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Bayesian correction of *H*(*z*) data uncertainties

J. F. Jesus,[1,2]★ T. M. Gregório,[1] F. Andrade-Oliveira,[3,4] R. Valentim[5] and C. A. O. Matos[1]

[1]*Universidade Estadual Paulista (UNESP), Câmpus Experimental de Itapeva, Rua Geraldo Alckmin 519, 18409-010 Vila N. Sra. de Fátima, Itapeva, SP, Brazil*
[2]*Universidade Estadual Paulista (UNESP), Faculdade de Engenharia, Guaratinguetá, Departamento de Física e Química, Av. Dr. Ariberto Pereira da Cunha 333, 12516-410 Guaratinguetá - SP, Brazil*
[3]*Universidade Estadual Paulista (UNESP), Instituto de Física Teórica, São Paulo SP - 01140-070, Brazil*
[4]*Laboratório Interinstitucional de e-Astronomia - LIneA, Rua General José Cristino, 77, Rio de Janeiro, RJ 20921-400, Brazil*
[5]*Departamento de Física, Instituto de Ciências Ambientais, Químicas e Farmacêuticas - ICAQF, Universidade Federal de São Paulo (UNIFESP), Unidade José Alencar, Rua São Nicolau No. 210, 09913-030 Diadema, SP, Brazil*

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ABSTRACT

We compile 41 $H(z)$ data from literature and use them to constrain OΛCDM and flat ΛCDM parameters. We show that the available $H(z)$ suffers from uncertainties overestimation and propose a Bayesian method to reduce them. As a result of this method, using $H(z)$ only, we find, in the context of OΛCDM, $H_0 = 69.5 \pm 2.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_{\mathrm{m}} = 0.242 \pm 0.036$, and $\Omega_\Lambda = 0.68 \pm 0.14$. In the context of flat ΛCDM model, we have found $H_0 = 70.4 \pm 1.2\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $\Omega_{\mathrm{m}} = 0.256 \pm 0.014$. This corresponds to an uncertainty reduction of up to ≈30 per cent when compared to the uncorrected analysis in both cases.

Key words: cosmological parameters – dark energy – dark matter – cosmology: observations.

1 INTRODUCTION

Measurements of the expansion of the Universe are a central subject in the modern cosmology. In 1998, observations of type Ia supernovae (Riess et al. 1997; Perlmutter et al. 1999) gave strong evidences of a transition epoch between decelerated and accelerated expansion. Those evidences are also consistent with data from Baryon Acoustic Oscillations (BAO) measurements and the Cosmic Microwave Background Anisotropies (CMB).

Among the many viable candidates to explain the cosmic acceleration, the cosmological constant Λ explains very well great part of the current observations and it is also the simplest candidate. It gave to the model formed by cosmological constant plus cold dark matter, the ΛCDM model, the status of standard model in cosmology. On the other hand, the Λ term presents important conceptual problems in its core, e.g. the huge inconsistency of the quantum derived and the cosmological observed values of energy density, the so-called *cosmological constant problem* (Weinberg 1989). Hence, despite of its observational success, the composition and the history of the Universe are still a question that needs further investigation.

Precise measurements of the cosmic expansion may be obtained through the SNe observations. Although they furnish stringent cosmological constraints, they are not directly measuring the expansion rate $H(z)$ but its integral in the line of sight. Today, three distinct methods are producing direct measurements of $H(z)$ namely, through differential dating of the cosmic chronometers (Simon et al. 2005; Stern et al. 2010; Moresco et al. 2012; Zhang et al. 2012; Moresco 2015; Moresco et al. 2016), BAO techniques (Gaztañaga et al. 2009; Blake et al. 2012; Busca et al. 2012; Anderson et al. 2013; Font-Ribeira et al. 2013; Delubac et al. 2014), and correlation function of luminous red galaxies (LRGs) (Chuang & Wang 2013; Oka et al. 2014), which does not rely on the nature of space–time geometry between the observed object and us.

In this work, we treat the ΛCDM model expansion history as a generative model for the $H(z)$ data (Hogg, Bovy & Lang 2008). However, considering a goodness-of-fit criterion, we discuss a possible overestimation in the uncertainty in the current $H(z)$ data and we propose a new generative model to $H(z)$ data, in order to take into account this overestimation.

This article is structured as follows. In Section 2, we discuss the basic features of the ΛCDM model. In Section 3, we review the $H(z)$ data available on the literature and compile a sample with 41 data.

In Section 4, we discuss the goodness of fit of ΛCDM with $H(z)$ data and in Section 5 we discuss a method to treat $H(z)$ uncertainties and apply it to the ΛCDM with spatial curvature. In Subsection 5.1, we apply the same method to the flat ΛCDM. In Section 6, we compare corrected and uncorrected models by using a Bayesian

★ E-mail: jfjesus@itapeva.unesp.br

criterion and in Section 7 we compare our results with other $H(z)$ analyses. Finally, in Section 8, we summarize the results.

2 COSMIC DYNAMICS OF ΛCDM MODEL

We start by considering the homogeneous and isotropic FRW line element (with $c = 1$):

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \tag{1}$$

where a is the scale factor, (r, θ, ϕ) are comoving coordinates, and the spatial curvature parameter k can assume values -1, $+1$ or 0.

