SÃO PAULO STATE UNIVERSITY (UNESP)
"JÚLIO DE MESQUITA FILHO"
SCHOOL OF ENGINEERING
ILHA SOLTEIRA - SP

MARCO ANTONIO LEITE BETETO

LESS CONSERVATIVE CONDITIONS FOR THE ROBUST AND GAIN-SCHEDULED LQR-STATE DERIVATIVE CONTROLLERS DESIGN

Ilha Solteira
2019

MARCO ANTONIO LEITE BETETO

LESS CONSERVATIVE CONDITIONS FOR THE ROBUST AND GAIN-SCHEDULED LQR-STATE DERIVATIVE CONTROLLERS DESIGN

Dissertation presented to the São Paulo State University (UNESP) - School of Engineering - Campus of Ilha Solteira, in fulfilment of one of the requirements for obtaining the degree of Master of Science in Electrical Engineering.
Speciality: Automation.

Prof. Dr. Edvaldo Assunção
Advisor

Ilha Solteira
2019

FICHA CATALOGRÁFICA
Desenvolvido pelo Serviço Técnico de Biblioteca e Documentação

B562l

Beteto, Marco Antonio Leite.
Less conservative conditions for the robust and gain-scheduled lqr-state derivative controllers design / Marco Antonio Leite Beteto. -- Ilha Solteira: [s.n.], 2019
74 f. : il.

Dissertação (mestrado) - Universidade Estadual Paulista. Faculdade de Engenharia de Ilha Solteira. Área de conhecimento: Automação, 2019

Orientador: Edvaldo Assunção
Inclui bibliografia

1. Regulador linear quadrático (LQR). 2. Desigualdades lineares matriciais (LMIS). 3. *Gain Scheduling* (GS). 4. Realimentação derivativa. 5. Controle de vibrações. 6. Estabilidade robusta.

Raiane da Silva Santos
Raiane da Silva Santos
Supervisora Técnica de Seção
Seção Técnica de Referência, Atendimento ao usuário e Documentação
Diretoria Técnica de Biblioteca e Documentação
CRB/8 - 9999

UNIVERSIDADE ESTADUAL PAULISTA

Câmpus de Ilha Solteira

CERTIFICADO DE APROVAÇÃO

TÍTULO DA DISSERTAÇÃO: Less Conservative Conditions for the Robust and Gain Scheduled LQR-State Derivative

AUTOR: MARCO ANTONIO LEITE BETETO
ORIENTADOR: EDVALDO ASSUNÇÃO

Aprovado como parte das exigências para obtenção do Título de Mestre em ENGENHARIA ELÉTRICA, área: Automação pela Comissão Examinadora:

Prof. Dr. EDVALDO ASSUNÇÃO
Departamento de Engenharia Elétrica / Faculdade de Engenharia de Ilha Solteira

Prof. Dr. JEAN MARCOS DE SOUZA RIBEIRO
Departamento de Engenharia Elétrica / Faculdade de Engenharia de Ilha Solteira

Cristiano Quevedo Andrea
Prof. Dr. CRISTIANO QUEVEDO ANDREA
Centro de Ciências Exatas e Tecnologia / Universidade Federal de Mato Grosso do Sul

Ilha Solteira, 05 de fevereiro de 2019

I dedicate this work to my parents, Luciana and Claudemir;

To my sister Emanuela;

To my girlfriend Ana Paula;

for all love, support, trust and encouragement at all times.

ACKNOWLEDGEMENTS

My thanks to all the relatives, friends, professors and employees of FEIS-UNESP, who directly or indirectly contributed to the accomplishment of this work. In particular, I give my thanks:

- To God, for giving me strength and health to get here;
- To my relatives, especially my parents, my sister and my girlfriend, present at all times;
- To my advisor, Dr Edvaldo Assunção, for the friendship, the teachings, the patience and, mainly for the opportunity, incentive and confidence;
- To professor Dr Marcelo Carvalho Minhoto Teixeira, for the friendship, suggestions and conversations, besides all the help and contributions for this work;
- To Dr Rodrigo Cardim, for the follow-up in the examining boards, suggestions and incentive;
- To Dr Emerson Ravazzi Pires da Silva, for the companionship, friendship, encouragement, teachings, besides the suggestions and contributions for this work;
- To the friends of the Research Laboratory in Control (LPC), Bruno Sereni, Leonardo Carniato, Gilberto, Douglas, Lázaro, Adalberto, Leidy, Hadamez, Marco, Gustavo, for the friendship and the contribution, that directly or indirectly helped;
- To the fomentation agencies, once that this study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq);
- To the Foundation for Research Support of the State of São Paulo - FAPESP (Case number 2011 / 17610-0)

”The task is not so much to see what no one has seen yet, but to think what nobody has thought yet, about what everybody sees.”

Arthur Schopenhauer (1788 - 1860)

”We know only too well that what we are doing is nothing more than a drop in the ocean. But if the drop were not there, the ocean would be missing something.”

Madre Teresa de Calcutá (1910 - 1997)

RESUMO

Neste trabalho é proposta a resolução do problema do regulador linear quadrático (*Linear Quadratic Regulator* - LQR) via desigualdades matriciais lineares (*Linear Matrix Inequalities* - LMIs) para sistemas lineares e invariantes no tempo sujeitos a incertezas politópicas, bem como para sistemas lineares sujeitos a parâmetros variantes no tempo (*Linear Parameter Varying* - LPV). O projeto dos controladores é baseado na realimentação derivativa. A escolha da realimentação derivativa se dá devido à sua fácil implementação em certas aplicações como, por exemplo, no controle de vibrações. Os sinais usados na realimentação são aceleração e velocidade, sendo obtidos por meio de acelerômetros. Por meio do método proposto é possível obter condições LMIs para a síntese de controladores que garantam a estabilização do sistema em malha fechada, sendo que os controladores possuem desempenho otimizado. Para a formulação das condições LMIs, uma função de Lyapunov do tipo quadrática é utilizada. Exemplos teóricos e simulações são utilizados como forma de validação dos métodos propostos, além de mostrar que os novos resultados apresentam condições menos conservadoras. Além disso, ao final é apresentada uma implementação prática em um sistema de suspensão ativa, produzida pela Quanser®.

Palavras-chave: Regulador linear quadrático (LQR). Desigualdades lineares matriciais (LMIs). *Gain scheduling* (GS). Realimentação derivativa. Controle de vibrações. Estabilidade robusta. Lema de Finsler. Incertezas politópicas. Parâmetros variantes.

ABSTRACT

The resolution of linear quadratic regulator (LQR) problem via linear matrix inequalities (LMIs) for linear time-invariant systems subject to polytopic uncertainties, as linear systems subjects to linear parameter varying (LPV), is proposed in this work. The controllers' designs are based on the state derivative feedback. The aim to the choice of the state derivative feedback is your easy implementation in a class of mechanical systems, such as in vibration control, for example. The signals used for feedback are acceleration and velocity, it is obtained by means of accelerometers. Through the proposed method it is possible to obtain LMIs conditions for the synthesis of controllers that guarantee the stabilisation of the closed-loop system, being that the controllers have optimised performance. For the LMIs conditions formulations, a Lyapunov function of type quadratic is used. As a form of validation, theoretical examples and simulations are performed, besides to show that the new results are less conservative. Furthermore, a practical implementation in an active suspension system, produced by Quanser®, is performed.

Keywords: Linear quadratic regulator (LQR). Linear matrix inequalities (LMIs). Gain scheduling (GS). State derivative feedback. Vibration control. Robust stability. Finsler's lemma. Polytopic uncertainties. Parameter-varying.

LIST OF FIGURES

LIST OF TABLES

LIST OF ABBREVIATIONS AND ACRONYMS

ARE	Algebraic Ricatti Equation
CQLFs	Commom Quadratic Lyapunov Functions
DoF	Degrees of Freedom
4WD4WS	Four-Wheel Driving Four-Wheel Steering
FLFs	Fuzzy Lyapunov Functions
GS	Gain Scheduling
LMIs	Linear Matrix Inequalities
LPV	Linear Parameter Varying
LQR	Linear Quadratic Regulator
LVDT	Linear Variable Differential Transformer
MatLab®	MATrix LABoratory
PDLFs	Parameter-Dependent Lyapunov Functions
PSO	Particle Swarm Optimisation Algorithm
PLFs	Piecewise Lyapunov Functions
PPDLFs	Polynominal Parameter-Dependent Lyapunov Functions
PID	Proportional-Integral-Derivative
RLV	Reusable Launch Vehicle
RLQR	Robust Linear Quadratic Regulator
TS	Takagi-Sugeno

LIST OF SYMBOLS

©	Copyright
$diag(.,.,...,.)$	Denotes a diagonal matrix with properly dimensions
$^{-T}$	Denotes the inverse of a transpose matrix
$*$	Denotes the transpose block of a symmetric matrix
^{T}	Denotes the transpose of a vector or matrix
$x_0(\beta)$	Represents an initial conditions polytope
0	Represents a null matrix with properly dimensions
$\mathfrak{A}$	Represents an unitary simplex, $\left\{\sum_{i=1}^{j}, \ (.) \geq 0, \ i = 1, ..., j\right\}$
$\mathfrak{F}$	Represents an unitary simplex similar to $\mathfrak{A}$
$M < 0$ ($M \leq 0$)	Represents negative definite matrices (semi-definite)
$M > 0$ ($M \geq 0$)	Represents positive definite matrices (semi-definite)
I	Represents the identity matrix with properly dimensions
r	Represents the polytope vertices
$J_D(\dot{x}(t), u(t))$	Represents the quadratic cost associated with the energy of the states derivative and the control input signal
$\mathbb{R}$	Represents the set of real numbers
μ	Represents the specified upper bound for J_D
$\sum_{i=1}^{j}(.)$	Represents the sum from $i = 1$ to j
$\mathcal{R}$	Represents the weighting matrix for the control input vector
$\mathcal{Q}$	Represents the weighting matrix for the state derivative vector
®	Trademark

BRIEF

1 INTRODUCTION

Over the past few years, the state derivative feedback has been widely investigated in many practical engineering problems, such as vibration control in mechanical systems and flexible structures where the accelerometer is the only sensor (ABDELAZIZ, 2012). Then, using accelerometers, the state derivative vector is easier to obtain than the state vector. By means of the acceleration signal, it is possible to reconstruct the velocity signal with good accuracy, but the same does not occur with the displacement signal (ABDELAZIZ; VALÁŠEK, 2004). Then, the signals used in the feedback are velocity and acceleration. Due to your simple structure and low cost, accelerometers have been used to solve a great number of engineering problems, such as in Sabato et al. (2016), that the authors developed a novel accelerometer board to achieve more accurate wireless vibration measurements; in Kasprzyk et al. (2017), that the authors presented a comparison between accelerometers and Linear Variable Differential Transformer (LVDT) sensors, besides that they intend to study if a combination of those two types of sensors improves the results in future researches; in Zhu et al. (2018), that the authors presented an active suspension controller which is appropriate to implement in practice and is low cost.

Many works use the state derivative feedback for closed-loop control system designs problems. For instance, in the following applications: Abdelaziz (2007) developed an efficient solution to the pole assignment problem using state derivative feedback; Abdelaziz (2008) performed a gradient flow approach for computing the robust controller for linear systems using state derivative feedback; Abdelaziz (2009) introduced a technique for computing robust controller for multivariable time-invariant linear systems via state derivative feedback; Michiels et al. (2009) presented a study of the stabilizability and stability robustness of a linear controllable system using state derivative feedback control; Moreira et al. (2010) developed a study of the observability and stabilisability of systems with state derivative feedback; Abdelaziz (2012) presented a complete parametric approach for solving the eigenstructure assignment problem using state derivative feedback for multivariable linear systems; Wu and Fang (2012) introduced the delayed-state derivative feedback into the existing delayed consensus protocol for improving the robustness against communication delay and the convergence speed of reaching the consensus simultaneously; Basturk and Krstic (2014) designed an adaptive backstepping controller to cancel sinusoidal disturbances using the state derivative feedback; Araújo et al. (2016) presented a comparative study of sensitivity to parameter variation in two feedback techniques applied in second-order linear systems: state feedback technique and the less conventional state derivative feedback technique; Bekhiti et al. (2016) performed a new algorithm which can assign block-roots and block-eigenvectors in order to achieve desired objectives with latent structure specifications using two types of control, the state and the state derivative feedback; Yazici and Sever (2017) designed a robust L_2 gain state derivative feedback controller for an active suspension system; Beteto et al. (2018a, 2018b) presented a robust LQR-derivative (linear quadratic regulator) controller via linear matrix inequalities (LMIs); Rossi et al. (2018) presented a design of a robust state derivative feedback control law in discrete time.

Still, in the specialised literature, it is possible to find several works that deal with the state derivative feedback via linear matrix inequalities (LMIs) to solve several problems, for example: Assunção et al. (2007) presented a design of a robust state derivative feedback via LMIs for multivariable linear systems; Cardim et al. (2009) performed the design of a digital state derivative controller from a know digital state feedback controller; Faria et al. (2009) designed a robust state derivative feedback for multivariable linear systems considering the $\mathfrak{D}$-stability; Jing et al. (2009) presented the problem of state derivative feedback with exponential convergence rate for a linear system with a constant time-delay; Faria et al. (2010) performed a design of a robust state derivative controller via LMIs for linear descriptor systems; Silva et al. (2011) developed a design of a robust state derivative controller via LMIs using parameter-dependent Lyapunov functions (PDLFs) instead of a common quadratic Lyapunov functions (CQLFs); Silva et al. (2012) presented a less conservative approach to the controller design via state derivative feedback using LMIs, PDLFs and Finsler's Lemma; Fallah et al. (2013) presented the control of the attitude and motion of a vehicle using LMIs and state derivative feedback; Zhao et al. (2015) performed a ship dynamic positioning control using the state derivative control law; Llins et al. (2017) designed a gain scheduling control via state derivative feedback; and others (YAZICI; SEVER, 2017; BETETO et al., 2018a, 2018b). The advantages of use LMIs are the possibility to include performance index in approaching the problem and the facility in the treatment of uncertainties present in the system model (BOYD et al., 1994). Also, when feasible, LMIs can be easily solved by some software present in the mathematical programming literature, such as MatLab® (GAHINET et al., 1994).

As aforementioned, LMIs can solve several problems that deal with the state derivative feedback. Moreover, LMIs can be utilised in the resolution of many problems (WANG et al., 2015b; OMIDI; MAHMOODI, 2016; ASSUNÇÃO et al., 2016; YAZICI; SEVER, 2016; MOBAYEN; BALEANU; TCHIER, 2017). One of this problem takes in hand the linear quadratic regulator (LQR). According to Das et al. (2013), the LQR problem may be reduced to the algebraic Riccati equation (ARE) and its solution results in the state feedback gains for a chosen set of weighting matrices which regulates the penalties on the deviation in the trajectories of the state space and control signal. Thus, with adequate weighting matrices the designer can choose which are more important in the control law, the states or the control signal, to achieve an appropriate transient and performance (OLALLA et al., 2009). Due to it, the LQR has been widely applied to several problems: Choi et al. (1998) designed the gains of the controller using a relation between the weighting matrix Q in LQR and an eigenstructure of the desired closed-loop system; Ge, Chiu and Wang (2002) designed a robust PID (proportional-integral-derivative) using an LQR-LMI framework; Olalla et al. (2009) developed a robust LQR control for power converters using LMIs; Kanieski, Gründlinge and Cardoso (2010) presented an application of the LQR in power quality conditioning devices; Das et al. (2013) performed an optimal analogue and discrete PID controllers using the continuous and discrete time LQR theory; Jose et al. (2015) presented a comparison between the conventional PID controller and the LQR controller applied in an inverted pendulum; Rijnen, Saccon and Nijmeijer (2015) developed an LQR controller to solve the trajectory tracking problem in mechanical systems with unilateral constraints; Wang et al. (2015a) used the LQR controller to obtain the integrated lateral force and yam moment (according to their respective reference values) for a four-wheel driving and four-wheel steering (4WD4WS) electric ground vehicles; Kumar, Raaja and Jerome (2016) developed an LQR controller to the attitude tracking control problem for a 2 *DoF* (degrees of freedom) laboratory helicopter; Rojas et al. (2016) performed a system of control of a robotic arm with two degrees of freedom using an LQR controller; Abdelaziz and Valášek (2005) and Abdelaziz (2010) presented the solution of the LQR problem using the state derivative feedback.

As can be seen, a great number of works uses the LQR to solve a large number of problems and each case the resolution of the ARE is made by a different technique. This work considers the formulation of the ARE via state derivative feedback, similar to Abdelaziz and Valášek (2005), Abdelaziz (2010), but the solution of the ARE is made by LMIs. Once that LMIs are based on Lyapunov theory (BOYD et al., 1994), there are many ways to formulate the problem: common quadratic Lyapunov functions (CQLFs) (ASSUNÇÃO et al., 2007; FARIA et al., 2009, 2010); fuzzy Lyapunov functions (FLFs) (MOZELLI; PALHARES; AVELLAR, 2009; LIU; CAO; CHANG, 2017); polynomial parameter-dependent Lyapunov functions (PPDLFs) (OLIVEIRA; PERES, 2006); parameter-dependent Lyapunov functions (PDLFs) (OLIVEIRA; SKELTON, 2001; OLIVEIRA; GEROMEL, 2005; SILVA et al., 2011, 2012; CAUN et al., 2016a, 2016b).

As mentioned previously, this work deal with the LQR problem via state derivative feedback and LMIs. In order to achieve less conservative results in the context of uncertain systems, this work proposes sufficient conditions based on Finsler's Lemma slack variables or on the results presented in (LIU; ZHANG, 2003; TEIXEIRA; ASSUNÇÃO; AVELLAR, 2003; CARNIATO et al., 2018). The uncertainties considered are polytopic uncertainties and, according to Boyd et al. (1994), these type of uncertainties can be included on the dynamic model of the system by a polytopic representation. However, in some cases, the uncertain parameter can vary with time. When it occurs, it is necessary to deal with properly, and if the variation of the uncertain parameter is known, that a control strategy named gain scheduling (GS) can be used (MERRY et al., 2014).

According to Montagner and Peres (2006), GS technique deal with systems subject to parametric variations, which can be linear systems with time-varying parameters and (or) nonlinear systems modelled as linear parameter-varying systems. From classical GS, several linear controllers are developed, and change the parameters of controller as specified by a function of operating conditions in a preprogrammed way (scheduled the controller gain using some interpolation method), i.e., the main idea is to vary the parameters of the controller according to operation conditions of the process (ÅSTRÖM; WITTENMARK, 2008; STILWELL; RUGH, 1999). Due to its nature, GS has been widely and successfully applied in fields ranging from aerospace to process control (LEITH; LEITHEAD, 2000). Oosterom and Babuska (2001) developed a fuzzy gain scheduling flight control law in order to automate the design procedure; Lee and Chung (2001) proposed a design method of gain-scheduled state feedback control using a minimum sensitivity eigenvalue assignment of the frozen parameter closed-loop system with quadratic stability checks; Masubuchi et al. (2004) performed a synthesis method of gain-scheduled controllers based on descriptor representations of linear parameter-varying (LPV) systems and its application to design of flight vehicle control; Sun et al. (2009) designed a flight control law based on neural network gain scheduling, and a modified particle swarm optimisation algorithm (PSO); Abu-Rmileh and Garcia-Gabin (2010) developed a controller with a feedforward loop to improve meal compensation, a gain scheduling scheme to account for different blood glucose levels, and an asymmetric cost function to reduce hypoglycaemic risk; Badihi, Zhang and Hong (2013), Liu et al. (2012) handle wind turbines problem with GS technique; Veselỳ and Ilka (2015) designed a novel approach to robust gain-scheduled controller based on the robust stability condition for an uncertain LPV system model also introduced by the paper; Chaiyatham and Ngamroo (2017) deal with the system inertia and synchronising coupling problem of large photovoltaic (PV) farms via fuzzy gain scheduling of proportional-integral-derivative (FGS-PID) controller; Llins et al. (2017) designed a gain-scheduled control via state derivative feedback; Sato (2018) designed a GS flight controller for the lateral-directional motions of an In-Flight Simulator (IFS) in a certain speed range; Zhang et al. (2018) approached the nonfragile H_∞ control problems for a class of discrete-time Takagi-Sugeno (TS) fuzzy systems with both randomly occurring gain variations (ROGVs) and channel fadings.

Recently, several papers have appeared dealing with the gain-scheduled control plus the LQR control. Tzes and Nikolakopoulos (2004) designed a gain scheduling LQR-output controller for a mobile networked controlled system; Masar and Stöhr (2011) developed a gain-scheduled LQR control for an autonomous airship; Ebli, Khani and Azizi (2013) study the missile flight control problem using a gain scheduling controller for the flight model and an LQR controller for the system response; Li et al. (2014) introduced an attitude controller design method which combines LQR with fuzzy gain scheduling for the Reusable Launch Vehicle (RLV); Aktaş, Sever and Yazici (2016) performed a gain scheduling LQR control in order to achieve position tracking with minimum swy angle for an overhead crane system; Wang et al. (2018) designed a gain-scheduled robust linear quadratic regulator (RLQR) in order to guarantee robustness with respect to uncertainties. Additionally, following the GS tendency, this work proposes a gain scheduling LQR-state derivative controller, in order to deal with the systems that have varying parameters in its model.

Moreover, to illustrate the efficiency of the new methodology, some numerical examples are performed, and a practical implementation to validate the technique is realised on an active suspension system, manufactured by Quanser®.

The work is organised as follows:

- Chapter 2: presents some fundamental concepts and properties that will be used over the text.

  ► Section 2.1: introduces the state derivative feedback based on the results of (ABDELAZIZ; VALÁŠEK, 2004).

  ► Section 2.2: presents the LQR-state derivative controller based on the results of (ABDELAZIZ; VALÁŠEK, 2005; KUNCEVIC; LYCAK, 1997; VESELỲ; ILKA, 2017).

  ► Section 2.3: presents the state derivative feedback for uncertain and time-invariant systems and the LQR-state derivative controller for uncertain and time-invariant systems. As the state derivative feedback is being used, then a polytope of initial conditions is presented.

- Chapter 3: presents new results for the stabilisation for uncertain linear and time-invariant systems.

  ► Section 3.1: investigates the asymptotic stability of uncertain linear systems considering the LQR-state derivative controller. Besides, using the results presented in (LIU; ZHANG, 2003; TEIXEIRA; ASSUNÇÃO; AVELLAR, 2003; CARNIATO et al., 2018), less conservative conditions are achieved. Furthermore, a comparison is performed to show the advantage of adding the LQR-state derivative to the problem approach. A theoretical example is performed to validate the results.

  ► Section 3.2: investigates the asymptotic stability of linear uncertain systems considering the LQR-state derivative controller and Finsler's Lemma (SKELTON; IWASAKI; GRIGORIADIS, 1998). Besides, using the results presented in (LIU; ZHANG, 2003; TEIXEIRA; ASSUNÇÃO; AVELLAR, 2003; CARNIATO et al., 2018), less conservative conditions are achieved. First, Finsler's Lemma slack variables for the whole domain of uncertainties are used. After, Finsler's Lemma slack variables for each vertex of the uncertainty polytope are used. A theoretical example is performed to validate the results.

