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ORIGINAL ARTICLE

Expansive movements in the development of mathematical modeling: analysis from an Activity Theory perspective

Jeannette Galleguillos[1] · Marcelo de Carvalho Borba[2]

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Abstract
This research was conducted during an online continuing education course for mathematics teachers, whose core focus was modeling and applications. We studied the interactions of one group of two teachers, who worked collectively in posing and solving a modeling problem through a closed group on the social network Facebook. The research question guiding this paper was how the development of mathematical modeling occurs from an Activity Theory perspective, recognizing tensions that occur, and its evolution in the process of posing and solving a modeling problem. The researchers took a qualitative approach, analyzing discursive manifestations in the modeling process. In the discussion, contradictions emerged in the group through events such as *dilemma* and *conflict*. The results indicate that the modeling task acted as an artifact that brought to light inner contradictions, and thus, allowed teachers to move from a *conflict* to the formulation of an open problem, and from a *dilemma* to the construction of a model and a pedagogic strategy.

Keywords Mathematics teacher education · Mathematical modelling · Cultural-historical Activity Theory · Online distance learning

1 Introduction

This work involves mathematical modeling and Activity Theory. mathematical modeling and applications generally include posing and solving problems situated in the real-world (Niss et al. 2007). However, there are diverse ways of viewing modeling. Our perspective sees modeling as a pedagogical approach in which the students are invited to help choose a topic to study and then propose a problem related to that topic (Borba and Villarreal 2005). The fact that the students choose a topic breaks with the traditional curriculum, where teachers and the school system itself are the only ones determining what will be studied in the classroom. The fact that students are involved in choosing a topic offers them an opportunity to engage in the construction of a problem and its solution. Through this process, student groups study an interesting topic taken from their own environments, and then they construct a problem and solve it. These situations are developed by means of multiple interrelationships between the students and their world, using cultural–historical artifacts to achieve their purpose. Nowadays, these social interactions are also developed through the Internet in virtual environments, where the outside world can be brought closer to students. In this way, virtual environments, through the means of information and communication, are becoming more relevant to education.

This study uses an analytic framework situated in Activity Theory (Engeström and Sannino 2011) that analyzes teachers' discussions to identify tensions or inner contradictions. The discussions were produced by mathematics teachers in a closed Facebook group focused on mathematics modeling. The present research was developed during the online extension course "Trends in mathematics education" offered to mathematics teachers-in-training. This online course has run since the year 2000 and, over the years, has used various virtual learning environments as its platform. The main characteristic of this course is that it establishes a class methodology based on active student participation through readings and discussions

✉ Jeannette Galleguillos
jeannette.galleguillos@uv.cl;
jeannette.galleguillos@gmail.com

Marcelo de Carvalho Borba
mborba@rc.unesp.br

[1] Universidad de Valparaíso, Valparaíso, Chile

[2] São Paulo State University (UNESP), Rio Claro, São Paulo, Brazil

in both synchronous and asynchronous modes, forming what Silva (2000) calls an *interactive class*. In this style of class, the teacher encourages students to raise questions that promote discussion and debate around topics covered in the course. Some results arising from research on previous versions of this course can be found in the book by Borba et al. (2010).

In 2014, this course was carried out within a closed Facebook group, addressing topics of mathematics modeling and applications. Working in small groups, participants selected a theme, then constructed and solved a problem, which, from our perspective, characterizes modeling. The present research analyzes how the mathematical modeling process occurs in this learning environment from the Activity Theory perspective, as presented by Engeström (1987). From this viewpoint, *inner contradictions* are opposite forces that produce tensions within a system, and which can trigger a transformation. Identifying these inner contradictions is therefore central to the analysis of the data, because successfully overcoming contradictions that emerge in the system produces an *expansion* situation, seen as learning or growth (Engeström 1987). This study reports the analysis of discussions of one group of two teachers in constructing and solving a modeling problem. In this analysis, we identify contradictions found in the process of mathematical modeling, and their evolution to becoming an *expansion* situation.

Others have studied Activity Theory in mathematics education. Ärlebäck (2009) designed modules on parameters of Activity Theory to introduce mathematical modeling in secondary school. Also using Activity Theory, Potari (2013) studied the relationship between theory and practice of mathematics teachers' professional development. In her work, she analyzed the activity of teaching and the activity of research, making comparisons between the two. Williams and Goos (2013) proposed Activity Theory as a theoretical framework integrating mathematical modeling and technologies. They reported an expansion situation in a mathematics class using technology. Anthony et al. (2014), also using Activity Theory, studied the nature and sustainability of a teacher's learning. In their work, they identified tensions in teachers' narratives regarding their practice, which later were transformed into an expansion situation. Souto and Borba (2016) studied the role of the Internet in an online mathematics education course, looking for expansion situations. We note that our work uses an analytical framework based on Activity Theory, a framework not previously used in mathematical modeling that includes collaborative problem-posing in an online environment.

The following sections provide details of our perspective of modeling and Activity Theory, followed by a methodology section, data analysis, and conclusions.

2 Mathematical modeling

Mathematical modeling began to be used in education in order to establish relationships between mathematics and reality (Blum 2012). In this way, modeling can help make mathematics more understandable by bringing contextualized situations to the classroom and by giving meaning to the mathematics that is taught and learned (Borba and Villarreal 2005). The international discussion of mathematics modeling shows a variety of perspectives of modeling. For example, Kaiser and Sriraman (2006) found six perspectives of modelling, namely, realistic, contextual, educational, socio-critical, epistemological, and cognitive modeling. Additional perspectives lie in other reaches of the international scenario. For example, Mexican authors such as Cordero (2006) present a socioepistemology approach that sees modeling as socially and historically bound. Also, Brazilian authors—such as Bassanezi (2002) (in Rosa and Orey 2013), Borba (2009), Caldeira (2009), and Araújo (2013)—present different twists regarding how much involvement students, teachers, and media have in the creation of the problem to be studied in the classroom.

Also, we note there are diverse ways of perceiving a problem. According to Blum and Niss (1991), a problem can be defined as "a situation which carries with it certain open questions that challenge somebody intellectually who is not in immediate possession of direct methods/procedures/algorithms etc. sufficient to answer the questions" (p. 37). These authors also indicate that this idea is relative to the person; thus, what may be a problem to one person may be an exercise for another.

The perspective of "problem" in this paper is consistent with that of Blum and Niss (1991). According to Saviani (1985) and Borba (1987), a problem has both subjective and objective components; the former involves the personal interest of the student, and the latter, a need linked to an obstacle that presents itself in a personal experience.

