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Exploring the Moon gravity to escape from the Earth–Moon system

S. H. S. Santana[1] · C. F. De Melo[2] · E. E. N. Macau[1] · O. C. Winter[3]

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Abstract Escape trajectories from the Earth–Moon system can be obtained through a single gravity assist maneuver of a spacecraft with the Moon. This paper presents a semi-analytical study of variations in the semi-major axis and the eccentricity of a spacecraft for a geocentric frame—before and after a close encounter of the spacecraft with the Moon. Studies on the swing-by dynamics between a spacecraft and the Moon were performed in order to establish a set of initial conditions in terms of eccentricities and semi-major axes of geocentric initial orbits. This way, it was possible identify which orbits, launched from Earth parking orbits, accomplish swing-bys with the Moon and generate escape trajectories.

Keywords Swing-by · Escape trajectories · Numerical simulations

Mathematics Subject Classification Primary 06B10; Secondary 06D05

1 Introduction

In astronautics, a gravity assist maneuver (also called as swing-by or fly-by) consists in the use of the gravitational field of a celestial body (planet or moon) to transfer energy for a spacecraft and to change its velocity and consequently its trajectory. This technique is quite effective when a mission aims to save fuel. The relevance of the swing-by maneuvers can be seen taking account the number of space missions that have already been launched or

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✉ S. H. S. Santana
saymonhss@gmail.com

1 Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, Brazil

2 Universidade Federal do ABC (UFABC), Santo André, Brazil

3 Universidade Estadual Paulista (UNESP), Guaratinguetá, Brazil

are scheduled to fly making use of this strategy. For example, missions such as Mariner-10 (Dunne and Burgess 1978), Voyager (Kholhase and Penzo 1977), Rosetta (Baillon et al. 1999) and New Horizons (Guo and Farquhar 2008) made use of this technique to gain energy and to achieve their goals.

The dynamics of swing-by has been studied since Laplace described its mechanism in the early nineteenth century (Laplace 1805). Subsequently, several analytical and numerical studies (Lawden 1954; Broucke 1988; Minovitch 1961; Downling et al. 1990, 1991) increased the knowledge on swing-by and they allowed many applications in designing of interplanetary trajectories.

In this work, a semi-analytical study on the changes in the geocentric orbital elements (semi-major axes and eccentricities) of spacecraft trajectories due to single close encounter with the Moon is presented. This analysis is made building a grid of initial conditions (semi-major axis and eccentricity) relative to the Earth at the instant that the spacecrafts are launched from an Earth parking orbit. For each initial condition, a final one (also in terms of semi-major axis and eccentricity) relative to the Earth after the spacecraft to accomplish a swing-by with the moon is obtained, and other grids are built. These new grids show how semi-major axes and eccentricities of the final geocentric orbits (after swing-by) vary as a function of the same quantities of the initial geocentric orbit.

This paper is organized as follows: Sect. 2 presents the basic mathematical formulation on the problem, and it will serve as input for the studied cases. Changes in semi-major axes and eccentricities for the different cases studied are shown in Sect. 3. Section 4 discusses the main results of this study, as well as the regions in the grids eccentricity versus semi-major axis where escape trajectories are observed. Conclusions and final comments are given in Sect. 5.

2 Basic mathematical formulation

Broucke (1988) studied space missions involving swing-by maneuvers considering a model in which a simple mission is divided in a number of two-body problems. Following this same assumption, we consider in this paper a system with three bodies, M_1, M_2 e M_3. M_1 is a massive body in the origin of a reference frame. M_2 body ($M_2 < M_1$) is in a Keplerian orbit around M_1. M_3 ($M_3 \ll M_1$) is a spacecraft also in a Keplerian orbit around M_1 and makes a close approach with M_2 resulting in a change of its initial orbit. A system formed by the Earth, the Moon and a spacecraft is a good example.

