UNIVERSIDADE ESTADUAL PAULISTA

DISSERTAÇÃO DE MESTRADO

CORREÇÕES EM NEXT-TO-LEADING ORDER EM REGRAS DE SOMA DA QCD

Autor:
Jesuel MARQUES LEAL Junior

Orientador:
Prof. Dr. Ricardo D'Elia Matheus

Banca Examinadora:
Prof. Dr. Ricardo D'Elia Matheus (UNESP)
Prof. Dr.ª Marina Nielsen (USP)
Prof. Dr. Raphael Moreira de Albuquerque (UERJ)

Dissertação apresentada ao Programa de Pós-Graduação do Instituto de Física Teórica da UNESP para obtenção do grau de Mestre em Física

17 de janeiro de 2019

L434c

Leal Junior, Jesuel Marques
Correções em next-to-leading order em regras de soma da QCD / Jesuel Marques Leral Junior. – São Paulo, 2019
80 f. : il.

Dissertação (mestrado) - Universidade Estadual Paulista (Unesp), Instituto de Física Teórica (IFT), São Paulo
Orientador: Ricardo D'Elia Matheus

1. Cromodinâmica quântica. 2. Partículas (Física nuclear). 3. Regras de soma (Física) I. Título.

Sistema de geração automática de fichas catalográficas da Unesp. Biblioteca do Instituto de Física Teórica (IFT), São Paulo. Dados fornecidos pelo autor(a).

"(. . .)
L'univers és infinit,
pertot acaba i comença,
i ençà, enllà, amunt i avall,
la immensitat és oberta,
i aon tu veus lo desert
eixams de mons formiguegen.
(. . .)."

"Plus Ultra", Jacint Verdaguer i Santaló

UNIVERSIDADE ESTADUAL PAULISTA

Abstract

Grupo de Fenomenologia de Partículas
Instituto de Física Teórica

Master in Physics

NEXT-TO-LEADING ORDER CORRECTIONS IN QCD SUM RULES

by Jesuel MARQUES LEAL Junior

The method of QCD Sum Rules is a non-perturbative method in QCD, which allows one to obtain hadronic observables from universal QCD quantities, like the masses of quarks and the condensates. In the calculations of this method, one has to introduce non-physical parameters and at the end of the computations, search for regions in the space of these parameters where there is little dependence of the physical quantities on them. Since this is not always possible, this dependence translates itself into theoretical uncertainty. One expects that by adding radiative corrections, that bring more physical information into the calculation, the dependence on the arbitrary parameters decreases. In order to do this, modern methods of solving Feynman integrals are welcome. The method of master integrals reduces the problem of calculating Feynman integrals to a linear combination of a basis of integrals, which can be solved, in turn, by means of coupled differential equations. We present the calculation of the relevant terms for the OPE for the J/Ψ sum rule up to two loops using reduction to master integrals and differential equations to solve them.

Keywords: Quantum Chromodynamics, Non-Perturbative Methods, QCD Sum Rules, Charmonium, Master Integrals, Differential Equations for Master Integrals.

UNIVERSIDADE ESTADUAL PAULISTA

Resumo

Grupo de Fenomenologia de Partículas
Instituto de Física Teórica

Mestre em Física

CORREÇÕES EM NEXT-TO-LEADING ORDER EM REGRAS DE SOMA DA QCD

por Jesuel MARQUES LEAL Junior

O método de regras de soma na QCD é um método não-perturbativo em QCD que permite obter observáveis hadrônicos a partir de quantidades universais da QCD, como as massas dos quarks e condensados. Ao longo dos cálculos desse método é necessário introduzir parâmetros não físicos e no final da conta, procurar por regiões do espaço desses parâmetros onde não há dependência tão forte das quantidades físicas neles. Como isso nem sempre é possível, essa dependência se traduz em incerteza teórica. Espera-se que adicionando correções radiativas, ou seja mais informação física no cálculo, essa dependência em parâmetros arbitrários diminua. Para tal métodos modernos de resolução de integrais de Feynman são bem-vindos. O método das integrais mestras reduz o problema de calcular integrais de Feynman a uma combinação linear de uma base de integrais, que podem ser resolvidas por equações diferenciais acopladas. Apresentamos o cálculo para os termos relevantes da OPE da regra de soma do J/Ψ até dois loops utilizando integrais mestras e equações diferenciais para resolvê-las.

Palavras Chave: Cromodinâmica Quântica, Métodos Não-perturbativos, Regras de Soma da QCD, Charmônio, Integrais Mestras, Equações Diferenciais para Integrais Mestras.

Agradecimentos

O primeiro agradecimento vai para minha namorada Amanda, que tem sido minha grande companheira nos últimos meses, tem me trazido tanta alegria e amor e tem me dado várias oportunidades de eu ser o meu melhor. Com ela meu mundo fica tão mais colorido e interessante.

Aos meus pais tenho que agradecer por me terem dado a melhor formação possível, me terem inspirado a curiosidade de como as coisas funcionam, o respeito ao ensino, a cultura e o conhecimento e sobretudo por serem meus pais, da melhor maneira que eles sabiam ser.

Quero agradecer a minha família por serem tão fortes e persistentes e ao mesmo tempo tão amáveis comigo. Tenho cada vez mais orgulho do meu sangue baiano e nordestino. Espero que eu possa levar o nome da minha família cada vez mais longe e não esqueça das minhas origens.

Sou grato a muitos amigos que compartilharam comigo momentos e sentimentos. Em especial aos meus amigos da graduação na USP, Bleider, Caique, Santiago e Vinícius, que me ajudaram de diversas formas com a física e me ajudando também a manter a sanidade mental falando de outras coisas que não são física. Além disso ao Theo e o Ricardo (Carioca) dos quais me aproximei mais tarde, que dividem comigo vários interesses, de humanidades além da física e matemática, e estão dispostos a conversar apesar dos fusos horários do mundo. Ao Alberto por vir me visitar, ajudar a trazer leite do mercado, roubar minha internet e me fazer chorar de rir às vezes. À Ana, por me aguentar reclamando tanto da vida, principalmente nos meus momentos mais difíceis, mas também por se divertir comigo falando besteira. Também sou grato ao Rafael (RJ), Black People (Paulo, Victor, Vincent, Maria Eugênia, Luca, Paula e Jordan), ao Oliveira e ao Gabriel Verbena. Ich muss mich bedanken bei dem Jörg und der Simone, meine deutsche Eltern, und der Aileen, die mich während mein Bachelorstudium so sehr unterstützt haben, und dann später gute Freunde geblieben sind.

Quero agradecer ao meu orientador Ricardo pela paciência em explicar várias vezes a mesma coisa, pelas conversas amigáveis, pela independência que me proporcionou e pelo exemplo de honestidade que me deu.

Agradeço ao grupo de QCD do IFT (especialmente ao professor Gastão Krein, Carlisson Miller, Fernando Serna, Enzo Sollis, Caroline Costa) por levar a frente os seminários de QCD semanais no instituto e assim proporcionar um ambiente de aprendizado e discussão, que para mim foi muito imporante. Ao Grhafite do IFUSP (especialmente ao professor Robilotta, que me mostrou que estudar física não significa estudar só física, ao professor Renato Higa que me introduziu ao mundo de física de hádrons e QCD e ao professor Jorge Noronha pelo entusiasmo contagiante que mostra ter com a física e por acreditar em mim), do qual participei como membro intruso por algum tempo. I would like to thank also professor SuHoung Lee and his students HyungJoo Kim, Aaron Park and SungSik No and his former student and my close research collaborator KieSang Jeong for the hospitality in Korea and the company during my stay there. It was a very pleasant time and this is due mostly to them.

Gostaria de agradecer minha psicanalista, Maria Fernanda do Valle, por me ajudar a aplicar minha inteligência no problema de saber quem eu sou e assim me abrir os horizontes do que eu posso ser, e por me ensinar a pensar de uma maneira completamente nova sobre tudo.

A minha professora de catalão Jèssica Ferrer Escandell, agradeço por colocar mais poesia na minha vida e ao mesmo tempo me mostrar que eu preciso de mais poesia na minha vida.

À FAPESP agradeço pelo apoio financeiro (processo 2016/02171-8), pelo respeito aos bolsistas e pela defesa da pesquisa pura. Também agradeço a CAPES pelo apoio financeiros nos primeiros meses do mestrado.

Contents

List of Figures

Aos meus pais.

Notation and Conventions

We use the mostly negative metric

$$g_{\mu\nu} = \mathrm{diag}(+,-,-,-) \tag{1}$$

and work in natural units, where $\hbar = c = 1$.
Integrals without identification of limits are to be understood as extending to all of the corresponding space.
The Fourier transform is

$$f(p) = \frac{1}{2\pi}\int \mathrm{d}x\, e^{-ip\cdot x} f(x), \tag{2}$$

and its inverse

$$f(x) = \int \mathrm{d}p\, e^{ip\cdot x} f(p), \tag{3}$$

where one can notice that the distinction between a function and its Fourier transform is made clear by the argument.

Momenta are usually labeled p, q and k and when more is needed, we use numbers in subscripts as in $p_1, p_2, \ldots$. If an integration extends over many variables of the same kind, we use the notation

$$\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\, f(x,y,z) = \int \mathrm{d}x,y,z\, f(x,y,z). \tag{4}$$

Color indices are put at the top of the field variables and Dirac indices at the bottom. Lorentz indices can be at the top or at the bottom, following the usual covariant and contravariant notation. Uppercase latin letters "A, $B, \ldots$" are adjoint representation indices and lowercase letters "$a, b, \ldots$" the fundamental indices. "$i, j, \ldots$" are Dirac indices and $\alpha, \ldots, \mu, \ldots$ Lorentz indices, so for example, the gluon field can be denoted by $B^A_\mu(x)$, a quark *charm* field is $c^a_i(x)$ and the generators of $\mathrm{SU}(3)$ are $t^{A\,ab}$. Einstein summation convention is used.