An example of modeling from such a perspective was presented in Borba et al. (2016). The authors show the development of this perspective with secondary school students from Argentina, where a group of students developed a topic related to the environment: the melting of glaciers. The students, after investigating the topic, proposed the problem of tracking the reduction rate of the Puncak Jaya glacier in Indonesia. Students found Internet data about the melting of this glacier and arranged a sequence of six images over an interval of 150 years. Then, with the help of Geogebra, they made a representation of the glacier's area according to the sequence of images, and then calculated it. Subsequently, they calculated the variation of the area over the years. From the data obtained, the students indicated that the glacier under analysis should have already disappeared, which was later verified by new information found on the Internet.

When students choose a theme and from it pose a problem and solve it, as in the previous example, they naturally take on an active role by engaging in the situation they pose, thus developing their creativity and ensuring their comprehension of the situation.

3 Activity theory

Cultural–historical Activity Theory has its origins in Soviet psychology, specifically in the work of Vygotsky (1978), who studied human development in order to understand mental processes. Vygotsky introduced the idea that learning, as a process of human development, is always mediated by artifacts in the subject-object relationship. According to Vygotsky, the artifacts are cultural-historical products, which have evolved and have meaning and value in a given context at one given time (Daniels 2003). These can be artifacts or expressions such as art, music, schemes, language, orality, writing, mathematics, and technology. Leont'ev (1981), a disciple of Vygotsky, introduced the notion of "collectivity" from the example of *primeval collective hunting*. In this example, one can appreciate the difference between the individual goal-directed action and the (organized) collective object-oriented activity, highlighting the potentiality of the collective work with respect to the individual action (Engeström 1987). The consideration of "collectivity" suggests the idea of multiple interrelationships of the subject with its environment in human development through cultural–historical artifacts.

Engeström (1987) organized human activity into a system (see this scheme in Fig. 1) that considers six components, establishing different interrelationships for the system's development: subject(s), artifacts, objects, community, rules, and division of labor. However, the system with its components must be seen as a whole, where the activity is

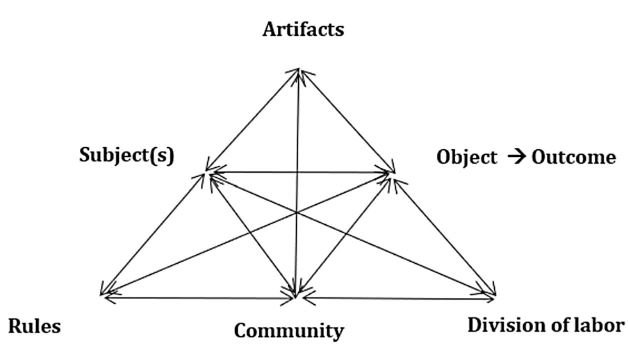

Fig. 1 Scheme of an activity system (adapted from Engeström 1987, p. 78)

the main unit of analysis. The activity is done collectively with a purpose, that is, it is intentional and oriented to the object. The *subjects* are the actors that have agency (power to act) (Souto and Borba 2016). The *artifacts* are tools or signs, either hypothetical or tangible. The *object* is seen by Engeström (1987) as raw material that transforms into an *outcome* or *product* by means of the activity. The *community* is comprised of all the subjects that share the same object. The *rules* correspond to norms and conventions that regulate the actions and the interactions within the activity system. The *division of labor* corresponds to the ways in which actions are organized in the system.

Engeström (2006) deepened the concept of *object* using a metaphor:

> If we think of a designer as the *subject* of her design work, the initial *object* would be an idea, order or assignment that triggers the design process. The initial object is necessarily ambiguous, requiring interpretation and conceptualization. Thus, the object is step-by-step invested with personal sense and cultural meaning. The object goes through multiple transformations until it stabilizes as a finished *outcome*, for example a prototype or even a commercial product. (p. 3).

The example shows the object as a designer's initial idea which goes through multiple transformations until it becomes a prototype or product, using mediational artifacts.

In an activity system, the subjects present multiple voices and contexts that make diverse opinions visible. The multiple opinions found in the system can conflict with one another, generating tensions caused by inner contradictions. Engeström highlights that the inner contradictions are not just tensions or problems; he considers that "a dialectical contradiction refers to a unity of opposites, opposite forces, or tendencies within such a moving system" (Engeström and Sannino 2011, p. 370). The importance of the inner contradictions in learning is viewed in two different ways. For Holzkamp (1993), the contradictions are undesirable situations that obstruct learning (Langemeyer 2005) and hence, should not occur; but, for Engeström (1987), they are opportunities for development and sources of transformation.

The inner contradictions found in a system could evolve, either in an expansion or in a contraction situation of the system. The *expansion* corresponds to one qualitative evolution that indicates development or learning of the system. The *contraction* is referred to as a loss of development opportunities or a deterioration of the system (Engeström 2000).

Engeström (1999) considers expansion as a *cycle of expansive learning* that begins with a questioning of the established practice in the system, and then, through analysis, the organized subjects develop a new model or a new "way of doing" in the system that resolves the inner contradictions. The organizational and institutional processes could exist over long periods of time (months or years) in which the expansive learning is developed; nevertheless, in teaching and learning processes, short periods (hours or weeks) can also be found (Engeström 1987), in which *movements* of expansion within the system can be identified.

One means of investigation used by Engeström (2000) involves the detection of inner contradictions and the study of their evolution. Engeström and Sannino (2011) developed an analytic framework of discursive data with the aim of detecting inner contradictions in the system through their manifestations. They clarified that the contradictions are not visible, but they can be detected by means of their manifestations in linguistic expressions or gestures. In our work, use these ideas to analyze data produced in discussions during mathematical modeling processes.

According to the study by Engeström and Sannino (2011), we must differentiate between tensions and contradictions. The contradictions produce tensions, but not all tension corresponds to a contradiction, because a contradiction involves opposing forces conflicting with one another to produce tension. However, tensions and contradictions can be analyzed in the scope of Activity Theory (Yamagata-Lynch 2010). In the study of mathematics education and Activity Theory, several researchers relate the emergence of tensions or inner contradictions, as well as opportunities for expansion.

In the learning processes of mathematics education, Goodchild and Jaworski (2005) found *contradictions* among mathematics teachers who were taking a course and participating in a project that consisted of setting up an online dialogue community; the teachers were required to propose new didactic approaches. In this course, contradictions related to the teachers emerged: teachers were to follow the school curriculum using innovative teaching strategies, yet these innovative strategies took up too much time. The two were incompatible. Another contradiction was found in the participating teachers themselves; some felt the need to learn innovative strategies but, at the same time, felt confident using the class style that they had been reproducing time and again.

