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A hybrid symbolic-numerical approach to the center-focus problem

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We propose a new hybrid symbolic-numerical approach to the center-focus problem. The method allowed us to obtain center conditions for a three-dimensional system of differential equations, which was previously not possible using traditional, purely symbolic computational techniques.

1. Introduction

1.1. Background

Determination of the local stability of an isolated singular point for a system of ordinary differential equations (ODEs) is one of the fundamental problems encountered across various branches of applied sciences and engineering. For a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{1}$$

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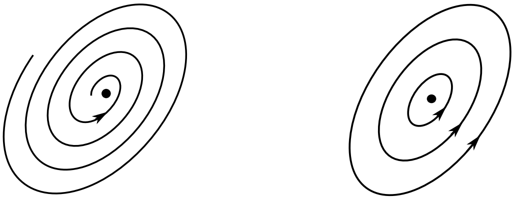

Fig. 1. An example of a stable focus (a) and a center (b).

where $\mathbf{f}: \mathbb{R}^n \supset \Delta \to \mathbb{R}^n$ is smooth, and x_0 is a singularity, i.e. $\mathbf{f}(\mathbf{x}_0) = 0$, the celebrated Hartman–Grobman theorem, e.g., see Chicone (2006), states that the linearization of (1) characterize the local qualitative behavior of the trajectories when x_0 is hyperbolic. That is, the set of eigenvalues $\lambda_1, \dots, \lambda_n$ of the Jacobian matrix $D\mathbf{f}(\mathbf{x}_0)$ describes the local behavior when the eigenvalues have non-zero real part, i.e., $Re(\lambda_j) \neq 0$. If $Re(\lambda_j) = 0$ for some j, then x_0 is called nonhyperbolic and the local stability is determined by the higher order terms.

One of the simplest and well-known stability questions is the *center-focus* (or *center*) problem, originally defined for planar polynomial differential systems, i.e., system (1) when $n = 2$ and $\mathbf{f}$ is a system of 2 polynomials in $\mathbb{R}[\mathbf{x}]$ of some degree m. It consists of obtaining conditions on the coefficients of $\mathbf{f}(\mathbf{x})$ to distinguish between a local focus (see Fig. 1(a)) or a center (see Fig. 1(b)), which has been the subject of intensive research, e.g., Żołądek (1994), Christopher (1994), Wang (1999), Romanovski and Shafer (2009), Valls (2015), Giné and Valls (2016), Algaba et al. (2014). Although the problem is open in its full generality, it has been solved for some important subclasses of planar polynomial vector fields. As an example, consider the quadratic system defined by

$$\begin{aligned} \dot{u} &= \quad v + a_1 u^2 + a_2 uv + a_3 v^2 \\ \dot{v} &= -u + a_4 u^2 + a_5 uv + a_6 v^2, \end{aligned} \tag{2}$$

where $a_1, \dots, a_6 \in \mathbb{R}$. The center conditions were established by Dulac (1908) and Kapteyn (1912). It is well-known (see e.g. Żołądek, 1994; Romanovski and Shafer, 2009) that, for system (2), the so-called Bautin ideal $\mathscr{B}$ is generated by the first three focus quantities of this system (Bautin, 1952). Moreover, the center variety $\mathbf{V}(\mathscr{B}) \subset \mathbb{R}^6$ decomposes into four irreducible components:

$$\mathbf{V}(\mathscr{B}) = \mathbf{V}(I_{Ham}) \cup \mathbf{V}(I_{sym}) \cup \mathbf{V}(I_{\triangle}) \cup \mathbf{V}(I_{con}),$$

corresponding to Hamiltonian systems, reversible systems, the Zariski closure of systems having three invariant lines, and the Zariski closure of systems having an invariant conic and an invariant cubic, respectively.

The center-focus problem can also be defined for higher dimensional systems and has recently been studied for a number of three-dimensional families (Edneral et al., 2012; Buică et al., 2011; García et al., 2013; Mahdi, 2013; Mahdi et al., 2011, 2013). We continue this study here by applying our new symbolic-numerical approach to a three-dimensional system presented in Sec. 1.3 with results presented in Theorems 1 and 3.

1.2. Computational challenges and the new approach

The process of solving the center-focus problem for a specific system of differential equations can be divided into three steps (Christopher and Li, 2007). The first step is to compute some finite number, say $p \in \mathbb{N}$, of *focus quantities* (also called *Lyapunov quantities*), which are polynomials in the parameters of the system. The second step is to compute the irreducible components of the solution set defined by these focus quantities. Since the vanishing of these finitely many polynomials is a necessary condition for a center, the third step is to check each component using additional conditions for the existence of a center. This typically involves the application of the Darboux theory of integrability or reduction to the center manifold.

Techniques for efficient computation of Lyapunov quantities has been motivated both by mathematical and engineering problems. Over the years, a number of algorithms have been developed (Wang, 1991; Romanovskii, 1993; Pearson et al., 1996; Gasull and Prohens, 1997; Gasull and Torregrosa, 2001; Lynch, 2005; Kuznetsov and Leonov, 2008; Yu and Chen, 2008). In this work, we used the approach described in Edneral et al. (2012) for computing the focus quantities for a system in dimension three, which is based on the equivalence of the existence of a center and a local analytic first integral in the neighborhood of a singular point (more details are provided in Sec. 2). The advantage of this approach is that it avoids center manifold approximation, which is especially important since its power series approximation of analytic or even polynomial systems need not converge (e.g., see Aulbach, 1985; Sijbrand, 1985; Mahdi et al., 2013).

From the computational point of view, the biggest obstacle in solving the center-focus problem for a specific system is the determination of the irreducible components of the variety (i.e., solution set) defined by a certain number of focus quantities. The most common approach (Aziz and Christopher, 2012; Giné et al., 2014; Ferčec et al., 2014) is the application of computer algebra algorithms for computing the primary decomposition of the ideal generated by the focus quantities such as Gianni–Trager–Zacharias (GTZ) (Gianni et al., 1988) or Shimoyama–Yokoyama (SY) (Shimoyama and Yokoyama, 1996), which have been implemented in various symbolic packages (e.g. Singular, Greuel et al., 2005, or Macaulay2, Grayson and Stillman, 2002). The computational difficulty related with Gröbner basis calculation over the field of characteristic zero was eased by implementation of modular arithmetics (Winkler, 1988; Edneral, 1997; Romanovski and Prešern, 2011), and successfully used in numerous problems (Ferčec et al., 2011; Han and Romanovski, 2012; Valls, 2015). Unfortunately, in practice, the application of algorithms that use Gröbner bases (also with modular arithmetics) is computationally very heavy and the center conditions can only be obtained for specific systems with few parameters. In this paper, we replace this particular step and find the common zeros of the polynomial systems formed by focus quantities using numerical algebraic geometry techniques (for more details, see Sec. 3 and the books, Bates et al., 2013b; Sommese and Wampler, 2005). The parallelizablity of numerical algebraic geometry together with a regeneration based approach (Hauenstein et al., 2011a; Hauenstein and Wampler, 2017) and exactness recovery (Bates et al., 2013a) provides a natural alternative to Gröbner basis methods. In particular, for the first time, we are able to solve the center-focus problem for a quadratic, three-dimensional system described next.

