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On local $\mathcal{H}_\infty$ switched controller design for uncertain T–S fuzzy systems subject to actuator saturation with unknown membership functions

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Abstract

This manuscript proposes a local $\mathcal{H}_\infty$ switched controller design for a class of uncertain nonlinear plants described by Takagi–Sugeno (T–S) fuzzy models with unknown membership functions. The control design requires only the lower and upper bounds of the system nonlinearities and of the system linear parameters, which can depend on uncertain parameters. The switched control law chooses a state-feedback controller gain, which belongs to a given set of gains, that minimizes the time derivative of a quadratic Lyapunov function. This procedure eliminates the necessity of finding the membership function expressions to implement the control law, guarantees an $\mathcal{H}_\infty$ performance and ensures that the state trajectory remains within a region in which the T–S fuzzy model is valid. Due to the $\mathcal{H}_\infty$ control design, that frequently results in very large control inputs, it is considered that the switched control law is subject to actuator saturation. Finally, two examples are presented. The first example studies the control of a chaotic Lorenz system. It shows that, for disturbances with large magnitude, the proposed procedures provided better results than the obtained with another recent method found in the literature, that considers full access to the membership functions. In the second example, a practical implementation of an active nonlinear suspension control system, considering an uncertain bounded mass and a fault in the actuator, confirms the effectiveness of the proposed approach.

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1. Introduction

A broad class of nonlinear systems can be exactly described by Takagi–Sugeno (T–S) fuzzy models [1,2] in a given operation region, through a combination of local linear models by means of nonlinear fuzzy membership functions. The operation region commonly is a bounded region in the state space and one can assure that the plant can be exactly represented by a T–S fuzzy model only in this region. Based on parallel distributed compensation [3] and linear matrix inequalities (LMIs), different control design techniques have been proposed over the past decades [2,4–19] for controlling T–S fuzzy models.

However, for a nonlinear system described by T–S fuzzy model, some problems can hinder a practical implementation. Usually, the control design procedure assumes that the membership functions are completely known. In practice, membership functions can depend on immeasurable premise variables or uncertain parameters. For this reason, methods that allow to use T–S fuzzy models with uncertainties in the membership functions or premise variables have been studied. An $\mathcal{H}_\infty$ filter design for T–S fuzzy systems with unknown or partially unknown membership functions is proposed in [20]. Considering again that the membership functions are unmeasurable or unknown, [21] addresses the fault detection problem for T–S fuzzy systems, based on a new fault detection filter with varying gains and [22] presents results regarding the design of $\mathcal{H}_\infty$ observer-based controllers for T–S fuzzy systems, with the premise variables estimation. A switched control design for a class of uncertain nonlinear plants, based on a minimum-type Lyapunov function, is proposed in [18] and this methodology eliminates the need to find the membership function expressions to implement the control law and the design conditions are given by bilinear matrix inequalities (BMIs).

A good performance of the controlled system can be prejudiced due to the inaccurate representation of the nonlinear system, because usually the operation region and practical constraints, inherent to the plants or actuators, are neglect in the control designs [23]. Thus, several papers have proposed methods that ensure that the state trajectories remain within the operation region in which the fuzzy model T–S is valid [24–26] and in [23,27–29] are also considered that the actuator is subject to saturation.

For nonlinear systems subject to external disturbances, the $\mathcal{H}_\infty$ control is a technique that mitigates the exogenous input effect in the system output and, consequently, improves the system performance. However, the actuator saturation is an inevitable problem for most $\mathcal{H}_\infty$ control designs. Then, [27] proposes an $\mathcal{H}_\infty$ fuzzy control design for nonlinear systems subject to actuator saturation. The design conditions ensure that the closed-loop system, subject to energy-bounded disturbance, has an $\mathcal{H}_\infty$ performance, but there is no assurance that the state trajectories will remain within the operation region. In [26], for a known nonlinear system with disturbance bounded by magnitude and energy, the proposed $\mathcal{H}_\infty$ fuzzy control guarantees an $\mathcal{H}_\infty$ performance and ensures that the state trajectories will not escape from the operation region.

In the literature there are different techniques to deal with the problem of control signal saturation. For instance, in [27,30] the saturation is described by a sector bounded condition. The method proposed in [31] presents a less conservative estimation of the domain of attraction by using a quadratic Lyapunov function and a saturation representation based on a polyhedral set. This methodology proved to be less conservative than existing conditions which are based on the circle criterion (see [31]), for instance. Moreover, the design conditions can be expressed in terms of LMIs, including the saturation constraints in the control design.

This manuscript proposes a local $\mathcal{H}_\infty$ switched controller design method for a class of uncertain nonlinear systems described by T–S fuzzy model subject to energy-bounded disturbances. For the representation of the uncertain nonlinear system, it is necessary only to know the lower and upper bounds of the system nonlinearities and linear terms that are calculated considering the operation region in the state space and the known set of uncertain parameters. Due to the $\mathcal{H}_\infty$ control design, that frequently results in very large control inputs, it is considered that the switched control law is subject to actuator saturation. The saturation representation is based on a polyhedral set, proposed in [31]. Using auxiliaries matrices, the switched control law selects a state-feedback controller gain that minimizes the time derivative of a quadratic Lyapunov function. This procedure eliminates the necessity to find the membership function expressions to implement the control law, which is an advantage because these terms may contain uncertain parameters or even have long and/or complex expressions. The control law also guarantees an $\mathcal{H}_\infty$ performance and ensures that the state trajectory remains within a region in which the T–S fuzzy model is valid.

Additionally, the proposed method is applied in two examples. The first example shows a comparison between the proposed switched control design and the method presented in [26]. The results make it clear that for an uncertain nonlinear system, subject to disturbances with large magnitude, the proposed procedure is the best option. Next, in the second example, an implementation demonstrates the practical effectiveness of the method, for controlling an active suspension system fabricated by Quanser® [32], considering a nonlinear spring, an uncertain bounded mass and an actuator fault.

The structure of this manuscript is as follows. In Section 2 the preliminary results on the T–S fuzzy model, the switched control law subject to saturation and the local $\mathcal{H}_\infty$ control problem are presented. Section 3 proposes a local $\mathcal{H}_\infty$ switched controller design for a class of uncertain nonlinear systems. Examples illustrate the proposed method performance in Section 4. Finally, Section 5 draws the conclusions.

For convenience, in some places, the following notation is used: $\mathbb{K}_r = \{1, 2, \ldots, r\}$, $r \in \mathbb{N}$, $x(t) = x$, $\alpha_i(z(t)) = \alpha_i$. For symmetric matrices, the symbol $(*)$ denotes each of their symmetric blocks and I represents the identity matrix with appropriate dimension. $\text{diag}\{M_1, M_2, \ldots, M_r\}$ represents a block-diagonal matrix in which the diagonal elements are M_1, M_2, …, M_r. $\text{co}\{a_1, \cdots, a_r\}$ is the convex hull of the vectors a_i, $i \in \mathbb{K}_r$. $M > 0$ ($M < 0$, $M \geq 0$ and $M \leq 0$) means that the matrix M is positive definite (negative definite, positive semi-definite and negative semi-definite, respectively). The space of square-integrable vector functions over $[0, \infty)$ is denoted by $\mathcal{L}_2[0, \infty)$, and for $w(t) \in \mathcal{L}_2[0, \infty)$, its norm is denoted by $\|w\|_2 = \sqrt{\int_{t=0}^{\infty} w(t)^T w(t) dt}$. Finally, E_z represents a generic matrix, such that

$$E_z = \sum_{i=1}^{r} \alpha_i E_i \quad \text{with} \quad \alpha = [\alpha_1\, \alpha_2\, \ldots\, \alpha_r]^T \in \Lambda_r = \left\{ \alpha \in \mathbb{R}^r : \alpha_i \geq 0,\ \sum_{i=1}^{r} \alpha_i = 1,\ i \in \mathbb{K}_r \right\}. \tag{1}$$

2. Problem statement and preliminaries

2.1. Takagi–Sugeno fuzzy systems

Consider an uncertain nonlinear system, subject to actuator saturation and energy-bounded disturbances, described by

$$\dot{x}(t) = f_1(z(t))x(t) + f_2(z(t))\,\text{sat}(u(t)) + f_3(z(t))w(t), \tag{2a}$$

$$y(t) = g_1(z(t))x(t) + g_2(z(t))\,\text{sat}(u(t)) + g_3(z(t))w(t), \tag{2b}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the input vector, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance input such that $w(t) \in \mathcal{L}_2[0, \infty)$ and $y(t) \in \mathbb{R}^{n_y}$ is the output vector. The uncertain nonlinear system dynamics is given by the nonlinear functions $f_1(\cdot) : \mathbb{R}^q \to \mathbb{R}^{n_x \times n_x}$, $f_2(\cdot) : \mathbb{R}^q \to \mathbb{R}^{n_x \times n_u}$, $f_3(\cdot) : \mathbb{R}^q \to \mathbb{R}^{n_x \times n_w}$, $g_1(\cdot) : \mathbb{R}^q \to \mathbb{R}^{n_y \times n_x}$, $g_2(\cdot) : \mathbb{R}^q \to \mathbb{R}^{n_y \times n_u}$ and $g_3(\cdot) : \mathbb{R}^q \to \mathbb{R}^{n_y \times n_w}$. $z(t) \in \mathbb{R}^q$ is a vector whose the entries $z_l(t)$, $l \in \mathbb{K}_q$, are the premise variables that depends on the state vector $x(t)$ and uncertain parameters or unknown variables, and $\text{sat}(u(t)) \in \mathbb{R}^{n_u}$ is the amplitude-bounded control input, such that

$$\text{sat}(u(t)) = \begin{bmatrix} \text{sat}(u_{(1)}(t)) \\ \vdots \\ \text{sat}(u_{(n_u)}(t)) \end{bmatrix}, \ \text{sat}(u_{(k)})(t) = \begin{cases} -\rho_{(k)}, & \text{if} \quad u_{(k)}(t) < -\rho_{(k)}, \\ u_{(k)}(t), & \text{if} \quad \left|u_{(k)}(t)\right| \leq \rho_{(k)}, \ \forall k \in \mathbb{K}_{n_u}, \\ \rho_{(k)}, & \text{if} \quad u_{(k)}(t) > \rho_{(k)}, \end{cases} \tag{3}$$

where $\rho_{(k)}$, $k \in \mathbb{K}_{n_u}$, are known positive constants.

Let an operation region $\mathcal{X}$ in the state space defined as follows [23]:

$$\mathcal{X} := \left\{ x(t) \in \mathbb{R}^{n_x} : \left|R_{(h)}x(t)\right| \leq \phi_{(h)},\ h \in \mathbb{K}_p \right\}, \tag{4}$$

where $R = \left[R_{(1)}^T \cdots R_{(p)}^T\right]^T \in \mathbb{R}^{p \times n_x}$ and $\phi = \left[\phi_{(1)} \cdots \phi_{(p)}\right]^T \in \mathbb{R}^p$ are known. Consider that in the region $\mathcal{X}$, the system (2) can be exactly represented by a T–S fuzzy model as described below in (5) and (6) [2,14,18,29]:

$$\underline{Rule\ i}: \quad \text{IF } z_1(t) \text{ is } \mu_{i1} \text{ and } \dots \text{ and } z_q(t) \text{ is } \mu_{iq}, \quad \text{THEN} \begin{cases} \dot{x}(t) = A_i x(t) + B_i \operatorname{sat}(u(t)) + H_i w(t), \\ y(t) = C_i x(t) + D_i \operatorname{sat}(u(t)) + G_i w(t), \end{cases} \tag{5}$$

where $i \in \mathbb{K}_r$, $l \in \mathbb{K}_q$, μ_{il} is a fuzzy set of the rule i corresponding to the function $z_l(t)$ and the following matrices are known: $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$, $H_i \in \mathbb{R}^{n_x \times n_w}$, $C_i \in \mathbb{R}^{n_y \times n_x}$, $D_i \in \mathbb{R}^{n_y \times n_u}$ and $G_i \in \mathbb{R}^{n_y \times n_w}$.

From [4] and the definitions in (1), $\dot{x}(t)$ and $y(t)$ given in (5) can be written as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(z(t))(A_i x(t) + B_i \operatorname{sat}(u(t)) + H_i w(t)) = A_z x(t) + B_z \operatorname{sat}(u(t)) + H_z w(t), \tag{6a}$$

$$y(t) = \sum_{i=1}^{r} \alpha_i(z(t))(C_i x(t) + D_i \operatorname{sat}(u(t)) + G_i w(t)) = C_z x(t) + D_z \operatorname{sat}(u(t)) + G_z w(t), \tag{6b}$$

where $\alpha_i(z(t)) = \dfrac{\mu_{i1}(z_1(t)) \times \cdots \times \mu_{iq}(z_q(t))}{\sum_{i=1}^{r}\left(\mu_{i1}(z_1(t)) \times \cdots \times \mu_{iq}(z_q(t))\right)}$, and $\mu_{il}(z_l(t))$ is the grade of membership, in terms of unknown variables or uncertain parameters, corresponding to the fuzzy term μ_{il}, $i \in \mathbb{K}_r$ and $l \in \mathbb{K}_q$. The entry α_i of the vector $\alpha = [\alpha_1\, \alpha_2 \dots \alpha_r]^T \in \Lambda_r$ given in (1) is the normalized weight of each local model system (A_i, B_i, C_i, D_i, G_i, H_i) defined in (5), for $i \in \mathbb{K}_r$.

Remark 1. To ensure that the uncertain nonlinear system (2) is exactly described by a T–S fuzzy model (6), it is adopted a procedure presented in [14,18,29]. The lower and upper bounds of the system nonlinearities and uncertain linear terms are calculated considering the given operation region of the state vector and also the known set of the plant uncertain parameters. The obtained T–S fuzzy models with this procedure, can exactly represent uncertain nonlinear systems described in (2) by a T–S fuzzy model (6), that present known local models and unknown normalized weights. This method was applied for controlling a magnetic levitator system [18,29], a ball-and-beam system [29] and an active suspension system described in the Example 2 of this manuscript.

Considering the quadratic Lyapunov function candidate

$$V(x(t)) = x(t)^T P x(t), \tag{7}$$

where $P \in \mathbb{R}^{n_x \times n_x}$ is a symmetric positive definite matrix. For a positive constant v_0, then it is defined the ellipsoid

$$\mathcal{E}(V, v_0) := \left\{ x \in \mathbb{R}^{n_x} : x^T P x \le v_0 \right\}. \tag{8}$$

2.2. Switched control law subject to saturation

The proposed procedure uses auxiliary symmetric matrices $\bar{Q}_j$, $j \in \mathbb{K}_r$, that are responsible for determining the value of the switching index σ, as described in (9). This index σ selects a state-feedback controller gain, which belongs to the set of gains $\{K_j \in \mathbb{R}^{n_u \times n_x},\ j \in \mathbb{K}_r\}$. The switched control law is defined as follows:

$$u(t) = u_\sigma(t) = -K_\sigma x(t), \qquad \sigma = \arg^* \min_{j \in \mathbb{K}_r} \{x(t)^T \bar{Q}_j x(t)\}, \tag{9}$$

where $\arg^* \min_{j \in \mathbb{K}_r} \{x(t)^T \bar{Q}_j x(t)\}$ denotes the smallest index $\sigma \in \mathbb{K}_r$, such that $x(t)^T \bar{Q}_\sigma x(t) = \min_{j \in \mathbb{K}_r} \{x(t)^T \bar{Q}_j x(t)\}$.

