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Synthetic charts to control bivariate processes with autocorrelated data

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ABSTRACT

In this study, we propose the use of simultaneous $\overline{X}$ charts to control bivariate processes with autocorrelated data. The first set of $\overline{X}$ charts is side-sensitive with regard to the same variable (SV $\overline{X}$ charts) and the second one is side-sensitive with regard to both variables (BV $\overline{X}$ charts). The Markov chain approach was used to obtain the steady-state properties of the $\overline{X}$ charts. In comparison with the standard synthetic T^2 chart, the SV and the BV charts signal faster in a wide variety of disturbances, except when the variables are high correlated. The BV charts are simpler and signal faster than the SV charts.

1. Introduction

The Hotelling T^2 chart is the usual chart for detecting changes in the mean vector of multivariate processes. However, it is not always easy to convince practitioners accustomed to work with $\overline{X}$ values to consider a more complex statistic. The T^2 statistic is not only more complex in terms of computation but also with regard to its interpretation. If the statistical process control demands the monitoring of only two quality characteristics, the practitioner might prefer to work with two $\overline{X}$ charts, even knowing that the single T^2 chart was designed to control more than one quality characteristic. In comparison with the bivariate T^2 chart, the joint $\overline{X}$ charts have a better overall performance in signaling changes in the mean vector of correlated variables (Machado & Costa, 2008).

In a growing number of multivariate processes, the variables are cross-correlated and their observations are autocorrelated. Leoni, Machado, and Costa (2014) evaluated the effect of the cross-correlation and the autocorrelation on the performance of two combined $\overline{X}$ charts and on the performance of the Hotelling's T^2 chart. The overall conclusion is that the speed with which the charts signal reduces when the variable affected by the assignable cause is autocorrelated.

Leoni, Costa, and Machado (2015) obtained the cross covariance matrix of the rational sample mean vectors and investigated the joint effect of the correlation and autocorrelation on the T^2 chart's performance. Leoni, Costa, Franco, and Machado (2015) and Leoni, Machado, and Costa (in press), respectively, considered the skipping and the mixed sampling strategies to reduce the negative effect of the autocorrelation on the T^2 chart's performance. The skipping strategy was proposed by Costa and Castagliola (2011), and the mixed sampling strategy was proposed by Franco, Castagliola, Celano, and Costa (2013).

Wu and Spedding (2000) proposed to change the standard $\overline{X}$ chart's signaling rule of one point in the action region by the synthetic rule. Davis and Woodall (2002) obtained the steady-state properties of the $\overline{X}$ chart with the synthetic rule with the aim to prove its faster mean shift detection. The results of their studies motivated other researchers to consider the synthetic rule as an alternative to enhance the control charts' performance. A recent list includes the works of Haridy, Wu, Khoo, and Yu (2012), Calzada and Scariano (2013), Khoo, Wu, Castagliola, and Lee (2013), Haridy, Wu, Abhary, Castagliola, and Shamsuzzaman (2014), Chong, Khoo, and Castagliola (2014), Lee and Khoo (2014), Yeong, Khoo, Lee, and Rahim (2014), Guo, Wang, and Cheng (2015), Chew, Khoo, Teh, and Castagliola (2015), and Bajirao and Parasharam (2015). Machado and Costa (2014) proved

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that the side-sensitive version of the synthetic rule enhances the $\overline{X}$ chart's performance. Considering a wide range of mean shifts, the side sensitive feature reduces in 23%, on average, the time to detect the out-of-control condition. Costa and Machado (2015, 2016) considered the Markov chain approach to obtain the properties of the double sampling $\overline{X}$ charts and the variable sample size $\overline{X}$ charts, both with the synthetic and with the side sensitive synthetic rules. Celano and Castagliola (2016) investigated the synthetic rules and the gain in speed with which the charts, used to control the ratio of two normal variables, signal an out-of-control condition. Haq, Brown, and Moltchanova (2015, 2016) proposed new synthetic charts for monitoring process mean and dispersion. You, Khoo, Lee, and Castagliola (2015), Guo et al. (2015) and Yeong, Khoo, Yanjing, and Castagliola (2015) investigated the performance of several synthetic charts when the process parameters are estimated.

In this article, we propose the following synthetic charts to control bivariate processes with autocorrelated data: the T^2 chart and two simultaneous univariate $\overline{X}$ charts (the joint $\overline{X}$ charts). We consider the T^2 chart with the standard synthetic rule (Syn T^2 chart) and the $\overline{X}$ charts with two kinds of side-sensitive synthetic rules. The first one is side-sensitive with regard to the same variable (SV $\overline{X}$ charts) and the second one is side-sensitive with regard to both variables (BV $\overline{X}$ charts). When the $\overline{X}$ charts are in use, the synthetic rules depend on the values of the two sample means $(\overline{X}_1, \overline{X}_2)$. With the first side-sensitive rule (SV rule), the joint $\overline{X}$ charts signal in two cases: case (I) when the $\overline{X}_1$ and the $\overline{X}_2$ values of the same sample fall beyond their control limits; case (II) when the $\overline{X}_1$ or the $\overline{X}_2$ values of two different samples, not far from each other, fall beyond their control limits, except if the two points in the warning region are from the same variable and located on opposite sides of the center line. With the second side-sensitive rule (BV rule), the joint $\overline{X}$ charts signal in two cases: case (I) when the $\overline{X}_1$ and the $\overline{X}_2$ values of the same sample fall beyond their control limits; case (II) when the $\overline{X}_1$ or the $\overline{X}_2$ values of two different samples, not far from each other, fall beyond their upper (lower) control limits.

The paper is organized as follows: Section 2 is devoted to the presentation of the multivariate first order autoregressive model, VAR (1), and the bivariate cross-covariance matrix of the sample

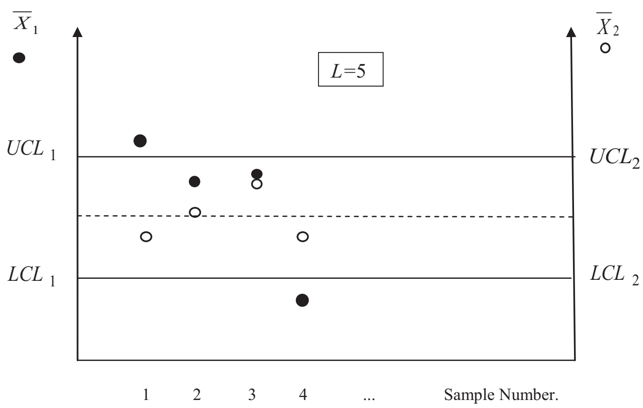

Fig. 1. Two points of the same variable in different warning regions.

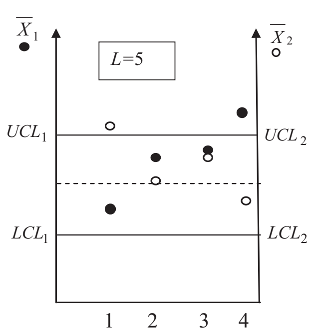

Fig. 2a. Two points in the same warning regions.

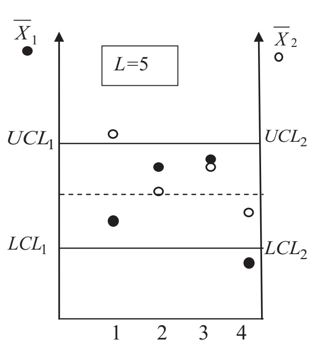

Fig. 2b. Two points in different warning regions.

Table 1
The $\overline{X}_1$ and $\overline{X}_2$ positions and the corresponding sample codes (SV $\overline{X}$ charts).

Sample codes	$\overline{X}_1$ position	$\overline{X}_2$ position
$\underline{2}$	$LCL_1 < \overline{X}_1 < UCL_1$	$\overline{X}_2 < LCL_2$
$\underline{1}$	$\overline{X}_1 < LCL_1$	$LCL_2 < \overline{X}_2 < UCL_2$
0	$LCL_1 < \overline{X}_1 < UCL_1$	$LCL_2 < \overline{X}_2 < UCL_2$
1	$\overline{X}_1 > UCL_1$	$LCL_2 < \overline{X}_2 < UCL_2$
2	$LCL_1 < \overline{X}_1 < UCL_1$	$\overline{X}_2 > UCL_2$

Table 2
The probabilities of the transition matrix (7).