1.3. An application

Consider a third-order differential equation of the form

$$\dddot{u} = \ddot{u} + \dot{u} + u + f(u, \dot{u}, \ddot{u}), \tag{3}$$

where $f = f(u, \dot{u}, \ddot{u}) \in \mathbb{R}[u, \dot{u}, \ddot{u}]$ is a polynomial of degree m. Following Mahdi (2013), we can equivalently write

$$\dot{u} = -v + h(u, v, w), \qquad \dot{v} = u + h(u, v, w), \qquad \dot{w} = -w + h(u, v, w), \tag{4}$$

where $h(u, v, w) = f(-u + w, v - w, u + w)/2$, which we call the *standard form* of system (3). Note that the origin of (4) is a nonhyperbolic singularity at which the associated Jacobian has two purely imaginary eigenvalues $\lambda_{1,2} = \pm i$ and $\lambda_3 = -1$. Various dynamic aspects of systems of the form (4) have recently been considered, including the center conditions (Buică et al., 2011; Dias and Mello, 2010; Edneral et al., 2012; Mahdi et al., 2011), limit cycle bifurcations (Wang et al., 2010; Mahdi et al., 2013), Lie symmetries (García et al., 2013), and isochronicity (Romanovski et al., 2013). In particular, the center conditions on the local center manifold for system (4), where

$$h(u, v, w) = a_1 u^2 + a_2 v^2 + a_3 w^2 + a_4 uv + a_5 uw + a_6 vw, \tag{5}$$

were studied in Mahdi (2013). Although it was possible to compute the first eight focus quantities, standard symbolic algorithms (e.g. GTZ and SY) were not able to provide the decomposition of the Bautin ideal into primes for a general six-parameter system, even over fields with non-zero characteristics. On the other hand, the application of our hybrid approach using numerical algebraic geometry to decompose described in this paper, allowed us to obtain the center conditions for a general six-parameter system (4).

Theorem 1. *The system (4) with $h(u, v, w)$ as in (5) admits a center on the local center manifold if and only if one of the following holds:*

(1) $a_1 = a_2 = a_4 = 0$
(2) $a_1 - a_2 = a_3 = a_5 = a_6 = 0$
(3) $a_1 + a_2 = a_3 = a_5 = a_6 = 0$
(4) $a_1 + a_2 = 2a_2 - a_3 + a_6 = a_3 - a_4 - 2a_5 = 2a_4 + 3a_5 + a_6 = 0$
(5) $2a_1 - a_6 = 2a_2 + a_5 = 2a_3 - a_5 + a_6 = a_4 + a_5 + a_6 = 0$
(6) $a_1 - a_2 = 2a_2 + a_6 = a_4 = a_5 + a_6 = 0$
(7) $2a_1 + a_2 = 2a_2 + a_6 = 4a_3 + 5a_6 = a_4 = 2a_5 - a_6 = 0.$

We leave the proof of the theorem to the end of Sec. 4. Nonetheless, an easy conclusion is that each irreducible component of the center variety (i.e., the variety of the Bautin ideal generated by the focus quantities) of system (4) for quadratic h (5) are vector subspaces of its six-dimensional parameter space, which was conjectured in Mahdi (2013).

1.4. Outline

The rest of the paper is organized as follows. Section 2 summarizes focus quantities and their computation. Section 3 summarizes the numerical algebraic geometric solving approach along with exactness recovery method used to prove Theorem 3 in Section 4. Appendix A presents the Dulac–Kapteyn criterion of quadratic planar systems with Appendix B summarizing Darboux theory of integrability. Appendix C performs a step-by-step computation of the center conditions for an illustrative example.

2. Focus quantities computation in $\mathbb{R}^3$

This section is a review of the method described in Edneral et al. (2012) (see also Mahdi, 2013; Mahdi et al., 2013) for studying the center problem on a center manifold for vector fields in dimension three. Let $X : \mathbb{R}^3 \supset U \to \mathbb{R}^3$ be a real analytic vector field, such that $DX(0)$ has one non-zero and two purely imaginary eigenvalues. By an invertible linear change of coordinates and a possible rescaling of time, the system of differential equations $\dot{\mathbf{u}} = X(\mathbf{u})$ can be written in the form

$$\begin{aligned} \dot{u} &= -v + P(u, v, w) \\ \dot{v} &= u + Q(u, v, w) \\ \dot{w} &= \beta w + R(u, v, w), \end{aligned} \tag{6}$$

where β is a non-zero real number. Let $X = (-v + P)\partial/\partial u + (u + Q)\partial/\partial v + (\beta w + R)\partial/\partial w$ denote the corresponding vector field. A *local first integral* of system (6) is a nonconstant differentiable function $H : \mathbb{R}^3 \supset U \to \mathbb{R}$ that is constant on trajectories of (6), equivalently, H satisfies on $U \subset \mathbb{R}^3$ the equality

$$XH := (-v + P)\frac{\partial H}{\partial u} + (u + Q)\frac{\partial H}{\partial v} + (\beta w + R)\frac{\partial H}{\partial w} \equiv 0. \tag{7}$$

A *formal first integral* for system (6) is a non-constant formal power series H in u, v and w such that when P, Q, and R are expanded in power series, every coefficient in the formal power series in (7) is zero.

Recall that system (6) admits a local center manifold W^c_{loc} at the origin (Kuznetsov, 2004, Thm. 5.1). One of the main tools for detecting a center on a center manifold is the following theorem:

Theorem 2. *The following statements are equivalent.*

(a) *The origin is a center for $X\,|_{W^c_{loc}}$.*
(b) *System (6) admits a local analytic first integral at the origin.*
(c) *System (6) admits a formal first integral at the origin.*

For a proof see Bibikov (1979), Edneral et al. (2012). In fact, a real analytic local first integral from statement (b) (as well as a formal first integral from statement (c)) can always be chosen to be of the form $H(u, v, w) = u^2 + v^2 + \cdots$ where the dots mean higher order terms in a neighborhood of the origin in $\mathbb{R}^3$.

