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The *S* chart with variable charting statistic to control bi and trivariate processes

Antonio Fernando Branco Costa [a,*], Antonio Faria Neto [b]

[a] Universidade Estadual Paulista (Unesp), Faculdade de Engenharia, Câmpus de Guaratinguetá, Departamento de Produção, Brazil
[b] Universidade Estadual Paulista (Unesp), Faculdade de Engenharia, Câmpus de Guaratinguetá, Departamento de Engenharia Elétrica, Brazil

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ABSTRACT

In this article, we propose the *S* chart with variable charting statistic to control the covariance matrix as an alternative to the use of the bivariate |*S*| chart and the trivariate VMAX chart. As usual, samples are regularly taken from the process, but only one of the two quality characteristics, *X* or *Y*, is measured and only one of the two statistics (S_x, S_y) is computed. The statistic in use and the position of the current sample point on the chart define the statistic for the next sample. If the current point is the standard deviation of the *X* values and it is in the central region (warning region), then the statistic for the next sample will be the standard deviation of the *Y* values (*X* values). If the current point is the standard deviation of the *Y* values and it is in the central region (warning region), then the statistic for the next sample will be the standard deviation of the *X* values (*Y* values). For the trivariate case, when the sample point falls in the central region, the charting statistic for the next sample changes from S_x to S_y, or from S_y to S_z, or yet, from S_z to S_x. The *VCS* chart is not only operationally simpler than the bivariate |*S*| and trivariate VMAX charts but also signals faster even with less measurements per sample.

1. Introduction

In many production lines, samples are regularly taken from the process with the purpose to obtain information about different quality characteristics of the products, which are used to calculate the plotting statistic of multivariate control charts. Signs of interest in bivariate control charts are acknowledged by numerous publications on this subject (Amiri, Allahyari, & Sogandi, 2015; Bersimis, Sachlas, & Castagliola, 2016; He, Wang, Tsung, & Shang, 2016; Ho & Costa, 2015; Leoni & Costa, 2017; Leoni, Machado, & Costa, 2015, 2016; Melo, Ho, & Medeiros, 2016; Osei-Aning, Abbasi, & Riaz, 2017; Saghir, 2015; Saghir, Chakraborti, & Ahmad, 2017; Saghir, Khan, & Chakraborti, 2016; Simões, Leoni, Machado, & Costa, 2016; Tran, 2016; Yang, 2016).

With the introduction of the generalized variance statistic |S|, Alt (1985) emphasized the importance to control the covariance matrix Σ of multivariate processes. In 1999, Aparisi, Jabaloyes and Carrión studied the distribution of the generalized variance and, in 2001, they introduced the generalized variance chart with variable sample size, see Aparisi, Jabaioyes, and Carrion (1999) and Jabaloyes, Carrión, and Aparisi (2001). Subsequently, Grigoryan and He (2005) developed the double sampling |S| chart.

Control charts more efficient than the generalized variance |*S*| chart have been proposed. Costa and Machado (2008a, 2008b), Machado and Costa (2008), and Machado, Costa, and Rahim (2009) considered the VMAX statistic to control the covariance matrix of multivariate processes. The points plotted on the VMAX chart correspond to the maximum of the sample variances of the *p* quality characteristics. Costa and Machado (2011) also considered the RMAX statistic to control the covariance matrix of multivariate processes. The points plotted on the RMAX chart are the maximum of the sample ranges of the *p* quality characteristics. The user's familiarity with sample ranges is a point in favor of the RMAX chart. In terms of the ability to detect an out-of-control condition, the VMAX chart performs better than the RMAX chart.

In this article, we show that the *S* chart with variable charting statistic (*VCS* chart) is an excellent device to control the variance-covariance matrix of bi and trivariate processes; the *VCS* chart is operationally simpler than its competitors: the generalized variance |*S*| chart, the VMAX chart, and the multivariate S chart proposed by Alt (1985). Moreover, the *VCS* chart has the best relation between signaling delays and measurements per sample. The basic idea is to work with larger samples, once only one of the two quality characteristics, *X* or *Y* (trivariate case: *X*, *Y* or *Z*), is measured and only one of the two charting statistics, S_x or S_y

* Corresponding author.
E-mail addresses: fbranco@feg.unesp.br (A.F.B. Costa), antfarianeto@gmail.com (A.F. Neto).

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is computed (trivariate case: S_x or S_Y, or S_z, is computed). If the current charting statistic is S_x and the current sample point falls in the central region (warning region), then the charting statistic for the next sample will change to S_y (will be the same, that is, S_x). Alternatively, if the current charting statistic is S_y and the current sample point falls in the central region (warning region), then the charting statistic for the next sample will change to S_x (will be the same, that is, S_y). For the trivariate case, when the sample point falls in the central region, the charting statistic for the next sample switches from S_x to S_y, or from S_y to S_z, or from S_z to S_x).

The idea of working with only one quality characteristic per time was introduced by Leoni and Costa (2017). They proposed to control the mean vector of bivariate (*X*, *Y*) and trivariate (*X*, *Y*, *Z*) processes by alternating the charting statistics $(\bar{x}, \bar{y}, \bar{z})$ of the Shewhart chart. The *ACS* chart, as it was named, is substantially easier to operate and faster than the Hotelling chart in signaling changes in the mean vector. Even with fewer measurements per sample, the *ACS* chart outperforms the Hotelling chart.