Note that it is not necessary to use the membership functions to implement the control law (9). Therefore, the membership functions may depend on uncertain and unknown parameters. It allows the application of this control strategy for controlling uncertain nonlinear plants exactly described by T–S fuzzy models with known local models and unknown normalized weights, using the procedure proposed in [14,18,29] (see Remark 1 for more details).

Let $\mathcal{F}$ be the set composed by diagonal matrices $F_s \in \mathbb{R}^{n_u \times n_u}$, $s \in \mathbb{K}_{2^{n_u}}$, whose diagonal elements are either 0 or 1 [33]. For example, if $n_u = 2$, then

$$\mathcal{F} = \left\{ F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, F_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, F_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}, \tag{10}$$

and for all $s \in \mathbb{K}_{2^{n_u}}$, F_s^- denotes the element of $\mathcal{F}$ associated with F_s, such that $F_s^- = I - F_s$. For convenience, the following notation is used:

$$F_\lambda = \sum_{s=1}^{2^{nu}} \lambda_s F_s, \quad F_\lambda^- = \sum_{s=1}^{2^{nu}} \lambda_s F_s^- \quad \text{with} \quad \lambda \in \Lambda_{2^{n_u}} = \left\{ \lambda \in \mathbb{R}^{2^{n_u}} : \lambda_s \geq 0, \sum_{s=1}^{2^{nu}} \lambda_s = 1, s \in \mathbb{K}_{2^{n_u}} \right\}. \tag{11}$$

Let the polyhedral set $\mathcal{L}(L_j)$ given by

$$\mathcal{L}(L_j) := \{x \in \mathbb{R}^{n_x} : \left|L_{j(k)} x\right| \leq \rho_{(k)},\ j \in \mathbb{K}_r,\ k \in \mathbb{K}_{n_u}\}, \tag{12}$$

where $L_j = \left[L_{j(1)}^T \cdots L_{j(n_u)}^T\right]^T \in \mathbb{R}^{n_u \times n_x}$ and $\rho = \left[\rho_{(1)} \cdots \rho_{(n_u)}\right]^T \in \mathbb{R}^{n_u}$ is a known vector.

For $x(t) \in \mathcal{L}(L_j)$, $\forall j \in \mathbb{K}_r$, then $x(t) \in \mathcal{L}(L_\sigma)$. According to [31,33], it follows that $\text{sat}(u(t)) = \text{sat}(-K_\sigma x(t)) \in \text{co}\{F_s(-K_\sigma x(t)) + F_s^- L_\sigma x(t)\}$. Consequently, using the notations (11), $\text{sat}(u(t))$ can be represented as:

$$\text{sat}(u(t)) = \sum_{s=1}^{2^{nu}} \lambda_s \left(F_s(-K_\sigma x(t)) + F_s^- L_\sigma x(t)\right) = \left(-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right) x(t). \tag{13}$$

Substituting (13) into (6) and using the notations (1) and (11), the closed-loop system can be rewritten as follows

$$\dot{x}(t) = A_z x(t) + B_z \left(-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right) x(t) + H_z w(t), \tag{14a}$$

$$y(t) = C_z x(t) + D_z \left(-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right) x(t) + G_z w(t). \tag{14b}$$

2.3. Local $\mathcal{H}_\infty$ control problem

As is well known, besides stability, a controller must assure a good performance of the closed-loop system. In this sense, an important index is the $\mathcal{H}_\infty$ norm, which is related to the capacity of the controlled system to reject energy-bounded disturbances [11,34]. Thus, consider the energy-bounded disturbance $w(t) \in \mathcal{W}$ and a positive constant ϵ, such that

$$\mathcal{W} := \left\{ w(t) \in \mathbb{R}^{n_w} : \int_0^\infty w(t)^T w(t) dt \leq \epsilon \right\}. \tag{15}$$

Based on [26], considering a positive slack variable φ and a positive constant ϵ_0, the local $\mathcal{H}_\infty$ control problem consists in determining a control law that satisfies the following statements:

1. for $w(t) = 0$, $t \geq 0$, the zero equilibrium point of (9) and (14) is locally asymptotically stable and the ellipsoid $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ is an invariant subset of the domain of attraction [31] (i.e., if $x(0)$ belongs to this set, then $x(t), t > 0$, will also stay in this set). Fig. 1a shows this propriety;
2. for $w(t) \in \mathcal{W}$, any trajectory with initial condition within $\mathcal{E}(V, \epsilon_0)$ (i.e., $x(0)^T P x(0) \leq \epsilon_0$) will not escape the ellipsoid $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$, for all $t \geq 0$. Fig. 1b illustrates this propriety;
3. for $w(t) \in \mathcal{W}$ and $x(0) = 0$, the closed-loop T–S fuzzy system (14) has an $\mathcal{H}_\infty$ guaranteed cost $\gamma > 0$, satisfying the following inequality

$$\|y(t)\|_2^2 \leq \gamma^2 \|w(t)\|_2^2, \quad \forall \alpha \in \Lambda_r. \tag{16}$$

3. Main result

This section presents a switched control design to deal with the local $\mathcal{H}_\infty$ control problem defined previously. For convenience, considering (1) and (11) and $j \in \mathbb{K}_r$, define the following matrix functions:

$$\mathcal{M}_{z\lambda}^{(\dot{x})}(P, K_j, L_j) = \begin{bmatrix} A_z^T P + P A_z + P B_z \left[-F_\lambda K_j + F_\lambda^- L_j\right] + \left[-F_\lambda K_j + F_\lambda^- L_j\right]^T B_z^T P & * \\ H_z^T P & -\varphi^{-1} I \end{bmatrix}, \tag{17}$$

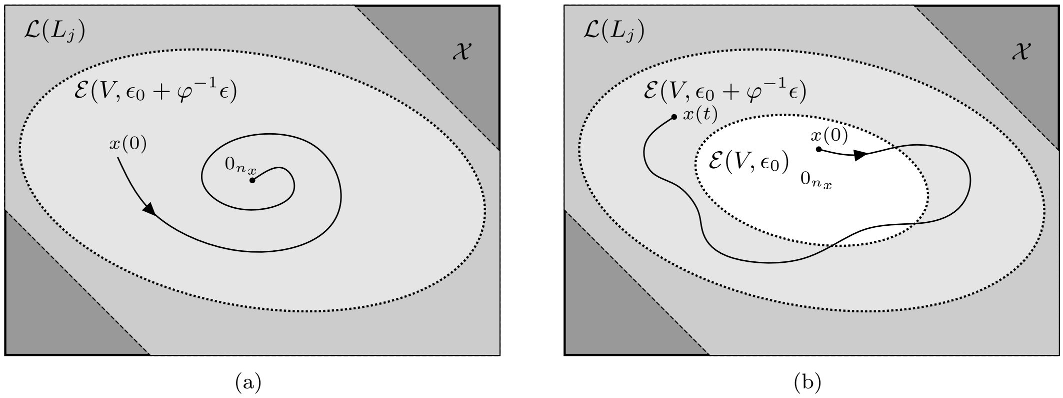


Fig. 1. (a) The inclusion relations among the sets $\mathcal{X}$, $\mathcal{L}(L_j)$ and $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$, and the state trajectory $x(t)$ for $w(t) = 0$, for all $t \geq 0$; (b) The inclusion relations among the sets $\mathcal{X}$, $\mathcal{L}(L_j)$, $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ and $\mathcal{E}(V, \epsilon_0)$, and the state trajectory $x(t)$ for $w(t) \in \mathcal{W}$, for all $t \geq 0$.

$$\mathcal{N}_{z\lambda}^{(\dot{x})}(P, K_j, L_j) = PB_z\left[-F_\lambda K_j + F_\lambda^- L_j\right] + \left[-F_\lambda K_j + F_\lambda^- L_j\right]^T B_z^T P + \varphi P H_z H_z^T P, \tag{18}$$

$$\mathcal{M}_{z\lambda}^{(\dot{x}y)}(P, K_j, L_j) = \begin{bmatrix} A_z^T P + PA_z + PB_z\left[-F_\lambda K_j + F_\lambda^- L_j\right] + \left[-F_\lambda K_j + F_\lambda^- L_j\right]^T B_z^T P & * \\ H_z^T P & -\varphi^{-1} I \end{bmatrix}$$
$$+ \begin{bmatrix} \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)^T \\ G_z^T \end{bmatrix} \varphi^{-1}\gamma^{-2} I \begin{bmatrix} \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)^T \\ G_z^T \end{bmatrix}^T, \tag{19}$$

$$\mathcal{N}_{z\lambda}^{(\dot{x}y)}(P, K_j, L_j) = PB_z\left[-F_\lambda K_j + F_\lambda^- L_j\right] + \left[-F_\lambda K_j + F_\lambda^- L_j\right]^T B_z^T P$$
$$+ \varphi^{-1}\gamma^{-2}\left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)^T \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)$$
$$+ \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T (C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right])\right)^T \left(\varphi^{-1} I - \varphi^{-1}\gamma^{-2} G_z^T G_z\right)^{-1}$$
$$\times \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T (C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right])\right). \tag{20}$$

The following lemmas will be used in the proof of the main result of this manuscript. Lemmas 1 and 2 are based on [35].

Lemma 1. *Consider* $\mathcal{M}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma)$ *and* $\mathcal{N}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma)$ *given in (19) and (20), respectively, and suppose that* $-\varphi^{-1} I + \varphi^{-1}\gamma^{-2} G_z^T G_z < 0$. *Then, the following condition*

$$\sup_{w \in \mathcal{L}_2} \left\{ \begin{bmatrix} x^T & w^T \end{bmatrix} \mathcal{M}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma) \begin{bmatrix} x^T & w^T \end{bmatrix}^T \right\} = x^T \left\{ A_z^T P + PA_z + \mathcal{N}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma) \right\} x \tag{21}$$

holds and the optimal solution of the left side of (21) is

$$w^* = \left(\varphi^{-1} I - \varphi^{-1}\gamma^{-2} G_z^T G_z\right)^{-1} \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T \left(C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]\right)\right) x. \tag{22}$$

Proof. Defining the function

$$f(w, x) = \begin{bmatrix} x^T & w^T \end{bmatrix} \mathcal{M}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma) \begin{bmatrix} x^T & w^T \end{bmatrix}^T, \tag{23}$$

the partial derivative of the function $f(w, x)$ with respect to w is

$$\frac{\partial f(w, x)}{\partial w} = 2\left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T \left(C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]\right)\right) x - 2\left(\varphi^{-1} I - \varphi^{-1}\gamma^{-2} G_z^T G_z\right) w. \tag{24}$$

Thus, for $\frac{\partial f(w,x)}{\partial w} = 0$ the critical point w^* (22) is obtained. Moreover, note that

$$\frac{\partial^2 f(w,x)}{\partial w^2} = -2\left(\varphi^{-1} I - \varphi^{-1}\gamma^{-2} G_z^T G_z\right) < 0, \tag{25}$$

and then $f(w^*, x)$ is a maximum point. Now, replacing w^* in $f(w, x)$

$$\begin{aligned}
f(w^*, x) = & \begin{bmatrix} x^T & w^{*T} \end{bmatrix} \mathcal{M}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma) \begin{bmatrix} x^T & w^{*T} \end{bmatrix}^T \\
= & \, x^T \Big\{ A_z^T P + P A_z + P B_z \left[-F_\lambda K_j + F_\lambda^- L_j\right] + \left[-F_\lambda K_j + F_\lambda^- L_j\right]^T B_z^T P \\
& + \varphi^{-1}\gamma^{-2} \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)^T \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right) \Big\} x \\
& + x^T \Big\{ P H_z + \varphi^{-1}\gamma^{-2} \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)^T G_z \Big\} w^* \\
& + w^{*T} \Big\{ H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right) \Big\} x \\
& + w^{*T} \Big\{ -\varphi^{-1} I + \varphi^{-1}\gamma^{-2} G_z^T G_z \Big\} w^* \\
= & \, x^T \Big\{ A_z^T P + P A_z + P B_z \left[-F_\lambda K_j + F_\lambda^- L_j\right] + \left[-F_\lambda K_j + F_\lambda^- L_j\right]^T B_z^T P \\
& + \varphi^{-1}\gamma^{-2} \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)^T \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right) \Big\} x \\
& + x^T \Big\{ \left(P H_z + \varphi^{-1}\gamma^{-2} \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)^T G_z\right) \left(\varphi^{-1} I - \varphi^{-1}\gamma^{-2} G_z^T G_z\right)^{-1} \\
& \times \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T \left(C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]\right)\right) \Big\} x \\
& + x^T \Big\{ \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T \left(C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]\right)\right)^T \left(\varphi^{-1} I - \varphi^{-1}\gamma^{-2} G_z^T G_z\right)^{-T} \\
& \times \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T \left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)\right) \Big\} x \\
& + x^T \Big\{ \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T \left(C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]\right)\right)^T \left(\varphi^{-1} I - \varphi^{-1}\gamma^{-2} G_z^T G_z\right)^{-T} \\
& \times \left(-\varphi^{-1} I + \varphi^{-1}\gamma^{-2} G_z^T G_z\right) \left(\varphi^{-1} I - \varphi^{-1}\gamma^{-2} G_z^T G_z\right)^{-1} \\
& \times \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T \left(C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]\right)\right) \Big\} x
\end{aligned} \tag{26}$$

one obtains the right side of (21), and the proof is concluded. □

Lemma 2. *Consider* $\mathcal{M}_{z\lambda}^{(\dot{x})}(P, K_\sigma, L_\sigma)$ *and* $\mathcal{N}_{z\lambda}^{(\dot{x})}(P, K_\sigma, L_\sigma)$ *given in* (17) *and* (18)*, respectively and suppose that* $\varphi > 0$*. Then, the following condition*

$$\sup_{w \in \mathcal{L}_2} \left\{ \begin{bmatrix} x^T & w^T \end{bmatrix} \mathcal{M}_{z\lambda}^{(\dot{x})}(P, K_\sigma, L_\sigma) \begin{bmatrix} x^T & w^T \end{bmatrix}^T \right\} = x^T \left\{ A_z^T P + P A_z + \mathcal{N}_{z\lambda}^{(\dot{x})}(P, K_\sigma, L_\sigma) \right\} x \tag{27}$$

holds and the optimal solution of the left side of (27) *is*

$$w^* = \varphi H_z^T P x. \tag{28}$$

Proof. Defining the function

$$f(w, x) = \begin{bmatrix} x^T & w^T \end{bmatrix} \mathcal{M}_{z\lambda}^{(\dot{x})}(P, K_\sigma, L_\sigma) \begin{bmatrix} x^T & w^T \end{bmatrix}^T, \tag{29}$$

the partial derivative of the function $f(w, x)$ with respect to w is

$$\frac{\partial f(w,x)}{\partial w} = 2 H_z^T P x - 2\varphi^{-1} w. \tag{30}$$

Thus, for $\frac{\partial f(w,x)}{\partial w}=0$ the critical point w^* (28) is obtained. Moreover, note that

$$\frac{\partial^2 f(w,x)}{\partial w^2} = -2\varphi^{-1} I < 0, \tag{31}$$

and then $f(w^*, x)$ is a maximum point. Now, replacing w^* in $f(w,x)$ one obtains the right side of (27), and the proof is concluded. □

Lemma 3. *Assume that there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices Z_i, $Q_j \in \mathbb{R}^{n_x \times n_x}$, matrices M_j, $N_j \in \mathbb{R}^{n_u \times n_x}$, for all $i, j \in \mathbb{K}_r$ and $s \in \mathbb{K}_{2^{n_u}}$, such that the condition holds:*

$$\begin{bmatrix} B_i\left[-F_s M_j + F_s^- N_j\right] + \left[-F_s M_j + F_s^- N_j\right]^T B_i^T - Z_i - Q_j & * & * \\ H_i^T & -\varphi^{-1} I & * \\ C_i X + D_i\left[-F_s M_j + F_s^- N_j\right] & G_i & -\varphi\mu I \end{bmatrix} < 0. \tag{32}$$

Then, considering (1), (9), (18) and (20), for $x \neq 0$, the conditions below also hold

$$x^T\left\{PB_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^T P\right\}x < x^T \bar{Z}_z x + x^T \bar{Q}_z x, \tag{33a}$$

$$x^T \mathcal{N}_{z\lambda}^{(\dot{x})}(P, K_\sigma, L_\sigma)x < x^T \bar{Z}_z x + x^T \bar{Q}_z x, \tag{33b}$$

$$x^T \mathcal{N}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma)x < x^T \bar{Z}_z x + x^T \bar{Q}_z x, \tag{33c}$$

where $P = X^{-1}$, $\bar{Z}_i = X^{-1} Z_i X^{-1}$, $\bar{Q}_i = X^{-1} Q_i X^{-1}$, $K_j = M_j X^{-1}$, $L_j = N_j X^{-1}$ and $\gamma^2 = \mu > 0$.