The equivalence of statements (a) and (b) is called the *Lyapunov Center Theorem* with a proof presented in, e.g., Bibikov (1979). By this theorem, we can restrict our efforts to investigate the conditions for the existence of a first integral H which is equivalent to determine necessary and sufficient conditions for the existence of a center or a focus on the local center manifold.

From now on, we assume that P, Q and R in (6) are polynomials of degree at most n. We begin by introducing the complex variable $x = u + iv$. The first two equations in (6) are equivalent to a single equation $\dot{x} = ix + \cdots$, where the dots represent a sum of homogeneous polynomials of degrees between 2 and n. Let $\bar{x}$ denote the complex conjugate of x. We add to this equation its complex conjugate, replacing $\bar{x}$ everywhere by y which is regarded as an independent complex variable and replacing w by z simply as a notational convenience. This yields the following complexification of (6):

$$\begin{aligned}\dot{x} &= ix + \sum_{p+q+r=2}^{n} a_{pqr} x^p y^q z^r,\\ \dot{y} &= -iy + \sum_{p+q+r=2}^{n} b_{pqr} x^p y^q z^r,\\ \dot{z} &= \beta z + \sum_{p+q+r=2}^{n} c_{pqr} x^p y^q z^r,\end{aligned} \tag{8}$$

where $b_{qpr} = \bar{a}_{pqr}$ and c_{pqr} are such that $\sum_{p+q+r=2}^{n} c_{pqr} x^p \bar{x}^q w^r$ is real for all $x \in \mathbb{C}$ and $w \in \mathbb{R}$. Let X be the corresponding vector field of system (8) on $\mathbb{C}^3$. Existence of a first integral $H(u, v, w) = u^2 + v^2 + \cdots$ for system (6) is equivalent to the existence of a first integral for system (8), denoted again by H, of the form

$$H(x, y, z) = xy + \sum_{j+k+\ell \ge 3} v_{jkl} x^j y^k z^\ell. \tag{9}$$

We now investigate the existence of a first integral H for system (8) by computing the coefficients of XH and equating them to zero. When H has the form (9), the coefficient $g_{jk\ell}$ of $x^j y^k z^\ell$ in XH can be calculated explicitly (see Edneral et al., 2012). Except when $j = k$ and $\ell = 0$, the equation $g_{jk\ell} = 0$ can be solved uniquely for $v_{jk\ell}$ in terms of the known quantities $v_{\alpha\beta\gamma}$ with $\alpha + \beta + \gamma < j + k + \ell$. A formal first integral H thus exists if and only if $g_{KK0} = 0$ for all $K \in \mathbb{N}$. Thus, an obstruction to the existence of the formal series H occurs if some g_{KK0} is non-zero. This coefficient is the Kth *focus quantity* and it can be expressed as

$$g_{KK0} = \sum_{\substack{j+k=2\\ j\ge 0, k\ge 0}}^{2K-1} \left(j\, a_{K-j+1,K-k,0} + k\, b_{K-j,K-k+1,0}\right) v_{jk0} \;+\; \sum_{\substack{j+k=2\\ j\ge 0, k\ge 0}}^{2K-2} c_{K-j,K-k,0}\, v_{jk1}, \tag{10}$$

where we have made the natural assignments $v_{\alpha\beta\gamma} = 0$ for $\alpha + \beta + \gamma = 2$ except $v_{110} = 1$. It is easy to verify that $g_{110} = 0$. The coefficient g_{220} is uniquely determined but the remaining ones depend on the choices made for v_{KK0}, $K \in \mathbb{N}_{\geq 2}$. Hence, once such an assignment is made, H is determined and satisfies

$$XH(x, y, z) = g_{220}(xy)^2 + g_{330}(xy)^3 + \cdots .$$

In summary, the vanishing of all focus quantities, i.e.,

$$g_{KK0} = 0 \qquad \text{for} \qquad K \geq 2 \tag{11}$$

is both a necessary and sufficient condition for the existence of a center on the center manifold, otherwise there is a focus (see Edneral et al., 2012).

By Hilbert's basis theorem, there exists $K_0 \geq 2$ such that the set of solutions of $g_{KK0} = 0$ for all $2 \leq K \leq K_0$ is equivalent to the set defined by an infinite system (11). Since such a K_0 is not known *a priori*, we will apply an iterative approach that solves $g_{KK0} = 0$ for $2 \leq K \leq M + 1$ given the solution set of $g_{KK0} = 0$ for $2 \leq K \leq M$.

3. Numerical algebraic geometry

As shown above, we are faced with computing the solution set to a system consisting of finitely many polynomial equations yielding a problem in computational algebraic geometry which consists of two general approaches: symbolic and numerical methods. Symbolic methods, such as Gröbner basis techniques, take an algebraic viewpoint for solving systems of polynomial equations. In broad terms, they manipulate equations to obtain new relations describing the solution set. Alternatively, numerical algebraic geometry follows a geometric viewpoint by manipulating solution sets which are represented by *witness sets* described below. A more detailed comparison of symbolic and numerical approaches is provided in Bates et al. (2014) with the books (Bates et al., 2013b; Sommese and Wampler, 2005) providing more details about the following discussion. These computations can be performed using Bertini (Bates et al., 2006).

Following the notation of Sec. 2, we want to solve $F_M := \{g_{220}, \ldots, g_{MM0}\} = 0$ for some given $M \geq 2$. To do this, we will follow a regenerative intersection approach developed in Hauenstein and Wampler (2013, 2017) which builds on the diagonal intersection (Sommese et al., 2004) and the regenerative cascade (Hauenstein et al., 2011a, 2011b). The first step is to solve $F_2 = 0$, which in the numerical algebraic geometric context means to compute witness sets for the irreducible components of this solution set, a so-called *numerical irreducible decomposition*.

Geometrically, for any polynomial system G, $\mathcal{V}(G)$ can be decomposed into a union of irreducible components $\mathcal{V}(G) = \cup_{i=1}^{r} V_i$. This decomposition corresponds algebraically to a prime decomposition of the radical ideal generated by G, namely $\sqrt{I(G)} = \cap_{i=1}^{r} I(V_i)$. A numerical irreducible decomposition is simply a collection of witness sets, one for each irreducible component V_i of $\mathcal{V}(G)$. In the commonly used algorithms to compute a numerical irreducible decomposition, one first computes witness sets for the pure-dimensional components of $\mathcal{V}(G)$. Each pure-dimensional component is then decomposed into its irreducible components using monodromy and a trace test.