An EWMA version of the *VCS* chart was also compared with the multivariate EWMA chart developed by Hawkins and Maboudou-Tchao (2008). For the bivariate case, they are similar in performance. The *VCS* scheme is especially recommended for cases where different types of equipment are used to obtain the *X* and *Y* observations. In this context, working with only one quality characteristic per time facilitates the measuring operation.

2. The generalized variance |*S*| chart

The generalized variance |*S*| chart is the usual Shewhart-type chart to control the variance-covariance matrix of multivariate processes. With the generalized variance |*S*| chart in use, samples of size *n* are regularly collected and the quality characteristics (*X*, *Y*, *Z*...) of the selected items are measured. The two common assumptions to study the performance of the generalized variance |*S*| charts are: (I) The *p* quality characteristics (*X*, *Y*, *Z*...) follow a multivariate normal distribution with the covariance matrix given as $\mathbf{\Sigma} = (\Sigma_{11} = a_1^2\sigma_x^2; \Sigma_{12} = a_1a_2\sigma_{xy}; \Sigma_{22} = a_2^2\sigma_y^2;\ \Sigma_{13} = a_1a_3\sigma_{xz}; \Sigma_{23} = a_2a_3\sigma_{yz}; \Sigma_{33} = a_3^2\sigma_z^2; \ldots)$, being $\Sigma_{ij} = \Sigma_{ji}$, (II) The in-control covariance matrix $\mathbf{\Sigma}_0$ has all $a_i = 1$ and the out-of-control covariance matrix $\mathbf{\Sigma}_1$ has at least one $a_i > 1$. The way the covariance matrix was defined, assures that the out-of-control conditions do not change the correlations, consequently the power of the generalized variance |*S*| chart is independent of the *XY*, *XZ*, *YZ*..., correlations - $\rho_{ij} = a_ia_j\sigma_{ij}(a_i^2a_j^2\sigma_i^2\sigma_j^2)^{-1/2} = \sigma_{ij}(\sigma_i^2\sigma_j^2)^{-1/2}, i \neq j \in \{x, y, z\}$.

In this paper, we deal with the most common case where the assignable cause increases the variances ($a_i > 1$). The particular case where $a_i < 1$, requires the use of more sophisticated charts, such as the multivariate EWMA charts, see Hawkins and Maboudou-Tchao (2008).

When $p = 2$, and the process is in control $2(n-1)|S|^{1/2}|\Sigma_0|^{-1/2}$ is distributed as chi-square variable with $2n - 4$ degrees of freedom, where $S = (S_x^2; S_{xy}; S_{yx}; S_y^2)$, being S_x^2 the sample variance of the *X* observations, S_y^2 the sample variance of the *Y* observations, and $S_{xy} = S_{yx}$ the sample covariance between *X* and *Y*. The control limit for the |S| chart is

$$CL = \frac{(\chi^2_{2n-4,\alpha})^2|\Sigma_0|}{4(n-1)^2} \tag{1}$$

The power of the |S| chart is function of a_1^2 and a_2^2, the out of-control values of the process variances (without losing generality, it is assumed that $\sigma_x^2 = \sigma_y^2 = 1$)

$$p_{|S|} = Pr\left(\chi^2_{2n-4,\alpha} \geqslant \frac{\chi^2_{2n-4,\alpha}}{\sqrt{a_1^2a_2^2}}\right) \tag{2}$$

The exact distribution of the monitoring statistic |S| for more than two variables is complex. To control trivariate processes, the |S| control chart is based on the mean and variance of the sample generalized variance |S|, and the property that most of the probability distribution of |S| is contained in the interval $E(|S|) \pm 3\sqrt{Var(|S|)}$. According to Alt and Smith (1988), $E(|S|) = b_1|\Sigma_0|$ and $Var(|S|) = b_2|\Sigma_0|^2$, where

$$b_1 = \frac{1}{(n-1)^p}\prod_{i=1}^{p}(n-i), \quad \text{and} \qquad b_2 = b_1\left[\prod_{i=1}^{p}\frac{n-i+2}{(n-1)^p} - b_1\right] \tag{3}$$

The |S| chart's control limits are obtained by using the asymptotic normality of |S|.

$$UCL = |\Sigma_0|(b_1 + k\sqrt{b_2}) \qquad LCL = |\Sigma_0|(b_1 - k\sqrt{b_2}) \tag{4}$$

where $k > 0$ is the control limit coefficient of the |S| chart. The value of *k* depends on the desired value of α, the false alarm risk, that is, $k = \Phi^{-1}(1-\alpha)$. However, the asymptotic normality of |S| is not applicable with small samples. For instance, with $p = 3$, $n = 4$; and $\Sigma_0 = \{\Sigma_{ii} = 1.0, i = 1, 2, 3;\ \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0\}$, simulated results proved that the use of $k = 3.0$ doesn't lead to $ARL_0 = 370.4$, but to an $ARL_0 \cong 304.0$. The main drawback of the |S| chart is the requirement that the size of the samples should be larger than *p*, that means, with three variables the smallest sample size is four. As the number of observations per sample is given by *np*, the minimum *np* is twelve.

The VMAX chart proposed by Costa and Machado (2008a, 2008b) to control trivariate processes doesn't require samples of size *n* larger than *p*. As they obtained the exact distribution of the monitoring statistic VMAX, the false alarm risk and the power of the VMAX chart can be determined with accuracy. They also proved that the VMAX chart is operationally simpler and more efficient than the |S| chart. All these points in favor of the chart proposed by Costa and Machado (2008a, 2008b) led us to compare the trivariate *VCS S* chart with the VMAX chart.

