



## Stochastics and Statistics

The effect of the autocorrelation on the performance of the  $T^2$  chart

Roberto Campos Leoni\*, Antonio Fernando Branco Costa,  
Marcela Aparecida Guerreiro Machado

Production Department, São Paulo State University (UNESP), Guaratinguetá, SP, Brazil

## ARTICLE INFO

## Article history:

Received 17 September 2014

Accepted 28 May 2015

Available online 6 June 2015

## Keywords:

Quality control

Statistical process control

Hotelling  $T^2$  chart

VAR (1) model

## ABSTRACT

In this article, we consider the  $T^2$  control chart for bivariate samples of size  $n$  with observations that are not only cross-correlated but also autocorrelated. The cross covariance matrix of the sample mean vectors is derived with the assumption that the observations are described by a multivariate first order autoregressive model – VAR (1). The combined effect of the correlation and autocorrelation on the performance of the  $T^2$  chart is also investigated. Earlier studies proved that changes in only one variable are detected faster when the variables are correlated. This result extends to the case that one or both variables are also autocorrelated.

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## 1. Introduction

In most literature on statistical process control, it is assumed that observations are independent over time. However, in many applications the dynamics of the process induce correlation in observations that are closely spaced in time. The autocorrelation has a serious impact on the performance of conventional control charts (Kalgonda & Kulkarni, 2004). Recently, Du and Lv (2013) proposed a control chart based on the minimal Euclidian distance to detect mean shifts in autocorrelated processes. Lin, Chou, Wang, and Liu (2012) considered the economic design of the autoregressive moving average (ARMA) control chart, which has been shown to be suitable for monitoring series of autocorrelated data. Franco, Costa, and Machado (2012) considered the first-order autoregressive model, AR (1), to describe the wandering behavior of the process mean, and they used Duncan's model to obtain the optimum values of the  $\bar{X}$  chart parameters. Costa and Machado (2011) also used the AR (1) model to investigate the effect of the wandering behavior of the process mean on the performance of the variable parameter  $\bar{X}$  chart and on the performance of the double sampling  $\bar{X}$  chart.

To reduce the negative effect of autocorrelation on the statistical performance of the Shewhart  $\bar{X}$  control chart, Costa and Castagliola (2011) proposed a sampling strategy denoted 's-skip'. According to this strategy, samples are obtained by collecting one item from the production line and then skipping  $s$  consecutive items before selecting the next one. Alternatively, Franco, Castagliola, Celano, and Costa (2013) introduced the *mixed* sampling strategy where the samples

are composed with units selected from the last two rational subgroups. A numerical analysis shows that the *mixed* sampling outperforms the *skip* sampling strategy for high levels of autocorrelation. Recently, Franco, Celano, Castagliola, and Costa (2014) investigated the economic-statistical design of the  $\bar{X}$  chart with the *skip* sampling strategy.

Modern manufacturing systems with advanced technology and high production rates results in processes that are multidimensional, cross and serially correlated. Hotelling (1947) suggested the use of the  $T^2$  statistic for monitoring the mean vector of multivariate processes. Seven decades later, Chen (2007) and Chen and Hsieh (2007) considered adaptive sampling schemes to enhance the performance of the  $T^2$  chart. Pan and Jarrett (2011) described some processes that are both multivariate and serially correlated. According to Huang, Bisgaard, and Xu (2014), autocorrelation increases the false-alarm risk, while cross-correlation reduces the power of the control chart. The combined effect of autocorrelation and cross-correlation on the performance of the  $T^2$  chart is worth to be investigated.

In many modern processes the observations of correlated quality characteristics gathered in rational subgroups of size  $n$  are not only cross-correlated but also autocorrelated. In this context, and under the assumption that the observations are given by a multivariate first order autoregressive model, VAR (1), in this paper we obtain the cross-covariance matrix of the sample mean vector ( $\Gamma_{\bar{X}}$ ). The cross-covariance matrix  $\Gamma_{\bar{X}}$  is important for the investigation of the effect of cross-correlation and autocorrelation on the performance of the  $T^2$  chart.

The paper is organized as follows. Section 2 and Appendix A describe the VAR (1) model and the bivariate cross-covariance matrix of the sample mean vector. Section 3 investigates the effects of autocorrelation and cross-correlation on the performance of the  $T^2$

\* Corresponding author. Tel.: +55 24 988032057.

E-mail addresses: [rleoni@yahoo.com.br](mailto:rleoni@yahoo.com.br) (R.C. Leoni), [franco@feg.unesp.br](mailto:franco@feg.unesp.br) (A.F.B. Costa), [marcela@feg.unesp.br](mailto:marcela@feg.unesp.br) (M.A.G. Machado).

**Table 1**  
The inverse cross-covariance matrix values;  $a$  and  $b \in (0, 0.5, 0.7)$ ;  $\rho \in (0, 0.7)$ .

	$\rho$	0.0								0.7							
	$a$	0.0	0.0	0.2	0.0	0.5	0.0	0.7	0.0	0.0	0.2	0.0	0.5	0.0	0.7		
	$b$	0.0	0.2	0.2	0.5	0.5	0.7	0.7	0.0	0.2	0.2	0.5	0.5	0.7	0.7		
$n = 2$	$\theta_{11}$	2.00	2.00	1.60	2.00	1.00	2.00	0.60	3.92	3.80	3.14	3.24	1.96	2.73	1.18		
	$\theta_{22}$	2.00	1.60	1.60	1.00	1.00	0.60	0.60	3.92	3.04	3.14	1.62	1.96	0.82	1.18		
	$\theta_{12}$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-2.75	-2.34	-2.20	-1.42	-1.37	-0.77	-0.82		
$n = 4$	$\theta_{11}$	4.00	4.00	2.86	4.00	1.45	4.00	0.75	7.84	7.70	5.60	6.87	2.85	5.86	1.48		
	$\theta_{22}$	4.00	2.86	2.86	1.45	1.45	0.75	0.75	7.84	5.50	5.50	2.50	2.85	1.10	1.48		
	$\theta_{12}$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-5.49	-4.51	-3.92	-2.68	-2.00	-1.43	-1.03		
$n = 10$	$\theta_{11}$	10.0	4.00	6.68	10.0	2.88	10.0	1.23	19.61	19.46	13.09	18.45	5.66	16.65	2.41		
	$\theta_{22}$	10.0	6.68	6.68	2.88	2.88	1.23	1.23	19.61	12.99	13.09	5.32	5.66	2.04	2.41		
	$\theta_{12}$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-13.73	-11.09	-9.17	-6.71	-3.96	-3.69	-1.68		

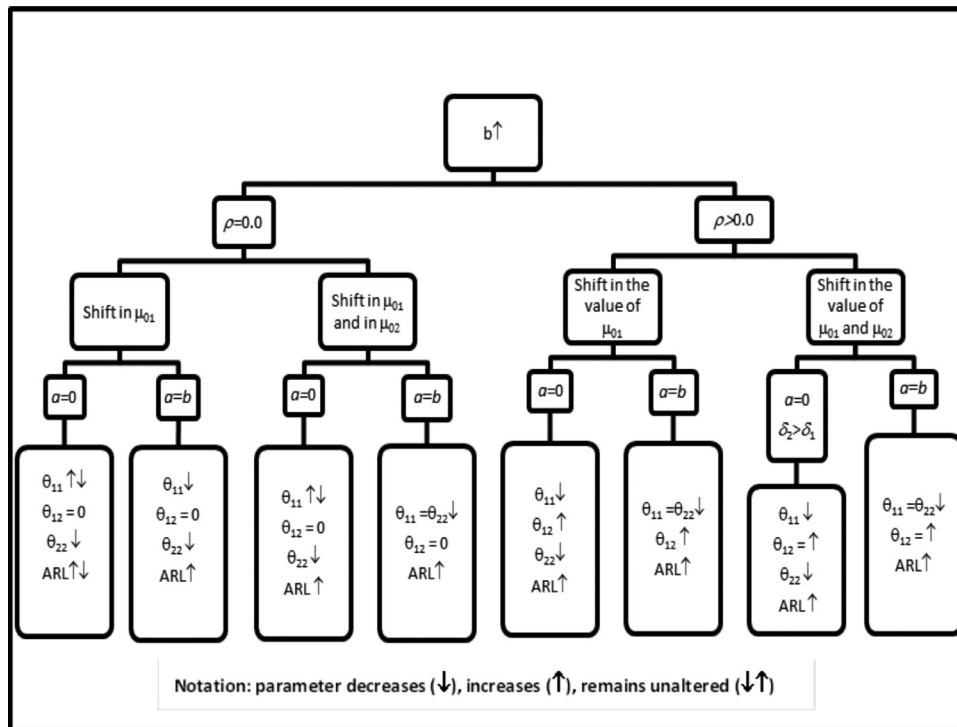


Fig. 1. The effect of  $b$  on  $(\theta_{11}, \theta_{22}, \theta_{12})$  and on the ARL.

chart. Section 4 presents an illustrative example. Section 5 gives the conclusion and suggests for future research. In Appendix B, we investigate the level of the autocorrelations for which the detection power of the  $T^2$  chart for individual observations is equal to the detection power of the  $T^2$  chart for samples of size  $n$ .