In this background, the Einstein Field Equations (EFEs) with a cosmological constant are given by

$$8\pi G\rho = 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} - \Lambda \tag{2}$$

$$-8\pi G p = 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda, \tag{3}$$

where ρ and p are total density and pressure of the cosmological fluid and Λ is cosmological constant. We may write the Friedmann equation (2) in terms of the observable redshift z, which relates to scale factor as $a = \frac{a_0}{1+z}$:

$$H^2 = \frac{8\pi G(\rho + \rho_\Lambda)}{3} - k(1+z)^2, \tag{4}$$

where $\rho_\Lambda = \frac{\Lambda}{8\pi G}$ and $H \equiv \frac{\dot{a}}{a}$ is the expansion rate. The EFEs include energy conservation, so we may deduce the continuity equation from equations (2) and (3):

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0, \tag{5}$$

where (ρ_i, p_i) stand for each fluid, be it dark matter, baryons, radiation, neutrinos, cosmological constant or anything else that does not exchange energy. For dark matter and baryons, we have $p_i \sim 0$, so they evolve with $\rho_i \propto a^{-3}$, the cosmological constant has a constant ρ_Λ and radiation and neutrinos follow $\rho_i \propto a^{-4}$, so they may be neglected in our work, as we are interested in low redshifts (up to $z \sim 2$). So, we may write for our components of interest:

$$\rho_m = \rho_{m0}(1+z)^3 \tag{6}$$

$$\rho_\Lambda = \rho_{\Lambda 0}, \tag{7}$$

where ρ_m stands for dark matter+baryons. So, the Friedmann equation can be written as

$$\left(\frac{H}{H_0}\right)^2 = \frac{8\pi G\rho_{m0}(1+z)^3}{3H_0^2} + \frac{8\pi G\rho_{\Lambda 0}}{3H_0^2} - \frac{k(1+z)^2}{H_0^2} \tag{8}$$

and by defining the density parameters $\Omega_i \equiv \frac{\rho_{i0}}{\rho_{c0}}$, where $\rho_{c0} \equiv \frac{3H_0^2}{8\pi G}$ and $\Omega_k \equiv -\frac{k}{a_0^2 H_0^2}$, we may write

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\rm m}(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda, \tag{9}$$

from which we deduce the normalization condition $\Omega_{\rm m} + \Omega_\Lambda + \Omega_k = 1$, or $\Omega_k = 1 - \Omega_{\rm m} - \Omega_\Lambda$, so we actually have three free parameters on this equation $(\Omega_{\rm m}, \Omega_\Lambda, H_0)$. Finally, we may write for $H(z)$

$$H(z) = H_0\left[\Omega_{\rm m}(1+z)^3 + (1 - \Omega_{\rm m} - \Omega_\Lambda)(1+z)^2 + \Omega_\Lambda\right]^{\frac{1}{2}}. \tag{10}$$

As usual, we will call this model, where we allow for spatial curvature, OΛCDM. The standard, concordance flat ΛCDM model has $\Omega_k = 0$, thus:

$$H(z) = H_0\left[\Omega_{\rm m}(1+z)^3 + 1 - \Omega_{\rm m}\right]^{\frac{1}{2}}. \tag{11}$$

3 $H(z)$ DATA

Hubble parameter data as a function of redshift yields one of the most straightforward cosmological tests because it is inferred from astrophysical observations alone, not depending on any background cosmological models.

At the present time, the most important methods for obtaining $H(z)$ data are[1] (i) through 'cosmic chronometers', for example, the differential age of galaxies (Simon et al. 2005; Stern et al. 2010; Moresco et al. 2012; Zhang et al. 2012; Moresco 2015; Moresco et al. 2016), (ii) measurements of peaks of acoustic oscillations of baryons (BAO) (Gaztañaga et al. 2009; Blake et al. 2012; Busca et al. 2012; Anderson et al. 2013; Font-Ribeira et al. 2013; Delubac et al. 2014), and (iii) through a correlation function of LRGs (Chuang & Wang 2013; Oka et al. 2014).

The data we work here are a combination of the compilations from Sharov & Vorontsova (2014) and Moresco et al. (2016). Sharov & Vorontsova (2014) add six $H(z)$ data in comparison to Farooq & Ratra (2013) compilation, which had 28 measurements. Moresco et al. (2016), on their turn, have added seven new $H(z)$ measurements in comparison to Sharov & Vorontsova (2014). By combining both data sets, we arrive at 41 $H(z)$ data, as can be seen in Table 1 and Fig. 1.

From these data, we perform a χ^2-statistics, generating the χ^2 function of free parameters:

$$\chi^2 = \sum_{i=1}^{41}\left[\frac{H_0 E(z_i, \Omega_{\rm m}, \Omega_\Lambda) - H_i}{\sigma_{Hi}}\right]^2, \tag{12}$$

where $E(z) \equiv \frac{H(z)}{H_0}$ and $H(z)$ is given by equation (10).

4 DATA ANALYSIS AND GOODNESS OF FIT

In order to minimize the χ^2 function (12) and find the constraints over the free parameters $(H_0, \Omega_{\rm m}, \Omega_\Lambda)$, we have sampled the likelihood $\mathcal{L} \propto e^{-\chi^2/2}$ through Monte Carlo Markov Chain (MCMC) analysis. A simple and powerful MCMC method is the so-called Affine Invariant MCMC Ensemble Sampler by Goodman & Weare (2010), which was implemented in `Python` language with the `emcee` software by Foreman et al. (2013). This MCMC method has the advantage over simple Metropolis-Hasting (MH) methods of depending on only one scale parameter of the proposal distribution and on the number of walkers, while MH methods in general depend on the parameter covariance matrix, that is, it depends on $n(n+1)/2$ tuning parameters, where n is dimension of parameter space. The main idea of the Goodman–Weare affine-invariant sampler is the so-called 'stretch move', where the position (parameter vector in parameter space) of a walker (chain) is determined by the position of the other walkers. Foreman-Mackey et al. (2013) modified this method, in order to make it suitable for parallelization, by splitting the walkers in two groups, then the position of a walker in one group is determined by *only* the position of walkers of the other group.[2]

[1] See Lima et al. (2012) for a review.

[2] See Allison & Dunkley (2014) for a comparison among various MCMC sampling techniques.