A witness set for $V \subset \mathbb{C}^N$, a pure-dimensional component of $\mathcal{V}(G)$ for some polynomial system G, is the triple $\{G, \mathcal{L}, W\}$ where $\mathcal{L} \subset \mathbb{C}^N$ is general linear subspace of codimension $d = \dim V$ and $W = V \cap \mathcal{V}(\mathcal{L})$ so that $|W| = \deg V$. Here, the definition of general means that $\mathcal{L}$ intersects V transversely, which is a Zariski open condition on the Grassmannian of codimension d linear subspaces in $\mathbb{C}^N$.

For the particular application, since $F_2 = \{g_{220}\}$, $\mathcal{V}(F_2)$ is a hypersurface, which is pure-dimensional. In fact, since F_2 is an irreducible polynomial, $\mathcal{V}(F_2)$ is an irreducible hypersurface. By restricting to a line, a witness set for $\mathcal{V}(F_2)$ can easily by computed by solving a univariate polynomial.

Given a numerical irreducible decomposition of $\mathcal{V}(F_{k-1})$, the regenerative intersection approach computes a numerical irreducible decomposition of $\mathcal{V}(F_k) = \mathcal{V}(F_{k-1}) \cap \mathcal{V}(g_{kk0})$ as follows. Suppose that V is an irreducible component of $\mathcal{V}(F_{k-1})$ with witness set $\{F_{k-1}, \mathcal{L}, W\}$. We first need to test if $V \subset \mathcal{V}(g_{kk0})$ which, due to the genericity of $\mathcal{L}$, is equivalent to $g_{kk0}(w) = 0$ for $w \in W$. When $V \subset \mathcal{V}(g_{kk0})$, then V is an irreducible component of F_k with witness set $\{F_k, \mathcal{L}, W\}$.

If V is not contained in $\mathcal{V}(g_{kk0})$, then $V \cap \mathcal{V}(g_{kk0})$ is either empty or pure-dimensional of dimension $d - 1$ where $d = \dim V$. Suppose that $\mathcal{L} = \mathcal{K} \cap \mathcal{H}$ where $\mathcal{K}$ is a general linear space of codimension

$d-1$ and $\mathcal{H}$ is a general hyperplane. We compute $V \cap \mathcal{V}(g_{kk0}) \cap \mathcal{K}$ from $V \cap \mathcal{H} \cap \mathcal{K}$ using regeneration as follows. Let $e = \deg g_{kk0}$ and select general hyperplanes $\mathcal{H}_1, \ldots, \mathcal{H}_e$. The homotopy

$$V \cap (t \cdot \mathcal{H} + (1-t) \cdot \mathcal{H}_i) \cap \mathcal{K}$$

deforms from the known points $V \cap \mathcal{H} \cap \mathcal{K}$ at $t = 1$ to yield $V \cap \mathcal{H}_i \cap \mathcal{K}$ at $t = 0$. We then deform from $\cup_{i=1}^{e} \mathcal{H}_i$ to $\mathcal{V}(g_{kk0})$, both hypersurfaces of degree e, via the homotopy:

$$V \cap (t \cdot \cup_{i=1}^{e} \mathcal{H}_i + (1-t) \cdot \mathcal{V}(g_{kk0})) \cap \mathcal{K}.$$

In summary, this yields $W' = V \cap \mathcal{V}(g_{kk0}) \cap \mathcal{K}$ with $\{F_k, \mathcal{K}, W'\}$ forming a witness set for the pure-dimensional algebraic set $V \cap \mathcal{V}(g_{kk0})$. We can then decompose this into its irreducible components using monodromy and a trace test.

The last step in computing a numerical irreducible decomposition of $\mathcal{V}(F_k)$ is to remove redundant and superfluous components. In particular, by using a membership test (see Bates et al., 2013b, § 8.4), we can compute the inclusion maximal collection of the irreducible components identified in this regenerative intersection which yields a numerical irreducible decomposition for $\mathcal{V}(F_k)$. As an example, regenerative intersection in Step 4 of Appendix C yields a point, namely 0, which is contained in another irreducible component, namely defined by (C.2), and thus $\{0\}$ is not an irreducible component.

By repeating this regeneration, we can compute a numerical irreducible decomposition for $\mathcal{V}(F_M)$. We can then increase $M \geq 2$ until $\mathcal{V}(F_M) = \mathcal{V}(F_{M+1})$ yielding a reasonable guess on when the ideal has stabilized as in Hilbert's basis theorem. Along the way in this process, we can analyze the irreducible components to possibly help simplify the computation. For example, since only real points are of interest to the center problem, we can ignore all irreducible components which do not contain real points. The approach of Hauenstein (2013) uses critical points conditions of the distance function to determine if an irreducible component V, represented by a witness set, contains real points. Thus, if $V \cap \mathbb{R}^N = \emptyset$, then we can disregard this component.

When the input polynomials have exact coefficients, e.g., are rational numbers, one often would like exact output. Although the internal computations and witness sets rely upon numerical approximations, there exist techniques for recovering exact answers. The resulting exact answers can be verified using exact symbolic methods, which is typically computationally inexpensive. For the problems at hand here, we use the exactness recovery technique described in Bates et al. (2013a) which uses a sufficiently accurate numerical approximation of a sufficiently general point on V to compute polynomials with integer coefficients that vanish on V, which is based on using a lattice-base reduction technique such as LLL (Lenstra et al., 1982) or PSLQ (Ferguson and Bailey, 1991).

4. Center conditions for a three dimensional quadratic system

Here we provide a proof of Theorem 1. Without loss of generality, we can always assume that either $a_6 = 0$ or $a_6 = 1$. The latter follows immediately by the change of variables $(u, v, w) \mapsto (x/a_6, y/a_6, z/a_6)$ and rescaling of time $dt = a_6 d\tau$. Thus, the seven cases in Theorem 1 can be split into ten cases, five each for $a_6 = 0$ and $a_6 = 1$. After showing these ten cases, we then related them to the seven cases of Theorem 1.

Theorem 3. *Consider system (4) with $h(u, v, w)$ as in (5).*

When $a_6 = 0$, system (4) admits a center on the local center manifold if and only if one of the following holds:

(a) $a_1 - a_2 = a_3 = a_5 = 0$
(b) $a_1 + a_2 = a_3 = a_5 = 0$
(c) $a_1 = a_2 = a_4 = 0$;
(d) $a_1 + a_2 = 2a_1 + a_3 = 6a_1 - a_4 = 4a_1 + a_5 = 0$
(e) $a_1 = a_2 + a_3 = 2a_2 - a_4 = 2a_2 + a_5 = 0$.