The gamma function $\Gamma(z)$ has an integral representation given by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, \mathrm{d}x, \tag{5}$$

and the beta function $B(z_1, z_2)$ is represented by the integral

$$B(z_1, z_2) = \int_0^1 x^{z_1 - 1}(1 - x)^{z_2 - 1}\, \mathrm{d}x, \tag{6}$$

which can be expressed as a combination of gamma functions

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}. \tag{7}$$

The polygamma function is

$$\psi^{(m)}(z) \equiv \frac{d^{m+1}}{dz^{m+1}} \log\left(\Gamma(z)\right), \tag{8}$$

and the polylogarithm is a generalization of the logarithm defined via

$$\mathrm{Li}_m(z) \equiv \sum_{k=1}^{\infty} \frac{z^k}{k^m}. \tag{9}$$

The Euler gamma constant is denoted by γ_E

$$\gamma_E \equiv \lim_{n \to \infty} \left(-\log(n) + \sum_{k=1}^{n} \frac{1}{k} \right). \tag{10}$$

The Riemann zeta function is defined by

$$\zeta(z) \equiv \sum_{k=1}^{\infty} \frac{1}{k^z}. \tag{11}$$

1 Introduction

Quantum chromodynamics, in short QCD, is one of the most successful theories to date. In the high-energy regime, its predictions, based on perturbation theory, are in complete agreement with the theory. In its low-energy regime however, one has to resort to other methods of calculation in order to obtain results, since standard perturbation theory does not apply there. One of these methods, and the one which is in the spotlight today, is lattice QCD (Gattringer and Lang, 2010). In this method it is possible to obtain hadron masses, which are intrinsic non-perturbative quantities, to a very good precision (Durr, 2008). Lattice QCD is based on the discretization of an euclidean space-time. Quark and gluon fields are then embedded in this quantized space-time and the relevant quantities can be extracted from simulations performed on computers, varying the spacing of the lattice and extrapolating to the continuum limit. But, although lattice QCD has provided us, through its success in obtaining non-perturbative quantities, confidence that QCD is everything that is needed to explain how quarks and gluons interact in all energy regimes, the mechanisms which are responsible for the inclusion of the non-perturbative effects are not clear.

QCD Sum Rules (or SVZ Sum Rules, from the names of Shifman-Vainshtein-Zakharov, the first to propose it (Shifman, Vainshtein, and Zakharov, 1979)) is another method to study the properties of hadrons and obtain, for example, their masses. It is based on the Operator Product expansion by Wilson (Wilson, 1969) and it includes the non-perturbative effects by condensates of combinations of quark and gluon fields, which are expectation values of the operators appearing in the OPE. With this, it is possible to directly express the mass of a ρ meson, for example (Pascual and Tarrach, 1984), in terms of the quark or chiral condensate. The straightforward extension of the original sum rules for mesons to baryons (Ioffe, 1981), allows one to directly link the mass of the nucleon to the chiral condensate, in what was one of the main successes of this method in the early 80s.

Another ingredient of QCD Sum Rules is the so-called phenomenological Ansatz. Usually one is restricted to obtain the properties of the lowest-lying state in a channel and then one has to find ways of suppressing the contributions from the higher mass resonances. In this process we have the introduction of a foreign mass scale, the Borel mass. The usual way one deals with that is by searching regions in the plots of physical quantities against this Borel mass where the values do not change much, that is basically searching for maxima or minima.

It is believed that the dependence on the Borel mass is related to the truncation of the OPE. Only a relatively small number of operators are taken into account in the expansion, since the OPE is an asymptotic series (Hilger, 2012) and the truncation has to be done at a low order in order to get a meaningful result. It is hopeless to try to increase the number of operators in order to get a better result, since actually the opposite may occur. This is an intrinsic limitation of the method.

The coefficients of the OPE are calculated in perturbation theory, which also calls for a truncation. In many calculations only the zeroth order in α_S is taken into account, since it is already at least a one-loop calculation for mesons. Going further into the perturbative series means increasing the numbers of loops and consequently increasing the difficulty in the calculation. The combination of the truncation of the OPE with the truncation of the perturbative series may result in a strong dependence on the unphysical Borel mass.

The aim of this dissertation is to present a way of incorporating a modern method for the solution of loop integrals into QCD Sum Rules and thus go further in the perturbative expansion of the coefficients. The method of master integrals (Henn, 2015) is already in use to calculate amplitudes in perturbative QCD and it has already been applied to QCD Sum Rules (Wang et al., 2017), although the knowledge about it in the field in not widespread. There is a whole community dedicated to developing this method, which is systematic and extremely versatile (Henn, 2015; Henn and Smirnov, 2013; Smirnov, 2012; Grozin, 2005). With its aid it should be possible to add corrections to the original Sum Rules and also to extensions of it, like the Sum Rules for the decays of hadrons, including the decays of exotic particles, which are to date still plagued by the immense uncertainties arisen from the strong dependence on the Borel mass. Consequently, it is possible that one will be able to distinguish between currents and discard models for compact or molecular configurations for tetra- and pentaquarks.

There has been recent developments in the calculation of these master integrals through the differential equations that they obey (Henn, 2015; Henn and Smirnov, 2013). The results are all analytic and the whole process of solving them is easily implemented on symbolic calculation softwares like *Mathematica* and in some cases one can rely on the community to provide answers to the relevant basis of master integrals, if they are already available, and in these cases what is left to do is to obtain the coefficients needed for the calculation at hand through a process of reduction to that basis.

The dissertation is divided as follows: Chapter 2 presents an overview about quantum chromodynamics, the theory of the strong force, based on Peskin and Schroeder, 1995; Muta, 1998. The beginning of Chapter 3 introduces the main topics and arguments of the method of QCD Sum Rules in a nutshell, based on Pascual and Tarrach, 1984; Colangelo and Khodjamirian, 2001; Cohen et al., 1995; Matheus, 2003 and the subchapters therein present a more detailed and deeper account of each of these topics. After that, the method for solving Feynman integrals with the use of Master Integrals is shown in 4. As as example of applying master integrals to QCD Sum Rules we calculate the needed integrals up to one-loop order in Chapter 5, the more complicated two-loop result specifically being shown in Chapter 5.3. Conclusions are presented in Chapter 6.

2 Quantum Chromodynamics

A specific quantum field theory is defined when we specify the degrees of freedom, that is the fields, the symmetry groups under which these fields will transform and then for each field fix the representation of the symmetry groups to which it belongs. Of course, historically the narrative is much more rich and at the same time more confusing than this, people have played around with different theories and different principles, guided by experimental observations, until a form for the lagrangian was obtained which could explain the experimental data satisfactorily and was moreover, able to predict phenomena and observations which were later confirmed experimentally. This is true, for example, for QED, which is the most successful quantum field theory, being able to, at once, accommodate known experimental results and predict the outcome of future experiments to a great precision with very few parameters. The degrees of freedom of QED are charged fermions and the photon, and its parameters are the mass of these fermions and a single universal coupling of the fermionic fields to the photon field, the electric charge.

Nowadays QED is considered to be part of the electroweak sector of a still larger theory, realized in the lagrangian of the Standard Model. The Standard Model is a sort of collage of the electroweak lagrangian with the lagrangian of QCD, which is the quantum field theory containing the description of quarks, gluons and their interactions. In the Standard Model lagrangian of particle physics there are links between the QCD or strong sector and the electroweak sector, which describe the interplay between the fields, for example, explaining the quarks bare masses through the Higgs mechanism.

The Quantum Chromodynamics, QCD, resembles and was inspired from quantum electrodynamics, QED, as the name bears witness (Peskin and Schroeder, 1995). The guiding principle to construct the QCD lagrangian, following the line of construction of the QED lagrangian, is the gauge principle [1]. We postulate the existence of fermionic fields, transforming in the

[1] The gauge principle is not really a principle but a guiding line in constructing gauge theories self-consistently in the Minkowski space. (Weinberg, 2005)

fundamental representation of an SU(3) symmetry group. The charge associated to this transformation, in parallel to the electric charge of the corresponding U(1) symmetry of QED, is denominated *color*, and can assume six different values: red, green, blue, anti-red, anti-green and anti-blue, where the first three are associated to quarks and the latter three are reserved to the anti-quarks. Technically speaking, this means that the complex conjugate fundamental representation in SU(3) is different from the original fundamental representation.

The fact is that SU(3) is the only candidate for the symmetry group of strong interactions between the semi-simple Lie groups, having the aforementioned property of the fundamental representation being different from its complex conjugate; the property that the number of colors (and anti-colors) is three, which are fixed by experimental observations already and having asymptotic freedom, also fixed by experimental observations, to be explained below (Muta, 1998).