Hardman (2005) identified *contradictions* during the introduction of computers to the mathematics classroom, which the author had seen as an opportunity to experiment with a new teaching tool. Zevenbergen and Lerman (2007) found *tensions* with the introduction of the interactive whiteboard to the mathematics classroom; these tensions were produced between the *artifacts* and the *division of labor*. Williams and Goos (2013) reported a *breakdown* situation in a mathematics class, as a group of students attempted to determine when a colony of bacteria would become extinct and then equated an exponential equation to zero by means of a calculator. The teacher used this situation to conduct a classroom discussion of the resulting mathematics explanation, thus transforming this scene into a situation of *expansion*.

Ärlebäck (2009) used Activity Theory in the design and implementation of modeling modules based on open problems in secondary school. *Tensions* were found the moment the work team considered options and made decisions during the design and development process of the modules. Roth (2013) observed *contradictions* and *uncertainty* in the graphical interpretations of a mathematical function made by a group of scientists. Soares and Souto (2014) saw *tensions* in the analysis process of a malaria contagion model in a course of differential and integral calculus. In addition, Souto and Borba (2015) found *tensions, movements, stagnations*, and *transformations* in an online mathematics education course, in which problems related to conics were developed. These results show different situations occurring due to the dynamic character of an activity system in diverse movements.

The notion of learning in Activity Theory distances itself from the traditional view that emphasizes an individual learning style and the absence of context in what one learns. And, as Roth and Radford (2011) point out, learning does not correspond to a numerical result of the evaluation of the difference in test scores before and after the transmission/reception of content. In the Activity Theory perspective, the subject learns through interaction with the world, by being part of a collective that receives the influence of its surroundings, and, as a result of this process, is transformed by it. But additionally, the subject reciprocates by influencing its surroundings and transforming it. Activity Theory sees contradictions as a means to generate expansive learning, because the contradictions are understood, in the dialectic view, as an opportunity for development or learning. This way of viewing learning is in harmony with our way of understanding modeling: students engage in a situation of their own interest, giving meaning to the mathematics they know, and learning the mathematics they did not know (Borba and Villarreal 2005).

In addition, the possibility of the student choosing a topic of interest, from which s/he collaboratively constructs a modeling problem, aligns with the idea that students are involved in the search for a theme that they themselves have chosen, and they solve a problem which they themselves had viewed and/or built. In this paper, we analyze the work of a group of two teachers in the process of choosing a theme to model during a continuing education course. The analyzed group is seen as a system of activity, in which we observed the dynamics of change, focusing to detect contradictions and the eventual resolution.

4 Methods

The present research has as its setting the online extension course "Trends in mathematics education" offered to mathematics teachers. This course has regularly taken place at the UNESP, a Brazilian university, since the year 2000, and is taught by the second author of this paper. A collaborating instructor of the course, part of the team of instructors, also helped substantially in solving the various tasks that teachers developed online, but he is not related to this research. In the course under analysis, the discussions centered around mathematical modeling and applications, for which a closed group of the social network Facebook was used between the months of September and October of 2014. The course consisted of 32 h of synchronous and asynchronous work, and up to 20 mathematics teachers were allowed to sign up for it. At the end, 17 teachers from different Brazilian states, two teachers from Colombia, and one teacher from Venezuela signed up for the course.

To complete the course, students needed to complete a final task. The task consisted of small groups (pairs or groups of three) proposing or constructing a modeling problem from a topic of the group's interest and then solving it. Students formed seven groups, each of which needed to propose and solve a problem (five teachers did not engage in the "final task" of the course). The groups had 2 weeks to develop their task. At the end of the first week, they were to show their chosen topics to the entire class, and at the end of the second week, they had to present their results, that is, the chosen theme, posed problem, and its solution. Three of the groups demonstrated vigorous discussions during the development of their task, and the discussions of one group are reported here.

In this paper, the results of the group "Cell phone providers", composed of two teachers, are reported. The analyzed data consist of the following: (1) discussions the teachers carried out by means of posts in a closed group of Facebook while proposing a problem and its solution; (2) the written modeling task, with its corresponding online presentation using the *Adobe Connect* platform (the presentation was video recorded); (3) interviews with the teachers at the end of the course, and (4) personal introductions of the teachers done prior to the beginning of the course.

For analysis, we use the analysis framework of Engeström and Sannino (2011), which is based on the idea that contradictions are not visible by themselves; they are visible by means of their manifestation in expressions or gestures. Engeström and Sannino propose detecting contradictions by means of their discursive manifestations. In their study they found four manifestations: dilemma, conflict, critical conflict and double bind.

Engeström and Sannino (2011) discuss the notion of a *dilemma*, which refers to a situation in which socially shared ideologies or beliefs give rise to individually opposing themes. Contrary opinions represent material about which people have the opportunity to think, argue, and take a stance. In this way, Engeström and Sannino establish that a dilemma occurs when opposite stances on an issue arise, that is, incompatible or opposing evaluations of the same situation. They specify that a dilemma situation can be identified in the data through expressions such as: "on the one hand […], on the other […]," "[…], but," "yes, but […]," or similar. A dilemma is usually repeated, often through denial and reformulation, rather than solved.

Engeström and Sannino (2011) indicate that a *conflict* occurs when an individual or group feels negatively affected by another individual or group. A conflict is seen when resistance, disagreement, argumentation, or criticism arise. Expressions that show conflict are "no", "I don't agree", "this isn't true", or similar. Conflict can be resolved when the individuals come to an agreement, submit themselves to an authority, or comply with the view of the majority.

Engeström and Sannino (2011) define *critical conflict*s as situations where people face inner doubts in the midst of a contradictory situation that is unsolvable by the subjects themselves. Critical conflicts can appear in strong expressions, for example when people use metaphors in their narratives, which indicates that they feel mistreated or abused.

Finally, a *double bind* corresponds to a situation that presents two equally undesirable alternatives. When double bind occurs, individuals express a sense of helplessness.

Engeström and Sannino (2011) organized discursive manifestation to show features and linguistic cues for each manifestation (see summary in Table 1). The linguistic cues can indicate the possible presence of a manifestation of contradiction. These expressions can help identify manifestations, but they do not mean that one identified cue automatically corresponds to a manifestation; the situation must be analyzed to determine the existence of a manifestation. For example, a simple "but" is not an indicator of a dilemma; however, a cluster of "buts" in a discussion could be.