When $a_6 = 1$, system (4) admits a center on the local center manifold if and only if one of the following holds:

(f) $a_1 = a_2 = a_4 = 0$
(g) $2a_1 - 1 = a_4 + a_5 + 1 = 2a_2 + a_5 = 2a_3 - a_5 + 1 = 0$
(h) $2a_1 + 1 = 2a_2 + 1 = a_4 = a_5 + 1 = 0$
(i) $a_1 + a_2 = 4a_2 - a_5 + 3 = 6a_2 + a_4 + 5 = 2a_2 - a_3 + 1 = 0$
(j) $4a_1 - 1 = 2a_2 + 1 = 4a_3 + 5 = a_4 = 2a_5 - 1 = 0.$

Necessary conditions.

We first consider $a_6 = 0$ and take $(a_1, \ldots, a_5) \in \mathbb{P}^4$. Using the notation from Sec. 3, $\mathcal{V}(F_2)$ and $\mathcal{V}(F_3)$ are irreducible of codimension 1 and 2 of degree 2 and 8, respectively. Now, $\mathcal{V}(F_4)$ has codimension 3 and decomposes into the following irreducible components:

- 5 linear spaces, 3 of multiplicity 1 and 2 of multiplicity 3, and
- an irreducible algebraic set of degree 39.

The three linear spaces of multiplicity 1 are (a), (b), and (c). The other two linear spaces are complex conjugates of each other with their union is defined in $\mathbb{P}^4$ by

$$a_1 + a_2 = 4a_2^2 + a_4^2 = a_5 = 0.$$

Since the real points on this union are contained in (c), we only need to further investigate the degree 39 component, say V, which is not contained in $\mathcal{V}(g_{550})$. Regenerating from V to compute $V \cap \mathcal{V}(g_{550})$ yields 189 distinct points in $\mathbb{P}^4$, of which 19 correspond to real points. There are 14 real points that do not lie on (a), (b), or (c) of which only 2 satisfy $g_{660} = 0$, namely (d) and (e). We note that (e) has multiplicity 2 with respect to F_5.

We next consider $a_6 = 1$ and take $(a_1, \ldots, a_5) \in \mathbb{C}^5$. Similar to the case above, $\mathcal{V}(F_2)$ and $\mathcal{V}(F_3)$ are irreducible of codimension 1 and 2 of degree 2 and 8, respectively. Also, $\mathcal{V}(F_4)$ has codimension 3 and decomposes into the following components:

- 3 linear spaces, one having multiplicity 1 and 2 having multiplicity 3, and
- an irreducible algebraic set of degree 41.

The linear space of multiplicity 1 is (f) while the two linear spaces of multiplicity 3 are complex conjugates of each other with their union defined in $\mathbb{C}^5$ by

$$a_1 + a_2 = 4a_2^2 + a_4^2 = 2a_2 + a_4 a_5 = 2a_2 a_5 - a_4 = a_5^2 + 1 = 0.$$

Since there are no real points on this union, we only need to further investigate the degree 41 components, denoted V, which is not contained in $\mathcal{V}(g_{550})$. Regenerating V yields 4 irreducible components of $V \cap \mathcal{V}(g_{550})$ not contained in (f) or the hyperplane $a_5^2 + 1 = 0$. Three of these are the lines (g), (h), and (i) with the fourth being an irreducible curve of degree 244, denoted V', not contained in $\mathcal{V}(g_{660})$. Regenerating V' yields 71 distinct real points not contained in the hyperplane $a_5^2 + 1 = 0$ nor satisfying (f), (g), (h), or (i). Of these, only one satisfies $g_{770} = 0$, namely (j).

Therefore, we have shown that the real points which satisfy $g_{220} = \cdots = g_{770} = 0$ are contained in (a)–(e) when $a_6 = 0$ and (f)–(j) when $a_6 = 1$.

Sufficient conditions.

Cases (a) and (b). If the condition (a) (resp. (b)) holds, system (4) reduces to

$$\begin{aligned}
\dot{u} &= -v + a_1 u^2 + a_2 v^2 + a_4 u v, \\
\dot{v} &= u + a_1 u^2 + a_2 v^2 + a_4 u v, \\
\dot{w} &= -w + a_1 u^2 + a_2 v^2 + a_4 u v,
\end{aligned}$$

with $a_2 = a_1$ (resp. $a_2 = -a_1$). Note that by Theorem 2, it is enough to show that this system admits a local analytic first integral at the origin. Since the first two equations are decoupled from the third we only need to show that

$$\dot{u} = -v + a_1 u^2 + a_2 v^2 + a_4 uv, \quad \dot{v} = u + a_1 u^2 + a_2 v^2 + a_4 uv, \tag{12}$$

admits a local analytic first integral. In fact, if $a_4 \neq 0$ and $a_2 = a_1$, system (12) has the inverse integrating factor

$$V(u, v) = -a_4 + a_4 (a_4 + 2a_1)(x - y) + a_1 (a_4 + 2a_1)^2 \left(x^2 + yx + y^2\right).$$

As $V(0, 0) = -a_4$, it follows that system (12) has a first integral defined at the origin. If $a_4 = 0$ and $a_2 = a_1$, applying Theorem 4(ii) with $a = c = a_1$, $b = d = -a_1$, $A = 2a_1$ and $B = -2a_1$, we have that (12) has a center at the origin and so it is integrable. The case $a_2 = -a_1$ (i.e. case (b)) is analogous, since

$$V(u, v) = 1 + (2a_1 - a_4)x + (2a_1 + a_4)y - a_1 a_4 x^2 - a_4{}^2 xy + a_1 a_4 y^2,$$

is an inverse integrating factor for system (12), which is also nonzero at the origin.

Case (c). In this case system (4) becomes

$$\begin{aligned} \dot{u} &= -v + a_3 w^2 + a_5 uw, \\ \dot{v} &= \quad u + a_3 w^2 + a_5 uw, \\ \dot{w} &= -w + a_3 w^2 + a_5 uw. \end{aligned}$$

Note that $w = 0$ is invariant and is a center manifold for this system. Moreover, the restriction of the associated vector field to $w = 0$ gives rise to a linear center.