Figure 1 shows a representation of a swing-by maneuver based in the system with three-body described above. This representation is adapted from de Almeida Prado (2001), Ref. de Almeida Prado (2001). Figure 1a shows M_3 in an arc of a conic relative to M_2 during the maneuver. $\mathbf{V}_2$ is the velocity of M_2 relative to M_1, $\mathbf{V}_\infty^-$ and $\mathbf{V}_\infty^+$ are the velocity of M_3 before and after his encounter with M_2, respectively. r_p is the smallest distance of M_3 relative to M_2. The angular variables involved are δ, called angle of curvature, and the angle Ψ between M_1–M_2 line and the periapse line, and they are shown in more detail in Fig. 1b.

The mathematical description of a swing-by maneuver consists in dividing the space into parts, and by associating a sphere of influence for each body involved. At the first stage, the gravitational influence of M_2 is disregarded and, consequently, M_3 is assumed to be in a Keplerian orbit around M_1. At second stage, M_3 penetrates in the sphere of lunar influence with velocity V_∞^- relative to M_2, and from this moment M_3 describes a conic orbit around M_2, in general, a hyperbole. The gravitational field of M_2 acts on M_3 and deflecting it so

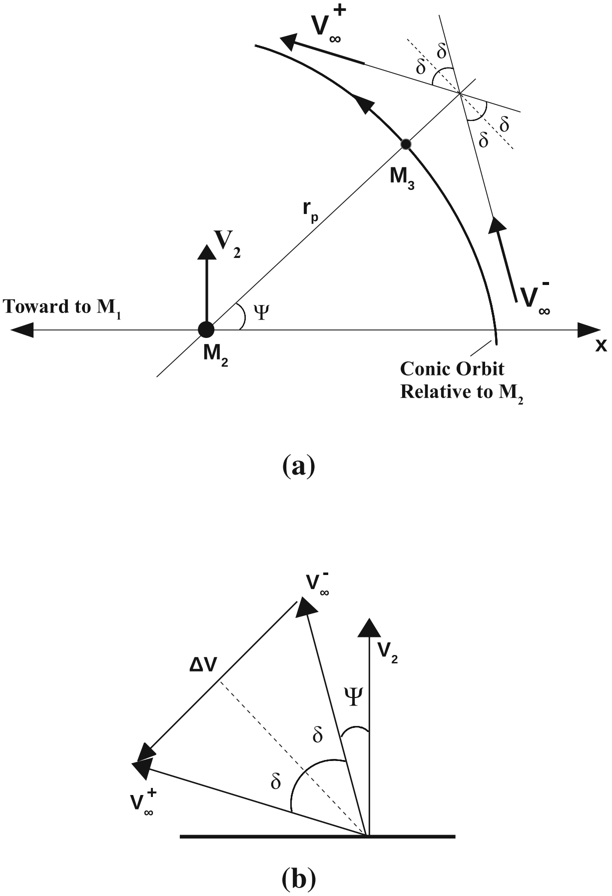


Fig. 1 **a** A scheme of a swing-by maneuver between M_3 and M_2. **b** Diagram of velocities and angular variables involved in the maneuver. These figures were adapted from de Almeida Prado (2001)

that its escape velocity is V_∞^+ relative to M_2. As a result, M_3 will have a new Keplerian orbit around M_1.

This work analyzes the resulting orbit of M_3 around M_1 after the swing-by maneuver. According to the diagram shown in Fig. 1b the magnitude of velocity variation relative to M_1, can be written as

$$\Delta V = 2V_\infty \sin\delta \tag{1}$$

The variation in the magnitude of the angular momentum, **C**, and in the specific energy, E, both also relative to the Earth, are given, respectively, by

$$\Delta C = -\frac{2V_2 V_\infty \sin\delta \sin\Psi}{\omega} \tag{2}$$

$$\Delta E = -2V_2 V_\infty \sin\delta \sin\Psi, \tag{3}$$

where ω is the angular velocity of M_2 around the baricenter of M_1–M_2 system.