The procedure for data analysis in our study was to (1) analyze the background of the subjects (2) find linguistic cues in the discussions based on those in Table 1, and, from them, identify one determined manifestation, and lastly, (3) identify possible inner contradictions. Thus, once the manifestations of the contradictions in our data were identified, their evolution was studied as a possible situation of *expansion* in the system.

The format of the data in this work includes numbered lines of posts generated by the participants from the closed Facebook group that correspond to discussions carried out in the construction of the problem and its resolution (1, 2, 3, …, etc.). In addition, lines were extracted from the

Table 1 Summary of discursive manifestation of contradiction, based on Engeström and Sannino (2011)

Manifestation of contradiction	Features	Linguistic cues
Dilemma	The subjects present different evaluations of a situation	"on the one hand […], on the other […]," "[…], but," "yes, but […]."
Conflict	Criticism, defensiveness, and argumentation	"no", "I disagree", "this is not true"
Critical conflict	A person feels violated or guilty	personal, emotional, moral accounts, narrative structure, vivid metaphors
Double bind	Facing pressing or unacceptable alternatives	"we", "us", "we must", "we have to", pressing rhetorical questions, expressions of helplessness

presentation session of the final group task, which was video recorded (V1, V2, V3, …, etc.).

5 Analysis and results

The subjects in the analyzed group are John and Peterson. We obtained the background of the subjects, that is, the historical–cultural aspects of the subjects themselves that complement the data analysis, through the participants' personal introductions. The two participating subjects are mathematics teachers in a professional development course, and although both teach mathematics, they have different areas of expertise and specialty. John is a secondary school mathematics teacher specializing in mathematics education. Peterson has a master's degree in mathematics and teaches engineering-degree students and some classes at the primary and secondary levels. Both teachers propose a problem with different characteristics. The following texts (translated from the Portuguese) show the initial proposals of each subject:

1. John: I'll post some ideas so we can develop them: (1) We could comment [in class] about the advantage that a certain provider offers for its prepaid, and from this, we could ask about what they thought of this. (2) With the discussion started, we could ask which provider each student uses, and which one is the most advantageous. (3) We could divide the students into groups (by provider) and ask them to list the advantages of each one of them (October 24 at 1:02).
2. Peterson: I […] thought of another proposal. […] John's proposal is interesting to make the students think about […] but I imagined that we could ask the following questions: (1) "A person changes their cell phone on an average every 2 years. So being, how many phones does one buy during one's lifetime? " (2) "With the cell phones one discards, what would be the impact on nature?" (3) "Taking into account the problem, how many cell phones would be acquired and discarded by the Brazilian population in 50 years?" (4) What is the size of the environmental impact of that? (October 24 at 16:30).
3. John: I understood your question, but for the modeling situation, it is closed. It is, in effect, an exercise. Do you understand? (October 24 at 16:56)
4. John: We have to work on questions or raise questions that lead them to find [solutions] (October 24 at 16:56)
5. Peterson: But the modeling would come when the person sees population growth that is variable (October 24 at 16:58)
6. Peterson: Hence, there is a modeling for a possible reason for growth. (October 24 at 16:58).

In these expressions, the subjects mentioned the word "but" three times (lines 2, 3 and 5). Here the subjects proposed two different problems around the same topic, "Cell phone providers". John suggested a problem that involved discussion, with the idea that students could discover which provider offered the best advantages (line 1). For his part, Peterson proposed calculating the number of cell phones a person discards when changing, on average every 2 years, guiding the students to calculate the environmental impact of the discarding of cell phones during the lifetime of a Brazilian (line 2).

John wrote above in line 3, "I understood your question, but for the modeling situation, it is closed. It is, in effect, an exercise", which is a criticism of Peterson's proposal. In response, Peterson defended his proposal saying that "the modeling would come when the person sees population growth that is variable" (line 5). These initial proposals show two opposing ways of thinking about the problem. In John's opinion, the problem proposed by Peterson has characteristics of an exercise, more "closed" in style, while he (John) posed an open question with the aim of generating discussion. John's perspective is consistent with the view of problem expressed by authors such as Blum and Niss (1991) and Borba (2012). The divergent opinions of the teachers demonstrate a *dilemma* situation in this system in the first few lines, that is, differing opinions of the proposed problems, according to Engeström and Sannino (2011) (see Table 1).

7. John: Ok! This is because we are working with two modeling perspectives (October 24 at 16:58)

8. John: You [work] with applied [mathematics], which seeks a model, and I [work] on the issue of a [learning] environment that discusses mathematics, without necessarily resulting in a mathematical model to resolve the question (October 24 at 16:59).
9. John: Fine… and now, how do we resolve this? (October 24 at 17:01).
10. John: I suggest that we begin with the initial discussions that I proposed, and as the activity develops, we raise these questions (October 24 at 17:02).
11. John: In the question that I proposed initially, the idea was for them to discover which provider is the one that offers more advantages. The result would produce a good discussion and even a change of provider by some of them (October 24 at 17:04)
12. Peterson: Your proposal has become more interesting than mine, because the students will discuss and analyze a question that they can reach immediately. With my proposal, the students, when imagining the number of cell phones purchased, would have to remember the growth rate of the population, the increase of a Brazilian's life expectancy, and the purchasing power of the population over the years. In other words, they would have to be involved with a much greater number of variables, giving rise to the reflection of various aspects. Therefore, your proposal is the most viable to work with children and young people (October 24 at 17:07).

Lines 7 to 10 suggest that John understood that both ideas present different modeling perspectives ("You [work] with applied [mathematics]", line 8); then, he made suggestions and argued in favor of his proposal (line 11). In line 12, Peterson, on one hand, uses a compliment indicating that John's proposal is better (because the students would reach the result immediately) and, on the other hand, highlights the qualities of his own proposal ("with a much greater number of variables", line 12). These expressions show a mutual criticism and defense/argumentation, that represent a *conflict*, according to Engeström and Sannino (2011) (see Table 1).