- Chapter 4: investigates the asymptotic stability of linear systems which have a time-varying parameter considering the gain scheduling LQR-state derivative controller. A theoretical example is performed to validate the results.
- Chapter 5: presents a practical implementation on an active suspension system, produced by Quanser®, to validate the proposed techniques. First, an implementation considering the LQR-state derivative controller is performed. After, an implementation considering the gain scheduling LQR-state derivative controller is performed.
- In the end, conclusions and future perspectives are presented.

2 FUNDAMENTAL CONCEPTS AND PROPERTIES

This Chapter presents a review of some concepts used throughout the text.

Lemma 1 (Schur Complement)**.** *Consider the LMI*

$$\begin{bmatrix} M_1(x) & M_2(x) \\ M_2(x)^T & M_3(x) \end{bmatrix} > 0, \tag{1}$$

where $M_1(x) = M_1(x)^T$, $M_3(x) = M_3(x)^T$ and $M_2(x)$ depend affinely on x, is equivalent to

$$M_3(x) > 0,\ M_1(x) - M_2(x)(M_3(x))^{-1}M_2(x)^T > 0, \tag{2}$$

or,

$$M_1(x) > 0,\ M_3(x) - M_2(x)^T(M_1(x))^{-1}M_2(x) > 0. \tag{3}$$

Proof. See Boyd et al. (1994). □

Propriety 1. *A matrix M is invertible if $M + M^T < 0$ for any nonsymmetric matrix M $(M \neq M^T)$ (BOYD et al., 1994).*

Lemma 2 (Finsler's Lemma)**.** *Consider $\mathscr{W} \in \mathbb{R}^{2n}$, $\mathscr{D} \in \mathbb{R}^{2n\times 2n}$ and $\mathscr{B} \in \mathbb{R}^{n\times 2n}$ with rank $(\mathscr{B}) < n$ and $\mathscr{B}^{\perp}$ a basis for the null space of $\mathscr{B}$ (in other words $\mathscr{B}\mathscr{B}^{\perp} = 0$). Then, the following conditions are equivalents:*

(i) $\mathscr{W}^T\mathscr{D}\mathscr{W} < 0, \forall \mathscr{W} \neq 0, \mathscr{B}\mathscr{W} = 0,$

(ii) $\mathscr{B}^{\perp^T}\mathscr{D}\mathscr{B}^{\perp} < 0,$

(iii)$\exists \rho \in \mathbb{R} : \mathscr{D} - \rho\mathscr{B}^T\mathscr{B} < 0,$

(iv) $\exists \mathscr{X} \in \mathbb{R}^{2n\times n} : \mathscr{D} + \mathscr{X}\mathscr{B} + \mathscr{B}^T\mathscr{X}^T < 0,$

where ρ and $\mathscr{X}$ are extra variables (or multipliers).

Proof. See (SKELTON; IWASAKI; GRIGORIADIS, 1998; OLIVEIRA; SKELTON, 2001). □

Lemma 3. *(TANAKA; IKEDA; WANG, 1998) If the following LMIs*

$$\Upsilon_{ii} < 0,\ \ i = 1, 2, ..., r, \tag{4}$$

$$\Upsilon_{ij} + \Upsilon_{ji} < 0, \;\; 1 \leq i < j \leq r, \tag{5}$$

are feasible, then the inequality

$$\begin{aligned} \sum_{i=1}^{r} \alpha_i \sum_{j=1}^{r} \alpha_j (\Upsilon_{ij}) < 0, \\ \Upsilon(\alpha) < 0, \end{aligned} \tag{6}$$

holds, where α *belongs to the unitary simplex* $\mathfrak{A}$*,*

$$\mathfrak{A} = \left\{ \sum_{i=1}^{r} \alpha_i = 1, \;\; \alpha_i \geq 0, \;\; i = 1, \ldots, r, \right\}, \tag{7}$$

where r represents the polytope vertices.

Using the results presented in (TANAKA; IKEDA; WANG, 1998; LIU; ZHANG, 2003), the following lemma presents conditions to achieve less conservative results, considering the state derivative feedback.

Lemma 4. *If the following LMIs*

$$\Upsilon_{ii} < Z_{ii}, \;\; i = 1, 2, \ldots, r, \tag{8}$$

$$\Upsilon_{ij} + \Upsilon_{ji} < Z_{ij} + Z_{ij}^T, \;\; 1 \leq i < j \leq r, \tag{9}$$

are feasible, then the inequality

$$\begin{aligned} \sum_{i=1}^{r} \alpha_i \sum_{j=1}^{r} \alpha_j (\Upsilon_{ij}) < \sum_{i=1}^{r} \alpha_i \sum_{j=1}^{r} \alpha_j (Z_{ij}), \\ \Upsilon(\alpha) < Z(\alpha), \quad Z(\alpha) = \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} Z_{11} & \ldots & Z_{1r} \\ \vdots & \ddots & \vdots \\ Z_{r1} & \ldots & Z_{rr} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix} < 0, \end{aligned} \tag{10}$$

holds, where α *belongs to the unitary simplex* $\mathfrak{A}$ (7).

Proof. Consider the following definitions

$$\sum_{i=1}^{r} \alpha_i^2 = \Sigma_i^2, \tag{11}$$

$$\sum_{i=1}^{r} \alpha_i = \Sigma_i, \tag{12}$$

$$\sum_{j=1}^{r} \alpha_j = \Sigma_j, \tag{13}$$

$$\sum_{i=1}^{r-1}\alpha_i\sum_{j=i+1}^{r}\alpha_j=\Sigma_{ij}, \tag{14}$$

$$\sum_{i=1}^{r}\alpha_i\sum_{j=1}^{r}\alpha_j=\sum_{i=1}^{r}\alpha_i^2+2\sum_{i=1}^{r-1}\alpha_i\sum_{j=i+1}^{r}\alpha_j\Leftrightarrow\Sigma_i\Sigma_j=\Sigma_i^2+2\Sigma_{ij}, \tag{15}$$

$$\begin{gathered}\sum_{i=1}^{r}\alpha_i\sum_{j=1}^{r}\alpha_j(M_iN_j)=\sum_{i=1}^{r}\alpha_i^2(M_iN_i)+\sum_{i=1}^{r-1}\alpha_i\sum_{j=i+1}^{r}\alpha_j(M_iN_j+M_jN_i),\\ \Sigma_i\Sigma_j(M_iN_j)=\Sigma_i^2(M_iN_i)+\Sigma_{ij}(M_iN_j+M_jN_i).\end{gathered} \tag{16}$$

Then, multiplying (8) by α_i^2, and summing all terms in i, with $i=1,2,...,r$,

$$\Sigma_i^2(\Upsilon_{ii})<\Sigma_i^2(Z_{ii}), \tag{17}$$

and multiplying (9) by $\alpha_i\alpha_j$, and summing all terms in i, with $i=1,2,...,r-1$, and in j, with $j=i+1,...,r$,

$$\Sigma_{ij}(\Upsilon_{ij}+\Upsilon_{ji})<\Sigma_{ij}(Z_{ij}+Z_{ij}^T). \tag{18}$$

Now, summing (17) and (18),

$$\Sigma_i\Sigma_j(\Upsilon_{ij})<\Sigma_i\Sigma_j(Z_{ij}), \tag{19}$$

or,

$$\Upsilon(\alpha)<Z(\alpha). \tag{20}$$

If the set of LMIs is feasible, then $Z(\alpha)<0$. Thus,

$$\begin{gathered}\Upsilon(\alpha)<Z(\alpha)<0,\\ \Upsilon(\alpha)<0.\end{gathered} \tag{21}$$

□

Following the same concept from (TANAKA; IKEDA; WANG, 1998), the next lemma presents conditions to deal with the cross product between three variables.

Lemma 5. *If the following LMIs*

$$\Upsilon_{iii}<0,\;\; i=1,2,...,r, \tag{22}$$

$$\Upsilon_{iij}+\Upsilon_{iji}+\Upsilon_{jii}<0,\;\;\begin{cases} i,j=1,2,...,r,\\ \quad i\neq j,\end{cases} \tag{23}$$

$$\Upsilon_{ijk} + \Upsilon_{ikj} + \Upsilon_{jik} + \Upsilon_{jki} + \Upsilon_{kij} + \Upsilon_{kji} < 0, \quad \begin{cases} i = 1,2,...,r-2, \\ j = i+1,...,r-1, \\ k = j+1,...,r, \end{cases} \tag{24}$$

are feasible, then the inequality

$$\begin{gathered} \sum_{i=1}^{r} \alpha_i \sum_{j=1}^{r} \alpha_j \sum_{j=k}^{r} \alpha_k(\Upsilon_{ijk}) < 0, \\ \Upsilon(\alpha) < 0, \end{gathered} \tag{25}$$

holds, where α *belongs to the unitary simplex* $\mathfrak{A}$ (7).

Proof. Consider the definitions (12) and (13), and the following definitions

$$\sum_{k=1}^{r} \alpha_k = \Sigma_k \tag{26}$$

$$\sum_{i=1}^{r} \alpha_i^3 = \Sigma_i^3, \tag{27}$$

$$\sum_{i=1}^{r} \alpha_i^2 \sum_{j \neq i}^{r} \alpha_j = \Sigma_{ij}^{i \neq j}, \tag{28}$$

$$\sum_{i=1}^{r-2} \alpha_i \sum_{j=i+1}^{r-1} \alpha_j \sum_{k=j+1}^{r} \alpha_k = \Sigma_{ijk}, \tag{29}$$

$$\begin{gathered} \sum_{i=1}^{r} \alpha_i \sum_{j=1}^{r} \alpha_j \sum_{k=1}^{r} \alpha_k = \sum_{i=1}^{r} \alpha_i^3 + 3\sum_{i=1}^{r} \alpha_i^2 \sum_{j \neq i}^{r} \alpha_j + 6\sum_{i=1}^{r-2} \alpha_i \sum_{j=i+1}^{r-1} \alpha_j \sum_{k=j+1}^{r} \alpha_k \\ \Sigma_i \Sigma_j \Sigma_k = \Sigma_i^3 + 3\Sigma_{ij}^{i \neq j} + 6\Sigma_{ijk} \end{gathered} \tag{30}$$

$$\begin{aligned} \sum_{i=1}^{r} \alpha_i \sum_{j=1}^{r} \alpha_j \sum_{k=1}^{r} \alpha_k (L_i M_j N_k) &= \sum_{i=1}^{r} \alpha_i^3 (L_i M_i N_i) + \sum_{i=1}^{r} \alpha_i^2 \sum_{j \neq i}^{r} \alpha_j (L_i M_i N_j + L_i M_j N_i + L_j M_i N_i) + \\ &+ \sum_{i=1}^{r-2} \alpha_i \sum_{j=i+1}^{r-1} \alpha_j \sum_{k=j+1}^{r} \alpha_k (L_i M_j N_k + L_i M_k N_j + L_j M_i N_k + L_j M_k N_i + L_k M_i N_j + L_k M_i N_j) \\ \Sigma_i \Sigma_j \Sigma_k = \Sigma_i^3 (L_i M_i N_i) + \Sigma_{ij}^{i \neq j} (L_i M_i N_j &+ L_i M_j N_i + L_j M_i N_i) + \Sigma_{ijk} (L_i M_j N_k + L_i M_k N_j + L_j M_i N_k + \\ &+ L_j M_k N_i + L_k M_i N_j + L_k M_i N_j) \end{aligned} \tag{31}$$

Then, multiplying (22) by α_i^3, and summing all terms in i, with $i = 1,2,...,r$,

$$\Sigma_i^3 (\Upsilon_{iii}) < 0, \tag{32}$$

multiplying (23) by $\alpha_i^2 \alpha_j$ and summing all terms in $i,\ j$ with $i, j = 1,2,...,r,\ i \neq j$,

$$\Sigma_{ij}^{i\neq j}(\Upsilon_{iij}+\Upsilon_{iji}+\Upsilon_{jii})<0, \tag{33}$$

and multiplying (24) by $\alpha_i\alpha_j\alpha_k$, and summing all terms in i, with $i=11,2,...,r-2$, j with $j=i+1,...,r-1$, and k, with $k=j+1,...,r$,

$$\Sigma_{ijk}(\Upsilon_{ijk}+\Upsilon_{ikj}+\Upsilon_{jik}+\Upsilon_{jki}+\Upsilon_{kij}+\Upsilon_{kji})<0 \tag{34}$$

Now, summing (32)-(34),

$$\Sigma_i\Sigma_j\Sigma_k(\Upsilon_{ijk})<0, \tag{35}$$

or,

$$\Upsilon(\alpha)<0. \tag{36}$$

□

In this work, for the numerical solution of the LMIs was used the MatLab® software, together with ”LMILab” solver and the YALMIP interface (LOFBERG, 2004). For the simulations, the ”ODE45” function from MatLab®/Simulink software was used. In some cases, the number of solver's iterations was increased. The command used was:

```
sdpsettings('solver','lmilab','lmilab.maxiter',1000);
```

The next section presents the feedback technique used in this work, the state derivative feedback.

2.1 STATE DERIVATIVE FEEDBACK

Consider a controllable, linear and time-invariant system described by:

$$\dot{x}(t)=Ax(t)+Bu(t),\ \ x(t_0)=x_0, \tag{37}$$

where $A\in\mathbb{R}^{n\times n}$ is the matrix that describes the system's behaviour, $B\in\mathbb{R}^{n\times m}$ is the input matrix, $x(t)\in\mathbb{R}^{n\times 1}$ is the state vector and $u(t)\in\mathbb{R}^{m}$ is the control input vector.

Admitting that the matrix A is non-singular ($\det(A)\neq 0$) (ABDELAZIZ; VALÁŠEK, 2004), to enable the state derivative feedback and the matrix $(I+BK)$ to be invertible by being feedback by the control input

$$u(t)=-K\dot{x}(t), \tag{38}$$

it is necessary to find a matrix $K \in \mathbb{R}^{m\times n}$ that satisfies this condition in closed-loop.

Thus, the closed-loop system becomes

$$\dot{x}(t) = Ax(t) - BK\dot{x}(t) \Leftrightarrow \dot{x}(t) = (I+BK)^{-1}Ax(t), \tag{39}$$

where $I \in \mathbb{R}^{n\times n}$ is an identity matrix. For more details see (ABDELAZIZ; VALÁŠEK, 2004).

Next section presents the LQR controller based on state derivative feedback.

2.2 LQR-STATE DERIVATIVE CONTROLLER

Consider the system (37), the LQR-state derivative control with good dynamic behaviour is achieved by means of the control design that minimises a quadratic cost or performance index of the type $J(\dot{x}(t), u(t))$ (ABDELAZIZ; VALÁŠEK, 2005). Through the Bellman-Lyapunov inequality (VESELÝ; ILKA, 2017) with a slight modification

$$\dot{V}(x(t)) < -J(\dot{x}(t), u(t)), \tag{40}$$

it is possible to find a controller with guaranteed cost.

Considering that a candidate Lyapunov function

$$V(x(t)) = x(t)^T P x(t) > 0, \tag{41}$$

is used, the derivative of the Lyapunov function candidate is

$$\dot{V}(x(t)) = \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) < 0. \tag{42}$$

Replacing (42) into (40), and considering $J(\dot{x}(t), u(t)) = \dot{x}(t)^T \mathcal{Q}\dot{x}(t) + u(t)^T \mathcal{R} u(t)$,

$$\dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) < -(\dot{x}(t)^T \mathcal{Q}\dot{x}(t) + u(t)^T \mathcal{R} u(t)) \tag{43}$$

Remark 1. *Matrix $\mathcal{Q}$ is a positive definite $n \times n$ symmetric matrix and matrix $\mathcal{R}$ is a positive definite $m \times m$ symmetric matrix. The matrices $\mathcal{Q}$ e $\mathcal{R}$ are the weighting matrices of state derivative and control signal, respectively.*

Integrating both sides of (43), from zero to infinity,

$$\int_0^\infty (\dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t))dt < -\int_0^\infty (\dot{x}(t)^T \mathcal{Q}\dot{x}(t) + u(t)^T \mathcal{R} u(t))dt \tag{44}$$

Regarding that $\dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) = d/dt(x(t)^T P x(t))$, (44) becomes

$$\begin{aligned}\int_0^\infty (d/dt(x(t)^T Px(t)))dt &< -\int_0^\infty (\dot{x}(t)^T \mathcal{Q}\dot{x}(t) + u(t)^T \mathcal{R}u(t))dt,\\ x(t)^T Px(t)\big|_0^\infty &< -\int_0^\infty (\dot{x}(t)^T \mathcal{Q}\dot{x}(t) + u(t)^T \mathcal{R}u(t))dt,\\ x(\infty)^T Px(\infty) - x(0)^T Px(0) &< -\int_0^\infty (\dot{x}(t)^T \mathcal{Q}\dot{x}(t) + u(t)^T \mathcal{R}u(t))dt.\end{aligned} \tag{45}$$

Assuming that the closed-loop system is asymptotically stable, that is, all eigenvalues of (39) have negative real parts, then $x(\infty) \to 0$. Thus, the performance index converges to the optimal value when

$$\begin{aligned}-x(0)^T Px(0) &< -\int_0^\infty (\dot{x}(t)^T \mathcal{Q}\dot{x}(t) + u(t)^T \mathcal{R}u(t))dt,\\ J_D(\dot{x}(t), u(t)) &= \int_0^\infty (\dot{x}(t)^T \mathcal{Q}\dot{x}(t) + u(t)^T \mathcal{R}u(t))dt < x(0)^T Px(0).\end{aligned} \tag{46}$$

The performance index $J_D(\dot{x}(t), u(t))$ can be obtained in terms of the initial conditions $x(0)$ and Lyapunov matrix P. The LQR-state derivative controller can be seen in greater details in (ABDELAZIZ; VALÁŠEK, 2005). To facilitate the reading, $x(0) = x_0$ will be used throughout the text.

Remark 2. *Note that this formulation is similar to the original LQR one as the performance index is based on state derivatives instead of states.*

2.3 STATE DERIVATIVE FEEDBACK FOR UNCERTAIN AND TIME-INVARIANT SYSTEMS

All of the systems are subject to some failure or interruption during its operation. Such a fact can occur due to the natural wear of some component, breakage by external factors, breakage by incorrect handling, among others (ISERMANN; BALLÉ, 1997; ISERMANN, 2006). These events are known as structural failures and can be included in the dynamic model of the system by polytopic uncertainties (BOYD et al., 1994). The polytopic representation can describe the physical parameter uncertainty without any conservatism since the interval of uncertain is well-known (KARIMI; KHATIBI; LONGCHAMP, 2007). Taking this into account, the state derivative for uncertain and time-invariant systems are described below.

Consider a controllable, linear, time-invariant and uncertain system described as a convex combination of the polytope vertices:

$$\dot{x}(t) = \sum_{i=1}^{r} \alpha_i (A_i x(t) + B_i u(t)) = A(\alpha)x(t) + B(\alpha)u(t), \tag{47}$$

where r represents the number of polytope vertices. The parameters $\alpha_i, i = 1, 2, ..., r$ are constant and unknown real numbers belonging to unitary simplex $\mathfrak{A}$ given by

$$\mathfrak{A} = \left\{ \sum_{i=1}^{r} \alpha_i = 1, \;\; \alpha_i \geq 0, \;\; i = 1, ..., r, \right\}. \tag{48}$$

Admitting that the matrix $A(\alpha)$ from system (47) be non-singular $(det(A(\alpha)) \neq 0)$ and replacing the control law (38) in (47), the robust system in closed-loop is given by

$$\dot{x}(t) = (I + B(\alpha)K)^{-1} A(\alpha) x(t). \tag{49}$$

Assuming that the system (47) can have varying parameters, sufficiently small rates of variation are allowed, ensuring the necessary accommodation time for disturbances arising from such variations, even α not explicitly time-depending (DAHLEH; DAHLEH, 1991; LEITE; MONTAGNER; PERES, 2004).

As this work deal with the state derivative feedback and the initial conditions vector x_0 is not available, i.e., we do not have access to x_0, then a polytope of initial conditions is used,

$$x_0(\beta) = \sum_{l=1}^{p} \beta_l x_{0l}, \tag{50}$$

similar to the unitary simplex (48), being p the vertices of the initial conditions polytope.

Next section, the LQR-state derivative for uncertain systems is proposed.

3 ROBUST LQR-STATE DERIVATIVE CONTROLLER

In this chapter, sufficient conditions for the robust LQR-state derivative controller are proposed.

3.1 STABILITY CONDITION

Considering the candidate Lyapunov function (41), and the state derivative feedback, Theorem 1 proposes sufficient conditions for the stabilisation of closed-loop system (49) through the LQR-state derivative controller.

Theorem 1. *Let A_i non-singular ($det(A_i) \neq 0$) and given $\mathcal{Q} \in \mathbb{R}^{n\times n}$, $\mathcal{R} \in \mathbb{R}^{m\times m}$ and $x_{0l} \in \mathbb{R}^{n\times 1}$, the uncertain system* (49) *is stable and has optimised performance if there exist symmetric matrix $X > 0 \in \mathbb{R}^{n\times n}$ and matrix $Y \in \mathbb{R}^{m\times n}$, satisfying:*

$$\underset{X=X^T>0,Y}{min}\ \mu$$

Subject to

$$\begin{bmatrix} \mu & x_{0l}^T \\ x_{0l} & X \end{bmatrix} > 0, \quad l = 1,2,...,p, \tag{51}$$

$$\begin{bmatrix} \Gamma_{ii} & A_iX & A_iY^T \\ XA_i^T & -\mathcal{Q}^{-1} & 0 \\ YA_i^T & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0, \quad i = 1,2,...,r, \tag{52}$$

$$\begin{bmatrix} \Gamma_{ij}+\Gamma_{ji} & A_iX+A_jX & A_iY^T+A_jY^T \\ XA_i^T+XA_j^T & -2\mathcal{Q}^{-1} & 0 \\ YA_i^T+YA_j^T & 0 & -2\mathcal{R}^{-1} \end{bmatrix} < 0, \quad \begin{cases} i = 1,2,...,r-1, \\ j = i+1,...,r, \end{cases} \tag{53}$$

where $\Gamma_{ii} = A_iX + XA_i^T + B_iYA_i^T + A_iY^TB_i^T$, $\Gamma_{ij} + \Gamma_{ji} = A_iX + XA_i^T + A_jX + XA_j^T + B_jYA_i^T + A_iY^TB_j^T + B_iYA_j^T + A_jY^TB_i^T$. *The state derivative feedback gain can be given by*

$$K_{T1} = YX^{-1}. \tag{54}$$

Proof. Supposing the set of LMIs feasible.

