SÃO PAULO STATE UNIVERSITY (UNESP)
"JÚLIO DE MESQUITA FILHO"
Câmpus Ilha Solteira - SP

BRUNO SERENI

# STATIC OUTPUT FEEDBACK CONTROL FOR LPV AND UNCERTAIN LTI SYSTEMS

Ilha Solteira
2019

BRUNO SERENI

# STATIC OUTPUT FEEDBACK CONTROL FOR LPV AND UNCERTAIN LTI SYSTEMS

Dissertation presented to the São Paulo State University (UNESP) - School of Engineering - Campus of Ilha Solteira, in fulfillment of one of the requirements for obtaining the degree of Master of Science in Electrical Engineering.
Specialty: Automation.

Prof. Dr. Edvaldo Assunção
Advisor

Ilha Solteira
2019

FICHA CATALOGRÁFICA
Desenvolvido pelo Serviço Técnico de Biblioteca e Documentação

S483s

Sereni, Bruno.
Static output feedback control for LPV and uncertain LTI systems / Bruno Sereni. -- Ilha Solteira: [s.n.], 2019
79 f. : il.

Dissertação (mestrado) - Universidade Estadual Paulista. Faculdade de Engenharia de Ilha Solteira. Área de conhecimento: Automação, 2019

Orientador: Edvaldo Assunção
Inclui bibliografia

1. Controle via realimentação estática de saída. 2. Controle gain-scheduling. 3. Controle robusto. 4. Desigualdades Matriciais Lineares (LMIs).

Raiane da Silva Santos

Raiane da Silva Santos
Supervisora Técnica de Seção
Seção Técnica de Referência, Atendimento ao usuário e Documentação
Diretoria Técnica de Biblioteca e Documentação
CRB/8 - 9999

unesp

UNIVERSIDADE ESTADUAL PAULISTA

**Câmpus de Ilha Solteira**

## CERTIFICADO DE APROVAÇÃO

TÍTULO DA DISSERTAÇÃO: Static Output Feedback Control for LPV and Uncertain LTI Systems

**AUTOR: BRUNO SERENI**
**ORIENTADOR: EDVALDO ASSUNÇÃO**

Aprovado como parte das exigências para obtenção do Título de Mestre em ENGENHARIA ELÉTRICA, área: Automação pela Comissão Examinadora:

Prof. Dr. EDVALDO ASSUNÇÃO
Departamento de Engenharia Elétrica / Faculdade de Engenharia de Ilha Solteira

Prof. Dr. MARCELO CARVALHO MINHOTO TEIXEIRA
Departamento de Engenharia Elétrica / Faculdade de Engenharia de Ilha Solteira

Profa. Dra. GABRIELA WERNER GABRIEL
Divisão de Engenharia Eletrônica / Instituto Tecnológico de Aeronáutica - ITA

Ilha Solteira, 07 de fevereiro de 2019

*To my beloved parents, Fatima and Antonio.*

## ACKNOWLEDGMENTS


Getting at the end of this journey was not easy. However, I could not feel more happy and accomplished by concluding this important step towards the edification of my career, doing what I love the most: teaching and learning.

Evidently, anything would not have been achieved without the crucial support that I had during this last years. Therefore, I have to start this dissertation acknowledging those who, in some way, aided me in this challenge.

First of all, I thank Thee, God Almighty, for the gift of life, for the opportunities received, and for the blessings bestowed upon me.

I thank my advisor, Edvaldo Assunção, first, for the guidance received during my studies, but also for the constant solicitude, for the numerous advice given, and for hours of pleasant conversation. For being an example to me as a teacher, researcher, human being and Christian. For the trust placed in me and for the honor of the built friendship.

I leave a special thanks to professors Marcelo Minhoto, Ricardo Cardim, and Jean Ribeiro for the collaboration and for contributing in the development of this work with important advice and suggestions.

I could not forget to mention my friends at the LPC. Undoubtedly, these past two years would not have been the same without the partnership that we build. I am profoundly thanked for all support given and for the fantastic teamwork. Every one of you has a significant role in my life, but I would like to give a special thanks to my dear friends Adalberto Lazarini, Douglas Bizarro, Gilberto Santos, Hadamez Kuzminskas, Igor Minari, Leidy Wolmuth, Leonardo Carniato, and Marco Beteto. You are the best.

There are no words in the world that could express the gratitude that I have for my family. Mom, dad, I am here today because of your efforts and the love you gave me. With little, you made all. Here I leave my humble, but sincere thank you.

And, I would like to thank the fomentation agencies, once that this study was financed in party by the “Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001”, the “Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil (CNPq)” and the “Fundação de Amparo à Pesquisa do Estado de São Paulo - Brasil (FAPESP)” - Grant 2011/17610-0.

*"You have your choices*

*and these are what make a man great,*

*his ladder to the stars."*

***Mumford & Sons***

# ABSTRACT

The static output feedback (SOF) control applied to linear parameter-varying (LPV) and uncertain linear time-invariant (LTI) systems are addressed in this work. The approach chosen for the design of SOF gains is based on the two-stage method, which consists in obtaining a state feedback gain at first, and then using that information for deriving the desired SOF gain at the second stage. The solutions for the investigated problems are presented in terms of linear matrix inequalities (LMIs), obtained by means of the application of the Finsler's Lemma. Based on previous papers found in literature, this work proposes a relaxation strategy in order to achieve a less conservative method for obtaining robust SOF gains for uncertain LTI systems. In the proposed strategy, the Finsler's Lemma additional variables are considered to be parameter-dependent along with the use of parameter-dependent Lyapunov functions (PDLFs). A study evaluating the effectiveness of the proposed strategy in providing a larger feasibility region for a given problem is presented. The results were used in a comparison with a relaxation method based only on PDLFs. Another contribution of this work lies in the proposal of a solution for the control of LPV systems via the design of a gain-scheduled SOF controller. The methods proposed for both control problems were applied on the design of controllers for an active suspension system. In the experiments, it was assumed that only one of its four system's states were available for measurement. The dynamic performance achieved with the practical implementation of the derived controllers attests to the potential of the proposed strategies in both robust and gain-scheduling SOF control.

**Keywords:** Static output feedback control. Gain-scheduling control. Robust control. Linear matrix inequalities (LMIs).

# RESUMO

Este trabalho aborda o controle via realimentação estática de saída aplicado à sistemas lineares com parâmetro variante (LPV) e lineares incertos invariantes no tempo (LIT). O projeto de ganhos de realimentação estática de saída apresentado neste trabalho é baseado no método dos dois estágios, o qual consiste em primeiramente obter um ganho de realimentação de estados, e então, utilizar esta informação no segundo estágio para obter-se o ganho de realimentação estática de saída desejado. As soluções para os problemas investigados são apresentadas na forma de desigualdades matriciais lineares (no inglês, *linear matrix inequalities*, LMIs), obtidas por meio da aplicação do Lema de Finsler. Baseado em resultados anteriores encontrados na literatura, este trabalho propõe uma estratégia de relaxação de forma a obter um método menos conservador para obtenção de ganhos robustos de realimentação estática de saída para sistemas incertos LTI. Na estratégia proposta, as variáveis adicionais do Lema de Finsler são consideradas como dependentes de parâmetro, juntamente com o uso de funções de Lyapunov dependentes de parâmetro (no inglês, *parameter-dependent Lyapunov functions*, PDLFs). É apresentado um estudo avaliando a eficácia da estratégia proposta em fornecer uma maior região de factibilidade para um dado problema. Os resultados foram utilizados em uma comparação com um método de relaxação baseado apenas no uso de PDLFs. Uma segunda contribuição deste trabalho consiste na proposta de uma solução para o controle de sistemas LPV por meio do projeto de um controlador gain-scheduled via realimentação estática de saída. As soluções propostas para ambos os problemas de controle foram aplicadas no projeto de controladores para um sistema de suspensão ativa. Nos experimentos, foi assumido que apenas um dos quatro estados do sistema estava disponível para medição. A performance dinâmica alcançada com a aplicação prática dos controladores projetados atesta o potencial dos métodos propostos, tanto em controle robusto como em controle gain-scheduling.

**Palavras-chave:** Controle via realimentação estática de saída. Controle gain-scheduling. Controle robusto. Desigualdades matriciais lineares (LMIs).

**LIST OF FIGURES**

## LIST OF ABBREVIATIONS AND ACRONYMS

| | |
|---|---|
| CQLF | Common quadratic Lyapunov functions |
| DC | Direct current |
| DOF | Dynamic output feedback |
| GS | Gain-scheduling/Gain-scheduled |
| LMI | Linear matrix inequalities |
| LPV | Linear parameter-varying |
| LTI | Linear time-invariant |
| PDFV | Parameter dependent Finsler variables |
| PDLF | Parameter dependent Lyapunov functions |
| SISO | Single input single output |
| SOF | Static output feedback |
| T-S | Takagi-Sugeno |

## LIST OF SYMBOLS

| | |
|---|---|
| $M^{\perp}$ | Basis for the null space of $M$. |
| $\|M\|_2$ | Euclidean norm of the real matrix $M$. |
| $\|v\|$ | Euclidean norm of the real vector $v$. |
| $M^{-1}$ | Inverse of the real matrix $M$. |
| $\lambda_{max(min)}(M)$ | Maximum (minimum) eigenvalue of the real matrix $M$. |
| $\sigma_{max(min)}(M)$ | Maximum (minimum) singular value of the real matrix $M$. |
| $M(\alpha)$ | $M(\alpha) = \sum_{i=1}^{N} \alpha_i M_i, \quad \alpha \in \wedge_N$. |
| $\mathbb{R}^{n \times m}$ | Set of the real matrices with $n$ rows and $m$ columns. |
| $*$ | Symmetric block of a symmetric matrix. |
| $M \geq (>)0$ | Symmetric positive semidefinite (definite) matrix $M$. |
| $M \leq (<)0$ | Symmetric negative semidefinite (definite) matrix $M$. |
| $M'$ | Transpose of the real matrix $M$. |
| $\wedge_N$ | Unitary simplex $\wedge_N = \left\{\alpha \in \mathbb{R}^n : \sum_{i=1}^{N} \alpha_i = 1; \alpha_i \geq 0; i = 1, ..., N\right\}$. |
| $\wedge_M$ | Unitary simplex $\wedge_M = \left\{\alpha(t) \in \mathbb{R}^n : \sum_{i=1}^{N} \alpha_i(t) = 1; \alpha_i(t) \geq 0; i = 1, ..., N; \forall t\right\}$. |

CONTENTS

# 1 INTRODUCTION

The state feedback is well-known as a classic and standard control technique. It consists in the design of a controller that uses the information provided by the system's state for composing a feedback loop. The proper operation of a state feedback closed-loop system relies on the complete information of the state vector, *i.e.* it is assumed that all of its entries are available for measurement.

However, in several practical problems, the complete state observation is unlikely (KIMURA, 1975). Therefore, in most of the cases, the state feedback technique cannot be directly applied in practice. Under such circumstances, an immediate solution for dealing with this issue is the design of a controller that relies only on the available system's state variables: the output feedback controller (ZHANG; SHI; MEHR, 2011).

The output feedback control can be addressed from two different perspectives. The first one consists in the design of a dynamic output feedback (DOF) controller. In this case, the feedback loop comprises dynamic elements (AGULHARI; OLIVEIRA; PERES, 2012; CAIGNY et al., 2012; SADABADI; KARIMI, 2015; WANG; XIE; SOUZA, 1992; ZHANG; LEWIS; DAS, 2011), and thus, the controller forms a whole separated system itself. Basically, the DOF uses the plant's output vector as an input signal. Then, it will estimate the system's state variables that are not available for measurement. The control signal is then composed as a function of the system's estimated state vector. In fact, the study of dynamic output feedback is a subject of major interest and has achieved relevant progress in the past decades (SADABADI; PEAUCELLE, 2016; SYRMOS et al., 1997).

A second approach to the output feedback problem is the static output feedback (SOF) control. This alternative strategy is based on the synthesis of a simple static feedback gain. In this case, the control signal is a linear function of the system's output vector (BERNSTEIN, 1992). Therefore, the SOF control technique proposes a simple and more affordable solution in practical terms. Moreover, it is interesting to remark that in the case where a dynamic controller is used, the plant-compensator set can be described as an augmented dynamic system, and then it may be brought back to the static output feedback case (BERNSTEIN, 1992; SYRMOS et al., 1997).

In spite of the aforementioned advantages, the design of SOF controllers has been shown to be a rather difficult and challenging task. In fact, some authors consider it as one the most important open questions in control engineering (ADEGAS, 2013; CRUSIUS; TROFINO, 1999; SADABADI; PEAUCELLE, 2016). One of the main reasons for the static output feedback problem to be so difficult to deal with is that the definition of necessary and sufficient conditions ends up in not numerically tractable solutions (ARZELIER et al., 2010; CRUSIUS; TROFINO, 1999). Nevertheless, several approaches have been developed in the past years, intended to provide sufficient conditions for the design of static output feedback controllers, as it will be properly discussed in the sequence.

Within those two main approaches, several papers have been published on the specialized literature proposing new output feedback designs for different types of systems and applications. Despite the fact that this work is focused on linear control, it is important to remark the relevance of output feedback applied to nonlinear systems in the control research community. In fact, a diversity of papers addressing the output feedback for nonlinear systems can be found in the literature, such as robust SOF control (CHADLI; GUERRA, 2012) and observer-based fuzzy mixed $\mathcal{H}_2/\mathcal{H}_\infty$ output feedback controller design technique, which deals with nonlinear systems through fuzzy T-S linear approximated models (CHEN; TSENG; UANG, 2000; TEIXEIRA; ASSUNÇÃO; AVELLAR, 2003); a strategy for the design of an output feedback controller for SISO nonlinear systems consisted of a high-gain observer (KHALIL, 1996); the synthesis of linear robust $\mathcal{H}_\infty$ dynamic output feedback controller for uncertain nonlinear time-varying systems represented in terms of a state space linear model with the addition of unknown nonlinear functions (WANG; XIE; SOUZA, 1992); control approaches for the design of observer-based adaptive fuzzy output feedback controllers applied to uncertain stochastic nonlinear systems (TONG et al., 2011); and also the design of robust $\mathcal{H}_\infty$ static output feedback controller for networked nonlinear systems with unreliable communication links between plant and controller (QIU; FENG; GAO, 2010).

With respect to linear systems, a first and simpler approach in a control design considers that the dynamic system is modeled through a state-space representation, where the matrices that describe the system's behavior are assumed to be constant and known. In this perspective, much have been done in the past several years, but its noteworthy to cite Trofino (2009), Crusius and Trofino (1999), which brought contributions for both static and dynamic output feedback problems, proposing sufficient conditions for the controller design, and to cite Trofino and Kucera (1993), where the authors derived necessary and sufficient conditions for designing SOF gains.

Although, in practical applications, it is not possible to obtain a precisely exact model, since the system's parameters are often obtained experimentally, with an intrinsic measurement error margin, for example. These are often referred as parametric uncertainties. Furthermore, several dynamic systems may be subject to input uncertainty, represented by disturbances that can compromise the system integrity, underlining the security factor in a control design problem (BARMISH, 1985). The robust control comes to this extent for dealing with these practical issues. Several papers addresses the robust control in the output feedback optics regarding LTI systems. To cite a few, in Agulhari, Oliveira and Peres (2010), the authors proposed a strategy for the design of robust SOF controllers for linear time-invariant systems, extensible to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm bounds problems; Arzelier, Peaucelle and Salhi (2003) presents sufficient conditions for robust SOF controller design (extensible to $\mathcal{H}_2$ norm bound) based on the quadratic stability; Dong and Yang (2013) presents a design technique based on a line search for a robust SOF controller and Chang, Park and Zhou (2015) studies the robust SOF $\mathcal{H}_\infty$ for continuous-time and discrete-time uncertain linear systems.

Another problem of great interest is related to linear dynamic systems subject to time-varying parametric uncertainty, i.e, linear parameter-varying (LPV) systems. In practice, an LPV system can be interpreted from two perspectives. The first is based on an LTI robust control problem, regarding the time-varying parameter as an uncertainty. The second is centered on the assumption that the referred parameter is available for measurement, no longer being an uncertainty, a fact which may produce enhanced performances (APKARIAN; GAHINET; BECKER, 1995). In the former approach, one can find very conservative results, or even end up with non-feasible solutions, whereas the latter can be addressed from a promising perspective in terms of the compromise between robustness and performance: the gain-scheduling control (APKARIAN; GAHINET, 1995). Notwithstanding one can find a handful of reports on scheduling techniques in military aircraft applications back in the 1970s, gain-scheduling has gained a substantial attention in the scientific community especially from the 1990s onwards (RUGH; SHAMMA, 2000). This particular crescent interest is justified for its effectiveness in coping with both nonlinearities and time-varying dynamics when compared to other traditional approaches (WEI et al., 2014).

The classical gain scheduling technique is based on the construction of a family of linear time-invariant subsystems to cover the range of operation of a nonlinear or/and time-varying plant. Then, an LTI compensator is designed for each of these operation points. And, based on the measurement of the time-varying parameters, the controller has its gains scheduled via some interpolation technique, in order to provide a global compensator to work within these operation points (MONTAGNER; PERES, 2006; RUGH; SHAMMA, 2000; SHAMMA, 1988). However, obtaining a good performance in some practical cases requires a refinement of the grid on the parametric space, at the price of higher complexity and computational cost. Even though, this effort does not eliminate the questioning about the stability, performance and robustness guarantee on the switching zone between the local stable points (APKARIAN; GAHINET, 1995; MONTAGNER; PERES, 2006; SHAMMA, 1988). Particularly regarding SOF control problem, gain-scheduling strategies for different control problems have been developed.

For instance, recently, methods for saturated LPV systems (NGUYEN; CHEVREL; CLAVEAU, 2018), guaranteed upper bound $\mathcal{H}_\infty$ performance for continuous-time (AL-JIBOORY; ZHU, 2018) and discrete-time (SADEGHZADEH, 2017), discretization and digital output feedback control (BRAGA et al., 2015) have been published in literature.

In this work, the parametric uncertainties and time-varying parameters are represented in polytopic domains. This choice was made regarding two facts: first, problems of this nature are efficiently addressed via quadratic stabilizability conditions (MONTAGNER; PERES, 2006); second, the convex nature of a polytopic system allows this issue to be handled as an optimization problem. For that reason, the formulation of Linear Matrix Inequalities (LMIs) is of particular interest, since this strategy has been considered as a powerful tool in linear control problems (SCHERER; GAHINET; CHILALI, 1997). This can be said given that the solution of an LMI is presented as a convex optimization problem (SCHERER; GAHINET; CHILALI, 1997), which can be easily programmed via simple interfaces, such as the MATLAB® LMI Control Toolbox or the YALMIP (Yet Another LMI Parser) (LOFBERG, 2004), and the solution can be efficiently obtained via solvers such as the MATLAB® LMILab (AGULHARI; OLIVEIRA; PERES, 2011).

One of the main contributions of this work is the proposal of new relaxed LMI conditions for the design of robust static output feedback controllers. These results are based on the work presented in Manesco (2013). In the refereed work, the author addresses the SOF problem using the two-stage method (AGULHARI; OLIVEIRA; PERES, 2011; MEHDI; BOUKAS; BACHELIER, 2004). This strategy consists in first deriving a state feedback gain $K$. Then, this information is used as an input parameter for the second stage, in which the desired SOF gain is calculated. The formulation of the proposed LMIs was obtained by means of the Finsler's Lemma technique, which has been widely used for dealing with non-convex problems through the insertion of additional variables (MOZELLI; PALHARES; MENDES, 2010).

Moreover, in Manesco (2013), the author discusses the flexibility problems inherent of LMI-based formulations (DABBOUSSI; ZRIDA, 2012). Particularly, the use of common quadratic Lyapunov functions (CQLF) results in restrictive effects when dealing with uncertain systems (OLIVEIRA; PERES, 2006). For tackling this issue, Manesco (2013) presents an LMI formulation based on parameter-dependent Lyapunov functions (PDLF), which results in less conservative conditions (APKARIAN; GAHINET, 1995; OLIVEIRA; GEROMEL, 2005). In this work, aiming to provide even less conservative LMI conditions, an additional relaxation method that considers the Finsler's slack variables as parameter-dependent, i.e. parameter-dependent Finsler variables (PDFV) is proposed.

And, with a more solid technical foundation on SOF achieved through the study of the robust SOF control, new LMI conditions for deriving gain-scheduled static output feedback controllers were also obtained and are proposed in this work. The LMI approach in the gain-scheduling problem enables the synthesis of gain-scheduled controllers that operate in a smoothly form, differently from the problematic classic grid strategy mentioned before. Furthermore, the relaxation strategy based on PDFV explored in the robust SOF case was extended to the gain-scheduling static output feedback (GS-SOF) design. The results obtained had its importance highlighted due to the publishing of two papers in national and international conferences on control (SERENI; ASSUNÇÃO; TEIXEIRA, 2017; SERENI et al., 2018).

This work is organized into four chapters. Chapter 2 addresses the synthesis of robust static output feedback controllers. At, first the approaches based on CQLFs, and on PDLFs, presented by Manesco (2013) are discussed. Then, the proposal of a new relaxation strategy considering parameter-dependent Finsler Lemma's additional variables is presented. In the end, the results of a comparison between new relaxation strategy and the CQLF and PDLF approaches are showed and discussed. A practical implementation example is also explored in order to verify the method's efficiency and performance in a real control problem.

In Chapter 3 the gain-scheduled static output feedback control design is investigated. New LMI conditions are proposed addressing the CQLF and PDFV approaches. An application example in an active suspension system is presented to evaluate the practical applicability of the proposed method.

Finally, the concluding remarks and future perspectives on the subject at matter are presented in Chapter 4.

# 2 ROBUST STATIC OUTPUT FEEDBACK CONTROL

In this chapter, the problem of the stabilization of uncertain linear time-invariant (LTI) systems via output feedback is addressed.