Table 1. 41 Hubble parameter versus redshift data.

z	$H(z)$	σ_H	Reference
0.070	69	19.6	Zhang et al. (2012)
0.090	69	12	Simon et al. (2005)
0.120	68.6	26.2	Zhang et al. (2012)
0.170	83	8	Simon et al. (2005)
0.179	75	4	Moresco et al. (2012)
0.199	75	5	Moresco et al. (2012)
0.200	72.9	29.6	Zhang et al. (2012)
0.240	79.69	6.65	Gaztañaga et al. (2009)
0.270	77	14	Simon et al. (2005)
0.280	88.8	36.6	Zhang et al. (2012)
0.300	81.7	6.22	Oka et al. (2014)
0.350	82.7	8.4	Chuang & Wang (2013)
0.352	83	14	Moresco et al. (2012)
0.3802	83	13.5	Moresco et al. (2016)
0.400	95	17	Simon et al. (2005)
0.4004	77	10.02	Moresco et al. (2016)
0.4247	87.1	11.2	Moresco et al. (2016)
0.430	86.45	3.68	Gaztañaga et al. (2009)
0.440	82.6	7.8	Blake et al. (2012)
0.4497	92.8	12.9	Moresco et al. (2016)
0.4783	80.9	9	Moresco et al. (2016)
0.480	97	62	Stern et al. (2010)
0.570	92.900	7.855	Anderson et al. (2013)
0.593	104	13	Moresco et al. (2012)
0.6	87.9	6.1	Blake et al. (2012)
0.68	92	8	Moresco et al. (2012)
0.73	97.3	7.0	Blake et al. (2012)
0.781	105	12	Moresco et al. (2012)
0.875	125	17	Moresco et al. (2012)
0.88	90	40	Stern et al. (2010)
0.9	117	23	Simon et al. (2005)
1.037	154	20	Moresco et al. (2012)
1.300	168	17	Simon et al. (2005)
1.363	160	22.6	Moresco (2015)
1.43	177	18	Simon et al. (2005)
1.53	140	14	Simon et al. (2005)
1.75	202	40	Simon et al. (2005)
1.965	186.5	50.4	Moresco (2015)
2.300	224	8	Busca et al. (2012)
2.34	222	7	Delubac et al. (2014)
2.36	226	8	Font-Ribeira et al. (2013)

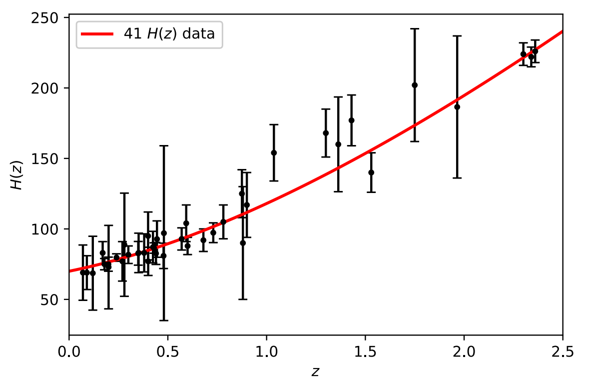

Figure 1. 41 $H(z)$ data and corresponding best-fitting ΛCDM model.

We used the freely available software `emcee` to sample from our likelihood in our three-dimensional parameter space. We have used flat priors over the parameters. In order to plot all the constraints in the same figure, we have used the freely available software `getdist`,[3] in its `Python` version. The results of our statistical analyses from equation (12) correspond to the red lines in Fig. 3 and Table 2. From this analysis, we have obtained $\chi^2_\nu = \frac{\chi^2_{\rm min}}{\nu} = 18.551/38 = 0.488\,19$, where $\nu = n - p$ is the number of degrees of freedom.

As it is well known (Vuolo 1996; Bevington & Robinson 2003), when one analyses the probability distribution of χ^2_ν it has an expected value $\chi^2_\nu = 1$. χ^2_ν values very far from this are unlikely. High χ^2_ν values may indicate underestimation of uncertainties or poor fitting of the model, while low values of χ^2_ν indicate, in general, overestimation of uncertainties. The χ^2_ν distribution is given by

$$h_\nu(\chi^2_\nu) = \frac{\nu^{\frac{\nu}{2}}(\chi^2_\nu)^{\frac{1}{2}(\nu-2)}e^{-\frac{\nu}{2}\chi^2_\nu}}{2^{\nu/2}\Gamma(\nu/2)}, \tag{13}$$

where Γ is complete gamma function. It can be shown that the mean χ^2_ν is given by $\overline{\chi^2_\nu} = 1$, while the mode is given by $\widehat{\chi^2_\nu} = 1 - \frac{2}{\nu}$. In the limit of a large sample and few parameters, both converge to the same value $\chi^2_\nu \approx 1$. From (13), we may also define the cumulative distribution function (cdf) or probability of obtaining a value of χ^2_ν as low as Q as

$$P(\chi^2_\nu < Q) \equiv \int_0^Q h_\nu(Q')\mathrm{d}Q'. \tag{14}$$

In order to illustrate the untypical small value of the χ^2_ν value we have obtained, namely, $\chi^2_\nu = 0.488\,19$, we have plotted the pdf $h_\nu(\chi^2_\nu)$ (13) and the cdf (14) for $\nu = 38$ in Fig. 2.

As one may see in the Fig. 2, the probability of obtaining χ^2_ν as low as $\chi^2_\nu = 0.488$ for $\nu = 38$ is quite small. In fact, by calculating the integral (14), we have obtained $P(\chi^2_\nu < 0.488\,19) = 0.3342$ per cent. Hence, it is a very small and unlikely χ^2 value, which, in turn, from equation (12) indicates overestimated $H(z)$ uncertainties.