3. The *S* chart with variable charting statistic

The *S* chart with variable charting statistic is the Shewhart-type chart specially designed to control the covariance matrix of bi and trivariate processes. When the *S* chart with variable charting statistic (*VCS S* chart) is used to control bivariate processes, samples of size *n* are regularly taken from the process, but only one of the two quality characteristics, *X* or *Y*, is measured and only one of the two statistics (S_x, S_y) is computed. The statistic in use and the position of the current point define the statistic for the next sample. If the statistic in use is S_x and the sample point falls in the central region (warning region), then the statistic for the next sample will change to S_y (will be the same, that is, S_X). Alternatively, If the statistic in use is S_y and the current sample point falls in the central region (warning region), then the statistic for the next sample will change to S_X (will be the same, that is, S_y).

When the *VCS* chart is used to control trivariate processes, only one of the three quality characteristics: *X*, *Y* or Z, is measured and only one of the three statistics: S_x, or S_y, or S_z, is computed. With three quality characteristics, the charting statistic changes from S_x to S_y, or from S_y to S_z, or yet, from S_z to S_x. Fig. 1 illustrates the

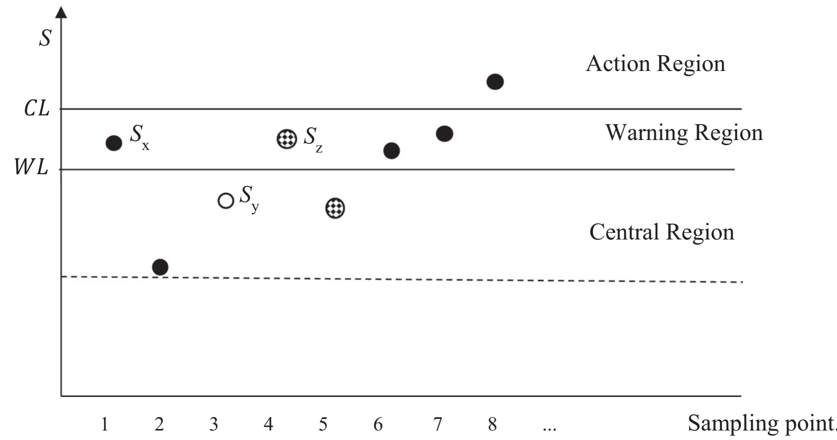

Fig. 1. The *VCS S* chart.

S chart with variable charting statistic. The S_x points are followed by the S_x points as long as they fall in the warning region, however, a S_x point in the central region changes the quality characteristic, that is, the charting statistic for the next sample will be the standard deviation of the *Y* observations - S_y. The same way, the S_y points are followed the S_y points as long as they fall in the warning region, however, a S_y point in the central region changes the quality characteristic, that is, the charting statistic for the next sample will be the standard deviation of the *Z* observations - S_z. Finally, the S_z points are followed the S_z points as long as they fall in the warning region, however, an S_z point in the central region changes the quality characteristic, that is, the charting statistic for the next sample will be S_x - the standard deviation of the *X* observations.

The Control Limit (*CL*) of the *S* chart depends on the desired value of α, the false alarm risk, that is, $(n-1)CL^2 = \chi^2_{n-1,\alpha}$. If $\delta > \alpha$, is the in-control probability to obtain a point outside the central region, then $(n-1)WL^2 = \chi^2_{n-1,\delta}$, where (WL) is the Warning Limit of the *S* chart.

The power of the *VCS S* chart depends on the statistic in use, if the current statistic is S_x, the power is given by $p_{S_x} = Pr(a_1^2\chi^2_{n-1} \geqslant \chi^2_{n-1,\alpha})$. Similarly, with the statistics S_y and S_z, the power is given by $p_{S_y} = Pr(a_2^2\chi^2_{n-1} \geqslant \chi^2_{n-1,\alpha})$ and $p_{S_z} = Pr(a_3^2\chi^2_{n-1} \geqslant \chi^2_{n-1,\alpha})$, respectively.

The speed with the control charts signal is measured by the Average Run Length – *ARL*. Expression (7) gives the *ARLs* of the *VCS S* chart, see Appendix A for details.

$$ARL = \frac{3(a+b+c) - [2b + c(1-p_y)]p_x - [2c + a(1-p_z)]p_y - [2a + b(1-p_x)]p_z}{3(p_x + p_y + p_z - p_xp_y - p_xp_z - p_yp_z)} \tag{5}$$

In expression (5), $a = 1/(1-g_x)$, $b = 1/(1-g_y)$, $c = 1/(1-g_z)$, where g_x is the probability of a sample point S_x falling in the warning region, and p_x is the probability of a sample point S_x falling in the action region, constrained by the restriction that the point did not fall in the warning region. The definitions of the two pair of probabilities, (g_y, p_y) and (g_z, p_z), are similar to the definitions of g_x and p_x:

$$g_x = Pr[a_1^2\chi^2_{n-1} < \chi^2_{n-1,\alpha}] - Pr[a_1^2\chi^2_{n-1} < \chi^2_{n-1,\delta}] \tag{6a}$$

$$g_y = Pr[a_2^2\chi^2_{n-1} < \chi^2_{n-1,\alpha}] - Pr[a_2^2\chi^2_{n-1} < \chi^2_{n-1,\delta}] \tag{6b}$$