Case (d). For $a_2 = -a_1$, $a_3 = -2a_1$, $a_4 = 6a_1$ and $a_5 = -4a_1$ the vector field associated to system (4) has the invariant algebraic surface

$$F(u, v, w) = w + a_1(u - v)^2 - 2a_1(v - w)^2 = 0$$

with cofactor $K(u, v, w) = -1$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. To determine the dynamics on it, we first use the change of coordinates $(u, v, w) \mapsto (x + z, y + z, z)$ that transforms the system into

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + 2z, \\ \dot{z} &= -z + a_1 x^2 + 6a_1 xy + 4a_1 xz - a_1 y^2 + 4a_1 yz. \end{aligned} \tag{13}$$

The center manifold in the new variables is defined by

$$F(x, y, z) = z + a_1(x - y)^2 - 2a_1 y^2 = 0.$$

The restriction of system (13) to $F = 0$ is given by

$$\dot{x} = -y, \quad \dot{v} = x - 2a_1 x^2 + 4a_1 xy + 2a_1 y^2.$$

Since this system has the following inverse integrating factor (nonzero at the origin)

$$V(u, v) = 1 - 4a_1 (x - y) + 4a_1^2 \left(x^2 - 2xy - y^2\right),$$

system (4) has a center on the center manifold.

Case (e). For $a_1 = 0$, $a_3 = -a_2$, $a_4 = 2a_2$ and $a_5 = -2a_2$ system (4) has the invariant algebraic surface $F(u, v, w) = w - a_2(y - z)^2 = 0$ with cofactor $K(u, v, w) = -1$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. To determine the dynamics on it first we use the change of coordinates $(u, v, w) \mapsto (x, y + z, z)$ that transforms the system into

$$\begin{aligned} \dot{x} &= -y - z + a_2 y^2 + 2a_2 xy + 2a_2 yz, \\ \dot{y} &= x + z, \\ \dot{z} &= -z + a_2 y^2 + 2a_2 yz + 2a_2 xy. \end{aligned} \tag{14}$$

The center manifold $F = 0$ in the new variables writes as $F(x, y, z) = z - a_2 y^2 = 0$. The restriction of system (14) to $F = 0$ is

$$\dot{x} = -y + 2a_2 xy + 2a_2^2 y^3, \quad \dot{y} = x + a_2 y^2.$$

This system is invariant by the change of variables $(x, y, t) \mapsto (x, -y, -t)$ so that it has a center at the origin this shows that system (4) restricted to (e) has a center on the center manifold.

Case (f). In this case, system (4) becomes

$$\begin{aligned} \dot{u} &= -v + a_3 w^2 + a_5 uw + vw, \\ \dot{v} &= u + a_3 w^2 + a_5 uw + vw, \\ \dot{w} &= -w + a_3 w^2 + a_5 uw + vw. \end{aligned}$$

It is clear that the plane $w = 0$ is invariant and is a center manifold for this system. Moreover, the restriction of the associated vector field to $w = 0$ gives rise to a linear center.

Case (g). If $a_1 = 1/2$, $a_3 = -a_2 - 1/2$, $a_4 = 2a_2 - 1$ and $a_5 = -2a_2$, then system (4) has the invariant algebraic surface $F(u, v, w) = -2w + (u - w)^2 + 2a_2(v - w)^2 = 0$ with cofactor $K(u, v, w) = -1$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. To determine the dynamics on it first we use the change of coordinates $(u, v, w) \mapsto (x + z, y + z, z)$ that transforms system (4) into

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + 2z, \\ \dot{z} &= -z + x^2/2 + (2a_2 - 1)xy + a_2 y^2 + 4a_2 yz. \end{aligned} \tag{15}$$

The center manifold in the new variables is $F(x, y, z) = -2z + x^2 + 2a_2 y^2 = 0$ while the restriction of system (15) to $F = 0$ is

$$\dot{x} = -y, \quad \dot{v} = x + x^2 + 2a_2 y^2.$$

As this system is invariant under $(x, y, t) \mapsto (x, -y, -t)$, it follows that it has a center at the origin, i.e., system (4) under the conditions (g) has a center on the center manifold.

Case (h). If $a_1 = -1/2$, $a_2 = -1/2$, $a_4 = 0$ and $a_5 = -1$. Then the vector field associate to system (4) has the invariant algebraic surface

$$F(u, v, w) = w + \left[(u + w)^2 + (v - w)^2\right]/2 - w^2 (1 + a_3) = 0$$

with the cofactor $K(u, v, w) = -1 - 2u + 2a_3 w$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. To determine the dynamics on it, first we use the change of coordinates $(u, v, w) \mapsto (x - z, y + z, z)$ that transforms system (4) into

$$\begin{aligned} \dot{x} &= -y - 2z + 2(1 + a_3)z^2 - x^2 - y^2, \\ \dot{y} &= x, \\ \dot{z} &= -z + (1 + a_3)z^2 - x^2/2 - y^2/2. \end{aligned} \tag{16}$$

The center manifold $F = 0$ in the new variables is given by

$$F(x, y, z) = z - (1 + a_3)z^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 = 0.$$

The restriction of system (16) to $F = 0$ gives rise to a linear center.

Case (i). For $a_2 = -a_1$, $a_3 = -2a_1 + 1$, $a_4 = 6a_1 - 5$ and $a_5 = -4a_1 + 3$ system (4) admits an invariant algebraic surface

$$F(u,v,w) = w + (a_1-1)(u-w)^2 + (1-2a_1)(u-w)(v-w) + (1-a_1)(v-w)^2 = 0$$

with cofactor $K(u,v,w) = -1$. Since $F = 0$ is tangent to $w = 0$ at the origin, it is a center manifold for this system. The change of coordinates $(u,v,w) \mapsto (x+z, y+z, z)$ transforms system (4) under the conditions (i) into

$$\begin{aligned}
\dot{x} &= -y,\\
\dot{y} &= x + 2z,\\
\dot{z} &= -z + a_1x^2 + (6a_1-5)xy + 2(2a_1-1)xz - a_1y^2 + 4(a_1-1)yz.
\end{aligned} \tag{17}$$

Again, in the new variables the center manifold is given by

$$F(x,y,z) = z + (a_1-1)x^2 + (1-2a_1)xy + (1-a_1)y^2 = 0$$

and the restriction of (17) to $F = 0$ reduces to

$$\dot{x} = -y, \quad \dot{v} = x + 2(1-a_1)x^2 + 2(2a_1-1)xy + 2(a_1-1)y^2.$$

This system has the following inverse integrating factor (nonzero at the origin)

$$\begin{aligned}
V(u,v) = {} & 1 + 4(1-a_1)x + 2(2a_1-1)y + 4(a_1-1)^2x^2\\
& -4(a_1-1)(2a_1-1)xy - 4(a_1-1)^2y^2.
\end{aligned}$$

Hence system (4) has a center on the center manifold.