## 2.1 SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider the uncertain LTI system

$$\begin{aligned} \dot{x}(t) &= A(\alpha)x(t) + B(\alpha)u(t) \\ y(t) &= C(\alpha)x(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the measured output vector, and $u(t) \in \mathbb{R}^m$ is the control input vector. Moreover, $A(\alpha) \in \mathbb{R}^{n\times n}$, $B(\alpha) \in \mathbb{R}^{n\times m}$, and $C(\alpha) \in \mathbb{R}^{p\times n}$ are uncertain matrices that describe the system's dynamics, and that can be represented in a polytopic domain $\mathscr{D}$ defined as

$$\mathscr{D} = \left\{ (A,B,C)(\alpha) : (A,B,C)(\alpha) = \sum_{i=1}^{N} \alpha_i (A,B,C)_i, \quad \alpha \in \wedge_N \right\}, \tag{2}$$

where $A_i$, $B_i$, and $C_i$ denote the $i$-th polytope vertex, and $N$ is the number of vertices of the polytope. Furthermore, $\mathscr{D}$ is parameterized in terms of a vector $\alpha = (\alpha_1, ..., \alpha_N)$, whose parameters $\alpha_i$ are unknown constants belonging to the unitary simplex set $\wedge_N$, defined as

$$\wedge_N = \left\{ \alpha \in \mathbb{R}^n : \sum_{i=1}^{N} \alpha_i = 1; \alpha_i \geq 0; i = 1, ..., N \right\}. \tag{3}$$

Supposing that the feedback loop is composed by the following control law

$$u(t) = Ly(t), \tag{4}$$

then the system (1) in closed-loop assumes the form

$$\dot{x}(t) = \Big[ A(\alpha) + B(\alpha)LC(\alpha) \Big] x(t). \tag{5}$$

In these terms, the objective is to find a robust static output feedback gain $L \in \mathbb{R}^{m\times p}$ that asymptotically stabilizes (5).

The approach chosen in this work to tackle the SOF control problem has its mathematical formulation based on the application of the Finsler's Lemma. Therefore, before properly addressing the stabilization via static output feedback, the Finsler's Lemma is formally presented in the sequence.

**Lemma 2.1** (Finsler's Lemma). *Consider $\mathscr{W} \in \mathbb{R}^n$, $\mathscr{S} \in \mathbb{R}^{n\times n}$, and $\mathscr{R} \in \mathbb{R}^{m\times n}$ with rank $(\mathscr{R}) < n$, where $\mathscr{R}^{\perp}$ is a basis for the null space of $\mathscr{R}$ (i.e. $\mathscr{R}\mathscr{R}^{\perp} = 0$).*

*Then, the following conditions are equivalent:*

*(i)* $\mathscr{W}'\mathscr{S}\mathscr{W} < 0, \ \forall \ \mathscr{W} \neq 0, \ \mathscr{R}\mathscr{W} = 0,$

*(ii)* $\mathscr{R}^{\perp'}\mathscr{S}\mathscr{R}^{\perp} < 0,$

*(iii)* $\exists \eta \in \mathbb{R} : \mathscr{S} - \eta\mathscr{R}'\mathscr{R} < 0,$

*(iv)* $\exists \chi \in \mathbb{R}^{2n\times n} : \mathscr{S} + \chi\mathscr{R} + \mathscr{R}'\chi' < 0,$

*where $\eta$ and $\chi$ are additional variables (or multipliers).*

**Proof:** See Skelton, Iwasaki and Grigoriadis (1997) and Oliveira and Skelton (2001). ■

## 2.2 ROBUST STABILIZATION VIA STATIC OUTPUT FEEDBACK

As mentioned in Chapter 1, the stabilization of dynamic systems via output feedback is a major still-opened problem in control theory. As a consequence, a vast number of controller design approaches for this particular problem can be found in the literature.

The developed studies and contributions proposed in this work are based upon the strategy presented in Manesco (2013) and in Mehdi, Boukas and Bachelier (2004). This approach consists of a two-stage control design which is detailed in the sequence.

The first stage consists in a preliminary state feedback control design. Thus, considering the control law as

$$u(t) = Kx(t), \tag{6}$$

then the system (1) in closed-loop is represented by

$$\dot{x}(t) = \Big[A(\alpha) + B(\alpha)K\Big]x(t). \tag{7}$$

Therefore, the first goal is to obtain a robust state feedback gain $K$ that asymptotically stabilizes the system (7).

As well known in the scientific community (BOYD et al., 1994), a sufficient condition to (7) be quadratically stable is that there exist matrices $W \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{m\times n}$, such that

$$\begin{gathered} W = W' > 0 \\ A_iW + WA_i' + B_iZ + Z'B_i' < 0 \end{gathered} \tag{8}$$

for $i = 1,2,...,N$ . If (8) is satisfied, then $K$ is given by $K = ZW^{-1}$.

In the second stage, the obtained gain $K$ is used as an input parameter for the design of the robust output feedback gain $L \in \mathbb{R}^{m\times p}$. Manesco (2013) proposed sufficient LMI conditions for deriving the desired SOF controller, as stated in Theorem 1.

**Theorem 1.** *(MANESCO, 2013) Assuming that there exists a state feedback gain $K$ such that $A(\alpha)+B(\alpha)K$ is asymptotically stable, then there exists a stabilizing static output feedback gain $L$ such that $A(\alpha)+B(\alpha)LC(\alpha)$ is asymptotically stable if there exist a symmetric matrix $P > 0$ and matrices $F$, $G$, $H$, and $J$ such that*

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK & P - F + A_i'G' + K'B_i'G' & FB_i + C_i'J' - K'H' \\ P - F' + GA_i + GB_iK & -G - G' & GB_i \\ B_i'F' + JC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0, \tag{9}$$

*for $i = 1,2,...,N$.*

*In the affirmative case, the robust static output feedback gain is given by $L = H^{-1}J$.*

**Proof:** According to Boyd et al. (1994), if $M$ is a non-symmetric matrix, and if $M + M' < 0$, then $M$ is invertible. Therefore, supposing that (9) has a solution, one can verify that $H$ is invertible.

Multiplying (9) by $\alpha_i$, and summing every term in $i$, from $i = 1$ to $i = N$, and regarding that $\sum_{i=1}^{N} \alpha_i = 1$, we obtain

$$\begin{bmatrix} A'(\alpha)F' + FA(\alpha) + K'B'(\alpha)F' + FB(\alpha)K & * & * \\ P - F' + GA(\alpha) + GB(\alpha)K & -G - G' & * \\ B'(\alpha)F' + JC(\alpha) - HK & B'(\alpha)G' & -H - H' \end{bmatrix} < 0. \tag{10}$$

Pre and post multiplying (10) by $T(\alpha)$ and $T'(\alpha)$, following the idea presented in Mehdi, Boukas and Bachelier (2004), where $T(\alpha)$ is defined as

$$T(\alpha) = \begin{bmatrix} I & 0 & S'(\alpha) \\ 0 & I & 0 \end{bmatrix}, \tag{11}$$

it follows that

$$\begin{bmatrix} \Psi(\alpha) & \Phi(\alpha) \\ * & -G-G' \end{bmatrix} < 0, \tag{12}$$

where

$$\begin{aligned} \Psi(\alpha) = {} & A'(\alpha)F' + FA(\alpha) + K'B'(\alpha)F' + FB(\alpha)K + S'(\alpha)B'(\alpha)F' + S'(\alpha)JC(\alpha) \\ & - S'(\alpha)HK + C'(\alpha)J'S(\alpha) - K'H'S(\alpha) + FB(\alpha)S(\alpha) + S'(\alpha)(-H-H')S(\alpha) \end{aligned} \tag{13}$$

and,

$$\Phi(\alpha) = P - F + A'(\alpha)G' + K'B'(\alpha)G' + S'(\alpha)B'(\alpha)G'. \tag{14}$$

Replacing $S(\alpha) = H^{-1}JC(\alpha) - K$, in (13) and (14), then $\Psi(\alpha)$ and $\Phi(\alpha)$ may be rewritten as

$$\begin{aligned} \Psi(\alpha) = {} & A'(\alpha)F' + FA(\alpha) + K'B'(\alpha)F' + FB(\alpha)K + (C'(\alpha)J'H^{-1'} - K')B'(\alpha)F' \\ & + C'(\alpha)J'(H^{-1}JC(\alpha) - K) - K'H'(H^{-1}JC(\alpha) - K) + (C'(\alpha)J'H^{-1'} - K')JC(\alpha) \\ & - (C'(\alpha)J'H^{-1'} - K')HK + FB(\alpha)(H^{-1}JC(\alpha) - K) \\ & + (C'(\alpha)J'H^{-1'} - K')(-H-H')(H^{-1}JC(\alpha) - K) \end{aligned} \tag{15}$$

and,

$$\Phi(\alpha) = P - F + (A(\alpha) + B(\alpha)H^{-1}JC(\alpha))'G'. \tag{16}$$

Expanding the products in (15), it follows that

$$\begin{aligned} \Psi(\alpha) = {} & A'(\alpha)F' + FA(\alpha) + K'B'(\alpha)F' + FB(\alpha)K + C'(\alpha)J'H^{-1'}B'(\alpha)F' - K'B'(\alpha)F' \\ & + C'(\alpha)J'H^{-1'}JC(\alpha) - K'JC(\alpha) - C'(\alpha)J'H^{-1'}HK + FB(\alpha)H^{-1}JC(\alpha) + K'HK \\ & + K'H'K - K'H'K + C'(\alpha)J'H^{-1}JC(\alpha) - K'HK - FB(\alpha)K - C'(\alpha)J'K \\ & - K'H'H^{-1}JC(\alpha) - C'(\alpha)J'H^{-1'}HH^{-1}JC(\alpha) - C'(\alpha)J'H^{-1'}H'H^{-1}JC(\alpha) \\ & + C'(\alpha)J'H^{-1'}HK + C'(\alpha)J'H^{-1'}H'K + K'HH^{-1}JC(\alpha) + K'H'H^{-1}JC(\alpha). \end{aligned} \tag{17}$$

In addition, considering the following equivalent relation

$$H^{-1}H = HH^{-1} = I = H^{-1'}H' = H'H^{-1'}, \tag{18}$$

then, (17) assumes the upcoming form

$$\Psi(\alpha) = (A(\alpha) + B(\alpha)H^{-1}JC(\alpha))'F' + F(A(\alpha) + B(\alpha)H^{-1}JC(\alpha)). \tag{19}$$

Now, making $L = H^{-1}J$ in (19) and (16), we obtain

$$\Psi(\alpha) = (A(\alpha) + B(\alpha)LC(\alpha))'F' + F(A(\alpha) + B(\alpha)LC(\alpha)), \tag{20}$$

and,

$$\Phi(\alpha) = P - F + (A(\alpha) + B(\alpha)LC(\alpha))'G'. \tag{21}$$

Considering (20) and (21), we can rewrite (12) in terms of a sum of two matrices:

$$\begin{bmatrix} F(A(\alpha)+B(\alpha)LC(\alpha)) & P-F \\ P+G(A(\alpha)+B(\alpha)LC(\alpha)) & -G \end{bmatrix} + \begin{bmatrix} (A(\alpha)+B(\alpha)LC(\alpha))'F' & (A(\alpha)+B(\alpha)LC(\alpha))'G' \\ -F' & -G' \end{bmatrix} < 0. \tag{22}$$

Moreover, splitting the first matrix in (22) as follows

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \begin{bmatrix} F(A(\alpha)+B(\alpha)LC(\alpha)) & -F \\ G(A(\alpha)+B(\alpha)LC(\alpha)) & -G \end{bmatrix} + \begin{bmatrix} (A(\alpha)+B(\alpha)LC(\alpha))'F' & (A(\alpha)+B(\alpha)LC(\alpha))'G' \\ -F' & -G' \end{bmatrix} < 0, \tag{23}$$

and, manipulating (23) properly, we have

$$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} A(\alpha)+B(\alpha)LC(\alpha) & -I \end{bmatrix} + \begin{bmatrix} (A(\alpha)+B(\alpha)LC(\alpha))' \\ -I \end{bmatrix} \begin{bmatrix} F' & G' \end{bmatrix} < 0. \tag{24}$$

Considering the following definitions:

$$\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \chi, \tag{25}$$

$$\mathscr{S} = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}, \quad \text{and,} \tag{26}$$

$$\mathscr{R}(\alpha) = \begin{bmatrix} A(\alpha)+B(\alpha)LC(\alpha) & -I \end{bmatrix}, \tag{27}$$

we can rewrite (24) as

$$\exists \chi \in \mathbb{R}^{2n\times n},\ \mathscr{S} + \chi\mathscr{R}(\alpha) + \mathscr{R}'(\alpha)\chi' < 0. \tag{28}$$

Observe that (28) corresponds to the condition $(iv)$ of Finsler's Lemma, as stated on Lemma 1. Hence, considering the condition $(i)$ of Finsler's Lemma, which is

$$\mathscr{W}'\mathscr{S}\mathscr{W} < 0,\ \forall \mathscr{W} \neq 0,\ \mathscr{R}(\alpha)\mathscr{W} = 0, \tag{29}$$

and assuming that $\mathscr{W} = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$, we can derive, regarding (27), the expressions

$$\begin{bmatrix} A(\alpha)+B(\alpha)LC(\alpha) & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = 0 \quad \text{and} \tag{30}$$

$$\begin{bmatrix} x'(t) & \dot{x}'(t) \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} < 0. \tag{31}$$

Therefore, according to (30) we have

$$\dot{x}(t) = \begin{bmatrix} A(\alpha)+B(\alpha)LC(\alpha) \end{bmatrix} x(t), \tag{32}$$

which corresponds to the considered system's equation (5).

Furthermore, (31) leads to

$$\dot{x}'(t)Px(t) + x'(t)P\dot{x}(t) < 0. \tag{33}$$

And finally, making $V(x(t)) = x'(t)Px(t)$, we can conclude that (33) becomes

$$\dot{V}(x(t)) < 0, \tag{34}$$

which is Lyapunov's equation for stability (BOYD et al., 1994). ■

## 2.3 DECAY RATE BOUNDING IN ROBUST SOF CONTROLLER DESIGN

Theorem 1, presented in Section 2.1, proposes an approach in order to guarantee robust stability in LTI systems via static output feedback. However, in many practical applications, a dynamic system must show, besides stability, improved performance during its operation.

A basic performance index associated to the system's transient duration is the decay rate. According to Boyd et al. (1994), the decay rate can be defined as the highest $\gamma$ such that

$$\lim_{t\to\infty} e^{\gamma t}\|x(t)\| = 0 \tag{35}$$

holds for all trajectories of the system's states $x(t)$.

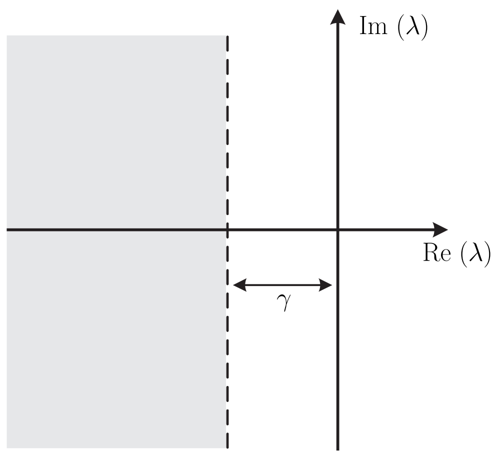
Considering the quadratic Lyapunov's function $V(x(t)) = x'(t)Px(t)$, a lower bound on the decay rate can be established if

$$\dot{V}(x(t)) \leq -2\gamma V(x(t)) \tag{36}$$

holds for all trajectories of the system's states $x(t)$ (BOYD et al., 1994).

An equivalent interpretation of this new design requirement can be given in terms of the system's eigenvalues ($\lambda$) placement. The condition described in (36) configures a relative stability criterion, forcing the system's eigenvalues to be placed in the left half plane shifted in $\gamma$ units along the real axis, as illustrated in Figure 1. Therefore, the greater the decay rate specified in the control design, the farther the system's eigenvalues will be from the imaginary axis, into the left half plane.

Figure 1 - Geometric interpretation of the minimum decay rate requirement.




Source: Adapted from Silva (2012).

Regarding the aforementioned concept, the two-stage method presented in Section 2.2 can be applied in order to design a SOF controller that guarantee robust stability, and also considers the inclusion of a lower bound in the closed-loop system's decay rate.

In these terms, a minimum decay rate is considered in both first and second stages of project, namely $\beta$ and $\gamma$, respectively. At this point, it is important to emphasize that the decay rate specification in each stage of project can assume different values. However, as it will be addressed in further stance, the decay rate specified in the first stage acts as a limiting factor on the feasibility of the LMIs for the design of the SOF gain $L$, in the second stage.

Therefore, for the first stage, a sufficient condition (BOYD et al., 1994) to (7) be quadratically stable, considering the inclusion of a minimum decay rate $\beta > 0$, is that there exist matrices $W \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{m\times n}$, such that

$$\begin{gathered} W = W' > 0 \\ A_iW + WA_i' + B_iZ + Z'B_i' + 2\beta W < 0 \end{gathered} \tag{37}$$

for $i = 1, 2, ..., N$ .

If (37) is satisfied, then $K$ is given by

$$K = ZW^{-1}. \tag{38}$$

And, for the second stage, Manesco (2013) presents an extension of Theorem 1, in terms of sufficient LMI conditions for the design of a robust SOF gain, regarding the specification of a minimum decay rate $\gamma > 0$, as stated in Theorem 2.

**Theorem 2.** *(MANESCO, 2013) Assuming that there exists a state feedback gain $K$ such that $A(\alpha)+B(\alpha)K$ is asymptotically stable, then there exists a stabilizing static output feedback gain $L$ such that $A(\alpha)+B(\alpha)LC(\alpha)$ is asymptotically stable, with a decay rate greater than or equal to $\gamma > 0$, if there exist a symmetric matrix $P > 0$ and matrices $F$, $G$, $H$, and $J$ such that*

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK + 2\gamma P & * & * \\ P - F' + GA_i + GB_iK & -G - G' & * \\ B_i'F' + JC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0 \tag{39}$$

*for $i = 1, 2, ..., N$.*

*In the affirmative case, the robust static output feedback gain is given by $L = H^{-1}J$.*

**Proof:** The demonstration of Theorem 2 follows similarly as for Theorem 1, taking into account that the quadratic stability condition considering a lower bound $\gamma$ for the system decay rate is as defined in (36). ■

## 2.4 STUDY ON THE TWO-STAGE METHOD FOR SOF DESIGN

The strategy presented in Theorems 1 and 2 for addressing the SOF control problem is based on the two-stage method. Since the derivation of the SOF gain $L$ is dependent on the existence of a state feedback gain $K$ (as properly discussed in the previous sections), the LMIs (9) and (39) represent only sufficient conditions.

Regarding this scope, it is reasonable to assume the existence of a more suitable choice for the first stage gain $K$. In fact, one can observe that in the particular case where $K = K_0 = L_0C$, and $L_0$ is a solution for (9), the output feedback system presents the same dynamic representation as in the state feedback case. Therefore, to examine this fact, a study assuming that the first stage gain is designed as $K = K_0 = L_0C$ was executed. Later, considering a more realistic scenario, the design of a first stage gain $K = L_0C + \Delta K$ was also investigated.

In fact, in the studies performed regarding this issue, it was observed that if the first stage gain $K$ is properly designed, then we can assure that a solution for LMIs (9) exists. The first results of these studies, with $K = L_0 C$, are formalized in Theorem 3.

**Remark 1.** *For simplifying the notation in the analysis, the matrices were assumed to be all known (i.e. $(A,B,C)(\alpha) = (A,B,C)$). The obtained results can be directly extended to the robust case.*

**Theorem 3.** *Assuming that there exist a symmetric matrix $X > 0$ and a matrix $L_0$ such that*

$$K = L_0 C \quad \textit{and} \quad M = KX \tag{40}$$

*with*

$$XA' + AX + BM + M'B' < 0, \tag{41}$$

*where $L_0$ is a stabilizing SOF gain for $A + BL_0C$, then, LMIs* (9) *are feasible.*

**Proof:** Let us define the matrix

$$\mathscr{F} = A'X^{-1} + X^{-1}A + K'B'X^{-1} + X^{-1}BK. \tag{42}$$

Conveniently rewriting $\mathscr{F}$ as

$$\mathscr{F} = X^{-1}\left(XA' + AX + XK'B' + BKX\right)X^{-1}, \tag{43}$$

and observing the initial hypothesis (40), which states that $M = KX$, we obtain

$$\mathscr{F} = X^{-1}\left(XA' + AX + M'B' + BM\right)X^{-1}. \tag{44}$$

And, regarding that, by (41), $XA' + AX + BM + M'B' < 0$, we can conclude that

$$\mathscr{F} < 0. \tag{45}$$

Now, we define a second matrix

$$\mathscr{G} = (A' + K'B')\frac{1}{2}I(A + BK). \tag{46}$$

Observe that (46) corresponds to a product of the kind $Y'Y$, which is always symmetric and positive semi-definite, regarding that $\det(Y'Y) \neq 0$. Therefore, we have that

$$\mathscr{G} > 0. \tag{47}$$

Note that $\mathscr{F}$ and $\mathscr{G}$ are contained within the following ranges

$$\lambda_{min}(\mathscr{F})I \leq \mathscr{F} \leq \lambda_{max}(\mathscr{F})I \quad \text{and} \quad \lambda_{min}(\mathscr{G})I \leq \mathscr{G} \leq \lambda_{max}(\mathscr{G})I \tag{48}$$

or even

$$-\lambda_{min}(\mathscr{F})I \geq -\mathscr{F} \geq -\lambda_{max}(\mathscr{F})I \quad \text{and} \quad -\lambda_{min}(\mathscr{G})I \geq -\mathscr{G} \geq -\lambda_{max}(\mathscr{G})I, \tag{49}$$

where $\lambda(\mathscr{F},\mathscr{G})$ denotes the eigenvalues of the matrix $(\mathscr{F},\mathscr{G})$.