$$g_z = Pr[a_3^2\chi^2_{n-1} < \chi^2_{n-1,\alpha}] - Pr[a_3^2\chi^2_{n-1} < \chi^2_{n-1,\delta}] \tag{6c}$$

$$p_x = p_{S_x}/(1-g_x) \tag{7a}$$

$$p_y = p_{S_y}/(1-g_y) \tag{7b}$$

$$p_z = p_{S_z}/(1-g_z) \tag{7c}$$

For the bivariate case, Eq. (5) reduces to Eq. (8)

$$ARL = \frac{a + b + a(1-p_y) + b(1-p_x)}{2(p_x + p_y - p_xp_y)} \tag{8}$$

4. Comparing the charts performance

In this section, the performance of the bivariate *VCS S* chart is compared with the performance of the generalized variance |S| chart, and the performance of the trivariate *VCS S* chart is compared with the performance of the VMAX chart. The *ARLs* tables were built for an in-control *ARL* of 370.4 (α = 0.0027). Tables 1 and 2 presents the bivariate case (p = 2); (n, m) are, respectively, the size of the samples and the number of measurements per sample. The control limits (*CL*) of the generalized variance |S| charts were computed for $\rho = 0.5$. When the *VCS S* chart in use $m = n$, but with the generalized variance |S| chart in use, $m = pn$. The *VCS S* chart requires less measurements per sample to overcome the generalized variance |S| chart. The *VCS S* chart with $m = 3$ ($n = 3$), signals faster (when $a_1 \neq a_2$), or with the same speed (when $a_1 = a_2$) than of the |S| chart with $m = 6$ ($n = 3$). In other words, with samples of size $n = 3$, the measurement of only one quality characteristic (*X* or *Y*) per time is always simpler and better than the measuring of the *X* and *Y* quality characteristics all the time. Table 2, presents the charts' performance when only one of the two variables is affected by the disturbance ($a_1 = 1.0$; $a_2 > 1.0$); the *VCS S* chart signals these types of out-of-control situations much faster than the generalized variance |S| chart with twice observations per sample.

In Table 3, the *VCS S* chart is compared with the VMAX chart, the percentage *ARL* reduction by the use of the *S* chart is given by expression (9). Table 3 shows that, even with smaller number of measurements per sample, the *VCS S* chart is superior to the VMAX chart.

$$PR = 100 * \frac{ARL_{VMAX} - ARL_{VCS}}{ARL_{VMAX}} \tag{9}$$

The *VCS S* chart is especially recommended for cases where the selection of larger samples is easy but the measuring operation is not so easy, because different types of equipment are used to obtain the *X*, *Y* and *Z* observations, and they might be far from each other. In this context, working with only one quality characteristic

Table 1
Comparing the *VCS* and |S| charts – $a_1, a_2 \in \{1, 1.25, 1.5, 1.75, 2\}$.

			VCS	|S|	VCS	|S|	VCS	|S|	VCS	|S|
		n	3	3	5	4	7	5	9	6
		m	3	6	5	8	7	10	9	12
		CL	2.432	6.559	2.016	5.502	1.829	4.717	1.717	4.168
a_1	a_2	WL	1.00		0.866		1.155		1.541	
1	1		370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
1.25	1.25		44.05		29.25		21.90		17.40	
1.5	1.25		20.02	23.44	11.63	14.29	8.30	10.19	6.49	7.85
1.5	1.5		13.86		8.03		5.61		4.29	
1.75	1.25		10.69	14.94	5.94	8.71	4.19	6.09	3.35	4.66
1.75	1.5		9.03	9.52	5.13	5.40	3.58	3.77	2.79	2.91
1.75	1.75		6.90		3.89		2.74		2.16	
2	1.25		6.95	10.65	3.97	6.07	2.82	4.23	2.33	3.25
2	1.5		6.39	7.18	3.72	4.05	2.62	2.85	2.10	2.23
2	1.75		5.34	5.42	3.06	3.07	2.19	2.20	1.76	1.77
2	2		4.39		2.52		1.85		1.52	

Table 2
Comparing the *VCS* and |S| charts – ($a_1 = 1.0$; $a_2 > 1.0$).

			VCS	|S|	VCS	|S|	VCS	|S|	VCS	|S|
		n	3	3	4	4	5	5	6	6
a_1	a_2	m	3	6	4	8	5	10	6	12
1	1		370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
1	1.25		71.23	113.48	55.87	88.72	44.72	74.13	41.36	63.98
1	1.5		21.98	51.57	15.43	35.10	11.65	26.70	10.11	21.45
1	1.75		10.61	29.36	7.31	18.41	5.75	13.32	4.69	10.35
1	2		6.67	19.25	4.69	11.48	3.94	8.11	3.07	6.21
1	2.25		4.89	13.86	3.56	8.03	3.18	5.61	2.42	4.29
1	2.5		3.95	10.65	2.98	6.07	2.80	4.23	2.09	3.25
1	2.75		3.39	8.59	2.64	4.85	2.59	3.40	1.92	2.63
1	3		3.03	7.18	2.43	4.05	2.45	2.85	1.81	2.23

Table 3
Comparing the *VCS* and VMAX charts.