Probabilities
$U_i = \Pr[\overline{X}_i > UCL_i, LCL_j < \overline{X}_j < UCL_j],\ i \neq j \in \{1,2\}$
$L_i = \Pr[\overline{X}_i < LCL_i, LCL_j < \overline{X}_j < UCL_j],\ i \neq j \in \{1,2\}$
$C = \Pr[LCL_1 < \overline{X}_1 < UCL_1, LCL_2 < \overline{X}_2 < UCL_2]$
$A = 1 - (C + U_1 + U_2 + L_1 + L_2)$
$B_i = 1 - (C + L_i),\ i \in \{1,2\}$
$D_i = 1 - (C + U_i),\ i \in \{1,2\}$

Table 3
The $\overline{X}_1$ and $\overline{X}_2$ positions and the corresponding sample codes (BV $\overline{X}$ charts).

Sample codes	$\overline{X}_1$ position	$\overline{X}_2$ position
$\underline{1}$	$LCL_1 < \overline{X}_1 < UCL_1$	$\overline{X}_2 < LCL_2$
	$\overline{X}_1 < LCL_1$	$LCL_2 < \overline{X}_2 < UCL_2$
0	$LCL_1 < \overline{X}_1 < UCL_1$	$LCL_2 < \overline{X}_2 < UCL_2$
1	$\overline{X}_1 > UCL_1$	$LCL_2 < \overline{X}_2 < UCL_2$
	$LCL_1 < \overline{X}_1 < UCL_1$	$\overline{X}_2 > UCL_2$

Table 4
The probabilities of the transition matrix (8).

Probabilities
$U = \sum_{i=1}^{2} \Pr[\overline{X}_i > LCL_i, LCL_j < \overline{X}_j < UCL_j],\ j \neq i \in \{1,2\}$
$L = \sum_{i=1}^{2} \Pr[\overline{X}_i < UCL_i, LCL_j < \overline{X}_j < UCL_j],\ j \neq i \in \{1,2\}$
$C = \Pr[LCL_1 < \overline{X}_1 < UCL_1, LCL_2 < \overline{X}_2 < UCL_2]$
$A = 1 - (C + U + L);\ B = 1 - (C + U);\ D = 1 - (C + L)$

Table 5
The *SSARLs* and the *MEQL* of the synthetic charts; $a = b$; $L = 3$.

$\rho = 0.3$

$(a;b)$			(0.3; 0.3)			(0.5; 0.5)			(0.7; 0.7)		
$\delta1$	$\delta2$		T^2	SV $k = 1.94157$	BV $k = 1.89838$	T^2	SV $k = 2.08558$	BV $k = 2.03919$	T^2	SV $k = 2.22027$	BV $k = 2.17088$
0	0	$n = 2$	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0	0.5		**126.63**	132.51	130.49	**142.27**	147.81	146.04	**156.19**	161.36	159.85
0	1		**24.14**	27.07	25.63	**29.73**	33.23	31.54	**35.5**	39.51	37.6
0	1.5		**7.25**	8.02	7.65	**8.9**	9.91	9.42	**10.7**	11.97	11.34
0.5	0.5		92.34	67.28	**57.21**	106.39	77.98	**66.36**	119.36	88.12	**75.11**
0.5	1		26.03	21.07	**18.53**	31.99	25.69	**22.47**	38.12	30.42	**26.51**
0.5	1.5		8.27	7.44	**6.87**	10.2	9.12	**8.35**	12.3	10.94	**9.94**
1	1		14.58	11.47	**10.24**	18.1	14.01	**12.39**	21.82	16.67	**14.65**
1	1.5		6.82	5.74	**5.3**	8.35	6.95	**6.36**	10.03	8.26	**7.5**
1.5	1.5		4.64	3.89	**3.65**	5.57	4.65	**4.32**	6.59	5.46	**5.04**
		MEQL	118.21	104.74	**96.30**	141.84	125.11	**114.61**	166.18	145.99	**133.38**

$\rho = 0.5$

$(a;b)$			(0.3; 0.3)			(0.5; 0.5)			(0.7; 0.7)		
$\delta1$	$\delta2$		T^2	SV $k = 1.99219$	BV $k = 1.96319$	T^2	SV $k = 2.13995$	BV $k = 2.10881$	T^2	SV $k = 2.27815$	BV $k = 2.24500$
0	0	$n = 2$	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0	0.5		**106.39**	153.12	156.76	**121.26**	169.16	173.33	**134.76**	183.09	187.65
0	1		**18.1**	32.27	31.81	**22.41**	39.81	39.42	**26.94**	47.45	47.19
0	1.5		**5.57**	9.02	8.82	**6.75**	11.28	11.02	**8.06**	13.77	13.45
0.5	0.5		106.39	74.91	**67.6**	121.26	86.28	**77.94**	134.76	96.95	**87.73**
0.5	1		27.71	24.5	**22.5**	34	29.86	**27.34**	40.43	35.3	**32.27**
0.5	1.5		**7.87**	8.41	7.96	**9.69**	10.41	9.8	**11.67**	12.58	11.8
1	1		18.1	13.17	**12.18**	22.41	16.1	**14.82**	26.94	19.16	**17.57**
1	1.5		7.87	6.47	**6.13**	9.69	7.9	**7.43**	11.67	9.43	**8.83**
1.5	1.5		5.57	4.31	**4.12**	6.75	5.2	**4.94**	8.06	6.15	**5.82**
		MEQL	174.25	132.74	**121.91**	203.77	156.56	**143.73**	232.72	180.24	**165.55**

$\rho = 0.7$

$(a;b)$			(0.3; 0.3)			(0.5; 0.5)			(0.7; 0.7)		
$\delta1$	$\delta2$		T^2	SV $k = 2.08360$	BV $k = 2.07057$	T^2	SV $k = 2.23814$	BV $k = 2.22415$	T^2	SV $k = 2.38268$	BV $k = 2.36779$
0	0	$n = 2$	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0	0.5		**70.86**	195.69	205.8	**83.15**	211.94	222.31	**94.8**	225.49	235.88
0	1		**10.14**	44.91	46.19	**12.56**	55.84	57.79	**15.16**	66.77	69.46
0	1.5		**3.53**	11.27	11.3	**4.12**	14.43	14.51	**4.79**	17.97	18.12
0.5	0.5		119.36	88.64	**85.02**	134.76	101.04	**97.01**	148.58	112.52	**108.14**
0.5	1		**24.82**	31.87	30.73	**30.54**	38.73	37.34	**36.44**	45.59	43.95
0.5	1.5		**5.98**	10.49	10.23	**7.28**	13.24	12.89	**8.71**	16.23	15.79
1	1		21.82	16.46	**15.89**	26.94	20.15	**19.42**	32.25	23.97	**23.06**
1	1.5		8.26	7.96	**7.75**	10.19	9.84	**9.55**	12.28	11.85	**11.49**
1.5	1.5		6.59	5.12	**5.01**	8.06	6.26	**6.11**	9.67	7.49	**7.3**
		MEQL	**117.19**	151.05	149.92	**140.50**	181.31	179.86	**164.53**	212.25	210.52