**2. Autoregressive model**

Autocorrelation measures the dependence between observations of the same quality characteristic, while cross-correlation measures the dependence between observations of the monitored quality characteristics. The multivariate autoregressive model for cross and serially correlated data has been adopted in recent studies dealing with multivariate control charts (Huang et al., 2014; Huang, Xu, & Bisgaard, 2013; Hwang & Wang, 2010; Pan & Jarrett, 2007).

$$\mathbf{X}_t - \boldsymbol{\mu} = \Phi(\mathbf{X}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\varepsilon}_t \tag{1}$$

where  $\mathbf{X}_t \sim N_p(\boldsymbol{\mu}, \Gamma)$  is the  $(p \times 1)$  vector of observations at time  $t$  ( $p$  is the number of variables),  $\boldsymbol{\mu}$  is the mean vector,  $\boldsymbol{\varepsilon}_t$  is an independent multivariate normal random vector with a mean vector  $\mathbf{0} = (0, \dots, 0)'$  and covariance matrix  $\Sigma$ , and  $\Phi$  is a  $(p \times p)$  matrix of

autocorrelation parameters. A recent study dealing with the estimation of the mean vector is in (Wang, Huwang, & Yu, 2015). According to Kalgonda and Kulkarni (2004), the cross covariance matrix  $\Gamma$  of  $\mathbf{X}_t$  has the following property:  $\Gamma = \Phi \Gamma \Phi' + \Sigma$ . After some algebra we obtain

$$\text{Vec } \Gamma = (I_{p^2} - \Phi \otimes \Phi)^{-1} \text{Vec } \Sigma \tag{2}$$

where  $\otimes$  is the Kronecker product and  $\text{Vec}$  is the operator that transform a matrix into a vector by stacking its columns.

To study the effects of the auto- and cross-correlation on the performance of the  $T^2$  chart, we consider the case of a bivariate process ( $p = 2$ )

$$\Phi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_{e_x}^2 & \rho \sigma_{e_x} \sigma_{e_y} \\ \rho \sigma_{e_x} \sigma_{e_y} & \sigma_{e_y}^2 \end{pmatrix} = \begin{pmatrix} \sigma_{e_x}^2 & \rho \sigma_{e_x} \sigma_{e_y} \\ \rho \sigma_{e_x} \sigma_{e_y} & \sigma_{e_y}^2 \end{pmatrix} \tag{3}$$

where  $\rho$  is the correlation of  $X$  and  $Y$ . We did not find examples in SPC where the observations of one variable at time  $(i)$  are dependent of the observations of the other variable at time  $(i-1)$ . The assumption of a diagonal matrix  $\Phi$  is very realistic.

**Table 2**  
The ARL values for the  $T^2$  control chart,  $\delta_1 = \delta_2$ .

(a, b)		(0, 0)		(0.0, 0.2)		(0.2, 0.2)		(0.0, 0.7)		(0.7, 0.7)	
$\delta_1$	$\delta_2$	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL
0.0	0.0	0.00	370.40	0.00	370.40	0.00	370.40	0.00	370.40	0.00	370.40
$\rho = 0.0$											
0.5	0.5	1.41	27.73	1.31	34.43	1.20	43.87	1.09	55.16	0.61	159.45
0.75	0.75	2.12	7.74	1.96	9.99	1.79	13.43	1.63	17.96	0.92	80.52
1.0	1.0	2.83	3.06	2.62	3.89	2.39	5.22	2.18	7.07	1.23	41.01
1.25	1.25	3.54	1.68	3.27	2.02	2.99	2.60	2.72	3.43	1.53	21.85
1.5	1.5	4.24	1.21	3.93	1.36	3.59	1.63	3.27	2.03	1.84	12.35
$\rho = 0.3$											
0.5	0.5	1.24	39.83	1.15	48.07	1.05	60.48	1.02	64.48	0.54	187.11
0.75	0.75	1.86	11.92	1.73	15.06	1.57	20.25	1.53	22.03	0.81	103.84
1.0	1.0	2.48	4.62	2.31	5.87	2.10	8.05	2.04	8.83	1.08	56.91
1.25	1.25	3.10	2.34	2.88	2.89	2.62	3.88	2.55	4.24	1.34	31.98
1.5	1.5	3.72	1.51	3.46	1.77	3.14	2.25	3.06	2.43	1.61	18.70
$\rho = 0.7$											
0.5	0.5	1.08	55.82	1.02	64.12	.92	81.12	1.01	65.51	0.47	214.55
0.75	0.75	1.63	18.24	1.53	21.87	1.38	30.04	1.52	22.51	0.71	130.24
1.0	1.0	2.17	7.19	2.04	8.76	1.83	12.50	2.02	9.04	0.94	76.85
1.25	1.25	2.71	3.48	2.55	4.20	2.29	6.00	2.53	4.34	1.18	45.73
1.5	1.5	3.25	2.06	3.07	2.42	2.75	3.33	3.04	2.48	1.41	27.90

**Table 3**  
The ARL values for the  $T^2$  control chart,  $\delta_1$  or  $\delta_2 = 0$ .

(a, b)		(0, 0)		(0.0, 0.2)		(0.2, 0.2)		(0.0, 0.7)		(0.7, 0.7)	
$\delta_1$	$\delta_2$	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL
0.0	0.0	0.00	370.40	0.00	370.40	0.00	370.40	0.00	370.40	0.00	370.40
$\rho = 0.0$											
0.0	0.5	1.00	67.32	0.85	95.27	0.85	95.27	0.43	230.33	0.43	230.33
0.5	0.0	1.00	67.32	1.00	67.32	0.85	95.27	1.00	67.32	0.43	230.33
0.0	0.75	1.50	23.34	1.27	37.58	1.27	37.58	0.65	147.06	0.65	147.06
0.75	0.0	1.50	23.34	1.50	23.34	1.27	37.58	1.50	23.34	0.65	147.06
0.0	1.0	2.00	9.41	1.69	16.19	1.69	16.19	0.87	90.63	0.87	90.63
1.0	0.0	2.00	9.41	2.00	9.41	1.69	16.19	2.00	9.41	0.87	90.63
0.0	1.25	2.50	4.51	2.11	7.85	2.11	7.85	1.08	55.87	1.08	55.87
1.25	0.0	2.50	4.51	2.50	4.51	2.11	7.85	2.50	4.51	1.08	55.87
0.0	1.5	3.00	2.57	2.54	4.31	2.54	4.31	1.30	35.03	1.30	35.03
1.5	0.0	3.00	2.57	3.00	2.57	2.54	4.31	3.00	2.57	1.30	35.03
$\rho = 0.3$											
0.0	0.5	1.05	60.48	0.89	87.08	0.89	86.92	0.45	224.59	0.45	221.26
0.5	0.0	1.05	60.48	1.05	60.61	0.89	86.92	1.03	62.91	0.45	221.26
0.0	0.75	1.57	20.25	1.33	33.13	1.33	33.05	0.67	140.79	0.68	137.24
0.75	0.0	1.57	20.25	1.57	20.31	1.33	33.05	1.55	21.33	0.68	137.24
0.0	1.0	2.10	8.05	1.77	13.99	1.77	13.94	0.89	85.39	0.91	82.48
1.0	0.0	2.10	8.05	2.09	8.08	1.77	13.94	2.06	8.52	0.91	82.48
0.0	1.25	2.62	3.88	2.21	6.73	2.21	6.71	1.12	51.96	1.14	49.82
1.25	0.0	2.62	3.88	2.62	3.89	2.21	6.71	2.58	4.09	1.14	49.82
0.0	1.5	3.14	2.25	2.66	3.72	2.66	3.71	1.34	32.25	1.36	30.74
1.5	0.0	3.14	2.25	3.14	2.26	2.66	3.71	3.09	2.36	1.36	30.74
$\rho = 0.7$											
0.0	0.5	1.40	28.53	1.17	46.04	1.18	45.00	0.52	192.22	0.61	161.55
0.5	0.0	1.40	28.53	1.39	29.27	1.18	45.00	1.21	42.47	0.61	161.55
0.0	0.75	2.10	8.00	1.76	14.27	1.78	13.86	0.79	108.49	0.91	82.17
0.75	0.0	2.10	8.00	2.08	8.24	1.78	13.86	1.82	12.90	0.91	82.17
0.0	1.0	2.80	3.15	2.35	5.55	2.37	5.39	1.05	60.27	1.21	42.09
1.0	0.0	2.80	3.15	2.78	3.24	2.37	5.39	2.42	5.01	1.21	42.09
0.0	1.25	3.50	1.72	2.93	2.75	2.96	2.67	1.31	34.22	1.52	22.51
1.25	0.0	3.50	1.72	3.47	1.75	2.96	2.67	3.03	2.51	1.52	22.51
0.0	1.5	4.20	1.23	3.52	1.70	3.55	1.66	1.57	20.16	1.82	12.75
1.5	0.0	4.20	1.23	4.16	1.24	3.55	1.66	3.63	1.58	1.82	12.75

From (2) and (3) it follows:

$$\Gamma = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix} = \begin{pmatrix} (1-a^2)^{-1}\sigma_{e_X}^2 & (1-ab)^{-1}\sigma_{e_{XY}} \\ (1-ab)^{-1}\sigma_{e_{XY}} & (1-b^2)^{-1}\sigma_{e_Y}^2 \end{pmatrix} \quad (4)$$

The cross covariance matrix  $\Gamma_{\bar{X}}$  of the sample mean vector  $\bar{X}$  for rational subgroups of size  $n$  is developed in Appendix A

$$\Gamma_{\bar{X}} = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix} \quad (5)$$

where

$$\zeta_{11} = \frac{\sigma_X^2}{n} \left[ 1 + \frac{2}{n} \sum_{j=1}^{n-1} (n-j)a^j \right];$$

$$\zeta_{22} = \frac{\sigma_Y^2}{n} \left[ 1 + \frac{2}{n} \sum_{j=1}^{n-1} (n-j)b^j \right];$$

$$\zeta_{12} = \frac{\sigma_{XY}}{n} \left[ 1 + \frac{1}{n} \sum_{j=1}^{n-1} (n-j)a^j + \frac{1}{n} \sum_{j=1}^{n-1} (n-j)b^j \right].$$

and where  $\zeta_{21} = \zeta_{12}$ .