Thus, we may state that

$$(-\mathscr{F}) + (-\mathscr{G}) \geq -\lambda_{max}(\mathscr{F})I - \lambda_{max}(\mathscr{G})I. \tag{50}$$

And therefore, since that, by (45) and (47), we have $\lambda_{max}(\mathscr{F}) < 0$ and $\lambda_{max}(\mathscr{G}) > 0$, we can observe that a sufficiently large scalar $\theta > 0$ such as

$$\theta > \frac{\lambda_{max}(\mathscr{G})}{-\lambda_{max}(\mathscr{F})}, \tag{51}$$

ensures that

$$-\theta\lambda_{max}(\mathscr{F})I - \lambda_{max}(\mathscr{G})I > 0. \tag{52}$$

For $\theta$ being a positive scalar, and $\mathscr{F} < 0$, we have that $\theta\mathscr{F} < 0$. So, comparing (52) with (50) one may observe that

$$\begin{gathered} -\theta\mathscr{F} - \mathscr{G} \geq -\theta\lambda_{max}(\mathscr{F})I - \lambda_{max}(\mathscr{G})I > 0 \Rightarrow \\ -\theta\mathscr{F} - \mathscr{G} > 0. \end{gathered} \tag{53}$$

Regarding the definitions (42) and (46), from (53) we have

$$-\theta(A'X^{-1} + X^{-1}A + K'B'X^{-1} + X^{-1}BK) - (A' + K'B')\frac{1}{2}I(A + BK) > 0, \tag{54}$$

which can be rearranged as

$$-\theta(A'X^{-1} + X^{-1}A + K'B'X^{-1} + X^{-1}BK) - (-A' - K'B')(\frac{1}{2}I)(-A - BK) > 0. \tag{55}$$

Then, applying the Schur's Complement in (55), we have that

$$\begin{bmatrix} -\theta(A'X^{-1} + X^{-1}A + K'B'X^{-1} + X^{-1}BK) & -A' - K'B' \\ -A - BK & 2I \end{bmatrix} > 0, \tag{56}$$

which multiplied by $-1$ gives

$$\Gamma < 0, \tag{57}$$

where

$$\Gamma = \begin{bmatrix} \theta(A'X^{-1} + X^{-1}A + K'B'X^{-1} + X^{-1}BK) & A' + K'B' \\ A + BK & -2I \end{bmatrix}, \tag{58}$$

concluding the first part of the proof.

Now, we define a new matrix

$$\Xi = \begin{bmatrix} \theta X^{-1}B \\ B \end{bmatrix} \begin{bmatrix} \theta B'X^{-1} & B' \end{bmatrix}, \tag{59}$$

which, similarly to (46), is always symmetric and positive semi-definite, and thus,

$$\Xi \geq 0. \tag{60}$$

Analogously as executed before, observe that (57) and (59) are contained within the following ranges

$$\lambda_{min}(\Gamma)I \leq \Gamma \leq \lambda_{max}(\Gamma)I \quad \text{and} \quad \lambda_{min}(\Xi)I \leq \Xi \leq \lambda_{max}(\Xi)I \tag{61}$$

and, consequently,

$$-\lambda_{min}(\Gamma)I \geq -\Gamma \geq -\lambda_{max}(\Gamma)I \quad \text{and} \quad -\lambda_{min}(\Xi)I \geq -\Xi \geq -\lambda_{max}(\Xi)I, \tag{62}$$

where $\lambda(\Gamma,\Xi)$ denotes the eigenvalues of the matrix $(\Gamma,\Xi)$.

Thus, we may state that

$$(-\Gamma) + (-\Xi) \geq -\lambda_{max}(\Gamma)I - \lambda_{max}(\Xi)I. \tag{63}$$

Therefore, note that a sufficiently small scalar $\delta > 0$, such as

$$\delta < \frac{\lambda_{max}(\Gamma)}{-\lambda_{max}(\Xi)}, \tag{64}$$

guarantees that

$$-\lambda_{max}(\Gamma)I - \delta\lambda_{max}(\Xi)I > 0, \tag{65}$$

regarding that $\lambda_{max}(\Gamma) < 0$ and $\lambda_{max}(\Xi) > 0$.

Since $\delta > 0$, we have that $\delta\lambda_{max}(\Xi) > 0$. Then, comparing (65) with (63), we can conclude that

$$\begin{gathered} -\Gamma - \delta\ \Xi \geq -\lambda_{max}(\Gamma)I - \delta\lambda_{max}(\Xi)I > 0 \Rightarrow \\ -\Gamma - \delta\ \Xi > 0. \end{gathered} \tag{66}$$

And, according to the definitions (58) and (59), we have

$$-\begin{bmatrix}\theta(A'X^{-1}+X^{-1}A+K'B'X^{-1}+X^{-1}BK) & A'+K'B' \\ A+BK & -2I\end{bmatrix}$$
$$-\delta\begin{bmatrix}\theta X^{-1}B \\ B\end{bmatrix}\begin{bmatrix}\theta B'X^{-1} & B'\end{bmatrix} > 0, \quad (67)$$

which can be conveniently represented as

$$-\begin{bmatrix}\theta(A'X^{-1}+X^{-1}A+K'B'X^{-1}+X^{-1}BK) & A'+K'B' \\ A+BK & -2I\end{bmatrix}$$
$$-\begin{bmatrix}-\theta X^{-1}B \\ -B\end{bmatrix}(\delta I)\begin{bmatrix}-\theta B'X^{-1} & -B'\end{bmatrix} > 0. \quad (68)$$

Observe that there exists a matrix $H$ such that

$$H = \frac{1}{2\delta}I \quad (69)$$

then, we have that

$$H+H = \frac{1}{\delta}I \quad \Rightarrow \quad (H+H')^{-1} = \delta I. \quad (70)$$

Which enables us to rewrite (68) as

$$-\begin{bmatrix}\theta(A'X^{-1}+X^{-1}A+K'B'X^{-1}+X^{-1}BK) & A'+K'B' \\ A+BK & -2I\end{bmatrix}$$
$$-\begin{bmatrix}-\theta X^{-1}B \\ -B\end{bmatrix}(H+H')^{-1}\begin{bmatrix}-\theta B'X^{-1} & -B'\end{bmatrix} > 0. \quad (71)$$

Now, applying the Schur's Complement in (71), we have

$$\begin{bmatrix}-\theta(A'X^{-1}+X^{-1}A+K'B'X^{-1}+X^{-1}BK) & -A'-K'B' & -\theta X^{-1}B \\ -A-BK & 2I & -B \\ -\theta B'X^{-1} & -B' & H+H'\end{bmatrix} > 0, \quad (72)$$

which multiplied by $-1$ gives

$$\begin{bmatrix}\theta(A'X^{-1}+X^{-1}A+K'B'X^{-1}+X^{-1}BK) & A'+K'B' & \theta X^{-1}B \\ A+BK & -2I & B \\ \theta B'X^{-1} & B' & -H-H'\end{bmatrix} < 0. \quad (73)$$

Remembering that $K = LC$ and that $L = H^{-1}J$, then we have that $K = H^{-1}JC$, and thus, that $C'J'H'^{-1} - K' = 0$. And, also, that $(C'J'H'^{-1} - K')H' = 0$. So, by defining

$F = F' = \theta X^{-1}$, we can see that

$$\begin{aligned}\theta X^{-1}B = FB &= FB + (C'J'H'^{-1} - K')H' \\ &= FB + C'J' - K'H'.\end{aligned} \tag{74}$$

And then, (73) becomes

$$\begin{bmatrix} A'F' + FA + K'B'F' + FBK & A' + K'B' & FB + C'J' - K'H' \\ A + BK & -2I & B \\ B'F' + JC - HK & B' & -H - H' \end{bmatrix} < 0. \tag{75}$$

Also, by considering $P = F$, and $G = G' = I$, we can make the following considerations

$$\begin{aligned} &A' + K'B' = 0 + A'I + K'B' = P - F + A'G' + K'B, \\ &-2I = -I - I' = -G - G', \quad \text{and} \\ &B = IB = GB, \end{aligned} \tag{76}$$

which, then, allows us to obtain

$$\begin{bmatrix} A'F' + FA + K'B'F' + FBK & P - F + A'G' + K'B'G' & FB + C'J' - K'H' \\ P - F' + GA + GBK & -G - G' & GB \\ B'F' + JC - HK & B'G' & -H - H' \end{bmatrix} < 0. \tag{77}$$

Finally, note that it is trivial to show (MANESCO, 2013) that by assuming that $(A,B,C) = (A,B,C)(\alpha)$, from (77), we obtain

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK & P - F + A_i'G' + K'B_i'G' & FB_i + C_i'J' - K'H' \\ P - F' + GA_i + GB_iK & -G - G' & GB_i \\ B_i'F' + JC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0, \tag{78}$$

for $i = 1, ..., N$, which corresponds to LMIs (9). ■

One can see that the proof of Theorem 3 establishes the existence of a convenient gain $K$ to be chosen in the first stage of the two-stage approach for SOF design. We see that, regarding the hypothesis of $K$ being such that $K = L_0C$, the discussion proves that the LMIs (9) are feasible, since there exists at least one solution in the variables $P, F, G, H$, with adequate values for $\theta$ and $\delta$.

A further step can be given by assuming that the designed gain $K$ in the first stage deviates slightly from the gain $K_0 = L_0C$. We assumed that $K = L_0C + \Delta K$, with a sufficiently small $\Delta K$. The goal is to evaluate how much the gain $K$ can deviate from $L_0K$ without compromising the feasibility of (9), in other words, how close to $L_0K$ the gain $K$ must be placed in order to guarantee the feasibility in the second stage. Theorem 4 presents the proposed conditions such that the second stage still providing a solution for the SOF problem, in the considered circumstances.

**Theorem 4.** *Assuming that there exist a symmetric matrix $X > 0$ and a matrix $L_0$ such that*

$$K = K_0 + \Delta K \quad \text{and} \quad M = KX \tag{79}$$

*with*

$$K_0 = L_0 C \tag{80}$$

*and*

$$XA' + AX + BM + M'B' < 0, \tag{81}$$

*where $L_0$ is a stabilizing SOF gain for $A + BL_0C$. Then, if $\Delta K$ is sufficiently small, the LMIs* (9) *are feasible.*

**Proof:** Define a matrix

$$\mathscr{C} = A'X^{-1} + X^{-1}A + K'B'X^{-1} + X^{-1}BK. \tag{82}$$

Regarding that $K = K_0 + \Delta K$, then

$$\mathscr{C} = A'X^{-1} + X^{-1}A + K_0'B'X^{-1} + X^{-1}BK_0 + \Delta K'B'X^{-1} + X^{-1}B\Delta K, \tag{83}$$

and considering that $\Delta K$ is such that (81) holds, then, making similar manipulations as in the proof of Theorem 3, we can conclude that

$$\mathscr{C} < 0. \tag{84}$$

Moreover, we also can define a matrix

$$\mathscr{V} = (A' + K_0'B' + \Delta K'B')\frac{1}{2}I(A + BK_0 + B\Delta K), \tag{85}$$

with $\mathscr{V} > 0$.

Now, note that if there exists a sufficiently small constant $\epsilon > 0$ such that

$$||\Delta K||_2^2 = \sigma_{max}(\Delta K'\Delta K) < \epsilon^2, \tag{86}$$

then, the inequality

$$-\zeta\lambda_{max}(\mathscr{C})I - \lambda_{max}(\mathscr{V})I > 0, \tag{87}$$

holds for a sufficiently large scalar $\zeta > 0$, such as

$$\zeta > \frac{\lambda_{max}(\mathscr{V})}{-\lambda_{max}(\mathscr{C})}, \tag{88}$$

since we have that $\lambda_{max}(\mathscr{C}) < 0$ and $\lambda_{max}(\mathscr{V}) > 0$.

Thus, similarly as in the proof of Theorem 3, one may observe that

$$-\zeta\mathscr{C} - \mathscr{V} > 0, \tag{89}$$

may also hold. And, by the definitions in (83) and (85), we have, equivalently,

$$\begin{aligned} &-\zeta(A'X^{-1} + X^{-1}A + K_0'B'X^{-1} + X^{-1}BK_0 + \Delta K'B'X^{-1} + X^{-1}B\Delta K) \\ &\qquad - (-A' - K_0'B' - \Delta K'B')\frac{1}{2}I(-A - BK_0 - B\Delta K) > 0. \end{aligned} \tag{90}$$

Applying the Schur's Complement in (90), and making the appropriate manipulations, we obtain

$$\Theta < 0, \tag{91}$$

where

$$\Theta = \begin{bmatrix} \zeta(A'X^{-1} + X^{-1}A + K_0'B'X^{-1} + X^{-1}BK_0 + \Delta K'B'X^{-1} + X^{-1}B\Delta K) & * \\ A + BK_0 + B\Delta K & -2I \end{bmatrix}. \tag{92}$$

In the sequence, we define a new matrix $\Pi$ such as

$$\Pi = \begin{bmatrix} -X^{-1}B\zeta + \Delta K'H' \\ -B \end{bmatrix} (\xi I) \begin{bmatrix} -B'X^{-1}\zeta + H\Delta K & -B' \end{bmatrix}, \tag{93}$$

where $\xi > 0$ is a scalar.

One may see that there is a $\Delta K$ such that

$$\Delta K = 2\xi\mathscr{X} \tag{94}$$

where $\mathscr{X}$ is a constant matrix.

Moreover, a matrix $H$ such as

$$H = \frac{1}{2\xi}I \tag{95}$$

would produce

$$H\Delta K = \frac{1}{2\xi}I2\xi\mathscr{X} = \mathscr{X}, \tag{96}$$

and thus, implying in a constant $\Delta K$ in (93). Then,

$$\Pi = \begin{bmatrix} -\zeta X^{-1}B + \mathscr{X}' \\ -B \end{bmatrix} (\xi I) \begin{bmatrix} -\zeta B'X^{-1} + \mathscr{X} & -B' \end{bmatrix}. \tag{97}$$

Additionally, by the definition presented in (95), we have that $H + H' = \frac{1}{\xi}I$, and thus,

$(H+H')^{-1} = \xi I$, implying in

$$\Pi = \begin{bmatrix} -X^{-1}B\zeta + \mathscr{X}' \\ -B \end{bmatrix} (H+H')^{-1} \begin{bmatrix} -B'X^{-1}\zeta + \mathscr{X} & -B' \end{bmatrix}. \tag{98}$$

With this analysis, we see that $\Delta K$ may be such that it will not interfere on the positiveness of $\Pi$, enabling the inequality

$$-\lambda_{max}(\Theta)I - \xi\lambda_{max}(\Pi)I > 0, \tag{99}$$

to hold with a sufficient small $\xi$ such as

$$\xi < \frac{\lambda_{max}(\Gamma)}{-\lambda_{max}(\Xi)}, \tag{100}$$

regarding that $-\Theta > 0$ and $-\Pi \leq 0$, since $\Pi$ is in the form $Y'Y$.

Consequently, since $-\Theta - \xi\Pi \geq -\lambda_{max}(\Theta)I - \xi\lambda_{max}(\Pi)I > 0$, then

$$-\Theta - \xi\Pi > 0. \tag{101}$$

Now, considering the definitions in (92) and (93), we have that (101) becomes

$$\begin{aligned} -&\begin{bmatrix} \zeta(A'X^{-1} + X^{-1}A + K_0'B'X^{-1} + X^{-1}BK_0 + \Delta K'B'X^{-1} + X^{-1}B\Delta K) & * \\ A + BK_0 + B\Delta K & -2I \end{bmatrix} \\ &- \begin{bmatrix} -X^{-1}B\zeta + \Delta K'H' \\ -B \end{bmatrix} (H+H')^{-1} \begin{bmatrix} -B'X^{-1}\zeta + H\Delta K & -B' \end{bmatrix} > 0, \end{aligned} \tag{102}$$

which holds with the existence of properly values for $\zeta$ and $\xi$ in (88) and (100), respectively.

And, strategically rewriting (102) as

$$\begin{aligned} &\begin{bmatrix} -\zeta(A'X^{-1} + X^{-1}A + K_0'B'X^{-1} + X^{-1}BK_0 + \Delta K'B'X^{-1} + X^{-1}B\Delta K) & * \\ -A - BK_0 - B\Delta K & 2I \end{bmatrix} \\ &- \begin{bmatrix} -X^{-1}B\zeta + \Delta K'H' \\ -B \end{bmatrix} (H+H')^{-1} \begin{bmatrix} -B'X^{-1}\zeta + H\Delta K & -B' \end{bmatrix} > 0, \end{aligned} \tag{103}$$

and, applying the Schur's Complement to (103), one can obtain

$$\begin{bmatrix} -\zeta(A'X^{-1} + X^{-1}A + K_0'B'X^{-1} + X^{-1}BK_0 + \Delta K'B'X^{-1} + X^{-1}B\Delta K) \\ -A - BK_0 - B\Delta K \\ -B'X^{-1}\zeta + H\Delta K \end{bmatrix}$$

$$\left.\begin{matrix} -A'-K_0'B'-\Delta K'B' & -X^{-1}B\zeta+\Delta K'H' \\ 2I & -B \\ -B' & H+H' \end{matrix}\right] < 0, \quad (104)$$

which multiplied by $-1$ gives

$$\begin{bmatrix} \zeta(A'X^{-1}+X^{-1}A+K_0'B'X^{-1}+X^{-1}BK_0+\Delta K'B'X^{-1}+X^{-1}B\Delta K) & A'+K_0'B'+\Delta K'B' & X^{-1}B\zeta-\Delta K'H' \\ A+BK_0+B\Delta K & -2I & B \\ B'X^{-1}\zeta-H\Delta K & B' & -H-H' \end{bmatrix} < 0. \quad (105)$$

Remembering that $K_0 = L_0C$ and that $L_0 = H^{-1}J$, then we have that $K_0 = H^{-1}JC$, and thus, that $C'J'H'^{-1} - K_0' = 0$. And, also, that $(C'J'H'^{-1} - K_0')H' = 0$. So, by defining $F = F' = \zeta X^{-1}$, we can see that

$$\begin{aligned} X^{-1}B\zeta - \Delta K'H' = FB - \Delta K'H' &= FB - \Delta K'H' + (C'J'H'^{-1} - K_0')H' \\ &= FB - \Delta K'H' + C'J' - K_0'H' \\ &= FB - C'J' - \Delta K'H' - K_0'H' \\ &= FB + C'J' - (K_0 + \Delta K)'H' \\ &= FB + C'J' - K'H'. \end{aligned} \quad (106)$$

And then, (105) becomes

$$\begin{bmatrix} A'F'+FA+K'B'F'+FBK & A'+K'B' & FB-K'H' \\ A+BK & -2I & B \\ B'F'-HK & B' & -H-H' \end{bmatrix} < 0. \quad (107)$$

Also, by defining $P = F$, and $G = G' = I$, we can make the following considerations

$$\begin{aligned} &A' + K'B' = 0 + A'I + K'B' = P - F + A'G' + K'B, \\ &-2I = -I - I' = -G - G', \quad \text{and} \\ &B = IB = GB, \end{aligned} \quad (108)$$

which allows us to finally obtain

$$\begin{bmatrix} A'F'+FA+K'B'F'+FBK & P-F+A'G'+K'B'G' & FB+C'J'-K'H' \\ P-F'+GA+GBK & -G-G' & GB \\ B'F'+JC-HK & B'G' & -H-H' \end{bmatrix} < 0. \quad (109)$$

And, analogously as executed in the proof of Theorem 3, by assuming that $(A,B,C) = (A,B,C)(\alpha)$, from (109), we have

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK & P - F + A_i'G' + K'B_i'G' & FB_i + C'J' - K'H' \\ P - F' + GA_i + GB_iK & -G - G' & GB_i \\ B_i'F' + JC - HK & B_i'G' & -H - H' \end{bmatrix} < 0, \tag{110}$$

for $i = 1, ..., N$, which corresponds to LMIs (9). ■

One can see that it is possible that $K$ diverges from $L_0C$ by $\Delta K$ and the LMIs will still have a solution. For that, $\Delta K$ must be small, say, there exists a sufficiently small scalar $\epsilon > 0$ such that $||\Delta K||_2^2 < \epsilon^2$. Additionally, in the second part of the analysis, we see that if $\Delta K = 2\xi\mathscr{X}$ where $\mathscr{X}$ is a constant matrix and $\xi > 0$ is a small scalar, the $\Delta K$ will not influence on the LMIs feasibility. So, in sum, if in the first stage the state feedback gain $K$ is chosen to be sufficiently near to $K_0 = L_0C$, then we ought to have more success in obtaining the desired static output feedback gain $L$ in the second stage.

### 2.4.1 COMPUTATIONAL ANALYSIS

In this subsection, an experiment is presented in order to investigate how the feasibility of the LMIs for obtaining a SOF gain is affected when the first stage gain $K$ diverges from $K_0 = L_0C$ by $\Delta K$.

For that, we consider a linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

where $A = \begin{bmatrix} 1 & -4 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -10 \\ 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

A first analysis consisted in obtaining a SOF gain $L_0$. This was accomplished by obtaining a state feedback gain $K = MX^{-1}$ using the conventional LMI strategy given in (8), rewritten in (81). Then, $L_0$ was obtained by solving the LMIs in Theorem 1, using $K$ as input parameter. In the sequence, the strategy for obtaining the SOF gain was tested by using the state feedback gain $K = L_0C + \Delta K$ as input parameter. The LMIs in Theorem 1 were solved regarding the restrictions imposed to the matrices $P, F, G$, and $H$, as presented in the proofs of Theorems 3 and 4. The matrix $X$, solution for the first stage executed in order to obtain $L_0$, was also used as input parameter, since $F = X^{-1}\zeta$. In this test, the elements in $\Delta K = [\Delta K_1 \;\; \Delta K_2]$ lied within a specified range around the nominal values of $K_0 = [K_1 \;\; K_2]$.

So, firstly, a state feedback gain

$$K = [0.0543 \quad -0.8304], \tag{111}$$

and

$$X = \begin{bmatrix} 1.5978 & -0.3551 \\ -0.3571 & 0.5326 \end{bmatrix} < 0, \tag{112}$$

were obtained by solving

$$\begin{gathered} X > 0 \\ XA' + AX + BM + M'B' < 0 \end{gathered} \tag{113}$$

for $X$ and $M$, where $K = MX^{-1}$.

