ representation, which is given by [1,33]

$$\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A3}$$

It is worth mentioning that Elko spinors form a complete set of eigenspinors of C with positive (S) and negative (A) eigenvalues. Another important aspect of these spinors is that they are constructed in such a way that they carry dual helicity.

From the last considerations, the explicit form of Elko spinors, in an arbitrary referential, is

$$\lambda^S_{\{\pm,\mp\}}(\boldsymbol{p}) = \Lambda_\pm(p^\mu)\lambda^S_{\{\pm,\mp\}}(\boldsymbol{0}) \tag{A4}$$

and

$$\lambda^A_{\{\pm,\mp\}}(\boldsymbol{p}) = \Lambda_\pm(p^\mu)\lambda^A_{\{\pm,\mp\}}(\boldsymbol{0}). \tag{A5}$$

The action of the boost operator $\Lambda_\pm(p^\mu)$ over the spinors, in the rest-frame, is given by

$$\Lambda_\pm(p^\mu)\lambda^{S/A}_{\{\pm,\mp\}}(\boldsymbol{0}) = \underbrace{\sqrt{\frac{E+m}{2}}\left(1 \pm \frac{p}{E+m}\right)}_{\text{boost factor}}\lambda^{S/A}_{\{\pm,\mp\}}(\boldsymbol{0}). \tag{A6}$$

To summarize the notation, we choose to define the boost factor by

$$\Upsilon_\pm = \sqrt{\frac{E+m}{2}}\left(1 \pm \frac{p}{E+m}\right). \tag{A7}$$

The four rest spinors are shaped by two *self-conjugate*

$$\lambda^S_{\{-,+\}}(\boldsymbol{0}) = \begin{pmatrix} +i\Theta[\phi^+_L(\boldsymbol{0})]^* \\ \phi^+_L(\boldsymbol{0}) \end{pmatrix},$$
$$\lambda^S_{\{+,-\}}(\boldsymbol{0}) = \begin{pmatrix} +i\Theta[\phi^-_L(\boldsymbol{0})]^* \\ \phi^-_L(\boldsymbol{0}) \end{pmatrix}, \tag{A8}$$

and the other two are *anti-self-conjugate*

$$\lambda^A_{\{-,+\}}(\mathbf{0}) = \begin{pmatrix} -i\Theta[\phi^+_L(\mathbf{0})]^* \\ \phi^+_L(\mathbf{0}) \end{pmatrix},$$

$$\lambda^A_{\{+,-\}}(\mathbf{0}) = \begin{pmatrix} -i\Theta[\phi^-_L(\mathbf{0})]^* \\ \phi^-_L(\mathbf{0}) \end{pmatrix}. \tag{A9}$$

The associated components are defined by imposing them to be eigenfunctions of the helicity operator, resulting in the left-hand components given by

$$\phi^+_L(\mathbf{0}) = \sqrt{m}\begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix},$$

$$\phi^-_L(\mathbf{0}) = \sqrt{m}\begin{pmatrix} -\sin(\theta/2)e^{-i\phi/2} \\ \cos(\theta/2)e^{i\phi/2} \end{pmatrix} \tag{A10}$$

and the right-hand components presented by

$$\phi^+_R(\mathbf{0}) = \sqrt{m}\begin{pmatrix} -i\sin(\theta/2)e^{-i\phi/2} \\ i\cos(\theta/2)e^{i\phi/2,} \end{pmatrix},$$

$$\phi^-_R(\mathbf{0}) = \sqrt{m}\begin{pmatrix} -i\cos(\theta/2)e^{-i\phi/2} \\ -i\sin(\theta/2)e^{i\phi/2} \end{pmatrix}. \tag{A11}$$

Once having addressed the formal structure for the Elko spinors, we can now define the dual by the following general formula:

$$\overset{\neg S/A}{\lambda}_h(\mathbf{p}) = [\Xi(\mathbf{p})\lambda^{S/A}_h(\mathbf{p})]^\dagger\gamma_0, \tag{A12}$$

with the operator $\Xi(\mathbf{p})$ defined as

$$\Xi(\mathbf{p}) \equiv \frac{1}{2m}\Big(\lambda^S_{\{+-\}}(\mathbf{p})\bar{\lambda}^S_{\{+-\}}(\mathbf{p}) + \lambda^S_{\{-+\}}(\mathbf{p})\bar{\lambda}^S_{\{-+\}}$$
$$(\mathbf{p}) - \lambda^A_{\{+-\}}(\mathbf{p})\bar{\lambda}^A_{\{+-\}}(\mathbf{p}) - \lambda^A_{\{-+\}}(\mathbf{p})\bar{\lambda}^A_{\{-+\}}(\mathbf{p})\Big), \tag{A13}$$

which in a matrix form reads

$$\Xi(\mathbf{p}) = \begin{pmatrix} \frac{ip\sin(\theta)}{m} & \frac{-i(E+p\cos(\theta))e^{-i\phi}}{m} & 0 & 0 \\ \frac{i(E-p\cos(\theta))e^{i\phi}}{m} & \frac{-ip\sin(\theta)}{m} & 0 & 0 \\ 0 & 0 & \frac{-ip\sin(\theta)}{m} & \frac{-i(E-p\cos(\theta))e^{-i\phi}}{m} \\ 0 & 0 & \frac{i(E+p\cos(\theta))e^{i\phi}}{m} & \frac{ip\sin(\theta)}{m} \end{pmatrix}, \tag{A14}$$

where $\Xi^2(\mathbf{p}) = \mathbb{I}$, and $\Xi^{-1}(\mathbf{p})$ indeed exists and is equal to $\Xi(\mathbf{p})$ itself [29]. Now, the explicit form for the dual can be readily written as

$$\overset{\neg S/A}{\lambda}_{\{-,+\}}(\mathbf{p}) = +i[\lambda^{S/A}_{\{+,-\}}(\mathbf{p})]^\dagger\gamma_0, \tag{A15}$$

$$\overset{\neg S/A}{\lambda}_{\{+,-\}}(\mathbf{p}) = -i[\lambda^{S/A}_{\{-,+\}}(\mathbf{p})]^\dagger\gamma_0. \tag{A16}$$

For more details as regards the whole construction, please see [29].