Multiplying by β_l, $l = 1,2,...,p$, summing all terms,

$$\begin{bmatrix} \sum_{l=1}^{p} \mu & \sum_{l=1}^{p} x_{0l}^T \\ \sum_{l=1}^{p} x_{0l} & \sum_{l=1}^{p} X \end{bmatrix} > 0, \tag{55}$$

applying Lemma 1 in LMI (51), and considering $P = X^{-1}$, it has been

$$\begin{gathered} \mu - x_0(\beta)^T X^{-1} x_0(\beta) > 0, \\ \mu > x_0(\beta)^T P x_0(\beta), \\ x_0(\beta)^T P x_0(\beta) < \mu \end{gathered} \tag{56}$$

In many practical situations, the objective (46) can be modified by (56), where μ is the specified upper bound, as seen in (GE; CHIU; WANG, 2002). Then,

$$\begin{gathered} J_D(\dot{x}(t), u(t)) < x_0(\beta)^T P x_0(\beta) < \mu \\ J_D(\dot{x}(t), u(t)) < \mu \end{gathered} \tag{57}$$

If the LMI is feasible, then the index $J_D(\dot{x}(t), u(t))$ will be always lesser than the upper bound μ.

Now, multiplying LMI (52) by α_i^2, and summing all terms in i, with $i = 1, 2, ..., r$,

$$\begin{bmatrix} \Sigma_i^2(\Gamma_{ii}) & \Sigma_i^2(A_i X) & \Sigma_i^2(A_i Y^T) \\ \Sigma_i^2(X A_i^T) & \Sigma_i^2(-\mathcal{Q}^{-1}) & 0 \\ \Sigma_i^2(Y A_i^T) & 0 & \Sigma_i^2(-\mathcal{R}^{-1}) \end{bmatrix} < 0, \tag{58}$$

and multiplying (53) by $\alpha_i \alpha_j$, and summing all terms in i, with $i = 1, 2, ..., r-1$, and in j, with $j = i+1, ..., r$,

$$\begin{bmatrix} \Sigma_{ij}(\Gamma_{ij} + \Gamma_{ji}) & \Sigma_{ij}(A_i X + A_j X) & \Sigma_{ij}(A_i Y^T + A_j Y^T) \\ \Sigma_{ij}(X A_i^T + X A_j^T) & \Sigma_{ij}(-2\mathcal{Q}^{-1}) & 0 \\ \Sigma_{ij}(Y A_i^T + Y A_j^T) & 0 & \Sigma_{ij}(-2\mathcal{R}^{-1}) \end{bmatrix} < 0. \tag{59}$$

Applying Lemma 3, (60) follows.

$$\begin{bmatrix} \Sigma_i\Sigma_j(\Gamma_{ij}) & \Sigma_i\Sigma_j(A_i X) & \Sigma_i\Sigma_j(A_i Y^T) \\ \Sigma_i\Sigma_j(X A_i^T) & \Sigma_i\Sigma_j(-\mathcal{Q}^{-1}) & 0 \\ \Sigma_i\Sigma_j(Y A_i^T) & 0 & \Sigma_i\Sigma_j(-\mathcal{R}^{-1}) \end{bmatrix} < 0, \tag{60}$$

where $\Gamma_{ij} = A_i X + X A_i^T + B_j Y A_i^T + A_i Y^T B_j^T$, or,

$$\begin{bmatrix} A(\alpha)X + XA(\alpha)^T + B(\alpha)YA(\alpha)^T + A(\alpha)Y^T B(\alpha)^T & A(\alpha)X & A(\alpha)Y^T \\ XA(\alpha)^T & -\mathcal{Q}^{-1} & 0 \\ YA(\alpha)^T & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0. \qquad (61)$$

Applying Lemma 1 recursively in (60), replacing the variable $Y = K_{T1}X$ and organising,

$$A(\alpha)(X + B(\alpha)K_{T1}X)^T + (X + B(\alpha)K_{T1}X)A(\alpha)^T + A(\alpha)XK_{T1}^T\mathcal{R}K_{T1}XA(\alpha)^T + A(\alpha)X\mathcal{Q}XA(\alpha)^T < 0,$$
$$\Updownarrow$$
$$A(\alpha)X(I + B(\alpha)K_{T1})^T + (I + B(\alpha)K_{T1})XA(\alpha)^T + A(\alpha)X(K_{T1}^T\mathcal{R}K_{T1} + \mathcal{Q})XA(\alpha)^T < 0. \qquad (62)$$

Now, applying Propriety 1 in (62) it is concluded that matrices $(I + B(\alpha)K_{T1})$, $A(\alpha)$ and X are invertible. Then, premultiplying by $X^{-1}A(\alpha)^{-1}$ and posmultiplying by $A(\alpha)^{-T}X^{-1}$, with $X^{-1} = P$:

$$(I + B(\alpha)K_{T1})^T A(\alpha)^{-T}P + PA(\alpha)^{-1}(I + B(\alpha)K_{T1}) + K_{T1}^T\mathcal{R}K_{T1} + \mathcal{Q} < 0. \qquad (63)$$

Premultiplying by $A(\alpha)^T(I + B(\alpha)K_{T1})^{-T}$, posmultiplying by $(I + B(\alpha)K_{T1})^{-1}A(\alpha)$, (64) follows.

$$P(I + B(\alpha)K_{T1})^{-1}A(\alpha) + A(\alpha)^T(I + B(\alpha)K_{T1})^{-T}P +$$
$$+ A(\alpha)^T(I + B(\alpha)K_{T1})^{-T}(K_{T1}^T\mathcal{R}K_{T1} + \mathcal{Q})(I + B(\alpha)K_{T1})^{-1}A(\alpha) < 0. \qquad (64)$$

Premultiplying by $x(t)^T$, posmultiplying by $x(t)$ and replacing $A_{cl}(\alpha) = (I + B(\alpha)K_{T1})^{-1}A(\alpha)$:

$$x(t)^T A_{cl}(\alpha)^T Px(t) + x(t)^T PA_{cl}(\alpha)x(t) + x(t)^T A_{cl}(\alpha)^T(K_{T1}^T\mathcal{R}K_{T1} + \mathcal{Q})A_{cl}(\alpha)x(t) < 0. \qquad (65)$$

Replacing $\dot{x}(t) = A_{cl}(\alpha)x(t) = (I + B(\alpha)K_{T1})^{-1}A(\alpha)x(t)$:

$$\begin{gathered} \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) < -\dot{x}(t)^T(K_{T1}^T\mathcal{R}K_{T1} + \mathcal{Q})\dot{x}(t), \\ \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) < -J(\dot{x}(t), u(t)). \end{gathered} \qquad (66)$$

As $J(\dot{x}(t), u(t)) > 0$, than,

$$\dot{V}(x(t)) = \dot{x}(t)^T Px(t) + x(t)^T P\dot{x}(t) < 0 \qquad (67)$$

The proof of Theorem 1 is completed. □

Now, in order to relax the feasibility of the LMIs from Theorem 1, based on the results presented in (LIU; ZHANG, 2003), less conservative conditions are proposed in Theorem 2. The seek for less conservative conditions occurs due to the attempt to obtain a larger range of values for $\mathcal{Q}$ and $\mathcal{R}$, in addition to decreasing the guaranteed cost μ.

Theorem 2. *Let A_i non-singular ($det(A_i) \neq 0$) and given $\mathcal{Q} \in \mathbb{R}^{n\times n}$, $\mathcal{R} \in \mathbb{R}^{m\times m}$ and $x_{0l} \in \mathbb{R}^{n\times 1}$, the uncertain system* (49) *is stable and has optimised performance if there exist symmetric matrices $X > 0 \in \mathbb{R}^{n\times n}$, $Z_{ii} \in \mathbb{R}^{2n+m\times 2n+m}$, and matrices $Y \in \mathbb{R}^{m\times n}$, and $Z_{ij} \in \mathbb{R}^{2n+m\times 2n+m}$ satisfying:*

$$\underset{X=X^T>0,Y}{min\ \ \mu}$$

Subject to

$$\begin{bmatrix} \mu & x_{0l}^T \\ x_{0l} & X \end{bmatrix} > 0, \;\; l = 1,2,...,p, \tag{68}$$

$$\begin{bmatrix} \Gamma_{ii} & A_iX & A_iY^T \\ XA_i^T & -\mathcal{Q}^{-1} & 0 \\ YA_i^T & 0 & -\mathcal{R}^{-1} \end{bmatrix} < Z_{ii}, \;\; i = 1,2,...,r, \tag{69}$$

$$\begin{bmatrix} \Gamma_{ij}+\Gamma_{ji} & A_iX+A_jX & A_iY^T+A_jY^T \\ XA_i^T+XA_j^T & -2\mathcal{Q}^{-1} & 0 \\ YA_i^T+YA_j^T & 0 & -2\mathcal{R}^{-1} \end{bmatrix} < Z_{ij}+Z_{ij}^T, \;\; \begin{cases} i = 1,2,...,r-1, \\ j = i+1,...,r, \end{cases} \tag{70}$$

$$\begin{bmatrix} Z_{11} & \dots & Z_{1r} \\ \vdots & \ddots & \vdots \\ Z_{r1} & \dots & Z_{rr} \end{bmatrix} < 0, \tag{71}$$

where $\Gamma_{ii} = A_iX + XA_i^T + B_iYA_i^T + A_iY^TB_i^T$, $\Gamma_{ij} + \Gamma_{ji} = A_iX + XA_i^T + A_jX + XA_j^T + B_jYA_i^T + A_iY^TB_j^T + B_iYA_j^T + A_jY^TB_i^T$. The state derivative feedback gain can be given by

$$K_{T2} = YX^{-1}. \tag{72}$$

Proof. Supposing the set of LMIs feasible.

The proof of LMI (68) is similar to proof of LMI (51).

For the LMI (69), multiplying by α_i^2, and summing all terms in i, with $i = 1,2,...,r$,

$$\begin{bmatrix} \Sigma_i^2(\Gamma_{ii}) & \Sigma_i^2(A_iX) & \Sigma_i^2(A_iY^T) \\ \Sigma_i^2(XA_i^T) & \Sigma_i^2(-\mathcal{Q}^{-1}) & 0 \\ \Sigma_i^2(YA_i^T) & 0 & \Sigma_i^2(-\mathcal{R}^{-1}) \end{bmatrix} < \Sigma_i^2 Z_{ii}, \tag{73}$$

and multiplying (70) by $\alpha_i\alpha_j$, and summing all terms in i, with $i = 1,2,...,r-1$, and in j, with $j = i+1,...,r$,

$$\begin{bmatrix} \Sigma_{ij}(\Gamma_{ij}+\Gamma_{ji}) & \Sigma_{ij}(A_iX+A_jX) & \Sigma_{ij}(A_iY^T+A_jY^T) \\ \Sigma_{ij}(XA_i^T+XA_j^T) & \Sigma_{ij}(-2\mathcal{Q}^{-1}) & 0 \\ \Sigma_{ij}(YA_i^T+YA_j^T) & 0 & \Sigma_{ij}(-2\mathcal{R}^{-1}) \end{bmatrix} < \Sigma_{ij}(Z_{ij}+Z_{ij}^T). \tag{74}$$

Applying Lemma 3, (75) follows.

$$\begin{bmatrix} \Sigma_i\Sigma_j(\Gamma_{ij}) & \Sigma_i\Sigma_j(A_iX) & \Sigma_i\Sigma_j(A_iY^T) \\ \Sigma_i\Sigma_j(XA_i^T) & \Sigma_i\Sigma_j(-\mathcal{Q}^{-1}) & 0 \\ \Sigma_i\Sigma_j(YA_i^T) & 0 & \Sigma_i\Sigma_j(-\mathcal{R}^{-1}) \end{bmatrix} < \Sigma_i\Sigma_j(Z_{ij}) < 0, \tag{75}$$

where $\Gamma_{ij} = A_iX + XA_i^T + B_jYA_i^T + A_iY^TB_j^T$, or,

$$\begin{bmatrix} A(\alpha)X+XA(\alpha)^T+B(\alpha)YA(\alpha)^T+A(\alpha)Y^TB(\alpha)^T & A(\alpha)X & A(\alpha)Y^T \\ XA(\alpha)^T & -\mathcal{Q}^{-1} & 0 \\ YA(\alpha)^T & 0 & -\mathcal{R}^{-1} \end{bmatrix} < Z(\alpha) < 0. \tag{76}$$

Thus,

$$\begin{bmatrix} A(\alpha)X+XA(\alpha)^T+B(\alpha)YA(\alpha)^T+A(\alpha)Y^TB(\alpha)^T & A(\alpha)X & A(\alpha)Y^T \\ XA(\alpha)^T & -\mathcal{Q}^{-1} & 0 \\ YA(\alpha)^T & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0. \tag{77}$$

From here, the proof of Theorem 2 follows similar steps from proof of Theorem 1.

Remark 3. *Note that, if we consider* $\Upsilon(\alpha) = A(\alpha)^T(I+B(\alpha)K_{T2})^{-T}P + P(I+B(\alpha)K_{T2})^{-1}A(\alpha)+\Omega(\alpha)$, *than*

$$\Upsilon(\alpha) = A(\alpha)^T(I+B(\alpha)K_{T2})^{-T}P + P(I+B(\alpha)K_{T2})^{-1}A(\alpha)+\Omega(\alpha) < Z(\alpha) \tag{78}$$

Choosing $\Omega(\alpha) = A(\alpha)^T(I+B(\alpha)K_{T2})^{-T}(K_{T2}^T\mathcal{R}K_{T2}+\mathcal{Q})(I+B(\alpha)K_{T2})^{-1}A(\alpha)$, *premultiplying by* $x(t)^T$ *and posmultiplying by* $x(t)$, (79) *follows.*

$$x(t)^TA(\alpha)^T(I+B(\alpha)K_{T2})^{-T}Px(t)+x(t)^TP(I+B(\alpha)K_{T2})^{-1}A(\alpha)x(t)+$$

$$+x(t)^T A(\alpha)^T (I+B(\alpha)K_{T2})^{-T} (K_{T2}^T \mathcal{R} K_{T2} + \mathcal{Q})(I+B(\alpha)K_{T2})^{-1} A(\alpha)x(t) < x(t)^T Z(\alpha)x(t) \tag{79}$$

Replacing $\dot{x}(t) = (I+B(\alpha)K_{T2})^{-1}A(\alpha)x(t)$,

$$\begin{gathered}\dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) + \dot{x}(t)^T (K_{T2}^T \mathcal{R} K_{T2} + \mathcal{Q})\dot{x}(t) < x(t)^T Z(\alpha)x(t), \\ \dot{V}(x(t)) + J(\dot{x}(t), u(t)) < x(t)^T Z(\alpha)x(t).\end{gathered} \tag{80}$$

Remembering, if the set of LMIs is feasible, then

$$Z(\alpha) = \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} Z_{11} & \dots & Z_{1r} \\ \vdots & \ddots & \vdots \\ Z_{r1} & \dots & Z_{rr} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix} < 0, \tag{81}$$

which means that

$$\begin{gathered}\dot{V}(x(t)) + J(\dot{x}(t), u(t)) < x(t)^T \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix}^T \begin{bmatrix} Z_{11} & \dots & Z_{1r} \\ \vdots & \ddots & \vdots \\ Z_{r1} & \dots & Z_{rr} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \vdots \\ \alpha_r I \end{bmatrix} x(t) < 0 \\ \dot{V}(x(t)) + J(\dot{x}(t), u(t)) < 0.\end{gathered} \tag{82}$$

As $J(\dot{x}(t), u(t)) > 0$, *then*

$$\begin{gathered}\dot{V}(x(t)) < -J(\dot{x}(t), u(t)), \\ \dot{V}(x(t)) < 0.\end{gathered} \tag{83}$$

□

Theorem 3. *If the conditions given in Theorem 1 hold, then the conditions given in Theorem 2 also hold.*

Proof. Considering Υ_{ii} and $\Upsilon_{ij} + \Upsilon_{ji}$, where

$$\Upsilon_{ii} = \begin{bmatrix} A_i X + X A_i^T + B_i Y A_i^T + A_i Y^T B_i^T & A_i X & A_i Y^T \\ X A_i^T & -\mathcal{Q}^{-1} & 0 \\ Y A_i^T & 0 & -\mathcal{R}^{-1} \end{bmatrix}, \tag{84}$$

$$\Upsilon_{ij} + \Upsilon_{ji} = \begin{bmatrix} A_i X + X A_i^T + A_j X + X A_j^T + B_j Y A_i^T + A_i Y^T B_j^T + B_i Y A_j^T + A_j Y^T B_i^T & * & * \\ X A_i^T + X A_j^T & -2\mathcal{Q}^{-1} & 0 \\ Y A_i^T + Y A_j^T & 0 & -2\mathcal{R}^{-1} \end{bmatrix}, \tag{85}$$

if (52) and (53) hold, then there exist $Z_{ii} = \Upsilon_{ii} + \varepsilon I$, where $\varepsilon > 0$ is sufficiently small such that, $Z_{ii} < 0$, for $i = 1, 2, ..., r$, and $Z_{ij} = 0$ for $i, j = 1, 2, ..., r,\ i \neq j$, such that (69) and (70) hold.

$\square$

To verify the efficiency of the proposed techniques, next section presents some examples.

3.1.1 Illustrative Examples

Example 1. *Three mass-spring-dashpots system*

Consider a three mass-spring-dashpots system, borrowed from (ABDELAZIZ, 2013). System's schematic can be seen in Figure 1. The system can be represented in state space form as follow

$$\begin{bmatrix} \dot{x}_1(t) \\ \ddot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_2(t) \\ \dot{x}_3(t) \\ \ddot{x}_3(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \\ x_2(t) \\ \dot{x}_2(t) \\ x_3(t) \\ \dot{x}_3(t) \end{bmatrix} + B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \tag{86}$$

with

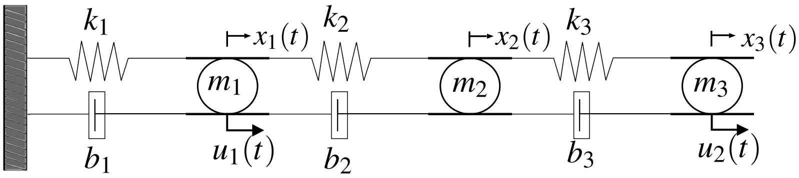
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-(k_1+k_2)}{m_1} & \frac{-(b_1+b_2)}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & \frac{-(k_2+k_3)}{m_2} & \frac{-(b_2+b_3)}{m_2} & \frac{k_3}{m_2} & \frac{b_3}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_3}{m_3} & \frac{b_3}{m_3} & \frac{-k_3}{m_3} & \frac{-b_3}{m_3} \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_3} \end{bmatrix}. \tag{87}$$

Table 1 shows the system parameters values.

Note in Table 1 that the value of spring stiffness k_2 is equal to 5 N/m. However, due to wear over time, the stiffness may vary between your nominal value and half the nominal value. Than, the parameter k_2 belongs to the interval $2.5 \leq k_2 \leq 5\ (N/m)$.

- Spring stiffness $k_2 = 2.5\ N/m$

Figure 1 - Three mass-spring-dashpots system.



Font: Adapted from (ABDELAZIZ, 2013).

Table 1 - Three dashpots system parameters.

Parameters	**Symbol**	**Value**
Mass (Kg)	m_1	0.8
Mass (Kg)	m_2	0.5
Mass (Kg)	m_3	1
Spring stiffness constant (N/m)	k_1	5
Spring stiffness constant (N/m)	k_2	5
Spring stiffness constant (N/m)	k_3	20
Damping coefficient ($N.s/m$)	b_1	2
Damping coefficient ($N.s/m$)	b_2	0.5
Damping coefficient ($N.s/m$)	b_3	2

Font: Borrowed from (ABDELAZIZ, 2013).

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -9.375 & -3.125 & 3.125 & .0625 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 1 & -45 & -5 & 40 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 20 & 2 & -20 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 1.25 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{88}$$

- Spring stiffness $k_2 = 5\ Kg$

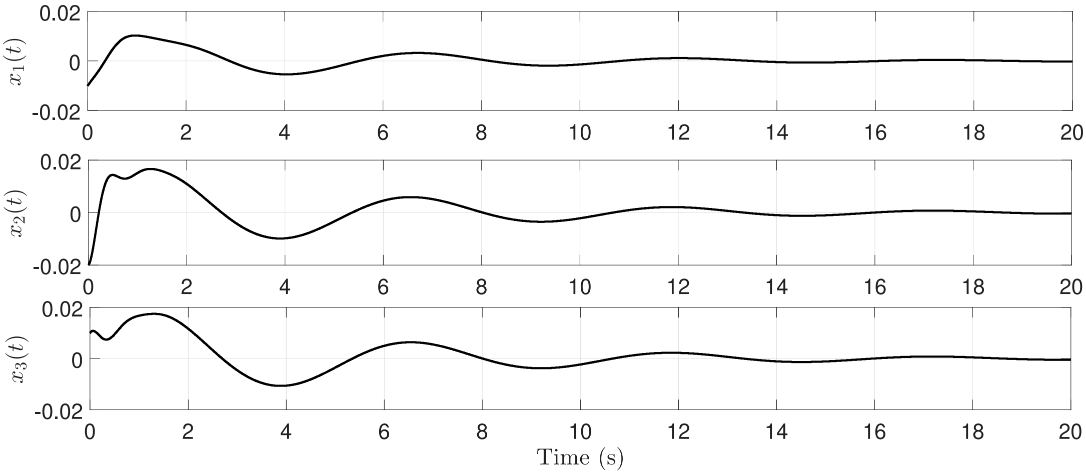
$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -12.5 & -3.125 & 6.25 & .0625 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 10 & 1 & -50 & -5 & 40 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 20 & 2 & -20 & -2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1.25 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{89}$$

For the initial states, the range are: $-0.05 \leq x_{01}, x_{02}, x_{03} \leq 0.05$ (m), and $-0.06 \leq$

$\dot{x}_{01}, \dot{x}_{02}, \dot{x}_{03} \leq 0.06\ (m/s)$. Therefore, the initial conditions polytope has sixty four vertices ($p = 64$).

First, a simulation without control is performed, considering $x_0 = [-0.01\ \ 0.03\ \ -0.02\ \ 0.02$
$0.01\ \ 0.03]^T$ as initial condition, and $k_2 = 5\ (N/m)$. The result can be seen in Figure 2.

Figure 2 - Behaviour of the system without control.



Font: Author's own result.