5 $H(z)$ UNCERTAINTIES CORRECTION

How one may try to correct uncertainties? Ideally, at the moment of data acquisition, a better control of systematic uncertainties is desirable and new methods less prone to errors are to be used. In fact, in general, data coming from BAO and Lyman α have smaller errors than data coming from differential ages. However, not being able to reobtain the measurements or reanalyze them through new methods, we are left with the available data. Then, can nothing be done? From the Bayesian viewpoint, not necessarily. In fact, we may view the data as a collection of (z_i, H_i, σ_{Hi}). Very often, we are interested in a likelihood given by $\mathcal{L} = Ne^{-\chi^2/2}$, where N is only a normalization constant and one is interested in maximizing the likelihood, which is equivalent to maximizing the χ^2. Let us recall from where this expression comes from.

As discussed in Hogg et al. (2008), the likelihood may be seen as an *objective function*, that is, a function that represents monotonically the quality of the fit. Given a scientific problem at hand as fitting a model to the data, one must define some objective function that represents this 'goodness of fit', then try to optimize it in order to determine the best set of free parameters of the model that describe the data.

[3] `getdist` is part of the great MCMC sampler and CMB power spectrum solver `COSMOMC`, by Lewis & Bridle (2002).

Table 2. Mean values of parameters of OΛCDM model from $H(z)$ data, without uncertainties correction and with uncertainties correction factor f. Uncertainties correspond to 68 per cent c.l.

	$H(z)$ only		$H(z)+H_0$	
Parameter	Uncorrected	Corrected	Uncorrected	Corrected
H_0	69.1 ± 3.5	69.5 ± 2.5	72.4 ± 1.5	72.5 ± 1.1
Ω_m	0.237 ± 0.051	0.242 ± 0.036	0.267 ± 0.038	0.268 ± 0.028
Ω_Λ	0.66 ± 0.20	0.68 ± 0.14	$0.825^{+0.11}_{-0.095}$	0.831 ± 0.073
f	–	$0.723^{+0.084}_{-0.085}$	–	$0.728^{+0.067}_{-0.098}$

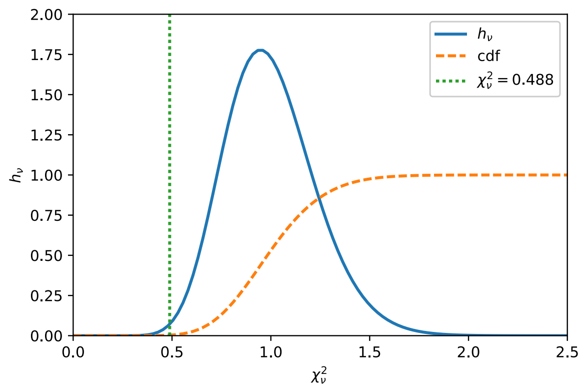

Figure 2. $h_\nu(\chi^2_\nu)$ and corresponding cdf for $\nu = 38$.

Hogg et al. (2008) argue that the only choice of the objective function that is truly justified, in the sense that it leads to probabilistic inference, is to make a *generative model* for the data. We may think of the generative model as a parametrized statistical procedure to reasonably generate the given data.

For instance, assuming Gaussian uncertainties in one dimension, we can create the following generative model: Imagine that the data really come from a function $y = f(x, \theta)$ given by the model and that the only reason for any data point deviates from this model is that to each of the true y values a small y-direction offset has been added, where that offset was drawn from a Gaussian distribution of zero mean and known variance σ_y^2. In this model, given an independent position x_i, an uncertainty σ_{yi}, and free parameters θ, the frequency distribution $p(y_i|x_i, \sigma_{yi}, \theta)$ for y_i is

$$p(y_i|x_i, \sigma_{yi}, \theta) = \frac{1}{(2\pi)^{1/2}\sigma_{yi}} \exp\left[-\frac{(y_i - f(x_i, \theta))^2}{2\sigma_{yi}^2}\right], \quad (15)$$

Thus, if the data points are independently drawn, the likelihood $\mathcal{L}$ is the product of conditional probabilities

$$\mathcal{L} = \prod_{i=1}^{n} p(y_i|x_i, \sigma_{yi}, \theta). \quad (16)$$

Taking the logarithm,

$$\ln \mathcal{L} = -\frac{1}{2}\sum_{i=1}^{n}\left[\frac{(y_i - f(x_i, \theta))^2}{\sigma_{yi}^2} + \ln(2\pi\sigma_{yi}^2)\right]. \quad (17)$$

In equation above, the second term $-\frac{1}{2}\sum_i \ln(2\pi\sigma_{yi}^2)$ is in general absorbed in the likelihood normalization constant, because the variances σ_{yi}^2 are considered fixed by the data. Here, we consider σ_i as parameters to be obtained by optimization of the objective function $\mathcal{L}$. As discussed in Hogg et al. (2008), it can be considered a correct procedure from the Bayesian point of view, although an involved one, and the obtained σ_i can be quite prior dependent.

In order to avoid having more free parameters than data, here we consider the σ_i to be all overestimated by a constant factor f, thus, $\sigma_{i,\,\mathrm{true}} = f\sigma_i$. This can be seen just as a simplifying hypothesis. More elaborated methods could be gather the data in some groups, then correct the σ_i for each group. However, as discussed in Hogg et al. (2008), it is not an easy task to separate good data from bad data, and not necessarily the bad data are the ones with bigger uncertainties. So, we limit ourselves here with just one overall correction factor and then we investigate if this is a good approximation. Taking f as a free parameter, we constrain it in a joint analysis with the cosmological parameters, similar to what is made in SNe Ia analyses (Amanullah et al. 2010; Suzuki et al. 2012; Betoule et al. 2014). For ΛCDM model, our set of free parameters now is $\theta = (H_0, \Omega_m, \Omega_\Lambda, f)$. A simpler but less justified hypothesis would be to simply find the value for f which provides $\chi^2_\nu \equiv 1$. However, as we expect χ^2_ν to have some variance, such a procedure is not much trustworthy. With f as a free parameter, it may include some uncertainty into the analysis, when compared to the standard, uncorrected analysis, but at the same time, it may also reduce the cosmological parameter uncertainties.