Since $0^\circ < \delta < 90^\circ$, $\sin\delta$ is always positive, the only parameter that can imply a signal change of ΔE is Ψ. For $0^\circ < \Psi < 180^\circ$ the swing-by occurs in front of M_2 (in the first or second quadrant of the a reference frame fixed in M_2) and the energy decreases so that the maximum decrease occurs for $\Psi = 90°$. In the range $180° < \Psi < 360°$ the swing-by occurs behind M_2 (in the third or fourth quadrant of a reference frame fixed in M_2) and the energy increases so that its maximum value takes place for $\Psi = 270°$.

Our analysis consists in calculating the variation of two-body (Earth–spacecraft) energy when the spacecraft (M_3) in a Keplerian orbit relative to the Earth (M_1) penetrates into the lunar sphere of influence and accomplishes a swing-by with the Moon (M_2). This general methodology is the same adopted in the Refs. Broucke (1988), de Almeida Prado (2001) and Araujo et al. (2012), for instance. In our specific investigation, a suitable close encounter (a swing-by) between a spacecraft and the Moon can increase the orbital energy of the first relative to the Earth becoming it positive, for example. In other words, this maneuver can provide as result an escape trajectory from the Earth–Moon system.

The energy gain relative to the Earth, given by Eq. (3), is also observed in scenarios where the orbits are gotten by the dynamics of the restricted three-body problem (RTBP), for example. Figure 2a shows a typical escape trajectory predicted by the RTBP Earth–Moon particle. This trajectory is derived from the periodic orbits around the Lagrangian equilibrium point L1 (Broucke 1968), and its energy gain due to the swing-by with the Moon can be obtained considering the Eqs. (1) and (3). Figure 2b shows the variation in the two-body (Earth–spacecraft) energy as function of the time. Note that the spacecraft leaves the Earth's neighborhood; and about 14 days later it has a close encounter with the Moon, and its energy relative to the Earth increases enough to reach a positive value. As a result the spacecraft escapes from the Earth–Moon system.

3 Results

Our goal is to seek for a set of initial conditions of type (a_0, e_0), where a_0 and e_0 are the initial semi-major axis and the eccentricity of an initial geocentric orbit that generates a close encounter with the Moon. The other orbital elements of the initial geocentric orbit are considered: inclination, i_0, equal to 0 degrees, longitude of the ascending node, Ω_0, equal to 0 degrees, argument of the perigee, ω_0, equal to 180 degrees and the time of perigee passage equal to 0. Figure 3 illustrates an example of an initial geocentric orbit. Note that they are tangent to Earth parking orbits, and at the instant in which the spacecrafts are launched, a ΔV is need to generate them. We are also considered in our simulations: $\mu_1 = GM_1 = 398479.14\,\mathrm{km^3 s^{-2}}$ and $\mu_2 = GM_2 = 44888.44\,\mathrm{km^3 s^{-2}}$ where $G = 6.67\times10^{-20}\,\mathrm{kg\,km^3 s^{-2}}$ is the Universal Gravitational Constant.

From the geometry shown in Fig. 1b

$$\sin\delta = \left[1 + \frac{r_p V_\infty^2}{\mu_2}\right]^{-1} \tag{4}$$

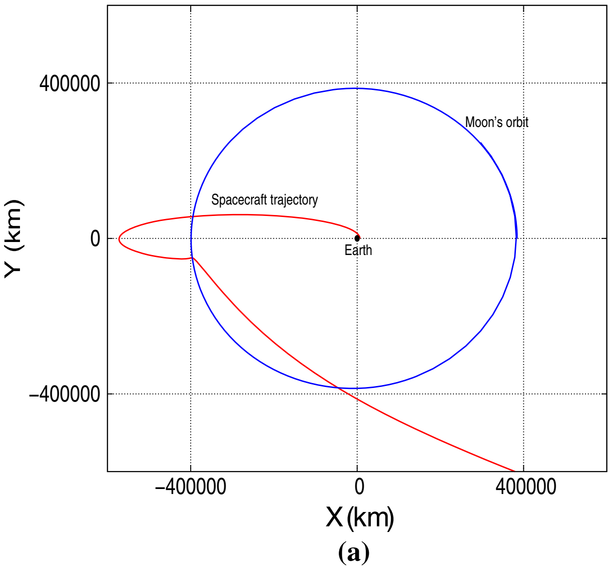
Therefore, from Eq. (1), the magnitude of the velocity variation can be written as

$$\Delta V = \frac{2V_\infty}{1 + \frac{r_p V_\infty^2}{\mu_2}} \tag{5}$$

Hence, the maximum value for ΔV is obtained solving the equation

$$\frac{\partial}{\partial V_\infty}(\Delta V) = 0 \tag{6}$$

taking r_p constant.