Note that the system is naturally stable, but has oscillations that may cause a system damage, and has high settling time. In order to reduce these oscillations/vibrations, the robust LQR-state derivative controller is used. Besides the robust LQR-state derivative controller, some results from (ASSUNÇÃO et al., 2007) are used to make a comparison between the techniques, Figure 3, and show the influence of the addition of LQR-state derivative on the system approach. Then, considering LMIs (51)-(53) from Theorem 1, the weighting matrices $\mathcal{R} = diag(1,1)$, and $\mathcal{Q} = diag(100,1,1,1,1,1)$, the gain K_{T1} (90) follows.

$$K_{T1} = YX^{-1} = \begin{bmatrix} 0.1998 & -2.4867\times10^{-5} & 0.1998 & 0.0002 & 0.1998 & 0.0002 \\ 0.0348 & -0.0071 & 0.1129 & -0.1517 & 0.1401 & -0.2191 \end{bmatrix}$$

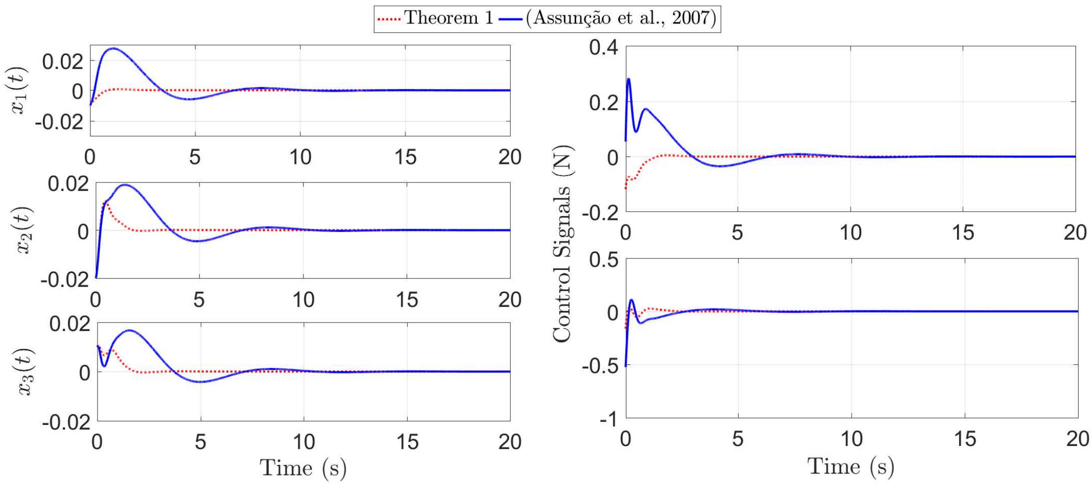
$$\times \begin{bmatrix} 87.3451 & 5.3462 & -39.9250 & -7.2587 & -5.9376 & -3.7058 \\ 5.3462 & 12.0227 & -2.9326 & -0.5808 & 0.0489 & -0.3702 \\ -39.9250 & -2.9326 & 209.1115 & 20.3493 & -170.4188 & -21.5121 \\ -7.2587 & -0.5808 & 20.3493 & 6.1402 & -12.7689 & -2.2672 \\ -5.9376 & 0.0489 & -170.4188 & -12.7689 & 177.0537 & 25.8682 \\ -3.7058 & -0.3702 & -21.5121 & -2.2672 & 25.8682 & 8.4920 \end{bmatrix}, \qquad (90)$$

$$K_{T1} = \begin{bmatrix} 8.2848 & 0.4914 & -0.2478 & 0.0649 & 0.1443 & 0.1316 \\ -0.4216 & -0.0545 & -0.0097 & -0.1751 & 1.6250 & -0.4482 \end{bmatrix},$$

Using Theorem 3 from (ASSUNÇÃO et al., 2007), the gain K_f (91) follows.

$$K_f = YX^{-1} = \begin{bmatrix} -0.9894 & 0.3026 & -1.9874 & 5.0201 & -1.6964 & 3.8640 \\ 0.9451 & -0.7950 & 1.2944 & -0.4737 & 1.6754 & -3.2743 \end{bmatrix}$$
$$\times \begin{bmatrix} 9.6581 & 2.2218 & -10.7791 & -0.4262 & 1.5826 & -0.0932 \\ 2.2218 & 0.7195 & -2.9614 & -0.1654 & 0.8090 & 0.0249 \\ -10.7791 & -2.9614 & 19.6221 & 1.3991 & -8.8412 & -0.5916 \\ -0.4262 & -0.1654 & 1.3991 & 0.4112 & -0.7940 & 0.0587 \\ 1.5826 & 0.8090 & -8.8412 & -0.7940 & 8.1162 & 0.9409 \\ -0.0932 & 0.0249 & -0.5916 & 0.0587 & 0.9409 & 0.4102 \end{bmatrix}, \quad (91)$$
$$K_f = \begin{bmatrix} 7.3542 & 1.7982 & -9.4912 & 1.2289 & 2.1311 & 1.5589 \\ -3.4323 & -0.9532 & 4.0271 & -0.1775 & 0.3018 & -0.6682 \end{bmatrix}.$$

Figure 3 - Comparison between controllers K_{T1} and K_f.



Font: Author's own result.

Several tests were performed, and choosing the weighting matrices properly it is possible to achieve similar or better system performance and lower control signal than the results presented in (ASSUNÇÃO et al., 2007). It is important to notice that if the state derivative vector is prioritised than the control vector suffers a penalty and vice versa. It is worth mentioning that the results considering a bound on the output peak and (or) a bound on the state derivative feedback gain, both from (ASSUNÇÃO et al., 2007), are used, and compared with the robust LQR-state derivative controller. It was noted that when a bound on the output peak is added, the response peak decreases and the control signal increases, but with robust LQR-state derivative controller it is possible to decrease the peak response with a low control signal, choosing the weighting matrices properly. The same occurs with the bound on the state derivative feedback gain, the control signal decreases, but the system's performance suffers a penalty.

Example 2. *Comparison between Theorems 1 and 2.*

Consider an uncertain system described as (47) that can be represented by the convex combination of the following vertices

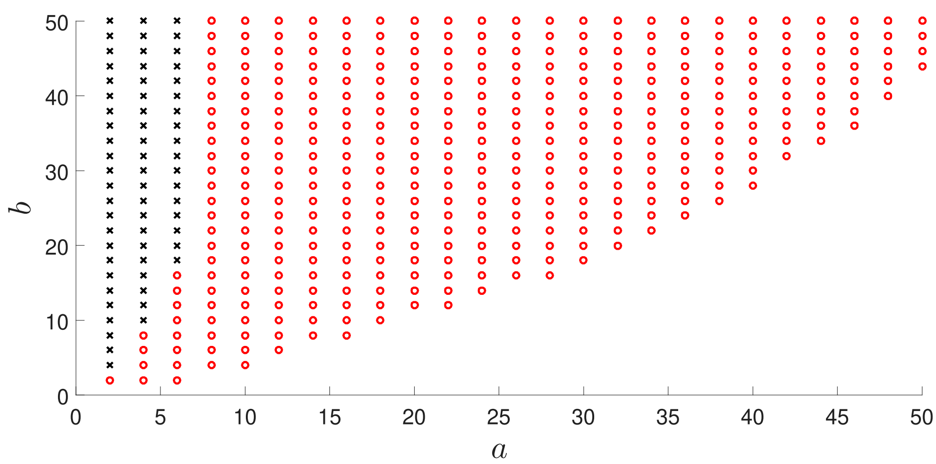
$$[A_1 \mid A_2] = \left[\begin{array}{cc|cc} -1 & -60 & a & -100 \\ 10 & 110 & 10 & 110 \end{array}\right], \; [B_1 \mid B_2] = \left[\begin{array}{c|c} 1 & b \\ 10 & 0 \end{array}\right]. \tag{92}$$

where $2 \leq a, b \leq 50$.

Remark 4. *For each value of a, and b, there is a different polytope composed by its respective vertices.*

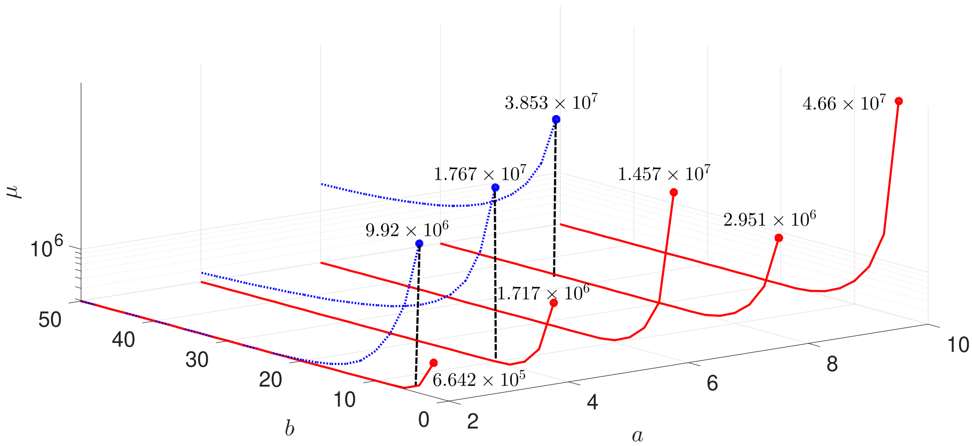
The feasibility region can be seen in Figure 4. In Figure 5, it is possible to see the variation of the guaranteed cost μ when the system's parameters are varied. The range of the initial states are: $-1 \leq x_{01}, x_{02} \leq 1$, which means that the polytope of initial conditions has four vertices $(p = 4)$. The weighting matrices chosen were $\mathcal{Q} = diag(1,\ 1)$ and $\mathcal{R} = 1$. The variation of parameter a is $2 \leq a \leq 10$, and for parameter b is $2 \leq b \leq 50$, for Figure 5. Note that the feasible region of Theorem 2 is higher than the feasible region of Theorem 1.

Figure 4 - Feasible regions obtained via Theorem 1 ($\times$), and Theorem 2 ($\times$,$\circ$).



Font: Author's own result.

Figure 5 - Guaranteed cost μ obtained via Theorem 1 (blue dotted line), and Theorem 2 (red solid line).



Font: Author's own result.

3.1.2 Partial Conclusions

This section presented sufficient conditions based on LMIs to obtain robust LQR-state derivative controllers for the stabilisation with optimised performance of uncertain linear time-invariant systems. A comparison was made to show the influence of the addition of LQR-state derivative on the system approach. Then, choosing the weighting matrices, $\mathcal{Q}$ and $\mathcal{R}$, properly is possible to achieve a good system performance with low control signals. With the results presented in (LIU; ZHANG, 2003; TEIXEIRA; ASSUNÇÃO; AVELLAR, 2003; CARNIATO et al., 2018) was possible to achieve less conservative conditions, which is proved by the comparison between Theorems 1 and 2. It is worth mentioning that the computational cost exponentially increases, considering the LMIs from Theorem 2. The next section addresses sufficient conditions for the design of robust controllers using Finsler's Lemma slack variables in order to achieve less conservative results and with low computational cost.

3.2 STABILITY CONDITION - FINSLER'S LEMMA

In the last years, Finsler's Lemma was used to achieve less conservative results (GUERRA; MANZO; LENDEK, 2015; MANZO; LENDEK; GUERRA, 2016; LLINS et al., 2017; YAN et al., 2018). With Finsler's Lemma, it is possible to avoid cross product of uncertain matrices, use slack variables, multiple Lyapunov matrices or reduce the number of LMIs (MOZELLI; PALHARES; MENDES, 2010; CHADLI; GUERRA, 2012).

Consider a candidate Lyapunov function, $\forall x(t) \neq 0$,

$$
\begin{aligned}
V(x(t)) &= x(t)^T P(\alpha) x(t) > 0, \quad P = P^T \in \mathbb{R}^{n\times n}, \\
\dot{V}(x(t)) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) < 0,
\end{aligned} \tag{93}
$$

and the following vectors and matrices:

$$
\mathscr{D}(\alpha) = \begin{bmatrix} A(\alpha)Y^T B(\alpha)^T + B(\alpha)YA(\alpha)^T + A(\alpha)X\mathcal{Q}XA(\alpha)^T + A(\alpha)Y^T\mathcal{R}YA(\alpha)^T & X \\ X & 0 \end{bmatrix},
$$

$$
\mathscr{X} = \begin{bmatrix} F \\ G \end{bmatrix}, \ \mathscr{B}(\alpha) = \begin{bmatrix} A(\alpha)^T & -I \end{bmatrix}, \ \mathscr{B}(\alpha)^{\perp} = \begin{bmatrix} I \\ A(\alpha)^T \end{bmatrix}. \tag{94}
$$

Therefore, Theorem 4 proposes sufficient LMIs conditions for the stabilisation of the system (49) through the LQR-state derivative controller.

Theorem 4. *Let A_i non-singular ($det(A_i) \neq 0$) and given $\mathcal{Q} \in \mathbb{R}^{n\times n}$, $\mathcal{R} \in \mathbb{R}^{m\times m}$ and $x_{0l} \in \mathbb{R}^{n\times 1}$, the uncertain system* (49) *is stable and has optimised performance if there exist symmetric matrix $X > 0 \in \mathbb{R}^{n\times n}$, and matrices $Y \in \mathbb{R}^{m\times n}$, $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times n}$, satisfying:*

$$
\min_{X = X^T > 0, Y, F, G} \ \mu
$$

Subject to

$$
\begin{bmatrix} \mu & x_{0l}^T \\ x_{0l} & X \end{bmatrix} > 0, \ \ l = 1, 2, ..., p, \tag{95}
$$

$$
\begin{bmatrix} FA_i^T + A_iF^T + A_iY^TB_i^T + B_iYA_i^T & X + A_iG^T - F & A_iX & A_iY^T \\ X + GA_i^T - F^T & -(G+G^T) & 0 & 0 \\ XA_i^T & 0 & -\mathcal{Q}^{-1} & 0 \\ YA_i^T & 0 & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0, \ \ i = 1, 2, ..., r, \tag{96}
$$

$$
\begin{bmatrix} \Phi_{ij} + \Phi_{ji} & * & * & * \\ 2X + GA_i^T + GA_j^T - 2F^T & -2(G+G^T) & * & * \\ XA_i^T + XA_j^T & 0 & -2\mathcal{Q}^{-1} & * \\ YA_i^T + YA_j^T & 0 & 0 & -2\mathcal{R}^{-1} \end{bmatrix} < 0, \ \ \begin{cases} i = 1, 2, ..., r-1, \\ j = i+1, ..., r, \end{cases} \tag{97}
$$

where $\Phi_{ij} + \Phi_{ji} = FA_i^T + A_iF^T + FA_j^T + A_jF^T + A_iY^TB_j^T + B_jYA_i^T + A_jY^TB_i^T + B_iYA_j^T$. The state derivative feedback gain can be given by

$$
K_{T4} = YX^{-1}. \tag{98}
$$

Proof. Supposing the set of LMIs feasible.

The proof of LMI (95) follows similar steps from proof of LMI (51).

For the LMI (96), multiplying by α_i^2, and summing all terms in i, with $i = 1,2,...,r$,

$$\begin{bmatrix} \Sigma_i^2(FA_i^T + A_iF^T + A_iY^TB_i^T + B_iYA_i^T) & \Sigma_i^2(X + A_iG^T - F) & \Sigma_i^2(A_iX & A_iY^T) \\ \Sigma_i^2(X + GA_i^T - F^T) & \Sigma_i^2(-(G+G^T)) & 0 & 0 \\ \Sigma_i^2(XA_i^T) & 0 & \Sigma_i^2(-\mathcal{Q}^{-1}) & 0 \\ \Sigma_i^2(YA_i^T) & 0 & 0 & \Sigma_i^2(-\mathcal{R}^{-1}) \end{bmatrix} < 0, \tag{99}$$

and multiplying (97) by $\alpha_i\alpha_j$, and summing all terms in i, with $i = 1,2,...,r-1$, and in j, with $j = i+1,...,r$,

$$\begin{bmatrix} \Sigma_{ij}(\Phi_{ij}+\Phi_{ji}) & * & * & * \\ \Sigma_{ij}(2X + GA_i^T + GA_j^T - 2F^T) & \Sigma_{ij}(-2(G+G^T)) & * & * \\ \Sigma_{ij}(XA_i^T + XA_j^T) & 0 & \Sigma_{ij}(-2\mathcal{Q}^{-1}) & * \\ \Sigma_{ij}(YA_i^T + YA_j^T) & 0 & 0 & \Sigma_{ij}(-2\mathcal{R}^{-1}) \end{bmatrix} < 0. \tag{100}$$

Applying Lemma 3, (101) follows.

$$\begin{bmatrix} \Sigma_i\Sigma_j(\Phi_{ij}) & \Sigma_i\Sigma_j(X + A_iG^T - F) & \Sigma_i\Sigma_j(A_iX) & \Sigma_i\Sigma_j(A_iY^T) \\ \Sigma_i\Sigma_j(X + GA_i^T - F^T) & \Sigma_i\Sigma_j(-2(G+G^T)) & 0 & 0 \\ \Sigma_i\Sigma_j(XA_i^T) & 0 & \Sigma_i\Sigma_j(-\mathcal{Q}^{-1}) & 0 \\ \Sigma_i\Sigma_j(YA_i^T) & 0 & 0 & \Sigma_i\Sigma_j(-\mathcal{R}^{-1}) \end{bmatrix} < 0, \tag{101}$$

where $\Phi_{ij} = A_iF^T + FA_i^T + B_jYA_i^T + A_iY^TB_j^T$, or,

$$\begin{bmatrix} \Phi(\alpha) & X + A(\alpha)G^T - F & A(\alpha)X & A(\alpha)Y^T \\ X + GA(\alpha)^T - F^T & -(G+G^T) & 0 & 0 \\ XA(\alpha)^T & 0 & -\mathcal{Q}^{-1} & 0 \\ YA(\alpha)^T & 0 & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0, \tag{102}$$

where $\Phi(\alpha) = A(\alpha)F^T + FA(\alpha)^T + B(\alpha)YA(\alpha)^T + A(\alpha)Y^TB(\alpha)^T$.

Applying Lemma 1 recursively in (102)

$$\begin{bmatrix} \Phi(\alpha) + A(\alpha)X\mathcal{Q}XA(\alpha)^T + A(\alpha)Y^T\mathcal{R}YA(\alpha)^T & X + A(\alpha)G^T - F \\ X + GA(\alpha)^T - F^T & -(G+G^T) \end{bmatrix} < 0, \tag{103}$$

Reorganising (103), (104) follows.

$$\begin{bmatrix} A(\alpha)Y^T B(\alpha)^T + B(\alpha)YA(\alpha)^T + A(\alpha)X\mathcal{Q}XA(\alpha)^T + A(\alpha)Y^T\mathcal{R}YA(\alpha)^T & X \\ X & 0 \end{bmatrix} + \\ + \begin{bmatrix} FA(\alpha)^T & -F \\ GA(\alpha)^T & -G \end{bmatrix} + \begin{bmatrix} A(\alpha)F^T & A(\alpha)G^T \\ -F^T & -G^T \end{bmatrix} < 0, \quad (104)$$

or,

$$\begin{bmatrix} A(\alpha)Y^T B(\alpha)^T + B(\alpha)YA(\alpha)^T + A(\alpha)X\mathcal{Q}XA(\alpha)^T + A(\alpha)Y^T\mathcal{R}YA(\alpha)^T & X \\ X & 0 \end{bmatrix} + \\ + \begin{bmatrix} F \\ G \end{bmatrix} \begin{bmatrix} A(\alpha)^T & -I \end{bmatrix} + \begin{bmatrix} A(\alpha) \\ -I \end{bmatrix} \begin{bmatrix} F^T & G^T \end{bmatrix} < 0, \quad (105)$$

Considering (94) and Lemma 2, $\mathscr{B}^{\perp^T}\mathscr{D}\mathscr{B}^{\perp} < 0$ is equivalent to $\mathscr{D}(\alpha) + \mathscr{X}\mathscr{B}(\alpha) + \mathscr{B}(\alpha)^T\mathscr{X}^T < 0$, due to $\mathscr{B}(\alpha)\mathscr{B}^{\perp} = 0$. Then, if the set of LMIs of Theorem 4 is feasible, exists a matrix $P = P^T > 0$ that satisfies the Lyapunov conditions (93) for the system (49). Thus, the system (49) is asymptotic stable considering the gain (98).

□

Theorem 5. *If the conditions given in Theorem 1 hold, then the conditions given in Theorem 4 also hold.*

Proof. Consider the following definitions

$$\Xi_{ii} = \begin{bmatrix} \xi_{ii} & A_iX & A_iY^T \\ XA_i^T & -\mathcal{Q}^{-1} & 0 \\ YA_i^T & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0, \quad (106)$$

$$\Xi_{ij} + \Xi_{ji} = \begin{bmatrix} \xi_{ij} + \xi_{ji} & A_iX + A_jX & A_iY^T + A_jT^T \\ XA_i^T + XA_j^T & -2\mathcal{Q}^{-1} & 0 \\ YA_i^T + YA_j^T & 0 & -2\mathcal{R}^{-1} \end{bmatrix} < 0, \quad (107)$$

where $\xi_{ii} = A_iX + XA_i^T + B_iYA_i^T + A_iY^TB_i^T$ and $\xi_{ij} + \xi_{ji} = A_iX + XA_i^T + A_jX + XA_j^T + B_jYA_i^T + A_iY^TB_j^T + B_iYA_j^T + A_jY^TB_i^T$, and

$$\Theta_{ii} = \begin{bmatrix} \theta_{ii} & X + A_iG^T - F & A_iX & A_iY^T \\ X + GA_i^T - F^T & -(G + G^T) & 0 & 0 \\ XA_i^T & 0 & -\mathcal{Q}^{-1} & 0 \\ YA_i^T & 0 & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0, \quad (108)$$

$$\Theta_{ij}+\Theta_{ji}=\begin{bmatrix}\theta_{ij}+\theta_{ji} & * & * & * \\ 2X+GA_i^T+GA_j^T-2F^T & -2(G+G^T) & * & * \\ XA_i^T+XA_j^T & 0 & -2\mathcal{Q}^{-1} & * \\ YA_i^T+YA_j^T & 0 & 0 & -2\mathcal{R}^{-1}\end{bmatrix}<0. \tag{109}$$

where $\theta_{ii}=FA_i^T+A_iF^T+A_iY^TB_i^T+B_iYA_i^T$ and $\theta_{ij}+\theta_{ji}=FA_i^T+A_iF^T+FA_j^T+A_jF^T+A_iY^TB_j^T+B_jYA_i^T+A_jY^TB_i^T+B_iYA_j^T$.