$\rho = 0.3$

$(a;b)$			(0.3; 0.3)			(0.5; 0.5)			(0.7; 0.7)		
$\delta1$	$\delta2$		T^2	SV $k = 1.36780$	BV $k = 1.33733$	T^2	SV $k = 1.60649$	BV $k = 1.57075$	T^2	SV $k = 1.88963$	BV $k = 1.84759$
0	0	$n = 5$	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0	0.5		**61.35**	66.87	64.45	**88.36**	94.47	92.03	**120.84**	126.8	124.7
0	1		**8.48**	9.44	8.97	**13.68**	15.35	14.52	**22.28**	25	23.66
0	1.5		**3.13**	3.34	3.27	**4.41**	4.79	4.63	**6.72**	7.42	7.09
0.5	0.5		39.7	29.26	**25.22**	60.26	43.8	**37.39**	87.27	63.49	**53.99**
0.5	1		9.14	7.78	**7.12**	14.78	12.28	**11.02**	24.04	19.52	**17.22**
0.5	1.5		3.43	3.18	**3.06**	4.94	4.52	**4.28**	7.66	6.9	**6.4**
1	1		5.33	4.46	**4.15**	8.3	6.79	**6.21**	13.44	10.63	**9.52**
1	1.5		3.01	2.56	**2.47**	4.19	3.59	**3.4**	6.33	5.35	**4.96**
1.5	1.5		2.42	1.87	**1.81**	3.09	2.54	**2.43**	4.35	3.64	**3.43**
		MEQL	49.50	44.45	**41.75**	73.13	65.47	**60.89**	110.34	97.90	**90.18**

(*continued on next page*)

Table 5 (*continued*)

$\rho = 0.5$

$(a;b)$			(0.3;0.3)			(0.5;0.5)			(0.7;0.7)		
$\delta 1$	$\delta 2$		T^2	SV $k = 1.40341$	BV $k = 1.38298$	T^2	SV $k = 1.64836$	BV $k = 1.62438$	T^2	SV $k = 1.93889$	BV $k = 1.91067$
0	0	$n = 5$	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0	0.5		**48.06**	80.06	80.79	**71.41**	111.67	113.68	**100.96**	147.05	150.48
0	1		**6.45**	10.71	10.47	**10.24**	17.89	17.5	**16.68**	29.73	29.27
0	1.5		**2.66**	3.53	3.5	**3.55**	5.2	5.12	**5.19**	8.3	8.12
0.5	0.5		48.06	33.42	**30.31**	71.41	49.53	**44.71**	100.96	70.85	**63.92**
0.5	1		9.74	8.86	**8.33**	15.77	14.19	**13.18**	25.6	22.7	**20.87**
0.5	1.5		3.31	3.39	**3.31**	**4.73**	4.95	4.77	**7.28**	7.77	7.37
1	1		6.45	4.98	**4.73**	10.24	7.72	**7.25**	16.68	12.2	**11.31**
1	1.5		3.31	2.75	**2.68**	4.73	3.95	**3.8**	7.28	6.01	**5.7**
1.5	1.5		2.66	1.96	**1.92**	3.55	2.73	**2.65**	5.19	4.02	**3.85**
		MEQL	50.72	49.79	**48.12**	75.26	74.19	**71.29**	113.82	111.77	**106.77**

$\rho = 0.7$

$(a;b)$			(0.3;0.3)			(0.5;0.5)			(0.7;0.7)		
$\delta 1$	$\delta 2$		T^2	SV $k = 1.46780$	BV $k = 1.45863$	T^2	SV $k = 1.72400$	BV $k = 1.71322$	T^2	SV $k = 2.02786$	BV $k = 2.01518$
0	0	$n = 5$	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4	370.4
0	0.5		**28.21**	111.05	116.93	**44.18**	149.99	158.34	**66.51**	189.34	199.3
0	1		**3.97**	13.63	13.7	**5.92**	23.91	24.24	**9.36**	41.21	42.29
0	1.5		**2.14**	3.94	3.95	**2.55**	6.1	6.11	**3.34**	10.27	10.3
0.5	0.5		56.4	41.28	**39.59**	82.15	60.12	**57.6**	113.67	84.16	**80.7**
0.5	1		**8.72**	11.14	10.83	**14.07**	18.3	17.7	**22.91**	29.53	28.48
0.5	1.5		**2.78**	3.81	3.76	**3.76**	5.84	5.73	**5.56**	9.61	9.39
1	1		7.69	5.97	**5.83**	12.35	9.5	**9.23**	20.13	15.23	**14.71**
1	1.5		3.42	3.11	**3.07**	4.94	4.65	**4.56**	7.64	7.35	**7.17**
1.5	1.5		2.95	2.13	**2.11**	4.07	3.1	**3.05**	6.12	4.75	**4.65**
		MEQL	**49.31**	61.11	60.97	**72.68**	92.52	92.08	**109.39**	140.86	139.88

mean vector. Section 3 describes the proposed synthetic charts. Section 4 presents the Markov chain model used to compute the steady state average run length. In Section 5, the proposed charts are compared with the synthetic T^2 chart, in terms of their statistical performance. Section 6 brings an example, and in Section 7 are the concluding remarks.

2. The autoregressive model

The multivariate autoregressive model for cross and serially correlated data has been adopted in recent studies dealing with control charts (Huang, Bisgaard, & Xu, 2013; Hwarng & Wang, 2010; Kim, Jitpitaklert, & Sukchotrat, 2010; Leoni, Costa, Machado, et al., 2015):

$$\mathbf{X}_t - \boldsymbol{\mu} = \boldsymbol{\Phi}(\mathbf{X}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\varepsilon}_\mathbf{t} \tag{1}$$

where $\mathbf{X}_t \sim N_p(\boldsymbol{\mu}, \Gamma)$ is the $(p \times 1)$ vector of observations at time t (p is the number of variables), $\boldsymbol{\mu}$ is the mean vector, $\boldsymbol{\varepsilon}_\mathbf{t}$ is an independent multivariate normal random vector with a mean vector of zeros and covariance matrix $\boldsymbol{\Sigma}_\mathbf{e}$, and $\boldsymbol{\Phi}$ is a $(p \times p)$ matrix of autocorrelation parameters. According to Kalgonda and Kulkarni (2004), the cross covariance matrix of $\mathbf{X}_t$ has the following property: $\boldsymbol{\Gamma} = \boldsymbol{\Phi}\boldsymbol{\Gamma}\boldsymbol{\Phi}' + \boldsymbol{\Sigma}_\mathbf{e}$. After some algebra we obtain:

$$\text{Vec } \boldsymbol{\Gamma} = (\mathbf{I}_{p^2} - \boldsymbol{\Phi} \otimes \boldsymbol{\Phi})^{-1}\text{Vec } \boldsymbol{\Sigma}_\mathbf{e} \tag{2}$$

where $\otimes$ is the Kronecker product and Vec is the operator that transform a matrix into a vector by stacking its columns.

To study the effects of the auto- and cross-correlation on the performance of the synthetic charts we considered the bivariate case ($p = 2$) with:

$$\boldsymbol{\Phi} = \text{diag}(a, b) \tag{3}$$

$$\boldsymbol{\Sigma}_\mathbf{e} = \begin{pmatrix} \sigma_{e_1}^2 & \sigma_{e_{12}} \\ \sigma_{e_{12}} & \sigma_{e_2}^2 \end{pmatrix} = \begin{pmatrix} \sigma_{e_1}^2 & \rho\sigma_{e_1}\sigma_{e_2} \\ \rho\sigma_{e_1}\sigma_{e_2} & \sigma_{e_2}^2 \end{pmatrix} \tag{4}$$

where a and b are the autocorrelation parameter and ρ is the correlation of X_1 and X_2.

From (2), (3) and (4) it follows:

$$\boldsymbol{\Gamma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} (1-a^2)^{-1}\sigma_{e_1}^2 & (1-ab)^{-1}\sigma_{e_{12}} \\ (1-ab)^{-1}\sigma_{e_{12}} & (1-b^2)^{-1}\sigma_{e_2}^2 \end{pmatrix} \tag{5}$$

Leoni, Costa, Machado, et al. (2015) obtained the cross covariance matrix $\Gamma_{\overline{X}}$ of the sample mean vector $\overline{X}$ when the sample items are collected according to the rational subgroup concept:

$$\boldsymbol{\Gamma}_{\overline{\mathbf{X}}} = \begin{pmatrix} \zeta_1^2 & \zeta_{12} \\ \zeta_{12} & \zeta_2^2 \end{pmatrix} = \begin{pmatrix} \frac{\sigma_1^2}{n}\left[1 + \frac{2}{n}\sum_{j=1}^{n-1}(n-j)a^j\right] & \frac{\sigma_{12}}{n}\left[1 + \frac{1}{n}\sum_{j=1}^{n-1}(n-j)a^j + \frac{1}{n}\sum_{j=1}^{n-1}(n-j)b^j\right] \\ \frac{\sigma_{12}}{n}\left[1 + \frac{1}{n}\sum_{j=1}^{n-1}(n-j)a^j + \frac{1}{n}\sum_{j=1}^{n-1}(n-j)b^j\right] & \frac{\sigma_2^2}{n}\left[1 + \frac{2}{n}\sum_{j=1}^{n-1}(n-j)b^j\right] \end{pmatrix} \tag{6}$$

where n is the size of the samples.