				VCS	VMAX		VCS	VMAX	
		CL		2.016	6.980		1.693	5.475	
		WL		1.323			1.254		
		ρ			0.5			0.5	
			n	5	2		7	3	
a_1	a_2	a_3	m	5	6	PR	7	9	PR
1.00	1.00	1.00		370.34	370.4		370.4	370.4	
1.25	1.00	1.00		66.51	76.93	13.5	51.09	61.83	17.4
1.25	1.25	1.00		39.40	44.35	11.2	29.69	34.98	15.1
1.25	1.25	1.25		29.25	31.84	8.1	21.90	24.98	12.3
1.50	1.00	1.00		17.27	21.68	20.3	11.92	15.62	23.7
1.25	1.50	1.00		15.62	18.42	15.2	10.92	13.46	18.8
1.25	1.25	1.50		14.38	16.18	11.1	10.14	11.93	15.0
1.50	1.50	1.00		10.39	12.06	13.8	7.22	8.73	17.3
1.25	1.50	1.50		10.08	11.20	10.0	7.05	8.16	13.6
1.50	1.50	1.50		8.03	8.76	8.4	5.61	6.39	12.2
1.75	1.00	1.00		7.80	9.69	19.5	5.41	6.77	20.0
1.25	2.00	1.00		5.05	5.54	8.8	3.70	3.90	5.2
2.00	1.00	1.00		4.91	5.71	14.0	3.57	3.99	10.4
1.75	1.75	1.00		4.90	5.54	11.5	3.45	3.95	12.6
1.50	2.00	1.00		4.65	4.96	6.2	3.41	3.53	3.5
1.75	1.75	1.75		3.89	4.16	6.6	2.74	3.03	9.4
2.50	1.00	1.00		3.14	3.05	−**2.9**	2.53	2.21	−**14.6**
2.00	2.00	1.00		3.16	3.39	6.7	2.34	2.46	5.1
2.00	2.00	2.00		2.52	2.63	4.4	1.85	1.97	6.3
2.50	2.50	1.00		2.05	1.95	−**5.3**	1.67	1.51	−**10.7**
2.50	2.50	2.50		1.60	1.61	0.9	1.28	1.31	2.4

per time facilitates the measuring operation. That is, taking larger samples and inspecting only one quality characteristic per time is operationally simpler than taking smaller samples but having to inspect more than one quality characteristic. We are working here with the concept of rational subgroup, where the disturbance always occurs in between sampling points, and not during the sample collection.

The aim of this paper is to prove that the *VCS S* chart is not only simpler to operate than the |S| and VMAX charts, but also more efficient. The average run length (*ARL*) is the traditional metric to measure the performance of the control charts (Knoth, 2016) where the run length is the number of samples until the chart produces a signal. Alternatively, the percentage points of the run length distribution – *RL* (%), can also be used to evaluate the performance of the charts, see Wheeler (2014) and Woodall (2016). The following relation is observed: *RL* (50%) < *ARL* < *RL* (95%), that is, the run length distribution shrinks (the variance of the *RL* diminishes in magnitude), as the *ARL* decreases.

5. Comparing the *VCS* chart with the EWMA and S charts

Hawkins and Maboudou-Tchao (2008) developed a multivariate EWMA chart to control the covariance matrix. According to them, it is convenient to work with multistandardized data vectors rather than the "raw" process reading $\mathbf{X}_i$; for this, they find a matrix A with the property $\mathbf{A}\Sigma_0\mathbf{A}' = \mathbf{I}_p$, and transform to $\mathbf{U}_i = \mathbf{A}(\mathrm{X}_i - \mu_0)$. The statistic plotted on their EWMA chart (*HM* EWMA chart) for the *i*th samples is:

$$c_i = \mathrm{tr}(\mathbf{B}_i) - \log|\mathbf{B}_i| - p \tag{10}$$

If the value of c_i plots above the upper control limit CL, the control chart triggers an out-of-control signal. In expression (10), $\mathbf{B}_0 = \mathbf{I}_p$, and, for $i = 1, 2 \ldots$,

$$\mathbf{B}_i = (1-\lambda)\mathbf{B}_{i-1} + \lambda\mathbf{U}_i\mathbf{U}_i', \quad \text{with} \quad 0 < \lambda < 1. \tag{11}$$

For the bivariate case ($p = 2$), it follows that $(1-\rho)^{1/2}\mathbf{U}_i' = (X_i, -\rho X_i + Y_i)$, where (X_i, Y_i) are the (*X*, *Y*) observations of the *i*th sample of size one ($n = 1$).

In order to make a fair comparison with the control chart proposed by Hawkins and Maboudou-Tchao, we constructed the *VCS* EWMA chart with the c_i statistic but with $\mathbf{U}_i' = (X_i(1), X_i(2))$ or $\mathbf{U}_i' = (Y_i(1), Y_i(2))$, where $(X_i(1), X_i(2))$ and $(Y_i(1), Y_i(2))$ are, respectively, the *X* and the *Y* observations of the *i*th sample of size two ($n = 2$). If the current vector $\mathbf{U}_c$ is a vector of *X*s, that is, $\mathbf{U}_c' = (X_{1c}, X_{2c})$, then the variance S_c^2 of (X_{1c}, X_{2c}) defines the type of the next sample observations. If $S_c^2 > \mathrm{WL}$, it will be *X* observations, consequently, $\mathbf{U}_{c+1}' = (X_{c+1}(1), X_{c+1}(2))$. Otherwise, it will be *Y* observations, consequently, $\mathbf{U}_{c+1}' = (Y_{c+1}(1), Y_{c+1}(2))$. Similar procedure is followed when the current vector $\mathbf{U}_c$ is a vector of *Y*s. Table 4 shows that two EWMA charts, *HM* and *VCS*, are similar in performance. The *VCS* scheme is preferable when dealing with samples of size two combined with the measuring of only one quality characteristic is easier than dealing with samples of size one combined with the measuring of two quality characteristics.