**Table 4**  
The ARL values for the  $T^2$  control chart,  $\delta_1 \neq \delta_2$ .

(a, b)		(0, 0)		(0.0, 0.2)		(0.2, 0.2)		(0.0, 0.7)		(0.7, 0.7)	
$\delta_1$	$\delta_2$	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL
0.0	0.0	0.00	370.40	0.00	370.40	0.00	370.40	0.00	370.40	0.00	370.40
$\rho = 0.0$											
0.5	0.75	1.80	13.19	1.61	18.67	1.52	22.28	1.19	44.09	0.78	109.84
0.75	0.5	1.80	13.19	1.72	15.28	1.52	22.28	1.56	20.69	0.78	109.84
0.5	1.0	2.24	6.50	1.96	9.99	1.89	11.33	1.32	33.41	0.97	72.02
1.0	0.5	2.24	6.50	2.17	7.17	1.89	11.33	2.05	8.72	0.97	72.02
0.5	1.25	2.69	3.56	2.34	5.62	2.28	6.14	1.47	24.53	1.17	46.56
1.25	0.5	2.69	3.56	2.64	3.79	2.28	6.14	2.54	4.30	1.17	46.56
0.5	1.5	3.16	2.22	2.73	3.43	2.67	3.64	1.64	17.76	1.37	30.26
1.5	0.5	3.16	2.22	3.12	2.31	2.67	3.64	3.03	2.49	1.37	30.26
0.75	1.0	2.50	4.51	2.26	6.28	2.11	7.85	1.73	14.97	1.08	55.87
1.0	0.75	2.50	4.51	2.37	5.38	2.11	7.85	2.10	7.97	1.08	55.87
0.75	1.25	2.92	2.79	2.59	4.02	2.46	4.73	1.85	12.12	1.26	37.85
1.25	0.75	2.92	2.79	2.80	3.14	2.46	4.73	2.58	4.06	1.26	37.85
0.75	1.5	3.35	1.90	2.95	2.71	2.83	3.04	1.99	9.63	1.45	25.55
1.5	0.75	3.35	1.90	3.26	2.05	2.83	3.04	3.07	2.41	1.45	25.55
1.0	1.25	3.20	2.14	2.91	2.81	2.71	3.50	2.27	6.14	1.39	29.22
1.25	1.0	3.20	2.14	3.02	2.53	2.71	3.50	2.65	3.76	1.39	29.22
1.0	1.5	3.61	1.61	3.23	2.10	3.05	2.46	2.39	5.25	1.56	20.60
1.5	1.0	3.61	1.61	3.44	1.79	3.05	2.46	3.12	2.29	1.56	20.60
1.25	1.5	3.91	1.38	3.56	1.65	3.30	1.98	2.82	3.09	1.69	16.09
1.5	1.25	3.91	1.38	3.67	1.55	3.30	1.98	3.19	2.17	1.69	16.09
$\rho = 0.3$											
0.5	0.75	1.61	18.95	1.43	27.08	1.36	31.11	1.09	55.75	0.70	132.76
0.75	0.5	1.61	18.95	1.56	20.86	1.36	31.11	1.50	23.26	0.70	132.76
0.5	1.0	2.04	8.76	1.77	14.02	1.73	15.13	1.19	44.37	0.89	86.86
1.0	0.5	2.04	8.76	2.02	9.15	1.73	15.13	2.00	9.40	0.89	86.86
0.5	1.25	2.51	4.43	2.15	7.42	2.12	7.71	1.32	33.36	1.09	55.14
1.25	0.5	2.51	4.43	2.50	4.50	2.12	7.71	2.51	4.47	1.09	55.14
0.5	1.5	3.00	2.56	2.55	4.24	2.54	4.30	1.48	24.25	1.30	34.97
1.5	0.5	3.00	2.56	3.00	2.57	2.54	4.30	3.01	2.53	1.30	34.97
0.75	1.0	2.21	6.75	1.99	9.61	1.87	11.74	1.59	19.66	0.96	73.77
1.0	0.75	2.21	6.75	2.12	7.75	1.87	11.74	2.01	9.29	0.96	73.77
0.75	1.25	2.62	3.88	2.30	5.91	2.21	6.71	1.67	16.68	1.14	49.82
1.25	0.75	2.62	3.88	2.56	4.18	2.21	6.71	2.50	4.51	1.14	49.82
0.75	1.5	3.07	2.42	2.65	3.73	2.59	4.02	1.78	13.62	1.33	33.02
1.5	0.75	3.07	2.42	3.03	2.51	2.59	4.02	3.00	2.57	1.33	33.02
1.0	1.25	2.82	3.08	2.56	4.20	2.39	5.25	2.09	8.09	1.22	41.24
1.25	1.0	2.82	3.08	2.69	3.57	2.39	5.25	2.51	4.42	1.22	41.24
1.0	1.5	3.21	2.12	2.85	2.98	2.72	3.46	2.17	7.18	1.39	28.90
1.5	1.0	3.21	2.12	3.11	2.31	2.72	3.46	3.00	2.56	1.39	28.90
1.25	1.5	3.44	1.79	3.13	2.29	2.90	2.82	2.60	3.97	1.49	23.78
1.5	1.25	3.44	1.79	3.26	2.04	2.90	2.82	3.02	2.51	1.49	23.78
$\rho = 0.7$											
0.5	0.75	1.50	23.26	1.28	36.71	1.27	37.47	1.01	66.50	0.65	146.83
0.75	0.5	1.50	23.26	1.52	22.25	1.27	37.47	1.58	19.94	0.65	146.83
0.5	1.0	2.08	8.30	1.71	15.70	1.76	14.37	1.07	58.22	0.90	84.07
1.0	0.5	2.08	8.30	2.14	7.56	1.76	14.37	2.17	7.20	0.90	84.07
0.5	1.25	2.71	3.48	2.21	6.77	2.29	6.00	1.18	45.13	1.18	45.73
1.25	0.5	2.71	3.48	2.79	3.20	2.29	6.00	2.76	3.28	1.18	45.73
0.5	1.5	3.37	1.88	2.75	3.35	2.85	2.99	1.34	32.26	1.46	25.15
1.5	0.5	3.37	1.88	3.46	1.77	2.85	2.99	3.36	1.89	1.46	25.15
0.75	1.0	2.00	9.33	1.75	14.49	1.69	16.07	1.50	23.33	0.87	90.19
1.0	0.75	2.00	9.33	2.01	9.30	1.69	16.07	2.08	8.25	0.87	90.19
0.75	1.25	2.52	4.37	2.11	7.84	2.13	7.60	1.53	22.10	1.09	54.58
1.25	0.75	2.52	4.37	2.58	4.06	2.13	7.60	2.66	3.68	1.09	54.58
0.75	1.5	3.12	2.31	2.56	4.17	2.63	3.82	1.60	19.27	1.35	31.56
1.5	0.75	3.12	2.31	3.20	2.14	2.63	3.82	3.25	2.06	1.35	31.56
1.0	1.25	2.52	4.37	2.24	6.47	2.13	7.60	2.00	9.40	1.09	54.58
1.25	1.0	2.52	4.37	2.50	4.50	2.13	7.60	2.58	4.05	1.09	54.58
1.0	1.5	3.00	2.56	2.56	4.19	2.54	4.29	2.01	9.24	1.30	34.93
1.5	1.0	3.00	2.56	3.05	2.45	2.54	4.29	3.16	2.22	1.30	34.93
1.25	1.5	3.05	2.45	2.74	3.38	2.58	4.08	2.50	4.49	1.32	33.42
1.5	1.25	3.05	2.45	3.00	2.57	2.58	4.08	3.09	2.37	1.32	33.42

In the absence of autocorrelation ( $a = b = 0$ ),  $\Gamma_{\bar{X}}$  reduces to  $\Psi_{\bar{X}} = n^{-1}\Sigma$ . Let  $\Gamma_{\bar{X}}^{-1}$  be the inverse of matrix  $\Gamma_{\bar{X}}$  defined as

$$\Gamma_{\bar{X}}^{-1} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \quad (6)$$

Table 1 brings the values of  $\theta_{11}$ ,  $\theta_{22}$  and  $\theta_{12} = \theta_{21}$  for  $\rho \in (0, 0.7)$ ,  $n \in (2, 4, 10)$ , and several combinations of  $a$  and  $b$ . The elements of the main diagonal of  $\Gamma_{\bar{X}}^{-1}$  increase with the cross-correlation and with the sample size but decrease with the autocorrelation. When the variables are independent, the off-diagonal elements of  $\Gamma_{\bar{X}}^{-1}$  are zero, that