Then, using the LMIs in (9), and using the gain (111) as input parameter, the SOF gain

$$L_0 = 2.7749 \tag{114}$$

was designed.

So, a state feedback gain $K_0$ was then considered as input parameter in the form

$$K_0 = L_0 C = [2.7749 \quad 0], \tag{115}$$

and the LMIs in (9) were tested, regarding that $P = F = X^{-1}\zeta$, $G = I$, and $H = \frac{1}{2\xi}I$, with $\zeta$ and $\xi$ considered as scalar variables, by varying the elements in $K_0$ in the range

$$-8.7749 \leq K_1 \leq 12.7749 \quad -10 \leq K_2 \leq 10. \tag{116}$$

Figure 3 shows the feasibility region obtained in the first analysis. Each ($\times$) corresponds to a feasible point for a given $K = K_0 + \Delta K$ used as input parameter. One may see that, in fact, a solution was possible to be obtained, even when $K$ diverts from $K_0$. However, the feasibility region does not have $K_0$ in its center, which means that the growth of $\Delta K$ is limited depending on which direction it occurs. This may imply in a small parameter $\epsilon$, which is considered for guaranteeing feasibility when $K$ is different from $K_0$, since $K_0$ is near to the limits of the feasibility region.

In fact, in this experiment, according to the shape of the obtained feasibility region, the minimum $\epsilon$ such that the maximum singular value ($\sigma_{max}$) of $\Delta K'\Delta K$ meets the condition

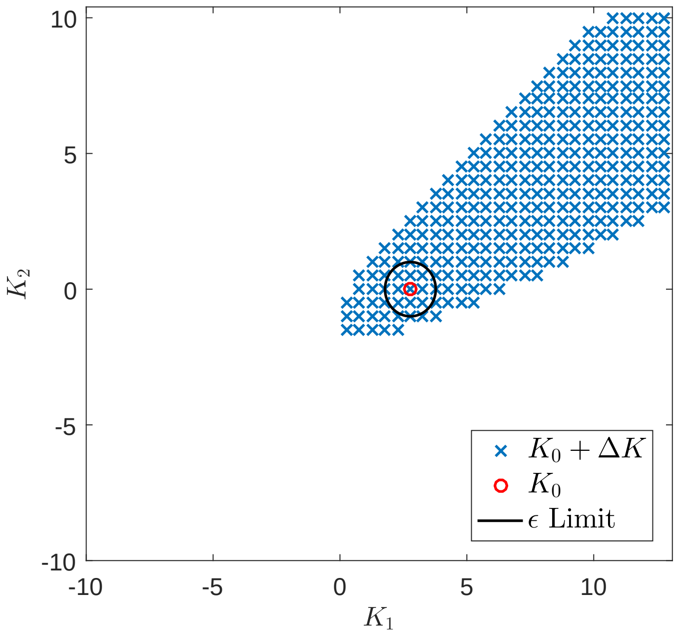
$$\sigma_{max}\left(\Delta K'\Delta K\right) < \epsilon^2, \tag{117}$$

and thus, guarantees the feasibility of the LMIs in (9) was found to be

$$\epsilon_{min} = 1,$$

indicating that $||\Delta K||_2 < 1$. The region for which $\Delta K$ is such that (117) holds is represented by the points inside the circle in Figure 2.

Figure 2 - Feasibility region: with restrictions imposed on the LMIs' variables for SOF gain search.




Source: Author's own results.

**Remark 2.** *The obtained value of* $\epsilon$ *depends on the step adopted for generating the values in* $\Delta K$*. With a more precise step equal to* 0.1*,* $\epsilon_{min}$ *was found to be equal to* 1.3*, and thus,* $||\Delta K||_{max} = 1.69$.

Now, introducing a second experiment, a solution $L_0$ for the LMIs (9) was derived using the normal procedure, *i.e.* the two-stage method, with $K$ in the first stage being obtained using the conventional LMI approach given in (8), and $L_0$ in the second stage by using the LMIs in Theorem 1. In the sequence, the SOF gain design strategy were tested by using a gain $K = L_0C + \Delta K$ as an input parameter. No restrictions on the matrices $P, F, G, H$ and $J$ were imposed. In this test, the elements in $\Delta K = [\Delta K_1 \quad \Delta K_2]$ lied within a specified range around the nominal values of $K_0 = [K_1 \quad K_2]$.

So, firstly, a state feedback gain

$$K = [0.0543 \quad -0.8304] \tag{118}$$

$K$ was obtained solving

$$\begin{aligned} X &> 0 \\ XA' + AX + BM + M'B' &< 0 \end{aligned} \tag{119}$$

for $X$ and $M$, where $K = MX^{-1}$.

Then, solving the LMIs in (9), using the gain (118) as input parameter, the SOF gain

$$L_0 = 2.7749 \tag{120}$$

was designed.

A state feedback gain $K_0$ was then considered as input parameter in the form

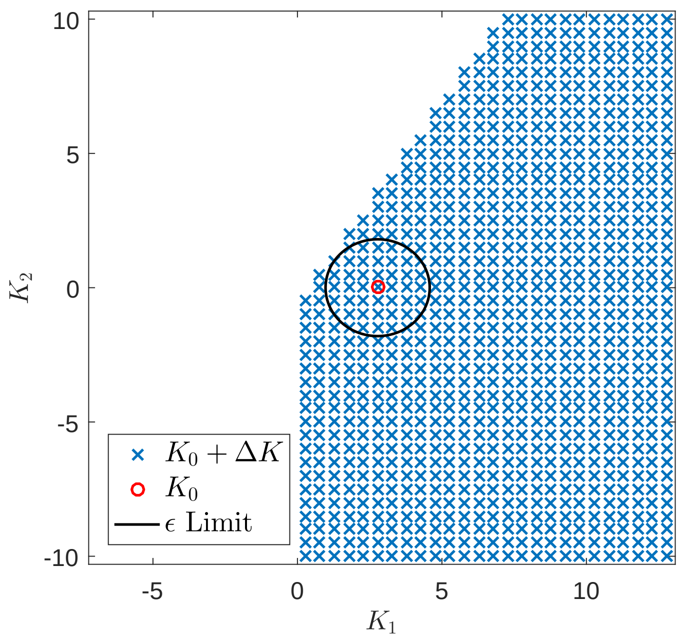
$$K_0 = L_0 C = [2.7749 \;\; 0], \tag{121}$$

and the LMIs in (9) were tested by varying the elements in $K_0$ in the range

$$-8.7749 \leq K_1 \leq 12.7749 \quad -10 \leq K_2 \leq 10. \tag{122}$$

Figure 3 shows the feasibility region obtained in this second analysis. One may see that, in fact, a solution was possible to be obtained, even when $K$ diverges from $K_0$. Note that the feasibility is limited by the value of $K_1$, since the region seems to grow as $K_1$ assumes higher values. Also observe that leaving the matrices $P, F, G$, and $H$ without restrictions resulted in a wider feasibility, as one may already have expected.

Figure 3 - Feasibility region: using standard LMIs for SOF gain search.




Source: Author's own results.

According to the shape of the obtained feasibility region, the minimum $\epsilon$ such that

$$\sigma_{max}(\Delta K' \Delta K) < \epsilon^2, \tag{123}$$

and thus, guarantees the feasibility of the LMIs in (9) was found to be

$$\epsilon_{min} = 1.8028.$$

This indicates that $||\Delta K||_2 < 3.2501$.

The region for which $\Delta K$ is such that (123) holds is represented by the points inside the circle in Figure 2. Note the minimum value for $\epsilon$ is greater when compared to the one observed in the first analysis. This may be attributed to the fact that since no restrictions were imposed on the problem variables, the LMIs assumed a more relaxed form, and thus, the feasibility region is such that we could guaranteed a solution for a slightly higher $||\Delta K||$.

With the presented analysis, one can conclude that when the first stage gain $K$ is properly designed, the LMIs in Theorem 1 are always feasible, given the imposed restrictions upon the matrices $P, F, G$, and $H$. Furthermore, it is possible to specify a maximum deviation $\Delta K$ from the suitable gain $K_0 = L_0 C$, by considering a restriction such as $\sigma_{max}(\Delta K' \Delta K) < \epsilon^2$, for a small $\epsilon$. The restrictions applied on the problem's variables implied on a slightly small range for which $||\Delta K||$ must lay in order to guarantee feasibility using a given $K = K_0 + \Delta K$ when compared to the results observed when the variables were considered to assume any form.

## 2.5 RELAXATION STRATEGIES FOR THE DESIGN OF ROBUST SOF CONTROLLERS

In this section, the development of more relaxed LMI conditions for obtaining the SOF controller is presented. The conservatism in the SOF control problem is reduce by considering the use of a parameter-dependent Lyapunov matrix, and Finsler Lemma's additional variables.

### 2.5.1 Parameter-Dependent Lyapunov Functions (PDLFs)

In the strategy discussed in Sections 2.3 and 2.4, the quadratic stability guarantee is based on the existence of a common quadratic Lyapunov function (CQLF), *i.e.* the Lyapunov's matrix $P$ is assumed to be fixed in Theorems 1 and 2. As discussed in Chapter 1, although CQLFs efficiently solve several convex optimization problems, its use results in restrictive effects when dealing with uncertain systems (OLIVEIRA; PERES, 2006).

With the intend of obtaining more relaxed LMI conditions, Manesco (2013) proposed a formulation based on parameter-dependent Lyapunov functions (PDLFs). Basically, the Lyapunov's matrix $P$ is considered to be dependent on the system's uncertain parameter $\alpha$, and thus, able to be represented in terms of convex combination of $N$ vertices of a polytope, as well as the system's matrices $(A,B,C)(\alpha)$. In sum, $P(\alpha)$ is defined as

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i, \quad \alpha \in \wedge_N, \tag{124}$$

where $\wedge_N$ is the unitary simplex defined in (3).

The extension to relaxed LMI conditions proposed in Manesco (2013) using PDLFs is presented in Theorem 5.

**Theorem 5.** *(MANESCO, 2013) Assuming that there exists a state feedback gain $K$ such that $A(\alpha)+B(\alpha)K$ is asymptotically stable, then there exists a stabilizing static output feedback gain $L$ such that $A(\alpha)+B(\alpha)LC(\alpha)$ is asymptotically stable, with a decay rate greater than or equal to $\gamma > 0$, if there exist symmetric matrices $P_i > 0$ and matrices $F$, $G$, $H$, and $J$ such that*

$$\begin{bmatrix} A_i'F'+FA_i+K'B_i'F'+FB_iK+2\gamma P_i & P_i-F+A_i'G'+K'B_i'G' & FB_i+C_i'J'-K'H' \\ P_i-F'+GA_i+GB_iK & -G-G' & GB_i \\ B_i'F'+JC_i-HK & B_i'G' & -H-H' \end{bmatrix} < 0 \tag{125}$$

*for $i = 1,2,...,N$.*

*In the affirmative case, the robust static output feedback gain is given by $L = H^{-1}J$.*

**Proof:** As mentioned in Theorem's 1 demonstration, the matrix $H$ is invertible if (125) has a solution.

Multiplying (125) by $\alpha_i$, and summing every term in $i$, from $i = 1$ to $i = N$, and regarding that $\sum_{i=1}^{N} \alpha_i = 1$, we obtain

$$\begin{bmatrix} A'(\alpha)F'+FA(\alpha)+K'B'(\alpha)F'+FB(\alpha)K+2\gamma P(\alpha) & * & * \\ P(\alpha)-F'+GA(\alpha)+GB(\alpha)K & -G-G' & * \\ B'(\alpha)F'+JC(\alpha)-HK & B'(\alpha)G' & -H-H' \end{bmatrix} < 0 \tag{126}$$

The remaining part of the proof follows similarly as Theorem's 1, taking into account that the quadratic stability condition considering a lower bound $\gamma$ for the system decay rate is as defined in (36). ■

The fact that using a Parameter-Dependent Lyapunov Function reduces the conservatism of the robust SOF control strategy presented in Theorem 2 can be proved by noticing that if the LMIs in Theorem 2 hold, then the LMIs in Theorem 5 will also hold. This relation is stated in Theorem 6.

**Theorem 6.** *Suppose that there exist a symmetric matrix $P$ and matrices $F$, $G$, $H$, and $J$ such that LMI conditions in Theorem 2* (39) *hold. Then, the LMI conditions* (125) *in Theorem 5 also hold.*

**Proof:** Let $P_i = P$, for $i = 1, ..., N$ in (125). With this consideration, note that (125) corresponds to

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK + 2\gamma P & P - F + A_i'G' + K'B_i'G' & FB_i + C_i'J' - K'H' \\ P - F' + GA_i + GB_iK & -G - G' & GB_i \\ B_i'F' + JC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0$$

for $i = 1, 2, ..., N$, which are feasible LMIs conditions in $P$, $F$, $G$, $H$, and $J$ by hypothesis. Therefore, if there is a solution for (39), then (125) will also have a solution. ∎

### 2.5.2 Parameter-Dependent Finsler's Variables

Aiming to provide even less conservative conditions for the design of robust SOF controllers, a second relaxation strategy is proposed in this work. In this new approach, the additional variables introduced via Finsler's Lemma are considered to be dependent on the uncertain parameter $\alpha$, as well as the Lyapunov's matrix ($P(\alpha)$).

Therefore, the Finsler's Lemma additional variables are assumed to be defined as

$$\begin{bmatrix} F(\alpha) \\ G(\alpha) \end{bmatrix} = \begin{bmatrix} \chi_1(\alpha) \\ \chi_2(\alpha) \end{bmatrix} = \chi(\alpha). \tag{127}$$

This new assumption is similar as the one proposed in the PDLF method. Setting the matrices $F$ and $G$ to be parameter-dependent enables the solution algorithm to search for different pair of matrices $F_i$ and $G_i$ for each of the $N$ vertices of the system's polytope. This procedure is more relaxed since it is easier to find a specific pair of matrices that satisfy the conditions imposed on the LMIs for each vertex, than to find a single pair that holds for the whole polytope.

However, this assumption will at some point imply on the product between two parameter-dependent matrices in the mathematical formulation of this control problem.

As in this work parameter-dependent matrices are being defined in terms of a polytopic representation, the presence of a term such as

$$M(\alpha)N(\alpha), \tag{128}$$

where $(M,N)(\alpha)$ are generic parameter-dependent matrices, will produce crossed-products between the $\alpha_i$ as evidenced bellow.

$$\sum_{i=1}^{N} \alpha_i M_i \sum_{i=1}^{N} \alpha_i N_i = \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j M_i N_j =$$

$$\begin{aligned} &\alpha_1\alpha_1 M_1 N_1 + \alpha_1\alpha_2 M_1 N_2 + \cdots + \alpha_1\alpha_N M_1 N_N + \cdots \\ &+\alpha_2\alpha_1 M_2 N_1 + \alpha_2\alpha_2 M_2 N_2 + \cdots + \alpha_2\alpha_N M_2 N_N + \cdots \\ &+\alpha_N\alpha_1 M_N N_1 + \alpha_N\alpha_2 M_N N_2 + \cdots + \alpha_N\alpha_N M_N N_N. \end{aligned} \tag{129}$$

In order to derive an equivalent representation for (129) one can observe the following property.

**Property 2.1.** *If the following LMIs*

$$\Upsilon_{ii} < 0, \quad i = 1, 2, ..., N, \tag{130}$$

*and,*

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \quad 1 \leq i < j \leq N \tag{131}$$

*holds, then it is true that*

$$\sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j \Upsilon_{ij} < 0. \tag{132}$$

**Proof:** See Tanaka, Ikeda and Wang (1998). ■

Note that the statement presented in Property 2.1 builds a bridge between the crossed-products and two sets of LMIs that does not depend on the $\alpha_i$.

In Theorem 7, based on Manesco (2013), the new and more relaxed LMI conditions obtained (SERENI et al., 2018) for the design of SOF controllers are proposed, as a result of the assumption made in (127) and the application of Property 2.1.

**Theorem 7.** *Assuming that there exists a state feedback gain $K$ such that $A(\alpha) + B(\alpha)K$ is asymptotically stable, then there exists a stabilizing static output feedback gain $L$ such that $A(\alpha) + B(\alpha)LC(\alpha)$ is asymptotically stable, with a decay rate greater than or equal to $\gamma > 0$, if there exists symmetric matrices $P_i > 0$ and $P_j > 0$, and matrices $F_i$, $F_j$, $G_i$, $G_j$, $H$ and $J$ such that* (133) *and* (134) *are satisfied.*

$$\begin{bmatrix} A_i'F_i' + F_iA_i + K'B_i'F_i' + F_iB_iK + 2\gamma P_i & * & * \\ P_i - F_i' + G_iA_i + G_iB_iK & -G_i - G_i' & * \\ B_i'F_i' + JC_i - HK & B_i'G_i' & -H - H' \end{bmatrix} < 0 \tag{133}$$

*for $i = 1, 2, ..., N$.*

$$\begin{bmatrix} \Omega_{ij}+\Omega_{ji} & * & * \\ P_i-F_i'+G_iA_j+G_iB_jK+P_j-F_j'+G_jA_i+G_jB_iK & -G_i'-G_i-G_j'-G_j & * \\ B_i'F_j'+JC_i+JC_j+B_j'F_i'-2HK & B_i'G_j'+B_j'G_i' & -2H-2H' \end{bmatrix} < 0 \tag{134}$$

*for* $i=1,2,...,N-1,$ *and* $j=i+1,i+2,...,N,$ *where*

$$\Omega_{ij}+\Omega_{ji} = A_i'F_j'+F_iA_j+K'B_i'F_j'+F_iB_jK+2\gamma P_i + A_j'F_i'+F_jA_i+K'B_j'F_i'+F_jB_iK+2\gamma P_j. \tag{135}$$

*In the affirmative case, the robust static output feedback gain is given by* $L=H^{-1}J$.

**Proof:** Multiplying (133) by $\alpha_i^2>0$, summing in $i$ from $i=1$ to $i=N$, and multiplying (134) by $\alpha_i\alpha_j>0$, summing in $i$ from $i=1$ to $i=N-1$ and summing in $j$ from $j=i+1$ to $j=N$, we obtain (136) and (137)

$$\begin{bmatrix} \sum_{i=1}^{N}\alpha_i^2(A_i'F_i'+F_iA_i+K'B_i'F_i'+F_iB_iK+2\gamma P_i) & * & * \\ \sum_{i=1}^{N}\alpha_i^2(P_i-F_i'+G_iA_i+G_iB_iK) & \sum_{i=1}^{N}\alpha_i^2(-G_i-G_i') & * \\ \sum_{i=1}^{N}\alpha_i^2(B_i'F_i'+JC_i-HK) & \sum_{i=1}^{N}\alpha_i^2(B_i'G_i') & \sum_{i=1}^{N}\alpha_i^2(-H-H') \end{bmatrix} < 0. \tag{136}$$

$$\begin{bmatrix} \sum_{i=1}^{N-1}\sum_{i=j+1}^{N}\alpha_i\alpha_j(\Omega_{ij}+\Omega_{ji}) & * & * \\ \sum_{i=1}^{N-1}\sum_{i=j+1}^{N}\alpha_i\alpha_j(P_i+P_j-F_i'-F_j'+G_iA_j+G_jA_i+G_iB_jK+G_jB_iK) & \sum_{i=1}^{N-1}\sum_{i=j+1}^{N}\alpha_i\alpha_j(-G_i'-G_j'-G_i-G_j) & * \\ \sum_{i=1}^{N-1}\sum_{i=j+1}^{N}\alpha_i\alpha_j(B_i'F_j'+B_j'F_i'+JC_i+JC_j-2HK) & \sum_{i=1}^{N-1}\sum_{i=j+1}^{N}\alpha_i\alpha_j(B_i'G_j'+B_j'G_i') & \sum_{i=1}^{N-1}\sum_{i=j+1}^{N}\alpha_i\alpha_j(-2H-2H') \end{bmatrix} < 0. \tag{137}$$

Now, summing (136) with (137), results in

$$\begin{bmatrix} \Gamma_{1,1} & * & * \\ \Gamma_{2,1} & \Gamma_{2,2} & * \\ \Gamma_{3,1} & \Gamma_{2,3} & \Gamma_{3,3} \end{bmatrix} < 0, \tag{138}$$

where

$$\begin{aligned} \Gamma_{1,1} = \sum_{i=1}^{N} \alpha_i^2 A_i' F_i' + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (A_i' F_j' + A_j' F_i') \\ + \sum_{i=1}^{N} \alpha_i^2 F_i A_i + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (F_i A_j' + F_j A_i) \\ + \sum_{i=1}^{N} \alpha_i^2 K' B_i' F_i' + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (K B_i' F_j' + K B_j' F_i') \\ + \sum_{i=1}^{N} \alpha_i^2 F_i B_i K + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (F_i B_j K + F_j B_i K) \\ + 2\gamma \sum_{i=1}^{N} \alpha_i^2 P_i + 2\gamma \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (t)(P_i + P_j), \end{aligned} \tag{139}$$

$$\begin{aligned} \Gamma_{2,1} = \sum_{i=1}^{N} \alpha_i^2 (P_i - F_i') + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (P_i + P_j - F_i' - F_j') \\ + \sum_{i=1}^{N} \alpha_i^2 (G_i A_i) + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (G_i A_j + G_j A_i) \\ + \sum_{i=1}^{N} \alpha_i^2 (G_i B_i K) + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (G_i B_j K + G_j B_i K), \end{aligned} \tag{140}$$

$$\begin{aligned} \Gamma_{3,1} = \sum_{i=1}^{N} \alpha_i^2 (B_i' F_i') + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (B_i' F_j' + B_j' F_i') \\ + \sum_{i=1}^{N} \alpha_i^2 (J C_i) + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (J C_i + J C_j) \\ - \sum_{i=1}^{N} \alpha_i^2 (H K) + 2 \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (H K), \end{aligned} \tag{141}$$

$$\Gamma_{2,2} = \sum_{i=1}^{N} \alpha_i^2 (-G_i - G_i') + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (-G_i - G_j - G_i' - G_j'), \tag{142}$$

$$\Gamma_{2,3} = \sum_{i=1}^{N} \alpha_i^2 (B_i' G_i') + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i \alpha_j (B_i' G_j' + B_j' G_i') \tag{143}$$

and,

$$\Gamma_{3,3} = \sum_{i=1}^{N} \alpha_i^2(-H-H') - 2\sum_{i=1}^{N-1}\sum_{i=j+1}^{N} \alpha_i\alpha_j(-2H-2H'). \tag{144}$$

Regarding Property 2.1 and making appropriate substitutions, (138) can be rewritten as (145).

$$\begin{bmatrix} \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(A_i'F_j'+F_iA_j+K'B_i'F_j'+F_iB_jK+2\gamma P_i) & * & * \\ \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(P_i-F_i'+G_iA_j+G_iB_jK) & \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(-G_i-G_i') & * \\ \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(B_i'F_j'+JC_j-HK) & \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(B_i'G_j') & \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(-H-H') \end{bmatrix} < 0 \tag{145}$$

Developing the terms in (145), and regarding that $\sum_{i=1}^{N}\alpha_i = \sum_{j=1}^{N}\alpha_j = 1$, we obtain

$$\begin{bmatrix} A'(\alpha)F'(\alpha)+F(\alpha)A(\alpha)+K'B'(\alpha)F'(\alpha)+F(\alpha)B(\alpha)K+2\gamma P(\alpha) & * & * \\ P(\alpha)-F'(\alpha)+G(\alpha)A(\alpha)+G(\alpha)B(\alpha)K & -G(\alpha)-G(\alpha)' & * \\ B'(\alpha)F'(\alpha)+JC(\alpha)-HK & B'(\alpha)G'(\alpha) & -H-H' \end{bmatrix} < 0. \tag{146}$$

The rest of the proof follows similarly as presented for Theorem 1, regarding the quadratic stability condition considering a lower bound $\gamma$ for the system decay rate is as defined in (36). ■

In a similarly analysis as the one executed before in this work, one can observe that the feasibility of the LMIs of Theorem 5 implies in the feasibility of the LMIs of Theorem 7. This fact is formally stated in Theorem 8.