Alt and Smith (1988) was the first to deal with the control of the covariance matrix. His multivariate **S** chart is a direct extension of the univariate S^2 control chart. The statistic plotted on the control chart for the *i*th samples is:

$$W_i = -pn + pn\ln(n) - n\ln(|A_i|/|\mathbf{\Sigma}_0|) + \mathrm{tr}(\mathbf{\Sigma}^{-1}A_i) \tag{12}$$

In expression (12), $A_i = (n-1)S_i$, S_i is the sample covariance matrix for sample *i*, and tr is the trace operator. If the value of W_i plots above the upper control limit CL, the chart triggers an out-of-control signal. Table 4 shows that, even with the same sample size, the *VCS* chart is superior to the bivariate **S** chart; except when the aim is to detect large shifts. At each sampling time, the *VCS* chart requires the inspection of only one of the two quality characteristics. That means, opting to the *VCS* chart, the number of observations per sample reduces to one-half.

According to Table 4, the *VCS* EWMA chart and the EWMA chart proposed by Hawkins and Maboudou-Tchao (2008) are similar in performance. However, there are some points in favor of the *VCS* version of the EWMA chart:

- First, the *VSC* version doesn't require the estimation of the *XY* correlation, consequently, less effort is required during phase I.
- Second, if the disturbance affects only one of the two variables, then the *VCS* version indicates which variable was affected; with the *HM* version, we never know which variable was affected.
- Third, with the *VCS* version, the U vector is the vector of the observations. With the *HM* version, the U vector is not the vector of the observations, but a function of them. In reality, a function that not only requires the estimation of the *XY* correlation, but also requires, from the user, a deep knowledge of matrices operations (matrix **A** with the property $\mathbf{A}S_0\hat{\mathbf{A}} = \mathrm{I}_p$, inverse-Cholesky root matrix ...). Remembering that the U vector is the one that "feeds" the monitoring statistic ($\mathrm{c_i}$) of the EWMA chart.
- Fourth, the *VCS* EWMA chart might work with the sample ranges (because with samples of size two, the sample variance is just the square of the sample range divided by two). Wheeler (2014) advocates the use of the sample ranges; according to him, the sample range is an intuitive measure of variability that is very easy to teach. In a very recent paper, Woodall (2016) commented two things: "The researcher advocate moving away from the use of the sample ranges, but practice has not change very much, if at all", "New methods based on Theory must be convincingly shown to be effective in practice and

Table 4
Comparing the *VCS* with the bivariate EWMA and S charts.

			EWMA		VCS	S
			VCS	HM		
		n	2	1	3	3
a_1	a_2	m	2	2	3	6
		CL	0.63	0.63	2.43	15.98
		WL	2.00		1.00	
1	1		370	370	370.37	370
1.25	1.25		31.52	31.24	44.05	104.6
1	1.5		21.57	20.1	21.98	41.22
1.5	1.5		10.71	10.69	13.86	20.98
1	1.75		11.21	11.13	10.61	15.97
1.75	1.75		6.12	6.12	6.90	8.08
1	2		7.38	7.63	6.67	8.34
2	2		4.26	4.27	4.39	4.34
2.5	1		4.47	4.74	3.95	3.81
2.5	2.5		2.71	2.72	2.98	2.19

adequately communicated before they can be accepted by practitioners. In addition, there is a considerable amount of inertia that needs to be overcome for a new method to be accepted or for an old method to be discarded". Based on all these comments, the *VCS* EWMA chart seems to be a better option for monitoring bivariate processes than the EWMA chart proposed by Hawkins and Maboudou-Tchao (2008).

6. An example of application

In this section, we consider the example given by Aparisi et al. (1999) to explain the operational simplicity of the *S* chart with variable charting statistic (*VCS*). This example was also considered by Costa and Machado (2008b) to illustrate the use of the VMAX chart. Fig. 2 shows the three quality characteristics: the distance between centers. X (cm), and the diameters Y (cm) and Z (cm). The in-control values of the mean vector, the covariance matrix and the correlations are:

$$\boldsymbol{\mu}_0 = \begin{pmatrix} 20 \\ 7 \\ 4 \end{pmatrix}; \ \boldsymbol{\Sigma}_0 = \begin{pmatrix} 0.04 & 0.02 & 0.01 \\ 0.02 & 0.02 & 0.011 \\ 0.01 & 0.011 & 0.01 \end{pmatrix}; \quad \boldsymbol{\rho} = \begin{pmatrix} \rho_{xy} = 0.707 \\ \rho_{xz} = 0.500 \\ \rho_{yz} = 0.778 \end{pmatrix}.$$

Following the variable charting policy, it was generated five samples of size $n = 5$. Until sample five, the data was simulated from an in-control process, after that, the data was simulated using the variance-covariance matrix $\boldsymbol{\Sigma}_1$, with:

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} 0.16 & 0.04 & 0.02 \\ 0.04 & 0.02 & 0.011 \\ 0.02 & 0.011 & 0.01 \end{pmatrix}$$

Table 5 presents the (X; Y; Z) data and the plotting points $S_x^* = S_x/\sigma_x$ $S_y^* = S_y/\sigma_y$, and $S_z^* = S_z/\sigma_z$, with $(\sigma_x; \sigma_y; \sigma_z) = (0.04; 0.01; 0.02)^{1/2}$. To assure an $ARL_0 = 370.4$, the control limit was fixed as 2.016, see Table 3. The warning limit doesn't affect the false alarm risk, it was fixed as 1.323.