Case (j). For $a_1 = 1/4$, $a_2 = -1/2$, $a_3 = -5/4$, $a_4 = 0$ and $a_5 = 1/2$ the vector field associated to system (4) admits a polynomial first integral

$$\begin{aligned}
H(x,y,z) = {} & x^2 + y^2 - \frac{1}{2}x^3 - \frac{1}{2}x^2y + 2x^2z - \frac{3}{2}xz^2 - y^2x + y^2z\\
& - \frac{1}{2}yz^2 - \frac{1}{2}x^3z + \frac{5}{4}x^2z^2 + \frac{1}{2}y^2x^2 - \frac{3}{2}xz^3 + \frac{1}{2}y^2z^2\\
& - yz^3 + xyz + \frac{1}{8}x^4 - x^2yz - y^2xz + 2yxz^2 + \frac{5}{8}z^4,
\end{aligned}$$

and so it has a center on the center manifold. □

Proof of Theorem 1.

Case (1). Follows from Cases (c) and (f) of Theorem 3.
Case (2). Follows from Case (a) of Theorem 3.
Case (3). Follows from Case (b) of Theorem 3.
Case (4). Follows from Cases (d) and (i) of Theorem 3.
Case (5). Follows from Cases (e) and (g) of Theorem 3.
Case (6). Follows from Cases (c) and (h) of Theorem 3.
Case (7). Follows from Cases (c) and (j) of Theorem 3. □

Appendix A. Dulac–Kapteyn criterion

The following theorem provides a criterion in order to determine when a quadratic planar polynomial system has a center at the origin. It was first proven by Dulac (1908) and Kapteyn (1912), but we present the version given in Coppel (1966).

Theorem 4 *(Quadratic center). The system*

$$\begin{aligned}
\dot{u} &= -v - bu^2 - (B+2c)uv - dv^2,\\
\dot{v} &= u + au^2 + (A+2b)uv + cv^2,
\end{aligned}$$

has a center at the origin if and only if at least one of the following three holds:

(i) $a+c=b+d$;
(ii) $A(a+c)-B(b+d)=aA^3-(3b+A)A^2B+(3c+B)AB^2-dB^3=0$;
(iii) $A+5b+5d=B+5a+5c=ac+bd+2a^2+2d^2=0$.

Appendix B. Basic Darboux theory of integrability

Since, by Poincaré theorem, the integrability is closely related to the existence of a center on a center manifold (also on the plane), we provide a short overview of the basic notions of the Darboux theory of integrability used in Sec. 4; for more information see Llibre (2000), Goriely (2001) and some applications see Gasull and Mañosa (2002), Llibre et al. (2013), Mahdi and Valls (2014).

We say that $F=F(x,y,z)\in\mathbb{C}[x,y,z]$ is a *Darboux polynomial* and $F=0$ is an *invariant algebraic surface* of the vector field X if and only if there exists a polynomial $K(x,y,z)\in\mathbb{C}[x,y,z]$, the *co-factor* of F, such that $XF=KF$. A the heart of the Darboux theory of integrability is the following result (Darboux, 1978): if there exists some number, say n, of pairs (F_j,K_j) for which there exists a nontrivial dependency relation $\sum\alpha_jK_j=0$ then $F_1^{\alpha_1}\cdots F_n^{\alpha_n}$ is a first integral of X.

Consider now the planar system

$$\dot{x}=P(x,y),\qquad \dot{y}=Q(x,y),\tag{B.1}$$

where $P,Q\in\mathbb{R}[x,y]$, and the associate vector field $X=P\partial/\partial x+Q\partial/\partial y$. Let U be an open subset of $\mathbb{R}^2$, and let $R,V:U\to\mathbb{R}$ be two analytic functions which are not identically zero on U. We say that R is an *integrating factor* of this polynomial system on U if one of the following three equivalent conditions holds

$$\frac{\partial RP}{\partial x}=-\frac{\partial RQ}{\partial x},\quad \mathrm{div}(RP,RQ)=0,\quad XR=-R\,\mathrm{div}(P,Q),$$

where div denotes the divergence. The first integral H associated to the integrating factor R can be easily obtained by

$$H(x,y)=\int R(x,y)P(x,y)dy+h(x),$$

where $h(x)$ is chosen such that it satisfies $\partial H/\partial x=-RQ$. Note that $\partial H/\partial y=RP$, so that $XH\equiv 0$. The function V is an *inverse integrating* factor of the polynomial system (B.1) on U if

$$P\frac{\partial V}{\partial x}+Q\frac{\partial V}{\partial y}=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)V.\tag{B.2}$$

We note that $\{V=0\}$ is formed by orbits of system (B.1) and $R=1/V$ defines on $U\setminus\{V=0\}$ an integrating factor of (B.1). We note that if P and Q are quadratic polynomials and the origin of system (B.1) is a center, then there always exits a polynomial function $V:\mathbb{R}^2\to\mathbb{R}$ of degree 3 or 5 satisfying equation (B.2), see Ferragut et al. (2007).

Appendix C. Illustrative example to solve a center-focus problem

The hybrid symbolic-numerical approach described above will be applied here to generate necessary and sufficient conditions for the existence of a center on a center manifold for a three-dimensional quadratic system. We illustrate step-by-step how to use our approach on a simple example with computations performed using the software Bertini, see Bates et al. (2006, 2013b). Although one could directly obtain a numerical irreducible decomposition defined by all three polynomials, there is benefit to describing each step of our approach to characterize the center conditions for the following system derived from Kuznetsov (2004), Edneral et al. (2012):

$$\begin{aligned}\dot{u}&=-v\\ \dot{v}&=\ u-uw\\ \dot{w}&=-w+c_1u^2+c_2uv+c_3v^2\end{aligned}$$

where $c_1,c_2,c_3\in\mathbb{R}$.

Step 1. Focus quantities. The first step is the computation of the focus quantities (see Sec. 2), also called Lyapunov polynomials:

$$\begin{aligned}
\mathtt{L}_1 &= 2\mathtt{c}_1 - \mathtt{c}_2 - 2\mathtt{c}_3 \\
\mathtt{L}_2 &= (-26\mathtt{c}_1^2 + 13\mathtt{c}_1\mathtt{c}_2 - 20\mathtt{c}_1\mathtt{c}_3 + 13\mathtt{c}_2\mathtt{c}_3 + 6\mathtt{c}_3^2)/20 \\
\mathtt{L}_3 &= (341082\mathtt{c}_1^3 - 138277\mathtt{c}_1^2\mathtt{c}_2 - 7782\mathtt{c}_1\mathtt{c}_2^2 - 4175\mathtt{c}_2^3 + 530378\mathtt{c}_1^2\mathtt{c}_3 - 188030\mathtt{c}_1\mathtt{c}_2\mathtt{c}_3 \\
&\quad -15682\mathtt{c}_2^2\mathtt{c}_3 + 381918\mathtt{c}_1\mathtt{c}_3^2 - 66453\mathtt{c}_2\mathtt{c}_3^2 + 161022\mathtt{c}_3^3)/300000
\end{aligned} \tag{C.1}$$

As a matter of course with numerical solving, we have trivially rescaled L_2 and L_3. The following computes $\mathcal{V}(L_1)$, $\mathcal{V}(L_1, L_2)$, and $\mathcal{V}(L_1, L_2, L_3)$ using Bertini. Each of these computations will take place in three folders called `L1`, `L12`, and `L123`, respectively.