Table 6
The *SSARLs* and the *MEQL* of the synthetic charts; $a \neq b$; $L = 3$.

$\rho = 0.3$															
$(a; b)$			(0.3; 0.7)			(0.7; 0.3)				(0.3; 0.7)			(0.7; 0.3)		
$\delta 1$	$\delta 2$		T^2	SV $k = 2.08439$	BV $k = 2.04328$	T^2	SV $k = 2.08440$	BV $k = 2.04328$		T^2	SV $k = 1.65327$	BV $k = 1.63611$	T^2	SV $k = 1.65330$	BV $k = 1.63611$
0	0	$n = 2$	370.4	370.4	370.4	370.4	370.4	370.4	$n = 5$	370.4	370.4	370.4	370.4	370.4	370.4
0	0.5		155.99	133.55	**128.31**	**126.44**	160.9	168.22		119.7	75.89	**73.44**	**60.55**	134.74	154.87
0	1		35.41	29.83	**28.04**	**24.08**	37.1	37.07		21.93	14.31	**13.91**	**8.35**	21.3	23.04
0	1.5		10.67	9.49	**9.04**	**7.23**	10.35	10.08		6.62	5.11	**5.05**	**3.1**	5.17	5.28
0.5	0		**126.44**	160.91	168.22	155.99	133.54	**128.31**		**60.55**	134.75	154.87	119.7	75.87	**73.44**
0.5	0.5		104.4	77.9	**67.74**	104.4	77.89	**67.74**		55.9	48.95	**45.85**	55.9	48.93	**45.85**
0.5	1		36.38	24.42	**21.76**	26.93	27.17	**24.19**		20.66	12.75	**12.2**	**9.7**	15.21	14.53
0.5	1.5		12.2	8.89	**8.26**	**8.29**	9.38	8.67		7.42	4.94	**4.85**	**3.41**	4.8	4.7
1	0		**24.08**	37.1	37.07	35.41	29.83	**28.04**		**8.35**	21.3	23.04	21.93	14.3	**13.91**
1	0.5		26.93	27.17	**24.19**	36.38	24.41	**21.76**		**9.7**	15.22	14.53	20.66	12.74	**12.2**
1	1		17.57	14.09	**12.65**	17.57	14.09	**12.65**		7.61	7.7	**7.39**	7.61	7.7	**7.39**
1	1.5		9.03	6.97	**6.47**	7.36	6.97	**6.44**		4.85	4.06	**3.97**	**3.33**	3.75	3.66
1.5	0		**7.23**	10.36	10.08	10.67	9.49	**9.04**		**3.1**	5.17	5.28	6.62	5.11	**5.05**
1.5	0.5		**8.29**	9.38	8.67	12.2	8.89	**8.26**		**3.41**	4.8	4.7	7.42	4.94	**4.85**
1.5	1		7.36	6.98	**6.44**	9.03	6.97	**6.47**		**3.33**	3.75	3.66	4.85	4.05	**3.97**
1.5	1.5		5.42	4.66	**4.37**	5.42	4.66	**4.37**		2.93	2.71	**2.66**	2.93	2.71	**2.66**
		MEQL	225.91	208.00	**195.40**	225.91	207.96	**195.40**	**MEQL**	**118.93**	120.38	120.80	**118.93**	120.34	120.80

$\rho = 0.5$															
$(a; b)$			(0.3; 0.7)			(0.7; 0.3)				(0.3; 0.7)			(0.7; 0.3)		
$\delta 1$	$\delta 2$		T^2	SV $k = 2.13447$	BV $k = 2.10708$	T^2	SV $k = 2.13447$	BV $k = 2.10708$		T^2	SV $k = 1.67901$	BV $k = 1.66806$	T^2	SV $k = 1.67901$	BV $k = 1.66806$
0	0	$n = 2$	370.4	370.4	370.4	370.4	370.39	370.4	$n = 5$	370.4	370.4	370.4	370.4	370.35	370.4
0	0.5		**134.1**	153.45	152.66	**105.78**	176.6	188.44		97.36	83.71	**82.61**	**45.8**	136.6	154.86
0	1		**26.7**	34.86	33.9	**17.93**	43.94	45.83		15.78	15.42	**15.22**	**6.14**	23.71	26.34
0	1.5		**7.99**	10.58	10.3	**5.52**	11.82	11.85		**4.95**	5.34	5.32	**2.59**	5.52	5.73
0.5	0		**105.78**	176.61	188.44	**134.1**	153.44	152.66		**45.8**	136.63	154.86	97.36	83.69	**82.61**
0.5	0.5		118.72	85.16	**78.04**	118.72	85.15	**78.04**		64.51	51.67	**49.68**	64.51	51.66	**49.68**
0.5	1		40.1	28.02	**25.96**	**27.47**	31	28.84		24.26	13.81	**13.44**	**9.23**	16.54	16.16
0.5	1.5		11.96	9.96	**9.47**	**7.62**	10.69	10.16		7.6	5.2	**5.14**	**3.1**	5.16	5.12
1	0		**17.93**	43.95	45.83	**26.7**	34.85	33.9		**6.14**	23.72	26.34	15.78	15.41	**15.22**
1	0.5		**27.47**	31	28.84	40.1	28.02	**25.96**		**9.23**	16.55	16.16	24.26	13.8	**13.44**
1	1		21.63	16.01	**14.9**	21.63	16	**14.9**		9.01	8.36	**8.15**	9.01	8.36	**8.15**
1	1.5		10.84	7.83	**7.44**	8.19	7.87	**7.46**		5.86	4.32	**4.27**	**3.42**	4.02	3.96
1.5	0		**5.52**	11.82	11.85	**7.99**	10.58	10.3		**2.59**	5.52	5.73	**4.95**	5.34	5.32
1.5	0.5		**7.62**	10.69	10.16	11.96	9.96	**9.47**		**3.1**	5.17	5.12	7.6	5.19	**5.14**
1.5	1		8.19	7.88	**7.46**	10.84	7.83	**7.44**		**3.42**	4.02	3.96	5.86	4.32	**4.27**
1.5	1.5		6.53	5.17	**4.95**	6.53	5.17	**4.95**		3.26	2.86	**2.82**	3.26	2.86	**2.82**
		MEQL	**224.23**	235.77	228.92	**224.23**	235.73	228.92	**MEQL**	**115.64**	128.58	130.49	**115.64**	128.52	130.49

$\rho = 0.7$															
$(a; b)$			(0.3; 0.7)			(0.7; 0.3)				(0.3; 0.7)			(0.7; 0.3)		
$\delta 1$	$\delta 2$		T^2	SV $k = 2.22355$	BV $k = 2.21146$	T^2	SV $k = 2.22355$	BV $k = 2.21145$		T^2	SV $k = 1.72055$	BV $k = 1.71644$	T^2	SV $k = 1.72057$	BV $k = 1.71644$
0	0	$n = 2$	370.4	370.4	370.4	370.4	370.4	370.4	$n = 5$	370.4	370.4	370.4	370.4	370.4	370.4
0	0.5		**93.12**	195.85	200.66	**69.43**	202.65	216.4		**58.24**	98.55	99.09	**24.03**	134.6	147.44
0	1		**14.77**	46.53	46.91	**9.88**	59.45	64.51		**7.98**	17.45	17.49	**3.53**	28.1	31.91

(*continued on next page*)