According to Fig. 3, after selecting the first sample, the quality characteristic X of the five sample items were measured. The standard deviation S_x of these five measurements was computed to obtain the first plotting point $S_x^* = S_x/\sigma_x$. As this point fell in the warning region, the charting statistic didn't change, that is, the quality characteristic X of the second sample items were measured. The standard deviation S_x of these five measurements was computed and used to obtain the second plotting point $S_x^* = S_x/\sigma_x$. As the second sample point fell in the central region, the charting statistic changed to $S_y^* = S_y/\sigma_y$, that is, the quality characteristic Y of the third sample items were measured. The standard deviation S_y of these five measurements was computed and used to obtain the third plotting point $S_y^* = S_y/\sigma_y$. As the third sample point also fell in the central region, the charting statistic changed again, now to $S_z^* = S_z/\sigma_z$. The monitoring with the *VCS* policy follows until the chart signals. According to Fig. 3, the signal was given when the charting statistic $S_x^* = S_x/\sigma_x$ was in use.

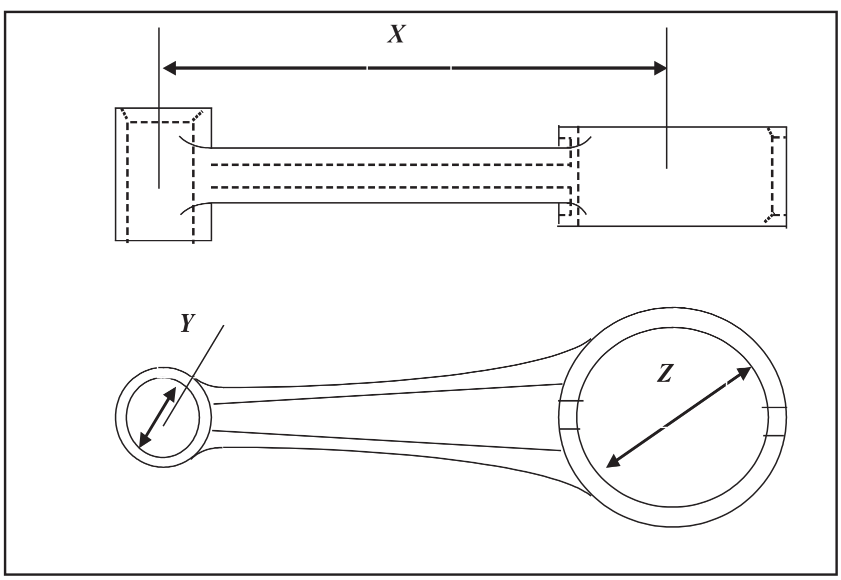

Fig. 2. Part of the example of application.

Table 5
The data and the values of the charting statistics ($S_x^*; S_y^*; S_z^*$).

Sample	Variable	Observations					S_x^*	S_y^*	S_z^*	Region
		1	2	3	4	5				
1	X	20.12	20.18	19.84	20.32	19.64	1.37			Warning
2	X	19.87	20.02	19.76	20.17	20.12	0.86			Central
3	Y	7.13	6.80	6.92	7.04	6.99		0.88		Central
4	Z	3.86	4.29	3.91	3.86	3.98			1.80	Warning
5	Z	3.92	4.12	3.93	3.89	3.99			0.91	Central
6	X	6.71	7.57	6.81	6.72	6.95	1.79			Warning
7	X	7.35	6.70	7.31	6.95	6.78	1.50			Warning
8	X	6.82	6.66	6.51	7.06	7.61	2.15			Action

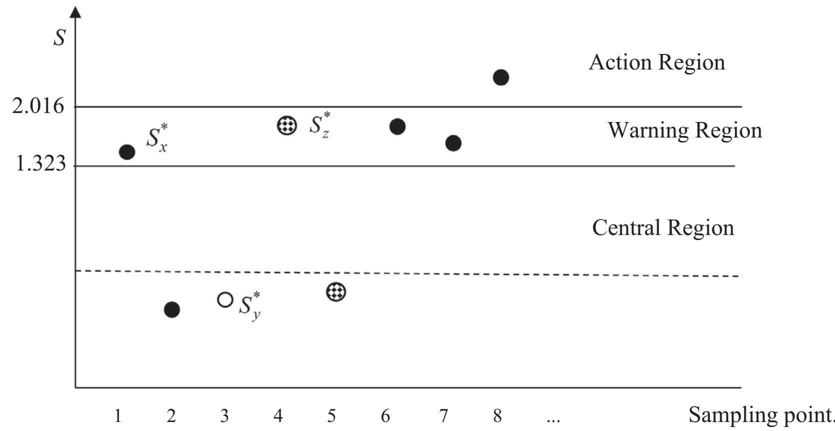

Fig. 3. The illustrative example of the *VCS S* chart.