The bound states of quarks, the hadrons, are all "white" in this color language, meaning that they are formed through a combination of quark fields in such a way that the net color is neutral, or more technically, they are singlets under the color group transformations. The hadrons are the only QCD objects which arrive at the detectors in collision experiments. The searches for free quarks have yielded negative results ever since they started, so it seems that it is a principle, possibly emerging from QCD itself, that only color singlets are observable, at least at zero temperature and chemical potential. This is called *confinement*. There is no clear theoretical picture explaining confinement to date, although some recent developments seem to have linked it to indications of violation of positivity in the quark spectral function (Alkofer and Smekal, 2001). It is believed that other phases of QCD matter can be realized at finite temperature and/or chemical potential, such as the quark-gluon plasma, in which quarks and gluons may be unbound.

In order to explain the pattern of bound states observed in the hadron spectrum as well as other observations from high-energy physics, people had to add different species or *flavours* of quarks. These flavours are divided into generations. The *down* and *up* flavours of quarks form the first generation, the second is formed by the *strange* and *charm* quarks and the third by the *bottom* and *top* quarks. The electrical charges of the quarks are fractionary. The *up*, *charm* and *top* quarks have charges $+\frac{2}{3}$ and the *down*, *strange* and *bottom* quarks have charges $-\frac{1}{3}$. Their masses vary a lot between generations and even within the same generation. Their values are listed in Table 2.1.

m_d	4.7 MeV
m_u	2.2 MeV
m_s	95 MeV
m_c	1.275 GeV
m_b	4.18 GeV
m_t	$\approx$ 170 GeV

TABLE 2.1: Masses of the Quarks as quoted in the Particle Data Group Tables. Details about the schemes in which these masses are define can be found in Patrignani, 2016

Following the gauge principle, one notices that in order to construct a self-consistent theory, that is to maintain the invariance of the lagrangian after an $\mathrm{SU}(3)$ gauge transformation, it is mandatory to include gauge fields transforming in the adjoint representation of $\mathrm{SU}(3)$, these are the gluonic fields. As a consequence of the non-abelianity of the symmetry group, we obtain gluonic self-interacting terms in the lagrangian, differently from what happens in the case of the photon field in QED. This last feature leads to a very different behaviour of the theory, as will be explained shortly.

The non-abelianity of the theory also causes the need for a further degree of freedom, related to the gauge fixing procedure of the theory. These are the ghosts, non-physical fields interacting with the gluons (Muta, 1998). We will not dwell on this subject, since for our interests at this dissertation, they will play no role.

The expression for the lagrangian without the ghost fields, which is, by construction, gauge-invariant, is

$$\mathcal{L}(x) = -\frac{1}{4}(G^A_{\mu\nu}(x))^2 + \bar{\psi}^{f,a}_i(x)(i\not{D} - m_f)^{ab}_{ij}\psi^{f,b}_j(x). \tag{2.1}$$

The quark fields $\psi^{f,a}_j$ are the fermionic matter fields. In the condensed notation used in the lagrangian, Eq.2.1, they carry many indices. The a, b indices are color indices running from 1 to 3. The Dirac indices i, j run from 0 to 3 and there is a flavour index f, running from 1 to 6.

When dealing with a specific flavour, we use the notation

$$\psi^{f,a}_i(x) = f^a_i(x), \tag{2.2}$$

so, for example, for the *charm* quark, $f = c$

$$\psi^{c,a}_i(x) = c^a_i(x). \tag{2.3}$$

The bar over the fermion field means

$$\bar{\psi} = \psi^\dagger \gamma^0 \tag{2.4}$$

where † means that the field is hermitian conjugated and the γ^0 is the Dirac gamma matrix γ_μ associated to the time direction, $\mu = 0$.

The field-strength $G^{ab}_{\mu\nu}(x)$ is now a matrix in color space. This means that it can be decomposed into a sum over the generators of the group of symmetry, $G^{ab}_{\mu\nu}(x) = G^{A}_{\mu\nu}(x) t^{A,ab}$. Each coefficient of this decomposition is $G^{A}_{\mu\nu}(x)$, where A is the adjoint index from SU(3), running from 1 to 8. Each of these coefficients is

$$G^{A}_{\mu\nu}(x) \equiv \partial_\mu B^A_\nu(x) - \partial_\nu B^A_\mu(x) + g f^{ABC} B^B_\mu(x) B^C_\nu(x). \tag{2.5}$$

$B^A_\mu(x)$ is the gluon field, and f^{ABC} is the structure constants of the group and g is the universal coupling constant of QCD.

The covariant derivative, also appearing in the lagrangian of QCD, Eq.2.1, is

$$D^{ab}_{\mu,ij} = \partial_\mu \delta_{ij} \delta^{ab} - i g B^A_\mu(x) \delta_{ij} t^{A\,ab}. \tag{2.6}$$

The generators of SU(3) are generally denoted by t^A matrices. They obey

$$[t^A, t^B] = i f^{ABC} t^C, \quad \mathrm{Tr}[t^A] = 0, \quad \mathrm{Tr}[t^A t^B] = \frac{\delta^{AB}}{2}, \tag{2.7}$$

where f^{ABC} is the structure constants from SU(3).

Other authors prefer the λ matrices and they are both related by

$$t^{A,ab} = \frac{\lambda^{A,ab}}{2} \tag{2.8}$$

An explicit representation of the λ matrices is provided by the Gell-Mann matrices, which are considered the standard representation, when one is needed.

The use of an index f for the flavours in the lagrangian of QCD, Eq.2.1, may give the impression that we want to express a symmetry between the different flavours, but this is not verified, as the masses given in Table 2.1 make explicit. Sometimes we can approximate in the Lagrangian the two lightest quarks, *up* and *down*, as having zero masses. Then one has explicit chiral symmetry at the lagrangian level and moreover, as the masses are equal between these two flavours, there is also a SU(2) isospin symmetry between them. This reflects itself in the hadronic spectrum, where one can observe that the mass of the neutron and the proton are almost the same, and also the masses of the different species of pions are approximately equal[2]. We can push this approximate symmetry even further, saying that the *strange* mass is negligible, and consider an $SU(3)$ symmetry in flavour space and add corrections to the quantities obtained in this approximation as powers of the *strange* quark mass over a characteristic mass scale of the theory $\Lambda_{\text{QCD}} \approx 200\text{MeV}$, to be explained below. This can be used, for example, to explain the masses of the hadrons present in the Eightfold Way of Gell-Mann. However when one adds the *charm* quark to the theory, one cannot say that its mass is negligible, since it is much larger than Λ_{QCD} and therefore the calculations have to take this fact into account from the very beginning.

Similarly to QED, where one has $\alpha = \frac{e^2}{4\pi}$ (Peskin and Schroeder, 1995), in QCD one can define

$$\alpha_S = \frac{g^2}{4\pi}. \tag{2.9}$$

The β-function which governs the running of the QCD coupling is, to one loop order,

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left(11 - \frac{2}{3}n_f\right), \tag{2.10}$$

where $n_f(\mu)$ is the numbers of active quarks.

Defining the coefficient of this order as

$$b_0 = 11 - \frac{2}{3}n_f, \tag{2.11}$$

we can write

$$\beta(g) = -\frac{g^3 b_0}{(4\pi)^2}. \tag{2.12}$$

With this, the running of the coupling constant is

$$\alpha_S(q^2) = \frac{\alpha_S}{1 + \frac{b_0\alpha_S}{2\pi}\log\left(\frac{q^2}{\mu^2}\right)}. \tag{2.13}$$

The coupling is small at large q^2 and in this regime perturbation theory

[2] In real life, with the masses of the *up* and *down* quarks are taken into account, isospin symmetry between the u and d quarks is explicitly broken at lagrangian level, and the mechanism of dynamically generating masses to the hadrons is actually responsible to make the symmetry emerge again in the hadron spectrum. So, although the masses of the down and up quark differ a lot, one has the masses of neutron and proton almost the same at the hadron spectrum level.

is meaningful and can be used to predict experimental results, such as deep-inelastic scattering observables or hadronic jet production amplitudes (Peskin and Schroeder, 1995; Muta, 1998). The fact that the coupling is small at large q^2 is referred to as the *asymptotic freedom*. At small q^2, conversely, the coupling is large and perturbation theory is meaningless. This behaviour can be traced back to the non-abelianity of the theory.

In order to remove the arbitrary renormalization scale from the formulas, we can define a scale Λ_{QCD} by

$$1 \equiv g^2 \frac{b_0}{8\pi^2} \log\left(\frac{\mu^2}{\Lambda_{\mathrm{QCD}}}\right). \tag{2.14}$$

Eq.2.13 can be then rewritten as

$$\frac{\alpha_S(q^2)}{\pi} = \frac{2}{b_0 \log\left(\frac{q^2}{\Lambda_{\mathrm{QCD}}}\right)}. \tag{2.15}$$

Λ_{QCD} is roughly the scale at which perturbation theory is no longer valid, that is the point at which non-perturbative effects take over, and its inverse correspond more or less to the size of light hadrons in natural units.

3 QCD Sum Rules

The method of QCD Sum Rules was a pioneer method proposed by the soviet physicists Shifman, Vainshtein and Zakharov (In short SVZ) in a paper published in 1979 (Shifman, Vainshtein, and Zakharov, 1979) with the aim of obtaining hadronic observables such as the mass of some states directly from the QCD lagrangian. QCD had been proposed in the late 60s and early 70s, and at that point it was already a very successful theory in the high-energy regime, that is the regime where perturbative methods were valid and could be applied, for example (Muta, 1998), in calculations of deep-inelastic lepton-hadron scattering, corrections to the cross section of e^+e^- annihilation processes and jets. In their first paper, SVZ used their method to extract properties of various mesonic states. Three years afterwards, in 1981, Reinders, Rubinstein and Yazaki (Reinders, Rubinstein, and Yazaki, 1980), working in England, reexamined the sum rules for the case of charmonium and bottomonium, and they were able to find good agreement with experiment for their masses of S and P charmonium states and the $\Upsilon - \eta_B$ splitting.