**Theorem 8.** *Suppose that there exist symmetric matrices $P_i$ and matrices $F$, $G$, $H$, and $J$, for $i = 1,...,N$ such that LMI conditions in Theorem 5* (125) *hold. Then, the LMI conditions* (133) *and* (134) *in Theorem 7 also hold.*

**Proof:** Let $F_i = F$ and $G_i = G$, for $i = 1,...,N$. Then, we have that (133) and becomes

$$\begin{bmatrix} A_i'F'+FA_i+K'B_i'F'+FB_iK+2\gamma P_i & * & * \\ P_i-F'+GA_i+GB_iK & -G-G' & * \\ B_i'F'+JC_i-HK & B_i'G' & -H-H' \end{bmatrix} < 0,$$

which holds for $i = 1,...,N$ as our initial hypothesis.

Additionally, one can see that (134) becomes

$$\begin{bmatrix} \Omega_{ij}+\Omega_{ji} & * & * \\ P_i-F'+GA_j+GB_jK+P_i-F'+GA_i+GB_iK & -G'-G-G'-G & * \\ B_i'F'+JC_i+JC_j+B_j'F'-2HK & B_i'G'+B_j'G' & -2H-2H' \end{bmatrix} < 0 \tag{147}$$

for $i = 1,2,...,N-1$, and $j = i+1, i+2,...,N$, with

$$\begin{aligned} \Omega_{ij}+\Omega_{ji} = A_i'F'+FA_j+K'B_i'F'+FB_jK+2\gamma P_i \\ +A_j'F'+FA_i+K'B_j'F'+FB_iK+2\gamma P_i. \end{aligned} \tag{148}$$

Now, since (125) holds for $i = 1,...,N$, note that

$$\begin{bmatrix} A_i'F'+FA_i+K'B_i'F'+FB_iK+2\gamma P_i & * & * \\ P_i-F'+GA_i+GB_iK & -G'-G & * \\ B_i'F'+JC_i-HK & B_i'G' & -H-H' \end{bmatrix} < 0 \tag{149}$$

holds for $i = 1,2,...,N-1$, and that

$$\begin{bmatrix} A_j'F'+FA_j+K'B_j'F'+FB_jK+2\gamma P_j & * & * \\ P_j-F'+GA_j+GB_jK & -G'-G & * \\ B_j'F'+JC_j-HK & B_j'G' & -H-H' \end{bmatrix} < 0 \tag{150}$$

holds for $j = i+1, i+2,...,N$. Thus, (147) can be rewritten as

$$\begin{aligned} &\begin{bmatrix} A_i'F'+FA_i+K'B_i'F'+FB_iK+2\gamma P_i & * & * \\ P_i-F'+GA_i+GB_iK & -G'-G & * \\ B_i'F'+JC_i-HK & B_i'G' & -H-H' \end{bmatrix} \\ &\quad + \begin{bmatrix} A_j'F'+FA_j+K'B_j'F'+FB_jK+2\gamma P_j & * & * \\ P_j-F'+GA_j+GB_jK & -G'-G & * \\ B_j'F'+JC_j-HK & B_j'G' & -H-H' \end{bmatrix} < 0, \end{aligned} \tag{151}$$

and will hold for $i = 1,2,...,N-1$, and $j = i+1, i+2,...,N$. So, if (125) is feasible, then (133) and (134) will also be feasible. ■

## 2.6 FEASIBILITY ANALYSIS

Intending to investigate the efficacy of the presented theorems, some numerical and practical experiments were executed. In this section, two feasibility analysis are presented in order to evaluate the efficiency of each relaxation strategy. Also, a practical experiment was performed using a robust SOF controller designed using Theorem 7 to improve the performance of an active suspension system. The results of the tests are presented and discussed at the end of this section.

***Experiment 2.1***

In this numerical experiment an uncertain system described according to (1) is considered. This system can be represented in terms of convex combination of the following vertices

- Vertex 1

$$A_1 = \begin{bmatrix} -1 & 10 \\ -1 & -1 \end{bmatrix}, \ B_1 = \begin{bmatrix} -9 \\ 0 \end{bmatrix}, \text{ and } \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{152}$$

- Vertex 2

$$A_2 = \begin{bmatrix} 1 & -4 \\ -2 & -3 \end{bmatrix}, \ B_2 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \text{ and } \ C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{153}$$

Observing the format of matrix $C$ one can see that in this example only the state variable $x_1(t)$ is measurable. In this case, a simple state feedback gain will not be able to be directly applied in a practical situation. Therefore, it is suitable to apply the output feedback strategy.

Following the two-stage method presented previously, first a state feedback gain $K$ must be derived. Intending to consider the specification of a minimum decay, the gain $K$ was obtained using (37). Theorems 2, 5, and 7 were tested for deriving the robust SOF gain $L$ in the second stage.

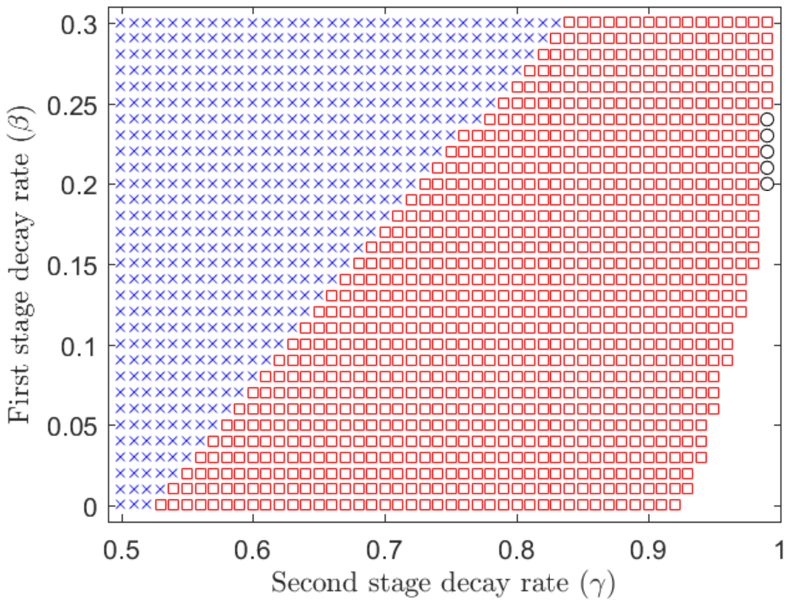
This first analysis was performed considering a range for the minimum decay rates in the first and second stage of the project as

$$0 \leq \beta \leq 0.3 \ \text{ and } \ 0.5 \leq \gamma \leq 1. \tag{154}$$

Programming the proposed LMIs via MATLAB® software, and solving via YALMIP interface (LOFBERG, 2004) and the standard MATLAB® LMI solver, the LMILab one can obtain the results presented in Figure 4, which represents the feasibility region obtained by each of the three considered theorems.

One can observe that by using Theorem 5, which considers the matrix $P$ as dependent on the uncertain parameter $\alpha$, results in a larger feasibility region when compared to Theorem 2, which is based on a fixed matrix $P$. Also, Figure 4 shows that Theorem 7 provides a slightly better result than Theorem 5.

Figure 4 - Feasibility region obtained with Theorem 2 ($\times$), Theorem 5 ($\times$ and $\square$), and Theorem 7 ($\times$, $\square$, and $\bigcirc$) for different minimum decay rates $\beta$ and $\gamma$.




Source: Author's own results.

***Experiment 2.2***

In Experiment 2.1, Theorem 7 relaxation strategy seems to provide little improvement in the feasibility region over the results provided by Theorem 5. This same numerical problem can be examined from the politopic uncertain perspective, which, in fact, is of uttermost relevance, since the decay rate bounding becomes useless if the system's uncertainties implies in severe restrictions to the control design.

With this purpose, a second feasibility test was executed, evaluating the perform of the theorems when the vertex (153) suffers a variation in matrices $A_2$ and $B_2$. For this second example, we assumed that the Vertex 1 remained the same as in (152), and

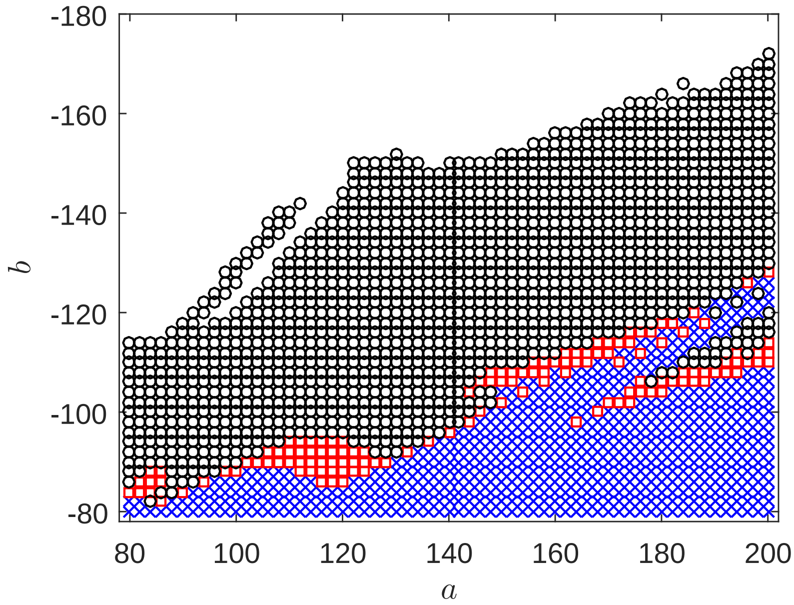
- Vertex 2

$$A_2 = \begin{bmatrix} a & -4 \\ -2 & -3 \end{bmatrix}, \; B_2 = \begin{bmatrix} b \\ 0 \end{bmatrix}, \text{ and } \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{155}$$

where $80 \leq a \leq 200$, and $-180 \leq b \leq -80$.

The proposed LMIs were solved similarly as in the previous analysis, but now fixing the decay rates of the first and second stages of the project as $\beta = \gamma = 0.6$, respectively. Regarding the values of $a$ and $b$ in the specified ranges, the obtained feasibility regions with each theorem is presented in Figure 5.

Figure 5 - Feasibility region obtained with Theorem 2 ($\times$), Theorem 5 ($\times$ and $\square$), and Theorem 7 ($\times$, $\square$, and $\bigcirc$) changing one of the polytope's vertex.




Source: Author's own results.

Figure 5 shows that the matrix $P(\alpha)$ assumed in Theorem 5 brings better results in terms of feasibility over Theorem 2. Furthermore, we can verify that using Theorem 7 resulted in a widely larger feasibility region, compared with the results obtained with Theorems 2 and 5. Thus, one can observe that considering the matrices $F$ and $G$ introduced with the Finsler's Lemma as functions of the uncertain parameter $\alpha$ clearly lead to less conservative restrictions, when dealing with politopic uncertainties.

It is important to notice that every point in Figure 5 represents a different polytope formed by its respective vertices, and consequently, possible different dynamic systems. Regarding this perspective, with the relaxation technique proposed in Theorem 7, the presented controller design strategy is able to be applied to a larger diversity of systems.

## 2.7 PRACTICAL EXPERIMENTS

Intending to study the practical applicability of the proposed strategy, the control design strategies presented in the previous sections were tested in a physical control problem. It consisted in developing a robust controller for an active suspension system. The performance of the designed robust SOF controller was evaluated through the analysis of the dynamic response of the system.

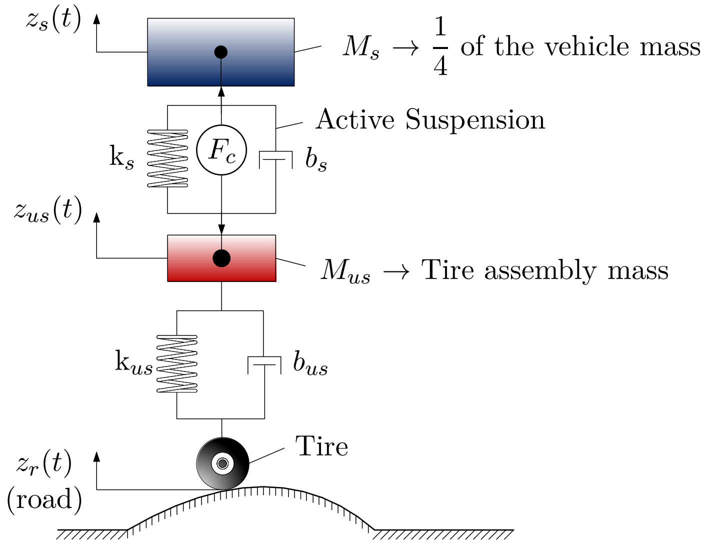
Figure 6 - QUANSER® Active Suspension. Property of the Laboratory of Research in Control at FEIS-UNESP.

Source: Own author.

The active suspension equipment used in the next experiments was a QUANSER® Active Suspension experimental system as shown in Figure 6. This system represents a quarter-car model, formed by three suspended plates, or floors. The top floor (vehicle body) is suspended over the middle plate (tire assembly) by springs and a DC motor that emulates an active suspension mechanism. The middle plate is in contact with the bottom plate though springs. The bottom plate is driven by a fast response DC motor in order to simulate different road profiles (QUANSER, 2009).

In Figure 7 the schematic diagram of the active suspension system is presented. $M_s$ is the sprung mass, representing 1/4 of vehicle body mass, $M_{us}$ is the unsprung mass that represents the tire of the quarter-car model, and $k_s$, $b_s$, $k_{us}$, and $b_{us}$, are the springs and dampers in the model assembly. $z_{us}(t)$ and $z_s(t)$ are the unsprung and sprung mass positions related to each shown reference level. At last, $z_r(t)$ is the road surface profile and $F_c(t)$ is the active suspension control command. These last two signals are the two inputs to the system in the considered approach. The objective is to properly control the active suspension in order to improve the passenger's comfort during the ride.

Figure 7 - Schematic diagram of an active suspension system.




Source: Adapted from Silva (2012).

A state-space model (QUANSER, 2009) of the system described in Figure 7 is given by

$$\dot{x}(t)=\begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-k_s}{M_s} & \frac{-b_s}{M_s} & 0 & \frac{b_s}{M_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{M_{us}} & \frac{b_s}{M_{us}} & \frac{-k_{us}}{M_{us}} & \frac{-(b_s+b_{us})}{M_{us}} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{\rho}{M_s} \\ 0 \\ \frac{-\rho}{M_{us}} \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t), \tag{156}$$

where $0 < \rho \leq 1$ is an uncertain parameter that represents a possible fault in the actuator. Additionally, the state and input vectors in (156) are defined as

$$x(t) = \begin{bmatrix} z_s(t) - z_{us}(t) \\ \dot{z_s}(t) \\ z_{us}(t) - z_r(t) \\ \dot{z_{us}}(t) \end{bmatrix} \quad \text{and} \quad u(t) = F_c(t) \tag{157}$$

The chosen state-space representation of the active suspension system describes the system's dynamics with respect to specifics performance measures, such as the suspension travel and the road handling. The former is associated to the relative movement between the vehicle body and the tire, which is constrained to a allowable workspace, and the later is directly dependent on the displacement between the tire and the road surface, *i.e.* tire deflection.

***Experiment 2.3***

In this experiment the controller design considers that the actuator may present a fault of up to 30% power loss (*i.e.* $0.7 \leq \rho \leq 1.0$). Also, it is suppose that only the state variable $\dot{z}_s(t)$ is available for online measurement. Taking that into account the parameters presented in Table 1 (QUANSER, 2009), and regarding that the control input will be represented by the signal $F_c(t)$, the active suspension may be described in terms of a polytope with two vertices:

- Vertex 1 (without fault)

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.347 & -3.061 & 0 & 3.061 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0.408 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}. \tag{158}$$

- Vertex 2 (with fault - 30% of power loss)

$$A_2 = A_1, B_2 = \begin{bmatrix} 0 \\ 0.2857 \\ 0 \\ -0.7 \end{bmatrix}, \quad \text{and} \quad C_2 = C_1. \tag{159}$$

Table 1 - Active Suspension Parameters

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $M_s$ | 2.45 kg | $k_{us}$ | 2500 N/m |
| $M_{us}$ | 1.0 kg | $b_s$ | 7.5 Ns/m |
| $k_s$ | 900 N/m | $b_{us}$ | 5.0 Ns/m |

Source: (QUANSER, 2009).

**Remark 3.** *Since it is not viable to damage the experimental equipment, the fault is emulated on the software that drives the DC motor through its power amplifier. The strategy consists in multiply the signal sent by the amplifier by* 0.7*. This procedure is executed online, internally in the computer software connected to the active suspension plant. With this, the nominal voltage which was supposed to drive the DC motor is reduced in* 30%*, producing the desired fault effect.*

Applying the proposed methodology, in the first stage of project, considering $\beta = 1$ in (37), we obtain the state feedback gain

$$K = \begin{bmatrix} -121.2615 & -11.0421 & 254.3442 & 0.2595 \end{bmatrix}. \tag{160}$$

And, in the second stage, using the LMIs in Theorem 7, and considering $\gamma = 1$ and (160), the designed static output feedback gain was

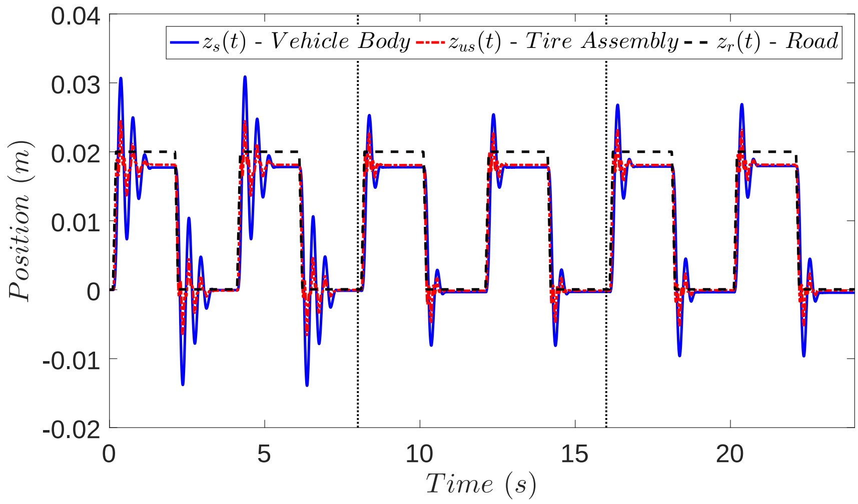
$$L = -24.6835. \tag{161}$$

**Remark 4.** *At this point, it is interesting to make an observation about the structure of the obtained robust SOF gain. As showed in* (161)*, the controller is a simple scalar gain. This is a direct consequence of the measured output $y(t)$ considered in the control design. Since it consists in a single state variable, the controller is able to work exclusively with that system state. Thus, applying LMIs in Theorem 6 produces a simple scalar gain outcome.*

*The aforementioned discussion underlines the fact that designing SOF controllers is a challenging task, though when a solution is found, the obtained controller will be simpler (a matrix with reduced dimensions) when compared to a full-state feedback controller.*

The observed behavior of the active suspension system with the designed robust SOF controller (161) is shown in Figure 8. During the experiment, the road profile $z_r(t)$ was set to a square wave with 0.02 m in amplitude and a period of 4 s. In the first 8 seconds of experiment, the system was operating in open loop. After that, the feedback control was switched on.

Figure 8 - System behavior with robust SOF controller ($\gamma = 1$): open-loop (0-8 s); closed-loop (8-16 s); fault: 30 % power loss in the actuator (16-24 s).




Source: Author's own results.

One can observe that the system presents a stable behavior in open loop. Although, both vehicle body and tire assembly shown accentuated oscillations. However, Figure 8 shows that the designed robust SOF gain was able to mitigate those oscillations, specially in the driver's seat position. Additionally, the system achieved a good performance even in the occurrence of a fault in the actuator, associated to a 30% of power loss, as seen in Figure 8 after 16 seconds of experiment.