Instead of equation (17), we must work here with the following objective function:

$$\ln \mathcal{L} = -\frac{1}{2}\sum_{i=1}^{n}\left\{\frac{[H_i - H(z_i, H_0, \Omega_m, \Omega_\Lambda)]^2}{f^2\sigma_{Hi}^2} + \ln(2\pi f^2\sigma_{Hi}^2)\right\}. \quad (18)$$

By maximizing the above likelihood, we find not only the best-fitting cosmological parameters, but also the best correction factor f which will furnish the best model to describe the data. By doing the same procedure of last section, now with the additional parameter f, we find the constraints shown by the black lines in Fig. 3.

From Fig. 3, we may already see the difference in the parameter space if we introduce the f parameter. The corrected contours (black lines) are narrower than the uncorrected contours (red lines). This can be quantified by the parameter constraints shown in Table 2.

As can be seen in Table 2, σ_{H0} has been reduced from 3.5 to 2.5, σ_{Ω_m} has been reduced from 0.051 to 0.036, and σ_{Ω_Λ} has been reduced from 0.20 to 0.14. The mean value for f was $f = 0.723^{+0.084}_{-0.085}$. An interesting feature we may see from Fig. 3 is that the f parameter is much uncorrelated to cosmological parameters (confidence contours quite aligned with parameter axes). As we show in the next section, the best fit for the cosmological parameters $(H_0, \Omega_m, \Omega_\Lambda)$ is independent from the best fit for f. On the other hand, this is not true for the likelihoods, that is, $\mathcal{L} \neq \mathcal{L}_1(H_0, \Omega_m, \Omega_\Lambda)\mathcal{L}_2(f)$, as one may see from equation (18). This small unequality explains the small shift on mean values of cosmological parameters from Table 2. Saying in another way, the central values of cosmological parameters are weakly dependent on overall shifts on H_i uncertainties, but their variances are directly affected by f.

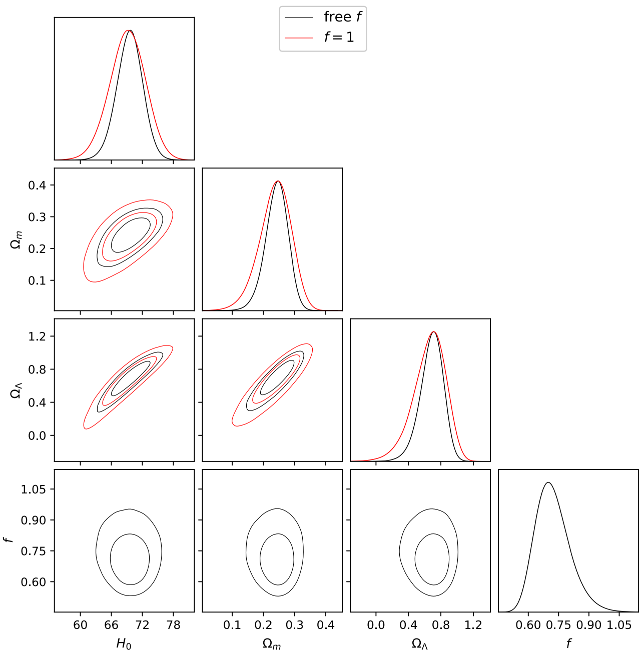

Figure 3. The results of statistical analysis for OΛCDM model. H_0 is in km s^{-1} Mpc^{-1}. **Diagonal:** Marginalized constraints from $H(z)$ data for each parameter. **Below diagonal:** Marginalized contour constraints for each indicated combination of parameters, with contours for 68.3 and 95.4 per cent confidence levels.

5.1 Flat ΛCDM

For completeness, as flat ΛCDM model is favoured from many observations, in this section we analyse this model similarly to OΛCDM. Equation (10) now reads

$$H(z) = H_0 \left[\Omega_m (1+z)^3 + 1 - \Omega_m\right]^{\frac{1}{2}} . \tag{19}$$

The results of this analysis may be seen in Fig. 4 and Table 3.

As one may see from Fig. 4, f is again uncorrelated to cosmological parameters, so it does not change their central values.

As one may see in Table 3, the H_0 uncertainty, for instance, is reduced from 1.7 to 1.2, which now corresponds to 1.7 per cent relative uncertainty. Ω_m uncertainty has reduced from 0.020 to 0.014.

5.2 Alternative analysis

In order to test the consistency of the above results, we have made an alternative analysis, considering only the data with lower redshifts and larger errors on $H(z)$. Namely, we have ignored the data with $z \geq 2.3$, which, although being distant, are reported with small uncertainties (3.15−3.57 per cent), when compared with lower redshift data with bigger uncertainties. Thus, here we use a new sample with 38 $H(z)$ data and $z < 2.3$. In the present analysis, we do not consider H_0 constraints, for simplicity.

As can be seen in Fig. 5 and Table 4, the result is that, without this 'anchor' at high redshift, the OΛCDM model is quite poorly constrained, mainly if we do not correct uncertainties. The result for Ω_m, for example, is compatible with the absence of dark matter at a 2.6σ c.l. ($\Omega_m \sim 0.04 \sim \Omega_b$ in its 2.6σ c.l. inferior limit). The constraints are slightly improved when we introduce the f correction ($\Omega_m \geq 0.04$ at a 3.7σ c.l.). Concerning the flat ΛCDM model (Fig. 6 and Table 4), the result already is good with no correction ($\sigma_{H0} = 2.3$) but is improved with the f correction ($\sigma_{H0} = 1.7$).