Proof. Suppose that there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices Z_i, $Q_j \in \mathbb{R}^{n_x \times n_x}$ and matrices M_j, $N_j \in \mathbb{R}^{n_u \times n_x}$, such that (32) holds, for all $i, j \in \mathbb{K}_r$ and $s \in \mathbb{K}_{2^{n_u}}$. From (1) and (11), remembering that $\alpha_i \geq 0$, $\sum_{i=1}^{r} \alpha_i = 1$, $\lambda_s \geq 0$ and $\sum_{s=1}^{2^{n_u}} \lambda_s = 1$, define $P = X^{-1}$, $K_\sigma = M_\sigma X^{-1}$, $L_\sigma = N_\sigma X^{-1}$, $\bar{Z}_z = X^{-1} Z_z X^{-1}$, $\bar{Q}_\sigma = X^{-1} Q_\sigma X^{-1}$ and $\mu = \gamma^2$. Pre- and post-multiplying (32) by diag$\{P, I, I\}$, replacing j by σ, multiplying the result by α_i and taking the sum from $i = 1$ to $i = r$, multiplying the result by λ_s and taking the sum from $s = 1$ to $s = 2^{n_u}$, then it follows the inequality

$$\begin{bmatrix} PB_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^T P - \bar{Z}_z - \bar{Q}_\sigma & * & * \\ H_z^T P & -\varphi^{-1} I & * \\ C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] & G_z & -\varphi\gamma^2 I \end{bmatrix} < 0, \tag{34}$$

and consequently

$$\begin{bmatrix} PB_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^T P - \bar{Z}_z - \bar{Q}_\sigma & * \\ H_z^T P & -\varphi^{-1} I \end{bmatrix} < 0, \tag{35}$$

$$PB_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^T P - \bar{Z}_z - \bar{Q}_\sigma < 0. \tag{36}$$

Applying the Schur complement to (35) and considering $\mathcal{N}_{z\lambda}^{\dot{x}}(P, K_\sigma, L_\sigma)$, given in (18), one obtains

$$\begin{aligned} &PB_z\left[-F_\lambda K_j + F_\lambda^- L_j\right] + \left[-F_\lambda K_j + F_\lambda^- L_j\right]^T B_z^T P + \varphi P H_z H_z^T P - \bar{Z}_z - \bar{Q}_\sigma \\ &= \mathcal{N}_{z\lambda}^{\dot{x}}(P, K_\sigma, L_\sigma) - \bar{Z}_z - \bar{Q}_\sigma < 0. \end{aligned} \tag{37}$$

Now, applying the Schur complement to (34), it follows that

$$\begin{bmatrix} PB_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^T P - \bar{Z}_z - \bar{Q}_\sigma & * \\ H_z^T P & -\varphi^{-1} I \end{bmatrix}$$
$$+\varphi^{-1}\gamma^{-2}\begin{bmatrix} (C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right])^T \\ G_z^T \end{bmatrix}\begin{bmatrix} (C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right])^T \\ G_z^T \end{bmatrix}^T$$

$$= \begin{bmatrix} PB_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^T P - \bar{Z}_z - \bar{Q}_\sigma \\ +\varphi^{-1}\gamma^{-2}(C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right])^T & * \\ \times(C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]) \\ H_z^T P + \varphi^{-1}\gamma^{-2}G_z^T(C_z + D_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]) & -\varphi^{-1}I + \varphi^{-1}\gamma^{-2}G_z^T G_z \end{bmatrix} < 0 \tag{38}$$

Considering $\mathcal{N}_{z\lambda}^{\dot{x}y}(P, K_\sigma, L_\sigma)$, given in (20), and applying the Schur complement to (38), one obtains

$$\begin{aligned} & PB_z\left[-F_\lambda K_j + F_\lambda^- L_j\right] + \left[-F_\lambda K_j + F_\lambda^- L_j\right]^T B_z^T P - \bar{Z}_z - \bar{Q}_\sigma \\ & +\varphi^{-1}\gamma^{-2}\left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right)^T\left(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right]\right) \\ & +\left(H_z^T P + \varphi^{-1}\gamma^{-2}G_z^T(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right])\right)^T\left(\varphi^{-1}I - \varphi^{-1}\gamma^{-2}G_z^T G_z\right)^{-1} \\ & \times\left(H_z^T P + \varphi^{-1}\gamma^{-2}G_z^T(C_z + D_z\left[-F_\lambda K_j + F_\lambda^- L_j\right])\right) \\ = & \mathcal{N}_{z\lambda}^{\dot{x}y}(P, K_\sigma, L_\sigma) - \bar{Z}_z - \bar{Q}_\sigma < 0. \end{aligned} \tag{39}$$

For $x \neq 0$, from (1) and (9), note that $x^T\bar{Q}_\sigma x = \min_{j\in\mathbb{K}_r}\{x^T\bar{Q}_j x\} \le \sum_{i=1}^{r}\alpha_i x^T\bar{Q}_i x = x^T\bar{Q}_z x$. Thus

$$\sum_{i=1}^{r}\alpha_i x^T\bar{Z}_i x + x^T\bar{Q}_\sigma x \le \sum_{i=1}^{r}\alpha_i x^T\left\{\bar{Z}_i + \bar{Q}_i\right\}x = x^T\left\{\bar{Z}_z + \bar{Q}_z\right\}x. \tag{40}$$

Therefore, for $x \neq 0$, from (40), observe that from (36), (37) and (39) it follows that

$$x^T\left\{PB_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^T P\right\}x < x^T\left\{\bar{Z}_z + \bar{Q}_\sigma\right\}x \le x^T\left\{\bar{Z}_z + \bar{Q}_z\right\}x, \tag{41}$$

$$x^T\mathcal{N}_{z\lambda}^{\dot{x}}(P, K_\sigma, L_\sigma)x < x^T\left\{\bar{Z}_z + \bar{Q}_\sigma\right\}x \le x^T\left\{\bar{Z}_z + \bar{Q}_z\right\}x, \tag{42}$$

$$x^T\mathcal{N}_{z\lambda}^{\dot{x}y}(P, K_\sigma, L_\sigma)x < x^T\left\{\bar{Z}_z + \bar{Q}_\sigma\right\}x \le x^T\left\{\bar{Z}_z + \bar{Q}_z\right\}x, \tag{43}$$

respectively. The conditions (41), (42) and (43) are equivalent to (33a), (33b) and (33c), respectively. Then the proof is concluded. □

Lemma 4. *[29,31,34] Let the sets $\mathcal{X}$, $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ and $\mathcal{L}(L_j)$ given in (4), (8) and (12), respectively. The constraints $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{X}$ and $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{L}(L_j)$ are enforced if the following conditions hold, respectively*

$$\begin{bmatrix} \frac{\phi_{(h)}^2}{\epsilon_0 + \varphi^{-1}\epsilon} & * \\ XR_{(h)}^T & X \end{bmatrix} \ge 0, \tag{44a}$$

$$\begin{bmatrix} \frac{\rho_{(k)}^2}{\epsilon_0 + \varphi^{-1}\epsilon} & * \\ N_{j(k)}^T & X \end{bmatrix} \ge 0, \tag{44b}$$

for all $h \in \mathbb{K}_p$, $j \in \mathbb{K}_r$ and $k \in \mathbb{K}_{n_u}$, where $X = P^{-1}$ and $N_{j(k)} = L_{j(k)}X$.

Proof. See [29]. □

In this context, considering the local $\mathcal{H}_\infty$ control problem, the main theorem is proposed.

3.1. Local $\mathcal{H}_\infty$ switched controller design

Theorem 1. *Consider an operation region $\mathcal{X}$ in which the nonlinear system subject to actuator saturation and energy-bounded disturbances (2) can be exactly described by (14), where $\rho \in \mathbb{R}^{n_u}$, $R \in \mathbb{R}^{p\times n_x}$, $\phi \in \mathbb{R}^p$, $\epsilon_0 \geq 0$, $\epsilon > 0$ and $\varphi > 0$ are known. Assume that there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n_x\times n_x}$, symmetric matrices $Z_i, Q_i \in \mathbb{R}^{n_x\times n_x}$, matrices $M_j, N_j \in \mathbb{R}^{n_u\times n_x}$ and scalar $\mu > 0$, such that the following optimization problem is feasible:*

$$\min \mu$$

subject to

$$\begin{bmatrix} B_i\left[-F_sM_j + F_s^- N_j\right] + \left[-F_sM_j + F_s^- N_j\right]^T B_i^T - Z_i - Q_j & * & * \\ H_i^T & -\varphi^{-1}I & * \\ C_iX + D_i\left[-F_sM_j + F_s^- N_j\right] & G_i & -\varphi\mu I \end{bmatrix} < 0, \tag{45a}$$

$$XA_i^T + A_iX + Z_i + Q_i < 0, \tag{45b}$$

$$\begin{bmatrix} \frac{\phi_{(h)}^2}{\epsilon_0+\varphi^{-1}\epsilon} & * \\ XR_{(h)}^T & X \end{bmatrix} \geq 0, \tag{45c}$$

$$\begin{bmatrix} \frac{\rho_{(k)}^2}{\epsilon_0+\varphi^{-1}\epsilon} & * \\ N_{j(k)}^T & X \end{bmatrix} \geq 0, \tag{45d}$$

for all $i, j \in \mathbb{K}_r$, $h \in \mathbb{K}_p$, $k \in \mathbb{K}_{n_u}$ and $s \in \mathbb{K}_{2^{n_u}}$, where $F_s \in \mathcal{F}$ and $F_s^- = I - F_s$. Then, $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{X}$, $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{L}(L_j)$ and the switched control law (9), where $P = X^{-1}$, $\bar{Q}_i = X^{-1}Q_iX^{-1}$ and the controller gains are given by $K_j = M_jX^{-1}$, ensures that:

1. *for $w(t) = 0$, $t \geq 0$, the uncertain nonlinear system (2) and (9) is locally asymptotically stable for all $x(0) \in \mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ and the ellipsoid $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ is an invariant subset of the domain of attraction (i.e., if $x(0)$ belongs to this set, then $x(t)$, $t > 0$, will also stay in this set). Fig. 1a shows this propriety;*
2. *for $w(t) \neq 0$, if $x(0) \in \mathcal{E}(V, \epsilon_0)$, then $x(t) \in \mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$, for all $t \geq 0$. Fig. 1b illustrates this propriety;*
3. *for $w(t) \neq 0$, if $x(0) = 0$, then the nonlinear system (2) and (9) has an $\mathcal{H}_\infty$ guaranteed cost $\gamma = \sqrt{\mu} > 0$, such that*

$$\|y(t)\|_2^2 \leq \gamma^2 \|w(t)\|_2^2, \tag{46}$$

and $x(t) \in \mathcal{E}(V, \varphi^{-1}\epsilon)$, for all $t \geq 0$.

Proof. Consider the Lyapunov function candidate (7). Preliminarily, according to Lemma 4, observe that (45c) and (45d) ensure that $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{X}$ and $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{L}(L_j)$, respectively. Pre- and post-multiplying (45b) by $P = X^{-1}$, changing the variables $\bar{Z}_i = X^{-1}Z_iX^{-1}$ and $\bar{Q}_i = X^{-1}Q_iX^{-1}$, multiplying the result by $\alpha_i \geq 0$, $i \in \mathbb{K}_r$, $\sum_{i=1}^r \alpha_i = 1$ and taking the sum from $i = 1$ to r, for $x \neq 0$, the following inequality is obtained

$$x^TA_z^TPx + x^TPA_zx + x^T\bar{Z}_zx + x^T\bar{Q}_zx < 0. \tag{47}$$

- First statement:

For the first statement, note that from Lemma 3, (45a) ensures that (33a) holds. For $x \neq 0$, from (33a) and (47), it follows

$$x^T\left\{PB_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^TP\right\}x < x^T\left\{\bar{Z}_z + \bar{Q}_z\right\}x < x^T\left\{-A_z^TP - PA_z\right\}x. \tag{48}$$

From (14a) and (7), if $w(t) = 0$ and $x(t) \in \mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$, remembering that $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{X}$ and $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{L}(L_j)$, the inequality (48) implies that $\dot{V}(x(t)) < 0$, for all $x(t) \in \mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)\backslash\{0\}$. Thus, for $w(t) = 0$, the nonlinear system (2) is locally asymptotically stable for all $x(0) \in \mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$.

- Second statement:

The second statement follows from Lemma 3. Considering (18), observe that (45a) ensures that (33b) holds. For $x \neq 0$, from (33b) and (47), it follows

$$x^T \mathcal{N}_{z\lambda}^{(\dot{x})}(P, K_\sigma, L_\sigma)x < x^T \left\{\bar{Z}_z + \bar{Q}_z\right\} x < x^T \left\{-A_z^T P - P A_z\right\} x. \tag{49}$$

For $x \neq 0$, considering (17) and (18), from Lemma 2 and (49), observe that

$$\begin{aligned}
0 &> x^T \left\{A_z^T P + P A_z + \mathcal{N}_{z\lambda}^{(\dot{x})}(P, K_\sigma, L_\sigma)\right\} x \geq \begin{bmatrix} x^T & w^T \end{bmatrix} \mathcal{M}_{z\lambda}^{(\dot{x})}(P, K_\sigma, L_\sigma) \begin{bmatrix} x^T & w^T \end{bmatrix}^T \\
&= \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A_z^T P + P A_z + P B_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^T P & * \\ H_z^T P & -\varphi^{-1} I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\
&= x^T \left\{A_z^T P + P A_z + P B_z\left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right] + \left[-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right]^T B_z^T P\right\} x \\
&\quad + x^T P H_z w + w^T H_z^T P x - w^T \varphi^{-1} w.
\end{aligned} \tag{50}$$

From (14a) and (7), if $w(t) \neq 0$ and $x(t) \in \mathcal{E}(V, \epsilon_0)$, remembering that $\mathcal{E}(V, \epsilon_0) \subset \mathcal{X}$ and $\mathcal{E}(V, \epsilon_0) \subset \mathcal{L}(L_j)$, the inequality (50) implies that

$$\dot{V}(x(t)) - \varphi^{-1} w(t)^T w(t) < 0, \tag{51}$$

for all $x(t) \in \mathcal{E}(V, \epsilon_0)\backslash\{0\}$.