***Experiment 2.4***

Another analysis were made aiming to evaluate the system performance when higher decay rates are specified in the controller design. To do so, Theorem 7 was applied, considering the same polytopic representation given in (158) and (159), and specifying $\beta = \gamma = 3$. With (37) one can found

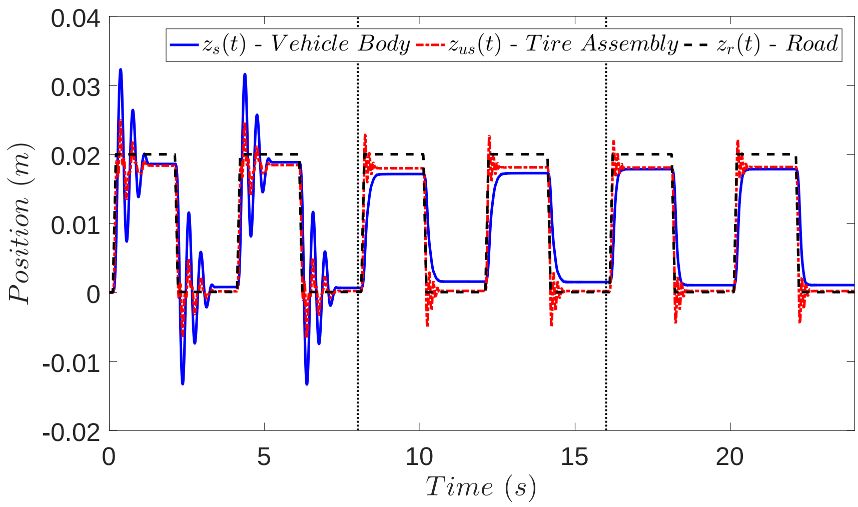
$$K = \begin{bmatrix} -2263.225 & -156.008 & -1630.615 & -19.517 \end{bmatrix}, \tag{162}$$

and, in sequence, (125) gives

$$L = -175.3377. \tag{163}$$

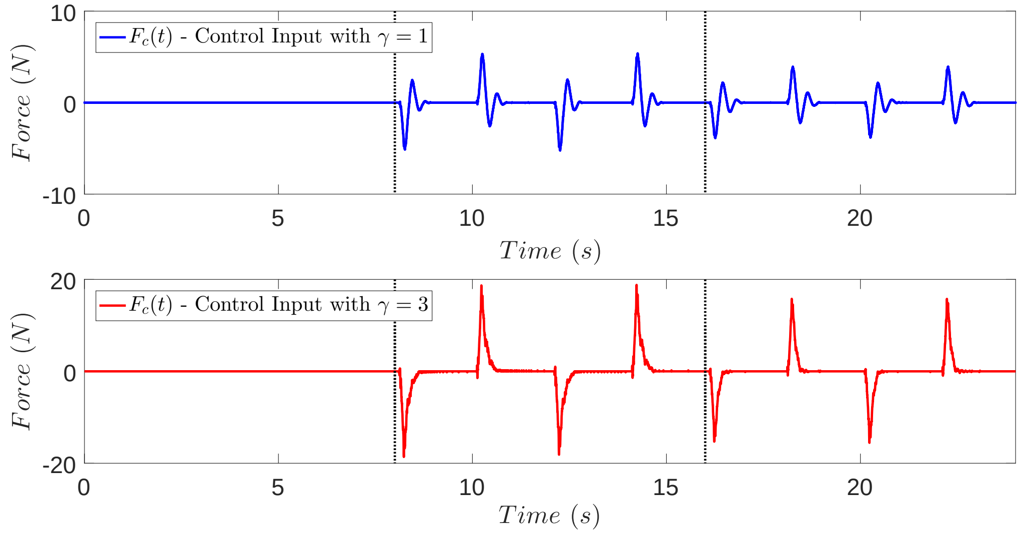
According to Figure 9, with a higher decay rate a even better dynamic response was achieved. Note that the vehicle body position have a soft progression as the road level suffers an abrupt variation. Also, Figure 10 shows that the control input $F_c(t)$ presented higher amplitudes when compared to the first experiment results. Although, the control signal remained within the saturation range of ±39.2 N (QUANSER, 2009).

Figure 9 - System behavior with robust SOF controller ($\gamma = 3$): open-loop (0-8 s); closed-loop (8-16 s); fault: 30 % power loss in the actuator (16-24 s).




Source: Author's own results.

Figure 10 - Control command $F_c(t)$ generated by the designed SOF controllers during the executed experiments with decay rates $\gamma = 1$ (top) and $\gamma = 3$ (bottom).




Source: Author's own results.

# 3 GAIN-SCHEDULING STATIC OUTPUT FEEDBACK CONTROL

This chapter has the purpose of presenting a new control solution for dealing with linear parameter-varying (LPV) systems by means of the design of a static output feedback gain-scheduled controller.

## 3.1 SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider the LPV system

$$\begin{aligned} \dot{x}(t) &= A(\alpha(t))x(t) + B(\alpha(t))u(t) \\ y(t) &= C(\alpha(t))x(t) \end{aligned} \tag{164}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the measured output vector, and $u(t) \in \mathbb{R}^m$ is the control input vector. Moreover, $A(\alpha(t)) \in \mathbb{R}^{n\times n}$, $B(\alpha(t)) \in \mathbb{R}^{n\times m}$, and $C(\alpha(t)) \in \mathbb{R}^{p\times n}$ are matrices where $\alpha(t) \in \mathbb{R}^n$ is a vector of measurable time-varying parameters. These matrices describes the system's dynamics and can be represented in a polytopic domain $\mathscr{B}$ defined as

$$\mathscr{B} = \left\{ (A,B,C)(\alpha(t)) : (A,B,C)(\alpha(t)) = \sum_{i=1}^{N} \alpha_i(t)(A,B,C)_i, \quad \alpha(t) \in \curlywedge_M; \ \ \forall t \right\}, \tag{165}$$

where $A_i$, $B_i$, and $C_i$ denote the $i$-th polytope vertex, and $N$ is the number of vertices of the polytope. Furthermore, $\mathscr{B}$ is parameterized in terms of the vector $\alpha(t)$, whose parameters $\alpha_i(t)$ belong to the unitary simplex set $\curlywedge_M$, defined as

$$\curlywedge_M = \left\{ \alpha(t) \in \mathbb{R}^n : \sum_{i=1}^{N} \alpha_i(t) = 1 \ \ ; \ \ \alpha_i(t) \geq 0 \ \ ; \ \ i = 1,...,N; \ \forall t \right\}. \tag{166}$$

Assuming that the feedback loop is composed by the following control law

$$u(t) = L(\alpha(t))y(t), \tag{167}$$

then the system (164) in closed-loop assumes the following form:

$$\dot{x}(t) = \Big[A(\alpha(t)) + B(\alpha(t))L(\alpha(t))C(\alpha(t))\Big]x(t). \tag{168}$$

In this new problem, the objective is to find a static output feedback gain-scheduled controller $L(\alpha(t)) \in \mathbb{R}^{m\times p}$ that asymptotically stabilizes (168). Since $L(\alpha(t))$ also depends on the time-varying parameter $\alpha(t)$, then it also can be described in terms of a convex combination of vertices $L_i$, such as

$$L(\alpha(t)) = \left\{ L(\alpha(t)) \in \mathbb{R}^{m\times p} : L(\alpha(t)) = \sum_{i=1}^{N} \alpha_i(t) L_i \quad ; \quad \alpha(t) \in \curlywedge_M \right\}. \tag{169}$$

## 3.2 STABILIZATION VIA GAIN-SCHEDULING SOF

The strategy proposed for the design of gain-scheduled SOF controller is the two-stage method, the same used on the design of robust SOF controllers, in Chapter 2. Therefore, the first step is to obtain a state feedback gain $K$ that asymptotically stabilizes the system (164), which may be derived using (37), as properly addressed in Section 2.2. Observe that, in this case, $K$ is not a function of $\alpha(t)$.

Since that in this case the SOF gain is a time-varying matrix, the second stage consists in deriving the $L_i$ constant matrices of the convex set in which $L(\alpha(t))$ lays in. However, similarly as occurred in de PDFV approach discussed in Subsection 2.5.2, considering that $L(\alpha(t))$ will result in a product between two parameter-dependent matrices. And, as properly addressed in the same Subsection 2.5.2, the mathematical formulation of the problem will lose its convex nature.

Intending to work around this issue, the strategy based on Property 2.1 is applied, making the properly considerations regarding that in this case $\alpha$ is a time-varying parameter.

Theorem 9 proposes new LMIs conditions for deriving a GS-SOF controller. The results obtained are based on the studies presented in Manesco (2013), which have been intensively discussed in Chapter 2. Moreover, Theorem 9 generalizes the results proposed in Sereni, Assunção and Teixeira (2017), by assuming that all system's matrices are affected by the time-varying parameter.

**Theorem 9.** *Assuming that there exists a state feedback gain $K$ such that $A(\alpha(t)) + B(\alpha(t))K$ is asymptotically stable, then there exists a stabilizing gain-scheduled static output feedback gain $L(\alpha(t))$ such that $A(\alpha(t)) + B(\alpha(t))L(\alpha(t))C(\alpha(t))$ is asymptotically stable if there exist a symmetric matrix $P > 0$ and matrices $F$, $G$, $H$, $J_i$, and $J_j$ such that* (170) *and* (171) *are satisfied.*

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK & * & * \\ P - F' + GA_i + GB_iK & -G - G' & * \\ B_i'F' + J_iC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0 \tag{170}$$

*for* $i = 1, 2, ..., N$.

$$\begin{bmatrix} \Xi_{ij} + \Xi_{ji} & * & * \\ 2P - 2F' + GA_j + GB_jK + GA_i + GB_iK & -2G' - 2G & * \\ B_i'F' + J_jC_i + J_iC_j + B_j'F' - 2HK & B_i'G' + B_j'G' & -2H - 2H' \end{bmatrix} < 0 \qquad (171)$$

*for* $i = 1, 2, ..., N-1$, *and* $j = i+1, i+2, ..., N$, *where*

$$\Xi_{ij} + \Xi_{ji} = A_i'F' + FA_j + K'B_i'F' + FB_jK + A_j'F' + FA_i + K'B_j'F' + FB_iK. \qquad (172)$$

*In the affirmative case, the gain-scheduled static output feedback gain with time-varying parameter is given by* $L(\alpha(t)) = H^{-1}J(\alpha(t))$, *where*

$$J(\alpha(t)) = \left\{ J(\alpha(t)) \in \mathbb{R}^{m \times p} : J(\alpha(t)) = \sum_{i=1}^{N} \alpha_i(t) J_i \quad ; \quad \alpha(t) \in \curlywedge_M \right\}. \qquad (173)$$

**Proof:** Multiplying (170) by $\alpha_i^2(t) > 0$, summing in $i$ from $i = 1$ to $i = N$, and multiplying (171) by $\alpha_i(t)\alpha_j(t) > 0$, summing in $i$ from $i = 1$ to $i = N-1$ and summing in $j$ from $j = i+1$ to $j = N$, we obtain (174) and (175)

$$\begin{bmatrix} \sum_{i=1}^{N} \alpha_i^2(t)(A_i'F' + FA_i + K'B_i'F' + FB_iK) & * & * \\ \sum_{i=1}^{N} \alpha_i^2(t)(P - F' + GA_i + GB_iK) & \sum_{i=1}^{N} \alpha_i^2(t)(-G - G') & * \\ \sum_{i=1}^{N} \alpha_i^2(t)(B_i'F' + J_iC_i - HK) & \sum_{i=1}^{N} \alpha_i^2(t)(B_i'G') & \sum_{i=1}^{N} \alpha_i^2(t)(-H - H') \end{bmatrix} < 0. \qquad (174)$$

$$\begin{bmatrix} \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(A_i'F' + FA_i + K'B_i'F' + FB_iK + A_j'F' + FA_j + K'B_j'F' + FB_jK) & * & * \\ \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(2P - 2F' + GA_j + GA_i + GB_jK + GB_iK) & \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(-2G' - 2G) & * \\ \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(B_i'F' + B_j'F' + J_jC_i + J_iC_j - 2HK) & \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(B_i'G' + B_j'G') & \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(-2H - 2H') \end{bmatrix} < 0. \qquad (175)$$

Now, summing (174) with (175), results in

$$\begin{bmatrix} \hat{\Delta}_{1,1} & * & * \\ \hat{\Delta}_{2,1} & \hat{\Delta}_{2,2} & * \\ \hat{\Delta}_{3,1} & \hat{\Delta}_{2,3} & \hat{\Delta}_{3,3} \end{bmatrix} < 0, \tag{176}$$

where

$$\begin{aligned} \hat{\Delta}_{1,1} = {} & \sum_{i=1}^{N} \alpha_i^2(t) A_i' F' + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(A_i'F' + A_j'F') \\ & + \sum_{i=1}^{N} \alpha_i^2(t) K' B_i' F' + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(KB_i'F' + KB_j'F') \\ & + \sum_{i=1}^{N} \alpha_i^2(t) F A_i + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(FA_j' + FA_i) \\ & + \sum_{i=1}^{N} \alpha_i^2(t) F B_i K + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(FB_jK + FB_iK), \end{aligned} \tag{177}$$

$$\begin{aligned} \hat{\Delta}_{2,1} = {} & \sum_{i=1}^{N} \alpha_i^2(t)(P - F') + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(2P - 2F') \\ & + \sum_{i=1}^{N} \alpha_i^2(t)(GB_iK) + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(GB_jK + GB_iK) \\ & + \sum_{i=1}^{N} \alpha_i^2(t)(GA_i) + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(GA_j + GA_i), \end{aligned} \tag{178}$$

$$\begin{aligned} \hat{\Delta}_{3,1} = {} & \sum_{i=1}^{N} \alpha_i^2(t)(B_i'F') + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(B_i'F' + B_j'F') \\ & + \sum_{i=1}^{N} \alpha_i^2(t)(J_iC_i) + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(J_jC_i + J_iC_j) \\ & - \sum_{i=1}^{N} \alpha_i^2(t)(HK) - 2\sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(HK), \end{aligned} \tag{179}$$

$$\hat{\Delta}_{2,2} = \sum_{i=1}^{N} \alpha_i^2(t)(-G - G') + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(-2G - 2G'), \tag{180}$$

$$\hat{\Delta}_{2,3} = \sum_{i=1}^{N} \alpha_i^2(t)(B_i'G') + \sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(B_i'G' + B_j'G') \tag{181}$$

and,

$$\hat{\Delta}_{3,3} = \sum_{i=1}^{N} \alpha_i^2(t)(-H - H') + 2\sum_{i=1}^{N-1} \sum_{i=j+1}^{N} \alpha_i(t)\alpha_j(t)(-2H - 2H'). \tag{182}$$

Regarding Property 2.1 and making appropriate substitutions, (176) can be rewritten as (183).

$$\begin{bmatrix} \sum_{i=1}^{N}\alpha_i(t)\sum_{j=1}^{N}\alpha_j(t)(A_i'F'+FA_j+K'B_i'F'+FB_jK) & * & * \\ \sum_{i=1}^{N}\alpha_i(t)\sum_{j=1}^{N}\alpha_j(t)(P-F'+GA_j+GB_jK) & \sum_{i=1}^{N}\alpha_i(t)\sum_{j=1}^{N}\alpha_j(t)(-G-G') & * \\ \sum_{i=1}^{N}\alpha_i(t)\sum_{j=1}^{N}\alpha_j(t)(B_i'F'+J_iC_j-HK) & \sum_{i=1}^{N}\alpha_i(t)\sum_{j=1}^{N}\alpha_j(t)(B_i'G') & \sum_{i=1}^{N}\alpha_i(t)\sum_{j=1}^{N}\alpha_j(t)(-H-H') \end{bmatrix} < 0. \quad (183)$$

Developing the terms in (183), and regarding that $\sum_{i=1}^{N}\alpha_i(t)=\sum_{j=1}^{N}\alpha_j(t)=1$, we obtain

$$\begin{bmatrix} A'(\alpha(t))F'+FA(\alpha(t))+K'B'(\alpha(t))F'+FB(\alpha(t))K & * & * \\ P-F'+GA(\alpha(t))+GB(\alpha(t))K & -G-G' & * \\ B'(\alpha(t))F'+J(\alpha(t))C(\alpha(t))-HK & B'(\alpha(t))G' & -H-H' \end{bmatrix} < 0. \quad (184)$$

The rest of the proof follows similarly as presented for Theorem 1. ■

## 3.3 DECAY RATE BOUNDING IN GAIN-SCHEDULED SOF CONTROLLER DESIGN

Similarly as presented for the robust SOF case, the inclusion of a performance improvement requirement via decay rate restriction may also be achieved in the design of gain-scheduled SOF controllers, with a slight change on the problem's LMIs, as state Theorem 10.

**Theorem 10.** *Assuming that there exists a state feedback gain $K$ such that $A(\alpha(t))+B(\alpha(t))K$ is asymptotically stable, then there exists a stabilizing gain-scheduled static output feedback gain $L(\alpha(t))$ such that $A(\alpha(t))+B(\alpha(t))L(\alpha(t))C(\alpha(t))$ is asymptotically stable, with a decay rate greater than or equal to $\gamma>0$, if there exists a symmetric matrix $P>0$ and matrices $F$, $G$, $H$, $J_i$, and $J_j$ such that* (185) *and* (186) *are satisfied.*

$$\begin{bmatrix} A_i'F'+FA_i+K'B_i'F'+FB_iK+2\gamma P & * & * \\ P-F'+GA_i+GB_iK & -G-G' & * \\ B_i'F'+J_iC_i-HK & B_i'G' & -H-H' \end{bmatrix} < 0 \quad (185)$$

*for $i=1,2,...,N$.*

$$\begin{bmatrix} X_{ij}+X_{ji} & * & * \\ 2P-2F'+GA_j+GB_jK+GA_i+GB_iK & -2G'-2G & * \\ B_i'F'+J_jC_i+J_iC_j+B_j'F'-2HK & B_i'G'+B_j'G' & -2H-2H' \end{bmatrix} < 0 \qquad (186)$$

*for* $i = 1,2,...,N-1,$ *and* $j = i+1, i+2, ..., N,$ *where*

$$X_{ij}+X_{ji} = A_i'F'+FA_j+K'B_i'F'+FB_jK+A_j'F'+FA_i+K'B_j'F'+FB_iK+4\gamma P. \qquad (187)$$

*In the affirmative case, the gain-scheduled static output feedback gain with time-varying parameter is given by* $L(\alpha(t)) = H^{-1}J(\alpha(t))$, *where*

$$J(\alpha(t)) = \left\{ J(\alpha(t)) \in \mathbb{R}^{m\times p} : J(\alpha(t)) = \sum_{i=1}^{N} \alpha_i(t) J_i \quad ; \quad \alpha(t) \in \curlywedge_M \right\}. \qquad (188)$$

**Proof:** The demonstration of Theorem 10 follows similarly as for Theorem 9, taking into account that the quadratic stability condition considering a lower bound $\gamma$ for the system decay rate is as defined in (36). ∎

## 3.4 RELAXATION STRATEGIES FOR THE DESIGN OF GAIN-SCHEDULED SOF CONTROLLER

As a consequence of the similarities in the LMI formulation of the robust SOF and gain-scheduling SOF problems, the relaxation technique based on the PDFV approach, as proposed on Chapter 2, can be extended for the design of gain-scheduled SOF controllers.

In Theorem 11 the LMI formulation to the SOF gain-scheduling problem with PDFV approach is proposed. This result was obtained by considering matrices $F(\alpha(t))$ and $G(\alpha(t))$ in order to achieve less conservative restrictions.

**Theorem 11.** *Assuming that there exists a state feedback gain* $K$ *such that* $A(\alpha(t)) + B(\alpha(t))K$ *is asymptotically stable, then there exists a stabilizing gain-scheduled static output feedback gain* $L(\alpha(t))$ *such that* $A(\alpha(t))+B(\alpha(t))L(\alpha(t))C(\alpha(t))$ *is asymptotically stable, with a decay rate greater than or equal to* $\gamma > 0$, *if there exist a symmetric matrix* $P > 0$, *and matrices* $F_i$, $F_j$, $G_i$, $G_j$, $H$, $J_i$, *and* $J_j$ *such that* (189) *and* (190) *are satisfied.*

$$\begin{bmatrix} A_i'F_i'+F_iA_i+K'B_i'F_i'+F_iB_iK+2\gamma P & * & * \\ P-F_i'+G_iA_i+G_iB_iK & -G_i-G_i' & * \\ B_i'F_i'+J_iC_i-HK & B_i'G_i' & -H-H' \end{bmatrix} < 0 \qquad (189)$$

*for* $i = 1, 2, ..., N$.

$$\begin{bmatrix} \Lambda_{ij} + \Lambda_{ji} & * & * \\ P! - F_i' - F_j' + G_i A_j + G_i B_j K + G_j A_i + G_j B_i K & -G_i' - G_j' - G_i - G_j & * \\ B_i' F_j' + J_j C_i + J_i C_j + B_j' F_i' - 2HK & B_i' G_j' + B_j' G_i' & -2H - 2H' \end{bmatrix} < 0 \tag{190}$$

*for* $i = 1, 2, ..., N-1$, *and* $j = i+1, i+2, ..., N$, *where*

$$\Lambda_{ij} + \Lambda_{ji} = A_i' F_i' + F_i A_j + K' B_i' F_j' + F_i B_j K + A_j' F_i' + F_j A_i + K' B_j' F_i' + F_j B_i K + 4\gamma P. \tag{191}$$

*In the affirmative case, the gain-scheduled static output feedback gain with time-varying parameter is given by* $L(\alpha(t)) = H^{-1} J(\alpha(t))$, *where*

$$J(\alpha(t)) = \left\{ J(\alpha(t)) \in \mathbb{R}^{m \times p} : J(\alpha(t)) = \sum_{i=1}^{N} \alpha_i(t) J_i \quad ; \quad \alpha(t) \in \curlywedge_M \right\}. \tag{192}$$

**Proof:** The demonstration of Theorem 11 is omitted since it follows similarly as the proof of Theorem 9, regarding the altered stability conditions when the minimum decay rate $\gamma$ is considered and observing the new crossed products involving the matrices $G(\alpha(t))$ and $B(\alpha(t))$. ■

As previously discussed in Chapter 2, it is interesting to evaluate the impact of a relaxation strategy in an LMI formulation. In order to do so, an analysis of the feasibility relation between Theorems 10 and 11 was performed. The results showed that the feasibility of the LMIs of Theorem 10 implies in the feasibility of the LMIs of Theorem 11, as formally stated in Theorem 12.