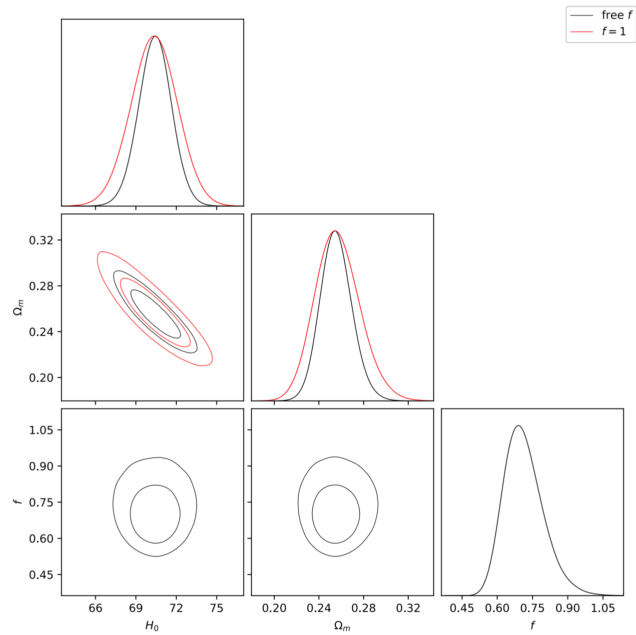

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Table 3. Mean values of parameters of flat ΛCDM model from $H(z)$ data, without uncertainties correction and with uncertainties correction factor f. Uncertainties correspond to 68 per cent c.l.

	$H(z)$ only		$H(z) + H_0$	
Parameter	Uncorrected	Corrected	Uncorrected	Corrected
H_0	70.3 ± 1.7	70.4 ± 1.2	71.8 ± 1.2	71.80 ± 0.89
Ω_m	0.257 ± 0.020	0.256 ± 0.014	$0.243^{+0.014}_{-0.015}$	0.242 ± 0.011
f	–	0.714 ± 0.082	–	$0.728^{+0.066}_{-0.096}$

Furthermore, the results for f are consistent with the ones we have obtained in the full 41 $H(z)$ sample data, which indicates some robustness of the method.

6 BAYESIAN CRITERION COMPARISON

Here, we use the Bayesian Information Criterion (BIC) (Schwarz 1978; Jesus, Valentim & Andrade-Oliveira 2017) in order to compare the models with uncertainties correction and without uncertainties correction. As an approximation for the Bayesian Evidence (BE) (Trotta 2008), BIC is useful because it is, in general, easier to calculate. As explained in, e.g. Kass & Raftery (1995), Trotta (2008), and Jesus et al. (2017), BE and BIC are great model comparison tools, because they incorporate the Ockham's razor principle, which penalizes models with excess of parameters due to their unnecessary complexity. They are different from other model selection tools, like Akaike Information Criterion (Akaike 1974), for instance, which does not take into account the excess of parameters. Let us discuss now for our case, if the introduction of the f parameter is necessary to better describe the $H(z)$ data. BIC is given by

$$\mathrm{BIC} = -2 \ln \mathcal{L}_{\max} + p \ln n, \tag{20}$$

where $\mathcal{L}_{\max}$ is the likelihood maximum and p is the number of free parameters. The two models we want to compare are: M_1: $f = 1$,

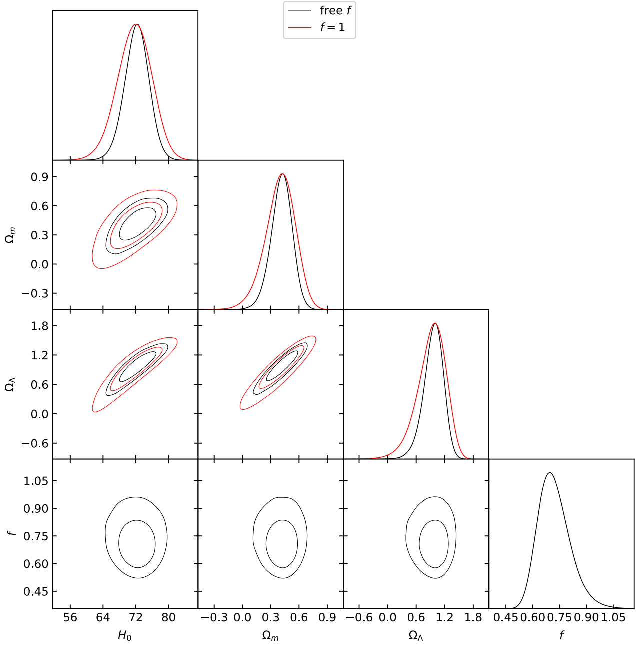

Figure 5. The results of statistical analysis for OΛCDM model with 38 $H(z)$ data with $z < 2.3$. H_0 is in km s^{-1} Mpc^{-1}. **Diagonal:** Marginalized constraints from $H(z)$ data for each parameter. **Below diagonal:** Marginalized contour constraints for each indicated combination of parameters, with contours for 68.3 and 95.4 per cent confidence levels.