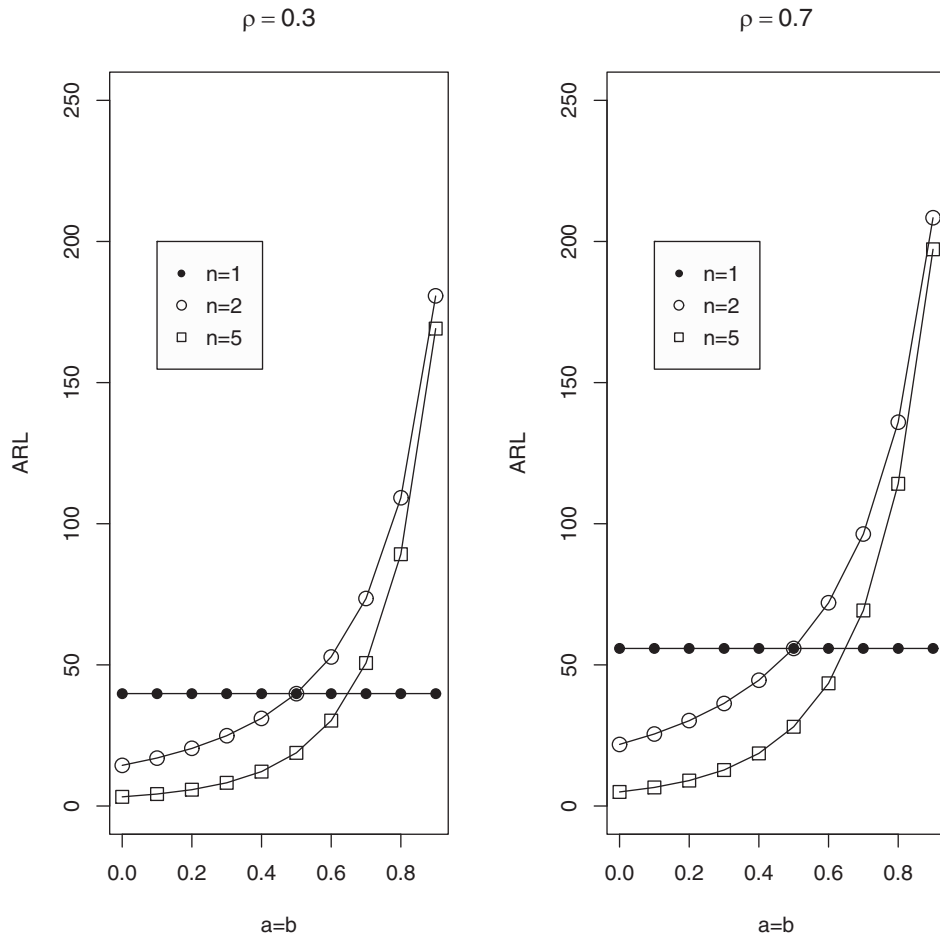


Fig. 2. ARL values ( $\delta_1 = \delta_2 = 1.0, a = b, n \in \{1, 2, 5\}$ ).

is,  $\theta_{12} = \theta_{21} = 0$ . In the presence of the cross-correlation, they become negative.

**3. The effect of the auto- and cross-correlation on the performance of the  $T^2$  control chart**

The Hotelling  $T^2$  control chart is the most referenced control scheme for detecting mean shifts in multivariate processes (Hotelling, 1947). This chart has been used in a wide variety of applications, for example, recently Faraz, Heuchenne, Saniga, and Foster (2013) considered the use of the  $T^2$  chart to control a delivery chain. When the in-control mean vector  $\mu_0 = (\mu_{01}, \mu_{02})$  and the matrices  $\Phi$  and  $\Sigma$  are known, the monitoring statistic for Hotelling’s  $T^2$  control chart is given by

$$T_i^2 = (\bar{X} - \mu_0)' \Gamma_{\bar{X}}^{-1} (\bar{X} - \mu_0) \tag{7}$$

The terms of the cross-covariance matrix  $\Gamma_{\bar{X}}$  of the sample mean vector  $\bar{X}$  only depend on the elements of the two matrices,  $\Phi$  and  $\Sigma$ , see expression (5). During the in-control period,  $T_i^2$  follows a chi-squared distribution with  $p$  degrees of freedom ( $\chi_p^2$ ). Similarly to the work of Champ, Jones-Farmer, and Rigdon (2005), the upper control limit (UCL) of the chart is chosen to be the  $(1 - \alpha)$ th quantile of the chi-squared distribution to achieve a desired in-control Average Run Length (ARL) of  $1/\alpha$ , i.e.  $UCL = \chi_{p,\alpha}^2$ . After the assignable cause occurrence the mean vector changes from  $\mu_0$  to  $\mu_1 = (\mu_{11}, \mu_{12})$  and the distribution of the monitoring statistic  $T_i^2$  changes to a non-central chi-squared distribution ( $\chi_{(p,\lambda)}^2$ ) with non-centrality parameter  $\lambda^2 = \delta' \Gamma_{\bar{X}}^{-1} \delta$ , where  $\delta = (\delta_1, \delta_2)' = (\frac{\mu_{11} - \mu_{01}}{\sigma_{eX}}, \frac{\mu_{12} - \mu_{02}}{\sigma_{eY}})'$  denotes the stan-

dardized mean vector shift, see (Wu & Makis, 2008) and (Franco et al., 2012). Without loss of generalization, we consider  $\mu_{01} = \mu_{02} = 0$ . In absence of autocorrelation, the non-centrality parameter is given by  $\lambda^2 = \delta' \Psi_{\bar{X}}^{-1} \delta$ . As  $\delta' \Gamma_{\bar{X}}^{-1} \delta < \delta' \Psi_{\bar{X}}^{-1} \delta$ , we conclude that autocorrelation reduces the ability of the chart to detect an assignable cause. According to expression (7),

$$T_i^2 = (\bar{X}_1 + \delta_1)^2 \theta_{11} + (\bar{X}_2 + \delta_2)^2 \theta_{22} + 2(\bar{X}_1 + \delta_1)(\bar{X}_2 + \delta_2) \theta_{12} \tag{8}$$

where  $\bar{X}_1$  and  $\bar{X}_2$  are the sample means of the two quality characteristics used to control the bivariate process.

To study the effects of auto- and cross-correlation on the performance of the  $T^2$  chart for bivariate process and we assume rational subgroups of size  $n = 4$ . The performance of the  $T^2$  chart is measured by the average run length (ARL). When the process is in-control, the ARL measures the rate of false alarms. A chart with a larger in-control ARL ( $ARL_0$ ) has a lower false alarm rate than other charts. A chart with a smaller out-of-control ARL has a better ability to detect process changes than other charts. In practice, a preliminary sample is taken to estimate the cross-covariance matrix  $\Gamma$  and, subsequently, the parameters  $a, b$  and  $\rho$ ; see Eqs. (3) and (4).

The ARL value is given by

$$ARL = \frac{1}{1 - \Pr(\chi_{(p,\lambda)}^2 < UCL)} \tag{9}$$

where UCL is the upper control limit of the  $T^2$  chart. Fig. 1 summarizes the effect of  $b$  on  $(\theta_{11}, \theta_{22}, \theta_{12})$  and on the ARL for  $a = 0$  and for  $a = b$ . Fig. 1 was built considering two uncorrelated and two correlated variables. In this figure, disturbances affecting either a single variable or both of them were also considered. It is interesting

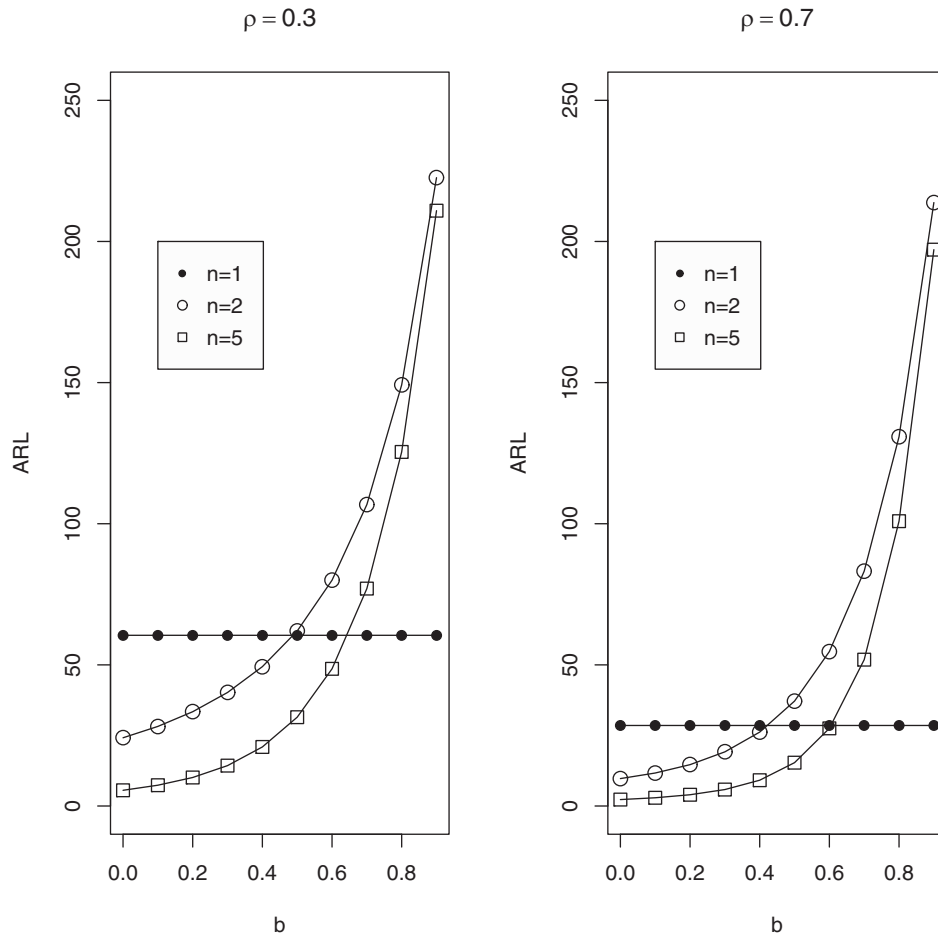


Fig. 3. ARL values ( $\delta_1 = 0, \delta_2 = 1.0, a = 0, b \neq 0, n \in \{1, 2, 5\}$ ).

to notice that the ARL is not affected by the autocorrelation if only one of the two independent variables is autocorrelated and robust to the occurrence of assignable causes ( $\rho = 0, a = 0, b \neq 0, \delta_2 = 0$ ). As

$\lambda$  is a function of  $\Gamma_{\bar{X}}$ , the performance of the  $T^2$  chart depends on the adopted model. An interesting subject of research is to study the  $T^2$  chart when the processes are described by VAR (p) models, with  $p > 1$ .