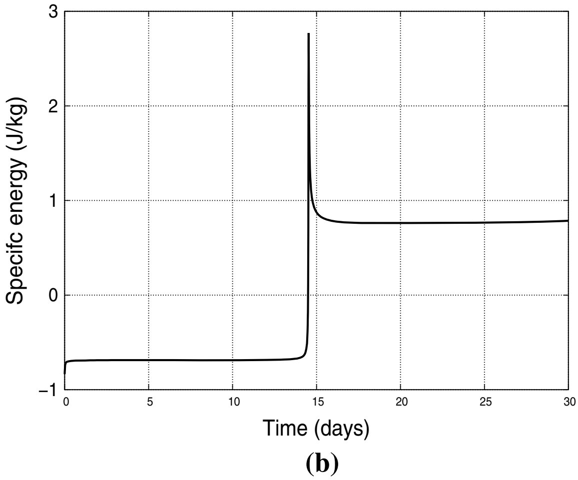


Fig. 2 **a** A typical escape trajectory predicted by the RTBP and whose escape occurs after a close encounter with the Moon, seem in a geocentric reference frame. **b** Earth–spacecraft two-body energy as a function of the time

Applying this condition, it is possible to find the maximum value for V_∞, It is the same value that has a spacecraft in circular orbit around M_2 with radius r_p, that is

$$V_\infty|_{\text{MAX}} = \sqrt{\frac{\mu_2}{r_p}} \tag{7}$$

As we are looking for the greatest possible energy variation, the result in Eq. (7) must be considered in (4) and (5). Thus, Eq. (3) can be re-written as

$$\Delta E|_{\text{MAX}} = -V_2 \left(V_\infty|_{\text{MAX}} \right) \sin \Psi = -V_2 \left(\sqrt{\frac{\mu_2}{r_p}} \right) \sin \Psi. \tag{8}$$

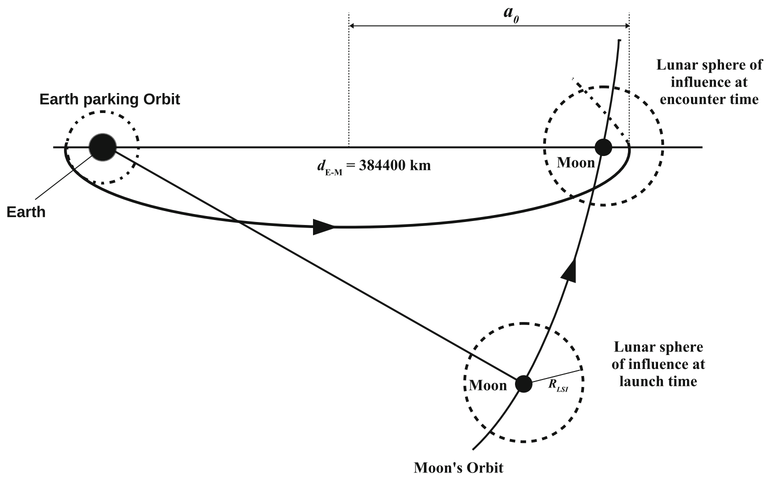


Fig. 3 Typical orbit with semi-major axis a_0 passing through the lunar sphere of influence