Table 6 (*continued*)

$\rho = 0.7$

$(a;b)$		(0.3;0.7)			(0.7;0.3)			(0.3;0.7)			(0.7;0.3)		
$\delta 1$	$\delta 2$	T^2	SV k = 2.22355	BV k = 2.21146	T^2	SV k = 2.22355	BV k = 2.21145	T^2	SV k = 1.72055	BV k = 1.71644	T^2	SV k = 1.72057	BV k = 1.71644
0	1.5	**4.69**	12.93	12.89	**3.46**	15.1	15.6	**3.01**	5.74	5.77	**2.06**	6.16	6.49
0.5	0	**69.43**	202.65	216.4	**93.12**	195.85	200.66	**24.03**	134.6	147.44	**58.24**	98.54	99.09
0.5	0.5	130.9	97.9	**94.58**	130.9	97.9	**94.58**	68.62	55.83	**55.12**	68.62	55.82	**55.12**
0.5	1	38.52	35.75	**34.6**	**22.73**	38.25	37.28	25.43	15.66	**15.53**	**6.63**	18.4	18.38
0.5	1.5	**9.27**	12.18	11.92	**5.51**	13.47	13.22	6.18	5.61	**5.61**	**2.42**	5.78	5.8
1	0	**9.88**	59.45	64.51	**14.77**	46.53	46.91	**3.53**	28.11	31.91	**7.98**	17.44	17.49
1	0.5	**22.73**	38.25	37.28	38.52	35.75	**34.6**	**6.63**	18.4	18.38	25.43	15.65	**15.53**
1	1	25.58	19.56	**18.96**	25.58	19.56	**18.96**	9.73	9.33	**9.26**	9.73	9.33	**9.26**
1	1.5	12.21	9.53	**9.3**	**7.94**	9.61	9.38	6.93	4.74	**4.73**	**3.03**	4.44	4.43
1.5	0	**3.46**	15.11	15.6	**4.69**	12.93	12.89	**2.06**	6.16	6.49	**3.01**	5.74	5.77
1.5	0.5	**5.51**	13.47	13.22	**9.27**	12.18	11.92	**2.42**	5.78	5.8	6.18	5.61	**5.61**
1.5	1	**7.94**	9.61	9.38	12.21	9.52	**9.3**	**3.03**	4.44	4.43	6.93	4.74	**4.73**
1.5	1.5	7.66	6.13	**6**	7.66	6.12	**6**	3.43	3.07	**3.06**	3.43	3.07	**3.06**
	MEQL	**200.68**	292.59	293.67	**200.68**	292.54	293.67	**100.23**	141.59	145.11	**100.23**	141.57	145.11

The cross covariance matrix $\Gamma_{\overline{X}}$ is important to study the performance of the synthetic charts by using Markov chains approach. Without loss of generalization, we consider an in-control mean vector $\boldsymbol{\mu}' = (\mu_{01}, \mu_{02}) = (0, 0)$, and after the assignable cause occurrence, the mean vector changes to $\boldsymbol{\mu}' = (\delta_1\sigma_{e_1}, \delta_2\sigma_{e_2})$.

3. The joint $\overline{X}$ charts with synthetic rules

In this article we propose the use of joint $\overline{X}$ charts with two kind of synthetic rules to control the mean vector of bivariate processes. These rules are based on the conforming run length (*CRL*) measure. The *CRL* is the number of conforming samples between two consecutive nonconforming samples plus the ending nonconforming one; in other words, the first of the two consecutive nonconforming samples is the reference to compute the *CRL*. A *CRL* lower than or equal to a specified positive integer *L* ($CRL \leqslant L$) triggers a signal. The sample is conforming when the two means of the quality characteristics ($\overline{X}_1, \overline{X}_2$) fall in the central region of their control charts; otherwise, the sample is nonconforming. The central regions of the $\overline{X}_i$ charts, with $i = 1, 2$, are defined by the lower and upper control limits ($LCL_i = \mu_{0i} - k\zeta_i$; $UCL_i = \mu_{0i} + k\zeta_i$), where k is the width coefficient of the control limits.

The first synthetic rule is side-sensitive with regard to the same variable (SV rule). The joint $\overline{X}$ charts with the SV rule (SV $\overline{X}$ charts) signal in two cases: case (I) when the sample is nonconforming with both, the $\overline{X}_1$ and the $\overline{X}_2$, in the warning region; case (II) when two consecutive nonconforming samples are not far from each other ($CRL \leqslant L$), except when the $\overline{X}$ values beyond the control limits are from the same quality characteristic and located on the opposite sides of the center line. Fig. 1 illustrates a case where $CRL \leqslant L$ (*CRL* = 3 and *L* = 5) and doesn't trigger a signal once the two points in the opposite warning regions are from the same quality characteristic, that is, from X_1.

The second side-sensitive rule is side-sensitive with regard to both variables (BV rule). The joint $\overline{X}$ charts with the BV rule (BV $\overline{X}$ charts) signal in two cases: case (I) when the sample is nonconforming with both, the $\overline{X}_1$ and the $\overline{X}_2$, in the warning region; case (II) when two consecutive nonconforming samples are not far from each other ($CRL \leqslant L$), except when their $\overline{X}$ values beyond the control limits are located on the opposite sides of the center line. Fig. 2 illustrates two cases where $CRL \leqslant L$ (*CRL* = 3 and *L* = 5). The set of mean values in Fig. 2a triggers a signal once the $\overline{X}_2$ from the first nonconforming sample is above the upper control limit (UCL_2) and the $\overline{X}_1$ from the second nonconforming sample is also above the upper control limit (UCL_1). The set of mean values in Fig. 2b doesn't trigger a signal once the two points in the warning regions are in opposite sides; the $\overline{X}_2$ from the first nonconforming sample is above the upper control limit (UCL_2) and the $\overline{X}_1$ from the second nonconforming sample is below the lower control limit (LCL_1).

The SV rule takes into account the assumption that the assignable cause shifts the mean of the first variable, or the mean of the second variable, or simultaneously shifts the mean of both variables, moving them from their target positions to higher or lower positions; after that, the two means remain unchanged until the assignable cause is eliminated. Because of that, the SV rule does not signal when the two $\overline{X}$ values, beyond the control limits and in opposite sides, are sample means of the same variable. This signaling constraint reduces the type I error without affecting, at least significantly, the power of the control chart, that is, the type II error. The following comments are useful to prove the type I error reduces without changing, at least significantly, the type II error. When the mean of the first variable moves to a higher position, the sample means computed with observations of the first variable have unbalanced probabilities to fall above the upper control limit and to fall below the lower control limit, being much higher the probability of falling above the upper control limit. By other hand, when the mean of the first variable moves to a lower position, the samples means also have unbalanced probabilities to fall above the upper control limit and to fall below the lower control limit, being much higher the probability of falling below the lower control limit. These comments extend to the second variable.

The BV rule was proposed under the assumption that, if the two variables are affected by the assignable cause, then the assignable cause always shifts the mean of the two variables, moving them, from their target positions, to a higher (lower) position. Because of that, if a $CRL \leqslant L$ is reached, then the BV rule signals if and only if the two $\overline{X}$ values, beyond the control limits, are also on the same side of the center line, that is, both of them are above their upper control limits, or both of them are below their lower control limits. This signaling constraint, similarly to the SV case, reduces the type I error without affecting, at least significantly, the power of the control chart, that is, the type II error.

In comparison with the SV $\overline{X}$ charts, the BV $\overline{X}$ charts signal faster assignable causes that increase (decrease) the two means of the mean vector or only increase (decrease) the mean of one variable, holding the other mean unaltered.