To investigate the effect of the autocorrelation level on the overall performance of the bivariate  $T^2$  charts, the ARLs in Tables 2, 3 and 4 were computed with Eq. (9). We considered uncorrelated variables,  $\rho = 0$ , and variables with low and high correlation,  $\rho = 0.3$  or  $0.7$ . Tables 2 and 4 consider the cases in which the two variables are affected by the assignable cause (in Table 2,  $\delta_1 = \delta_2$ , and in Table 4,  $\delta_1 \neq \delta_2$ ). Table 3 considers the cases in which only one variable is affected by the assignable cause ( $\delta_1$  or  $\delta_2 = 0$ ).

In the absence of autocorrelation ( $a = b = 0$ ) or in the presence of two variables with the same level of autocorrelation ( $a = b \neq 0$ ), higher dependence between the two variables improves the performance of the  $T^2$  chart in signaling shifts in only one variable ( $\delta_1 \neq 0; \delta_2 = 0$  or  $\delta_1 = 0; \delta_2 \neq 0$ ). For instance, in Table 3, if  $(a, b, \delta_1, \delta_2) = (0, 0, 0, 1.0)$  and the variables are uncorrelated ( $\rho = 0$ ), the  $T^2$  chart requires, on average, 9.41 samples to signal. This number decreases to 3.15 when the variables are highly correlated ( $\rho = 0.7$ ). When the assignable cause shifts both variables the ARLs increase with  $\rho$ . For instance, in Table 2, if  $(a, b, \delta_1, \delta_2) = (0.7, 0.7, 1.0, 1.0)$  and the variables are uncorrelated ( $\rho = 0$ ), the  $T^2$  chart requires, on average, 41.01 samples to signal. This number increases to 76.85 when the variables are highly correlated ( $\rho = 0.7$ ).

If only one variable is autocorrelated and only one variable is affected by the assignable cause ( $a \neq 0; b = 0$  or  $a = 0; b \neq 0$  and  $\delta_1 \neq 0; \delta_2 = 0$  or  $\delta_1 = 0; \delta_2 \neq 0$ ): a) higher dependence between the two variables improves the  $T^2$  chart's performance. For instance, in Table 3, if  $(a, b, \delta_1, \delta_2) = (0, 0.2, 1.0, 0)$  and the two variables are

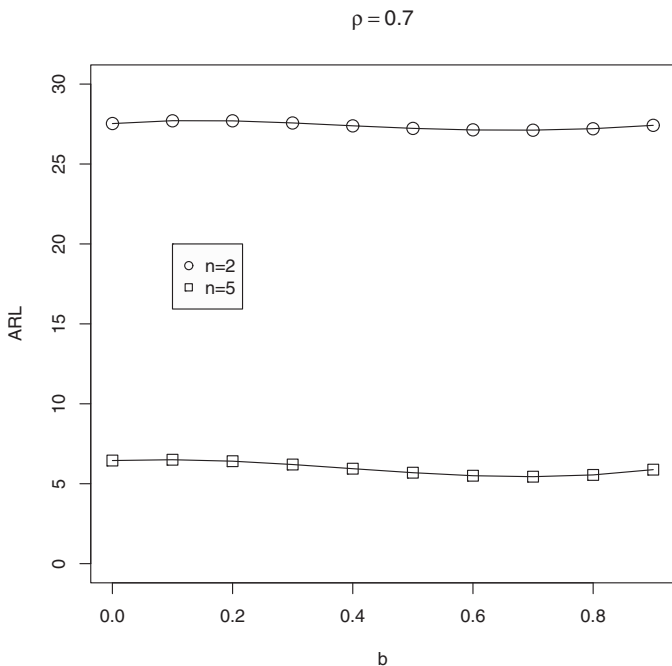


Fig. 4. ARL values ( $\delta_1 = 1.0, \delta_2 = 0.75, a = 0, b \neq 0, n \in \{2, 5\}$ ).



**Table 5**  
The values of  $(a^*, b^*)$  for which  $\lambda^2(1) = \lambda^2(n)$ .

$n$		2	3	5	10	20
$a = b$		0.5	0.579	$a^*$ 0.654	0.737	0.803
	$\rho$			$b^*$		
$\delta_1 = 0, a = 0$	<b>0.0</b>	0.501	0.58	0.655	0.737	0.803
	<b>0.3</b>	0.491	0.571	0.648	0.733	0.801
	<b>0.5</b>	0.469	0.553	0.635	0.725	0.793
	<b>0.7</b>	0.426	0.515	0.606	0.709	0.786

uncorrelated ( $\rho = 0$ ), the  $T^2$  chart requires, on average, 9.41 samples to signal. This number decreases to 3.24 when the variables are highly correlated ( $\rho = 0.7$ ). b) the ARLs are always higher when the variable affected by the assignable cause is the autocorrelated one. For instance, in Table 3, if  $\rho = 0.3$  and  $(a, b, \delta_1, \delta_2) = (0, 0.2, 0, 1.0)$  the  $T^2$  chart requires, on average, 13.99 samples to signal. This number decreases to 8.08 when  $(a, b, \delta_1, \delta_2) = (0, 0.2, 1.0, 0)$ .

The performance of the  $T^2$  chart depends on the inverse of the cross covariance matrix of the sample mean vector. Its parameters  $\theta_{11}$  and  $\theta_{22}$  increase with the size of the sample, but  $\theta_{11}$  decreases with  $a$ , and  $\theta_{22}$  decreases with  $b$ . Table 5 shows several combinations of  $\rho, a, b$ , and  $n$  for which  $\Gamma_{\bar{X}}^{-1} = \Sigma^{-1}$ ; in these cases, the detection power of the  $T^2$  chart based on individual observations is equal to the detection power of the  $T^2$  chart based on rational subgroups of size  $n$ , see Appendix B for details. Figs. 2 and 3 show for low and high cross-correlated variables,  $\rho = \{0.3, 0.7\}$ , the range of  $b$  values for which

the  $T^2$  chart for individual observations is superior to the  $T^2$  chart for rational subgroups of size two (or five). In Fig. 2,  $\delta_1 = \delta_2 = 1.0$ , and  $a = b$ ; in Fig. 3,  $\delta_1 = 0, \delta_2 = 1.0$ , and  $a = 0$ . Fig. 4 shows a case where the ARL is nearly independent of  $b$ .

To investigate the effect of the sample size on the performance of the  $T^2$  chart, Tables 6, 7 and 8 present the ARL values for  $n \in \{1, 2, 5\}$ ,  $\rho \in \{0.0, 0.3, 0.7\}$  and 96 combinations of  $(a, b, \delta_1, \delta_2)$ . When the two autocorrelated variables have the same high level of autocorrelation ( $a = b = 0.7$ ), the bivariate  $T^2$  chart for individual observations performs better than the bivariate  $T^2$  chart for rational subgroups of size two, or even five. When the observations of only one of the two variables are autocorrelated ( $a = 0, b = 0.5$  or  $a = 0.5, b = 0$ ), there are combinations of  $(\rho, \delta_1, \delta_2)$  for which the bivariate  $T^2$  chart for individual observations performs better than the bivariate  $T^2$  chart for rational subgroups of size two (see the ARLs in bold).

**4. An illustrative example**

The inner diameter ( $X$ ) and the thickness ( $Y$ ) of carbon fiber tubes are monitored during the drilling process. With the process in control, a preliminary sample was collected and the two quality characteristics were measured, see Table 9. This example was introduced by Santos Fernández (2012).

A study based on this database allowed to conclude that the two variables satisfactorily fits to a VAR (1) model. The adequacy of the model was tested using the goodness-of-fit Portmanteau test, for de-