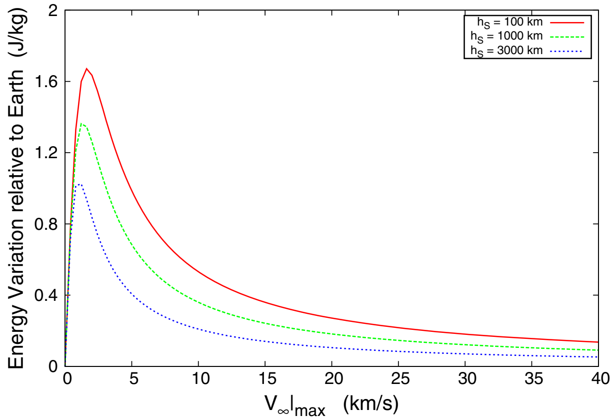


Fig. 4 Variation in spacecraft two-body energy relative to de Earth, ΔE, as a function of the $V_\infty|_{\text{MAX}}$ for different safety altitudes above the Moons surface, h_S

As predicted by Eq. (7), ΔE will reach the maximum value, $\Delta E|_{\text{MAX}} = 5.199$ J/kg, for the smallest value of r_p ($r_p|_{\text{MIN}} = 1738$ km), which corresponds to the medium radius of the Moon. In order to study the design of missions, or, in our case, the definition of a set of initial conditions that generates escape trajectories from the Earth–Moon system, it is suitable to introduce a safety altitude, h_S, to analysis ΔE. Thus, $r_p = 1738$ km$+h_S$. As example, we considered $V_2 = 1.023$ km/s (Moon's velocity around the Earth assuming circular orbit), $w = 2.663811 \times 10^{-6}$ rad/s as it angular velocity around the Earth–Moon baricenter, and $h_S = 100$, 1000 and 3000 km to show ΔE as a function of the $V_\infty|_{\text{MAX}}$ in Fig. 4.

Assuming that the two-body energy and the angular momentum of the final orbit of a spacecraft (M_3) final orbit (after the swing-by) relative to the Earth are $E_f = E_0 + \Delta E$ and $C_0 = C_i + \Delta C$, respectively. Where $E_0 = -\mu_1/2a_0$ is the initial value of energy and $C_0 = \sqrt{\mu_1 a_0(1 - e_0^2)}$ is the initial value of the angular momentum, (both before the encounter), as well as ΔE and ΔC are given by (8) and (2), respectively. So, it is possible to estimate the final semi-major, a_f, and the final eccentricity, e_f, such that.

$$a_f = -\frac{\mu_1}{2E_f} \tag{9}$$

and

$$e_f = \sqrt{1 - \frac{C_f^2}{a_f \mu_1}}. \tag{10}$$

In order to evaluate the effects of a simple close encounter between a spacecraft and the Moon, a set of initial conditions, in terms orbital elements $(a, e, i, \omega, \Omega, \tau) = (a_0, e_0, 0, 180°, 0, 0)$ of an initial geocentric orbit, was taken at the instant of departure from an Earth's initial parking orbit. These elements are considered constant until the moment when the spacecraft penetrates in the lunar sphere of influence.

A grid in terms of (a_0, e_0) was taken with 300 equally spaced steps, such $d_{E-M} - 1.5R_{LSI} \leq a_0 \leq d_{E-M} + 1.5R_{LSI}$, where $d_{E-M} = 384400$ km is the medium Earth–Moon distance and $R_{LSI} = 66181$ km is the radius of the lunar sphere of influence. For each value of a_0 in the adopted range, it was considered 100 equally spaced steps for e_0 in the range $0 \leq e_0 \leq 0.99$. Three safety altitudes above the Moons surface were considered: $h_S = 100$, 1000 and 3000 km. The results are shown in Figs. 5 and 6.

4 Result analysis

The first implication of the general method used in this study is the relation between the spacecraft energy gain relative to the Earth, ΔE, the smallest Moon–spacecraft distance, r_p, and the approach velocity, V_∞, in a swing-by maneuver, as predicted by Eqs. (7) and (8). For each specific smallest Moon–spacecraft distance, there is a V_∞ value that provides a maximum energy gain for a spacecrafts orbit. As a result, escape trajectories can be generated, and this can be observed in diagrams of Figs. 5 and 6.