Premultiplying (108) and (109) by Λ, and posmultiplying by Λ^T, (110) and (111) follow.

$$\Lambda\Theta_{ii}\Lambda^T=\begin{bmatrix}\theta_{ii} & A_iX & A_iY^T & X+A_iG^T-F \\ XA_i^T & -\mathcal{Q}^{-1} & 0 & 0 \\ YA_i^T & 0 & -\mathcal{R}^{-1} & 0 \\ X+GA_i^T-F^T & 0 & 0 & -(G+G^T)\end{bmatrix}<0, \tag{110}$$

$$\Lambda(\Theta_{ij}+\Theta_{ji})\Lambda^T=\begin{bmatrix}\theta_{ij}+\theta_{ji} & * & * & * \\ XA_i^T+XA_j^T & -2\mathcal{Q}^{-1} & * & * \\ YA_i^T+YA_j^T & 0 & -2\mathcal{R}^{-1} & * \\ 2X+GA_i^T+GA_j^T-2F^T & 0 & 0 & -2(G+G^T)\end{bmatrix}<0, \tag{111}$$

with

$$\Lambda=\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0\end{bmatrix}. \tag{112}$$

Assuming $F=F^T=X$, $G=\dfrac{1}{\varepsilon}S$ and $G^T=\dfrac{1}{\varepsilon}S^T$, with $\varepsilon>0$, and applying Lemma 1 in (110) and (111),

$$T\Theta_{ii}T^T=\begin{bmatrix}\xi_{ii}+\dfrac{1}{\varepsilon}\overbrace{A_iS^T(S+S^T)^{-1}SA_i^T}^{M_{ii}} & A_iX & A_iY^T \\ XA_i^T & -\mathcal{Q}^{-1} & 0 \\ YA_i^T & 0 & -\mathcal{R}^{-1}\end{bmatrix}<0. \tag{113}$$

$$T(\Theta_{ij}+\Theta_{ji})T^T=\begin{bmatrix}\xi_{ij}+\xi_{ji}+\dfrac{1}{\varepsilon}\overbrace{(A_iS^T+A_jS^T)\frac{1}{2}(S+S^T)^{-1}(SA_i^T+SA_j^T)}^{M_{ij}+M_{ji}} & * & * \\ XA_i^T+XA_j^T & -2\mathcal{Q}^{-1} & * \\ YA_i^T+YA_j^T & 0 & -2\mathcal{R}^{-1}\end{bmatrix}<0, \tag{114}$$

It is important to highlight that when it is assumed $F=F^T=X$, than θ_{ii} is equivalent to ξ_{ii}, and $\theta_{ij}+\theta_{ji}$ is equivalent to $\xi_{ij}+\xi_{ji}$. Moreover, note from (108) and (109) that $G>0$ ($S>0$) and, consequently, $M_{ii}>0$ and $M_{ij}+M_{ji}>0$. Thus,

$$\begin{aligned} \xi_{ii}+\frac{1}{\varepsilon}M_{ii}&<0,\\ \xi_{ii}&<-\frac{1}{\varepsilon}M_{ii}, \end{aligned} \tag{115}$$

$$\begin{aligned} \xi_{ij}+\xi_{ji}+\frac{1}{\varepsilon}(M_{ij}+M_{ji})&<0,\\ \xi_{ij}+\xi_{ji}&<-\frac{1}{\varepsilon}(M_{ij}+M_{ji}). \end{aligned} \tag{116}$$

With $\varepsilon>0$, ξ_{ii} and $\xi_{ij}+\xi_{ji}$ are negative definite. Then, the conditions of Theorem 4 are equivalents to the conditions of Theorem 1. Thereby, if the conditions (52) and (53) hold, then conditions (96) and (97) also hold.

□

Note that just a set of slack variables is used for uncertain hole domain, i.e., a single set of slack variables for hole vertices of the polytope. Next theorem proposes sufficient conditions considering a set of slack variables for each vertex of the polytope.

Theorem 6. *Let A_i non-singular ($det(A_i)\neq 0$) and given $\mathcal{Q}\in\mathbb{R}^{n\times n}$, $\mathcal{R}\in\mathbb{R}^{m\times m}$ and $x_{0l}\in\mathbb{R}^{n\times 1}$, the uncertain system* (49) *is stable and has optimised performance if there exist symmetric matrix $X>0\in\mathbb{R}^{n\times n}$, and matrices $Y\in\mathbb{R}^{m\times n}$, $F_i\in\mathbb{R}^{n\times n}$, $G_i\in\mathbb{R}^{n\times n}$, satisfying:*

$$\min_{X=X^T>0,Y,F_i,G_i} \mu$$

Subject to

$$\begin{bmatrix} \mu & x_{0l}^T \\ x_{0l} & X \end{bmatrix} > 0,\ \ l=1,2,...,p, \tag{117}$$

$$\begin{bmatrix} F_iA_i^T+A_iF_i^T+A_iY^TB_i^T+B_iYA_i^T & X+A_iG_i^T-F_i & A_iX & A_iY^T \\ X+G_iA_i^T-F_i^T & -(G_i+G_i^T) & 0 & 0 \\ XA_i^T & 0 & -\mathcal{Q}^{-1} & 0 \\ YA_i^T & 0 & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0,\ \ i=1,2,...,r, \tag{118}$$

$$\begin{bmatrix} \Psi_{ij}+\Psi_{ji} & * & * & * \\ \Theta_{ij}+\Theta_{ji} & -(G_i+G_j+G_i^T+G_j^T) & * & * \\ XA_i^T+XA_j^T & 0 & -2\mathcal{Q}^{-1} & * \\ YA_i^T+YA_j^T & 0 & 0 & -2\mathcal{R}^{-1} \end{bmatrix} < 0,\ \ \begin{cases} i=1,2,...,r-1, \\ j=i+1,...,r, \end{cases} \tag{119}$$

where $\Psi_{ij}+\Psi_{ji}=F_jA_i^T+A_iF_j^T+F_iA_j^T+A_jF_i^T+A_iY^TB_j^T+B_jYA_i^T+A_jY^TB_i^T+B_iYA_j^T$, and

$\Theta_{ij} + \Theta_{ji} = 2X + G_j A_i^T + G_i A_j^T - F_i^T - F_j^T$. *The state derivative feedback gain can be given by*

$$K_{T6} = YX^{-1}. \tag{120}$$

Proof. The proof of Theorem 6 follows similar steps from proof of Theorem 4, considering

$$\mathscr{X} = \begin{bmatrix} F(\alpha) \\ G(\alpha) \end{bmatrix}. \tag{121}$$

□

Theorem 7. *If the conditions given in Theorems 1 hold, then the conditions given in Theorem 6 also hold.*

Proof. The proof follows similar steps from proof of Theorem 5, assuming $F_i = F_i^T = X$, $G_i = \varepsilon S_i$ and $G_i^T = \varepsilon S_i^T$. □

An example to verify the efficiency of the proposed technique are presented in the next section.

3.2.1 Illustrative Example

To show how the proposed techniques are effective, an example is presented.

Example 3. *Comparison between Theorems 1, 4, and 6.*

Consider an uncertain system described as (47) that can be represented by the convex combination of the following vertices

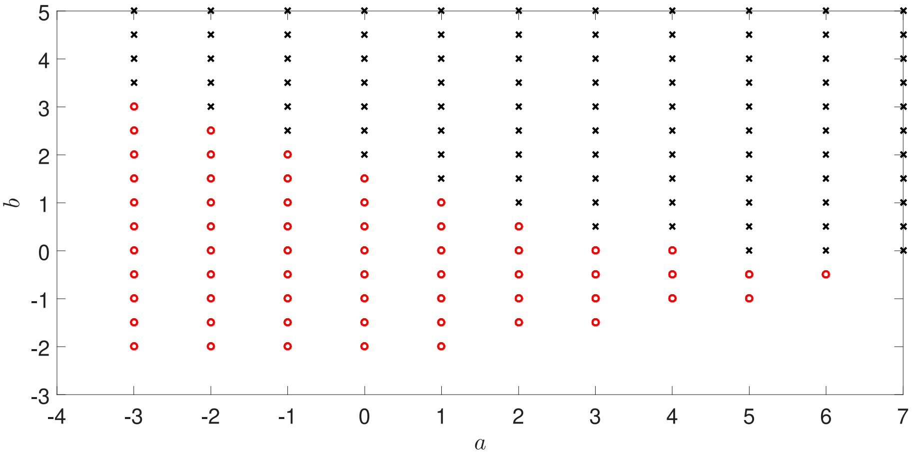
$$[A_1 \mid A_2] = \left[\begin{array}{cc|cc} 20 & -30 & a & -10 \\ 10 & 10 & 10 & 30 \end{array}\right], \quad [B_1 \mid B_2] = \left[\begin{array}{c|c} 10 & b \\ 10 & 10 \end{array}\right]. \tag{122}$$

where $-3 \leq a \leq 7$ and $-2 \leq b \leq 5$.

Remark 5. *For each value of a, and b, there is a different polytope composed by its respective vertices.*

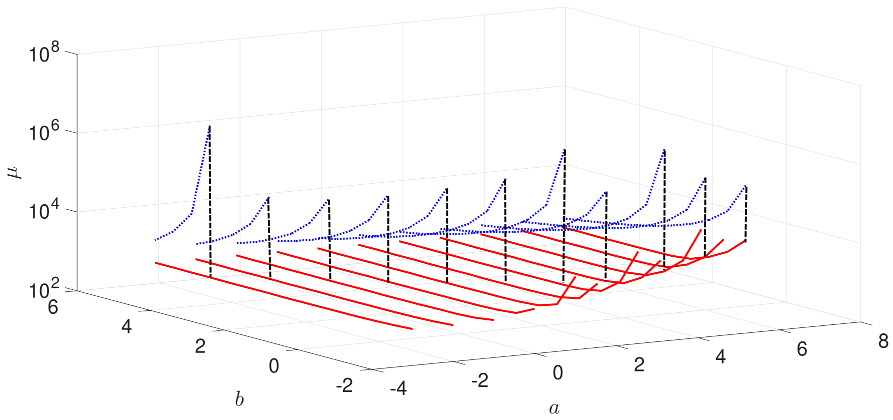
The feasibility region can be seen in Figure 6. In Figure 7, it is possible to see the variation of the guaranteed cost μ when the system's parameters are varied. The range of the initial states are: $-1 \leq x_{01}, x_{02} \leq 1$, which means that the polytope of initial conditions has four vertices $(p = 4)$. The weighting matrices chosen were $\mathcal{Q} = diag(1,\ 1)$ and $\mathcal{R} = 1$. Note that the feasible

Figure 6 - Feasible regions obtained via Theorem 1 ($\times$), Theorem 4 ($\times$), and Theorem 6 ($\times$,$\circ$).



Font: Author's own result.

Figure 7 - Guaranteed cost μ obtained via Theorem 1 (blue dotted line), and Theorem 6 (red solid line).

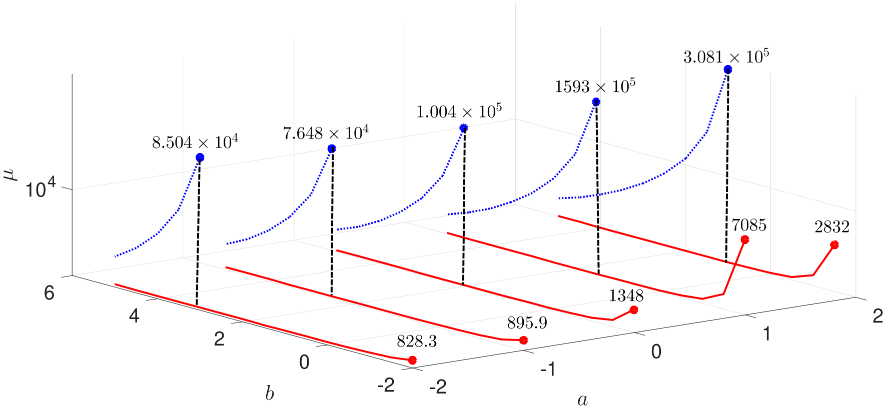


Font: Author's own result.

region of Theorem 6 is higher than the feasible region of Theorems 1 and 4. Also, note that the Theorems 1 and 4 had the same feasible region.

For a better view of the guaranteed cost behaviour, an interval of system's parameters is chosen (Figure 8). From it, it is possible to see that the conditions of Theorem 6 are more relaxed than the Theorem 1.

Figure 8 - Guaranteed cost μ obtained via Theorem 1 (blue dotted line), and Theorem 6 (red solid line).



Font: Author's own result.

3.2.2 Partial Conclusions

This section presented sufficient conditions based on LMIs to obtain robust LQR-state derivative controllers for the stabilisation with optimised performance of uncertain linear time-invariant systems. In order to achieve less conservative results, Finsler's Lemma with slack variables is used. Then, feasible tests were performed to show the efficiency of the proposed techniques. Through the example, it is possible to achieve less conservative results using a set of Finsler's Lemma slack variables for each vertex of the polytope.

4 GAIN SCHEDULING LQR-STATE DERIVATIVE CONTROLLER

Consider a controllable and linear system with a measurable and time-varying parameter, $\theta(t)$, described as a convex combination of the polytope vertices:

$$\dot{x}(t) = \sum_{i=1}^{r} \theta_i(t)(A_i x(t) + B_i u(t)) = A(\theta(t))x(t) + B(\theta(t))u(t),\ \theta(t) \in \mathfrak{F}, \tag{123}$$

where r represents the number of polytope vertices, and $\mathfrak{F}$ is the unitary simplex given by

$$\mathfrak{F} = \left\{ \sum_{i=1}^{r} \theta_i(t) = 1,\ \ \theta_i(t) \geq 0,\ \ i = 1, ..., r, \right\}. \tag{124}$$

The objective is to find a gain $K(\theta(t)) \in \mathbb{R}^{m\times n}$ such that the closed-loop system is asymptotically stable when feedback by the control input

$$u(t) = -\sum_{i=1}^{r} \theta_i(t) K_i \dot{x}(t) = -K(\theta(t))\dot{x}(t). \tag{125}$$

Then, the closed-loop system is given by (126).

$$\dot{x}(t) = \sum_{i=1}^{r} \theta_i(t)((I + B_i K_i)^{-1} A_i)x(t) = (I + B(\theta(t))K(\theta(t)))^{-1} A(\theta(t))x(t). \tag{126}$$

Remark 6. *In state derivative feedback it is assumed that the matrix $A(\theta(t))$ is non-singular ($det(A(\theta(t)) \neq 0$), and when the problem is feasible the gain $K(\theta(t))$ becomes the $(I + B(\theta(t))K(\theta(t))$ matrix invertible.*

Theorem 8 proposes sufficient LMIs conditions for the stabilisation of the system (123) through the gain scheduling LQR-state derivative controller.

Theorem 8. *Let A_i non-singular ($det(A_i) \neq 0$) and given $\mathcal{Q} \in \mathbb{R}^{n\times n}$, $\mathcal{R} \in \mathbb{R}^{m\times m}$ and $x_{0l} \in \mathbb{R}^{n\times 1}$, the system* (126) *is stable and has optimised performance if there exist symmetric matrix $X > 0 \in \mathbb{R}^{n\times n}$ and matrices $Y_i \in \mathbb{R}^{m\times n}$, satisfying:*

$$\min_{X = X^T > 0, Y} \ \mu$$

Subject to

$$\begin{bmatrix} \mu & x_{0l}^T \\ x_{0l} & X \end{bmatrix} > 0, \;\; l = 1,2,...,p, \tag{127}$$

$$\begin{bmatrix} \Psi_{iii} & A_iX & A_iY_i^T \\ XA_i^T & -\mathcal{Q}^{-1} & 0 \\ Y_iA_i^T & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0, \;\; i = 1,2,...,r, \tag{128}$$

$$\begin{bmatrix} \Psi_{iij}+\Psi_{iji}+\Psi_{jii} & (2A_i+A_j)X & A_iY_i^T+A_iY_j^T+A_jY_i^T \\ X(2A_i^T+A_j^T) & -3\mathcal{Q}^{-1} & 0 \\ Y_iA_i^T+Y_iA_j^T+Y_jA_i^T & 0 & -3\mathcal{R}^{-1} \end{bmatrix} < 0, \;\; \begin{cases} i = 1,2,...,r, \\ j = 1,2,...,r, \\ i \neq j, \end{cases} \tag{129}$$

$$\begin{bmatrix} \Psi_{ijk}+\Psi_{ikj}+\Psi_{jik}+\Psi_{jki}+\Psi_{kij}+\Psi_{kji} & * & * \\ X(2A_i^T+2A_j^T+2A_k^T) & -6\mathcal{Q}^{-1} & * \\ Y_iA_j^T+Y_iA_k^T+Y_jA_i^T+Y_jA_k^T+Y_kA_i^T+Y_kA_j^T & 0 & -6\mathcal{R}^{-1} \end{bmatrix} < 0, \;\; \begin{cases} i = 1,2,...,r-2, \\ j = i+1,...,r-1, \\ k = j+1,...,r, \end{cases} \tag{130}$$

where
$\Psi_{iii} = A_iX + XA_i^T + B_iY_iA_i^T + A_iY_i^TB_i^T$,
$\Psi_{iij} + \Psi_{iji} + \Psi_{jii} = X(2A_i^T + A_j^T) + (2A_i + A_j)X + B_iY_jA_i^T + B_jY_iA_i^T + B_iY_iA_j^T + A_iY_j^TB_i^T + A_jY_i^TB_i^T + A_iY_i^TB_j^T$,
$\Psi_{ijk}+\Psi_{ikj}+\Psi_{jik}+\Psi_{jki}+\Psi_{kij}+\Psi_{kji} = X(2A_i^T+2A_j^T+2A_k^T)+(2A_i+2A_j+2A_k)X+B_jY_kA_i^T + B_kY_jA_i^T + B_iY_kA_j^T + B_kY_iA_j^T + B_iY_jA_k^T + B_jY_iA_k^T + A_jY_k^TB_i^T + A_kY_j^TB_i^T + A_iY_k^TB_j^T + A_kY_i^TB_j^T + A_iY_j^TB_k^T + A_jY_i^TB_k^T$.

The state derivative feedback gain can be given by

$$K_{i_{T8}} = Y_iX^{-1}, \;\; i = 1,2,...,r. \tag{131}$$

Proof. Supposing the set of LMIs feasible.

The proof of LMI (127) follow similar steps from proof of LMI (51).

For the LMI (128), multiplying by α_i^3, and summing all terms in i, with $i = 1,2,...,r$,

$$\begin{bmatrix} \Sigma_i^3(\Psi_{iii}) & \Sigma_i^3(A_iX) & \Sigma_i^3(A_iY_i^T) \\ \Sigma_i^3(XA_i^T) & \Sigma_i^3(-\mathcal{Q}^{-1}) & 0 \\ \Sigma_i^3(Y_iA_i^T) & 0 & \Sigma_i^3(-\mathcal{R}^{-1}) \end{bmatrix} < 0, \tag{132}$$

multiplying (129) by $\alpha_i^2\alpha_j$ and summing all terms in $i, \; j$ with $i,j = 1,2,...,r, \; i \neq j$,

$$\begin{bmatrix} \Sigma_{ij}^{i\neq j}(\Psi_{iij}+\Psi_{iji}+\Psi_{jii}) & \Sigma_{ij}^{i\neq j}((2A_i+A_j)X) & \Sigma_{ij}^{i\neq j}(A_iY_i^T+A_iY_j^T+A_jY_i^T) \\ \Sigma_{ij}^{i\neq j}(X(2A_i^T+A_j^T)) & \Sigma_{ij}^{i\neq j}(-3\mathcal{Q}^{-1}) & 0 \\ \Sigma_{ij}^{i\neq j}(Y_iA_i^T+Y_iA_j^T+Y_jA_i^T) & 0 & \Sigma_{ij}^{i\neq j}(-3\mathcal{R}^{-1}) \end{bmatrix} < 0. \tag{133}$$

and multiplying (24) by $\alpha_i\alpha_j\alpha_k$, and summing all terms in i, with $i = 1,2,...,r-2$, j with $j = i+1,...,r-1$, and k, with $k = j+1,...,r$,

$$\begin{bmatrix} \Sigma_{ijk}(\Psi_{ijk}+\Psi_{ikj}+\Psi_{jik}+\Psi_{jki}+\Psi_{kij}+\Psi_{kji}) & * & * \\ \Sigma_{ijk}(X(2A_i^T+2A_j^T+2A_k^T)) & \Sigma_{ijk}(-6\mathcal{Q}^{-1}) & * \\ \Sigma_{ijk}(Y_iA_j^T+Y_iA_k^T+Y_jA_i^T+Y_jA_k^T+Y_kA_i^T+Y_kA_j^T) & 0 & \Sigma_{ijk}(-6R^{-1}) \end{bmatrix} < 0, \tag{134}$$

Applying Lemma 5, (135) follows.

$$\begin{bmatrix} \Sigma_i\Sigma_j(\Upsilon_{ijk}) & \Sigma_i\Sigma_j(A_iX) & \Sigma_i\Sigma_j(A_iY^T) \\ \Sigma_i\Sigma_j(XA_i^T) & \Sigma_i\Sigma_j(-\mathcal{Q}^{-1}) & 0 \\ \Sigma_i\Sigma_j(YA_i^T) & 0 & \Sigma_i\Sigma_j(-\mathcal{R}^{-1}) \end{bmatrix} < 0, \tag{135}$$

where $\Upsilon_{ijk} = A_iX + XA_i^T + B_jY_kA_i^T + A_iY_k^TB_j^T$, or,

$$\begin{bmatrix} A(\theta)X+XA(\theta)^T+B(\theta)Y(\theta)A(\theta)^T+A(\theta)Y(\theta)^TB(\theta)^T & A(\theta)X & A(\theta)Y^T \\ XA(\theta)^T & -\mathcal{Q}^{-1} & 0 \\ Y(\theta)A(\theta)^T & 0 & -\mathcal{R}^{-1} \end{bmatrix} < 0. \tag{136}$$

Applying Lemma 1 recursively in (136), replacing the variable $Y(\theta) = K_{T8}(\theta)X$ and organising,

$$\begin{gathered} A(\theta)(X+B(\theta)K_{T8}(\theta)X)^T+(X+B(\theta)K_{T8}(\theta)X)A(\theta)^T+A(\theta)XK_{T8}(\theta)^T\mathcal{R}K_{T8}(\theta)XA(\theta)^T+ \\ +A(\theta)X\mathcal{Q}XA(\theta)^T<0, \\ \Updownarrow \\ A(\theta)X(I+B(\theta)K_{T8}(\theta))^T+(I+B(\theta)K_{T8}(\theta))XA(\theta)^T+A(\theta)X(K_{T8}(\theta)^T\mathcal{R}K_{T8}(\theta)+\mathcal{Q})XA(\theta)^T<0. \end{gathered} \tag{137}$$

Now, applying Propriety 1 in (137) it is concluded that matrices $(I+B(\theta)K_{T8}(\theta))$, $A(\theta)$ and X are invertible. Then, premultiplying by $X^{-1}A(\theta)^{-1}$ and posmultiplying by $A(\theta)^{-T}X^{-1}$, with $X^{-1} = P$:

$$(I+B(\theta)K_{T8}(\theta))^TA(\theta)^{-T}P+PA(\theta)^{-1}(I+B(\theta)K_{T8}(\theta))+K_{T8}(\theta)^T\mathcal{R}K_{T8}(\theta)+\mathcal{Q}<0. \tag{138}$$

Premultiplying by $A(\theta)^T(I + B(\theta)K_{T8}(\theta))^{-T}$, posmultiplying by $(I + B(\theta)K_{T8}(\theta))^{-1}A(\theta)$, (139) follows.

$$A(\theta)^T(I+B(\theta)K_{T8}(\theta))^{-T}P+P(I+B(\theta)K_{T8}(\theta))^{-1}A(\theta)+$$

$$+A(\theta)^T(I+B(\theta)K_{T8}(\theta))^{-T}(K_{T8}(\theta)^T\mathcal{R}K_{T8}(\theta)+\mathcal{Q})(I+B(\theta)K_{T8}(\theta))^{-1}A(\theta)<0. \quad (139)$$

Premultiplying by $x(t)^T$, posmultiplying by $x(t)$ and replacing $A_{cl}(\theta)=(I+B(\theta)K_{T8}(\theta))^{-1}A(\theta)$:

$$x(t)^TA_{cl}(\theta)^TPx(t)+x(t)^TPA_{cl}(\theta)x(t)+x(t)^TA_{cl}(\theta)^T(K(\theta)^T_{T8}\mathcal{R}K(\theta)_{T8}+\mathcal{Q})A_{cl}(\theta)x(t)<0. \quad (140)$$

Replacing $\dot{x}(t)=A_{cl}(\theta)x(t)=(I+B(\theta)K_{T8}(\theta))^{-1}A(\theta)x(t)$:

$$\dot{x}(t)^TPx(t)+x(t)^TP\dot{x}(t)<-\dot{x}(t)^T(K_{T8}(\theta)^T\mathcal{R}K_{T8}(\theta)+\mathcal{Q})\dot{x}(t). \quad (141)$$

As $\dot{x}(t)^T(K_{T8}(\theta)^T\mathcal{R}K_{T8}(\theta)+\mathcal{Q})\dot{x}(t)$ is positive, than,

$$\dot{V}(x(t))=\dot{x}(t)^TPx(t)+x(t)^TP\dot{x}(t)<0 \quad (142)$$

The proof of Theorem 8 is completed. □

Example 4. *Vibration isolation example*

Consider the vibration isolation system, Figure 9, borrowed from (ABDELAZIZ, 2010). The system can be described as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \ddot{x}_1(t)\\ \ddot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k_1c_1 & -k_2c_2 & -b_1c_1 & -b_2c_2\\ -k_1c_2 & -k_2c_1 & -b_1c_2 & -b_2c_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ \dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\\ c_1 & c_2\\ c_2 & c_1\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}, \quad (143)$$

where $c_1=1/m_v+L^2/I_n$, $c_2=1/m_v-L^2/I_n$, m_v and I_n represent the mass and the inertia of the mass, k_1 and k_2 are the spring constants, b_1 and b_2 are the damper constants, $x_1(t)$ and $x_2(t)$ are the mass displacement from both sides, $2L$ is the distance between two supporting points, and $u_1(t)$ and $u_2(t)$ are the control input. The objective of the control law $u(t)$ is to suppress the vibration. In Table 2 is showed the value of system parameters.