**Table 6**  
The ARL values for the  $T^2$  control chart,  $\rho = 0.0$ .

$n = 1$				$n = 2$											
(a,b)		(0,0,0)		(0,0,0.5)		(0.5,0,0)		(0.3,0.3)		(0.5,0.5)		(0.7,0.7)			
$\delta_1$	$\delta_2$	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL		
0.0	0.0	0.0	370.4	0.0	370.4	0.0	370.4	0.0	370.40	0.0	370.4	0.0	370.40		
0.0	0.5	0.50	202.24	0.50	202.24	0.71	129.80	0.59	167.08	0.50	202.24	0.39	<b>250.86</b>		
0.0	1.0	1.00	67.32	1.00	67.32	1.41	27.73	1.18	45.02	1.00	67.32	0.77	<b>111.64</b>		
0.0	1.5	1.50	23.34	1.50	23.34	2.12	7.74	1.77	13.87	1.50	23.34	1.16	<b>47.15</b>		
0.5	0.0	0.50	202.24	0.71	129.80	0.50	202.24	0.59	167.08	0.50	202.24	0.39	<b>250.86</b>		
0.5	0.5	0.71	129.80	0.87	90.90	0.87	90.90	0.84	97.11	0.71	129.80	0.55	<b>183.34</b>		
0.5	1.0	1.12	51.88	1.22	41.18	1.50	23.34	1.32	33.47	1.12	51.88	0.87	<b>90.90</b>		
0.5	1.5	1.58	19.91	1.66	17.19	2.18	7.08	1.87	11.70	1.58	19.91	1.22	<b>41.18</b>		
1.0	0.0	1.00	67.32	1.41	27.73	1.00	67.32	1.18	45.02	1.00	67.32	0.77	<b>111.64</b>		
1.0	0.5	1.12	51.88	1.50	23.34	1.22	41.18	1.32	33.47	1.12	51.88	0.87	<b>90.90</b>		
1.0	1.0	1.41	27.73	1.73	14.99	1.73	14.99	1.67	16.71	1.41	27.73	1.10	<b>54.51</b>		
1.0	1.5	1.80	13.19	2.06	8.51	2.35	5.56	2.13	7.60	1.80	13.19	1.40	<b>28.75</b>		
1.5	0.0	1.50	23.34	2.12	7.74	1.50	23.34	1.77	13.87	1.50	23.34	1.16	<b>47.15</b>		
1.5	0.5	1.58	19.91	2.18	7.08	1.66	17.19	1.87	11.70	1.58	19.91	1.22	<b>41.18</b>		
1.5	1.0	1.80	13.19	2.35	5.56	2.06	8.51	2.13	7.60	1.80	13.19	1.40	<b>28.75</b>		
1.5	1.5	2.12	7.74	2.60	3.98	2.60	3.98	2.51	4.45	2.12	7.74	1.64	<b>17.69</b>		
$n = 1$				$n = 5$											
(a,b)		(0,0,0)		(0,0,0.5)		(0.5,0,0)		(0.3,0.3)		(0.5,0.5)		(0.7,0.7)			
$\delta_1$	$\delta_2$	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL		
0.0	0.0	0.0	370.4	0.0	370.4	0.0	370.4	0.0	370.40	0.0	370.4	0.0	370.40		
0.0	0.5	0.50	202.24	0.65	147.52	1.12	51.88	0.84	96.43	0.65	147.52	0.46	<b>221.08</b>		
0.0	1.0	1.00	67.32	1.30	35.24	2.24	6.50	1.68	16.52	1.30	35.24	0.91	<b>82.32</b>		
0.0	1.5	1.50	23.34	1.95	10.27	3.35	1.90	2.52	4.40	1.95	10.27	1.37	<b>30.66</b>		
0.5	0.0	0.50	202.24	1.12	51.88	0.65	147.52	0.84	96.43	0.65	147.52	0.46	<b>221.08</b>		
0.5	0.5	0.71	129.80	1.29	35.65	1.29	35.65	1.19	44.59	0.92	80.88	0.64	<b>149.32</b>		
0.5	1.0	1.12	51.88	1.71	15.52	2.33	5.69	1.88	11.56	1.45	25.71	1.02	<b>64.74</b>		
0.5	1.5	1.58	19.91	2.25	6.41	3.42	1.82	2.66	3.72	2.05	8.63	1.44	<b>26.37</b>		
1.0	0.0	1.00	67.32	2.24	6.50	1.30	35.24	1.68	16.52	1.30	35.24	0.91	<b>82.32</b>		
1.0	0.5	1.12	51.88	2.33	5.69	1.71	15.52	1.88	11.56	1.45	25.71	1.02	<b>64.74</b>		
1.0	1.0	1.41	27.73	2.59	4.05	2.59	4.05	2.38	5.33	1.84	12.44	1.29	<b>36.08</b>		
1.0	1.5	1.80	13.19	2.97	2.66	3.60	1.62	3.03	2.50	2.34	5.59	1.64	<b>17.77</b>		
1.5	0.0	1.50	23.34	3.35	1.90	1.95	10.27	2.52	4.40	1.95	10.27	1.37	<b>30.66</b>		
1.5	0.5	1.58	19.91	3.42	1.82	2.25	6.41	2.66	3.72	2.05	8.63	1.44	<b>26.37</b>		
1.5	1.0	1.80	13.19	3.60	1.62	2.97	2.66	3.03	2.50	2.34	5.59	1.64	<b>17.77</b>		
1.5	1.5	2.12	7.74	3.88	1.39	3.88	1.39	3.56	1.65	2.75	3.32	1.93	<b>10.56</b>		

**Table 7**  
The ARL values for the  $T^2$  control chart,  $\rho = 0.3$ .

$n = 1$				$n = 2$									
(a,b)		(0,0,0,0)		(0,0,0,5)		(0,5,0,0)		(0,3,0,3)		(0,5,0,5)		(0,7,0,7)	
$\delta_1$	$\delta_2$	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL
0.0	0.0	0.0	370.4	0.0	370.4	0.0	370.4	0.0	370.40	0.0	370.4	0.0	370.40
0.0	0.5	0.52	192.53	0.52	<b>194.74</b>	0.73	122.43	0.62	157.11	0.52	192.53	0.41	<b>242.52</b>
0.0	1.0	1.05	60.48	1.04	<b>61.99</b>	1.47	24.94	1.24	39.83	1.05	60.48	0.81	<b>102.65</b>
0.0	1.5	1.57	20.25	1.56	<b>20.92</b>	2.20	6.86	1.86	11.92	1.57	20.25	1.22	<b>41.78</b>
0.5	0.0	0.52	192.53	0.73	122.43	0.52	<b>194.74</b>	0.62	157.11	0.52	192.53	0.41	<b>242.52</b>
0.5	0.5	0.62	157.11	0.78	110.83	0.78	110.83	0.73	122.31	0.62	157.11	0.48	<b>210.35</b>
0.5	1.0	1.02	64.15	1.10	53.96	1.42	27.40	1.21	42.60	1.02	64.15	0.79	<b>107.50</b>
0.5	1.5	1.50	23.30	1.53	21.83	2.12	7.73	1.78	13.84	1.50	23.30	1.16	<b>47.08</b>
1.0	0.0	1.05	60.48	1.47	24.94	1.04	<b>61.99</b>	1.24	39.83	1.05	60.48	0.81	<b>102.65</b>
1.0	0.5	1.02	64.15	1.42	27.40	1.10	53.96	1.21	42.60	1.02	64.15	0.79	<b>107.50</b>
1.0	1.0	1.24	39.83	1.56	20.92	1.56	20.92	1.47	24.90	1.24	39.83	0.96	<b>73.49</b>
1.0	1.5	1.61	18.95	1.83	12.50	2.17	7.19	1.90	11.10	1.61	18.95	1.24	<b>39.46</b>
1.5	0.0	1.57	20.25	2.20	6.86	1.56	<b>20.92</b>	1.86	11.92	1.57	20.25	1.22	<b>41.78</b>
1.5	0.5	1.50	23.30	2.12	7.73	1.53	21.83	1.78	13.84	1.50	23.30	1.16	<b>47.08</b>
1.5	1.0	1.61	18.95	2.17	7.19	1.83	12.50	1.90	11.10	1.61	18.95	1.24	<b>39.46</b>
1.5	1.5	1.86	11.92	2.33	5.65	2.33	5.65	2.20	6.85	1.86	11.92	1.44	<b>26.25</b>

$n = 1$				$n = 5$									
(a,b)		(0,0,0,0)		(0,0,0,5)		(0,5,0,0)		(0,3,0,3)		(0,5,0,5)		(0,7,0,7)	
$\delta_1$	$\delta_2$	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL
0.0	0.0	0.0	370.4	0.0	370.4	0.0	370.4	0.0	370.40	0.0	370.40	0.0	370.40
0.0	0.5	0.52	192.53	0.68	138.95	1.16	46.84	0.88	88.02	0.68	137.69	0.48	<b>211.75</b>
0.0	1.0	1.05	60.48	1.35	31.46	2.33	5.68	1.76	14.23	1.36	30.93	0.95	<b>74.60</b>
0.0	1.5	1.57	20.25	2.03	8.97	3.49	1.72	2.64	3.78	2.04	8.79	1.43	<b>26.79</b>
0.5	0.0	0.52	192.53	1.16	46.84	0.68	138.95	0.88	88.02	0.68	137.69	0.48	<b>211.75</b>
0.5	0.5	0.62	157.11	1.17	46.20	1.17	46.20	1.04	61.37	0.81	104.25	0.56	<b>177.01</b>
0.5	1.0	1.02	64.15	1.52	22.56	2.24	6.50	1.72	15.44	1.33	33.22	0.93	<b>78.75</b>
0.5	1.5	1.50	23.30	2.04	8.87	3.37	1.88	2.52	4.39	1.95	10.25	1.37	<b>30.60</b>
1.0	0.0	1.05	60.48	2.33	5.68	1.35	31.46	1.76	14.23	1.36	30.93	0.95	<b>74.60</b>
1.0	0.5	1.02	64.15	2.24	6.50	1.52	22.56	1.72	15.44	1.33	33.22	0.93	<b>78.75</b>
1.0	1.0	1.24	39.83	2.34	5.58	2.34	5.58	2.08	8.22	1.61	18.83	1.13	<b>50.66</b>
1.0	1.5	1.61	18.95	2.62	3.86	3.37	1.87	2.70	3.53	2.09	8.18	1.46	<b>25.15</b>
1.5	0.0	1.57	20.25	3.49	1.72	2.03	8.97	2.64	3.78	2.04	8.79	1.43	<b>26.79</b>
1.5	0.5	1.50	23.30	3.37	1.88	2.04	8.87	2.52	4.39	1.95	10.25	1.37	<b>30.60</b>
1.5	1.0	1.61	18.95	3.37	1.87	2.62	3.86	2.70	3.53	2.09	8.18	1.46	<b>25.15</b>
1.5	1.5	1.86	11.92	3.51	1.70	3.51	1.70	3.12	2.29	2.42	5.04	1.69	<b>16.10</b>

tails see (Lütkepohl, 2005). The database was also used to estimate the in-control mean vector, the cross covariance matrix  $\Gamma$  and the autocorrelation matrix, see (Montgomery, 2009) for details

$$\hat{\mu}_0 = (\hat{\mu}_{0X}, \hat{\mu}_{0Y}) = (10.44, 30.00)$$

$$\hat{\Gamma} = \begin{pmatrix} \hat{\sigma}_x^2 & \hat{\sigma}_{xy} \\ \hat{\sigma}_{xy} & \hat{\sigma}_y^2 \end{pmatrix} = \begin{pmatrix} 0.4962 & 0.3741 \\ 0.3741 & 0.5888 \end{pmatrix}$$

$$\hat{\Phi} = \begin{pmatrix} \hat{a} & 0 \\ 0 & \hat{b} \end{pmatrix} = \begin{pmatrix} \frac{\hat{\sigma}_{X_i, X_{i-1}}}{\hat{\sigma}_x^2} & 0 \\ 0 & \frac{\hat{\sigma}_{Y_i, Y_{i-1}}}{\hat{\sigma}_y^2} \end{pmatrix} = \begin{pmatrix} 0.4820 & 0 \\ 0 & 0.4782 \end{pmatrix}$$

where  $\hat{\sigma}_{X_i, X_{i-1}}$  ( $\hat{\sigma}_{Y_i, Y_{i-1}}$ ) is the covariance of the X (Y) observations that are one time period apart.