Diagrams of Fig. 5 show that the initial conditions (a_0, e_0) that generate escape trajectories from the Earth–Moon system are above the line dividing the yellow and black bands. Thus, escapes can occur for any value of a_0 in the range considered, but only for e_0 values above this line. In general, it is also possible to conclude that how much above of this line, larger is the energy gain relative to the Earth, ΔE, with the swing-by between the spacecraft and the Moon. These characteristics can be observed in the three diagrams. However, as said in the anterior paragraph, it is also observed the major energy variations relative to the Earth occur for the less value of h_S (or r_p).

Diagrams of Fig. 6 show that the e_f values are dependents of a_0 e e_0. And just as in diagrams of Fig. 5, the highest values of e_f occur for the less value of h_S. However, the e_f values present dependence with the values of a_0 and e_0. It is also possible to observe in these diagrams which initial conditions $(a_0, e_0, 0, \omega_0 = 180°, 0, 0)$ generate escape trajectories from the Earth–Moon system. They are above the line $e_f = 1$ in the all diagrams of Fig. 6.

Matching the result of Figs. 5a and 6a, it is possible to conclude that the best swing-by maneuver between a spacecraft and the Moon in terms of ΔE will be obtained for $a_0 = 1.25d_{E-M}$ (or 480,500 km) and $e_0 = 0.98$, which corresponds to a hyperbole with eccentricity, e_f, equal to 3.5.

The white regions in Figs. 5 and 6 represent initial conditions $(a_0, e_0, 0, \omega_0 = 180°, 0, 0)$ for which a close encounter with the Moon it is not possible. In the lower left region are the cases for which the apogees of the initial orbits, r_{A0}, are internal to the Moons orbit $(r_{A0} < d_{E-M} - R_{LSI})$. While the low right region in the diagrams are the initial conditions

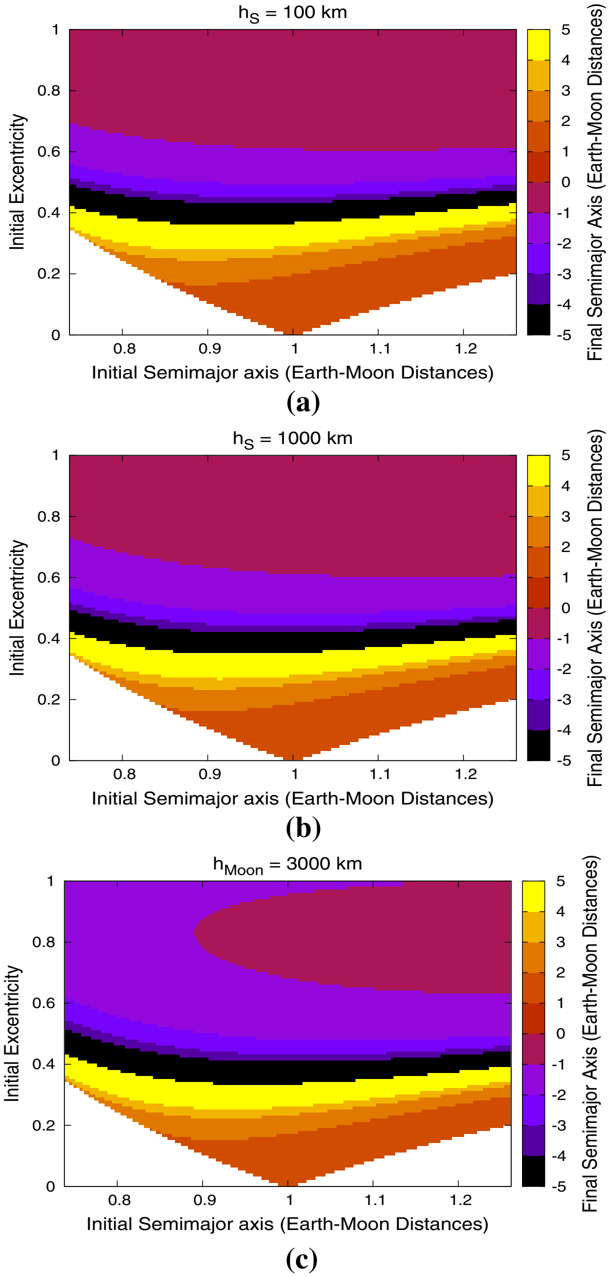


Fig. 5 Diagrams of the final geocentric semi-major axes in terms of (a_0, e_0) for a spacecraft after a swing-by with the Moon. **a** $h_S = 100$ km, **b** $h_S = 1000$ km *e* **c** $h_S = 3000$ km