For $x(0) \in \mathcal{E}(V, \epsilon_0)$ and from (15), one has $V(x(0)) \leq \epsilon_0$ and $\int_0^\infty w(t)^T w(t)dt \leq \epsilon$, respectively. Thus, integrating (51) from 0 to ∞, one obtains

$$V(x(\infty)) < V(x(0)) + \varphi^{-1} \int_0^\infty w(t)^T w(t)dt \leq V(x(0)) + \varphi^{-1}\epsilon \leq \epsilon_0 + \varphi^{-1}\epsilon, \tag{52}$$

concluding that $V(x(\infty)) < \epsilon_0 + \varphi^{-1}\epsilon$. Remembering that $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{X}$ and $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{L}(L_j)$, the inequality (51) ensures that a state trajectory started with an initial condition $x(0) \in \mathcal{E}(V, \epsilon_0)$ will remain within $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)\backslash\partial\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$, for all $t \geq 0$, where $\partial\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ is the boundary of $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$.

- Third statement:

For the third statement, considering (20) and Lemma 3, observe that (45a) ensures that (33c) holds. For $x \neq 0$, from (33c) and (47), it follows

$$x^T \mathcal{N}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma)x < x^T \left\{\bar{Z}_z + \bar{Q}_z\right\} x < x^T \left\{-A_z^T P - P A_z\right\} x. \tag{53}$$

Note that, from (45a), $\Psi = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} < 0$, where $\psi_1 = \begin{bmatrix} -\varphi^{-1} I & G_i^T \end{bmatrix}^T$ and $\psi_2 = \begin{bmatrix} G_i & -\varphi\mu I \end{bmatrix}^T$. Thus, considering the Schur Complement, $\Psi < 0$ is equivalent to $-\varphi^{-1} I + \varphi^{-1}\mu^{-1} G_i^T G_i < 0$. Therefore, remembering that $\mu = \gamma^2$, the condition supposed in Lemma 1 holds. For $x \neq 0$, considering (19) and (20), from Lemma 1 and (53), observe that

$$\begin{aligned}
0 &> x^T \left\{A_z^T P + P A_z + \mathcal{N}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma)\right\} x \geq \begin{bmatrix} x^T & w^T \end{bmatrix} \mathcal{M}_{z\lambda}^{(\dot{x}y)}(P, K_\sigma, L_\sigma) \begin{bmatrix} x^T & w^T \end{bmatrix}^T \\
&= \begin{bmatrix} x \\ w \end{bmatrix}^T \left\{ \begin{bmatrix} A_z^T P + P A_z + P B_z\left[-F_\lambda K_j + F_\lambda^- L_j\right] + \left[-F_\lambda K_j + F_\lambda^- L_j\right]^T B_z^T P & * \\ H_z^T P & -\varphi^{-1} I \end{bmatrix} \right.
\end{aligned}$$

$$+\left[\begin{array}{c}\left(C_z+D_z\left[-F_\lambda K_j+F_\lambda^- L_j\right]\right)^T\\ G_z^T\end{array}\right]\varphi^{-1}\gamma^{-2}I\left[\begin{array}{c}\left(C_z+D_z\left[-F_\lambda K_j+F_\lambda^- L_j\right]\right)^T\\ G_z^T\end{array}\right]^T\Bigg\}\begin{bmatrix}x\\ w\end{bmatrix}$$

$$=x^T\left\{A_z^T P+PA_z+PB_z\left[-F_\lambda K_\sigma+F_\lambda^- L_\sigma\right]+\left[-F_\lambda K_\sigma+F_\lambda^- L_\sigma\right]^T B_z^T P\right\}x+x^T PH_z w+w^T H_z^T Px$$
$$+\varphi^{-1}\gamma^{-2}\left(C_z x+D_z\left[-F_\lambda K_j+F_\lambda^- L_j\right]x+G_z w\right)^T\left(C_z x+D_z\left[-F_\lambda K_j+F_\lambda^- L_j\right]x+G_z w\right)$$
$$-\varphi^{-1}w^T w. \tag{54}$$

From (14), (7) and Statement 2, if $w(t)\neq 0$ and $x(0)=0$, then $x(t)\in\mathcal{E}(V,\varphi^{-1}\epsilon)\subset\mathcal{E}(V,\epsilon_0+\varphi^{-1}\epsilon)$, $\forall t\geq 0$. Remembering that $\mathcal{E}(V,\epsilon_0+\varphi^{-1}\epsilon)\subset\mathcal{X}$ and $\mathcal{E}(V,\epsilon_0+\varphi^{-1}\epsilon)\subset\mathcal{L}(L_j)$, the inequality (54) implies that

$$\dot V(x(t))+\varphi^{-1}\gamma^{-2}y(t)^T y(t)-\varphi^{-1}w(t)^T w(t)<0, \tag{55}$$

for all $x(t)\in\mathcal{E}(V,\epsilon_0+\varphi^{-1}\epsilon)\backslash\{0\}$.

Integrating (55) from 0 to ∞, one obtains

$$\varphi^{-1}\gamma^{-2}\int_0^\infty y(t)^T y(t)dt-\varphi^{-1}\int_0^\infty w(t)^T w(t)dt<V(x(0))-V(x(\infty))\leq V(x(0))=0. \tag{56}$$

Hence

$$\varphi^{-1}\gamma^{-2}\int_0^\infty y(t)^T y(t)dt-\varphi^{-1}\int_0^\infty w(t)^T w(t)dt<0$$
$$\int_0^\infty y(t)^T y(t)dt<\gamma^2\int_0^\infty w(t)^T w(t)dt$$
$$\|y(t)\|_2^2\leq\gamma^2\|w(t)\|_2^2. \qquad\square \tag{57}$$

Remark 2. In literature, different procedures are proposed to enlarge the size of $\mathcal{E}(V,1)$. According to [26], the boundary of $\mathcal{E}(V,1)$ can be enlarged inserting the restriction $\left\{x\in\mathbb{R}^{n_x}:x^T x\leq\beta\right\}\subset\mathcal{E}(V,1)$, that is enforced if

$$\begin{bmatrix}\beta^{-1}I & *\\ I & P\end{bmatrix}\geq 0 \tag{58}$$

holds, where β is a positive constant.

Remark 3. Usually, when the $\mathcal{H}_\infty$ norm is reduced, the controllers K_j present high values of gains, that could result in equipment damage. However, these high gains can be reduced inserting the restriction $\mathcal{E}(V,\epsilon_0+\varphi^{-1}\epsilon)\subset\mathcal{L}(K_j)$, that is enforced if

$$\begin{bmatrix}\frac{\rho_{(k)}^2}{\epsilon_0+\varphi^{-1}\epsilon} & *\\ M_{j(k)}^T & X\end{bmatrix}\geq 0 \tag{59}$$

holds, for all $j\in\mathbb{K}_r$ and $k\in\mathbb{K}_{n_u}$. Note that, $\mathcal{L}(K_j)$ is a polyhedral set similar to (12) [31].

Remark 4. For an appropriate control design, several parameters need to be adjusted. Note that, the parameters ϕ and ρ are related to physics limitations, such that ϕ and ρ are chosen according to the operation regions (4) and saturation region (12), respectively. The parameter ϵ_0 is related to the initial condition $x(0)$ and the symmetric positive definite matrix P, and it can be determined according to a possible region of initial conditions. From (8), observe that $x(0)^T Px(0)\leq\epsilon_0$. Therefore, additional constraints such as (58) should be added to enlarge the size of $\mathcal{E}(V(x(0)),\epsilon_0)$. The parameter ϵ is the disturbance energy. The parameter φ is a slack variable with no physical meaning. Together, the parameters ϵ_0, ϵ and φ are responsible for establishing the region $\mathcal{E}(V,\epsilon_0+\varphi^{-1}\epsilon)$ and the parameter β determines the volume of this region.

Remark 5. The control law, given by $u(t) = -Kx(t)$, is an alternative to solving the local $\mathcal{H}_\infty$ control problem, without the necessity to find the membership functions expressions. Using a quadratic Lyapunov function (7), the local $\mathcal{H}_\infty$ control design conditions can be enunciated similarly to Theorem 1.

Corollary 1. *Consider an operation region $\mathcal{X}$ in which the nonlinear system subject to actuator saturation and energy-bounded disturbances (2) can be exactly described by (5), where $\rho \in \mathbb{R}^{n_u}$, $R \in \mathbb{R}^{p\times n_x}$, $\phi \in \mathbb{R}^p$, $\epsilon_0 \geq 0$, $\epsilon > 0$ and $\varphi > 0$ are known. Assume that there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, matrices $M, N \in \mathbb{R}^{n_u\times n_x}$ and scalar $\mu > 0$, such that the following optimization problem is feasible:*

$$\min \mu$$

subject to

$$\begin{bmatrix} XA_i^T + A_iX + B_i\left[-F_sM + F_s^-N\right] + \left[-F_sM + F_s^-N\right]^T B_i^T & * & * \\ H_i^T & -\varphi^{-1}I & * \\ C_iX + D_i\left[-F_sM + F_s^-N\right] & G_i & -\varphi\mu I \end{bmatrix} < 0, \tag{60a}$$

$$\begin{bmatrix} \frac{\phi_{(h)}^2}{\epsilon_0+\varphi^{-1}\epsilon} & * \\ XR_{(h)}^T & X \end{bmatrix} \geq 0, \quad \begin{bmatrix} \frac{\rho_{(k)}^2}{\epsilon_0+\varphi^{-1}\epsilon} & * \\ N_{(k)}^T & X \end{bmatrix} \geq 0, \tag{60b}$$

for all $i \in \mathbb{K}_r$, $h \in \mathbb{K}_p$, $k \in \mathbb{K}_{n_u}$ and $s \in \mathbb{K}_{2^{n_u}}$, where $F_s \in \mathcal{F}$ and $F_s^- = I - F_s$. Then, $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{X}$, $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{L}(L)$ and the control law $u(t) = -Kx(t)$, where the controller gain is given by $K = MX^{-1}$, ensures that:

1. *for $w(t) = 0$, $t \geq 0$, the uncertain nonlinear system (2), with $u(t) = -Kx(t)$, is locally asymptotically stable for all $x(0) \in \mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ and the ellipsoid $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ is an invariant subset of the domain of attraction (i.e., if $x(0)$ belongs to this set, then $x(t)$, $t > 0$, will also stay in this set). Fig. 1a shows this propriety;*
2. *for $w(t) \neq 0$, if $x(0) \in \mathcal{E}(V, \epsilon_0)$, then $x(t) \in \mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$, for all $t \geq 0$. Fig. 1b illustrates this propriety;*
3. *for $w(t) \neq 0$, if $x(0) = 0$, then the nonlinear system (2), with $u(t) = -Kx(t)$, has an $\mathcal{H}_\infty$ guaranteed cost $\gamma = \sqrt{\mu} > 0$, such that*

$$\|y(t)\|_2^2 \leq \gamma^2 \|w(t)\|_2^2, \tag{61}$$

and $x(t) \in \mathcal{E}(V, \varphi^{-1}\epsilon)$, for all $t \geq 0$.

Proof. Suppose that there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n_x\times n_x}$ and matrices $M, N \in \mathbb{R}^{n_u\times n_x}$, for all $i \in \mathbb{K}_r$. Similarly to Lemma 3 and considering $K_\sigma = K$, $L_\sigma = L$, (18) and (20), for $x \neq 0$, if the condition (60a) holds, then the following conditions also hold

$$x^T\left\{PB_z\left[-F_\lambda K + F_\lambda^- L\right] + \left[-F_\lambda K + F_\lambda^- L\right]^T B_z^T P\right\}x < x^T\left\{-A_z^TP - PA_z\right\}x, \tag{62a}$$

$$x^T\mathcal{N}_{z\lambda}^{(\dot{x})}(P, K, L)x < x^T\left\{-A_z^TP - PA_z\right\}x, \tag{62b}$$

$$x^T\mathcal{N}_{z\lambda}^{(\dot{x}y)}(P, K, L)x < x^T\left\{-A_z^TP - PA_z\right\}x. \tag{62c}$$

Observe that, the inequalities (62a), (62b) and (62c) are equivalent to (48), (49) and (53), respectively, for $K_\sigma = K$ and $L_\sigma = L$. Hence, from (62) and following the steps of the proof of Theorem 1, the Statements 1, 2 and 3 of this corollary can be easily demonstrated. □

Remark 6. In [26] is proposed a local $\mathcal{H}_\infty$ control design that guarantees an $\mathcal{H}_\infty$ performance and ensures that the state trajectory remains within an ellipsoid $\mathcal{E}(V, 1 + \varphi^{-1}\epsilon) \subset \mathcal{X}$. However, note that, in [26]:

- The Lyapunov function is given by $V(x(t)) = x(t)^T P_z^{-1} x(t)$, where $z(t) = \mathcal{T}x(t) \in \mathbb{R}^q$, $\mathcal{T} \in \mathbb{R}^{q\times n_x}$ i.e., the premise variables are a linear combinations of the state variables, and the following sets are considered:

$$\mathcal{R} = \left\{x(t) \in \mathbb{R}^{n_x} : \mathcal{T}_l x(t) \in [-\xi_{l,\max}, \xi_{l,\max}], l \in \mathbb{K}_q\right\} \text{ and } \mathcal{H}(b) = \{x(t) \in \mathcal{R} : |\dot{\alpha}_i(z(t))| \leq b, i \in \mathbb{K}_r\}.$$

- It is considered that the continuous-time T–S system is subject a magnitude- and energy-bounded disturbances, such that $w(t)^T w(t) \leq \delta$ and $\int_0^\infty w(t)^T w(t)dt \leq \epsilon$. The restriction $w(t)^T w(t) \leq \delta$ is an auxiliary condition necessary to ensure that: if $w(t) \neq 0$ and $x(0) \in \mathcal{E}(V, 1+\varphi^{-1}\epsilon)$, then $\mathcal{E}(V, 1+\varphi^{-1}\epsilon) \subset \mathcal{H}(b)$.
- The control law considers full access to membership functions, such that $u(t) = K_z P_z^{-1} x(t)$.
- The control design does not consider that the actuator is subject to saturation. Therefore, $\mathcal{L}(L_j)$ is a region that does not exist in the proposed control project in [26].