The method of SVZ is based on two sides: the operator product expansion side and the phenomenological side. On the **operator product expansion**, in short OPE, proposed by Wilson in 1969 (Wilson, 1969; Hilger, 2012; Muta, 1998) to systematically organize and regularize divergences appearing in products of local operators in the short-distance limit. QCD Sum Rules uses this OPE to express the correlator of two-currents written in terms of QCD fields (quark fields and possibly gluonic fields as well), with the quantum numbers of the hadrons one wants to describe, in the limit where the space-time points where these currents are evaluated approach each other (Pascual and Tarrach, 1984).

Within this method the correlator can be decomposed in a sum of operators multiplied by coefficients, which can be, in turn, calculated in perturbation theory. So one gets an expansion in two directions: the direction of including more and more operators in the operator expansion and the direction of calculating each of the coefficients with more precision, including more orders in perturbation theory.

The OPE is valid as an expression of the correlator in a specific kinematic region, that is the region of large euclidean momentum, large negative q^2, corresponding to a short-distance expansion, as will be shown in Chapter 3.1. The imaginary part of the correlator of two-currents can be described differently in another kinematic region, which is the positive q^2 region. There one can observe, through manipulations analogous to the ones used to obtain the Källen-Lehmann representation of two-point functions in quantum field theory, that the correlator is basically a weighted sum of the lowest-lying hadronic state propagator with the propagators of higher-energy resonances (Colangelo and Khodjamirian, 2001). This is called the **phenomenological side**. The structure of the phenomenological side as well as the reasoning behind the Ansätze used in QCD Sum rules for them are shown in Chapter 3.2.

With the aid of a dispersion relation, and some further hypotheses and Ansätze, we can connect both descriptions in hope of obtaining hadronic parameters in terms of QCD quantities, that is, quantities related to the quantum chromodynamical theory, written in terms of QCD degrees of freedom and the QCD vacuum, and at the same time universal, the so-called condensates. They are the expectation values in the QCD vacuum of the operators appearing in the OPE, which are combinations of quark and gluon fields. The standard methods for estimates of the main condensates is presented in Chapter 3.6.

If one is interested in obtaining informations about the lowest-lying state contained in the spectral sum of the phenomenological side separately from the more massive resonances, we have to find a way of disentangling the contributions from the first state from the ones of these higher-energy excitations, which in QCD form a continuum and therefore are called the continuum contribution. So, in order to obtain, for example, the mass and decay constants of a ground-state hadron, we have to further manipulate the expressions of the OPE and the phenomenological side to enhance the contributions from this ground-state with respect to the continuum. This is done mathematically through the application of the Borel transform in these expressions, as will be seen in Chapter 3.4. This introduces a non-physical parameter, or at least a parameter which is not easily interpreted in terms of QCD or the hadron theory. This parameter has dimensions of mass and is called the Borel mass, or sometimes, the inverse of it, which has the dimensions of a length in natural units, is called Borel length.

Experience shows that it is not enough to subtract the contributions from the continuum in a QCD sum rule. In order to further reduce the higher-energy contamination of the sum rule, we apply the quark-hadron duality to it (Colangelo and Khodjamirian, 2001). This consists in equating the analytical structure of the perturbative part of the OPE to the one of the phenomenological side in the q^2 plane from infinity up to a point called the effective threshold, which bears this name because it is expected to be near to the physical threshold of the continuum in the spectral density representation of the correlator, but has not quite the same value due to the imperfection of this approximation. This point is rather technical and we will lay down the arguments, physical interpretation and mathematical description of it in Chapter 3.5.

The original approach to handle the fact that we now have two parameters not fixed a priori, which are the Borel mass and the effective threshold, is to find a stability region in the plots of physical observables against those quantities (Pascual and Tarrach, 1984; Colangelo and Khodjamirian, 2001). These stabilities should show up as plateaus in the plots of physical quantities in the "Borel mass"-"effective threshold" plane, and the value of the physical quantity in this plane is taken to be the hadron observable one wants to obtain.

We add as a last remark in this chapter that the method of QCD Sum Rules has been extremely successful, although it has recently (that is since the early 2000s) lost the spotlight to different approaches, such as lattice QCD and Dyson-Schwinger Equations, in explaining hadron observables from the QCD lagrangian. As already stated, the method of QCD Sum Rules was proposed originally for mesonic systems, but it has been extended to include baryonic systems as well as the exotic systems of tetra- and pentaquarks; effects of nuclear medium density, temperature and to explore three-point functions, in order to study decay matrix elements.

3.1 The Operator Product Expansion

The Operator Product Expansion, in short OPE, was proposed by Wilson (Wilson, 1969) to regularize products of local-operators $A(x)B(0)$ which were known to be divergent when $x \to 0$.

Wilson conjectured that, in order to organize this divergence, one could write this product as a sum over a complete basis of local operators

$$\lim_{x\to 0} A(x)B(0) = \sum_d c_d(x)\mathcal{O}_d(0), \tag{3.1}$$

this equation being understood to be true after sandwiching the operators with initial and final states, although the coefficients are state independent, depending only on the operators A and B used in the left-hand side (Hilger, 2012; Pascual and Tarrach, 1984). In this form, the divergence is absorbed by the coefficient $c_d(x)$ and the expectation values of local operators $\mathcal{O}_d(0)$ are finite. The $\mathcal{O}_d(0)$ are conveniently organized by their mass dimension and spin. The number of operators in this expansion is infinite, however if one is contented with a finite precision for x near 0 only a finite number of operators will contribute. The operators A and B can be any operator in a quantum field theory. In QCD Sum Rules one is usually interested in the case where these operators are currents with the quantum numbers of a chosen hadron.

In conformal theories, the OPE can be shown to hold and the singularity structure of $c_d(x)$ is completely determined by the scale invariance of the theory and its Lorentz transformation properties.

The OPE is a short-distance expansion. When one takes the Fourier transform, one sees that this corresponds to the limit $q^2 \to -\infty$, where q is the conjugate momentum of x in the Fouerier transformation. So we conclude that this expression is valid for large euclidean values of q. In this regime perturbation theory is valid for QCD, which is our interest here, and we can calculate the $c_d(x)$ using Feynman diagrams order by order in an expansion in α_S.

$$\Pi(q) = i \int \mathrm{d}^4x \, e^{iq\cdot x} \langle \Omega | T\{J(x)J^\dagger(0)\} | \Omega \rangle = \sum_d c_d(q) \langle \Omega | \mathcal{O}_d(0) | \Omega \rangle. \tag{3.2}$$

When we sandwich 3.1 between vacuum states, we see that only spin-0 operators will survive in the right-hand side, since we have Lorentz invariance, and in QCD in particular, only colorless operators will remain, since the vacuum has no color charge. Imposing CPT symmetry, we further restrict the possible operators in the expansion to scalar operators. Examples of the operators which survive, organized by their mass dimensions d, is given by $\mathcal{O}_0 = \mathbb{1}$, $\mathcal{O}_3 = \bar{\psi}_i^a \psi_i^a = \bar{\psi}\psi$, $\mathcal{O}_4 = G^a_{\mu\nu} G^{a\mu\nu}$, where $\mathbb{1}$ is the operator with lowest dimension, ψ_i^a is the quark field and $G^a_{\mu\nu}$ the field-strength tensor from QCD, Eq.2.5.

In standard perturbation theory only the operator $\mathbb{1}$ would appear, but we know that the non-perturbative effects of QCD generate a non-vanishing quark condensate, $\langle \Omega | \bar{\psi}_i^a \psi_i^a | \Omega \rangle$ for example, so we conjecture that these operators systematically add the non-perturbative effects of the theory and introduce corrections to the usual perturbative expansion. In addition to the quark condensate one will have gluon condensates, $\langle \Omega | G^a_{\mu\nu} G^{a\mu\nu} | \Omega \rangle$, quark-gluon condensates, four-quark condensates, etc.. These condensates will appear in any sum rule for any hadron, independent of the current used, unless the coefficient c_d for the said operator is 0, of course. The condensates are thus universal and can, in principle, be considered as phenomenological parameters to be obtained by doing many different sum rules for different hadrons and an universal best-fit. In practice, sometimes, as explained in Chapter 3.6, one takes the value of condensates, for instance the quark condensate, as an external input.

Another effect of the interactions of QCD is that it makes the expansion break down at some critical value of the operators dimension d (Pascual and Tarrach, 1984). Another way to say it is that the series present in the OPE is asymptotic in QCD, so there is an optimal truncation order.

As a final remark, we notice that the approach of QCD Sum Rules does not try to solve or really even understand the problems of non-perturbative QCD, such as confinement of the mechanisms of generation of a dynamical mass in the transition from current to constituent quarks. It simply absorbs these effects by parametrizing them with the condensates and see the results in the end at the hadronic spectrum level.

3.2 The Phenomenological Side

This section follows closely Matheus, 2003; Matheus, 2006.

For a generic current, $J(x)$, with the quantum numbers of the hadron one wants to describe, we can define the two-point correlator $\Pi(q)$ as

$$\Pi(q) \equiv i \int \mathrm{d}^4x\, e^{iq\cdot x} \langle \Omega | T\{J(x) J^\dagger(0)\} | \Omega \rangle, \tag{3.3}$$

where $|\Omega\rangle$ is the true vacuum of the theory.