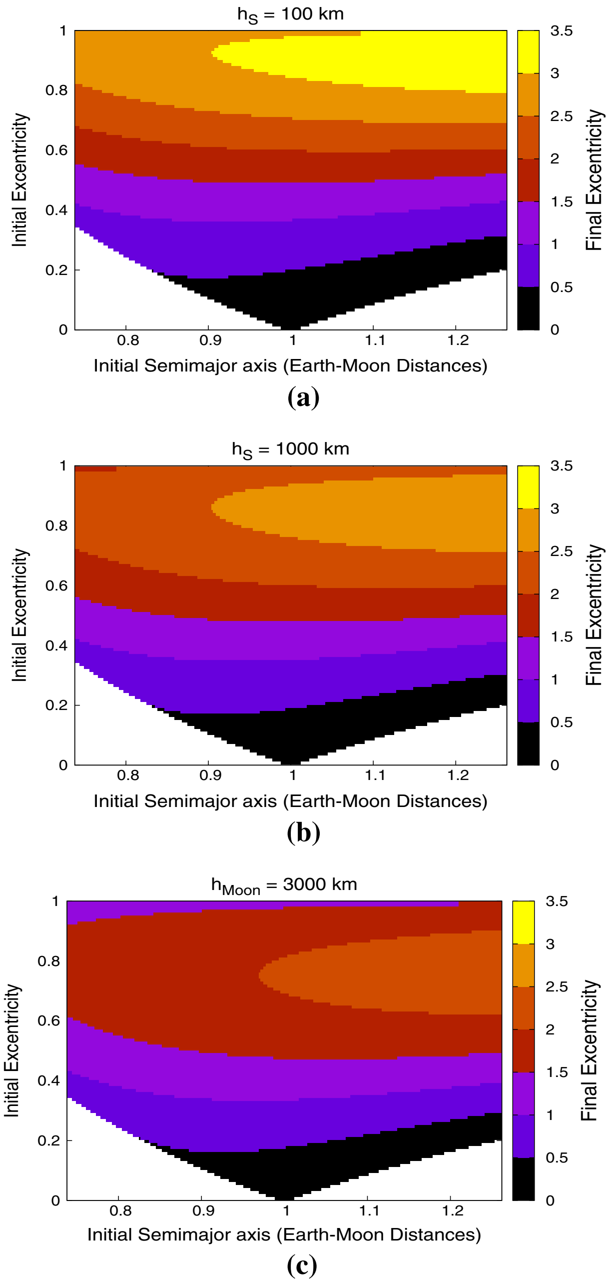


Fig. 6 Diagrams of the final geocentric eccentricity of a spacecraft orbit in terms of (a_0, e_0) after a swing-by with the Moon. **a** h_S = 100 km, **b** h_S = 1000 km *e* **c** h_S = 3000 km

$(a_0, e_0, 0, \omega_0 = 180°, 0, 0)$ that generate geocentric orbits that do not intercept the lunar sphere of influence and, therefore, they do not accomplish any encounter with the Moon.

5 Conclusions

This work presents a semi-analytical study on the two-body (Earth–spacecraft) energy, semi-major axes and eccentricities of final geocentric orbits obtained after close encounters between a spacecrafts and the Moon.

Studies on the dynamics of the swing-by between the spacecrafts and the Moon were performed in order to establish a set of initial conditions of geocentric orbits, launched from Earth's parking orbit, to accomplish swing-bys with the Moon and gain sufficient energy to escape from the Earth–Moon system, as shown in Figs. 5 and 6.

As future works, we can glimpse at the use of this dynamics to design missions to Near Earth Asteroids, for example, as well as studies on the fuel consumption associated with these maneuvers.

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