For this system, and the simulation, it is considered that the power of each actuator varies with time according to two sinusoidal signals

$$\text{Power in actuator 1 } (at_1)\rightarrow 0.75+0.25sin(2\pi t+\pi/2), \quad (144)$$

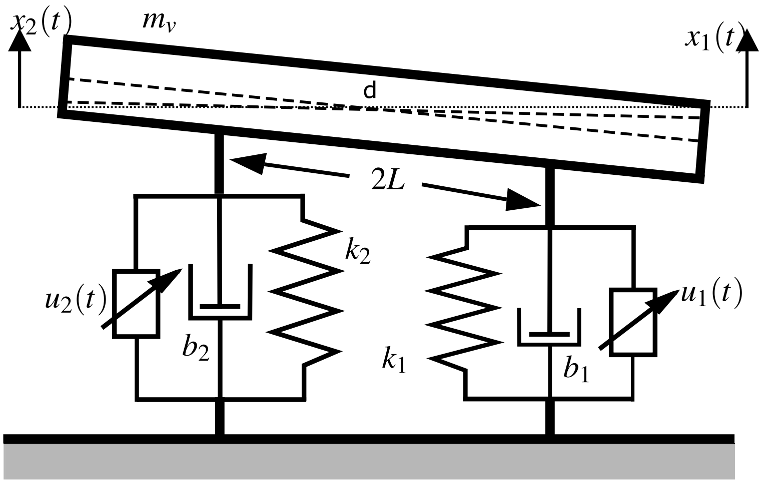
$$\text{Power in actuator 2 } (at_2)\rightarrow 0.85+0.15sin(2\pi 0.5t+\pi/2). \quad (145)$$

Table 2 - Vibration isolation parameters.

Parameters	**Symbol**	**Value**
Mass (Kg)	m_v	10
Inertia of the mass ($Kg\ m^2$)	I_n	1
Spring stiffness constant (N/m)	k_1	500
Spring stiffness constant (N/m)	k_2	600
Damping coefficient ($N.s/m$)	b_1	10
Damping coefficient ($N.s/m$)	b_2	15

Font: Borrowed from (ABDELAZIZ, 2010).

Figure 9 - Vibration isolation example.



Font: Adapted from (ABDELAZIZ, 2010).

In this way, the power of actuator at_1 varies between 50% and 100% of power, and the power of actuator at_2 varies between 70% and 100% of power. Then, the system can be represented by a polytope, which vertices are represented as follow.

- Vertex 1 - $at_1 \rightarrow 50\%$ and $at_2 \rightarrow 70\%$

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -550 & 540 & -11 & 13.5 \\ 450 & -660 & 9 & -16.5 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.55 & -0.63 \\ -0.45 & 0.77 \end{bmatrix}. \tag{146}$$

- Vertex 2 - $at_1 \rightarrow 50\%$ and $at_2 \rightarrow 100\%$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -550 & 540 & -11 & 13.5 \\ 450 & -660 & 9 & -16.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.55 & -0.9 \\ -0.45 & 1.1 \end{bmatrix}. \tag{147}$$

- Vertex 3 - $at_1 \rightarrow 100\%$ and $at_2 \rightarrow 70\%$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -550 & 540 & -11 & 13.5 \\ 450 & -660 & 9 & -16.5 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1.1 & -0.63 \\ -0.9 & 0.77 \end{bmatrix}. \tag{148}$$

- Vertex 4 - $at_1 \rightarrow 100\%$ and $at_2 \rightarrow 100\%$

$$A_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -550 & 540 & -11 & 13.5 \\ 450 & -660 & 9 & -16.5 \end{bmatrix}, B_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1.1 & -0.9 \\ -0.9 & 1.1 \end{bmatrix}. \tag{149}$$

For the initial states, the range are: $-0.06 \leq x_{01}, x_{02} \leq 0.06$ (m), and $-2 \leq \dot{x}_{01}, \dot{x}_{02} \leq 2$ (m/s). Therefore, the initial conditions polytope has sixteen vertices ($p = 16$).

The parametric components are $\sigma_1(t)$ and $\sigma_2(t)$ for at_1, and $\xi_1(t)$ and $\xi_2(t)$ for at_2.

$$\sigma_1(t) = 0.5 + 0.5sin(2\pi t + \pi/2), \tag{150}$$

$$\sigma_2(t) = 1 - \sigma_1(t), \tag{151}$$

$$\xi_1(t) = 0.5 + 0.5sin(2\pi 0.5t + \pi/2), \tag{152}$$

$$\xi_2(t) = 1 - \xi_1(t). \tag{153}$$

Considering the convex combinations,

$$at_1(\sigma(t)) = \sigma_1(t)at_{1min} + (1 - \sigma_1(t))at_{1max} = \sigma_1(t)at_{1min} + \sigma_2(t)at_{1max}, \tag{154}$$

$$at_2(\sigma(t)) = \xi_1(t)at_{2min} + (1 - \xi_1(t))at_{2max} = \xi_1(t)at_{2min} + \xi_2(t)at_{2max}, \tag{155}$$

we can rewrite as,

$$at_1(\sigma(t),\xi(t)) = (\xi_1(t)+\xi_2(t))(\sigma_1(t)at_{1min}+\sigma_2(t)at_{1max}),$$
$$at_1(\sigma(t),\xi(t)) = \xi_1(t)\sigma_1(t)at_{1min}+\xi_1(t)\sigma_2(t)at_{1max}+\xi_2(t)\sigma_1(t)at_{1min}+\xi_2(t)\sigma_2(t)at_{1max}, \tag{156}$$
$$at_2(\sigma(t),\xi(t)) = (\sigma_1(t)+\sigma_2(t))(\xi_1(t)at_{2min}+\xi_2(t)at_{2max}),$$
$$at_2(\sigma(t),\xi(t)) = \sigma_1(t)\xi_1(t)at_{2min}+\sigma_1(t)\xi_2(t)at_{2max}+\sigma_2(t)\xi_1(t)at_{2min}+\sigma_2(t)\xi_2(t)at_{2max}. \tag{157}$$

Thus, defining

$$\theta_1(t) = \xi_1(t)\sigma_1(t), \tag{158}$$
$$\theta_2(t) = \xi_1(t)\sigma_2(t), \tag{159}$$
$$\theta_3(t) = \xi_2(t)\sigma_1(t), \tag{160}$$
$$\theta_4(t) = \xi_2(t)\sigma_2(t), \tag{161}$$

at_1 and at_2 can rewritten as,

$$at_1(\theta(t)) = \theta_1(t)at_{1min}+\theta_2(t)at_{1max}+\theta_3(t)at_{1min}+\theta_4(t)at_{1max}, \tag{162}$$
$$at_2(\theta(t)) = \theta_1(t)at_{2min}+\theta_2(t)at_{2min}+\theta_3(t)at_{2max}+\theta_4(t)at_{2max}. \tag{163}$$

Now, using LMIs (128)-(130) from Theorem 8, the weighting matrices $\mathcal{R} = diag(1,1)$, and $\mathcal{Q} = diag(10000, 10000, 100, 1000)$, the gains $K_{1_{T8}}$ (164), $K_{2_{T8}}$ (165), $K_{3_{T8}}$ (166), and $K_{4_{T8}}$ (167) follow.

$$K_{1_{T8}} = Y_1X^{-1} = \begin{bmatrix} 1.0009\times10^{-3} & 3.9741\times10^{-7} & 1.3844\times10^{-7} & 2.495\times10^{-7} \\ 3.4274\times10^{-7} & 1.1669\times10^{-3} & -4.4879\times10^{-8} & -1.1864\times10^{-7} \end{bmatrix}$$
$$\times 10^5 \times \begin{bmatrix} 1.3464 & -0.0985 & 0.0660 & -0.0180 \\ -0.0985 & 1.8954 & -0.0246 & 0.2421 \\ 0.0660 & -0.0246 & 0.0171 & -0.0010 \\ -0.0180 & 0.2421 & -0.0010 & 0.0770 \end{bmatrix}, \tag{164}$$
$$K_{1_{T8}} = \begin{bmatrix} 134.7603 & -9.7766 & 6.6049 & -1.7926 \\ -11.4463 & 221.1733 & -2.8689 & 28.2511 \end{bmatrix},$$

$$K_{2_{T8}} = Y_2X^{-1} = \begin{bmatrix} 1.0024\times10^{-3} & -3.5806\times10^{-7} & 4.8164\times10^{-8} & 9.6863\times10^{-9} \\ -1.4076\times10^{-7} & 1.5026\times10^{-3} & 3.1980\times10^{-9} & 1.5372\times10^{-8} \end{bmatrix}$$

$$\times 10^5 \times \begin{bmatrix} 1.3464 & -0.0985 & 0.0660 & -0.0180 \\ -0.0985 & 1.8954 & -0.0246 & 0.2421 \\ 0.0660 & -0.0246 & 0.0171 & -0.0010 \\ -0.0180 & 0.2421 & -0.0010 & 0.0770 \end{bmatrix}, \tag{165}$$

$$K_{2_{T8}} = \begin{bmatrix} 134.9651 & -9.9399 & 6.6163 & -1.8154 \\ -14.8176 & 284.8129 & -3.6980 & 36.3811 \end{bmatrix},$$

$$K_{3_{T8}} = Y_3 X^{-1} = \begin{bmatrix} 1.6397 \times 10^{-3} & -1.5422 \times 10^{-7} & -3.3910 \times 10^{-8} & -4.5661 \times 10^{-8} \\ -2.9691 \times 10^{-7} & 1.1672 \times 10^{-3} & 2.5384 \times 10^{-8} & 3.1371 \times 10^{-8} \end{bmatrix}$$

$$\times 10^5 \times \begin{bmatrix} 1.3464 & -0.0985 & 0.0660 & -0.0180 \\ -0.0985 & 1.8954 & -0.0246 & 0.2421 \\ 0.0660 & -0.0246 & 0.0171 & -0.0010 \\ -0.0180 & 0.2421 & -0.0010 & 0.0770 \end{bmatrix}, \tag{166}$$

$$K_{3_{T8}} = \begin{bmatrix} 220.7675 & -16.1788 & 10.8215 & -2.9595 \\ -11.5356 & 221.2488 & -2.8738 & 28.2618 \end{bmatrix},$$

$$K_{4_{T8}} = Y_4 X^{-1} = \begin{bmatrix} 1.6399 \times 10^{-3} & 4.0479 \times 10^{-8} & -3.1542 \times 10^{-8} & -4.3913 \times 10^{-8} \\ 5.0953 \times 10^{-8} & 1.5026 \times 10^{-3} & 5.1072 \times 10^{-9} & 1.6556 \times 10^{-8} \end{bmatrix}$$

$$\times 10^5 \times \begin{bmatrix} 1.3464 & -0.0985 & 0.0660 & -0.0180 \\ -0.0985 & 1.8954 & -0.0246 & 0.2421 \\ 0.0660 & -0.0246 & 0.0171 & -0.0010 \\ -0.0180 & 0.2421 & -0.0010 & 0.0770 \end{bmatrix}, \tag{167}$$

$$K_{4_{T8}} = \begin{bmatrix} 220.7905 & -16.1437 & 10.8222 & -2.9551 \\ -14.7921 & 284.8181 & -3.6968 & 36.3816 \end{bmatrix},$$

Thereby, the controller $K(\theta(t))_{T8}$ is designed as follows

$$K_{T8}(\theta(t)) = \theta_1(t) K_{1_{T8}} + \theta_2(t) K_{2_{T8}} + \theta_3(t) K_{3_{T8}} + \theta_4(t) K_{4_{T8}}. \tag{168}$$

As a way of showing the influence of the addition of LQR-state derivative on the system approach, a comparison (Figure 10) between Theorem 8 and Theorem 2 from (LLINS et al., 2017) is performed. For the simulation, an initial condition of $x_0 = [0.04 \;\; 0.02 \;\; 0.01 \;\; -0.02]^T$ was used. The controller designed with Theorem 2 (LLINS et al., 2017) was

$$K_{1_L} = Z_1 W^{-T} = \begin{bmatrix} 0.66267 & 0.53774 & 1.2931 & 1.9731 \\ 0.10392 & 0.84624 & 0.52402 & 1.5378 \end{bmatrix}$$

$$\times 10^3 \times \begin{bmatrix} 0.0428 & 0.0278 & 1.5937 & -0.6491 \\ -0.0346 & -0.0174 & -1.8132 & 1.1012 \\ -0.0029 & -0.0012 & 0.0323 & -0.0117 \\ -0.0026 & -0.0034 & -0.0450 & 0.0285 \end{bmatrix}, \tag{169}$$

$$K_{1_L} = \begin{bmatrix} 823.3261 & -204.0960 & 16.1306 & -5.5759 \\ -135.0891 & 724.9186 & -2.3569 & 17.0411 \end{bmatrix},$$

$$K_{2_L} = Z_2 W^{-T} = \begin{bmatrix} 0.76756 & 0.70816 & 1.2009 & 2.5080 \\ 0.11205 & 0.6844 & 0.26994 & 1.3228 \end{bmatrix}$$

$$\times 10^3 \times \begin{bmatrix} 0.0428 & 0.0278 & 1.5937 & -0.6491 \\ -0.0346 & -0.0174 & -1.8132 & 1.1012 \\ -0.0029 & -0.0012 & 0.0323 & -0.0117 \\ -0.0026 & -0.0034 & -0.0450 & 0.0285 \end{bmatrix}, \tag{170}$$

$$K_{2_L} = \begin{bmatrix} 338.3339 & 545.5914 & 6.3735 & 12.9636 \\ -404.6222 & 951.4163 & -7.8919 & 22.8938 \end{bmatrix},$$

$$K_{3_L} = Z_3 W^{-T} = \begin{bmatrix} 0.56208 & 0.4616 & 0.93555 & 1.5354 \\ 0.40828 & 1.1208 & 1.0216 & 2.426 \end{bmatrix}$$

$$\times 10^3 \times \begin{bmatrix} 0.0428 & 0.0278 & 1.5937 & -0.6491 \\ -0.0346 & -0.0174 & -1.8132 & 1.1012 \\ -0.0029 & -0.0012 & 0.0323 & -0.0117 \\ -0.0026 & -0.0034 & -0.0450 & 0.0285 \end{bmatrix}, \tag{171}$$

$$K_{3_L} = \begin{bmatrix} 531.2268 & -33.0298 & 10.0803 & -1.4234 \\ 101.9076 & 785.6472 & 2.1137 & 18.2139 \end{bmatrix},$$

$$K_{4_L} = Z_4 W^{-T} = \begin{bmatrix} 0.67197 & 0.62545 & 1.0628 & 2.1482 \\ 0.40441 & 0.94944 & 0.83621 & 2.2828 \end{bmatrix}$$

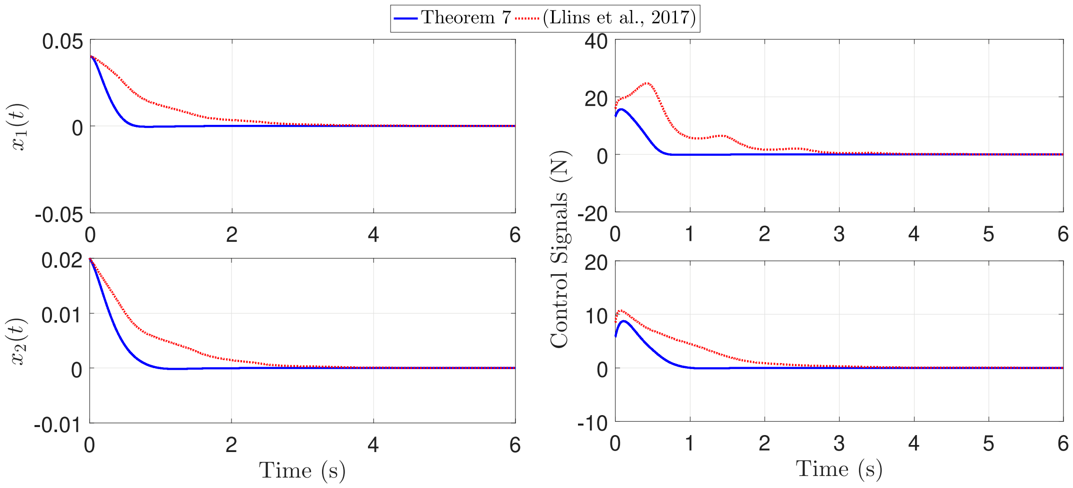
$$\times 10^3 \times \begin{bmatrix} 0.0428 & 0.0278 & 1.5937 & -0.6491 \\ -0.0346 & -0.0174 & -1.8132 & 1.1012 \\ -0.0029 & -0.0012 & 0.0323 & -0.0117 \\ -0.0026 & -0.0034 & -0.0450 & 0.0285 \end{bmatrix}, \tag{172}$$

$$K_{4_L} = \begin{bmatrix} 345.4923 & 404.4162 & 6.5017 & 9.4597 \\ -105.4379 & 967.1052 & -1.9930 & 23.0787 \end{bmatrix}.$$

Thereby, the controller $K(\theta(t))_L$ is designed as follows

$$K_L(\theta(t)) = \theta_1(t)K_{1_L} + \theta_2(t)K_{2_L} + \theta_3(t)K_{3_L} + \theta_4(t)K_{4_L}. \tag{173}$$

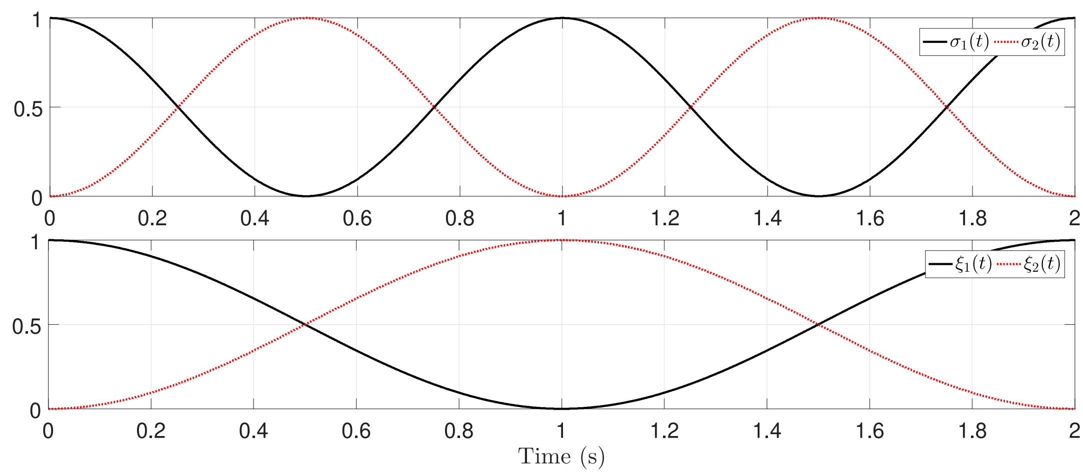
Figure 10 - Comparison between controller $K_{T8}(\theta(t))$ and $K_L(\theta(t))$.



Font: Author's own result.

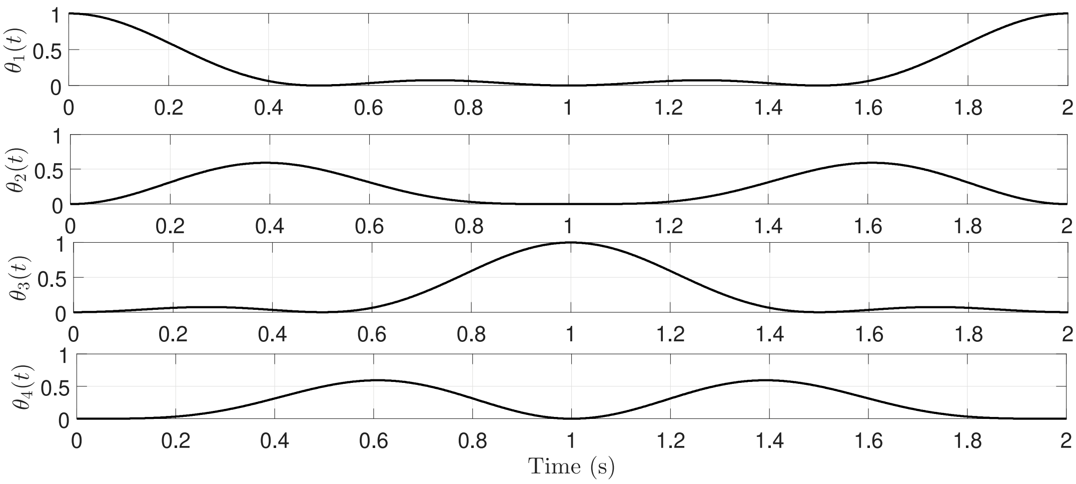
From Figure 10, note that the controller $K_{T8}(\theta(t))$ (168) got a better result, once that the settling time and the control signal are lesser than the result with controller $K_L(\theta)$ (173). In Figures 11 and 12, it is possible to see the variation of $\sigma(t)$ and $\xi(t)$, and $\theta(t)$.