Appendix B: VSR Elko spinors

As is extensively shown in [2,3], Elko spinors can be understood as objects carrying a linear representation of SIM(2) or HOM(2) Lorentz subgroups. In this vein, it would be important to explore a possible Clifford algebra basis deformation in this case, if necessary.

In order to illustrate the situation, we shall make use of the Elko spinors found in [3]. Notice that, in general, the helicity operator does not commute with the VSR boost. Therefore, one cannot freely choose the rest spinors as a basis for such an operator. This is the kernel of the subtle difference between the Elko spinors studied previously and the spinors here investigated.

For an arbitrary momentum, it is possible to see that in the VSR scope we have [3]

$$\chi^S_{\{-,+\}}(\mathbf{p}) = \sqrt{m}\begin{pmatrix} i\frac{p_x-ip_y}{\sqrt{m(p_0-p_z)}}e^{i\phi/2} \\ i\sqrt{\frac{p_0-p_z}{m}}e^{i\phi/2} \\ \sqrt{\frac{p_0-p_z}{m}}e^{-i\phi/2} \\ -\frac{p_x+ip_y}{\sqrt{m(p_0-p_z)}}e^{-i\phi/2} \end{pmatrix} \tag{B1}$$

as a prototype spinor and

$$\overset{\neg S}{\chi}_{\{-,+\}}(\mathbf{p}) = \sqrt{m}\begin{pmatrix} 0 & -i\frac{e^{-i\phi/2}}{\sqrt{\frac{p_0-p_z}{m}}} & \frac{e^{i\phi/2}}{\sqrt{\frac{p_0-p_z}{m}}} & 0 \end{pmatrix} \tag{B2}$$

as its dual, whose construction obeys the same previous prescription, i.e., $\chi^{S/A}_h(\mathbf{p}) = [\Xi(\mathbf{p})_{VSR}\chi^{S/A}_h(\mathbf{p})]^\dagger\gamma_0$. All the aspects investigated in the main text have a parallel here. The only differences are the general boost of VSR, which reads

$$\mathcal{V}_{VSR} = \begin{pmatrix} \sqrt{\frac{m}{p_0-p_z}} & \frac{p_x-ip_y}{\sqrt{m(p_0-p_z)}} & 0 & 0 \\ 0 & \sqrt{\frac{p_0-p_z}{m}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{p_0-p_z}{m}} & 0 \\ 0 & 0 & -\frac{p_x+ip_y}{\sqrt{m(p_0-p_z)}} & \sqrt{\frac{m}{p_0-p_z}} \end{pmatrix}, \tag{B3}$$

and the $\Xi(\mathbf{p})_{VSR}$ operator. In order to find this last operator, it is possible to see that starting from Eq. (A13), but replacing the usual spinors by VSR Elko spinors (just as (B1)) in the composition of $\Xi(\mathbf{p})_{VSR}$, one arrives at

$$\Xi(\boldsymbol{p})_{VSR} = \begin{pmatrix} \frac{i(p_x-ip_y)e^{i\phi}}{m} & -\frac{i(p_x-ip_y)^2 e^{i\phi}}{m(p_0-p_z)} - \frac{ime^{-i\phi}}{p_0-p_z} & 0 & 0 \\ \frac{i(p_0-p_z)e^{i\phi}}{m} & -\frac{i(p_x-ip_y)e^{i\phi}}{m} & 0 & 0 \\ 0 & 0 & -\frac{i(p_x+ip_y)e^{-i\phi}}{m} & \frac{-i(p_0-p_z)e^{-i\phi}}{m} \\ 0 & 0 & \frac{i(p_x+ip_y)^2 e^{-i\phi}}{m(p_0-p_z)} + \frac{ie^{i\phi}m}{p_0-p_z} & \frac{i(p_x+ip_y)e^{-i\phi}}{m} \end{pmatrix}. \tag{B4}$$

This operator acts in VSR spinors just as the $\Xi(\boldsymbol{p})$ operator does in the usual Elko spinors: it changes the spinor type, obeys $\Xi^2(\boldsymbol{p})_{VSR} = 1$, and has determinant equal to 1, ensuring the existence of the inverse.

With such ingredients, it is possible to see that all the previous steps are repeated here, namely: the FPK identities are not respected, except if the Clifford basis undergoes the same deformation (with $\Xi(\boldsymbol{p})_{VSR}$ replacing $\Xi(\boldsymbol{p})$); σ_{VSR} and K_{VSR}^{μ} are non-null real quantities, ω_{VSR} is zero, and the remaining bilinears are imaginary. Besides, as Eq. (23) still holds, the covariance structure is also the same, provided we use (B3) instead of the usual Lorentz transformation.

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