The tensions in the system reveal a possible contradiction activated by two opposing views of understanding the concepts of modeling and problem—on one hand seen as an exercise-style proposal, and, on the other, a more open style that involves discussion. The *object* of the system is understood as an initial idea that is transformed into a product by means of the activity (Engeström 2006). At this point, the *object* of the subjects is the posing of a modeling problem, but the subjects' ideas in relation to the problem diverged (problem-exercise dichotomy). The subjects' opposing views on modeling and problem correspond to the *objects* of the *subject* that collide with each other, thus fueling the discussion. These expressions reveal a contradiction that occurs between the *subjects* and their *objects*. We note that the *subjects* do not present tensions with respect to the task itself (*artifact*) as given by the teachers, but rather with their differing perspectives. In this way, the contradiction is not between *subjects* and *artifacts*. The situation under analysis shows, in Fig. 2, two activity systems interacting: that of the team of the instructors (Instructors system) and that of the group of teachers developing modeling using the cell phone theme (Modeling teachers system). The task of proposing a modeling problem, constructed by the team of instructors, intervened as an *artifact* and produced movement within the "Modeling teachers system", which eventually caused tensions. At this point, no *product* is evident in the system.

In response to Peterson's criticism that John's proposal is appropriate for children and young people (line 12), John indicated it might be possible to orient the problem to two different groups, one at the secondary level and the other at the university level, as a way of negotiating some parameters of the proposal (line 13); this idea is accepted by Peterson ("Yes, we can", line 14).

13. John: Actually, Peterson, we can think of activities for two different groups: […] students of primary and secondary education, and university students (October 24 at 7:08).
14. Peterson: Yes, we can. Thereafter, I can organize the idea of waste and consumerism with the environmental [engineering] students (October 24 at 17:26).
15. Peterson: And, what are the issues we should be addressing with regard to the providers? (October 27 at 21:11).
16. John: Like I said, that was just an idea, but I do not have anything concrete. Some questions: minimum value for credits? What are the advantages to calls to the same provider as opposed to another provider? In truth, it is the students who have to come up with this range of advantages (October 27 at 21:13).
17. Peterson: As well as the value of the data and voice package (October 27 at 21:13).
18. John: Yes. And then they would, as a group, reach conclusions on the best [provider] (October 27 at 21:16).
19. John: We could do some research into some providers just to see what advantages exist (October 27 at 21:16).
20. Peterson: I am just remembering this because teenagers are addicted to *whatsapp* and the Internet (October 27 at 21:16).
21. John: Exactly, but there are other advantages: as I said—calls to the same provider, number of calls, etc. (October 27 at 21:18).

After this negotiation, the *conflict* disappeared. Peterson resumed the conversation after three days with a conciliatory

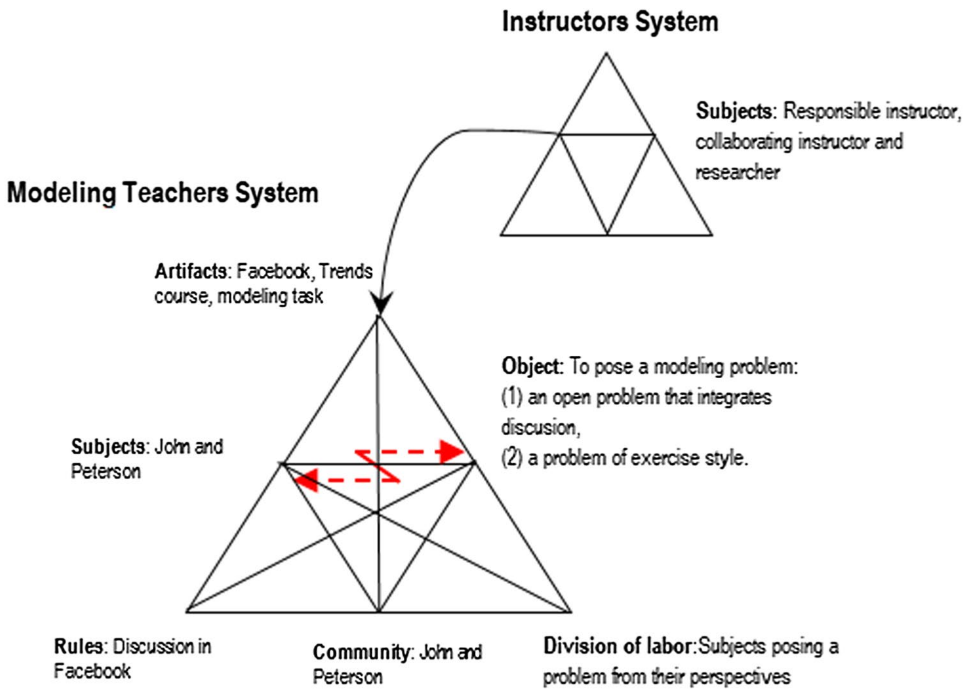

Fig. 2 Activity system in which the subjects discuss posing a problem: a dilemma and conflict emerge between *subjects* and *object*. No product is observed

question ("What are the issues we should be addressing with regard to the providers?" line 15), which shows self-reflection about the discussion that made him agree to John's proposal, thus allowing the dialogue to continue. The resolution of this contradiction was made through discussion of the teachers' differing points of view, a self-reflection on the points of view, which includes consent concerning the point of view of the other, and the negotiation of some aspects of the proposal (the educational level toward which the problem is to be oriented). After overcoming the tensions, the teachers then suggested variables concerning the cell phone providers, such as the least expensive package available, allowable minutes to cell phones of the same company and to those of other companies (line 16), and the cost of data and voice packages (line 17).

At this time, the collaborating instructor of the course participated in the discussion (line 23), suggesting the subjects consider what motivated them to think about the topic. That question made them ponder their own reasons for choosing the topic:

23. Instructor: One thing that could also be thought about is what motived you to think about that topic, if it was of private interest or for another reason (October 27 at 21:18).
24. Peterson: If one analyzes the motives behind why we acquire a particular provider's SIM card, it is, more often than not, to do with the number of contacts we have who use that particular provider, but that does not mean it is the cheapest (October 27 at 21:24).
25. Peterson: But we could also think about it in another way; what about if the student goes beyond the idea of best value, what [provider] offers the best services, that is, they do a cost and benefit analysis (October 27 to 21:53).
26. Peterson: […] Then, what should be a basic model to construct such argumentation. I thought along these lines: Amount spent per provider = (Calls made per day to the same provider × time spent) + (Calls made per day to other providers × time spent) + Use of the data package (October 27 at 22:13).
27. John: Ok! It would be a model that we make them arrive at, and not give to them already (October 27 at 22:14).
29. 29. Peterson: But John, it would be good to think of a mathematical model to have as a base or an ideal, so that they [the students] know […] which is the more advantageous provider (October 27 to 22:22).
30. 30. John: That is it, Peterson, as I said initially, and we can put these two aspects in the work, because I do not agree with the idea of "having to" present something. For me, the worthwhile thing is the mathematical discussions that will emerge from the initial problem. In the event that [the students] arrive at a model, fantastic, and if that is not the case, then that is fantastic too! (October 27 to 22:26).
31. 31. John: Let us take both conceptions. What do you think? (October 27 to 22:26).