4. The *SSARLs* of the proposed synthetic charts

The construction of the transition probabilities matrix (TP matrix) is the first step to obtain the steady-state *ARLs* (*SSARLs*) of the SV and BV $\overline{X}$ charts. The *SSARL* is given by $\mathbf{S}'(\mathbf{I}-\mathbf{R})^{-1}\mathbf{1}$, where $\mathbf{S}$ is the vector with the stationary probabilities of being in each non-absorbing state, $\mathbf{I}$ is an $(hL+1)$ by $(hL+1)$ identity matrix, $\mathbf{R}$ is the transition matrix, with the last row and last column removed, and $\mathbf{1}$ is an $(hL+1)$ by one vector of ones. If the $\overline{X}$ chart is side-sensitive with regard to the same variable (SV), $h=4$ and R is in (7), however, if the $\overline{X}$ chart is side-sensitive with regard to both variables (BV), $h=2$ and R is in (8). As was first observed by Wu and Spedding (2000), the speed with which the synthetic charts signal is slightly affected by the input parameter L; based on that, and taking into account the operational simplicity, we adopted $L=3$.

4.1. The SSARLs of the SV $\overline{X}$ charts

Table 1 presents the relation between the $\overline{X}_1$ and $\overline{X}_2$ positions on the SV charts and the sample codes. The codes of the last L samples define the transient states of the transition matrix (7). If the current state is state $(00\ldots0i)$, $(0\ldots0i0)\ldots(00i\ldots0)$, $(0i0\ldots0)$, or $(i00\ldots0)$, where $i=1$, 2, and the code of the next sample is "$\underline{i}$", the Markov chain moves to state $(0\ldots00\underline{i})$. The same way, if the current state is state $(0\ldots00\underline{i})$, $(0\ldots0\underline{i}0)\ldots(00\underline{i}\ldots0)$, $(0\underline{i}0\ldots0)$, or $(\underline{i}00\ldots0)$ and the code of the next sample is "i", the Markov chain moves to state $(0\ldots00i)$. Table 2 presents the probabilities of the transition matrix (7).

$$
\begin{array}{c|cccccccccccccccccc}
 & 100\ldots0 & \cdots & 0\ldots010 & 0\ldots001 & \underline{1}00\ldots0 & \cdots & 0\ldots0\underline{1}0 & 0\ldots00\underline{1} & 0\ldots0 & 0.00\underline{2} & 0\ldots0\underline{2}0 & \cdots & \underline{2}00\ldots0 & 0\ldots002 & 0\ldots020 & \cdots & 200\ldots0 & \text{Signal} \\
100\ldots0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & L_1 & C & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & B_1 \\
010\ldots0 & C & \cdots & 0 & 0 & 0 & \cdots & 0 & L_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & B_1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0\ldots001 & 0 & \cdots & C & 0 & 0 & \cdots & 0 & L_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & B_1 \\
\underline{1}00\ldots0 & 0 & \cdots & 0 & U_1 & 0 & \cdots & 0 & 0 & C & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & D_1 \\
0\underline{1}0\ldots0 & 0 & \cdots & 0 & U_1 & C & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & D_1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0\ldots00\underline{1} & 0 & \cdots & 0 & U_1 & 0 & \cdots & C & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & D_1 \\
0\ldots0 & 0 & \cdots & 0 & U_1 & 0 & \cdots & 0 & L_1 & C & L_2 & 0 & \cdots & 0 & U_2 & 0 & \cdots & 0 & A \\
0\ldots00\underline{2} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & C & \cdots & 0 & U_2 & 0 & \cdots & 0 & D_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0\underline{2}0\ldots0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & C & U_2 & 0 & \cdots & 0 & D_2 \\
\underline{2}00\ldots0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & C & 0 & 0 & \cdots & 0 & U_2 & 0 & \cdots & 0 & D_2 \\
0\ldots002 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & L_2 & 0 & \cdots & 0 & 0 & C & \cdots & 0 & B_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
020\ldots0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & L_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & C & B_2 \\
200\ldots00 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & C & L_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & B_2 \\
\text{Signal} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1
\end{array}
\tag{7}
$$

The **S** vector is obtained by solving the system of linear equations $\mathbf{S}'\mathbf{R_{adj}} = \mathbf{S}$, constrained to $\mathbf{S}'\mathbf{1} = 1$. The matrix $\mathbf{R_{adj}}$ is an adjusted version of **R**, where the probabilities in each row are divided by the complement of the last row probability, that is, the one in the "Signal" column; after that, the last row and the last column are removed. It follows that $S = d^{-1}\mathbf{N}$, where $d = 4f^L + 4\sum_{i=1}^{L} f^{i-1}(1-g)$, and $\mathbf{N} = [f^{L-1}(1-g), f^{L-2}(1-g), \ldots, f^0(1-g), f^{L-1}(1-g), f^{L-2}(1-g), \ldots, f^0(1-g), 4f^L, f^0(1-g), \ldots, f^{L-2}(1-g), f^{L-1}(1-g), f^0(1-g), \ldots, f^{L-2}(1-g), f^{L-1}(1-g)]$. When the process is in control $B_1 = B_2 = D_1 = D_2$, $f = C/(1-B_1)$ and $g = C/(1-A)$.

4.2. The SSARLs of the BV $\overline{X}$ charts

Table 3 presents the relation between the $\overline{X}_1$ and $\overline{X}_2$ positions on the BV charts and the sample codes. The codes of the last L samples define the transient states of the transition matrix (8). If the current state is state $(00\ldots01)$, $(0\ldots010)\ldots(001\ldots0)$, $(010\ldots0)$, or $(100\ldots0)$, and the code of the next sample is $\underline{1}$, the Markov chain moves to state $(0\ldots00\underline{1})$. The same way, if the current state is state $(0\ldots00\underline{1})$, $(0\ldots0\underline{1}0)\ldots(00\underline{1}\ldots0)$, $(0\underline{1}0\ldots0)$, or $(\underline{1}00\ldots0)$ and the code of the next sample is "1", the Markov chain moves to state $(0\ldots001)$. Table 4 presents the probabilities of the transition matrix (8).

$$\begin{array}{c|ccccccccccccc}
 & \underline{1}00\ldots0 & 0\underline{1}0\ldots0 & \ldots & 0\ldots0\underline{1}0 & 0\ldots00\underline{1} & 00\ldots00 & 0\ldots001 & 0\ldots010 & \ldots & 010\ldots0 & 100\ldots0 & \text{Signal} \\
\underline{1}00\ldots0 & 0 & 0 & \ldots & 0 & 0 & C & U & 0 & \ldots & 0 & 0 & B \\
0\underline{1}0\ldots0 & C & 0 & \ldots & 0 & 0 & 0 & U & 0 & \ldots & 0 & 0 & B \\
00\underline{1}\ldots0 & 0 & C & \ldots & 0 & 0 & 0 & U & 0 & \ldots & 0 & 0 & B \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0\ldots00\underline{1} & 0 & 0 & \ldots & C & 0 & 0 & U & 0 & \ldots & 0 & 0 & B \\
00\ldots00 & 0 & 0 & \ldots & 0 & L & C & U & 0 & \ldots & 0 & 0 & A \\
0\ldots001 & 0 & 0 & \ldots & 0 & L & 0 & 0 & C & \ldots & 0 & 0 & D \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
001\ldots0 & 0 & 0 & \ldots & 0 & L & 0 & 0 & 0 & \ldots & C & 0 & D \\
010\ldots0 & 0 & 0 & \ldots & 0 & L & 0 & 0 & 0 & \ldots & 0 & C & D \\
100\ldots0 & 0 & 0 & \ldots & 0 & L & C & 0 & 0 & \ldots & 0 & 0 & D \\
\text{Signal} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{array} \quad (8)$$

When the process is in control $B = D$, and the stationary probability vector is given by $\mathbf{S} = d^{-1}\mathbf{N}$, with $[N = f^{L-1}(1-g), f^{L-2}(1-g), \ldots, f^0(1-g), 2f^L, f^0(1-g), \ldots, f^{L-2}(1-g), f^{L-1}(1-g)]$, and $d = 2f^L + 2\sum_{i=1}^{L} f^{(i-1)}(1-g)$, being $f = C/(1-B)$ and $g = C/(1-A)$.