**Theorem 12.** *Suppose that there exist a symmetric matrix* $P$ *and matrices* $F$, $G$, $H$, *and* $J_i$, *for* $i = 1, ..., N$ *such that LMI conditions* (185) *and* (186) *in Theorem 10 hold. Then, the LMI conditions* (189) *and* (190) *in Theorem 11 also hold.*

**Proof:** Let $F_i = F$ and $G_i = G$, for $i = 1, ..., N$. Then, we have that (189) and becomes

$$\begin{bmatrix} A_i' F' + F A_i + K' B_i' F' + F B_i K + 2\gamma P & P - F + A_i' G' + K' B_i' G' & F B_i + C_i' J_i' - K' H' \\ P - F' + G A_i + G B_i K & -G - G' & G B_i \\ B_i' F' + J_i C_i - HK & B_i' G' & -H - H' \end{bmatrix} < 0,$$

which holds for $i = 1, ..., N$ as our initial hypothesis.

Additionally, one can see that (190) becomes

$$\begin{bmatrix} \Lambda_{ij} + \Lambda ji & * & * \\ P - F' + GA_j + GB_jK + P - F' + GA_i + GB_iK & -2G' - 2G & * \\ B_i'F' + J_jC_i + J_iC_j + B_j'F' - 2HK & B_i'G' + B_j'G' & -2H - 2H' \end{bmatrix} < 0 \quad (193)$$

for $i = 1, 2, ..., N-1$, and $j = i+1, i+2, ..., N$, with

$$\Lambda ij + \Lambda ji = A_i'F' + FA_j + K'B_i'F' + FB_jK + 2\gamma P \\ + A_j'F' + FA_i + K'B_j'F' + FB_iK + 2\gamma P, \quad (194)$$

which, due to the initial hypothesis, also holds for $i = 1, 2, ..., N-1$, and $j = i+1, i+2, ..., N$. Therefore, if (185) and (186) are satisfied, then (189) and (190) will also be satisfied. ■

**Remark 5.** *In order to investigate the proposed GS-SOF technique in terms of its LMIs conservatism, a feasibility study (similar to the one presented in Section 2.6) was performed, using the same numerical example, but now, in the parameter-varying case. For the particular example studied, no substantial improvements were observed comparing the results obtained with Theorems 10 and 11, as both provided feasible solutions for the very same points.*

*A brief examination of these results lead to the partial conclusion that regarding that in Theorems 10 and 11, the matrix $J(\alpha(t))$ is composed by a convex combination of N vertex $J_i$, the SOF problem is naturally more flexible, when compared to the performance in robust SOF case. Theorems 2, 5, and, 7, are considerably more conservative than Theorems 10 and 11, since the search for a solution in the former considers a single robust gain L for guaranteeing asymptotic stability for the whole system's polytope, whereas the later provides a SOF gain $L_i$ for each of the N uncertain system's vertices. Therefore, it is reasonable to assume that the PDFV technique may not produce huge improvements in the feasibility region (as observed in the robust SOF case) when dealing with the GS-SOF, since it is already significantly relaxed, due to the intrinsic nature of the gain-scheduled control. It is possible that with other systems the PDFV relaxation technique may produce improvements when compared to the CQLF strategy, specially when the uncertain system possesses severe uncertain conditions (e.g. a higher number of uncertain parameters, or wider uncertain range for a particular uncertain parameter).*

## 3.5 PRACTICAL EXPERIMENTS

Intending to illustrate a practical application of the proposed gain-scheduled control technique, the design of a gain-scheduled SOF controller for the QUANSER® Active Suspension system is presented.

However, the active suspension system does not present any uncertainty on its mathematical model related to a parameter susceptible to sufficiently fast variations. In other words, there is no natural variable that can be considered as a time-varying parameter on the scope of the gain-scheduling strategy.

Thus, in order to use the active suspension as a control example of the gain-scheduling SOF technique proposed in this work, a time-varying parameter is artificially inserted on the system's state-space model. The strategy is similar to the one applied on the examples in Section 2.7 (see Remark 1), where it is assumed that the active suspension actuator (*i.e* the DC Motor mechanism) is susceptible to experience a fault related to a power loss. Nevertheless, in this case, similarly as in presented in (LLINS, 2015), the amplifier signal is multiplied by a time-varying parameter $\rho(\alpha(t))$, defined as

$$\rho(\alpha(t)) = \alpha_1(t) + (1 - fault_{max})\alpha_2(t), \tag{195}$$

where

$$\alpha_1(t) + \alpha_2(t) = 1, \quad \alpha_1(t), \quad \alpha_2(t) \geq 0, \quad \text{and} \quad 0 < fault_{max} \leq 1.0. \tag{196}$$

Therefore, the effective force provided by the actuator is affected by a time-varying amplifier gain. With this, the outcome is an input signal as a function of $\alpha(t)$, which can be represented by

$$F_c(\alpha(t)) = F_c(t)\rho(\alpha(t)). \tag{197}$$

With this, the system's state-space representation becomes

$$\dot{x}(t) = Ax(t) + BF_c(t)\rho(\alpha(t)) \tag{198}$$

Incorporating the time-varying parameter in the matrix $B$, we have

$$\dot{x}(t) = Ax(t) + B(\alpha(t))F_c(t) \tag{199}$$

which is a LPV system. The gain-scheduling method may now be applied.

***Experiment 3.1***

The time-varying parameter adopted in this example is characterized as a sinusoidal wave with frequency $f = 0.05Hz$ that can described as

$$\rho(t) = 0.85 + 0.15sin\left(2\pi 0.05t + \frac{\pi}{2}\right). \tag{200}$$

Thus, the amplifier gain is set to be multiplied by a time-varying parameter that oscillates sinusoidally from 0.7 to 1.0.

Since the gain-scheduled strategy proposed in this work requires knowledge only of the extreme values that the time-varying parameter can achieve, a polytopic representation of the active suspension system can be build in order to apply the control strategy. The artificial time-varying parameter defined in (200) implies on the definition of second vertex as being the nominal matrix $B$ matrix of the system's space-state model multiplied by 0.7. Thus, in this new example we may consider the same polytope described earlier in Section 2.7, presented here again for the reader convenience in (201) and (202).

- Vertex 1 (amplifier's gain at 100%)

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.347 & -3.061 & 0 & 3.061 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0.408 \\ 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}. \tag{201}$$

- Vertex 2 (amplifier's gain at 70%)

$$A_2 = A_1, B_2 = \begin{bmatrix} 0 \\ 0.2857 \\ 0 \\ -0.7 \end{bmatrix}, \quad \text{and} \quad C_2 = C_1. \tag{202}$$

Observe that the matrices $C_1$ and $C_2$ assumed implies that only the state variable $x_2(t)$ is able to compose the feedback loop. Therefore, a state-feedback technique cannot be applied to solve this problem in practice, justifying the design of a static output feedback gain.

Also note that the parameter $\rho(t)$ can be rewritten as a convex combination of its extreme values, as

$$\rho(t) = \rho(\alpha(t)) = 0.7\alpha_1(t) + 1.0\alpha_2(t), \tag{203}$$

where

$$\alpha_1(t) = 0.5 + 0.5sin\left(2\pi 0.05t + \frac{\pi}{2}\right), \tag{204}$$

and,

$$\alpha_2(t) = (1 - \alpha_1(t)). \tag{205}$$

Therefore, with the appropriate measure of $\rho(\alpha(t))$, a gain-scheduled SOF controller $L(\alpha(t))$ describe as

$$L(\alpha(t)) = L_1\alpha_1(t) + L_2\alpha_2(t), \tag{206}$$

can be applied to solve this control problem. And, since the time-varying parameter is being measured online, the GS-SOF controller can provide a better performance than a single-gain robust controller.

**Remark 6.** *It is important to emphasize that, in Experiment 3.1, the time-varying parameter is artificially inserted on the system dynamic, with pure exemplification purposes, as properly explained in the beginning of this section. Thus, $\rho(\alpha(t))$ is not being actually measured, since it is created inside the computer's software that implements the controller and drives the system's actuator.*

So, continuing the two stage method, considering $\beta = 1$ in (37), the following state feedback gain may be derived

$$K = \begin{bmatrix} -121.2615 & -11.0421 & 254.3442 & 0.2595 \end{bmatrix}. \tag{207}$$

One can note that this is the same state feedback gain obtained in Section 2.7, regarding that LMIs (37) represents a robust control solution, and that the gain-scheduled technique considers only the extreme values. Thus, as this approach leads to the same polytopic representation, the same gain $K$ is derived.

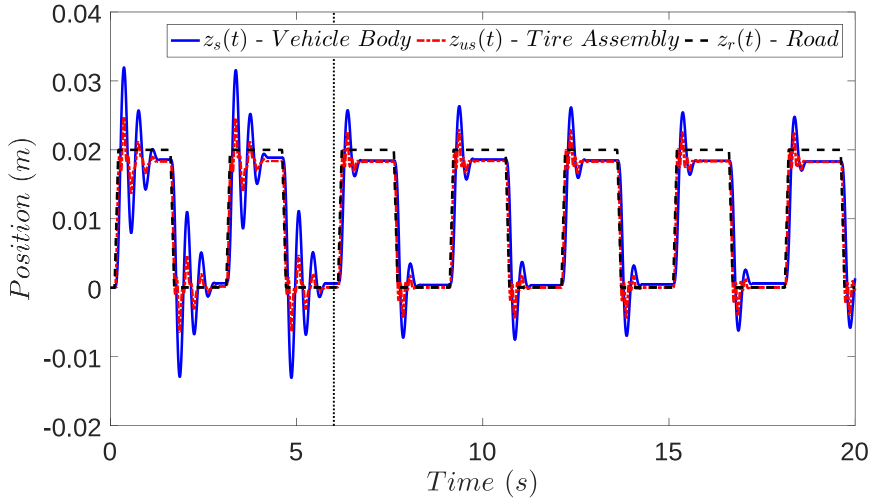
And, solving the LMIs enunciated in Theorem 11, considering $\gamma = 1$ and (207), the values of $L_1$ and $L_2$ for composing the desired GS-SOF are

$$L_1 = -32.4578 \text{ and } L_2 = -33.4116. \tag{208}$$

The obtained controller was implemented on the actual QUANSER® Active Suspension System. During the experiment, which had a total duration of 20 seconds, the road profile $z_r(t)$ was set to a square wave with 0.02 m in amplitude and a period of 3 s. The feedback loop was closed only after 6 seconds. The results are discussed regarding the observed behavior of the system's oscillations, the control signal applied to the plant and the controller gain.

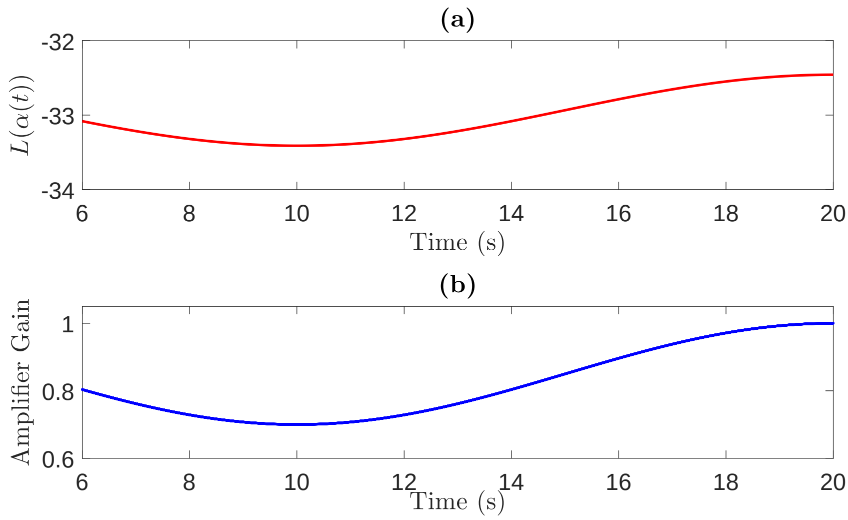
As one can verify through Figure 11, the system is naturally stable. However, it presents accentuated oscillations while operating in open-loop. After closing the feedback loop with the GS-SOF controller (208) the system's performance improved dramatically, since both vehicle body and tire assembly experienced oscillations with short duration and smaller amplitudes.

Figure 11 - System behavior with GS-SOF controller ($\gamma = 1$) subject to time-varying amplifier gain, oscillating at 0.05 Hz (0-6s: open loop; 6-20s closed-loop.)




Source: Author's own results.

Figure 12 - Gain-Scheduled SOF gain $L(\alpha(t))$ (a) ; and time-varying amplifier gain (b) during Experiment 3.1.



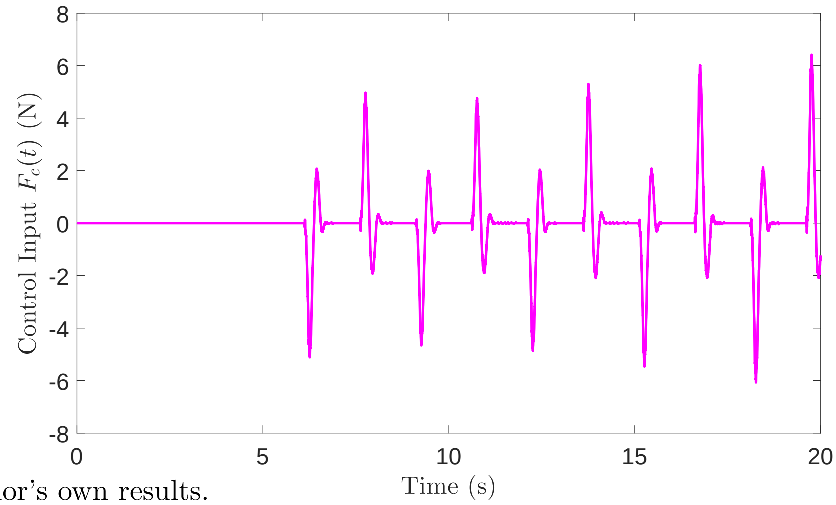

Source: Author's own results.

The variation of the gain-scheduling control can be verified in Figure 12 (a), which presents the behavior of the GS-SOF gain $L(\alpha(t))$. One can see that the controller gain changes accordingly to the instantaneous measured value of the time-varying amplifier gain imposed to the system on the experiment (Figure 12 (b)). With this, the system has the most suitable feedback gain for any of the operating conditions.

It is vital to emphasize that this dynamic performance was achieved using just one of the four system states. This fact corroborates the practical applicability of the output feedback strategy proposed in this work.

A better comprehension of the time-varying behavior of the amplifier gain can be obtained by investigating the control input $F_c(t)$ during the experiment, shown in Figure 13. The influence of the time-varying parameter $\rho(\alpha(t))$ present in matrix $B(\alpha(t))$ is observed in the variation of amplitude of the control input $F_c(t)$. The nominal amplitude of the control signal is compromised by the oscillating multiplier imposed on the amplifier gain (see Figure 12(b)). Observe that the signal reaches smaller amplitudes when $\rho(t)$ is at its maximum value. Therefore, this procedure emulates the condition where the DC motor have its efficiency in providing the required torque (directly related to the force $F_c(t)$) compromised at a given instant of time $t$.

Figure 13 - Control input $F_c(t)$ during Experiment 3.1.




Source: Author's own results.

***Experiment 3.2***

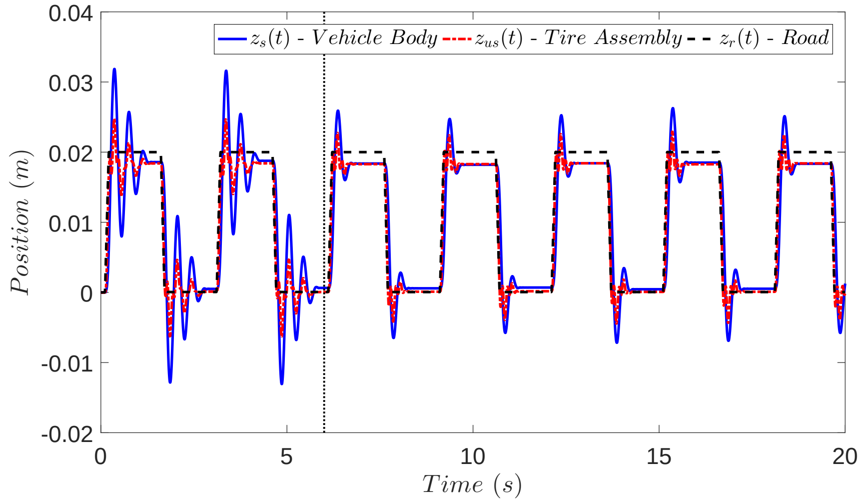
One interesting feature of the proposed methodology is that the derivation of the GS-SOF gains does not requires information about the variation rate of the system's time-varying parameter. To illustrate this feature, a second experiment was executed regarding that the time-varying amplifier gain have a frequency two times higher than in Experiment 3.1.

The feedback loop was composed by the same GS-SOF controller obtained with (208), and the time-varying amplifier gain was set to present a frequency $f = 0.1Hz$. Thus, in this second experiment

$$\rho(t) = 0.85 + 0.15sin\left(2\pi 0.1t + \frac{\pi}{2}\right). \tag{209}$$

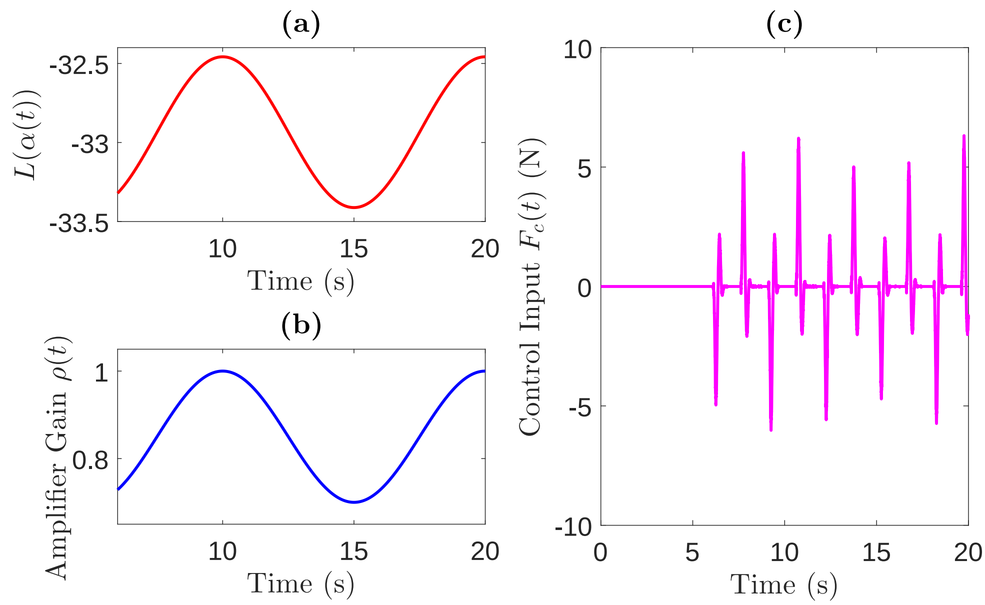
The obtained results shown that the dynamic response of the active suspension does not presented significant changes when compared to the results obtained with a lower-frequency time-varying amplifier gain. If fact, Figure 14 demonstrates that the oscillations were also significantly reduced in both floors when compared to the open-loop configuration.

Figure 14 - System behavior with GS-SOF controller ($\gamma = 1$) subject to time-varying amplifier gain, oscillating at 0.1 Hz (0-6s: open loop; 6-20s closed-loop).




Source: Author's own results.

Figure 15 - Gain-Scheduled SOF gain $L(\alpha(t))$ (a) ; time-varying amplifier gain (b); and control input $F_c(t)$ (c) during Experiment 3.2.




Source: Author's own results.

This occurred as the GS-SOF controller changed within its scheduled gain values according to the system's current operation condition, as shown in Figure 15(a). It is also possible to identify a slightly higher peak of oscillation near to the 6th and the 15th seconds of experiment. This, as also observed in the results of the first experiment (Figure 11), is a consequence of the amplifier gain achieving its worst condition (70% of the nominal value), which can be verified in Figure 15(b). At last, it is possible to observe the influence of the time-varying amplifier gain in the amplitude of the control input, as one can see in Figure 15(c).

***Experiment 3.3***

A better dynamic performance can be achieved by choosing higher values for the minimum decay rate in the control design, as previously verified in Section 2.6.2. examples.

For instance, by adopting $\beta = 2$ and $\gamma = 2$ in the stages one and two of the control design, one can derive through LMIs (37) and Theorem 11 LMIs the following state and gain-scheduled static output feedback gains $K = \begin{bmatrix} -1036.4 & -91.6 & -472.1 & -10.4 \end{bmatrix}$, and, $L_1 = -171.2959$ and, $L_2 = -184.6192$, respectively.