Figure 11 - Variation of $\sigma(t)$ and $\xi(t)$.



Font: Author's own result.

Figure 12 - Variation of $\theta(t)$.



Font: Author's own result.

4.1 PARTIAL CONCLUSIONS

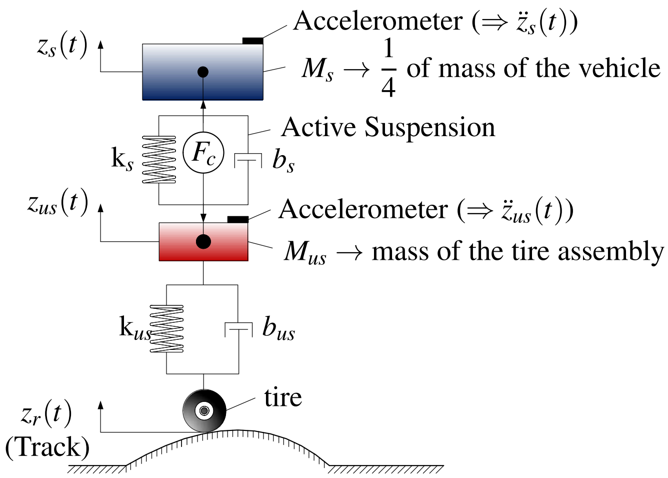
This section presented sufficient conditions based on LMIs to obtain gain scheduling LQR-state derivative controllers for the stabilisation with optimised performance of linear systems subjects to a time-varying parameter. Through the comparison was possible to note the influence of the LQR-state derivative on the system approach once that the behaviour of the system was improved, as well as the control signal. This improve occurs due to the addition of LQR-state derivative, but, specifically in the properly choice of the weighting matrices. As mentioned above, the LQR-state derivative controller is capable of to obtain good results with low control signals, becomes an excellent alternative for practical implementations.

However, it is necessary to take care with the value of weighting matrices, $\mathcal{Q}$ and $\mathcal{R}$, since this matrices are linked with the behaviour of the state derivatives and the control signals.

5 PRACTICAL IMPLEMENTATION

Consider a model of active suspension of a $1/4$ of vehicle, produced by Quanser®, whose schematic is presented in Figure 13. The system consists of two masses, M_s represents $1/4$ the mass of the vehicle body, and M_{us} represents the tire assembly of the vehicle. The mass M_s is supported by the spring k_s and the damper b_s. The mass M_{us} is supported by the spring k_{us} and the damper b_{us}. In order to reduce the oscillations caused by runway irregularities the active suspension system is used, which is composed of the masses M_s and M_{us}, and a motor (actuator) connected between them and controlled by force F_c.

Figure 13 - Model of active suspension of a $\frac{1}{4}$ of the vehicle.



Font: Adapted from (SILVA, 2012).

The dynamic of system is given by (QUANSER, 2009):

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\rho k_s & -\rho b_s & 0 & \rho b_s \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{M_{us}} & \frac{b_s}{M_{us}} & -\frac{k_{us}}{M_{us}} & -\frac{(b_s+b_{us})}{M_{us}} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & \rho \\ -1 & 0 \\ \frac{b_{us}}{M_{us}} & -\frac{1}{M_{us}} \end{bmatrix} u(t). \quad (174)$$

with $x(t) = \begin{bmatrix} z_s(t) - z_{us}(t) & \dot{z}_s(t) & z_{us}(t) - z_r(t) & \dot{z}_{us}(t) \end{bmatrix}^T$, $u(t) = \begin{bmatrix} \dot{z}_r(t) & F_c \end{bmatrix}^T$ and $\rho = \frac{1}{M_s}$. The value of constants can be seen in Table 3 (QUANSER, 2009).

Table 3 - Active suspension parameters.

Parameters	Symbol	Value
Mass of $\frac{1}{4}$ of the total body of vehicle (Kg)	M_s	2.45
Mass of the tire assembly (Kg)	M_{us}	1
Spring stiffness constant (N/m)	k_s	900
Spring stiffness constant (N/m)	k_{us}	2500
Damping coefficient (Ns/m)	b_s	7.5
Damping coefficient (Ns/m)	b_{us}	5

Font: Borrowed from (QUANSER, 2009).

Note in the state space representation of the system that there are two control inputs, one referring to the track surface velocity $(\dot{z}_r)$ and one referring to the force (F_c) applied to the active suspension actuator. For this work, only the control input (F_c) will be taken into account.

For this system, some implementations are performed. First, an implementation using the robust LQR-state derivative controller, and after using the gain scheduling LQR-state derivative controller.

ROBUST LQR-STATE DERIVATIVE CONTROLLER

For this implementation, Theorem 6 was used. Considering that the mass can be changed due to a two equal loads (each one has a weight of 0.4975 Kg), which make up the mass M_s. In this way, the mass M_s can belong to the range $1.455 \leq M_s \leq 2.45$ (Kg). Considering the parameter $\rho = 1/M_s$, the range can be modified to $1/M_{smax} \leq 1/M_s \leq 1/M_{smin} \Rightarrow \rho_{min} \leq \rho \leq \rho_{max} \Rightarrow 0.4082 \leq \rho \leq 0.6873$.

Still, will be considered a fault in the actuator, i.e., there will be a power loss of 50% in the motor. The power loss coming from the actuator fault is represented in the model by the constant k_{fault} (LI et al., 2012). Considering the fault channel from the controller to the actuator, we have

$$u(t) = F_c \Rightarrow u(t)_{fault} = k_{fault}F_c. \tag{175}$$

Thus, we have a four polytope vertices:

- Vertex 1 - $M_s = 1.455$ Kg and 50% of power loss,

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.35 & -3.0612 & 0 & 3.0612 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0.2041 \\ 0 \\ -0.5 \end{bmatrix}. \tag{176}$$

- Vertex 2 - $M_s = 1.455\ Kg$ and 100% of power

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.35 & -3.0612 & 0 & 3.0612 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.4082 \\ 0 \\ -1 \end{bmatrix}. \tag{177}$$

- Vertex 3 - $M_s = 2.45\ Kg$ and 50% of power loss

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.56 & -5.1546 & 0 & 5.1546 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0.3436 \\ 0 \\ -0.5 \end{bmatrix}. \tag{178}$$

- Vertex 4 - $M_s = 2.45\ Kg$ and 100% of power

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.56 & -5.1546 & 0 & 5.1546 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ 0.6873 \\ 0 \\ -1 \end{bmatrix}. \tag{179}$$

The range of the initial states are: $-0.02 \leq x_{01}, x_{03} \leq 0.02\ (m)$ and $-0.15 \leq x_{02}, x_{04} \leq 0.15\ (m/s)$. Therefore, the initial conditions polytope has sixteen vertices ($p = 16$).

The weighting matrices chosen were: $\mathcal{Q} = diag(10, 10, 10, 12)$ and $\mathcal{R} = 1$. Using Theorem 6, the following robust controller was designed:

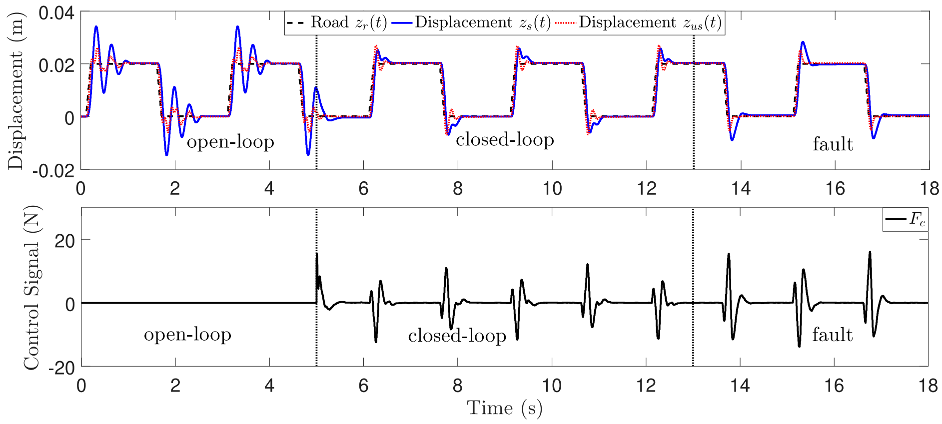
$$\begin{aligned} K_{T6} = YX^{-1} = 10^{-3} \times &\begin{bmatrix} 0.3302 & -0.0008 & -0.0813 & -0.0001 \end{bmatrix} \\ \times 10^6 \times &\begin{bmatrix} 0.2340 & 0.0046 & 0.0488 & -0.0008 \\ 0.0046 & 0.0006 & -0.0084 & -0.0001 \\ 0.0488 & -0.0084 & 1.0545 & 0.0129 \\ -0.0008 & -0.0001 & 0.0129 & 0.0006 \end{bmatrix}, \\ K_{T6} = &\begin{bmatrix} 73.2982 & 2.2074 & -69.5687 & -1.3069 \end{bmatrix}. \end{aligned} \tag{180}$$

As a reference signal (z_r) for practical implementation a square wave signal was adopted.

Such signal has an amplitude of 0.02 m, a frequency of $\frac{1}{3}$ Hz and pulse width of 50%. The sampling period was 1 ms. For the implementations a time interval of 0 to 18 seconds was considered, and until 4.99 seconds the system is in open-loop, in 5 seconds the system is in closed-loop with control law $u(t) = -K\dot{x}(t)$ and in 13 seconds the fault occurs.

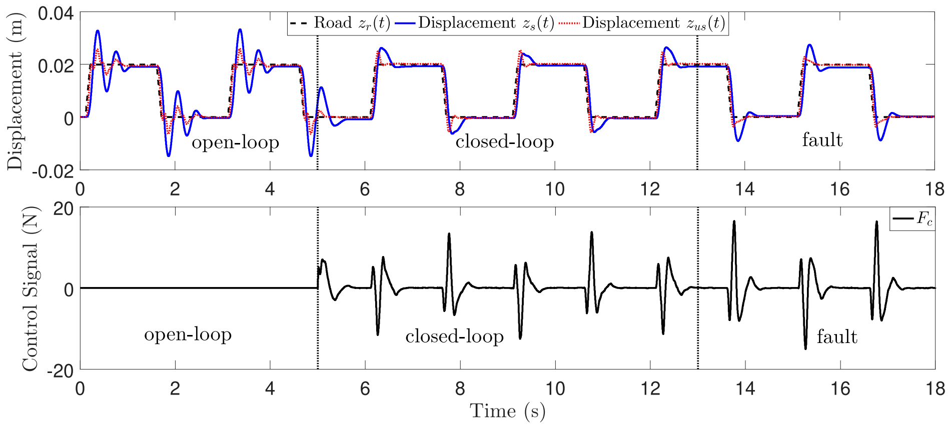
Two implementations are performed. The implementations consider the robust controller K_s with mass $M_s = 1.455\ kg$, or mass $M_s = 2.45\ kg$. The result of the implementations can be seen in Figures 14 and 15. Observe that the system is naturally stable, but presents high settling time and abrupt oscillations, which can cause damage and discomfort to the driver and passenger inside the vehicle. It is possible to decrease the oscillations taking into account the system controlled by force F_c.

Figure 14 - Real system behaviour and control signal for the controller K_{T6} and mass $M_s = 1.455\ kg$.



Font: Author's own result.

Figure 15 - Real system behaviour and control signal for the controller K_{T6} and mass $M_s = 2.45\ kg$.



Font: Author's own result.

Analysing the Figures 14 and 15, note that the designed robust controllers were able to mitigate the vibrations (attenuate the oscillations) considering the mass M_s uncertain and a fault occurrence in the actuator.

In addition, more tests were performed in the active suspension system to verifying the behaviour of the system with a variety of weighting matrices $\mathcal{Q}$ and $\mathcal{R}$. By varying the matrix $\mathcal{Q}$, good system's performance can be achieved, but signal control increases. For this system was observed that small values of matrix $\mathcal{Q}$ were enough to obtain a good result. Now, by varying the matrix $\mathcal{R}$, small control signals can be obtained, but the system performance decreases. Again, small values of $\mathcal{R}$ were enough to obtain small control signals. Thus, to achieve good behaviour of the system, it is necessary to conciliate the weighting matrices by adopting appropriate values for them.

GAIN SCHEDULING LQR-STATE DERIVATIVE CONTROLLER

Now, using Theorem 8, an implementation is performed. Unlike the previous implementation, it considers that the actuator power varies with time. This variation can be represented as

$$\theta(t) = 0.85 + 0.15sin(2\pi 0.5t + \pi/2), \tag{181}$$

which means that the actuator works between 70% and 100% of power.

Parameterising $\theta(t)$, we have

$$\theta_1(t) = 0.5 + 0.5sin(2\pi 0.5t + \pi/2), \tag{182}$$

$$\theta_2(t) = 1 - \theta_1(t). \tag{183}$$

In this way, the system can be represented by a polytope, which vertices are represented as follow.

- Vertex 1 - 70% of power

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -367.35 & -3.0612 & 0 & 3.0612 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 0.2857 \\ 0 \\ -0.7 \end{bmatrix}. \tag{184}$$

- Vertex 2 - 100% of power

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -367.35 & -3.0612 & 0 & 3.0612 \\ 0 & 0 & 0 & 1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0.4082 \\ 0 \\ -1 \end{bmatrix}. \tag{185}$$

The range of the initial states are: $-0.02 \leq x_{01}, x_{03} \leq 0.02$ (m) and $-0.15 \leq x_{02}, x_{04} \leq 0.15$ (m/s). Therefore, the initial conditions polytope has sixteen vertices ($p = 16$). Choosing the weighting matrices as $\mathcal{Q} = diag(10000, 10, 10, 12)$ and $\mathcal{R} = 1$, the designed gain scheduling LQR-state derivative controller was

$$\begin{aligned} K_{1_{T8}} = Y_1 X^{-1} &= \begin{bmatrix} 7.7793 \times 10^{-4} & -3.1317 \times 10^{-7} & 7.3528 \times 10^{-9} & 3.0566 \times 10^{-7} \end{bmatrix} \\ &\times 10^5 \times \begin{bmatrix} 1.5639 & 0.0369 & -0.7061 & -0.0175 \\ 0.0369 & 0.0074 & -0.0935 & -0.0021 \\ -0.7061 & -0.0935 & 9.1436 & 0.1268 \\ -0.0175 & -0.0021 & 0.1268 & 0.0067 \end{bmatrix}, \\ K_{1_{T8}} &= \begin{bmatrix} 121.6603 & 2.8728 & -54.9176 & -1.3637 \end{bmatrix}, \end{aligned} \tag{186}$$

$$\begin{aligned} K_{2_{T8}} = Y_2 X^{-1} &= \begin{bmatrix} 1.0172 \times 10^{-3} & 8.1525 \times 10^{-8} & -1.9390 \times 10^{-9} & -8.6280 \times 10^{-8} \end{bmatrix} \\ &\times 10^5 \times \begin{bmatrix} 1.5639 & 0.0369 & -0.7061 & -0.0175 \\ 0.0369 & 0.0074 & -0.0935 & -0.0021 \\ -0.7061 & -0.0935 & 9.1436 & 0.1268 \\ -0.0175 & -0.0021 & 0.1268 & 0.0067 \end{bmatrix}, \end{aligned} \tag{187}$$

$$K_{2_{T8}} = \begin{bmatrix} 159.0760 & 3.7569 & -71.8268 & -1.7836 \end{bmatrix} \tag{188}$$

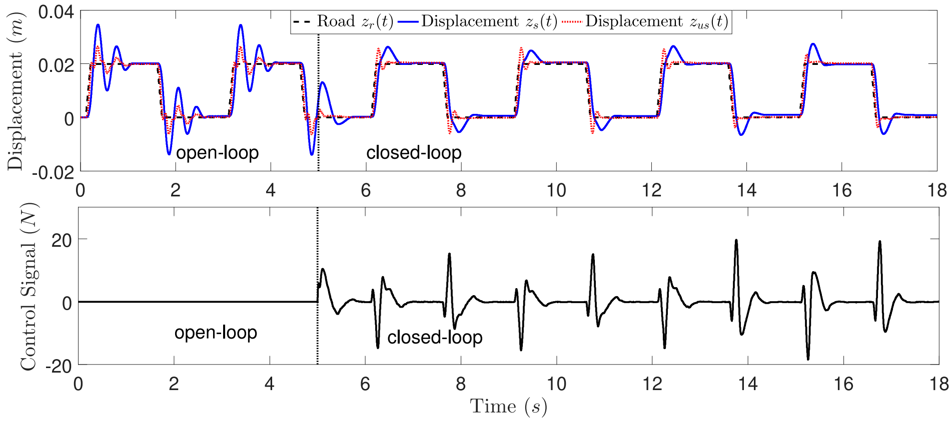
Thereby, the controller $K(\theta(t))_{T8}$ is designed as follows

$$K(\theta(t))_{T8} = \theta_1(t) K_{1_{T8}} + \theta_2(t) K_{2_{T8}}. \tag{189}$$

As a reference signal (z_r) for practical implementation a square wave signal was adopted. Such signal has an amplitude of 0.02 m, a frequency of $\frac{1}{3}$ Hz and pulse width of 50%. The sampling period was 1 ms. For the implementations a time interval of 0 to 18 seconds was considered, and until 4.99 seconds the system is in open-loop, and in 5 seconds the system is in closed-loop with control law $u(t) = -K(\theta(t))\dot{x}(t)$.

As seen by the Figure 16, it is possible to decrease the oscillations taking into account the system controlled by force F_c. So, using controller $K(\theta(t))_{T8}$, it is possible to increase the system response.

Figure 16 - Real system behaviour and control signal for the controller $K_{T8}(\theta(t))$.



Font: Author's own result.

6 CONCLUSIONS

In this work, less conservative conditions for the synthesis of optimal controllers for linear time-invariant systems subject to polytopic uncertainties are proposed. When feasible, LMIs conditions can be easily solved by software programs present in the mathematical programming literature, such as MatLab® (GAHINET et al., 1994). In case of this work, the software MatLab® was used together with your standard solver, ”LMILab”, and the interface for advanced modelling ”YALMIP toolbox”.

In controllers design, the control law used was based on state derivative feedback $(u(t) = -K\dot{x}(t))$. The use of state derivative feedback is more advantageous in some systems that have the acceleration and velocity signals for feedback due to the presence of accelerometers sensors. Considering the LQR-state derivative controller, the project was shown to be an important tool in the control of mechanical systems, especially in vibration control, since it is possible to mitigate the vibrations with a lower control signal choosing the weighting matrices properly, even when the system is subject to uncertainties.

Considering that the resolution of the LQR-state derivative via LMIS is a relatively new topic, the first conditions on specialised literature (BETETO et al., 2018a, 2018b) are conservatives. In this way, this work proposed less conservative conditions based on Finsler’s Lemma (SKELTON; IWASAKI; GRIGORIADIS, 1998; OLIVEIRA; SKELTON, 2001) and the results presented in (LIU; ZHANG, 2003; TEIXEIRA; ASSUNÇÃO; AVELLAR, 2003; CARNIATO et al., 2018). The feasible regions showed that the proposed techniques achieved better results, i.e., achieved less conservative results.

Besides the robust LQR-state derivative controller, and with the high interest on systems subjects to varying parameters, it still proposed in this work the gain scheduling LQR-state derivative controller. This technique is shown to be an effective method for vibration control, considering that there is a varying parameter on the system. Through the comparison, it was possible to see that the inclusion of LQR-state derivative on the problem approach it is advantageous, once that better performance with low control signal is achieved.

Throughout the text, simulations and practical implementations are shown to validate the proposed techniques, which showed that the proposed methods perform well, even when there are uncertainties in the system model or the system is subject to varying parameters.

6.1 FUTURE RESEARCH SUGGESTIONS

- Use the Projection Lemma or Reciprocal Projection Lemma as a starting point to formulate the LMIs conditions;
- Analyse the result of add one matrix $\mathcal{Q}$ and $\mathcal{R}$ for each vertex;
- Investigate less conservative conditions for the gain scheduling LQR-state derivative controller.

REFERENCES

ABDELAZIZ, T. H. S. Pole assignment by state-derivative feedback for single-input linear systems. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, London, v. 221, n. 7, p. 991–1000, 2007.

ABDELAZIZ, T. H. S. Robust pole assignment by state-derivative feedback using gradient flow approach. In: IEEE. INTERNATIONAL CONFERENCE ON COMPUTER ENGINEERING & SYSTEMS - ICCES - ICCES, 2008, Cairo. *Proceedings* [...] Pistacaway: IEEE, 2008. 2008. p. 375–380.

ABDELAZIZ, T. H. S. Robust pole assignment for linear time-invariant systems using state-derivative feedback. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, London, v. 223, n. 2, p. 187–199, 2009.

ABDELAZIZ, T. H. S. Optimal control using derivative feedback for linear systems. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, London, v. 224, n. 2, p. 185–202, 2010.

ABDELAZIZ, T. H. S. Parametric eigenstructure assignment using state-derivative feedback for linear systems. *Journal of Vibration and Control*, London, v. 18, n. 12, p. 1809–1827, 2012.

ABDELAZIZ, T. H. S. Robust pole placement for second-order linear systems using velocity-plus-acceleration feedback. *IET Control Theory & Applications*, IET, v. 7, n. 14, p. 1843–1856, 2013.

ABDELAZIZ, T. H. S.; VALÁŠEK, M. Pole-placement for SISO linear systems by state-derivative feedback. *IEE Proceedings-Control Theory and Applications,* Stevenage, v. 151, n. 4, p. 377–385, 2004.

ABDELAZIZ, T. H. S.; VALÁŠEK, M. State derivative feedback by LQR for linear time-invariant systems. *IFAC Proceedings Volumes,* Amsterdam, v. 38, n. 1, p. 435–440, 2005.

ABU-RMILEH, A.; GARCIA-GABIN, W. A gain scheduling model predictive controller for blood glucose control in type 1 diabetes. *IEEE transactions on biomedical engineering,* Pistacaway, v. 57, n. 10, p. 2478–2484, 2010.

AKTAŞ, A.; SEVER, M.; YAZICI, H. Gain scheduling LQR control of linear parameter varying overhead crane. In: NATIONAL CONFERENCE ON ELECTRICAL, ELECTRONICS AND BIOMEDICAL ENGINEERING - ELECO, 2016, Turkish. *Proceedings* [...] Pistacaway. 2016. p. 232–236.

ARAÚJO, J. M.; DÓREA, C. E. T.; GONÇALVES, L. M. G.; DATTA, B. N. State derivative feedback in second-order linear systems: A comparative analysis of perturbed eigenvalues under coefficient variation. *Mechanical Systems and Signal Processing,* London , v. 76, p. 33–46, 2016.

ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; BUZACHERO, L. F. S.; SILVA, E. R. P. da. Switched optimal robust control of a 3-DoF helicopter subject to a critical fault. In: IEEE INTERNATIONAL CONFERENCE ON INDUSTRY APPLICATIONS - INDUSCON, 12th., 2016, Curitiba. *Proceedings* [...] Piscataway: IEEE, 2016. 2016. p. 1–5.

ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; FARIA, F. A.; SILVA, N. A. P. D.; CARDIM, R. Robust state-derivative feedback LMI-based designs for multivariable linear systems. *International Journal of Control,* Abingdon, v. 80, n. 8, p. 1260–1270, 2007.

ÅSTRÖM, K. J.; WITTENMARK, B. *Adaptive control.* 2nd. Reading: Addison-Wesley, 2008. 2008.

BADIHI, H.; ZHANG, Y.; HONG, H. Fault-tolerant control design for a large off-shore wind turbine using fuzzy gain-scheduling and signal correction. In: IEEE AMERICAN CONTROL CONFERENCE - ACC, 2013, Washington. *Proceedings* [...] Piscataway: IEEE, 2013. 2013. p. 1448–1453.

BASTURK, H. I.; KRSTIC, M. State derivative feedback for adaptive cancellation of unmatched disturbances in unknown strict-feedback LTI systems. *Automatica,* Kidlington, v. 50, n. 10, p. 2539–2545, 2014.

BEKHITI, B.; DAHIMENE, A.; NAIL, B.; HARICHE, K. Block-structure relocation via state and state derivative feedback. *Journal of Control and Systems Engineering,* New York, v. 4, n. 1, p. 95–111, 2016.

BETETO, M. A.; ASSUNÇÃO, E.; TEIXEIRA, M. C.; SILVA, E. R.; BUZACHERO, L. F.; CAUN, R. P. New design of robust LQR-state derivative controllers via LMIs. *IFAC-PapersOnLine,* Amsterdam, v. 51, n. 25, p. 422–427, 2018.

BETETO, M. A. L.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; SILVA, E. R. P. da; BUZACHERO, L. F. S.; CAUN, R. P. Design of robust LQR-derivative controllers for the $\mathscr{D}$-stabilization of linear systems. In: CONGRESSO BRASILEIRO DE AUTOMÁTICA - CBA, 2018, João Pessoa. *Anais* [...] João Pessoa: UFPB; IFAC, 2018. 2018. p. 612–617.

BOYD, S.; GHAOUI, L. E.; FERON, E.; BALAKRISHNAN, V. ***Linear matrix inequalities in system and control theory***. Philadelphia: SIAM, 1994. 193 p.

CARDIM, R.; TEIXEIRA, M. C. M.; FARIA, F. A.; ASSUNÇÃO, E. LMI-based digital redesign of linear time-invariant systems with state-derivative feedback. In: IEEE CONTROL APPLICATIONS – CCA; INTELLIGENT CONTROL - ISIC, 2009, St. Petersburg. *Proceedings* [...] Piscataway: IEEE, 2009. 2009. p. 745–749.

CARNIATO, L. A.; CARNIATO, A. A.; TEIXEIRA, M. C. M.; CARDIM, R.; JUNIOR, E. I. M.; ASSUNÇÃO, E. Output control of continuous-time uncertain switched linear systems via switched static output feedback. *International Journal of Control,* Abingdon, p. 1–20, 2018.

CAUN, R. P.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; FARIA CAUN, A. P. Projeto de controladores LQR-LMI com restrição de taxa de decaimento e estabilidade estendida. CONGRESSO BRASILEIRO DE AUTOMÁTICA – CBA, 21., 2016, Vitória. *Anais* [...] Vitória: UFES, 2016, v. 1, p. 72–79, 2016.

CAUN, R. P.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; FARIA CAUN, A. P. Relaxação de controladores robustos LQR-LMI sujeito a restrição de taxa de decaimento via lema da projeção recíproca. INTERNATIONAL CONFERENCE ON INDUSTRY APPLICATIONS, 12th, 2016, Curitiba. *Proceedings* [...] New York: IEEE/IAS, 2016, v. 1, p. 1–7, 2016.

CHADLI, M.; GUERRA, T.-M. LMI solution for robust static output feedback control of discrete Takagi–Sugeno fuzzy models. *IEEE Transactions on Fuzzy Systems,* Piscataway, v. 20, n. 6, p. 1160–1165, 2012.

CHAIYATHAM, T.; NGAMROO, I. Improvement of power system transient stability by PV farm with fuzzy gain scheduling of PID controller. *IEEE Systems Journal,* Piscataway, v. 11, n. 3, p. 1684–1691, 2017.

CHOI, J. W.; SEE, Y. B.; YOO, W. S.; LEE, M. H. LQR approach using eigenstructure assignment with an active suspension control application. In: IEEE INTERNATIONAL CONFERENCE ON CONTROL APPLICATIONS, 1998, Trieste. *Proceedings* [...] Piscataway: IEEE, 1998. 1998. v. 2, p. 1235–1239.

DAHLEH, M.; DAHLEH, M. A. On slowly time-varying systems. *Automatica,* Amsterdam, v. 27, n. 1, p. 201–205, 1991.

DAS, S.; PAN, I.; HALDER, K.; DAS, S.; GUPTA, A. LQR based improved discrete PID controller design via optimum selection of weighting matrices using fractional order integral performance index. *Applied Mathematical Modelling,* Amsterdam, v. 37, n. 6, p. 4253–4268, 2013.

EBLI, H. G.; KHANI, A.; AZIZI, A. Gain scheduling controller for missile flight control problem by applying LQR. *Journal of World's Electrical Engineering and Technology*, Seoul, v. 2, p. 17–21, 2013.

FALLAH, S.; KHAJEPOUR, A.; FIDAN, B.; CHEN, S.-K.; LITKOUHI, B. Vehicle optimal torque vectoring using state-derivative feedback and linear matrix inequality. *IEEE Transactions on Vehicular Technology,* Piscataway, v. 62, n. 4, p. 1540–1552, 2013.

FARIA, F. A.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; CARDIM, R.; SILVA, N. A. P. D. Robust state-derivative pole placement LMI-based designs for linear systems. *International Journal of Control,* Abingdon, v. 82, n. 1, p. 1–12, 2009.

FARIA, F. A.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; CARDIM, R. Robust state-derivative feedback LMI-based designs for linear descriptor systems. *Mathematical Problems in Engineering*, Hindawi, v. 2010, 2010.

GAHINET, P.; NEMIROVSKII, A.; LAUB, A. J.; CHILALI, M. The LMI control toolbox. In: IEEE CONFERENCE ON DECISION AND CONTROL, 33rd., 1994, Lake Buena Vista. *Proceedings* [...] Piscataway: IEEE, 1994. 1994. v. 3, p. 2038–2041.

GE, M.; CHIU, M.-S.; WANG, Q.-G. Robust PID controller design via LMI approach. *Journal of process control,* Amsterdam, v. 12, n. 1, p. 3–13, 2002.

GUERRA, T. M.; MANZO, V. E.; LENDEK, Z. Observer design for Takagi–Sugeno descriptor models: An LMI approach. *Automatica,* Amsterdam, v. 52, p. 154–159, 2015.

ISERMANN, R. *Fault-diagnosis systems: an introduction from fault detection to fault tolerance*, Berlim. Springer Science & Business Media. 2006.

ISERMANN, R.; BALLÉ, P. Trends in the application of model-based fault detection and diagnosis of technical processes. *Control engineering practice,* Amsterdam, v. 5, n. 5, p. 709–719, 1997.

JING, Y.; SHEN, C.; DIMIROVSKI, G. M.; JIANG, N. Delay-dependent state-derivative feedback with an α-stability constraint for time delay systems. In: IEEE. *American Control Conference, 2009. ACC'09.* 2009. p. 1844–1848.

JOSE, A.; AUGUSTINE, C.; MALOLA, S. M.; CHACKO, K. Performance study of PID controller and LQR technique for inverted pendulum. *World Journal of Engineering and Technology,* Tupe Nagar, v. 3, n. 02, p. 76, 2015.

KANIESKI, J. M.; GRÜNDLINGE, H. A.; CARDOSO, R. A new LQR modeling approach for power quality conditioning devices. In: ANNUAL CONFERENCE ON IEEE INDUSTRIAL ELECTRONICS SOCIETY, 36th, 2010, Glendale. *Proceedings* [...] Piscataway: IEEE, 2010. 2010. p. 2001–2006.

KARIMI, A.; KHATIBI, H.; LONGCHAMP, R. Robust control of polytopic systems by convex optimization. *Automatica,* Kidlington, v. 43, n. 8, p. 1395–1402, 2007.

KASPRZYK, J.; KRAUZE, P.; BUDZAN, S.; RZEPECKI, J. Vibration control in semi-active suspension of the experimental off-road vehicle using information about suspension deflection. *Archives of Control Sciences,* Gliwice, v. 27, n. 2, p. 251–261, 2017.

KUMAR, E. V.; RAAJA, G. S.; JEROME, J. Adaptive PSO for optimal LQR tracking control of 2 DoF laboratory helicopter. *Applied Soft Computing,* Amsterdam, v. 41, p. 77–90, 2016.

KUNCEVIC, V. M.; LYCAK, M. M. Control systems designs using Lyapunov function approach. In: *Nauka, Moscow*. 1997. p. (in Russian).

LEE, C. H.; CHUNG, M. J. Gain-scheduled state feedback control design technique for flight vehicles. *IEEE Transactions on Aerospace and Electronic Systems,* Piscataway, v. 37, n. 1, p. 173–182, 2001.

LEITE, V. J. d. S.; MONTAGNER, V. F.; PERES, P. L. D. Alocaçao robusta de pólos através de realimentaçao de estados dependente de parâmetros. *Sba: Controle & Automação Sociedade Brasileira de Automatica,* Campinas, v. 15, n. 2, p. 127–134, 2004.

LEITH, D. J.; LEITHEAD, W. E. Survey of gain-scheduling analysis and design. *International journal of control,* Abingdon, v. 73, n. 11, p. 1001–1025, 2000.

LI, A.; YANG, L.; ZHANG, J.; HE, C. LQR and fuzzy gain-scheduling based attitude controller for RLV within large operating envelope. In: IEEE International Conference on Control Science and Systems Engineering – CCSSE, 2014. *Proceedings* [...] Piscataway. 2014. p. 51–56.

LI, H.; LIU, H.; GAO, H.; SHI, P. Reliable fuzzy control for active suspension systems with actuator delay and fault. *IEEE Transactions on Fuzzy Systems,* Pistacaway, v. 20, n. 2, p. 342–357, 2012.

LIU, G.; CAO, Y.-Y.; CHANG, X.-H. Fault detection observer design for fuzzy systems with local nonlinear models via fuzzy Lyapunov function. *International Journal of Control, Automation and Systems,* Heidelberg, v. 15, n. 5, p. 2233–2242, 2017.

LIU, X.; WANG, R.; ZHANG, X.; XU, D. Gain scheduling PD controller for variable pitch wind turbines. In: POWER ELECTRONICS AND MOTION CONTROL CONFERENCE – IPEMC, 2012, Harbin. *Proceedings* [...] Piscataway: IEEE, 2012. 2012. v. 3, p. 2162–2167.

LIU, X.; ZHANG, Q. Approaches to quadratic stability conditions and H_∞ control designs for T-S fuzzy systems. *IEEE Transactions on Fuzzy systems,* Piscataway, v. 11, n. 6, p. 830–839, 2003.

LLINS, L. I. H.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; CARDIM, R.; CADALSO, M. R. R.; OLIVEIRA, D. R. d.; SILVA, E. R. P. da. Design of gain scheduling control using state derivative feedback. *Mathematical Problems in Engineering*, Hindawi, v. 2017, 2017.

LOFBERG, J. YALMIP: A toolbox for modeling and optimization in MATLAB. In: IEEE INTERNATIONAL SYMPOSIUM ON COMPUTER AIDED CONTROL SYSTEMS DESIGN, 2004, Taipei. *Proceedings* [...] Piscataway: IEEE, 2004. 2004. p. 284–289.

MANZO, V. E.; LENDEK, Z.; GUERRA, T. M. Generalized LMI observer design for discrete-time nonlinear descriptor models. *Neurocomputing,* Amsterdam, v. 182, p. 210–220, 2016.

MASAR, I.; STÖHR, E. Gain-scheduled LQR-control for an autonomous airship. In: INTERNATIONAL CONFERENCE ON PROCESS CONTROL, 18th, 2011, Tatranská Lomnica. *Proceedings* [...] Piscataway: IEEE, 2011. 2011. p. 14–17.

MASUBUCHI, I.; KATO, J.; SAEKI, M.; OHARA, A. Gain-scheduled controller design based on descriptor representation of LPV systems: Application to flight vehicle control. In: IEEE CONFERENCE ON DECISION AND CONTROL (CDC), 2004, Nassau. *Proceedings* [...] Pistacaway: IEEE, 2004. 2004. v. 1, p. 815–820.

MERRY, R. J. E.; HOLIERHOEK, J. L.; MOLENGRAFT, M. J. G. van de; STEINBUCH, M. Gain scheduling control of a walking piezo actuator. *IEEE/ASME Transactions on mechatronics,* Piscataway, v. 19, n. 3, p. 954–962, 2014.

MICHIELS, W.; VYHLÍDAL, T.; HUIJBERTS, H.; NIJMEIJER, H. Stabilizability and stability robustness of state derivative feedback controllers. *SIAM Journal on Control and Optimization,* Philadelphia, v. 47, n. 6, p. 3100–3117, 2009.

MOBAYEN, S.; BALEANU, D.; TCHIER, F. Second-order fast terminal sliding mode control design based on LMI for a class of non-linear uncertain systems and its application to chaotic systems. *Journal of Vibration and Control*, London, v. 23, n. 18, p. 2912–2925, 2017.

MONTAGNER, V. F.; PERES, P. L. D. State feedback gain scheduling for linear systems with time-varying parameters. Journal of dynamic systems, measurement, and control, *American Society of Mechanical Engineers*, New York, v. 128, n. 2, p. 365–370, 2006.

MOREIRA, M. R.; JÚNIOR, E. I. M.; ESTEVES, T. T.; TEIXEIRA, M. C. M.; CARDIM, R.; ASSUNÇÃO, E.; FARIA, F. A. Stabilizability and disturbance rejection with state-derivative feedback. *Mathematical Problems in Engineering,* New York, v. 2010, 2010.

MOZELLI, L. A.; PALHARES, R. M.; AVELLAR, G. S. C. A systematic approach to improve multiple Lyapunov function stability and stabilization conditions for fuzzy systems. *Information Sciences,* Amsterdam, v. 179, n. 8, p. 1149–1162, 2009.

MOZELLI, L. A.; PALHARES, R. M.; MENDES, E. M. A. M. Equivalent techniques, extra comparisons and less conservative control design for Takagi–Sugeno (TS) fuzzy systems. *IET Control Theory & Applications,* Stevenage, v. 4, n. 12, p. 2813–2822, 2010.

OLALLA, C.; LEYVA, R.; AROUDI, A. E.; QUEINNEC, I. Robust LQR control for PWM converters: An LMI approach. *IEEE Transactions on industrial electronics,* Piscataway, v. 56, n. 7, p. 2548–2558, 2009.

OLIVEIRA, M. C. de; GEROMEL, J. C. A class of robust stability conditions where linear parameter dependence of the Lyapunov function is a necessary condition for arbitrary parameter dependence. *Systems & Control Letters,* Amsterdam, v. 54, n. 11, p. 1131–1134, 2005.

OLIVEIRA, M. C. de; SKELTON, R. E. Stability tests for constrained linear systems. In: *Perspectives in robust control,* Berlim, Springer. 2001. p. 241–257.

OLIVEIRA, R. C. L. F.; PERES, P. L. D. LMI conditions for robust stability analysis based on polynomially parameter-dependent Lyapunov functions. *Systems & Control Letters,* Amsterdam, v. 55, n. 1, p. 52–61, 2006.

OMIDI, E.; MAHMOODI, S. N. Vibration control of collocated smart structures using H_∞ modified positive position and velocity feedback. *Journal of Vibration and Control*, London, v. 22, n. 10, p. 2434–2442, 2016.

OOSTEROM, M.; BABUSKA, R. Fuzzy gain scheduling for flight control laws. In: IEEE INTERNATIONAL CONFERENCE ON FUZZY SYSTEMS, 10th., 2001, Melbourne. *Proceedings* [...] Piscataway: IEEE, 2001. 2001. v. 2, p. 716–719.

QUANSER. *Active Suspension - User's Manual,* Canada, Quanser Consulting Inc., 2009. 2009.

RIJNEN, M.; SACCON, A.; NIJMEIJER, H. On optimal trajectory tracking for mechanical systems with unilateral constraints. In: IEEE CONFERENCE ON DECISION AND CONTROL (CDC), 54th, 2015, Osaka. *Proceedings* [...] Piscataway: IEEE, 2015. 2015. p. 2561–2566.

ROJAS, J. H. C.; SERREZUELA, R. R.; LÓPEZ, J. A. Q.; PERDOMO, K. L. R. LQR hybrid approach control of a robotic arm two degrees of freedom. *International Journal of Applied Engineering Research,* Delhi, v. 11, n. 17, p. 9221–9228, 2016.

ROSSI, F. Q.; GALVÃO, R. K. H.; TEIXEIRA, M. C. M.; ASSUNÇÃO, E. Direct discrete time design of robust state derivative feedback control laws. *International Journal of Control,* Abingdon, v. 91, n. 1, p. 70–84, 2018.

SABATO, A.; FENG, M. Q.; FUKUDA, Y.; CARNÍ, D. L.; FORTINO, G. A novel wireless accelerometer board for measuring low-frequency and low-amplitude structural vibration. *IEEE Sensors Journal,* Piscataway, v. 16, n. 9, p. 2942–2949, 2016.

SATO, M. Robust gain-scheduled flight controller via a new formulation for over-bounding scheduling parameter errors. In: *IFAC-PapersOnLine,* Amsterdam. 2018. p. 163–168.

SILVA, E. R. P. d. *Controle de sistemas lineares incertos via realimentação derivativa utilizando Funções de Lyapunov dependentes de parâmetros*, Ilha Solteira. Universidade Estadual Paulista (UNESP), 2012.

SILVA, E. R. P. D.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; FARIA, F. A.; BUZACHERO, L. F. S. Parameter-dependent Lyapunov functions for state-derivative feedback control in polytopic linear systems. *International Journal of Control,* Abingdon, v. 84, n. 8, p. 1377–1386, 2011.

SILVA, E. R. P. D.; ASSUNÇÃO, E.; TEIXEIRA, M. C. M.; BUZACHERO, L. F. S. Less conservative control design for linear systems with polytopic uncertainties via state-derivative feedback. *Mathematical Problems in Engineering*, Hindawi, 2012.

SKELTON, R. E.; IWASAKI, T.; GRIGORIADIS, D. E. *A unified algebraic approach to control design*. : CRC Press, Boca Raton, 1998, 1998.

STILWELL, D. J.; RUGH, W. J. Interpolation of observer state feedback controllers for gain scheduling. *IEEE transactions on automatic control,* Piscataway, v. 44, n. 6, p. 1225–1229, 1999.

SUN, Y.; ZHANG, W.; ZHANG, M.; LI, D. Design of neural network gain scheduling flight control law using a modified PSO algorithm based on immune clone principle. In: INTERNATIONAL CONFERENCE ON INTELLIGENT COMPUTATION TECHNOLOGY AND AUTOMATION, 2nd., 2009, Changsha. *Proceedings* [...] Piscataway: IEEE, 2009. 2009. v. 1, p. 259–263.

TANAKA, K.; IKEDA, T.; WANG, H. O. Fuzzy regulators and fuzzy observers: relaxed stability conditions and lmi-based designs. *IEEE Transactions on fuzzy systems,* Piscataway, v. 6, n. 2, p. 250–265, 1998.

TEIXEIRA, M. C. M.; ASSUNÇÃO, E.; AVELLAR, R. G. On relaxed LMI-based designs for fuzzy regulators and fuzzy observers. *IEEE Transactions on Fuzzy Systems,* Piscataway, v. 11, n. 5, p. 613–623, 2003.

TZES, A.; NIKOLAKOPOULOS, G. LQR-output feedback gain scheduling of mobile networked controlled systems. In: AMERICAN CONTROL CONFERENCE, 2004, Boston. *Proceedings* [...] Piscataway: IEEE, 2004. 2004. v. 5, p. 4325–4329.

VESELỲ, V.; ILKA, A. Design of robust gain-scheduled PI controllers. *Journal of the Franklin Institute,* Amsterdam, v. 352, n. 4, p. 1476–1494, 2015.

VESELỲ, V.; ILKA, A. Generalized robust gain-scheduled PID controller design for affine LPV systems with polytopic uncertainty. *Systems & Control Letters,* Amsterdam, v. 105, p. 6–13, 2017.

WANG, R.; HU, C.; WANG, Z.; YAN, F.; CHEN, N. Integrated optimal dynamics control of 4WD4WS electric ground vehicle with tire-road frictional coefficient estimation. *Mechanical Systems and Signal Processing,* Amsterdam, v. 60, p. 727–741, 2015a.

WANG, R.; JING, H.; YAN, F.; KARIMI, H. R.; CHEN, N. Optimization and finite-frequency H_∞ control of active suspensions in in-wheel motor driven electric ground vehicles. *Journal of the Franklin Institute,* Amsterdam, v. 352, n. 2, p. 468–484, 2015b.

WANG, Z.; MONTANARO, U.; FALLAH, S.; SORNIOTTI, A.; LENZO, B. A gain scheduled robust linear quadratic regulator for vehicle direct yaw moment control. *Mechatronics,* Amsterdam, v. 51, p. 31–45, 2018.

WU, Z.; FANG, H. Delayed-state-derivative feedback for improving consensus performance of second-order delayed multi-agent systems. *International Journal of Systems Science,* Abingdon, v. 43, n. 1, p. 140–152, 2012.

YAN, S.; ZHANG, G.; LI, T.; SHEN, M.; LI, L. H∞ Static Output Control of Discrete-Time Networked Control Systems with an Event-Triggered Scheme. 2018.

YAZICI, H.; SEVER, M. Observer based optimal vibration control of a full aircraft system having active landing gears and biodynamic pilot model. *Shock and Vibration*, Hindawi, v. 2016, 2016.

YAZICI, H.; SEVER, M. L2 gain state derivative feedback control of uncertain vehicle suspension systems. *Journal of Vibration and Control*, London, p. 1077546317711335, 2017.

ZHANG, S.; WANG, Z.; DING, D.; WEI, G.; ALSAADI, F. E.; HAYAT, T. A gain-scheduling approach to nonfragile H_∞ fuzzy control subject to fading channels. *IEEE Transactions on Fuzzy Systems,* Piscataway, v. 26, n. 1, p. 142–154, 2018.

ZHAO, D.; DING, F.; ZHOU, L.; ZHANG, W.; XU, H. Robust control of neutral system with time-delay for dynamic positioning ships. *Mathematical Problems in Engineering*, Hindawi, v. 2015, 2015.

ZHU, Q.; LI, L.; CHEN, C.-J.; LIU, C.-Z.; HU, G.-D. A low-cost lateral active suspension system of the high-speed train for ride quality based on the resonant control method. *IEEE Transactions on Industrial Electronics,* Piscataway, v. 65, n. 5, p. 4187–4196, 2018.