5. Comparing the synthetic charts

According to Davis and Woodall (2002) the proper parameter to measure the performance of a synthetic chart is the steady-state average run length (*SSARL*), that is, the *ARL* value obtained when the process remains in-control for a long time before the occurrence of the assignable cause. When the process is in-control, the *SSARL* measures the rate of false alarms. A chart with a larger in-control *SSARL* ($SSARL_0$) has a lower false alarm rate than other charts. A chart with a smaller out-of-control *SSARL* has a better ability to detect process changes than other charts. The in-control *SSARL* is an input parameter, and the chart's parameter k is the adjusting parameter to obtain to the desired in-control *SSARL*, see Machado and Costa (2014).

There are other parameters to measure the performance of the control charts. For example, Reynolds and Stoumbos (2004) and Wu, Yang, Jiang, and Khoo (2008) used the Extra Quadratic Loss (*EQL*) to measure and compare the performance of the charts. The smaller the *EQL* value of a chart, the better the overall performance of the chart in the detection of process changes. The *EQL* was used in the literature only for the univariate case. We adapted the *EQL* for the multivariate case (*MEQL*):

$$MEQL = \frac{1}{d_{max} - d_{min}} \int_{d_{min}}^{d_{max}} E^2 SSARL(E) dE \quad (9)$$

where E^2 is the Euclidian distance based on δ_1 and δ_2; $d_{max} = 4.5 (d_{min} = 0.25)$ is the square root of the maximum (minimum) Euclidian distance; $SSARL(E)$ is the steady state *ARL* computed with δ_1 and δ_2, being $E^2 = \delta_1^2 + \delta_2^2$.

Tables 5 and 6 present the *SSARL*s and the *MEQL* of the synthetic T^2 chart (Syn T^2 chart), and the synthetic $\overline{X}$ charts with the SV and with the BV rules (SV $\overline{X}$ and BV $\overline{X}$ charts) for $ARL_0 = 370.4$, $n = 2$ or 5 and $L = 3$. The *SSARL* values of the best strategy are in bold. We considered variables with low ($\rho = 0.3$), moderate ($\rho = 0.5$) and high correlation ($\rho = 0.7$). Table 5 considers the cases where the two variables have the same level of autocorrelation, that is, $a = b$, and Table 6 considers the cases where $a \neq b$.

From Table 5, when the correlation is low or moderate ($\rho = 0.3$ or 0.5), the BV charts has the best overall performance, that is, the lowest *MEQL*. However, considering the particular case where only one variable is affected by the assignable cause, the Syn T^2 chart is faster than the Syn $\overline{X}$ charts. For instance, if $(a, b, \delta_1, \delta_2, n) = (0.7, 0.7, 0, 1.5, 2)$, and the variables are moderate correlated ($\rho = 0.5$), the Syn T^2 chart requires, on average, 8.06 samples to signal. The correspondent *SSARL* of the BV and SV charts are 13.45 and 13.77, respectively. When the two variables are high correlated ($\rho = 0.7$), the Syn T^2 chart has the lowest *MEQL*, that is, the high correlation enhances the Syn T^2 chart's overall performance.

However, considering the particular case where $\delta_1 = \delta_2$ or $\delta_1 \geqslant 1.0$ and $\delta_2 \geqslant 1.0$, the BV charts are the best option. For instance, if $(a, b, \delta_1, \delta_2, n) = (0.7, 0.7, 1.0, 1.5, 2)$, and the variables are high correlated ($\rho = 0.7$), the BV charts require, on average, 11.49 samples to signal. The correspondent *SSARL* of the Syn T^2 chart and the SV charts are 12.28 and 11.85, respectively. The charts' performance always improves with larger samples.

From Table 6, when the correlation is low ($\rho = 0.3$), the Syn $\overline{X}$ charts have a better overall performance. Figs. 3 and 4 present the contour plots for $n = 2$ and $\rho = 0.3$ and 0.7. According to these Figures, the BV charts perform better than the Syn T^2 chart when $\delta_1 \cong 0.5$ and $\delta_2 \cong 0.5$.

6. Illustrative example

Leoni et al. (in press) give an interesting example of a bivariate process with autocorrelated data. In their example, X_1 and X_2 are the amount of milk injected in milk containers of 990 mL by a filling machine with two filling heads. With the aim to illustrate the control of such processes with the SV and BV charts, let $(Z_i = (\overline{X}_i - \mu_{0i})/\zeta_i,\ i = 1, 2)$ be the standardized mean vector from samples with observations fitting to the autoregressive model described in Section 2. According to Table 5, if $L = 3$, $\rho = 0.7$, $a = b = 0.5$ and the signaling rule is the SV rule, then the k value that leads to an in-control *SSARL* of 370.4 is 1.724; switching to the BV rule the width coefficient k slightly decreases to 1.713. Figs. 5 and 6 illustrate, respectively, the joint $\overline{X}$ charts with the SV and with BV rules. The Z_1 and Z_2 values from 7 samples of size 5 are in Table 7.

The SV chart is depicted in Fig. 5 with a signal given by two nonconforming samples; the first one with a Z_2 value above the upper control limit, and the second one with a Z_1 value below the lower control limit. This kind of signal suggests that the two variables were affected by the assignable cause, that is, the mean of X_1 decreased and the mean of X_2 increased. The BV chart is depicted in Fig. 6 with a signal given by two nonconforming samples; now the first one and also the second one with Z_2 values above the upper control limit. This kind of signal suggests that only the second variable was affected by the assignable cause.

7. Conclusions

The aim of the proposed side-sensitive $\overline{X}$ charts was to offer a simpler charting method to control bivariate processes with autocorrelated data. The overall conclusion is that the side-sensitive