The Eqs. (3), (4) and (5) are used to obtain the estimated correlation, covariance matrix, and the cross covariance matrix  $\Gamma_{\bar{X}}$

$$\hat{\rho} = \frac{(1 - \hat{a}\hat{b})\hat{\sigma}_{xy}}{\sqrt{[(1 - \hat{a}^2)\hat{\sigma}_x^2][(1 - \hat{b}^2)\hat{\sigma}_y^2]}} = 0.6921$$

$$\hat{\Sigma} = \begin{pmatrix} \hat{\sigma}_{e_x}^2 & \hat{\sigma}_{e_{xy}} \\ \hat{\sigma}_{e_{xy}} & \hat{\sigma}_{e_y}^2 \end{pmatrix} = \begin{pmatrix} (1 - \hat{a}^2)\hat{\sigma}_x^2 & (1 - \hat{a}\hat{b})\hat{\sigma}_{xy} \\ (1 - \hat{a}\hat{b})\hat{\sigma}_{xy} & (1 - \hat{b}^2)\hat{\sigma}_y^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3809 & 0.2879 \\ 0.2879 & 0.4542 \end{pmatrix}$$

$$\hat{\Gamma}_{\bar{X}} = \begin{pmatrix} \hat{\zeta}_X^2 & \hat{\zeta}_{XY} \\ \hat{\zeta}_{YX} & \hat{\zeta}_Y^2 \end{pmatrix} = \begin{pmatrix} 0.2145 & 0.1612 \\ 0.1612 & 0.2529 \end{pmatrix}$$

The cross-covariance matrix  $\hat{\Gamma}_{\bar{X}}$  was estimated for rational subgroups of size five ( $n = 5$ ). Finally, the in-control mean vector  $\hat{\mu}_0$  and the inverse of the cross-covariance matrix  $\hat{\Gamma}_{\bar{X}}$  are used to compute the statistic  $T_i^2$ ,  $i = 1, 2, \dots$  for Hotelling's  $T^2$  chart

$$T_i^2 = (\bar{x}_i - 10.44 \quad \bar{y}_i - 30.00) \begin{pmatrix} 8.95 & -5.70 \\ -5.70 & 7.59 \end{pmatrix} \begin{pmatrix} \bar{x}_i - 10.44 \\ \bar{y}_i - 30.00 \end{pmatrix}$$

Fig. 5 brings the  $T^2$  chart with the  $T^2$  values of Table 10, and an upper control limit of 11.83 ( $ARL_0 = 370.4$ ). The samples values were simulated with an assignable cause that shifts the process mean vector from  $\hat{\mu}_0 = (10.44, 30.00)$  to  $\hat{\mu}_1 = (10.75, 30.67)$  that is,  $\delta_1 = 0.5$  and  $\delta_2 = 1.0$ . The  $T^2$  chart requires, on average, 29.25 samples to signal. This number reduces to 5.81 when the variables are non-autocorrelated.

**5. Conclusions**

In this article we evaluated the performance of the bivariate Hotelling's  $T^2$  chart for rational subgroups with observations that are



**Table 8**

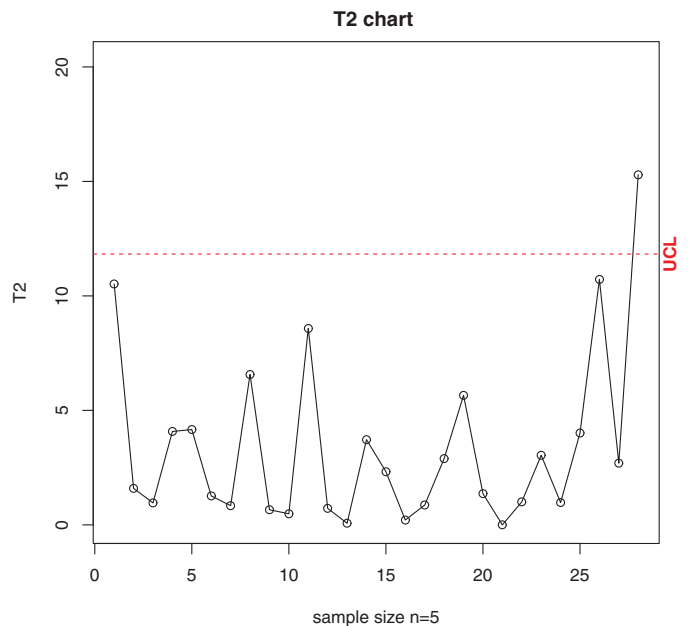
The ARL values for the  $T^2$  control chart,  $\rho = 0.7$ .

$n = 1$				$n = 2$									
(a,b)		(0,0,0.0)		(0,0,0.5)		(0.5,0.0)		(0.3,0.3)		(0.5,0.5)		(0.7,0.7)	
$\delta_1$	$\delta_2$	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL
0.0	0.0	0.0	370.4	0.0	370.4	0.0	370.4	0.0	370.40	0.0	370.4	0.0	370.40
0.0	0.5	0.70	131.83	0.64	<b>151.65</b>	0.90	84.20	0.83	98.93	0.70	131.83	0.54	<b>185.41</b>
0.0	1.0	1.40	28.53	1.27	<b>37.17</b>	1.80	13.26	1.66	17.24	1.40	28.53	1.08	<b>55.82</b>
0.0	1.5	2.10	8.00	1.91	<b>10.95</b>	2.70	3.53	2.49	4.60	2.10	8.00	1.63	<b>18.24</b>
0.5	0.0	0.70	131.83	0.90	84.20	0.64	<b>151.65</b>	0.83	98.93	0.70	131.83	0.54	<b>185.41</b>
0.5	0.5	0.54	185.41	0.71	128.52	0.71	128.52	0.64	149.93	0.54	185.41	0.42	<b>236.29</b>
0.5	1.0	1.04	61.81	1.01	<b>66.39</b>	1.49	23.68	1.23	40.83	1.04	61.81	0.80	<b>104.42</b>
0.5	1.5	1.69	16.32	1.53	<b>22.17</b>	2.36	5.44	2.00	9.48	1.69	16.32	1.31	<b>34.67</b>
1.0	0.0	1.40	28.53	1.80	13.26	1.27	<b>37.17</b>	1.66	17.24	1.40	28.53	1.08	<b>55.82</b>
1.0	0.5	1.04	61.81	1.49	23.68	1.01	<b>66.39</b>	1.23	40.83	1.04	61.81	0.80	<b>104.42</b>
1.0	1.0	1.08	55.82	1.42	27.23	1.42	27.23	1.28	36.36	1.08	55.82	0.84	<b>96.35</b>
1.0	1.5	1.50	23.26	1.62	18.39	2.16	7.31	1.78	13.82	1.50	23.26	1.16	<b>47.02</b>
1.5	0.0	2.10	8.00	2.70	3.53	1.91	<b>10.95</b>	2.49	4.60	2.10	8.00	1.63	<b>18.24</b>
1.5	0.5	1.69	16.32	2.36	5.44	1.53	<b>22.17</b>	2.00	9.48	1.69	16.32	1.31	<b>34.67</b>
1.5	1.0	1.50	23.26	2.16	7.31	1.62	18.39	1.78	13.82	1.50	23.26	1.16	<b>47.02</b>
1.5	1.5	1.63	18.24	2.13	7.58	2.13	7.58	1.93	10.66	1.63	18.24	1.26	<b>38.18</b>
$n = 1$				$n = 5$									
(a,b)		(0,0,0.0)		(0,0,0.5)		(0.5,0.0)		(0.3,0.3)		(0.5,0.5)		(0.7,0.7)	
$\delta_1$	$\delta_2$	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL	$\lambda$	ARL
0.0	0.0	0.0	370.4	0.0	370.4	0.0	370.4	0.0	370.40	0.0	370.4	0.0	370.40
0.0	0.5	0.70	131.83	0.86	92.28	1.48	24.28	1.18	45.74	0.91	82.54	0.64	<b>151.40</b>
0.0	1.0	1.40	28.53	1.72	15.36	2.96	2.67	2.35	5.50	1.82	12.84	1.27	<b>37.05</b>
0.0	1.5	2.10	8.00	2.58	4.08	4.44	1.15	3.53	1.69	2.73	3.42	1.91	<b>10.91</b>
0.5	0.0	0.70	131.83	1.48	24.28	0.86	92.28	1.18	45.74	0.91	82.54	0.64	<b>151.40</b>
0.5	0.5	0.54	185.41	1.12	51.26	1.12	51.26	0.91	82.18	0.70	130.68	0.49	<b>204.87</b>
0.5	1.0	1.04	61.81	1.35	31.93	2.48	4.61	1.74	14.66	1.35	31.75	0.95	<b>76.10</b>
0.5	1.5	1.69	16.32	1.96	10.08	3.93	1.36	2.83	3.05	2.19	6.98	1.53	<b>21.80</b>
1.0	0.0	1.40	28.53	2.96	2.67	1.72	15.36	2.35	5.50	1.82	12.84	1.27	<b>37.05</b>
1.0	0.5	1.04	61.81	2.48	4.61	1.35	31.93	1.74	14.66	1.35	31.75	0.95	<b>76.10</b>
1.0	1.0	1.08	55.82	2.25	6.40	2.25	6.40	1.82	12.75	1.41	28.07	0.99	<b>69.28</b>
1.0	1.5	1.50	23.26	2.33	5.71	3.56	1.65	2.52	4.38	1.95	10.23	1.37	<b>30.56</b>
1.5	0.0	2.10	8.00	4.44	1.15	2.58	4.08	3.53	1.69	2.73	3.42	1.91	<b>10.91</b>
1.5	0.5	1.69	16.32	3.93	1.36	1.96	10.08	2.83	3.05	2.19	6.98	1.53	<b>21.80</b>
1.5	1.0	1.50	23.26	3.56	1.65	2.33	5.71	2.52	4.38	1.95	10.23	1.37	<b>30.56</b>
1.5	1.5	1.63	18.24	3.37	1.88	3.37	1.88	2.73	3.40	2.11	7.86	1.48	<b>24.25</b>