Remark 7. The control design shown in [26] does not consider that the actuator is subject to saturation. Thus, in order to make a fair comparison among the methodologies addressed in this work, the switched control design proposed in Corollary 2 must not also consider that the control signal is subject to saturation. Then, preliminarily, consider a class of nonlinear system subject to energy-bounded disturbances, described by

$$\dot{x}(t) = f_1(z(t))x(t) + f_2(z(t))u(t) + f_3(z(t))w(t), \tag{63a}$$

$$y(t) = g_1(z(t))x(t) + g_2(z(t))u(t) + g_3(z(t))w(t). \tag{63b}$$

Hence, considering the operation region $\mathcal{X}$, given in (4), the nonlinear system subject to energy-bounded disturbances (63) can be exactly represented by a T–S fuzzy model as follows [2,4,14,18]:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i(z(t))(A_i x(t) + B_i u(t) + H_i w(t)) = A_z x(t) + B_z u(t) + H_z w(t), \tag{64a}$$

$$y(t) = \sum_{i=1}^{r} \alpha_i(z(t))(C_i x(t) + D_i u(t) + G_i w(t)) = C_z x(t) + D_z u(t) + G_z w(t). \tag{64b}$$

Corollary 2. *Consider an operation region $\mathcal{X}$ in which the nonlinear system subject to energy-bounded disturbances (63) can be exactly described by (64), where $R \in \mathbb{R}^{p\times n_x}$, $\phi \in \mathbb{R}^p$, $\epsilon_0 \geq 0$, $\epsilon > 0$ and $\varphi > 0$ are known. Assume that there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n_x\times n_x}$, symmetric matrices Z_i, $Q_i \in \mathbb{R}^{n_x\times n_x}$, matrices $M_j \in \mathbb{R}^{n_u\times n_x}$ and scalar $\mu > 0$, such that the following optimization problem is feasible:*

$$\min \mu$$

subject to

$$\begin{bmatrix} -B_i M_j - M_j^T B_i^T - Z_i - Q_j & * & * \\ H_i^T & -\varphi^{-1} I & * \\ C_i X - D_i M_j & G_i & -\varphi\mu I \end{bmatrix} < 0, \tag{65a}$$

$$X A_i^T + A_i X + Z_i + Q_i < 0, \tag{65b}$$

$$\begin{bmatrix} \frac{\phi_{(h)}^2}{\epsilon_0+\varphi^{-1}\epsilon} & * \\ X R_{(h)}^T & X \end{bmatrix} \geq 0, \tag{65c}$$

for all $i, j \in \mathbb{K}_r$ and $h \in \mathbb{K}_p$. Then, $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{X}$ and the switched control law (9), where $P = X^{-1}$, $\bar{Q}_i = X^{-1} Q_i X^{-1}$ and the controller gains are given by $K_j = M_j X^{-1}$, ensures that:

1. *for $w(t) = 0$, $t \geq 0$, the uncertain nonlinear system (63) and (9) is locally asymptotically stable for all $x(0) \in \mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ and the ellipsoid $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ is an invariant subset of the domain of attraction (i.e., if $x(0)$ belongs to this set, then $x(t)$, $t > 0$, will also stay in this set). Fig. 2a shows this propriety;*
2. *for $w(t) \neq 0$, if $x(0) \in \mathcal{E}(V, \epsilon_0)$, then $x(t) \in \mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$, for all $t \geq 0$. Fig. 2b illustrates this propriety;*
3. *for $w(t) \neq 0$, if $x(0) = 0$, then the nonlinear system (63) and (9) has an $\mathcal{H}_\infty$ guaranteed cost $\gamma = \sqrt{\mu} > 0$, such that*

$$\|y(t)\|_2^2 \leq \gamma^2 \|w(t)\|_2^2, \tag{66}$$

and $x(t) \in \mathcal{E}(V, \varphi^{-1}\epsilon)$, for all $t \geq 0$.

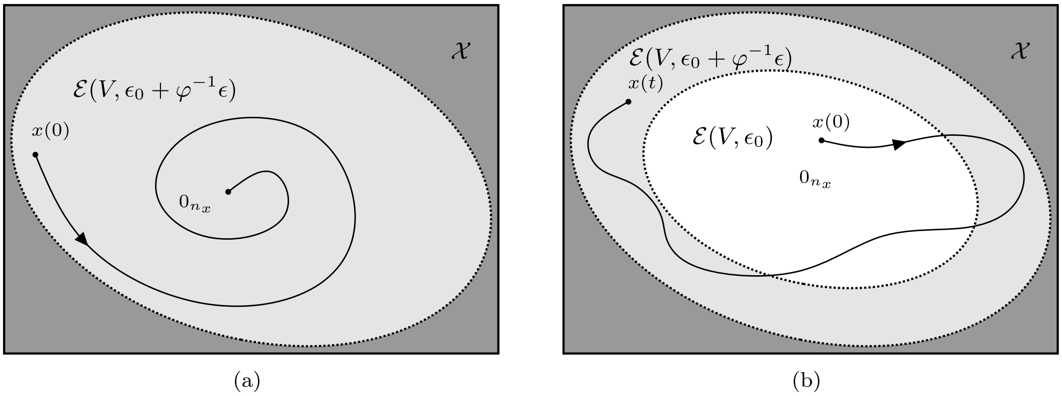


Fig. 2. (a) The inclusion relations among the sets $\mathcal{X}$ and $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$, and the state trajectory $x(t)$ for $w(t) = 0$, for all $t \geq 0$; (b) The inclusion relations among the sets $\mathcal{X}$, $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon)$ and $\mathcal{E}(V, \epsilon_0)$, and the state trajectory $x(t)$ for $w(t) \in \mathcal{W}$, for all $t \geq 0$.

Proof. First, note that the control design does not consider that the actuator is subject to saturation. Therefore, it is not necessary to take into account the region $\mathcal{L}(L_j)$ in this control design.

Consider the Lyapunov function candidate (7). Preliminarily, according to Lemma 4, observe that (65c) ensure that $\mathcal{E}(V, \epsilon_0 + \varphi^{-1}\epsilon) \subset \mathcal{X}$. Suppose that there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices Z_i, $Q_j \in \mathbb{R}^{n_x \times n_x}$ and matrices M_j, such that (65a) and (65b) hold, for all $i, j \in \mathbb{K}_r$. From (1), remembering that $\alpha_i \geq 0$ and $\sum_{i=1}^{r} \alpha_i = 1$, define $P = X^{-1}$, $K_\sigma = M_\sigma X^{-1}$, $\bar{Z}_z = X^{-1} Z_z X^{-1}$, $\bar{Q}_z = X^{-1} Q_z X^{-1}$, $\bar{Q}_\sigma = X^{-1} Q_\sigma X^{-1}$ and $\mu = \gamma^2$. Pre- and post-multiplying (65a) by diag$\{P, I, I\}$ and pre- and post-multiplying (65b) by P, changing the variables $\bar{Z}_i = X^{-1} Z_i X^{-1}$ and $\bar{Q}_i = X^{-1} Q_i X^{-1}$, replacing j by σ, multiplying the results by $\alpha_i \geq 0$, $i \in \mathbb{K}_r$, $\sum_{i=1}^{r} \alpha_i = 1$ and taking the sum from $i = 1$ to r, for $x \neq 0$, the following inequalities are obtained, respectively:

$$\begin{bmatrix} -PB_z K_\sigma - K_\sigma^T B_z^T P - \bar{Z}_z - \bar{Q}_\sigma & * & * \\ H_z^T P & -\varphi^{-1} I & * \\ C_z - D_z K_\sigma & G_z & -\varphi\gamma^2 I \end{bmatrix} < 0, \tag{67}$$

$$A_z^T P + PA_z + \bar{Z}_z + \bar{Q}_z < 0. \tag{68}$$

Similarly to Lemma 3 and considering (40), for $x \neq 0$, if the conditions (67) and (68) hold, then the following conditions also hold

$$x^T \left\{-PB_z K_\sigma - K_\sigma^T B_z^T P\right\} x < x^T \left\{\bar{Z}_z + \bar{Q}_\sigma\right\} x \leq x^T \left\{\bar{Z}_z + \bar{Q}_z\right\} x < x^T \left\{-A_z^T P - PA_z\right\} x, \tag{69a}$$

$$x^T \left\{-PB_z K_\sigma - K_\sigma^T B_z^T P + \varphi P H_z H_z^T P\right\} x < x^T \left\{\bar{Z}_z + \bar{Q}_\sigma\right\} x \leq x^T \left\{\bar{Z}_z + \bar{Q}_z\right\} x < x^T \left\{-A_z^T P - PA_z\right\} x, \tag{69b}$$

$$\begin{aligned} &x^T \big\{ -PB_z K_\sigma - K_\sigma^T B_z^T P + \varphi^{-1}\gamma^{-2} (C_z - D_z K_\sigma)^T (C_z - D_z K_\sigma) \\ &+ \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T (C_z - D_z K_\sigma)\right)^T \left(\varphi^{-1} I - \varphi^{-1}\gamma^{-2} G_z^T G_z\right)^{-1} \left(H_z^T P + \varphi^{-1}\gamma^{-2} G_z^T (C_z - D_z K_\sigma)\right) \big\} x \\ &< x^T \left\{\bar{Z}_z + \bar{Q}_\sigma\right\} x \leq x^T \left\{\bar{Z}_z + \bar{Q}_z\right\} x < x^T \left\{-A_z^T P - PA_z\right\} x. \end{aligned} \tag{69c}$$

The proof of Theorem 1 can be adapted to the case in which the control design does not consider that the actuator is subject to saturation. Note that, from (9) and (13), considering $F_\lambda = I$ and $F_\lambda^- = 0$, one obtains

$$\text{sat}(u(t)) = \left(-F_\lambda K_\sigma + F_\lambda^- L_\sigma\right) x(t) = -K_\sigma x(t) = u(t). \tag{70}$$

Then, for $F_\lambda = I$ and $F_\lambda^- = 0$, sat$(u(t)) = u(t)$ and the inequalities (69a), (69b) and (69c) are equivalent to (48), (49) and (53), respectively, considering that the actuator is not subject to saturation. Hence, from (69) and following the steps of the proof of Theorem 1, the Statements 1, 2 and 3 of this corollary can be demonstrated. □

Remark 8. In [18] is proposed a state-feedback switched controller design for a class of uncertain nonlinear systems. The design of switched controllers is based on a minimum-type Lyapunov function given by

$$V(x(t)) = \min_{j\in\mathbb{K}_N} \{x(t)^T P_j x(t)\}, \tag{71}$$

where P_j, $j \in \mathbb{K}_N$, are symmetric positive definite matrices.

By using the minimum-type Lyapunov function (71) it is possible to insert relaxing parameters such that the design conditions become less conservative. These design conditions are given by a kind of BMIs, which contain some products of full matrices and scalars, and the path-following method [18] has been used in order to find feasible solutions.

The procedure presented in [18] does not suppose that the nonlinear system is subject to disturbances and deals with the stability problem without the specification of the operation region $\mathcal{X}$. Furthermore, it is not considered that the actuator is subject to saturation.

Hence, Corollary 2 can be seen as an extension of the procedure presented in [18], to deal with the $\mathcal{H}_\infty$ control problem considering a given operating region and also that the nonlinear system is subject to disturbances. From (7) and (71), note that from the conditions of Corollary 2 $j \in \mathbb{K}_1$, avoiding design conditions given by BMIs.

4. Numerical examples

Example 1 *(A comparative example).* From (6), consider the chaotic Lorenz system described by the following T–S fuzzy model [26,36]:

$$\begin{gathered} A_1 = \begin{bmatrix} -\eta_1 & \eta_1 & 0 \\ \eta_2 & -1 & 20 \\ 0 & -20 & -\eta_3 \end{bmatrix}, A_2 = \begin{bmatrix} -\eta_1 & \eta_1 & 0 \\ \eta_2 & -1 & -30 \\ 0 & 30 & -\eta_3 \end{bmatrix}, B_1 = \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \\ \eta_3 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & -\eta_1 \\ -\eta_2 & 0 \\ 0 & \eta_3 \end{bmatrix}, \\ H_1 = H_2 = \begin{bmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_1 & 0 \\ 0 & 0 & \eta_1 \end{bmatrix}, \begin{matrix} C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ G_1 = G_2 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \\ D_1 = D_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}. \end{matrix} \end{gathered} \tag{72}$$

The energy-bounded disturbance input $w(t)$ is given by

$$\begin{cases} w(t) = \mathcal{A} \begin{bmatrix} \sin(\omega t) \\ \sin(\omega t) \\ \sin(\omega t) \end{bmatrix}, & t \in [0, t_f); \\ w(t) = 0_3, & t \in [t_f, \infty), \end{cases} \tag{73}$$

and the parameters $\mathcal{A}$, ω and t_f are chosen to satisfy the following conditions:

$$\int_0^{t_f} w(t)^T w(t) dt \le \int_0^{\infty} w(t)^T w(t) dt \le \epsilon \text{ and } w(t)^T w(t) \le \delta, \ \forall t \in [0, \infty). \tag{74}$$

From (73) and (74), observe that

$$w(t)^T w(t) = 3\mathcal{A}^2 \sin^2(\omega t) \le 3\mathcal{A}^2 \le \delta, \tag{75}$$

and

$$\int_0^{t_f} w(t)^T w(t) dt = \int_0^{t_f} 3\mathcal{A}^2 \sin^2(\omega t) dt = \int_0^{t_f} 3\mathcal{A}^2 \left(\frac{1-\cos(2\omega t)}{2}\right) dt = \frac{3\mathcal{A}^2 t_f}{2} - \frac{3\mathcal{A}^2 \sin(2\omega t_f)}{4\omega}. \tag{76}$$

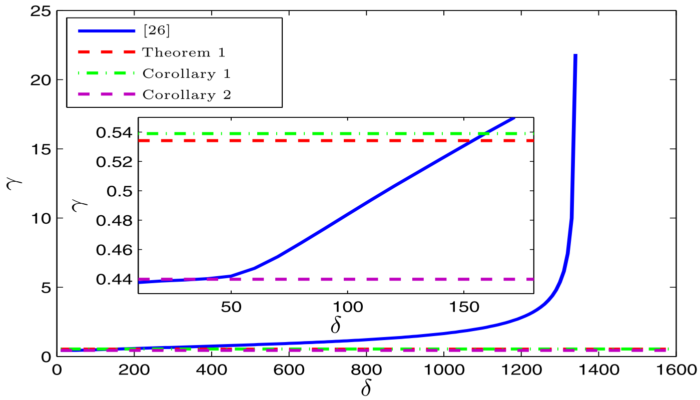


Fig. 3. Comparison among the methods proposed in Theorem 1 (LMIs (45) and (58)), Corollary 1 (LMIs (60) and (58)), Corollary 2 (LMIs (65) and (58)) and [26]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Consider the disturbance signal $w(t)$ given in (73), with $t_f = \dfrac{a\pi}{2\omega}$ and $a \in \{1, 2, \dots\}$. Therefore, from (76) one obtains

$$\int_0^{t_f} w(t)^T w(t)dt = \frac{3\mathcal{A}^2\pi a}{4\omega} - \frac{3\mathcal{A}^2 \sin(a\pi)}{4\omega} = \frac{3\mathcal{A}^2\pi a}{4\omega} \leq \epsilon, \quad a \in \{1, 2, \dots\}. \tag{77}$$

For $(\eta_1, \eta_2, \eta_3) = (5, 30, 2)$, $N = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\phi = 30$, $\rho = \begin{bmatrix} 100 & 100 \end{bmatrix}^T$, $\varphi = 110$, $\beta = 100$, $\epsilon_0 = 1$, $\epsilon = 15\pi$ and $10 \leq \delta \leq 1600$. A comparison among the methods proposed in Theorem 1 (LMIs (45) and (58)), Corollary 1 (LMIs (60) and (58)), Corollary 2 (LMIs (65) and (58)) and [26] is shown in Fig. 3.

According to Fig. 3, note that for $10 \leq \delta \leq 36$ the method proposed in [26] presents the best values of γ. However, for $\delta \geq 1443$, the conditions of this same method are not feasible. For Theorem 1, Corollary 1 and Corollary 2, one obtains $\gamma = 0.5342$, $\gamma = 0.5390$ and $\gamma = 0.4398$, respectively, and note that the variation of the parameter δ does not change the values of γ because the LMIs conditions proposed in these theorems are not influenced by the magnitude of the disturbance. Therefore, for $\delta > 36$, Corollary 2 presents the best values of γ. As in [26], the conditions of Corollary 2 do not consider that the control signal is subject to saturation. Hence, Corollary 2 presents fewer constraints and consequently better results than Theorem 1 and Corollary 1.