Writing explicitly the time ordering of Eq.3.3, one has

$$\begin{aligned} \Pi(q) = i \int \mathrm{d}^4x\, e^{iq\cdot x} \Big\{ &\theta(x_0) \langle \Omega | J(x) J^\dagger(0) | \Omega \rangle \\ &+ \theta(-x_0) \langle \Omega | J^\dagger(0) J(x) | \Omega \rangle \Big\}. \end{aligned} \tag{3.4}$$

We can consider a complete set of on-shell states, written as

$$\mathbb{1} = \sum_{i=0}^{\infty} \int \frac{d^3\vec{p}_{\alpha_i}}{2p^0_{\alpha_i}} \frac{1}{(2\pi)^3} |\alpha_i\rangle\langle\alpha_i|, \tag{3.5}$$

where we have an integration in phase space and a sum over the index i representing a sum over all other relevant quantum numbers, and we choose

$|\alpha\rangle$ as being eigenstates of the total momentum operator $\mathbb{P}$

$$\mathbb{P}|\alpha_i\rangle = p_{\alpha_i}|\alpha_i\rangle. \tag{3.6}$$

We can insert this complete set as intermediate states inside the matrix elements, between the two currents

$$\begin{aligned}\Pi(q) = i\int \mathrm{d}^4x\, e^{iq\cdot x}\int \frac{d^3\vec{p}_{\alpha_0}}{2p^0_{\alpha_0}(2\pi)^3}\Big\{&\theta(x_0)\langle\Omega|J(x)|\alpha_0\rangle\langle\alpha_0|J(0)^\dagger|\Omega\rangle \\ &+\theta(-x_0)\langle\Omega|J^\dagger(0)|\alpha_0\rangle\langle\alpha_0|J(x)|\Omega\rangle\Big\} \\ &+\text{Cont.},\end{aligned} \tag{3.7}$$

where Cont. represents the sum over everything which is not the lowest lying state which couples to the current

$$\begin{aligned}\text{Cont.} = i\int \mathrm{d}^4x\, e^{iq\cdot x}\sum_{i=1}^{\infty}\int \frac{d^3\vec{p}_{\alpha_i}}{2p^0_{\alpha_i}(2\pi)^3}\Big\{&\theta(x_0)\langle\Omega|J(x)|\alpha_i\rangle\langle\alpha_i|J(0)^\dagger|\Omega\rangle \\ &+\theta(-x_0)\langle\Omega|J^\dagger(0)|\alpha_i\rangle\langle\alpha_i|J(x)|\Omega\rangle\Big\}.\end{aligned} \tag{3.8}$$

Of course many of these states will have no overlapping with the ones the current $J(x)$ can excite from the vacuum of the theory, and consequently the matrix elements such as $\langle\Omega|J^\dagger(x)|\alpha_i\rangle$, in these cases, will be 0. However, by choosing a current with the correct quantum numbers of the hadron of interest, we ensure that the current will excite states from the vacuum of the theory which will have a non-vanishing overlapping with the states of these hadrons. From here on the index i will denote the excitations of the first state, with the lowest mass m_{α_0}, which has a non-vanishing matrix element with the states excited by the current $J(q)$, such that $m_{\alpha_0} < m_{\alpha_1} < \dots$. In QCD these higher excitations have a larger width than the first excitation, in the sense of their spectral representations, and they overlap with each other, and because of this people call these usually the *continuum* of QCD. This is a continuum in a different sense than the ones in other theories, like QED for example, where the electron and the positron in a positronium can have such high energies that they escape one another, forming in the spectral sum of the positronium a free electron-positron continuum, represented by a branch cut starting at the energy of $2m_e$. In QCD quarks are confined, so phenomenologically there is no such a thing as a free-quark continuum, and the name continuum is used for the overlapping higher order divergences somewhat abusively, although the hadrons contained in the sum may decay into other free hadrons, and in this case the continuum also contain these proper continuum states.

We do this separation of the first state and the continuum because usually in QCD Sum Rules and in this thesis in particular, we will be interested in studying the properties of the lowest-lying excitation only.

We can then apply a translation operator $U(a)$ to bring the currents to the same space-time point. The translation operator is unitary

$$U^{-1}(a)U(a) = 1. \tag{3.9}$$

It acts in the current operator $J(x)$ as

$$U(a)J(x)U^{-1}(a) = J(x+a), \tag{3.10}$$

and in the states $|\alpha_i\rangle$ as

$$U(a)|\alpha_i\rangle = e^{i\mathbb{P}\cdot a}|\alpha_i\rangle = e^{ip_{\alpha_i}\cdot a}|\alpha_i\rangle, \tag{3.11}$$

where we used the fact that $|\alpha_i\rangle$ are eigenstates of the momentum operator.

In the matrix elements we can use it to translate the current from x to 0, inserting the identity operators conveniently in the form of $U^{-1}U$, obtaining

$$\langle\Omega|J(x)|\alpha_0\rangle = \langle\Omega|U^{-1}(-x)U(-x)J(x)U^{-1}(-x)U(-x)|\alpha_0\rangle \tag{3.12}$$
$$= e^{-ip_{\alpha_0}\cdot x}\langle\Omega|J(0)|\alpha_0\rangle, \tag{3.13}$$

where we have used the fact that

$$\mathbb{P}|\Omega\rangle = 0 \Rightarrow U(a)|\Omega\rangle = |\Omega\rangle. \tag{3.14}$$

Similarly

$$\langle\alpha_0|J(x)|\Omega\rangle = e^{ip_{\alpha_0}\cdot x}\langle\alpha_0|J(0)|\Omega\rangle. \tag{3.15}$$

$\langle\Omega|J(0)|\alpha_0\rangle$ is now simply a number with dimensions of mass, and we can define

$$\langle\Omega|J(0)|\alpha_0\rangle = f_{\alpha_0}m_{\alpha_0}, \tag{3.16}$$

where the mass of the state has been used as a natural ruler of this quantity and the f_{α_0} is adimensional. The correlator $\Pi(q)$ is then

$$\Pi(q) = if_{\alpha_0}^2 m_{\alpha_0}^2 \int \mathrm{d}^4x\, e^{iq\cdot x} \int \frac{d^3\vec{p}_{\alpha_0}}{2p^0_{\alpha_0}(2\pi)^3}\left\{\theta(x_0)e^{-ip_{\alpha_0}\cdot x} + \theta(-x_0)e^{ip_{\alpha_0}\cdot x}\right\}$$
$$+\,\text{Cont.}, \tag{3.17}$$

and reverting the integration in the phase space to a d^4p integral

$$\Pi(q) = i f_{\alpha_0}^2 m_{\alpha_0}^2 \int \mathrm{d}^4x\, e^{iq\cdot x} \int \frac{\mathrm{d}^4 p_{\alpha_0}}{(2\pi)^4}\, e^{-ip_{\alpha_0}\cdot x} \frac{i}{p^2 - m_{\alpha_0}^2 + i\epsilon} + \text{Cont.}. \tag{3.18}$$

Solving the x integral and using the resulting $\delta^{(4)}(q - p_{\alpha_0})$ to solve the p_{α_0} integral, we arrive at

$$\Pi(q) = i f_{\alpha_0}^2 m_{\alpha_0}^2 \frac{i}{q^2 - m_{\alpha_0}^2 + i\epsilon} + \text{Cont.}. \tag{3.19}$$

In order to make the connection to the spectral function more clear, we can go back some steps, to Eq.3.17, but writing the continuum still as a sum over the i index. Doing the translation to the $J(x)$ operators we have

$$\begin{aligned}\Pi(q) = i \int \mathrm{d}^4x\, e^{iq\cdot x} \sum_i \int \frac{\mathrm{d}^4 p_{\alpha_i}}{(2\pi)^4} \{&\theta(x_0) e^{-ip_{\alpha_i}\cdot x} \langle\Omega|J(0)|\alpha_i\rangle\langle\alpha_i|J^\dagger(0)|\Omega\rangle \\ &+ \theta(-x_0) e^{ip_{\alpha_i}\cdot x}\langle 0|J^\dagger(0)|\alpha_i\rangle\langle\alpha_i|J(0)|\Omega\rangle\}.\end{aligned} \tag{3.20}$$

Inserting the following mathematical identity inside the sum

$$\int_0^\infty ds \int \mathrm{d}^4p\, \theta(p_0)\delta(p^2 - s)\delta^{(4)}(p - p_{\alpha_i}) = 1, \tag{3.21}$$

integrating over p and manipulating, we recognize the definition of the spectral function

$$\sum_i \int \frac{\mathrm{d}^4 p_{\alpha_i}}{(2\pi)^4} \langle\Omega|J^\dagger(0)|\alpha_i\rangle\langle\alpha_i|J(0)|\Omega\rangle(2\pi)^4\delta^{(4)}(p - p_{\alpha_i}) \equiv 2\pi\rho_J(p^2), \tag{3.22}$$

where the subscript J in $\rho_J(s)$ reminds us that each current will have different overlappings with the physical states and this will be reflected in the spectral function.