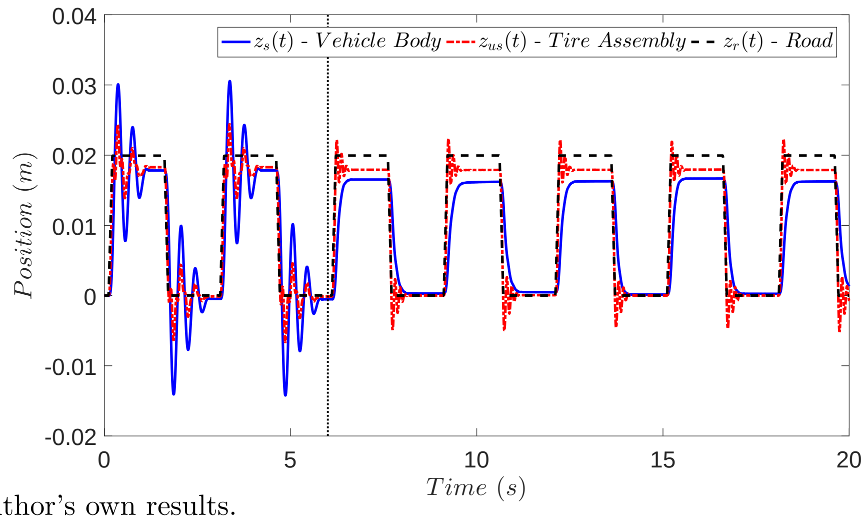
In a third experiment, executed similarly as the two earlier but now closing the feedback loop with the new designed GS-SOF gains, an expressive improvement could be observed in the system dynamic.

As shown if Figure 16, the transient of the plate positions were significantly shortened, as the oscillations due to the variation on the road profile are extinguished almost immediately. This result is a direct consequence of a more rigorous requirement on the system's decay rate, specified in the control design.

Also, as expected, the control input (see Figure 17 (c)) presents higher amplitudes when compared to the results found in Experiments 3.1 and 3.2. As the bounding on the decay rate value increases, the closed loop eigenvalues are allocated further in the left complex semi-plane. With this, the system response becomes faster, but it requires a more severe control signal. However, the peak values observed does not exceeds the actuator's saturation value of $\pm 39.2N$, which guarantees the safe operation of the system.

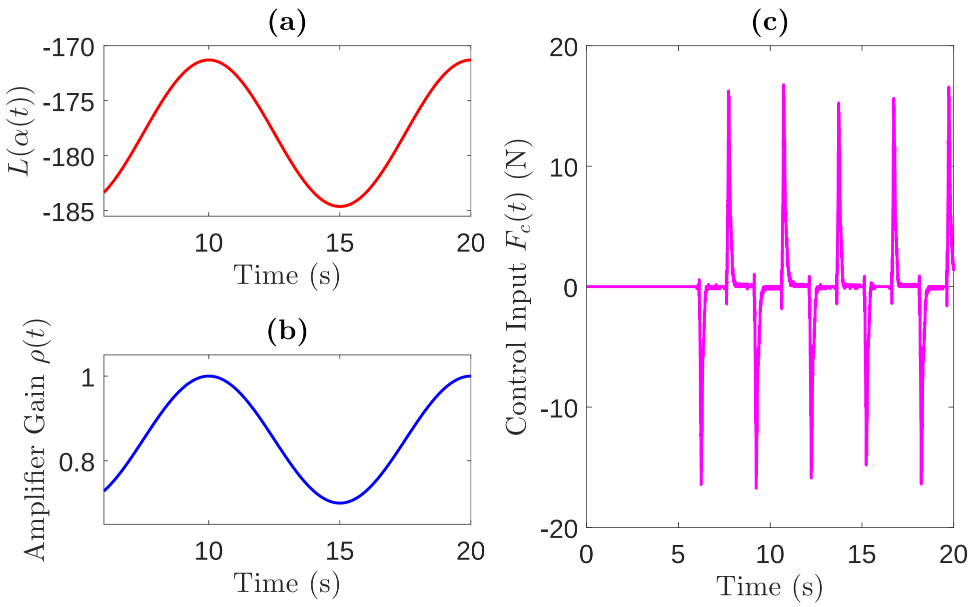
It is also interesting to point out that with a higher minimum decay rate, the obtained control exhibits larger gains. Figure 17 (a) is convenient to address the fact that in this new control problem the gain-scheduled SOF gain presents a longer excursion between its extreme values, when compared to the controller derived in Experiment 3.1.

Figure 16 - System behavior with GS-SOF controller ($\gamma = 2$) subject to time-varying amplifier gain, oscillating at 0.1 Hz (0-6s: open loop; 6-20s closed-loop).




Source: Author's own results.

Figure 17 - Gain-Scheduled SOF gain $L(\alpha(t))$ (a) ; time-varying amplifier gain (b); and control input $F_c(t)$ (c) during Experiment 3.3.




Source: Author's own results.

# 4 CONCLUSIONS AND PERSPECTIVES

This work proposes new results on gain-scheduled and static output feedback control. As an overall conclusion, the outcome of the performed studies corroborates with the fact that this line of research can provide interesting and useful contributions to control theory.

From the perspective of the static output feedback, Chapter 2 compiled the main results on the development of the work started in Manesco (2013), by granting less conservative LMI conditions for the design of robust SOF controllers. The feasibility tests shown that the relaxation strategy proposed in Theorem 7, which considers the additional variables in Finsler's Lemma as parameter dependent, can significantly improve the feasibility region, specially in terms of the variety of systems for which the technique found a control solution. The results of a practical implementation of a SOF controller designed via Theorem 7 for the QUANSER® Active Suspension serve as motivation to continue exploring the proposed method, since it was possible to obtain a SOF controller that relies on the feedback of only one of the four system's states. Furthermore, the controller was able to significantly suppress the oscillations on both suspension places, even during the occurrence of a fault on the system actuator.

Moreover, the analysis performed on the two-stage method provided an interesting information, since it proved that there exists a suitable state feedback gain to be derived in the first stage in order to ensure feasibility in the second stage. With this background, further research is encouraged to investigate the development of a new strategy which combines the two stages in a single and enhanced control design strategy for the SOF problem.

The development of the mathematical and control theory concepts obtained with the studies on robust SOF technique made possible the extension of the previous results to the gain-scheduled case, summarized in the theorems proposed in Chapter 3. The relaxation technique based on PDFV, which was applied on the robust control case, were able to be directly extended to the gain-scheduled method. The theoretical analysis proposed in Theorem 12 showed that the relaxed conditions of Theorem 11 provide results, in terms of feasibility, at least as good as the more conservative restrictions of Theorem 10. Considering the improvements with PDFV technique in the robust SOF case, it is reasonable to assume that the PDFV in GS-SOF can also enhance the feasibility performance of the GS-SOF proposed in this work technique. The practical applicability of this strategy was verified through the design and implementation of a GS-SOF controller for the QUANSER Active Suspension System.

In both the robust and the GS cases, the PDFV technique resulted in crossed-products between two parameter-dependent matrices in mathematical formulation. This fact lead to the presence of products of the nature of $\alpha_i(t)\alpha_j(t)$, with $i, j \in \mathbb{N}^+$ in the problem description. Therefore, in order to derive LMI conditions, Property 2.1 was applied. And, as a consequence, this procedure resulted in the increase of the total number of LMIs to be solved for obtaining the desired robust/GS-SOF controller.

As the output feedback still is an open problem in control literature, it is of great interest to investigate different approaches for addressing this issue. Even regarding that the proposed theorems only guarantees the sufficiency in solving a SOF problem, the presented results showed that it can be worthy to pursue further studies on the topic.

**Research Perspectives**

Studies in a more advanced stage are related to further improvements on the proposed LMI conditions. Regarding the gain-scheduling approach, a feasibility study on the impact of the design of a gain-scheduled state-feedback gain $K(\alpha(t))$ in the first stage of design, allied to the PDLF are also being discussed

Another topic worth of being explored is the optimization of the norm of the matrix $S = H^{-1}JC - K$ in order to force $S$ to be as close as possible to $\mathbf{0}$. This study is motivated by the fact that in the case of $S = \mathbf{0}$ we have $H^{-1}JC = LC = K$, and this means that $A + BLC$ is equal to $A + BK$. In other words, the controller $L$ is such that the overall dynamic of the systems in SOF is equivalent to a state feedback. With the analysis performed regarding the feasibility of the two-stage method, it was showed that a gain $K = LC$ lead to guaranteed feasibility in the second stage. Hence, there exists a great interest in investigating if such optimization may bring enhanced techniques for the SOF control design. It is also desired to study the idea of introducing information about the second stage on the design of the gain $K$, aiming to derived a more suitable first stage gain, or even come up with a new control design strategy, promoting the coupling of both of two stages.

Two other topics of interest are related to inserting a $\mathcal{D}$-stability pole-placement requirement to the design of the SOF controllers, and the robust $\mathscr{H}_\infty$ SOF control design. In Manesco (2013) the author investigated the $\mathcal{D}$-stability applied to the design of the first stage gain $K$, but remained open its application in the second stage. With this, a better control on the system dynamic may be achieved, as a detailed specification of the closed-loop eigenvalues placement will be possible. Moreover, the $\mathscr{H}_\infty$ SOF control addresses a extremely important practical issue: the presence of disturbances during the system's performance. Such strategy has being intensively studied by the control research community, which justifies the efforts in expanding the obtained LMIs to also cope with the $\mathscr{H}_\infty$ norm minimization problem.

## REFERENCES


ADEGAS, F. D. New sufficient LMI conditions for static output stabilization. *IFAC Proceedings Volumes*, Kidlington, v. 46, n. 2, p. 184–189, 2013.

AGULHARI, C. M.; OLIVEIRA, R. C. L. F.; PERES, P. L. D. Novas condições LMI para projeto de controladores estáticos mistos $\mathcal{H}_2/\mathcal{H}_\infty$ por realimentação de saída para sistemas politópicos contínuos invariantes no tempo. In: CONGRESSO BRASILEIRO DE AUTOMÁTICA - CBA, 18., 2010, São Paulo. *Anais* [...]. São Paulo: USP, 2010. p. 179–186.

AGULHARI, C. M.; OLIVEIRA, R. C. L. F.; PERES, P. L. D. Static output feedback control of polytopic systems using polynomial Lyapunov functions. In: IEEE CONFERENCE ON DECISION AND CONTROL - CDC, 49th., 2010, Atlanta. *Proceedings* [...]. Pistacaway, 2011. p. 6894–6901.

AGULHARI, C. M.; OLIVEIRA, R. C. L. F.; PERES, P. L. D. LMI relaxations for reduced-order robust $\mathcal{H}_\infty$ control of continuous-time uncertain linear systems. *IEEE Transactions on Automatic Control*, Pistacaway, v. 57, n. 6, p. 1532–1537, 2012.

AL-JIBOORY, A. K.; ZHU, G. Static output-feedback robust gain-scheduling control with guaranteed $\mathcal{H}_2$ performance. *Journal of the Franklin Institute*, Kidlington, v. 355, n. 5, p. 2221–2242, 2018.

APKARIAN, P.; GAHINET, P. A convex characterization of gain-scheduled $\mathcal{H}_\infty$ controllers. *IEEE Transactions on Automatic Control*, Pistacaway, v. 40, n. 5, p. 853–864, 1995.

APKARIAN, P.; GAHINET, P.; BECKER, G. Self-scheduled $\mathcal{H}_\infty$ control of linear parameter-varying systems: A design example. *Automatica*, Kidlington, v. 31, n. 9, p. 1251–1261, 1995.

ARZELIER, D.; GRYAZINA, E. N.; PEAUCELLE, D.; POLYAK, B. T. Mixed LMI/randomized methods for static output feedback control design. In: AMERICAN CONTROL CONFERENCE - ACC, 2010, Baltimore. *Proceedings* [...]. Pistacaway: IEEE, 2010. p. 4683–4688.

ARZELIER, D.; PEAUCELLE, D.; SALHI, S. Robust static output feedback stabilization for polytopic uncertain systems: Improving the guaranteed performance bound. *IFAC Proceedings Volumes*, Kidlington, v. 36, n. 11, p. 425–430, 2003.

BARMISH, B. R. Necessary and sufficient conditions for quadratic stabilizability of an uncertain system. *Journal of Optimization Theory and Applications*, New York, v. 46, n. 4, p. 399–408, 1985.

BERNSTEIN, D. S. Some open problems in matrix theory arising in linear systems and control. *Linear Algebra and its Applications*, Philadelphia, v. 162, p. 409–432, 1992.

BOYD, S.; GHAOUI, L. E.; FERON, E.; BALAKRISHNAN, V. Linear matrix inequalities in system and control theory. Philadelphia: SIAM, 1994.

BRAGA, M. F.; MORAIS, C. F.; TOGNETTI, E. S.; OLIVEIRA, R. C. L. F.; PERES, P. L. D. Discretization and event triggered digital output feedback control of LPV systems. *Systems & Control Letters*, Kidlington, v. 86, p. 54–65, 2015.

CAIGNY, J. D.; CAMINO, J. F.; OLIVEIRA, R. C. L. F.; PERES, P. L. D.; SWEVERS, J. Gain-scheduled dynamic output feedback control for discrete-time LPV systems. *International Journal of Robust and Nonlinear Control*, Oxford, v. 22, n. 5, p. 535–558, 2012.

CHADLI, M.; GUERRA, T.-M. LMI solution for robust static output feedback control of discrete Takagi–Sugeno fuzzy models. *IEEE Transactions on Fuzzy Systems*, Pistacaway, v. 20, n. 6, p. 1160–1165, 2012.

CHANG, X.-H.; PARK, J. H.; ZHOU, J. Robust static output feedback $\mathcal{H}_\infty$ control design for linear systems with polytopic uncertainties. *Systems & Control Letters*, Kidlington, v. 85, p. 23–32, 2015.

CHEN, B.-S.; TSENG, C.-S.; UANG, H.-J. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ fuzzy output feedback control design for nonlinear dynamic systems: an LMI approach. *IEEE Transactions on Fuzzy Systems*, Pistacaway, v. 8, n. 3, p. 249–265, 2000.

CRUSIUS, C. A.; TROFINO, A. Sufficient LMI conditions for output feedback control problems. *IEEE Transactions on Automatic Control*, Pistacaway, v. 44, n. 5, p. 1053–1057, 1999.

DABBOUSSI, K.; ZRIDA, J. Sufficient dilated LMI conditions for $\mathcal{H}_\infty$ static output feedback robust stabilization of linear continuous-time systems. *Journal of Applied Mathematics*, New York, v. 2012, 2012.

DONG, J.; YANG, G.-H. Robust static output feedback control synthesis for linear continuous systems with polytopic uncertainties. *Automatica*, Kidlington, v. 49, n. 6, p. 1821–1829, 2013.

KHALIL, H. K. Adaptive output feedback control of nonlinear systems represented by input-output models. *IEEE Transactions on Automatic Control*, Pistacaway, v. 41, n. 2, p. 177–188, 1996.

KIMURA, H. Pole assignment by gain output feedback. *IEEE Transactions on Automatic Control*, Pistacaway, v. 20, n. 4, p. 509–516, 1975.

LLINS, L. I. H. *Projeto de controlador gain scheduling usando realimentação derivativa via LMI.* 2015. 61 f. Dissertação (Mestrado em Automação) - Faculdade de Engenharia, Universidade Estadual Paulista - UNESP, Ilha Solteira, 2015.

LOFBERG, J. YALMIP: A toolbox for modeling and optimization in MATLAB. In: IEEE INTERNATIONAL SYMPOSIUM ON COMPUTER AIDED CONTROL SYSTEMS DESIGN, 2004, Taipei. *Proceedings* [...]. Pistacaway: IEEE, 2004. p. 284–289.

MANESCO, R. M. *Projeto de controladores robustos para sistemas sujeitos a falhas estruturais usando realimentação estática de saída*. 2013. 68 f. Dissertação (Mestrado em Automação) - Faculdade de Engenharia, Universidade Estadual Paulista (UNESP), Ilha Solteira, 2013.

MEHDI, D.; BOUKAS, E.; BACHELIER, O. Static output feedback design for uncertain linear discrete time systems. *Journal of Mathematical Control and Information*, Oxford, v. 21, n. 1, p. 1–13, 2004.

MONTAGNER, V. F.; PERES, P. L. State feedback gain scheduling for linear systems with time-varying parameters. *Journal of Dynamic Systems, Measurement, and Control*, New York, v. 128, n. 2, p. 365–370, 2006.

MOZELLI, L. A.; PALHARES, R. M.; MENDES, E. M. A. M. Equivalent techniques, extra comparisons and less conservative control design for Takagi-Sugeno (TS) fuzzy systems. *IET Control Theory & Applications*, Stavenage, v. 4, n. 12, p. 2813–2822, 2010.

NGUYEN, A.-T.; CHEVREL, P.; CLAVEAU, F. Gain-scheduled static output feedback control for saturated lpv systems with bounded parameter variations. *Automatica*, Kidlington, v. 89, p. 420–424, 2018.

OLIVEIRA, M. C. de; GEROMEL, J. C. A class of robust stability conditions where linear parameter dependence of the Lyapunov function is a necessary condition for arbitrary parameter dependence. *Systems & Control Letters*, Kidlington, v. 54, n. 11, p. 1131–1134, 2005.

OLIVEIRA, M. C. de; SKELTON, R. E. Stability tests for constrained linear systems. In: MOHEIMANI, S. O. R. (Ed.). *Perspectives in Robust Control*. New York: Springer, 2001. p. 241–257.

OLIVEIRA, R. C. L. F.; PERES, P. L. D. LMI conditions for robust stability analysis based on polynomially parameter-dependent Lyapunov functions. *Systems & Control Letters*, Kidlington, v. 55, n. 1, p. 52–61, 2006.

QIU, J.; FENG, G.; GAO, H. Fuzzy-model-based piecewise $\mathcal{H}_\infty$ static-output-feedback controller design for networked nonlinear systems. *IEEE Transactions on Fuzzy Systems*, Pistacaway, v. 18, n. 5, p. 919–934, 2010.

QUANSER. *Active Suspension System: User Manual*. 2009.

RUGH, W. J.; SHAMMA, J. S. Research on gain scheduling. *Automatica*, Kidlington, v. 36, n. 10, p. 1401–1425, 2000.

SADABADI, M. S.; KARIMI, A. Fixed-order control of LTI systems subject to polytopic uncertainty via the concept of strictly positive realness. In: AMERICAN CONTROL CONFERENCE - ACC, 2015, Chicago. *Proceedings* [...]. Pistacaway: IEEE, 2015. p. 2882–2887.

SADABADI, M. S.; PEAUCELLE, D. From static output feedback to structured robust static output feedback: A survey. *Annual reviews in control*, Kidlington, v. 42, p. 11–26, 2016.

SADEGHZADEH, A. Gain-scheduled static output feedback controller synthesis for discrete-time LPV systems. *International Journal of Systems Science*, Abingdon, v. 48, n. 14, p. 2936–2947, 2017.

SCHERER, C.; GAHINET, P.; CHILALI, M. Multiobjective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control*, Pistacaway, v. 42, n. 7, p. 896–911, 1997.

SERENI, B.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M. Projeto de controlador gain scheduling através de realimentação estática de saída. In: SIMPÓSIO BRASILEIRO DE AUTOMACÃO INTELIGENTE - SBAI, 2017, Porto Alegre. *Anais* [...]. Porto Alegre: UFRGS, 2017. p. 1138–1146.

SERENI, B.; MANESCO, R.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M. Relaxed lmi conditions for the design of robust static output feedback controllers. *IFAC-PapersOnLine*, Kidlington, v. 51, n. 25, p. 428–433, 2018.

SHAMMA, J. S. *Analysis and design of gain scheduled control systems*. 1988. Doctoral's (Dissertation) - Massachusetts Institute of Technology, Massachusetts, 1988.

SILVA, E. R.P. *Controle de sistemas lineares incertos via realimentação derivativa utilizando funções de Lyapunov dependentes de parâmetros*. 2012. 126 f. Tese (Doutorado em Automação) - Faculdade de Engenharia, Universidade Estadual Paulista (UNESP), Ilha Solteira, 2012.

SKELTON, R. E.; IWASAKI, T.; GRIGORIADIS, D. E. *A unified algebraic approach to control design*. 1997.

SYRMOS, V. L.; ABDALLAH, C. T.; DORATO, P.; GRIGORIADIS, K. Static output feedback – A survey. *Automatica*, Kidlington, v. 33, n. 2, p. 125–137, 1997.

TANAKA, K.; IKEDA, T.; WANG, H. O. Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs. *IEEE Transactions on Fuzzy Systems*, Pistacaway, v. 6, n. 2, p. 250–265, 1998.

TEIXEIRA, M. C. M.; ASSUNÇÃO, E.; AVELLAR, R. G. On relaxed LMI-based designs for fuzzy regulators and fuzzy observers. *IEEE Transactions on Fuzzy Systems*, Pistacaway, v. 11, n. 5, p. 613–623, 2003.

TONG, S.; LI, Y.; LI, Y.; LIU, Y. Observer-based adaptive fuzzy backstepping control for a class of stochastic nonlinear strict-feedback systems. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, Piscataway, v. 41, n. 6, p. 1693–1704, 2011.

TROFINO, A. Sufficient LMI conditions for the design of static and reduced order controllers. In: CONFERENCE ON DECISION AND CONTROL - CDC, 48th., 2009, Shangai. *Proceedings* [...]. Piscataway: IEEE, 2009. p. 6668–6673.

TROFINO, A.; KUCERA, V. Stabilization via static output feedback. *IEEE Transactions on Automatic Control*, Piscataway, v. 38, n. 5, p. 764–765, 1993.

WANG, Y.; XIE, L.; SOUZA, C. E. de. Robust control of a class of uncertain nonlinear systems. *Systems & Control Letters*, Kidlington, v. 19, n. 2, p. 139–149, 1992.

WEI, G.; WANG, Z.; LI, W.; MA, L. A survey on gain-scheduled control and filtering for parameter-varying systems. *Discrete Dynamics in Nature and Society*, New York, v. 2014, p. 1–10, 2014.

ZHANG, H.; LEWIS, F. L.; DAS, A. Optimal design for synchronization of cooperative systems: state feedback, observer and output feedback. *IEEE Transactions on Automatic Control*, Pistacaway, v. 56, n. 8, p. 1948–1952, 2011.

ZHANG, H.; SHI, Y.; MEHR, A. S. Robust static output feedback control and remote pid design for networked motor systems. *IEEE Transactions on Industrial Electronics*, Piscataway, v. 58, n. 12, p. 5396–5405, 2011.