Among the control designs proposed in this work, although the best numerical results are presented by Corollary 2, Theorem 1 will be used in the simulations because it is a control procedure that considers the saturation of the control signal. Solving the optimization problem given by the LMIs (45) and (58), using the parameters aforementioned, one obtains the following controller gains K_j, symmetric matrices $\bar{Q}_j$, $j \in \mathbb{K}_2$, and positive definite matrix P:

$$\begin{aligned} K_1 &= \begin{bmatrix} -5.1207 & -1.3967 & 1.0277 \\ 5.3067 & 1.8843 & 0.1022 \end{bmatrix}, \quad && \bar{Q}_1 = \begin{bmatrix} 0.0201 & -0.0050 & 0.0047 \\ -0.0050 & -0.0017 & 0.0027 \\ 0.0047 & 0.0027 & -0.0007 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -5.0252 & -1.3993 & 1.0316 \\ 5.1996 & 1.8793 & 0.0982 \end{bmatrix}, && \\ P &= \begin{bmatrix} 0.0090 & 0.0028 & -0.0008 \\ 0.0028 & 0.0011 & -0.0003 \\ -0.0008 & -0.0003 & 0.0002 \end{bmatrix}, && \bar{Q}_2 = \begin{bmatrix} -0.0226 & -0.0291 & 0.0049 \\ -0.0291 & -0.0149 & 0.0030 \\ 0.0049 & 0.0030 & -0.0007 \end{bmatrix}. \end{aligned} \tag{78}$$

- First Simulation:

The goal of the first simulation is to analyze the behavior of the chaotic Lorenz system (6) and (72) subject to an external disturbance $w(t)$ with large magnitude. Consider an initial condition $x(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ and a disturbance

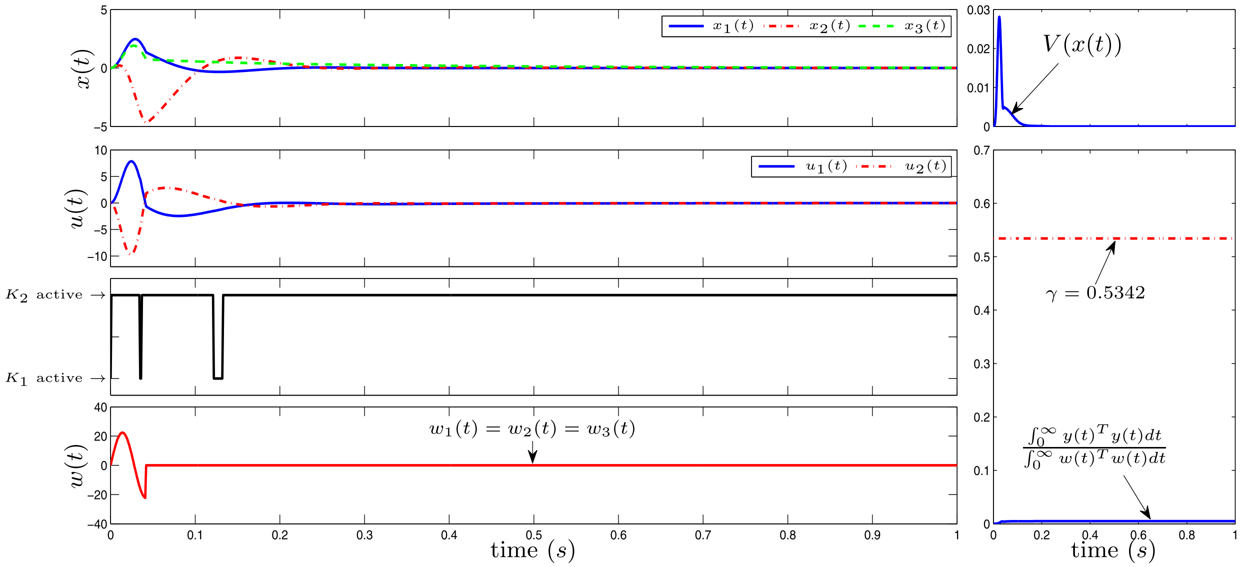


Fig. 4. The dynamic behavior of the controlled system (2), (9) and (72)–(73): the state trajectory $x(t)$, the switched control law (9), the energy of the system $V(x(t))$, the external disturbance $w(t)$ and the relation between $\int_0^\infty y(t)^T y(t)dt / \int_0^\infty w(t)^T w(t)dt$ and γ.

$w(t)$, given by (73), where

$$\mathcal{A} = 10\sqrt{5}, \quad \omega = 112{,}5 \text{ rad/s}, \quad a = 3 \quad \text{and} \quad t_f = \frac{3\pi}{225}\text{s}, \tag{79}$$

such that $\int_0^{t_f} w(t)^T w(t)dt = 10\pi \leq \epsilon = 15\pi$ and $w(t)^T w(t) \leq 1500$. The control law (9), with the set of gains and auxiliary matrices given in (78), was used to perform a simulation of the closed-loop system (2) and (72)–(73) using the software MatLab/Simulink®. The simulation result is shown in Fig. 4.

Note that, using the switched control law (9) and (78), for $x(0) = 0$, the trajectories of the controlled system remained within $\mathcal{E}(V, \varphi^{-1}\epsilon) = \mathcal{E}(V, 0.4284)$ and, consequently, $x(t) \in \mathcal{L}(L_j)$ and $x(t) \in \mathcal{X}$, for all $t \geq 0$. The $\mathcal{H}_\infty$ guaranteed cost ensured that $\int_0^\infty y(t)^T y(t)dt / \int_0^\infty w(t)^T w(t)dt \leq 0.0048 \leqslant \gamma = 0.5342$. Finally, note that $w(t)^T w(t) \leq 1500 = \delta$, thus, according to Fig. 3, the conditions of the method presented in [26] are not feasible.

- Second Simulation:

In the second implementation, the intention is to analyze the dynamics of the Lorenz chaotic system subject to external disturbance $w(t)$ with low frequency, in order to increase the value of the relation $\frac{\int_0^\infty y(t)^T y(t)dt}{\int_0^\infty w(t)^T w(t)dt}$. Consider an initial condition $x(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ and a disturbance $w(t)$, given by (73), where

$$\mathcal{A} = \frac{10}{3}\sqrt{6}, \quad \omega = 15 \text{ rad/s}, \quad a = 3 \quad \text{and} \quad t_f = \frac{3\pi}{30}\text{s}, \tag{80}$$

such that $\int_0^{t_f} w(t)^T w(t)dt = 10\pi \leq \epsilon = 15\pi$ and $w(t)^T w(t) \leq 200$. The control law (9), with the set of gains and auxiliary matrices given in (78), was used to perform a simulation of the closed-loop system (2) and (72)–(73). The simulation result is shown in Fig. 5.

The energy of the external disturbance is the same in both simulations ($\int_0^{t_f} w(t)^T w(t)dt = 10\pi$). In the second simulation, presented in Fig. 5, note that the frequency $\omega = 15$ rad/s increased the value of the relation $\int_0^\infty y(t)^T y(t)dt / \int_0^\infty w(t)^T w(t)dt$. However, the $\mathcal{H}_\infty$ guaranteed cost ensured that $\int_0^\infty y(t)^T y(t)dt / \int_0^\infty w(t)^T w(t)dt \leqslant 0.0473 \leqslant \gamma = 0.5342$.

In the first simulation, from the results presented in Fig. 4, note that the gains K_1 and K_2 commuted few times and the gain K_2 was used more frequently. Now, from the results of the second simulation in Fig. 5, observe that the commutations between the gains K_1 and K_2 increased as well as the time in which the gain K_1 was active. The

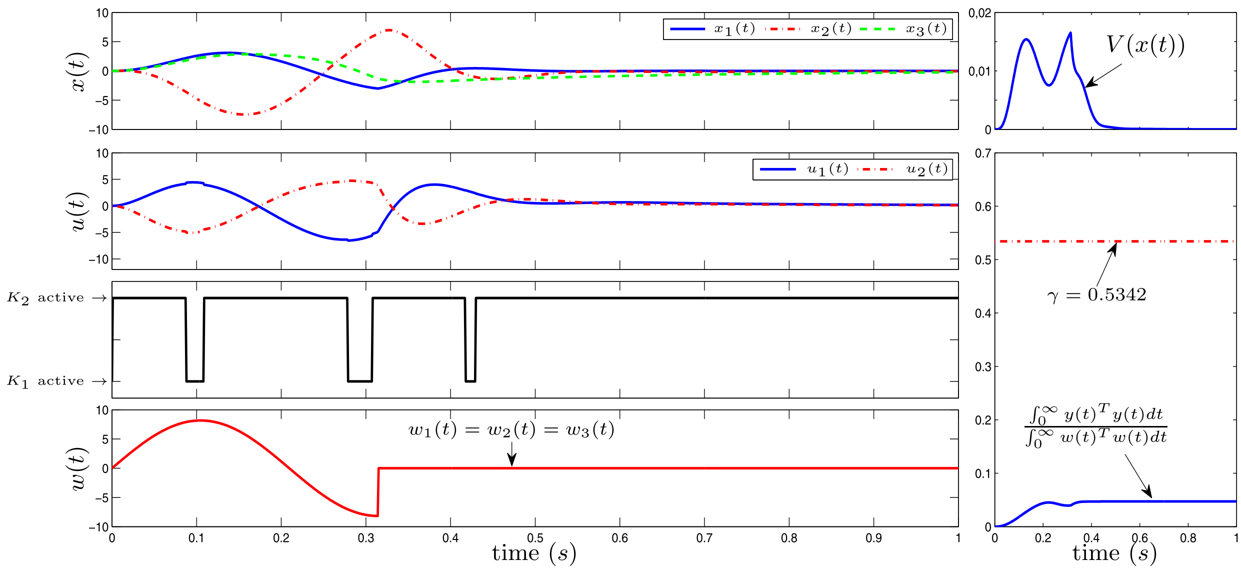


Fig. 5. The dynamic behavior of the controlled system (2), (9) and (72)–(73): the state trajectory $x(t)$, the switched control law (9), the energy of the system $V(x(t))$, the external disturbance $w(t)$ and the relation between $\int_0^\infty y(t)^T y(t)dt / \int_0^\infty w(t)^T w(t)dt$ and γ.

increase of the relation $\int_0^\infty y(t)^T y(t)dt / \int_0^\infty w(t)^T w(t)dt$ is one of the reasons why this fact can be observed. In other words, the frequency 15 rad/s is close to the resonant frequency of the system and the switched control law (9) uses more controller options to mitigate the influence of the disturbance.

Note that Corollary 1 and Corollary 2 could be applied to control this system. However, Theorem 1 was used in both simulations for the following reasons:

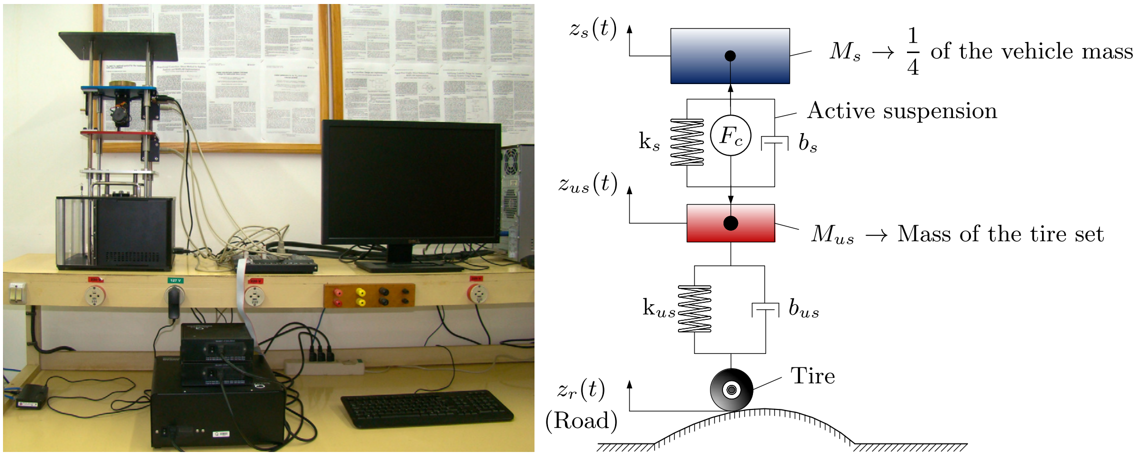
- Among the methods that consider saturation of the control signal, Theorem 1 presents the best numerical results;
- Corollary 1 presents numerical results with close values, but the choice of Theorem 1 is also justified by the switched control strategy that often provides a better dynamic response of the system, due to the switched control law (9) that chooses the gain K_i, $i \in \mathbb{K}_2$, that minimizes the time derivative of a quadratic Lyapunov function. For instance, in the second simulation, using Theorem 1 and Corollary 1, one obtains $\int_0^\infty y(t)^T y(t)dt / \int_0^\infty w(t)^T w(t)dt$ equal to 0.473 and 0.475, respectively;
- As in [26], the conditions of Corollary 2 do not consider that the control signal is subject to saturation. Therefore, Corollary 2 presents fewer constraints and consequently better results than Theorem 1 and Corollary 1.

Example 2 *(A practical implementation using an active suspension system with an actuator fault).* The purpose of the active suspension example is to design and implement a switched $\mathcal{H}_\infty$ regulator for a quarter-car model. Then, consider the active suspension system of a vehicle, manufactured by the Quanser® [32] and its schematic model represented in Fig. 6.

The system consists of a set of two masses, denoted by M_s and M_{us}. The mass M_s represents $\frac{1}{4}$ of the total vehicle body and is supported by the spring k_s and by the damper b_s. The mass M_{us} corresponds to the mass of the tire set and is supported by the spring k_{us} and by the damper b_{us}. The vibrations caused by irregularities in the road can be attenuated by the vehicle's active suspension system, represented by a motor (actuator) connected between the masses M_s and M_{us}, and controlled by the force F_c [37].

The schematic model provided by the Quanser® considers that the spring stiffness k_{us} is constant and equal to k_{us_0}. Although it constitutes a good approximation to model the spring, none of the springs has linear characteristics in real life and due to it is reasonable to model the suspension spring by a nonlinear function [38]. According to [38] the spring stiffness has a nonlinear behavior near the spring ends. Hence, based on [38], this manuscript considers a more general case with the stiffness given by

$$k_{us}(z_{us} - z_r, \Delta k_{us}) = k_{us_0}\left(1 + \Delta k_{us}\,|z_{us} - z_r|\right). \quad (81)$$



(a) (b)

Fig. 6. (a) Active suspension system from Quanser® belonging to the LPC–FEIS–UNESP (Brazil); (b) Schematic model of the active suspension system.

Observe that $\Delta k_{us} |z_{us} - z_r|$ can represent the nonlinearity and also parametric uncertainty related to the spring stiffness, where Δk_{us} is an uncertain parameter such that $0 \le \Delta k_{us} \le \Delta k_{us0}$, and Δk_{us0}, k_{us0} are known constants. Note that this definition of k_{us}, for $\Delta k_{us} = 0$, includes the nominal value $k_{us} = k_{us0}$, given in [32].