The expression for the correlator in terms of the spectral function is then

$$\begin{aligned}\Pi(q) = &\int \mathrm{d}^4x\, e^{iq\cdot x} \int_0^\infty ds\, \rho_J(s) \\ &\int \frac{\mathrm{d}^4p}{(2\pi)^3}\{i\theta(x_0)e^{-ip\cdot x}\theta(p_0)\delta(p^2 - s) \\ &\qquad + i\theta(-x_0)e^{ip\cdot x}\theta(p_0)\delta(p^2 - s)\}.\end{aligned} \tag{3.23}$$

Recognizing a Feynman propagator for a particle with mass s inside the spectral sum,

$$\Delta_F(x;s) = i\int \frac{\mathrm{d}^4p}{(2\pi)^3}\Big\{i\theta(x_0)e^{-ip\cdot x}\theta(p_0)\delta(p^2-s) \\ + i\theta(-x_0)e^{ip\cdot x}\theta(p_0)\delta(p^2-s)\Big\}, \tag{3.24}$$

and knowing that it can be written as

$$\Delta_F(x;s) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, e^{-ip\cdot x}\frac{i}{p^2-s+i\epsilon}, \tag{3.25}$$

we obtain for the correlator

$$\begin{aligned}\Pi(q^2) &= \int \mathrm{d}^4x\, e^{iq\cdot x}\int_0^\infty ds\,\rho_J(s)\Delta_F(x;s) \\ &= \int_0^\infty ds\,\rho_J(s)\frac{1}{s-q^2-i\epsilon}.\end{aligned} \tag{3.26}$$

Now we can make an Ansatz to the spectral function, based on what we expect it to look like. If we expect to have a first excitation with a relatively narrow width, we can approximate it for a δ-function. This can be followed by a continuum, as explained in the paragraph following Eq.3.8, starting at a threshold s_0, to which we will find a way later of dealing with, and which we can represent by a Heaviside θ-function. More complex forms for modelling the first excitation can and have been considered within QCD Sum Rules, for example a Breit-Wigner shape, but here we will stick to the simplest modelling with a δ-function:

$$\rho_J(s) = \lambda_{\alpha_0}\delta(s-m^2_{\alpha_0}) + \theta(s-s_0)\rho_J(s). \tag{3.27}$$

Substituting Eq.3.27 into Eq.3.26, we find

$$\Pi(q^2) = \frac{-\lambda_{\alpha_0}}{q^2-m^2_{\alpha_0}} + \int_{s_0}^\infty ds\,\frac{\rho_J(s)}{s-q^2-i\epsilon}. \tag{3.28}$$

We can identify the λ as the square of $\langle\Omega|J(0)|\alpha_0\rangle$ in Eq.3.16, and therefore

$$\lambda = m^2_{\alpha_0}f^2_{\alpha_0}. \tag{3.29}$$

Finally, we arrive at,

$$\Pi(q^2) = \frac{-m_{\alpha_0}^2 f_{\alpha_0}^2}{q^2 - m_{\alpha_0}^2 + i\epsilon} + \int_{s_0}^{\infty} ds \, \frac{\rho_J(s)}{s - q^2 - i\epsilon}, \tag{3.30}$$

which is basically the free-propagator of the hadron of interest plus the ones of the resonances.

3.3 Dispersion Relation

We have obtained two expressions for the correlator of two-currents, $\Pi(q^2)$, in Chapters 3.1 and 3.2. Using the OPE, we were able to obtain an expression for $\Pi(q^2)$ valid for large negative q^2. We have seen that there is a relation between the correlator and the spectral function $\rho_J(s)$, for $s > 0$ containing information about the hadrons with the quantum numbers of the currents $J(x)$ used to define $\Pi(q^2)$ in Eq.3.26.

There is yet another way of deriving a relation of the form of Eq.3.26 (Colangelo and Khodjamirian, 2001). Formally, we analytically extend $\Pi(q^2)$ to the complex q^2 plane. We can express the value of $\Pi(q^2)$ at an arbitrary complex point q^2 by a contour integral which avoids all poles of the $\Pi(q^2)$ function, where the q^2 point must lie inside the contour. That is

$$\Pi(q^2) = \frac{1}{2\pi i} \oint_{\Gamma} dz \, \frac{\Pi(z)}{z - q^2}. \tag{3.31}$$

Eq.3.31 is the Cauchy formula, and can be seen to be true by taking the residue of the integrand at the point where it has a pole, which is at $z = q^2$.

The contour is taken to be the circle of radius R, but when close to the real axis for positive q^2, we take a detour avoiding it and making a slit in the circle, as in the Figure 3.1. Here we have supposed that $\Pi(q^2)$ is an analytic function everywhere in the complex plane except for positive q^2 starting at a threshold $m_{\alpha_0}^2$, where we expect the discontinuity arising from the production of hadron states to appear.

The integration is then written as

$$\Pi(q^2) = \frac{1}{2\pi i} \oint_{\Gamma} dz \, \frac{\Pi(z)}{z - q^2} = \frac{1}{2\pi i} \oint_{|z|=R} dz \, \frac{\Pi(z)}{z - q^2}$$
$$+ \frac{1}{2\pi i} \int_0^R dz \, \frac{\Pi(z + i\epsilon) - \Pi(z - i\epsilon)}{z - q^2}. \tag{3.32}$$

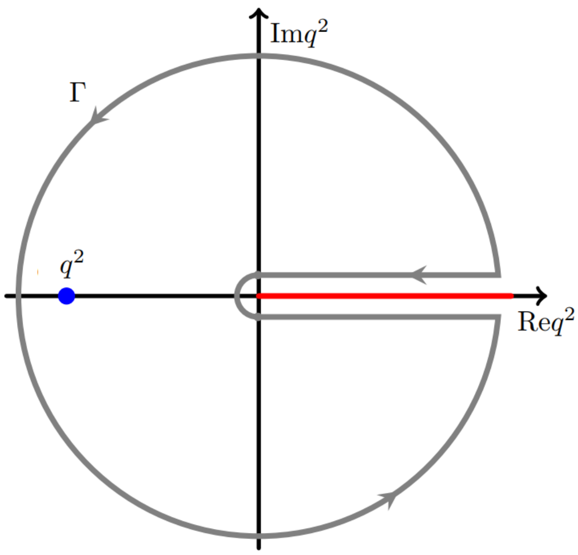

FIGURE 3.1: Contour of integration used for the dispersion relation. The outer circle has radius R. Figure taken from Hilger, 2012.

If we take the radius R going to ∞, and if $\Pi(z)$ vanishes sufficiently fast there, we can neglect the integration around the circle. If $\Pi(z)$ doesn't vanish sufficiently fast, we have to modify the equation a bit, as will be seen shortly.

Now we make use of the fact that $\Pi(q^2)$ is real for q^2 below the threshold $m^2_{\alpha_0}$. This implies that, for $q^2 > m^2_{\alpha_0}$:

$$\Pi(q^2 + i\epsilon) - \Pi(q^2 - i\epsilon) = 2i\,\mathrm{Im}\Pi(q^2), \tag{3.33}$$

which is the Schwarz reflection principle from complex analysis.

Hence, we have the dispersion relation

$$\Pi(q^2) = \frac{1}{\pi}\int_{m^2_{\alpha_0}}^{\infty} ds\,\frac{\mathrm{Im}\Pi(s)}{s - q^2 - i\epsilon} \tag{3.34}$$

Comparing Eq.3.34 to Eq.3.26, we may identify $\rho_J(s)$ over π with the imaginary part of $\Pi(q^2)$. The spectral function describes the discontinuity of this function, in a way reminiscent of the optical theorem. If we have a phenomenological model of $\rho_J(s)$ we can then calculate a $\Pi_{\text{phen.}}(q^2)$ and compare it with our $\Pi_{\text{OPE.}}(q^2)$ at some matching scale. However before doing this we need a way of suppressing the heavier resonances and this is provided by the Borel Transformation.

3.3.1 Subtracted Dispersion Relations

If $\Pi(q^2)$ is ultraviolet divergent, the imaginary part in Eq.3.34 does not vanish for $s \to \infty$ and the dispersion integral diverges (Colangelo and Khodjamirian, 2001). To cure this problem, one can subtract from $\Pi(q^2)$ the first terms of its Taylor expansion at 0 and define a subtracted correlator

$$\bar{\Pi}(q^2; N) = \Pi(q^2) - \sum_{n=0}^{N-1} \frac{\Pi^{(n)}(0)}{n!} \left(q^2\right)^n . \tag{3.35}$$

The dispersion relation is then modified to

$$\bar{\Pi}(q^2; N) = \frac{q^2}{\pi} \int_{m_{\alpha_0}^2}^{\infty} ds \left(\frac{q^2}{s}\right)^N \frac{\mathrm{Im}\Pi(s)}{(s - q^2) - i\epsilon}. \tag{3.36}$$

3.4 Borel Transformation

In order to suppress the contributions coming from higher-mass states in the phenomenological side, we apply the Borel transformation, or Borel transform, to the correlator. It is defined through the following operations

$$\Pi(M^2) \equiv \mathcal{B}_{M^2}\Pi(q^2) = \lim_{\substack{-q^2, n \to \infty \\ -q^2/n = M^2}} \frac{(-q^2)^{n+1}}{n!} \left(\frac{d}{dq^2}\right)^n \Pi(q^2). \tag{3.37}$$

The infinite number of derivatives act on polynomials killing them, which eliminates the subtraction factors needed for subtracted dispersion relations. It has the effect of transforming denominators of propagators, such as the ones in the phenomenological side of the Sum Rule, Eq.3.26 in exponential suppressions,

$$\mathcal{B}_{M^2}(q^2)^k = 0; \quad \mathcal{B}_{M^2}\left(\frac{1}{(m^2 - q^2)^k}\right) = \frac{1}{(k-1)!} \frac{e^{-m^2/M^2}}{M^{2(k-1)}}, \tag{3.38}$$

for $k > 0$.