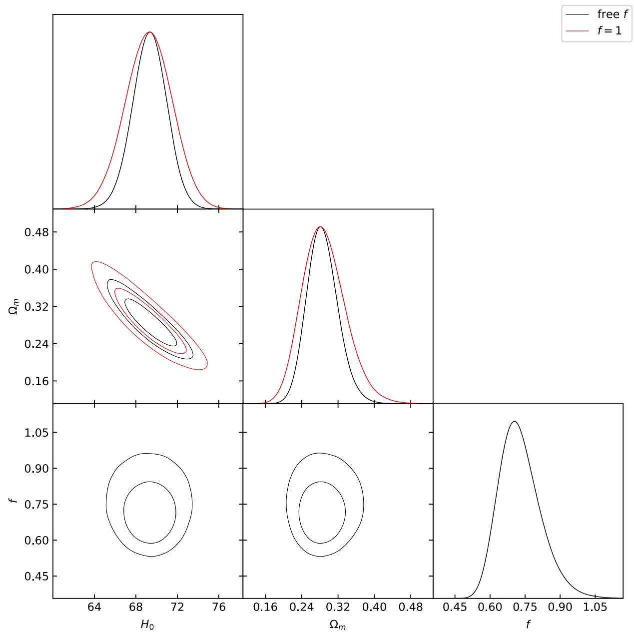

Figure 6. The results of statistical analysis for flat ΛCDM model with 38 $H(z)$ data with $z < 2.3$. H_0 is in km s^{-1} Mpc^{-1}. **Diagonal:** Marginalized constraints from $H(z)$ data for each parameter. **Below diagonal:** Marginalized contour constraints for each indicated combination of parameters, with contours for 68.3 and 95.4 per cent confidence levels.

Table 4. Mean values of parameters of OΛCDM and flat ΛCDM models from $H(z)$ data, without uncertainties correction and with uncertainties correction factor *f*. Uncertainties correspond to 68 per cent c.l.

	OΛCDM		Flat ΛCDM	
Parameter	Uncorrected	Corrected	Uncorrected	Corrected
H_0	71.7 ± 4.2	72.2 ± 3.0	69.2 ± 2.3	69.3 ± 1.7
Ω_m	$0.40^{+0.18}_{-0.14}$	$0.41^{+0.12}_{-0.10}$	$0.290^{+0.041}_{-0.053}$	$0.286^{+0.030}_{-0.037}$
Ω_Λ	$0.92^{+0.34}_{-0.23}$	$0.96^{+0.23}_{-0.17}$	–	–
f	–	$0.72^{+0.069}_{-0.10}$	–	$0.730^{+0.069}_{-0.10}$

that is, ΛCDM model without uncertainties correction is enough to describe the data; and M_2: $f \neq 1$ such that some correction f to uncertainties is necessary in order for the ΛCDM model to explain the $H(z)$ data. We may write the log-likelihood as

$$\ln \mathcal{L} = -\frac{1}{2}\left[\frac{\chi^2}{f^2} + \sum_{i=1}^{n} \ln(2\pi f^2 \sigma_i^2)\right], \tag{21}$$

where χ^2 is the uncorrected $\chi^2 \equiv \sum_{i=1}^{n} \frac{[H_i - H(z_i, H_0, \Omega_m, \Omega_\Lambda)]^2}{\sigma_{Hi}^2}$. In order to calculate BIC, we must find the maximum of $\ln \mathcal{L}$. By deriving (21) with respect to f:

$$\frac{\partial \ln \mathcal{L}}{\partial f} = -\frac{1}{f}\left[n - \frac{\chi^2}{f^2}\right]. \tag{22}$$

When it vanishes, we find the best fit:

$$\hat{f} = \sqrt{\frac{\chi^2_{\mathrm{min}}}{n}}. \tag{23}$$

From (20) and (23), we find:

$$\mathrm{BIC}_1 = \chi^2_{\mathrm{min}} + \sum_{i=1}^{n} \ln(2\pi\sigma_i^2) + p_1 \ln n \tag{24}$$

$$\mathrm{BIC}_2 = n + n \ln\left(\frac{2\pi\, \chi^2_{\mathrm{min}}}{n}\right) + \sum_{i=1}^{n} \ln(\sigma_i^2) + p_2 \ln n, \tag{25}$$

where p_j is the number of free parameters in M_j. So,

$$\Delta\mathrm{BIC} = \mathrm{BIC}_1 - \mathrm{BIC}_2$$

$$= \chi^2_{\mathrm{min}} - n \ln\left(\chi^2_{\mathrm{min}}\right) + (n - p_2 + p_1) \ln n - n. \tag{26}$$

For $p_1 = 3$ and $p_2 = 4$, it simplifies to

$$\Delta\mathrm{BIC} = \chi^2_{\mathrm{min}} - n \ln\left(\chi^2_{\mathrm{min}}\right) + (n-1) \ln n - n. \tag{27}$$

For $n = 41$ and $\chi^2_{\mathrm{min}} = 18.551$, it yields: $\Delta\mathrm{BIC} = 6.352$. As discussed in Jesus et al. (2017), for example, values of $\Delta\mathrm{BIC} > 5$ correspond to a decisive or strong statistical difference. That is, by this criterion, the model M_1 (no correction) may be discarded against model M_2 (with correction).

So, according to the BIC, the inclusion of the f parameter is necessary and, in the context of ΛCDM model, it leads to a more appropriate analysis of $H(z)$ data.

7 COMPARISON WITH OTHER $H(z)$ DATA ANALYSES

Farooq & Ratra (2013) have constrained OΛCDM model with 28 $H(z)$ data and two possible priors over H_0. With the most stringent prior, namely, the one from Riess et al. (2011), they have found, at 2σ, $0.20 \le \Omega_m \le 0.44$ and $0.62 \le \Omega_\Lambda \le 1.14$. We have found $0.13 \le \Omega_m \le 0.34$ and $0.23 \le \Omega_\Lambda \le 1.04$ for 41 $H(z)$ data without correction and $0.162 \le \Omega_m \le 0.31$ and $0.38 \le \Omega_\Lambda \le 0.96$ with the f correction. By considering the prior from Riess et al. (2011), namely, $H_0 = 73.8 \pm 2.4$ km s^{-1} Mpc^{-1}, we have found $0.18 \le \Omega_m \le 0.34$ and $0.57 \le \Omega_\Lambda \le 1.04$ without correction and $0.21 \le \Omega_m \le 0.32$ and $0.65 \le \Omega_\Lambda \le 0.99$ with the f correction.