Furthermore, in this example is considered an actuator fault that results in a power loss. The power loss is represented, in the mathematical model, by the function $k_{fault}(t)$ [13,37]. Supposing the fault channel from the controller to the actuator, it follows that

$$u(t)_{fault} = k_{fault}(t)\, u(t), \quad u(t) = F_c(t), \tag{82}$$

and one can consider three cases corresponding to three different actuator conditions [37]: (i) If $k_{fault}(t) = 0$, implies that the corresponding actuator $u(t)_{fault}$ has completely failed, or the active suspension system is in open-loop; (ii) If $k_{fault}(t) = 1$, represents the case of no fault in the actuator $u(t)_{fault}$; (iii) If $0 < k_{fault}(t) < 1$, means that there exists a partial fault in the corresponding actuator $u(t)_{fault}$. Thus an actuator fault can be considered as a parametric uncertainty. In the physical model of the active suspension system there is a payload mass, that consists of two identical weight units, resulting in the mass M_s. The M_s mass may assume values between 1.455 kg (without the two weight units) and 2.45 kg (with the two weight units). Hence, the M_s mass may be uncertain and belongs to the interval $1.455 \le M_s \le 2.45$ kg. Therefore, based on the modeling presented in [32] and considering (81), (82) and the vector of the premise variables $z(t) = \left[x(t)^T \ \Delta k_{us} \ k_{fault} \ M_s\right]^T$, the dynamic model of the active suspension system can be represented by

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{M_s} & -\frac{b_s}{M_s} & 0 & \frac{b_s}{M_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{M_{us}} & \frac{b_s}{M_{us}} & f_{43}(z(t)) & -\frac{(b_s+b_{us})}{M_{us}} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{k_{fault}(t)}{M_s} \\ 0 \\ -\frac{k_{fault}(t)}{M_{us}} \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ -1 \\ \frac{b_{us}}{M_{us}} \end{bmatrix} w(t), \\
x(t) &= \begin{bmatrix} z_s(t) - z_{us}(t) \\ \dot{z}_s(t) \\ z_{us}(t) - z_r(t) \\ \dot{z}_{us}(t) \end{bmatrix}, \quad y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t), \quad w(t) = \dot{z}_r, \\
f_{43}(z(t)) &= -\frac{k_{us0}\left(1 + \Delta k_{us} |z_{us} - z_r|\right)}{M_{us}}.
\end{aligned} \tag{83}$$

The respective values for the system parameters are shown in Table 1.

Table 1
Parameters of the active suspension system [32].

Parameters	Symbol	Value
Mass of $\frac{1}{4}$ of the total vehicle (kg)	M_s	$1.455 \leq M_s \leq 2.45$
Mass of the tire set (kg)	M_{us}	1
Stiffness constant of the spring (N/m)	k_s	900
Stiffness constant of the spring (N/m)	k_{us_0}	2500
Damping coefficient (Ns/m)	b_s	7.5
Damping coefficient (Ns/m)	b_{us}	5
Parameter of the spring (m^{-1})	Δk_{us_0}	60

Thus, to find the local models, the maximum and minimum values of the functions $k_{fault}(t)$ and $f_{43}(z(t))$ must be obtained. In this case, the methodology proposed in [14] will be used. Considering that the actuator fault can decrease from 0% to 20% the actuator power, then $0.8 \leq k_{fault}(t) \leq 1$. Due the physics restrictions of the spring length, the state variable $z_{us} - z_r$ is limited in the interval $-0.02 \leq z_{us} - z_r \leq 0.02$ m. Then, the domain D of the nonlinear function $f_{43}(z(t))$, $k_{fault}(t)$ and M_s is

$$D = \Big\{ z(t) = \Big[x(t)^T \; \Delta k_{us} \; k_{fault} \; M_s \Big]^T \in \mathbb{R}^7 : \\ -0.02 \leq z_{us} - z_r \leq 0.02, \quad 0 \leq \Delta k_{us} \leq 60, \quad 0.8 \leq k_{fault}(t) \leq 1, \quad 1.455 \leq M_s \leq 2.45 \Big\}. \tag{84}$$

The maximum and minimum values of the function $f_{43}(z(t))$, in the domain D, are the following:

$$a_{43_1} = \max_{z(t)\in D} \{f_{43}(z(t))\} = -2500, \qquad a_{43_2} = \min_{z(t)\in D} \{f_{43}(z(t))\} = -3000. \tag{85}$$

Thus, from (83), (84), (85) and Table 1, the following local models are obtained:

$$A_1 = A_3 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.35 & -3.0612 & 0 & 3.0612 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -3000 & -12.5 \end{bmatrix},$$
$$A_2 = A_4 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.35 & -3.0612 & 0 & 3.0612 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix},$$
$$A_5 = A_7 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.56 & -5.1546 & 0 & 5.1546 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -3000 & -12.5 \end{bmatrix},$$
$$A_6 = A_8 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.56 & -5.1546 & 0 & 5.1546 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix},$$
$$B_1 = B_2 = \begin{bmatrix} 0 & 0.32653 & 0 & -0.8 \end{bmatrix}^T, \qquad B_3 = B_4 = \begin{bmatrix} 0 & 0.40816 & 0 & -1 \end{bmatrix}^T,$$
$$B_5 = B_6 = \begin{bmatrix} 0 & 0.54983 & 0 & -0.8 \end{bmatrix}^T, \qquad B_7 = B_8 = \begin{bmatrix} 0 & 0.68729 & 0 & -1 \end{bmatrix}^T,$$
$$H_1 = H_2 = H_3 = H_4 = H_5 = H_6 = H_7 = H_8 = \begin{bmatrix} 0 & 0 & -1 & 5 \end{bmatrix}^T,$$
$$C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = C_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$D_1 = D_2 = D_3 = D_4 = D_5 = D_6 = D_7 = D_8 = 0_{2\times 1},$$
$$G_1 = G_2 = G_3 = G_4 = G_5 = G_6 = G_7 = G_8 = 0_{2\times 1}. \tag{86}$$

Consider that the operation region $\mathcal{X}$ (4) is given by: $p = 1$, $h \in \mathbb{K}_1$, $R = [\,0 \quad 0 \quad 1 \quad 0\,]$ and $\phi = 0.02$. The input signal $F_c(t)$ is limited between ± 39.2 N, according to the manufacturer's suggestions and the system restrictions [32]. This limitation is performed using a saturator via Simulink®. Then, the polyhedral set $\mathcal{L}(L_j)$ (12) has $n_u = 1$, $k \in \mathbb{K}_1$ and $\rho = 39.2$.

The reference $z_r(t)$ was chosen to produce a sine wave signal, with amplitude 0.0015 m and frequency $(f = 1 + t)$ Hz for $0.5 \le t \le 9.5$ s, which varies linearly from 1.5 to 10.5 Hz and $z_r(t) = 0$ for $0 \le t < 0.5$ s and $9.5 < t \le 10$ s. Observe that $w(t)$ has a finite energy. Then, consider that $x(0) = 0$, $x(0)^T P x(0) = 0$ and $\int_0^{10} w(t)^T w(t) dt \le \int_0^{\infty} w(t)^T w(t) dt \le 0.02$, such that $\epsilon_0 = 0$ and $\epsilon = 0.02$.

For $\varphi = 10$ and $\beta = 0.01$, the optimization problem with LMIs (45), given in Theorem 1, and (58)–(59) were solved using the MatLab® software and the modeling language YALMIP with the solver LMILab. A feasible solution was obtained with an $\mathcal{H}_\infty$ guaranteed cost $\gamma = 0.4587$ and the following controller gains K_j, symmetric matrices $\bar{Q}_j$, $j \in \mathbb{K}_8$, and positive definite matrix P:

$$
\begin{aligned}
K_1 &= \begin{bmatrix} 21.317 & 55.892 & -1519.125 & -25.726 \end{bmatrix}, & K_5 &= \begin{bmatrix} -279.307 & 43.373 & -1439.728 & -26.561 \end{bmatrix},\\
K_2 &= \begin{bmatrix} 19.501 & 55.759 & -1821.392 & -26.042 \end{bmatrix}, & K_6 &= \begin{bmatrix} -277.567 & 43.212 & -1680.111 & -26.284 \end{bmatrix},\\
K_3 &= \begin{bmatrix} 39.206 & 54.827 & -1586.110 & -26.951 \end{bmatrix}, & K_7 &= \begin{bmatrix} -229.815 & 43.310 & -1537.852 & -29.692 \end{bmatrix},\\
K_4 &= \begin{bmatrix} 34.657 & 54.621 & -1833.107 & -26.886 \end{bmatrix}, & K_8 &= \begin{bmatrix} -220.798 & 43.648 & -1761.310 & -29.029 \end{bmatrix},
\end{aligned}
$$

$$
\bar{Q}_1 = \begin{bmatrix} -16272.519 & -1825.561 & 114155.930 & 925.079 \\ -1825.561 & -36.668 & -2248.541 & -51.874 \\ 114155.930 & -2248.541 & 546811.233 & 7447.960 \\ 925.079 & -51.874 & 7447.960 & 91.543 \end{bmatrix},
$$

$$
\bar{Q}_2 = \begin{bmatrix} -16272.500 & -1825.557 & 114150.475 & 925.078 \\ -1825.557 & -36.668 & -2248.077 & -51.874 \\ 114150.475 & -2248.077 & 546732.075 & 7446.793 \\ 925.078 & -51.874 & 7446.793 & 91.543 \end{bmatrix},
$$

$$
\bar{Q}_3 = \begin{bmatrix} -16272.670 & -1825.572 & 114157.018 & 925.113 \\ -1825.572 & -36.632 & -2249.797 & -51.898 \\ 114157.018 & -2249.797 & 546851.740 & 7448.712 \\ 925.113 & -51.898 & 7448.712 & 91.557 \end{bmatrix},
$$

$$
\bar{Q}_4 = \begin{bmatrix} -16272.642 & -1825.573 & 114151.610 & 925.108 \\ -1825.573 & -36.632 & -2249.350 & -51.899 \\ 114151.610 & -2249.350 & 546775.471 & 7447.611 \\ 925.108 & -51.899 & 7447.611 & 91.557 \end{bmatrix},
$$

$$
\bar{Q}_5 = \begin{bmatrix} -16252.451 & -1823.735 & 114105.917 & 924.290 \\ -1823.735 & -36.536 & -2251.732 & -51.923 \\ 114105.917 & -2251.732 & 546872.720 & 7448.650 \\ 924.290 & -51.923 & 7448.650 & 91.544 \end{bmatrix},
$$

$$
\bar{Q}_6 = \begin{bmatrix} -16252.542 & -1823.744 & 114101.419 & 924.289 \\ -1823.744 & -36.536 & -2251.270 & -51.924 \\ 114101.419 & -2251.270 & 546796.656 & 7447.574 \\ 924.289 & -51.924 & 7447.574 & 91.544 \end{bmatrix},
$$

$$
\bar{Q}_7 = \begin{bmatrix} -16253.609 & -1823.739 & 114107.876 & 924.361 \\ -1823.739 & -36.488 & -2253.168 & -51.944 \\ 114107.876 & -2253.168 & 546912.743 & 7449.212 \\ 924.361 & -51.944 & 7449.212 & 91.551 \end{bmatrix},
$$

$$
\bar{Q}_8 = \begin{bmatrix} -16253.743 & -1823.758 & 114103.620 & 924.370 \\ -1823.758 & -36.489 & -2252.700 & -51.945 \\ 114103.620 & -2252.700 & 546838.514 & 7448.179 \\ 924.370 & -51.945 & 7448.179 & 91.553 \end{bmatrix},
$$

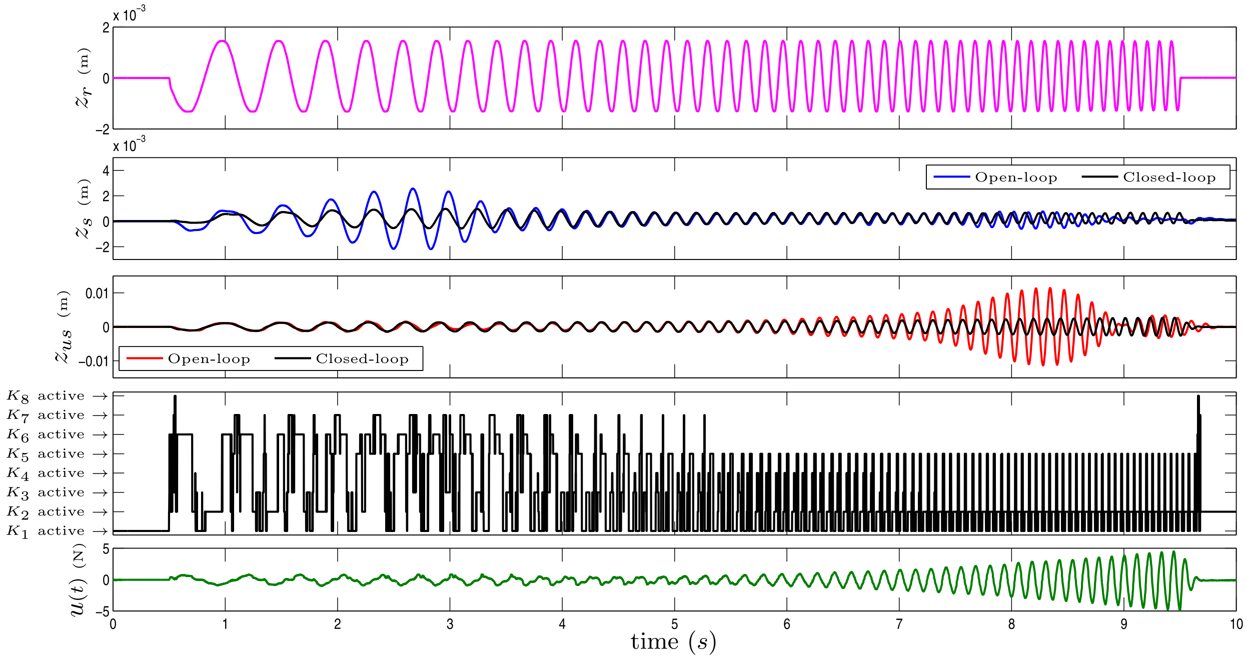


Fig. 7. The active suspension dynamic behavior for the open-loop and the closed-loop systems, with $M_s = 2.45$ kg. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$P = \begin{bmatrix} 2.0338 & 0.0761 & -0.6087 & 0.0258 \\ 0.0761 & 0.0070 & -0.1292 & -0.0002 \\ -0.6087 & -0.1292 & 9.7192 & 0.0909 \\ 0.0258 & -0.0002 & 0.0909 & 0.0031 \end{bmatrix}. \tag{87}$$

The goal of the practical implementation is to decrease the oscillations caused by the road surface ($z_r(t)$). Then, two cases of practical implementations will be presented. In the first case, one considers the two weight units coupled to the active suspension system ($M_s = 2.45$ kg). In the second case, no weight unit is considered coupled to the system ($M_s = 1.455$ kg).

For each case ($M_s = 2.45$ kg and $M_s = 1.455$ kg), the following practical implementations were performed:

- an open-loop implementation, such that $u(t) = 0$, for $0 \leq t \leq 10$ s;
- a closed-loop implementation, such that the switched $\mathcal{H}_\infty$ controller given in (9) and (87) was used, considering an actuator fault. To add a fault in the actuator without any physical change, a 20% decrease in power of the actuator is forced by inserting a gain $k_{fault} = 0.8$ acting directly on engine, as described in (82), using the MatLab/Simulink® software. Emphasizing that the complete closed-loop implementation was performed with the actuator fault ($k_{fault}(t) = 0.8$ in (82), for $0 \leq t \leq 10$ s).

Fig. 7 and Fig. 8 show the system dynamic behavior for $M_s = 2.45$ kg and $M_s = 1.455$ kg, respectively. From these figures, observe that the open-loop system responses are bounded, even without the control action. However, the system presents oscillations with large amplitudes, causing discomfort to the driver and high levels of mechanical stress, which can cause damage to the suspension components. Note that the closed-loop system reduced the maximum amplitudes of z_s and z_{us}, providing comfort and safety to the equipment. Finally, the $\mathcal{H}_\infty$ norm minimization mitigated the exogenous input effect in the output.