This suppresses the contributions from higher mass excitations with respect to the lowest mass state, since the threshold s_0 is larger than $m_{\alpha_0}^2$.

$$\Pi(M^2) = f_{\alpha_0}^2 m_{\alpha_0}^2 e^{\frac{-m{\alpha_0}^2}{M^2}} + \int_{s_0}^{\infty} ds\, \rho(s) e^{\frac{-s}{M^2}}. \tag{3.39}$$

Besides this, it has the good side-effect of making the OPE converge a little faster, since it transforms higher powers of $\frac{1}{q^2}$, associated to higher order operators, into inverse Borel masses terms divided by a factorial.

3.5 The quark-hadron duality

In order to estimate the integral over the continuum, we can use the quark-hadron duality (Colangelo and Khodjamirian, 2001). The condensates are suppressed with respect to the perturbative contribution for large q^2. If we neglect their contributions completely, which are corrections, we may approximate $\Pi(q^2) \to \Pi^{(\text{pert.})}(q^2)$, where $\Pi^{(\text{pert.})}(q^2)$ is the perturbative part of the OPE. The corresponding once subtracted dispersion relations are then

$$q^2 \int_{m_{\alpha_0}}^{\infty} ds \frac{\text{Im}(s)}{s(s-q^2)} \approx \int_{4m^2}^{\infty} ds \frac{\text{Im}^{(\text{pert.})}(s)}{s(s-q^2)} \tag{3.40}$$

at $q^2 \to -\infty$. In order for them to satisfy this relation, they must have the same asymptotic behaviour also

$$\text{Im}\Pi(s) \to \text{Im}\Pi^{\text{pert.}}(s) \quad \text{at} \quad s \to -\infty. \tag{3.41}$$

This is called a local (or semi-local) quark-hadron duality.

It is then further assumed that the agreement is not so bad if one consider large but finite $-q^2$ up to a point s_{eff}.

$$q^2 \int_{s_{\text{eff}}}^{\infty} ds \frac{\rho_J(s)}{s(s-q^2)} \approx \frac{1}{\pi} q^2 \int_{s_{\text{eff}}}^{\infty} ds \frac{\text{Im}^{(\text{pert.})}(s)}{s(s-q^2)}. \tag{3.42}$$

The s_{eff} is called the effective threshold. It is a parameter not fixed a priori and so one may search for stability of the sum rule of a physical quantity with respect to it also, or simply consider a good approximation for it to be the physical threshold of the continuum. As a remark, we notice that the process of taking the Borel transform will further reduce the importance of deviations from this approximation.

In order to make use of the quark-hadron duality in the calculations, we will have to express the perturbative coefficient of the OPE, the one accompanying the $\mathbb{1}$ operator, as a dispersion integral from the threshold of $4m^2$ to infinity (where m is the quark mass). We can then equate the OPE side with the phenomenological side, in which the continuum part of the dispersion relation will be expressed from the physical threshold, s_{eff} to infinity. Subtracting one from the other, we will have an integral from $4m^2$ to s_{eff}.

3.6 Estimating the Quark and Gluon Condensates

Usually the condensates are taken from different sources (Cohen et al., 1995). For example, the light quark condensate, defined as

$$\langle\Omega|\bar{\psi}_i^a(0)\psi_i^a(0)|\Omega\rangle = \langle\bar{\psi}\psi\rangle, \tag{3.43}$$

was already known before QCD Sum Rules because of its role in the mechanism of spontaneous breaking of chiral symmetry. It also is one of the most investigated order parameters in studies of restoration of chiral symmetry for high temperature and densities.

In the vacuum $\langle\bar{\psi}\psi\rangle$ can be obtained from the PCAC formula, relating the value of it to the masses of the pion, the *up* and *down* quarks and the decay constant of the pion

$$\langle\bar{\psi}\psi\rangle = -\frac{f_\pi^2 m_\pi^2}{m_u + m_d} \approx -(240 \pm 10\text{MeV})^3. \tag{3.44}$$

The value for the gluon condensate was obtained from sum rules for the charmonium spectrum in QCD Sum Rules itself and today the accepted value is the same that has been used since the seminal works in this field. It is

$$\langle\frac{\alpha_S}{\pi}G_{\mu\nu}^a G^{a\mu\nu}\rangle = 0.012\text{GeV}^4 \pm 30\%. \tag{3.45}$$

The condensate of heavy quarks can be neglected (Colangelo and Khodjamirian, 2001). It is possible to show that (Shifman, Vainshtein, and Zakharov, 1979; Reinders, Rubinstein, and Yazaki, 1980)

$$\langle\Omega|\bar{h}h|\Omega\rangle \propto \frac{1}{m_h}\langle\frac{\alpha_S}{\pi}G_{\mu\nu}^a G^{a\mu\nu}\rangle, \tag{3.46}$$

showing that the value of this condensate is inversely proportional to its mass. For the *charm* quark, for example, the value is small and does not need to be considered.

4 The Method of Master Integrals

The method of master integrals allows one to solve loop integrals common in quantum field theory. Multiloop scattering amplitudes and lower order and radiative corrections to the coefficients of OPEs in QCD Sum Rules, for example, can be solved through the application of the method. It consists of two processes. In the first process, one reduces the integrals at hand to a basis of integrals, the so-called master integrals. The second process is the one of solving these master integrals. The process of reduction of integral to a set of basis scalar integrals go back to the method of Passarino and Veltman (Passarino and Veltman, 1979), which treated the one-loop case.

In general master integrals are universal and depend only on the topology of the diagram. Thus, one can solve, for example, one-loop calculations with one external momentum entering the diagram, and the details of the structure of the loop will be contained in the coefficients of the master integrals expansion, the basis of masters will be the same.

The process of reduction to master integrals can be further divided into two parts. The first part is expressing the products of momenta in the numerator of the integrand as combinations of terms in the denominator. This is simple straightforward algebra. In the second part, one defines a family of integrals and uses relations between integrals in the same family to arrive at a set of irreducible integrals: the so-called master integrals. The integrals in this set cannot be expressed in terms of the others and so one has a sort of linear independence and a basis, into which one can decompose the other integrals of the family. The integrals appearing in the first part are all from the same family, so for a given Feynman integral, one has at the end their expression as a linear combination of integrals from the set of master integrals.

The process of solving the master-integrals condensates the difficulty of the problem, since the method of reduction to the master integrals is straightforward. There are many methods available to solve them (Smirnov, 2012). One can use the well known Feynman and Schwinger parametrizations, Mellin-Barnes representations, differential equations for master integrals, etc. In this dissertation, we will solve some simple integrals via Schwinger parametrization, but for the more complicated ones, we will restrict ourselves to the method of differential equations, since, in our opinion, this is best suited to the case at hand and it is a versatile method, which can be applied to problems with many scales and in principle, an arbitrarily high number of loops. It is, furthermore, easily implemented in symbolic manipulation softwares, such as *Mathematica*, and some packages have already been developed to aid the problem solving in these softwares. Recent developments in the field of differential equations to master integrals have been reported, and new discoveries made it interesting in itself, from the mathematical point of view (Henn, 2015).

4.1 Reduction of the Numerator

We will introduce the Integration By Parts Relations (IBP-Relations) through the example of a one-loop massive system with one external momentum. A typical one-loop Feynman integral arising in the calculations we are considering has the following form

$$\int \mathrm{d}^d p \frac{N(p,q)}{D_1 D_2} \tag{4.1}$$

with $D_1 = p^2 - m^2$, $D_2 = (p+q)^2 - m^2$ and $N(p,q)$ is a polynomial in p and q, containing for example, $p \cdot q$ and p^2. The first part of the calculation is the reduction of the numerator $N(p,q)$, by expressing these combinations in terms of the D_1, D_2 and the external parameters q and m as

$$p^2 = D_1 + m^2, \quad p \cdot q = \frac{1}{2}\left(D_2 - D_1 - q^2\right). \tag{4.2}$$

By doing this, we will get more terms, but also some cancellations between the numerator terms and the denominators in some of these terms[1]. If D_2 cancels, for example, we will have

$$\int \mathrm{d}^d p \frac{1}{D_1}, \tag{4.3}$$

which is a scalar tadpole integral.

[1] There may be products which are irreducible in terms of the denominators at higher loops, however this will pose no difficulty to the solution of the master integrals later. One has to consider only a generalization of the family to include these irreducible products in the numerator with arbitrary powers. We will see this happening when we calculate a two loop example in Chapter 5.3

We can define a family of integrals,

$$J(\text{oneloop}, n_1, n_2; q^2) \equiv \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^{n_1}} \frac{1}{((q+p)^2 - m^2)^{n_2}} \tag{4.4}$$

with general powers in the terms of the denominators. The process of reducing the numerator through cancellations against the terms in the denominator, lowers the power of one of these terms, but the integrals are still of the same family. The calculation of Dirac traces, which appear in loop diagrams with fermions, in d-dimensions and the cancellations between numerator and denominator can be implemented in *Mathematica* using the package FeynCalc (Shtabovenko, Mertig, and Orellana, 2016).

In terms of its graphical representation, when a power of the propagator goes to zero after a cancellation, it is as if one would take a propagator of the original Feynman Diagram and shrink it to a vertex, as shown in Fig.4.1.