Step 2. $L_1 = 0$. We first compute a numerical irreducible decomposition for $\mathcal{V}(L_1)$, which defines a plane. Working inside a folder called `L1`, we create the file `inputL1`:

```
CONFIG
 TrackType: 1; % compute a numerical irreducible decomposition
END;
INPUT
 variable_group c1,c2,c3;
 function L1;
 L1 = 2*c1 - c2 - 2*c3;
END;
```

Running Bertini with input file `inputL1` via the command

```
>> bertini inputL1
```

yields the following summary printed to the screen showing $\mathcal{V}(L_1)$ defines a plane:

```
************** Decomposition by Degree **************
Dimension 2: 1 classified component
-----------------------------------------------------
   degree 1: 1 component
*****************************************************
```

Step 3. $L_1 = L_2 = 0$. We next regenerate from $\mathcal{V}(L_1)$ to compute $\mathcal{V}(L_1, L_2)$ in the folder `L12` using the following file `inputL12`:

```
CONFIG
 TrackType: 7; % perform regenerative intersection
END;
INPUT
 variable_group c1,c2,c3;
 function L1,L2;
 L1 = 2*c1 - c2 - 2*c3;
 L2 = (-26*c1^2 + 13*c1*c2 - 20*c1*c3 + 13*c2*c3 + 6*c3^2)/20;
END;
```

Running Bertini with input file inputL12 via the command

```
>> bertini inputL12
```

yields several prompts, where we have put in red the text to be entered by the user: (for interpretation of the references to color please refer to the web version of this article)

```
Please enter the number of nontrivial components (-1 to quit): 1

Please enter the name of the corresponding input file or type
  'quit' or 'exit' (max of 255 characters): ../L1/inputL1

Please enter the name of the corresponding witness_data file or type
  'quit' or 'exit' (max of 255 characters): ../L1/witness_data

Please select a dimension to regenerate (-1 to quit): 2

Please select a component to regenerate (-1 to quit,
  -2 to regenerate all): 0
```

This following summary printed to the screen shows that $\mathcal{V}(L_1, L_2)$ consists of two lines:

```
************** Decomposition by Degree **************
Dimension 1: 2 classified components
-----------------------------------------------------
   degree 1: 2 components
*****************************************************
```

Step 4. $L_1 = L_2 = L_3 = 0$**.** Before regenerating, we first investigate the lines in $\mathcal{V}(L_1, L_2)$. The approximation of general points listed in the file main_data created by Bertini are to enough accuracy to use PSLQ (Ferguson and Bailey, 1991) to compute the defining equations. In more complicated examples, one may first want to utilize Bertini's sharpening module to yield numerical approximations which are computed to the user's accuracy requirement. In our example, the point (where $I = \sqrt{-1}$)

```
c1 = 2.968118932274116e-01 + I*1.520885450197431e+00
c2 = 1.187247572909648e+00 + I*6.083541800789725e+00
c3 = -2.968118932274119e-01 - I*1.520885450197432e+00
```

yields the line $c_1 + c_3 = c_2 - 4c_1 = 0$ while the point

```
c1 = -1.359028418458732e+00 + I*2.899781284193458e-01
c2 = -2.718056836917465e+00 + I*5.799562568386909e-01
c3 = -4.552348104595714e-16 - I*1.940963322531678e-16
```

yields the line

$$c_2 - 2c_1 = c_3 = 0. \tag{C.2}$$

The former is not contained in $\mathcal{V}(L_3)$ so it must be regenerated while the latter (defined by (C.2)) is indeed contained in $\mathcal{V}(L_3)$ yielding an irreducible component of $\mathcal{V}(L_1, L_2, L_3)$. We note that ordering by Bertini of the two lines can change with different runs. In our run, the line which needs to be regenerated was first, which Bertini calls Component 0.

We next use regeneration to compute the intersection of the first line with the hypersurface $\mathcal{V}(L_3)$ in the folder L123 using the following file inputL123:

```
CONFIG
 TrackType: 7; % perform regenerative intersection
END;
INPUT
 variable_group c1,c2,c3;
 function L1,L2,L3;
 L1 = 2*c1 - c2 - 2*c3;
 L2 = (-26*c1^2 + 13*c1*c2 - 20*c1*c3 + 13*c2*c3 + 6*c3^2)/20;
 L3 = (341082*c1^3 - 138277*c1^2*c2 - 7782*c1*c2^2 - 4175*c2^3 +
       530378*c1^2*c3 - 188030*c1*c2*c3 - 15682*c2^2*c3 +
       381918*c1*c3^2 - 66453*c2*c3^2 + 161022*c3^3)/300000;
END;
```

Running Bertini with input file `inputL123` via the command

```
>> bertini inputL123
```

yields several prompts, where we have put in red the text to be entered by the user: (for interpretation of the references to color please refer to the web version of this article)

```
Please enter the number of nontrivial components (-1 to quit): 1

Please enter the name of the corresponding input file or type
  'quit' or 'exit' (max of 255 characters): ../L12/inputL12

Please enter the name of the corresponding witness_data file or type
  'quit' or 'exit' (max of 255 characters): ../L12/witness_data

Please select a dimension to regenerate (-1 to quit): 1

Please select a component to regenerate (-1 to quit,
  -2 to regenerate all): 0
```

This computation yielded one point which was the limit of three solution paths, which is observed via the file `witness_superset`:

```
----------DIMENSION 0----------
SINGULAR SOLUTIONS
---------------
7.995887974832271e-25 1.110011689888151e-24
3.198355189932908e-24 4.440046759552608e-24
-7.995887974832270e-25 -1.110011689888152e-24
Multiplicity: 3
```

Since this point, namely the origin, is contained in the line defined by (C.2), we obtain that (C.2), which is equal to $\mathcal{V}(L_1, L_2, L_3)$, is the collection of necessary conditions for the existence of a center.

Step 5. Sufficient conditions. The last step is to show that the necessary conditions (C.2) are also sufficient. This can be obtained, for example, by using Darboux theory of integrability as summarized in Appendix B (see also Edneral et al., 2012).

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