32. 32. Peterson: Yes, maybe. I believe that a broader discussion on any theme helps us to investigate further (October 27 at 22:29).

Peterson thought about the motive behind his choice of topic, and this brought them to consider new variables in the analysis (line 24) and to define the problem (line 25), which together led them to propose an initial model that helped to answer their problem (line 26). In this way, what stimulated them to generate an initial model was the fact that they thought about the motive for, and the interest in, choosing the topic.

The *object* of the discussions was "posing a modeling problem", which was tensioned by the teachers' differing ideas over the problem that they wanted to propose. Now, after multiple discussions and agreements, the teachers posed the following problem: what cell phone provider offers the best cost benefit? (line 25). The posed and approved problem shows that the *object* of posing a modeling problem was achieved, and tensions disappeared and were transformed into a *product* (Engeström 2006). This situation is shown in Fig. 3.

In line 26, we see another focus of the activity of the group: the solution of the problem. Peterson proposed a basic model which includes some of the variables discussed by the teachers. Observing the discussion, one can see that the variables involved in the problem and the interest in choosing the topic were elements that stimulated the process of proposing a model. One can note John's immediate agreement with the posed model; he readily proposed a way to integrate that model into class, namely, challenging the students to construct the model rather than simply giving it to them ready-made (line 27) as a means of helping them. We now see the modeling teachers' new *object* has become constructing the model and constructing a pedagogic strategy that integrates the model into the class.

A new *dilemma* began at this point. John presented a way of integrating the model into the class based on the principle of *making the students construct the model* (line 27) while Peterson defended the idea of *giving students the model ready-made* so they had a basic model, which they could apply to then find the cell phone provider with the best value (line 29). These comments represent two opposing teaching principles for the integration of the model into a class: one as a principle of construction, and the other, as a principle of reproduction of the model (construction-reproduction dichotomy). The *object* of the teachers at this moment was "constructing the model and integrating the model into the class", for which the *subjects* presented two opposite ways of reaching it, revealing a contradiction between the *subjects* and their *objects* (see Fig. 4).

At this point, teachers quickly negotiated and agreed to take into account aspects of both principles and to encourage class discussion, although one of them did not approve of being required to present the model to the class ("I do not agree with the idea of 'having to' present something", line 30). The fluid negotiation and agreement, despite the subjects' differences, could have been created by the negotiating

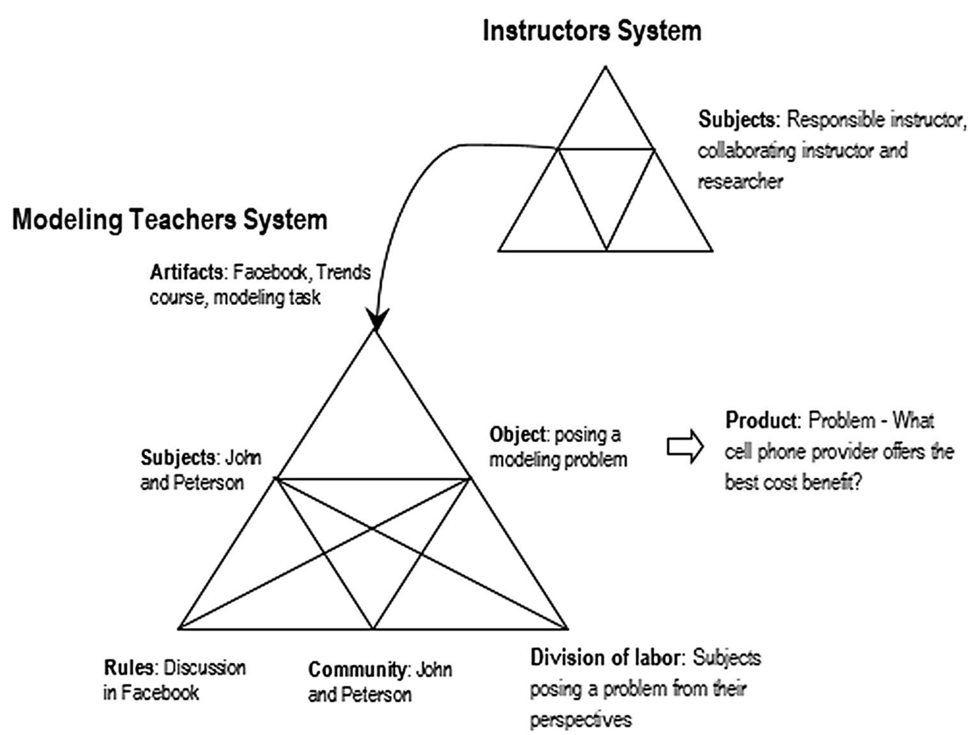

Fig. 3 Activity system in which the subjects discuss posing a problem: the tensions were solved and the product is the elaborated problem