With 34 $H(z)$ data, Sharov & Vorontsova (2014) find a more stringent result, namely, $H_0 = 70.26 \pm 0.32$, $\Omega_\mathrm{m} = 0.276^{+0.009}_{-0.008}$, and $\Omega_\Lambda = 0.769 \pm 0.029$. However, they have combined $H(z)$ data with SNe Ia and BAO data, which is beyond the scope of our present work. However, by comparing their result with our Table 2, we may see that both constraints are compatible at a 1σ c.l.

Moresco et al. (2016) have used their compilation of 30 $H(z)$ data combined with H_0 from Riess et al. (2011) to constrain the transition redshift from deceleration to acceleration, in the context of OΛCDM (Lima et al. 2012):

$$z_t = \left[\frac{2\Omega_\Lambda}{\Omega_\mathrm{m}}\right]^{1/3} - 1. \tag{28}$$

They have found $z_t = 0.64^{+0.11}_{-0.07}$. By using the present 41 $H(z)$ data, we find $z_t = 0.77 \pm 0.22$ without correction and $z_t = 0.78 \pm 0.15$ with the f correction. The results are in full agreement without the correction and are compatible at a 2σ c.l. with the f correction. We have mentioned the mean value for z_t, while Moresco et al. (2016) refer to the best-fitting value.

The constraints over H_0 are quite stringent today from many observations (Planck Collaboration XIII et al. 2016; Riess et al. 2016). However, there is some tension among H_0 values estimated from different observations (Bernal, Verde & Riess 2016), so we choose not to use H_0 in our main results here, Figs 3 and 4. We combine $H(z) + H_0$ only in Tables 2 and 3 and in the present section, using Riess et al. (2011) result, in order to compare with other earlier analyses.

8 CONCLUSION

In this work, we have compiled 41 $H(z)$ data and proposed a new method to better constrain models using $H(z)$ data alone, namely, by reducing overestimated uncertainties through a Bayesian approach. The BIC was used to show the need for correcting $H(z)$ data uncertainties. The uncertainties in the parameters were quite reduced when compared with methods of parameter estimation without correction and we have obtained an estimate of an overall correction factor in the context of OΛCDM and flat ΛCDM models.

Further investigations may include constraining other cosmological models or trying to optimally group $H(z)$ data and then correcting uncertainties.

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REFERENCES

Akaike H., 1974, Trans. Autom. Control, 19, 6
Allison R., Dunkley J., 2014, MNRAS, 437, 3918
Amanullah R. et al., 2010, ApJ, 716, 712
Anderson L. et al., 2013, MNRAS, 439, 83
Bernal J. L., Verde L., Riess A. G., 2016, J. Cosmology Astropart. Phys., 1610, 019
Betoule M. et al., 2014, A&A, 568, A22
Bevington P. R., Robinson D. K., 2003, Data Reduction and Error Analysis for the Physical Sciences. McGraw-Hill Book Company, New York, NY
Blake C. et al., 2012, MNRAS, 425, 405
Busca N. G. et al., 2013, A&A, 552, A96
Chuang C. H., Wang Y., 2013, MNRAS, 435, 255
Delubac T. et al., 2015, A&A, 574, A59
Farooq O., Ratra B., 2013, ApJ, 766, L7
Font-Ribera A. et al., 2013, J. Cosmology Astropart. Phys., 05, 027
Foreman-Mackey D., Hogg D. W., Lang D., Goodman J., 2013, PASP, 125, 306
Gaztañaga E., Cabré A., Hui L., 2009, MNRAS, 399, 1663
Goodman J., Weare J., 2010, Commun. Appl. Math. Comput. Sci., 5, 65
Hogg D. W., Bovy J., Lang D., 2008, preprint (arXiv:1008.4686)
Jesus J. F., Valentim R., Andrade-Oliveira F., 2017, J. Cosmology Astropart. Phys., 1709, 030
Kass R. E., Raftery A. E., 1995, J. Am. Stat. Assoc., 90, 773
Lewis A., Bridle S., 2002, Phys. Rev. D, 66, 103511
Lima J. A. S., Jesus J. F., Santos R. C., Gill M. S. S., 2012, preprint (arXiv:1205.4688)
Moresco M., 2015, MNRAS, 450, L16
Moresco M. et al., 2012, J. Cosmology Astropart. Phys., 8, 006
Moresco M. et al., 2016, J. Cosmology Astropart. Phys., 1605, 014
Oka A., Saito S., Nishimichi T., Taruya A., Yamamoto K., 2014, MNRAS, 439, 2515
Perlmutter S. et al., 1999, ApJ, 517, 565
Planck Collaboration XIII et al., 2016, A&A, 594, A13
Riess A. G. et al., 1997, AJ, 116, 1009
Riess A. G. et al., 2011, ApJ, 730, 119
Riess A. G. et al., 2016, ApJ, 826, 56
Schwarz G., 1978, Ann. Stat., 5, 461
Sharov G. S., Vorontsova E. G., 2014, J. Cosmology Astropart. Phys., 1410, 057
Simon J., Verde L., Jimenez R., 2005, Phys. Rev. D, 71, 123001
Stern D., Jimenez R., Verde L., Kamionkowski M., Stanford S. A., 2010, J. Cosmology Astropart. Phys., 02, 008
Suzuki N. et al., 2012, ApJ, 746, 85
Trotta R., 2008, Contemp. Phys., 49, 71
Vuolo J. H., 1996, Fundamentos da teoria de erros. Edgard Blücher, São Paulo (in Portuguese)
Weinberg S., 1989, Rev. Mod. Phys., 61, 1
Zhang C., Zhang H., Yuan S., Liu S., Zhang T.-J., Sun Y.-C., 2012, Res. Astron. Astrophys., 14, 1221

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