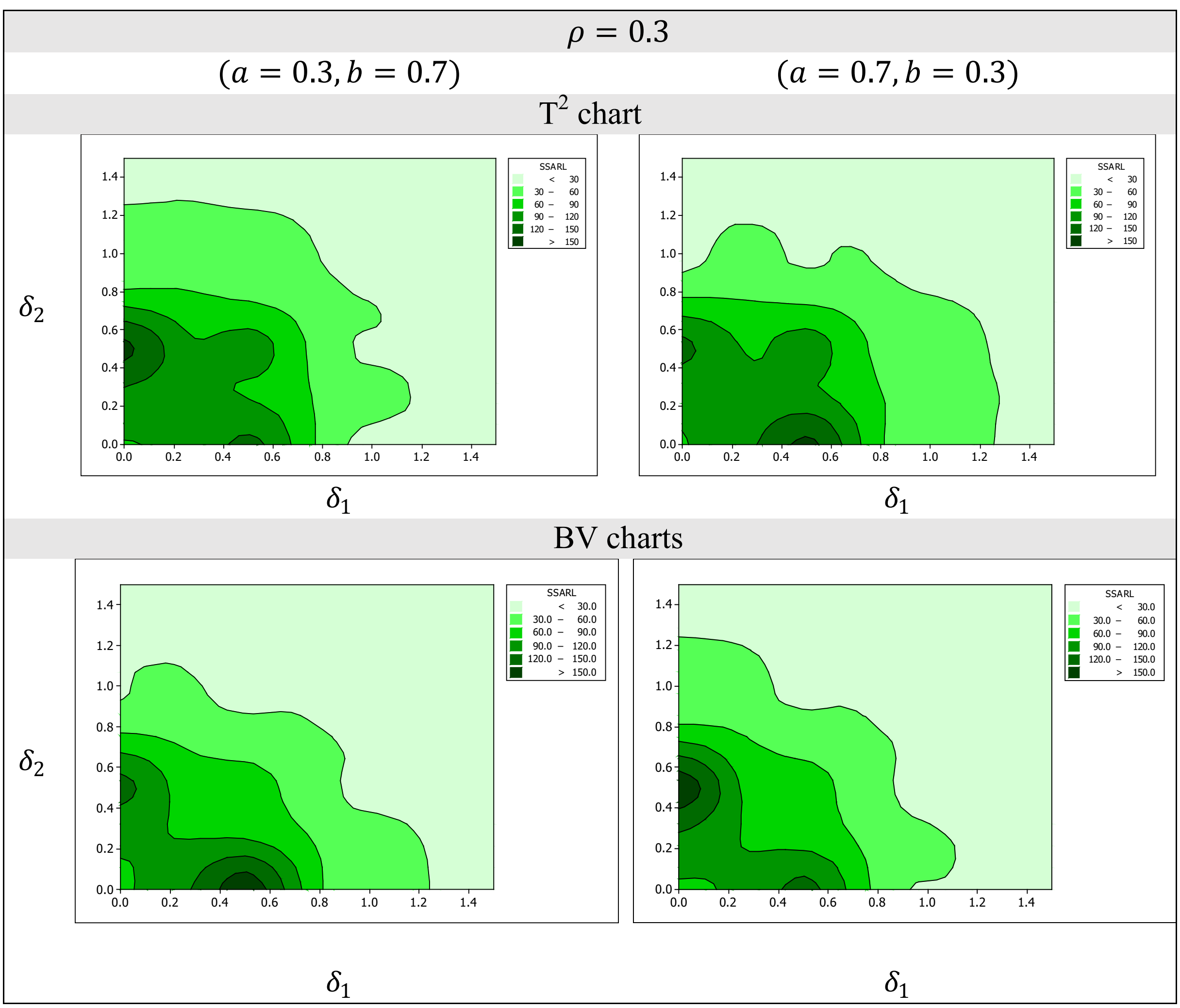

Fig. 3. The contour plots of *SSARL* with $n = 2$ and $\rho = 0.3$.

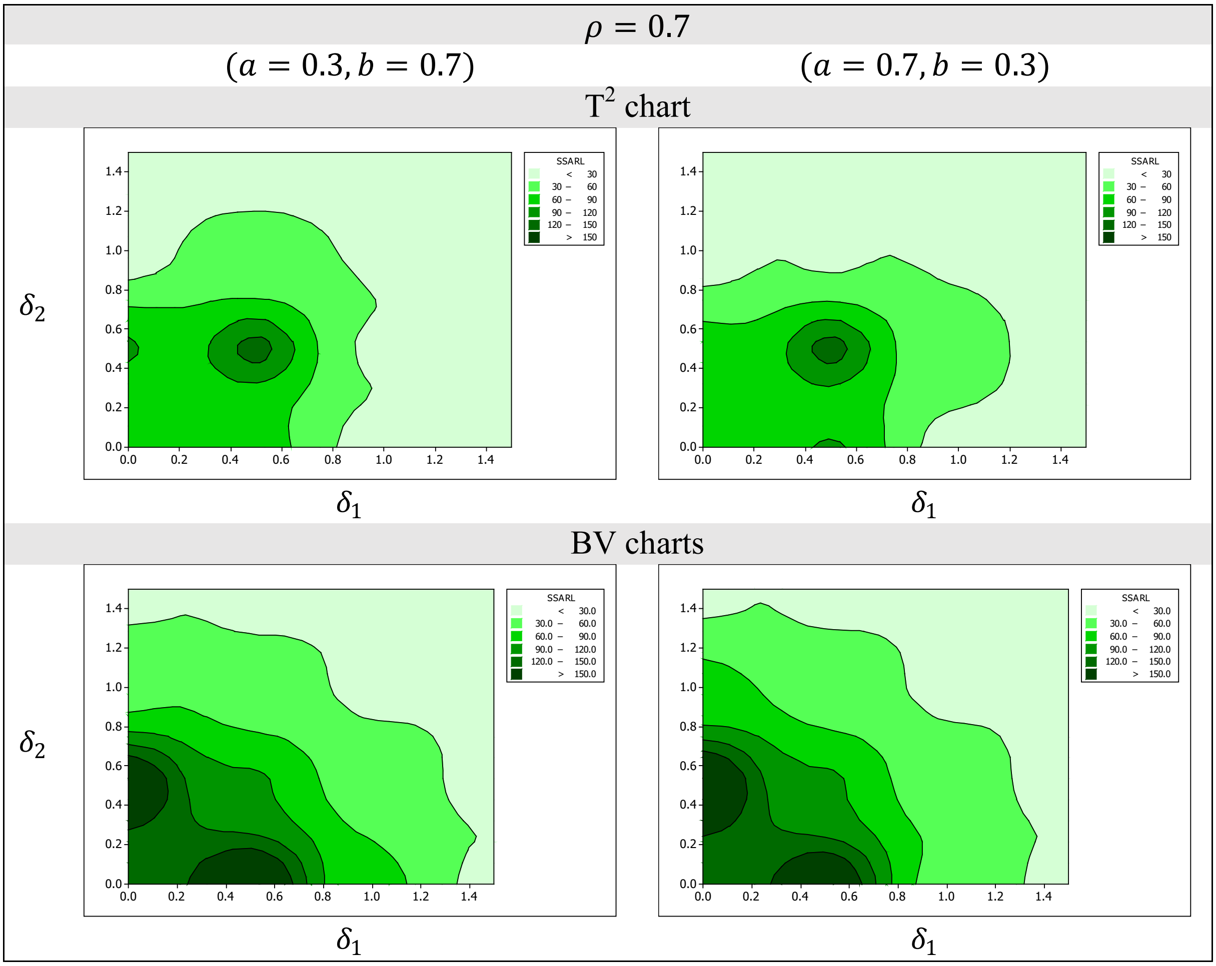

Fig. 4. The contour plots of *SSARL* with $n = 2$ and $\rho = 0.7$.

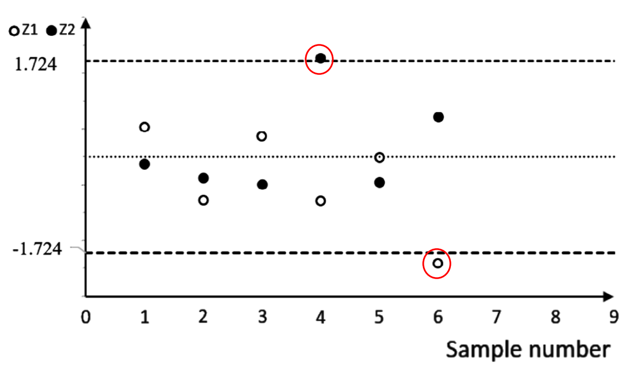

Fig. 5. The joint $\overline{X}$ charts with the SV rule.

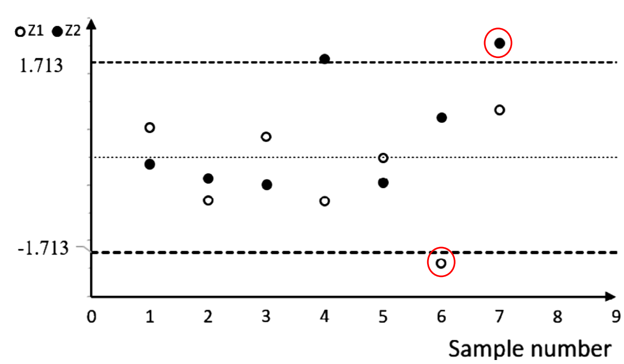

Fig. 6. The joint $\overline{X}$ charts with the BV rule.

rules reduce the delay with which the $\overline{X}$ charts signal an out-of-control condition. Consequently, the side-sensitive rules restore part of the lost performance of the $\overline{X}$ charts due to the autocorrelation. The BV $\overline{X}$ charts are the best option in terms of simplicity and overall performance, except when the variables are high correlated. The correlation has a negative effect on the performance of the side-sensitive $\overline{X}$ charts; their competitiveness with the synthetic T^2 chart is drastically affected by highly correlated variables.

Table 7
The standardized sample means.

Sample number	Z_1	Z_2	Sample	*CRL* (SV rule)	*CRL* (BV rule)
1	0.5377	−0.1092	Conforming	1	1
2	−0.7767	−0.3688	Conforming	2	2
3	0.3741	−0.4808	Conforming	3	3
4	−0.7776	1.7879	Nonconforming	4	4
5	−0.0025	−0.4476	Conforming	1	1
6	−1.9027	0.7192	Nonconforming	Signal	2
7	0.8591	2.0685	Nonconforming		Signal

Falls above (below) the upper (lower) warning region.

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