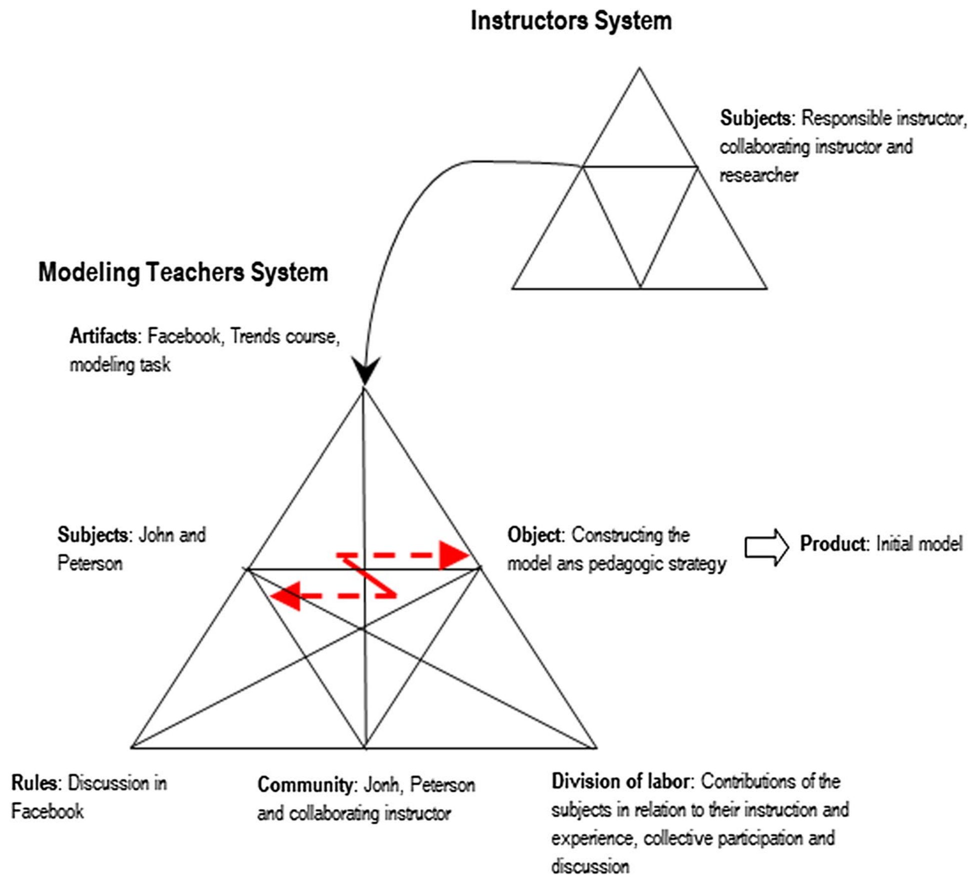

Fig. 4 Activity system of the construction of the model and pedagogic strategy: a dilemma emerges between *subjects* and *object*

climate that remained suffused within the group after they had resolved the previous contradiction. The group continued its work by constructing a teaching approach that integrated the problem and the model into the mathematics class, but also one that considered aspects of the two principles (lines 30 and 31).

In what follows, the teachers interacted with the Internet in search of real data that allowed them to outline their model. The teachers observed possible data that the students could have at their disposal for the development of their task. Those data included an online simulator of cell phone plans and real information on the costs of calls. After accessing the Internet to seek data to use in the model, the teachers finished their participation in the closed Facebook group, and after a few days, they presented their task in front of a virtual class, through the *Adobe Connect* video conference platform.

The participants of this group presented the problem, the model and a teaching strategy of integration of the model in class. The title of the topic was "Cell phone providers" and the constructed problem was "which provider offers the best cost benefit?". Figure 5 shows one of the slides of the presentation in the video-conference environment (on the left), the images of the instructors (responsible and collaborating instructor, respectively), and the dialogues of the modeling teachers.

Teachers were concerned about creating an approach that would help their students build the solution by means of steps (line V1, V2 and V3).

V1. We perceive that for us to work with the students of primary and secondary education, it would be more appropriate to use the context of the use of SIM cards from several providers, to make the students raise the following question […]. This way of thinking becomes simpler when one uses mathematical modeling with people who do not have much experience with this type of teaching methodology.
(Video of the task presentation, 6:08)

V2. At first, the simplest way is to make the students think what needs to be done to acquire a chip [SIM card] of a particular provider […]
(Video of the task presentation, 6:38)

V3. To acquire a chip [SIM card], we should consider = what the promotion is that we are acquiring + how many people I know who use the same provider.
(Video of the task presentation, 7:25).

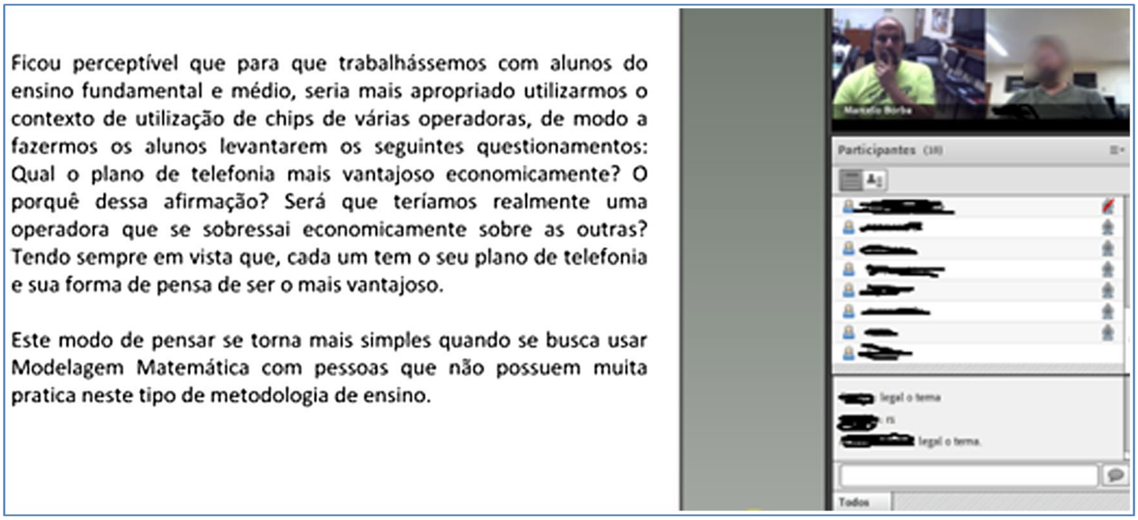

Fig. 5 Video-conference environment showing a slide, the image of the instructors, and dialogues of the teachers

Afterwards, the students would be led to think a bit more deeply about the costs, making them arrive at the expression of line V4.

V4. To acquire a chip, we have to consider = (price of each call) * (time spent on each call) + (price of the data package).
(Video of the task presentation, 7:29).

Finally, the participants presented a more elaborate model (line V5), which represents an ideal cell phone plan including the following variables: plan promotion, cost of calls to persons with the same provider and cost of calls to those with a different provider, the model (shown in the line V5), and parameters for the data package weighted by the connection speed. This shows a refinement of the model, albeit not complete.

V5. Ideal plan = (plan promotion) + (price of each call to persons with the same provider) × (time spent on each call) + (price of each call to persons with another provider) × (time spent on each call) + (price spent on the data package) × (connection speed).
(Video of the task presentation, 7:33).

The teaching approach proposed by the participants has four stages which can be summarized as (1) discussing a determined promotion of a cell phone provider, (2) finding advantages and disadvantages of providers with Internet data (in groups), (3) jointly constructing a panel in class, and (4) raising the question about which provider offers the best cost benefit (using steps V1–V5 if necessary).

Figure 6 shows the activity system in which the subjects construct a model and a pedagogic approach. Here, the subjects resolve the dilemma and discuss a pedagogic strategy to introduce the model to a classroom. That pedagogic strategy and the model correspond to the *product* of this system.