**Table 9**  
Preliminary sample.

Observation	X	Y
1	9.42	28.45
2	10.78	29.80
3	10.40	31.57
4	10.43	30.72
5	10.30	29.91
6	9.55	29.21
7	10.85	30.03
8	10.21	29.68
9	10.58	30.05
10	9.53	29.21
⋮	⋮	⋮
⋮	⋮	⋮
299	10.53	30.13
300	11.37	30.66

not only cross-correlated but also autocorrelated. The autocorrelation has a strong and negative impact on the performance of the  $T^2$  chart, especially when more frequent and/or larger shifts occur on the mean of the variable with the higher level of autocorrelation. It is also worth to note that, depending on the level of the autocorrelations the  $T^2$  chart based on individual observations signals faster than the  $T^2$  chart based on rational subgroups. For example, when the observations of two variables have the same level of autocorrelation and they are moderate or high ( $a = b > 0.5$ ), the bivariate  $T^2$  chart for individual observations performs better than the bivariate  $T^2$  chart for rational subgroups of size two. Regarding to the correlation, if the de-



**Fig. 5.** The Hotelling  $T^2$  control chart for the drilling process data.

pendence among  $X$  observations and among  $Y$  observations are the same (or close), that is,  $a \cong b$ ,  $T^2$  chart has a better overall performance when the variables  $X$  and  $Y$  are highly correlated.

**Table 10**  
The sample means and the  $T_i^2$  values for the drilling process data.

Sample	$\bar{x}_i$	$\bar{y}_i$	$T_i^2$
1	11.82	31.50	10.52
2	10.85	29.98	1.60
3	10.84	30.47	0.96
4	9.79	30.04	4.08
5	10.52	30.79	4.16
6	10.73	30.56	1.27
7	10.77	30.46	0.84
8	10.59	31.03	6.57
9	10.37	29.65	0.66
10	10.76	30.30	0.49
:	:	:	:
26	11.52	31.64	10.72
27	9.94	30.08	2.70
28	11.44	31.93	15.29

The authors are currently investigating the use of the skip sampling strategy proposed by Costa and Castagliola (2011) and the mixed sampling strategy proposed by Franco et al. (2013) to reduce the negative effect of the autocorrelations on the statistical performance of the  $T^2$  control chart.

**Acknowledgments**

The authors are thankful to the reviewers and editor for the useful comments to improve the initial version of the article. This work was supported by CNPq – National Council for Scientific and Technological Development (grant no. 306189/2011-9 and 301739/2010-2).

**Appendix A. Determining the cross-covariance matrix  $\Gamma_{\bar{X}}$  of the sample mean vector  $\bar{X}$**

If  $n$  units of the sample are collected close together in time, and the length of the sampling interval  $h$  is large enough to eliminate any dependence between samples, the vector with the observations of the  $j$ th unit is described by a first-order autoregressive model:

$$X_j = \sum_{i=1}^j \Phi^{j-i} \varepsilon_i; \quad j = 1, 2, \dots, n. \tag{A.1}$$

where  $\varepsilon_i \sim N_2(\mu; \Sigma)$  with  $\Sigma = (a_{ij})_{2 \times 2}$ ;  $a_{ij} = 1$  if  $i = j$ ;  $a_{ij} = \rho$  if  $i \neq j$  and  $\Phi = \text{diag}(a; b)$ .

According to the multivariate first-order autoregressive model:

$$\text{Var}(X_j) = (\Phi^{j-1})\Gamma(\Phi^{j-1})' + (\Phi^{j-2})\Sigma(\Phi^{j-2})' + \dots + \Sigma \tag{A.2}$$

where  $\Gamma = \Phi\Gamma\Phi' + \Sigma$ . The cross-covariance matrix of the sample mean vector ( $\Gamma_{\bar{X}}$ ) is given by

$$\Gamma_{\bar{X}} = \text{Var}\left[\frac{X_1 + X_2 + \dots + X_n}{n}\right] \tag{A.3}$$

From A.1, A.2 and A.3, it follows that:

$$\Gamma_{\bar{X}} = \frac{1}{n^2} \times \left[ \begin{aligned} &(I + \Phi + \Phi^2 + \dots + \Phi^{n-1})\Gamma(I + \Phi + \Phi^2 + \dots + \Phi^{n-1})' + \\ &(I + \Phi + \Phi^2 + \dots + \Phi^{n-2})\Sigma(I + \Phi + \Phi^2 + \dots + \Phi^{n-2})' + \\ &\dots + (I + \Phi)\Sigma(I + \Phi)' + \Sigma \end{aligned} \right] \tag{A.4}$$

After some manipulation

$$\Gamma_{\bar{X}} = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix} \tag{A.5}$$

where  $\zeta_{11} = \frac{\sigma_x^2}{n} [1 + \frac{2}{n} \sum_{j=1}^{n-1} (n-j)a^j]$ ;  $\zeta_{22} = \frac{\sigma_y^2}{n} [1 + \frac{2}{n} \sum_{j=1}^{n-1} (n-j)b^j]$ ;  $\zeta_{12} = \frac{\sigma_{xy}}{n} [1 + \frac{1}{n} \sum_{j=1}^{n-1} (n-j)a^j + \frac{1}{n} \sum_{j=1}^{n-1} (n-j)b^j]$ ;  $\zeta_{21} = \zeta_{12}$ ;  $\sigma_x^2 = \frac{\sigma_{\varepsilon_x}^2}{1-a^2}$ ;  $\sigma_y^2 = \frac{\sigma_{\varepsilon_y}^2}{1-b^2}$ ;  $\sigma_{xy} = \frac{\sigma_{\varepsilon_{xy}}}{1-ab}$ .

**Appendix B. The values of  $a^*$  and  $b^*$  that makes  $\lambda^2(1) = \lambda^2(n)$**

The detection power of the control chart is function of  $\lambda^2(n)$ , the non-centrality parameter of the chi-square distribution; consequently, if  $\lambda^2(1) = \lambda^2(n)$  the detection power of the  $T^2$  chart for individual observations is equal to the detection power of  $T^2$  chart for samples of size  $n$ .

When  $a = b$ , the two elements of the main diagonal of matrix  $\Gamma_{\bar{X}}$  are equal; consequently,  $\theta_{11} = \theta_{22}$  and  $\lambda^2(n) = (\delta_1^2 + \delta_2^2) \theta_{11} + 2\delta_1\delta_2\theta_{12}$ . For individual observations,  $\lambda^2(1) = (\delta_1^2 + \delta_2^2)(1 - \rho^2)^{-1} - 2\delta_1\delta_2\rho(1 - \rho^2)^{-1}$ , and for  $n \geq 2$ ,

$$\lambda^2(n) = [(\delta_1^2 + \delta_2^2) - 2\rho\delta_1\delta_2n^2(1 - a^2)] \times \left\{ (1 - \rho^2) \left[ n + 2 \sum_{j=1}^{n-1} (n-j)a^j \right] \right\}^{-1} \tag{B.1}$$

The value of  $a^*$  that makes  $\lambda^2(1) = \lambda^2(n)$  is the  $a$ -value, restrict to  $\{a \in \mathbb{R} : 0 < a < 1\}$ , that makes  $n^2(a - 1)^3(a + 1) = -n + 2a + a^2n - 2a^{n+1}$ . Table 5 provides the values of  $a^*$  for  $n \in \{2, 3, 5, 10, 20\}$ . The value of  $a^*$  is independent of  $\delta_1, \delta_2$  and  $\rho$ .

When  $a = 0$  and  $b \neq 0$ ,  $\lambda^2(n) = \delta_1^2\theta_{11} + \delta_2^2\theta_{22} + 2\delta_1\delta_2\theta_{12}$  and the general expressions of  $\theta_{11}, \theta_{22}$  and  $\theta_{12}$  are very complex. For individual observations,  $\lambda^2(1) = (\delta_1^2 + \delta_2^2)(1 - \rho^2)^{-1} - 2\delta_1\delta_2\rho(1 - \rho^2)^{-1}$ , and for  $n \geq 2$  with  $\delta_1 = 0$ ,  $\lambda^2(n)$  reduces to

$$\lambda^2(n) = n^3\delta_2^2(b - 1)^4(b + 1) [b^3(n^2\rho^2 - n^2 + 2n\rho^2 + \rho^2) - b^2(n^2\rho^2 - n^2 + 2n\rho^2 - 2n) + b^{2n+2}\rho^2 - 2b^{n+3}\rho^2 - 2b^{n+2}\rho^2 + b^{2n+3}\rho^2 - 2b^{n+1}n + 2b^{n+2}n + n^2\rho^2 - n^2 + 2b^{n+1}n\rho^2 - 2b^{n+3}n\rho^2]^{-1} \tag{B.2}$$

Table 5 also provides the values of  $b^*$  that make  $\lambda^2(1) = \lambda^2(n)$  for ( $\delta_1 = 0, a = 0$ ). The value of  $b^*$  is independent of  $\delta_2$  but dependent on  $\rho$ . When  $a = b > a^*$  or when  $a = \delta_1 = 0$  and  $b > b^*$ , the  $T^2$  chart for individual observations performs better than the  $T^2$  chart for samples of size  $n$ .

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