Remark 9. Now, considering the active suspension example, some considerations will be presented relating the methods proposed in this work (Theorem 1, Corollary 1 and Corollary 2) and [26]. In this example, implementing the method proposed in Theorem 1 is justified because:

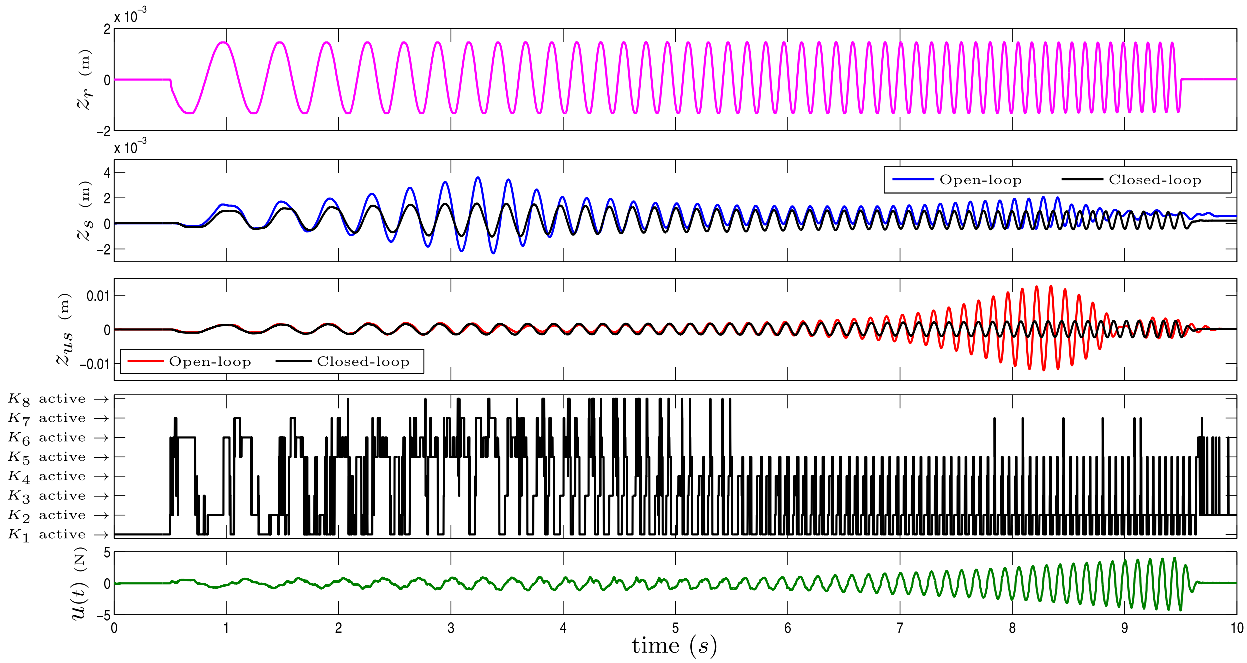


Fig. 8. The active suspension dynamic behavior for the open-loop and the closed-loop systems, with $M_s = 1.455$ kg. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

- the control design of Theorem 1 considers that the control signal is subject to saturation, unlike Corollary 2 and [26];
- in this example, it is considered that the active suspension system has an uncertain mass (M_s) and a nonlinearity related to the spring stiffness k_{us}. After obtaining the state-feedback gains solving the optimization problem of Theorem 1, to implement the switched control law (9) it is not necessary to calculate the membership functions of the nonlinear system. On the other hand, to implement the control law presented in [26] it is necessary to calculate the membership function. Therefore, due to the uncertain mass (M_s) and the nonlinearity k_{us}, it is advantageous to use the method proposed in Theorem 1;
- considering (86) and the aforementioned design parameters ($\phi = 0.02$, $R = [\,0 \quad 0 \quad 1 \quad 0\,]$, $\rho = 39.2$, $\varphi = 10$, $\beta = 0.01$, $\epsilon_0 = 0$ and $\epsilon = 0.02$), the conditions of the optimization problem with LMIs (60), given in Corollary 1, and (58)–(59) are not feasible. Thus, using the method proposed in Corollary 1, it was not possible to design only one state-feedback gain to control the active suspension system (83)–(86).

5. Conclusions

An $\mathcal{H}_\infty$ switched controller design for a class of uncertain nonlinear systems, subject to actuator saturation and energy-bounded disturbances, is proposed. The switched control law ensures that the state trajectory remains within a region in which the uncertain nonlinear system can be exactly described by a T–S fuzzy model. The two main advantages of the proposed method are: i) To perform the design and implementation of the control law is not necessary to find the membership functions expressions. Thus, the proposed procedure allows the membership functions depend on uncertain or unknown parameters and also avoids complex calculations that are often related to these expressions; ii) The design conditions do not depend on the magnitude of the disturbance, unlike the method presented [26] that only is applied in systems subject to magnitude-bounded disturbances. Considering a chaotic Lorenz system, presented in the first example, the switched control design proposed in Theorem 1 presented relaxation when compared to the result obtained with the control design proposed in Corollary 1, that uses only one controller. For disturbances with large magnitude, the proposed methods provided better results, even when compared to the procedure presented in [26], that considers full access to the membership functions. As in [26], the conditions of Corollary 2 do not consider that the control signal is subject to saturation. Therefore, Corollary 2 presents fewer constraints and consequently better results than Theorem 1 and Corollary 1. A practical implementation using an active suspension system illustrated the practical effectiveness of the proposed methodology as well as showed that the $\mathcal{H}_\infty$ switched control can be able to mitigate the action of an exogenous input in system output, even with an actuator fault and different operating conditions related to the mass M_s. Finally, considering the control design specifications and the conditions presented in Corollary 1, it was not possible to find a solution with only one state-feedback gain to control the same active suspension system.

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References

[1] T. Takagi, M. Sugeno, Fuzzy identification of systems and its applications to modeling and control, IEEE Trans. Syst. Man Cybern. SMC-15 (1) (1985) 116–132, https://doi.org/10.1109/TSMC.1985.6313399.
[2] T. Taniguchi, K. Tanaka, H. Ohatake, H.O. Wang, Model construction, rule reduction, and robust compensation for generalized form of Takagi–Sugeno, IEEE Trans. Fuzzy Syst. 9 (4) (2001) 525–538.
[3] H.O. Wang, K. Tanaka, M.F. Griffin, An approach to fuzzy control of nonlinear systems: stability and design issues, IEEE Trans. Fuzzy Syst. 4 (1) (1996) 14–23, https://doi.org/10.1109/91.481841.
[4] K. Tanaka, T. Ikeda, H.O. Wang, Fuzzy regulators and fuzzy observers: relaxed stability conditions and lmi-based designs, IEEE Trans. Fuzzy Syst. 6 (2) (1998) 250–265, https://doi.org/10.1109/91.669023.
[5] M.C.M. Teixeira, S.H. Żak, Stabilizing controller design for uncertain nonlinear systems using fuzzy models, IEEE Trans. Fuzzy Syst. 7 (2) (1999) 133–142, https://doi.org/10.1109/91.755395.
[6] K. Tanaka, H.O. Wang, Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach, John Wiley & Sons, Inc., New York, NY, USA, 2001.
[7] H. Tuan, P. Apkarian, T. Narikiyo, Y. Yamamoto, Parameterized linear matrix inequality techniques in fuzzy control system design, IEEE Trans. Fuzzy Syst. 9 (2) (2001) 324–332, https://doi.org/10.1109/91.919253.
[8] X.D. Liu, Q. Zhang, New approaches to $\mathcal{H}_\infty$ controller designs based on fuzzy observers for T–S fuzzy systems via LMI, Automatica 39 (9) (2003) 1571–1582, https://doi.org/10.1016/S0005-1098(03)00172-9.
[9] C.-H. Fang, Y.-S. Liu, S.-W. Kau, L. Hong, C.-H. Lee, A new LMI-based approach to relaxed quadratic stabilization of T–S fuzzy control systems, IEEE Trans. Fuzzy Syst. 14 (3) (2006) 386–397, https://doi.org/10.1109/TFUZZ.2006.876331.
[10] F. Delmotte, T.M. Guerra, M. Ksantini, Continuous Takagi–Sugeno's models: reduction of the number of LMI conditions in various fuzzy control design technics, IEEE Trans. Fuzzy Syst. 15 (3) (2007) 426–438, https://doi.org/10.1109/TFUZZ.2006.889829.
[11] V.F. Montagner, R.C.L.F. Oliveira, P.L.D. Peres, Convergent LMI relaxations for quadratic stabilizability and $\mathcal{H}_\infty$ control of Takagi–Sugeno fuzzy systems, IEEE Trans. Fuzzy Syst. 17 (4) (2009) 863–873, https://doi.org/10.1109/TFUZZ.2009.2016552.
[12] Y.J. Chen, H. Ohtake, K. Tanaka, W.J. Wang, H.O. Wang, Relaxed stabilization criterion for T–S fuzzy systems by minimum-type piecewise-Lyapunov-function-based switching fuzzy controller, IEEE Trans. Fuzzy Syst. 20 (6) (2012) 1166–1173, https://doi.org/10.1109/TFUZZ.2012.2196049.
[13] H. Li, H. Liu, H. Gao, P. Shi, Reliable fuzzy control for active suspension systems with actuator delay and fault, IEEE Trans. Fuzzy Syst. 20 (2) (2012) 342–357, https://doi.org/10.1109/TFUZZ.2011.2174244.
[14] M.P.A. Santim, M.C.M. Teixeira, W.A. de Souza, E. Assunção, R. Cardim, Design of a Takagi–Sugeno fuzzy regulator for a set of operation points, Math. Probl. Eng. 2012 (2012) 1–17, https://doi.org/10.1155/2012/731298.
[15] V.C.S. Campos, F.O. Souza, L.A.B. Tôrres, R.M. Palhares, New stability conditions based on piecewise fuzzy Lyapunov functions and tensor product transformations, IEEE Trans. Fuzzy Syst. 21 (4) (2013) 748–760, https://doi.org/10.1109/TFUZZ.2012.2230178.
[16] M. Chadli, H.R. Karimi, Robust observer design for unknown inputs Takagi–Sugeno models, IEEE Trans. Fuzzy Syst. 21 (1) (2013) 158–164, https://doi.org/10.1109/TFUZZ.2012.2197215.
[17] M. Liu, X. Cao, P. Shi, Fault estimation and tolerant control for fuzzy stochastic systems, IEEE Trans. Fuzzy Syst. 21 (2) (2013) 221–229, https://doi.org/10.1109/TFUZZ.2012.2209432.
[18] W.A. de Souza, M.C.M. Teixeira, R. Cardim, E. Assunção, On switched regulator design of uncertain nonlinear systems using Takagi–Sugeno fuzzy models, IEEE Trans. Fuzzy Syst. 22 (6) (2014) 1720–1727, https://doi.org/10.1109/TFUZZ.2014.2302494.
[19] R. Márquez, T.M. Guerra, M. Bernal, A. Kruszewski, Asymptotically necessary and sufficient conditions for Takagi–Sugeno models using generalized non-quadratic parameter-dependent controller design, Fuzzy Sets Syst. 306 (2017) 48–62, https://doi.org/10.1016/j.fss.2015.12.012.
[20] X.J. Li, G.H. Yang, Switching-type $\mathcal{H}_\infty$ filter design for T–S fuzzy systems with unknown or partially unknown membership functions, IEEE Trans. Fuzzy Syst. 21 (2) (2013) 385–392, https://doi.org/10.1109/TFUZZ.2012.2212908.
[21] X.J. Li, G.H. Yang, Fault detection for T–S fuzzy systems with unknown membership functions, IEEE Trans. Fuzzy Syst. 22 (1) (2014) 139–152, https://doi.org/10.1109/TFUZZ.2013.2249519.
[22] M.H. Asemani, V.J. Majd, A robust observer-based controller design for uncertain T–S fuzzy systems with unknown premise variables via LMI, Fuzzy Sets Syst. 212 (2013) 21–40, https://doi.org/10.1016/j.fss.2012.07.008.

[23] M. Klug, E.B. Castelan, V.J. Leite, L.F. Silva, Fuzzy dynamic output feedback control through nonlinear Takagi–Sugeno models, Fuzzy Sets Syst. 263 (2015) 92–111, https://doi.org/10.1016/j.fss.2014.05.019.
[24] M. Klug, E.B. Castelan, D. Coutinho, Control of nonlinear discrete-time systems subject to energy bounded disturbances using local T–S fuzzy models, in: 52nd IEEE Conference on Decision and Control, 2013, pp. 7426–7431.
[25] D.H. Lee, Y.H. Joo, M.H. Tak, LMI conditions for local stability and stabilization of continuous-time T–S fuzzy systems, Int. J. Control. Autom. Syst. 13 (4) (2015) 986–994, https://doi.org/10.1007/s12555-014-0327-6.
[26] D.H. Lee, J.B. Park, Y.H. Joo, S.K. Kim, Local $\mathcal{H}_\infty$ controller design for continuous-time T–S fuzzy systems, Int. J. Control. Autom. Syst. 13 (6) (2015) 1499–1507, https://doi.org/10.1007/s12555-014-0439-z.
[27] C.-S. Tseng, B.-S. Chen, $\mathcal{H}_\infty$ fuzzy control design for nonlinear systems subject to actuator saturation, in: 2006 IEEE International Conference on Fuzzy Systems, 2006, pp. 783–788.
[28] A. Nguyen, M. Dambrine, J. Lauber, Lyapunov-based robust control design for a class of switching non-linear systems subject to input saturation: application to engine control, IET Control Theory Appl. 8 (17) (2014) 1789–1802, https://doi.org/10.1049/iet-cta.2014.0398.
[29] U.N.L.T. Alves, M.C.M. Teixeira, D.R. de Oliveira, R. Cardim, E. Assunção, W.A. de Souza, Smoothing switched control laws for uncertain nonlinear systems subject to actuator saturation, Int. J. Adapt. Control Signal Process. 30 (8–10) (2016) 1408–1433, https://doi.org/10.1002/acs.2671.
[30] F. Wu, B. Lu, Anti-windup control design for exponentially unstable LTI systems with actuator saturation, Syst. Control Lett. 52 (3) (2004) 305–322, https://doi.org/10.1016/j.sysconle.2004.02.007.
[31] T. Hu, Z. Lin, B.M. Chen, An analysis and design method for linear systems subject to actuator saturation and disturbance, Automatica 38 (2) (2002) 351–359, https://doi.org/10.1016/S0005-1098(01)00209-6.
[32] Quanser Innovate Educate, Active Suspension LQG Control Using Quarc-Instructor Manual, 2010, pp. 1–48.
[33] Y.-Y. Cao, Z. Lin, Robust stability analysis and fuzzy-scheduling control for nonlinear systems subject to actuator saturation, IEEE Trans. Fuzzy Syst. 11 (1) (2003) 57–67, https://doi.org/10.1109/TFUZZ.2002.806317.
[34] S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, Stud. Appl. Math., vol. 15, SIAM – Soc. Ind. Appl. Math., Philadelphia, PA, 1994.
[35] G.S. Deaecto, G.C. Santos, State feedback $\mathcal{H}_\infty$ control design of continuous-time switched affine systems, IET Control Theory Appl. 9 (10) (2015) 1511–1516, https://doi.org/10.1049/iet-cta.2014.0153.
[36] L. Wang, X. Liu, Local analysis of continuous-time Takagi–Sugeno fuzzy system with disturbances bounded by magnitude or energy: a Lagrange multiplier method, Inf. Sci. 248 (2013) 89–102, https://doi.org/10.1016/j.ins.2013.06.023.
[37] E.R.P. da Silva, E. Assunção, M. Teixeira, R. Cardim, Robust controller implementation via state-derivative feedback in an active suspension system subjected to fault, in: Control and Fault-Tolerant Systems (SysTol), 2013 Conference on, 2013, pp. 752–757.
[38] C. Onat, I. Kucukdemiral, S. Sivrioglu, I. Yuksek, G. Cansever, LPV gain-scheduling controller design for a nonlinear quarter-vehicle active suspension system, Trans. Inst. Meas. Control 31 (1) (2009) 71–95, https://doi.org/10.1177/0142331208090630.