In summary, the dilemmas and conflict revealed contradictions between the *subjects* and their *objects*. These contradictions were resolved through discussion, self-reflection, negotiation, and consent, which leads us to the argument that the modeling teachers system experienced two expansive *movements*. It was identified that this system was transformed from a *conflict* situation (Fig. 2) to the formulation of the open problem (Fig. 3) and from a *dilemma* (Fig. 4) to the construction of a model and a pedagogic strategy (Fig. 6).

6 Conclusions and discussion

The group under analysis worked collectively using multiple interrelationships. The interrelationships were established between the modeling teachers, the online course (with its readings, problems and discussion), the instructors, and the Internet. In this process, by means of the modeling task, the teachers collectively constructed a modeling problem, a solution, and a pedagogic strategy. The introduction of the modeling task as a new artifact to the central system (modeling teachers system) revealed contradictions between the subjects and their objects, which is in agreement with the following expression:

> When an activity system adopts a new element from the outside (for example, a new technology or a new object), it often leads to an aggravated secondary contradiction where some old element (for example, the

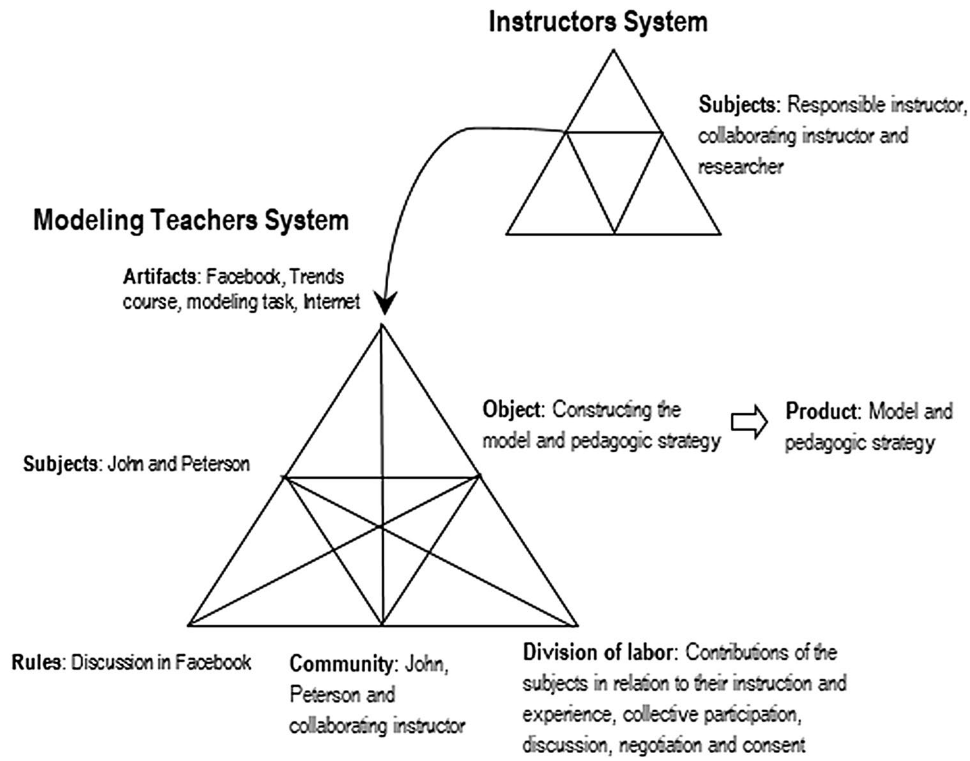

Fig. 6 Activity system of the construction of the model and pedagogic strategy: the dilemma disappeared and the subjects constructed a model and a pedagogic strategy (product)

rules or the division of labor) collides with the new one (Engeström 2001, p. 137).

In the case of the group under analysis, tensions arose concerning the modeling task, due to the teachers' contrary ideas that collided with each other. The contradictions proved to be sources of change and transformation in the system, which is in keeping with ideas expressed by Engeström (1987).

In the development of events, two situations were observed. In the first, during the constructing of the problem, a *dilemma* emerged that evolved into a *conflict*: each teacher perceived a modeling problem in an opposing way; one involved a closed question, and the other, an open situation. The *conflict* triggered an *expansive* situation, that, by means of discussion, self-reflection, consent, and negotiation, was transformed into an open problem. We think it important to include open problems in the class and we agree with Bonotto (2007), that the "teacher has to be ready to create and manage open situations, that are continuously transforming and of which he/she cannot foresee the final evolution or result" (p. 191). Thus, teachers can develop abilities to create links between the real world and the mathematical content, and in so doing, observe, in a more tangible way, the mathematics involved in the real world. One way for teachers to develop these capabilities is by proposing open problems and solving them, as we experienced in this course. Considering students, Ärlebäck (2009) concluded that open problems, in the form of realistic Fermi problems, may provide a good and potentially fruitful opportunity to introduce mathematical modeling at the secondary level.

The intensive discussion of the participants shows the difficulties that teachers face when letting go of a practice they have been reproducing over time, one that perhaps they had adopted from their teachers or their own teaching experience (Cunha 1989). In addition, their insistence on defending their stance shows how the teachers, despite manifesting the need to adopt innovative ways of learning, trust the way they had been traditionally developing their classes (Goodchild and Jaworski 2005).

A second situation emerged in the group on how to integrate the constructed model into a mathematics class. During this process, a *dilemma* arose when teachers suggested opposing strategies with regards to integrating the model into a class; one used a reproductive approach (giving the students a model for them to use), and the other used a construction approach (having the students themselves construct the model). This *dilemma* triggered an *expansion* situation, in which a pedagogic strategy, begun as a reproductive approach, was transformed into one that integrated both principles and perhaps was a means of preventing students from remaining on a dead-end path in the modeling process. It was necessary for the teachers to construct a teaching approach that would lead (future) students to think about the variables involved in the problem. Then students would consider a basic model using these variables, and after that, find data from actual cell phone plans on the Internet that would allow them to refine the model. Teachers' prior knowledge on how to guide students in the construction of a model can be seen here as imperative.

These results suggest that mathematical modeling—from the perspective of participants choosing a topic by collectively developing and proposing the problem—may reveal contradictions related to the practice of modeling. Once the contradictions become visible, there are opportunities for interventions to be made on these practices in order to induce expansive movements within the system. We believe that analyzing the dynamics of groups of teachers using Activity Theory enabled us to obtain a balanced view about such processes, in which there is neither "good" nor "bad", and where effective interventions can be a means to resolve tensions in modeling activity.

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