7. Conclusions

In this article, we introduced the idea of varying the charting statistic. According to the *VCS* policy, the *X*. *Y* and *Z* observations are not from the same samples, consequently, the dependence between quality characteristics doesn't affect the performance of the *VCS* chart. This feature is an additional advantage of the *VCS* chart, once the design of the competing VMAX and |S| charts require accurate estimation of the correlation between each pair of quality characteristics. It is worthwhile the superiority of *VCS* chart against the competing |S| chart, but the main reason to advocate the use of the *VCS* chart lies in the fact that standard deviations are much easier to compute than the sample variance-covariance matrices and their determinants. Regarding to the monitoring of trivariate processes, the *VCS S* chart is superior to the VMAX chart, even when it works with smaller number of measurements per sample.

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Appendix A

A.1. The ARL of the S chart with variable charting statistic

The speed with which the bivariate *VCS S* chart signals an out-of-control condition depends on the charting statistic that is in use, just after the assignable cause occurrence. This charting statistic has a fifty-fifty probability of being S_x - the sample standard deviation of the *X* observations or S_y - the sample standard deviation of the *Y* observations. Thus, the *ARL* is computed as $(ARL_X + ARL_Y)/2$, where ARL_X (ARL_Y) is the *ARL* value when S_x (S_y) is the charting statistic in use, just after the assignable cause occurrence. When the monitoring is initialized, the first quality characteristic to be measured is randomly selected with fifty-fifty probability of being the *X* or the *Y* quality characteristic. After a false alarm, the same procedure decides the next quality characteristic to be measured.

When the charting statistic S_x is in use the expected number of points until the first one falling outside the warning region is $E(M_X) = a$.

$$a = \sum_{i=1}^{\infty} i q_x^{i-1}(1-g_x) = \frac{1}{1-g_x} \tag{A1}$$

If *CL* and *WL* are respectively the control and the warning limits of the *VCS S* chart, then $g_X = Pr[CL < S_X < WL]$. When the charting statistic S_y is in use, the expected number of points until the first one falling outside the warning region is $E(M_Y) = b$:

$$b = \sum_{i=1}^{\infty} i q_y^{i-1}(1-g_y) = \frac{1}{1-g_y} \tag{A2}$$

In expression (A2) $g_y = Pr[CL < S_y < WL]$.

The ARL_X is the expected number of points on the chart until the chart signals, given that the first point on the chart, just after the assignable cause or the false alarm occurrences is the standard deviation of *X* values:

$$\begin{aligned} ARL_X &= ap_x + (a+b)(1-p_x)p_y + (2a+b)(1-p_x)(1-p_y)p_x \\ &\quad + (2a+2b)(1-p_x)^2(1-p_y)p_y \\ &\quad + (3a+2b)(1-p_x)^2(1-p_y)^2 p_x \\ &\quad + (3a+3b)(1-p_x)^3(1-p_y)^2 p_y + \cdots \\ &= \sum_{m=1}^{\infty} m(a+b)[(1-p_X)(1-p_y)]^{m-1}[p_x + (1-p_x)p_y] \\ &\quad - \sum_{m=1}^{\infty} b[(1-p_x)(1-p_y)]^{m-1} p_x = \frac{a+b-bp_x}{p_x+p_y-p_xp_y} \end{aligned} \tag{A3}$$

In expression (A3):

$$p_x = \frac{Pr[S_x > CL]}{1 - Pr[CL < S_x < WL]} \tag{A4}$$

$$p_y = \frac{Pr[S_y > CL]}{1 - Pr[CL < S_y < WL]} \tag{A5}$$

Similar development leads to the expression for the ARL_Y:

$$ARL_Y = \frac{a+b-ap_y}{p_x+p_y-p_xp_y} \tag{A6}$$

From (A3) and (A6). it follows that:

$$ARL = \frac{ARL_X + ARL_Y}{2} = \frac{a+b+a(1-p_y)+b(1-p_x)}{2(p_x+p_y-p_xp_y)} \tag{A7}$$

Extending to the trivariate case:

$$\begin{aligned} ARL_X &= \sum_{i=1}^{\infty}[(m-b-c)p_x + (m-c)(1-p_X)p_y + m(1-p_x)(1-p_y)p_z](1-q)^{i-1} \\ &= \frac{a+b+c-bp_x-cp_y-cp_x(1-p_y)}{1-q} \end{aligned} \tag{A8}$$

In Expression (A8) $q = p_x + p_y + p_z - p_x p_z - p_x p_z + p_x p_y p_z$ and $m = i(a + b + c)$, with $c = 1/(1 - g_Z)$. The probability g_x and p_z are related to the charting statistic S_z, that is, $g_z = Pr[CL < S_z < WL]$ and

$$p_z = \frac{Pr[S_z > CL]}{1 - Pr[CL < S_z < WL]} \quad \text{(A9)}$$

Similar developments lead to the expressions for the ARL_Y and for the ARL_Z:

$$ARL_Y = \frac{a + b + c - cp_y - ap_z - ap_y(1 - p_z)}{1 - q} \quad \text{(A10)}$$

$$ARL_Z = \frac{a + b + c - ap_z - bp_x - bp_z(1 - p_x)}{1 - q} \quad \text{(A11)}$$

The *ARL* of the trivariate *VCS S* chart is given by the average of the (ARL_X. ARL_Y. ARL_Z):

$$ARL = \frac{ARL_X + ARL_Y + ARL_Z}{3} = \frac{3(a + b + c) - [2b + c(1 - p_y)]p_x - [2c + a(1 - p_z)]p_y - [2a + b(1 - p_x)]p_z}{3(1 - q)} \quad \text{(A12)}$$

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