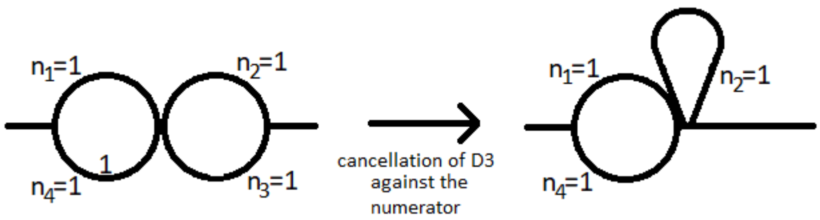

FIGURE 4.1: Graphic depiction of the process of cancelling a term in the denominator against the denominator. The bubble on the right side of the picture turns to a tadpole after one of its denominators has its power lowered to zero.

4.2 Integration by Parts Relations

Now we make the observation that integrals of the divergence of a function within a region has to vanish if the function vanishes in the boundary. Taking the function to be the integrand of Eq.4.4 times p^μ, we have

$$0 = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\partial}{\partial p_\mu} \left(p^\mu \frac{1}{(p^2 - m^2)^{n_1}} \frac{1}{((q+p)^2 - m^2)^{n_2}} \right). \tag{4.5}$$

Applying the derivative to the integrand explicitly, we get the following relation

$$\begin{aligned}0 =&(d-2n_1-n_2)J(\text{oneloop},n_1,n_2,q^2)-n_2J(\text{oneloop},n_1-1,n_2+1,q^2)\\&+2m^2n_1J(\text{oneloop},n_1+1,n_2,q^2)\\&+\left(2m^2-q^2\right)n_2J(\text{oneloop},n_1,n_2+1,q^2).\end{aligned} \tag{4.6}$$

Besides that, we can notice that the integral in Eq.4.4 is symmetric with respect to $n_1 \leftrightarrow n_2$, so that we get a similar relation exchanging n_1 by n_2. With these relations, for $n_1 \neq 0$ and $n_2 \neq 0$, we have two equations relating the integrals $J(\text{oneloop}, n_1+1, n_2, q^2)$ and $J(\text{oneloop}, n_1, n_2+1, q^2)$ to other integrals which have indices which add up to n_1+n_2, so that we will always be able to express integrals with indices adding up to n_1+n_2+1 as integrals with indices adding up to n_1+n_2.

For the special case in which one of the indices is 0, we have

$$0 = (d-2n_1)J(\text{oneloop},n_1,0,q^2)+2m^2n_1J(\text{oneloop},n_1+1,0,q^2), \tag{4.7}$$

so any $J(\text{oneloop}, n_1, 0, q^2)$ integral can be expressed in terms of $J(\text{oneloop}, 1, 0, q^2)$. This can also be checked explicitly.

We conclude that any integral can be decomposed in terms of $J(\text{oneloop}, 1, 0, q^2)$ and $J(\text{oneloop}, 1, 1, q^2)$: these are the canonical master integrals and the IBP relations implement the reduction to them for any integral of the family displayed in Eq.4.4.

In general, to obtain the all IBP relations, one uses

$$0 = \int \mathrm{d}^d p_1, \ldots, p_i, \ldots, p_M \frac{\partial}{\partial p_i^\mu} p_{ij}^\mu \prod_{k=1}^{N} \frac{1}{(p_{kl}^2-m^2)^{n_k}}, \tag{4.8}$$

where $p_{ij} = p_i - p_j$ and some of the p_i may correspond actually to external momenta and not be integrated over.

The *Mathematica* package LiteRed (Lee, 2012) finds IBP relations and with them reduces any integral to a canonical basis.

The choice of basis is not unique,and one can choose another basis on the grounds of convenience for solving them later. To find a new basis, one has simply to find a new set of linearly independent integrals.

4.3 Differential Equations for Master Integrals

After obtaining the decomposition of the integral of interest in terms of some master integrals, one is worried about the solution in terms of master integrals. The solution may already be available in the literature, such as the base for one loop integrals with one external momentum in Hooft and Veltman, 1973; Ellis and Zanderighi, 2008, but sometimes this may not be the case and one is faced with the need to obtain the master integrals and to this end, many methods are available. We will focus on the method of differential equations for master integrals, which, in our opinion is the most versatile method and well suited to our needs.

The scalar master integrals one obtains are finite in quantity and satisfy differential equations (Henn, 2015). To find out what differential equations these are, we take the derivative with respect to the external momenta or combinations thereof, like the Mandelstam variables. The derivatives will act on the integrands generating new integrals. These new integrals will still be part of the family of integrals which can be decomposed in the basis of master integrals and one may do this decomposition. In the end of this process, what one finds is that the derivative of the integral can be expressed as a linear combination of the same integrals, that is the differential equation obtained can always be written as

$$\partial_x \vec{f}(x;\varepsilon) = A(x;\varepsilon)\vec{f}(x;\varepsilon). \tag{4.9}$$

where $\vec{f}(x;\varepsilon)$ is the vector of master integrals, x is a generic variable or set of variables, constructed as a combination of the external momenta, and $A(x;\varepsilon)$ is the matrix with the coefficients of the differential equations.

To solve the differential equations, one can expand the master integrals as a Laurent series in the dimensional regularization parameter $\varepsilon = \frac{4-d}{2}$, $\vec{f}(x;\varepsilon) = \sum_i \varepsilon^i \vec{f}_i(x)$, and solve them order by order in ε. Each order should depend only on the answers of the previous orders of other master integrals. To solve a differential equation one needs a boundary condition and in this case, one has to solve the differential integral through other means in some limiting case, such as the massless limit, the limit where the incoming momentum vanishes or the threshold. Solving these integrals in these limits is much easier, in general, since there one has simply one scale and the dependence on this scale can be determined simply by dimensional analysis.

The *Mathematica* package LiteRed (Lee, 2012) includes a command to obtain the derivative of a master integral with respect to a variable constructed from the external momenta.

As already stated, the basis of integrals is not unique, and thus one may choose a different set of master integrals $\vec{g}(x;\varepsilon)$ and obtain a different matrix for the coefficients of the differential equation $B(x;\varepsilon)$. A particularly desirable choice of basis is one which factors out the ε, $B(x;\varepsilon) = \varepsilon B(x)$ and moreover makes the poles in x, which correspond to the physical singularity structure (s- or t-channel pole), manifest. Using this, one may write the differential equations in terms of forms

$$d\vec{g}(x;\varepsilon) = \varepsilon(d\tilde{B})\vec{g}(x;\varepsilon), \tag{4.10}$$

where

$$\tilde{B} = \left[\sum_k A_k \log\left(\alpha_k(x)\right)\right], \tag{4.11}$$

and

$$\alpha_k(x) = x - x_k, \tag{4.12}$$

x_k being the position of the singularities.

The answer is given then as Chen iterated integrals

$$\vec{g}(x;\varepsilon) = \mathbb{P}\exp\left[\varepsilon \int_\gamma d\tilde{B}\right]\vec{g}_0(\varepsilon), \tag{4.13}$$

where $\mathbb{P}$ denotes path ordering for integration along γ and $\vec{g}_0(\varepsilon)$ is a vector of initial conditions. For this dissertation, these more advanced techniques were not needed. Details can be seen in Henn, 2015.

Example of the reduction to a basis of master integrals and their solution will be presented in Chapters 5.2 and 5.3.

5 OPE for the Charmounium

5.1 Obtaining the Relevant Terms of the OPE

5.1.1 First Order Perturbative Contribution

We start the calculation with the Fourier transform of the correlator between two currents at points x and 0. Later as we take the Fourier transform and consider $q^2 \to -\infty$, we will be taking effectively the short-limit expansion.

$$\Pi_{\Psi}^{\mu\nu}(q) = i \int \mathrm{d}^4x \, e^{iq\cdot x} \langle\Omega| T\left\{J_{\Psi}^{\mu}(x) J_{\Psi}^{\nu}(0)\right\}|\Omega\rangle \tag{5.1}$$

The vacuum one uses is the interacting QCD vacuum and in the limit of high euclidean q^2 we expect that the perturbative contribution will be dominant and the non-perturbative effects will be included in this calculation through the inclusion of the condensates.

The current which has the correct quantum numbers for the J/Ψ meson ($J^{PC} = 1^{--}$) is

$$J_{\Psi}^{\mu} = \bar{c}(x)\gamma^{\mu}c(x), \tag{5.2}$$

which is a vector current, transforming like a vector through Lorentz

$$J_{\Psi}^{\mu} \to \Lambda_{\mu}^{\alpha} J_{\Psi}^{\mu}, \tag{5.3}$$

and parity transformations $J_{\Psi}^{\mu} \to (-1)^{\mu} J_{\Psi}^{\mu}$, where $(-1)^{\mu}$ means that the time component does not get a minus sign, whereas the spatial components do get one. Application of the charge conjugation makes $J^{\mu} \to -J^{\mu}$.

To solve the loop integrals which will appear later, we will use dimensional regularization, so at this point we generalize our integration to d space-time dimensions and keep track of the dimension wherever it appears in the calculation:

$$\Pi_{\Psi}^{\mu\nu}(q) = i \int \mathrm{d}^dx \, e^{iq\cdot x} \langle\Omega| T\left\{J_{\Psi}^{\mu}(x) J_{\Psi}^{\nu}(0)\right\}|\Omega\rangle. \tag{5.4}$$

Since the current we are using J^μ_Ψ is also the conserved electromagnetic current, from the Ward identity the correlator is necessarily transverse

$$\Pi^{\mu\nu}_\Psi(q) \equiv (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi_\Psi(q^2). \tag{5.5}$$

Contracting 5.5 with $g_{\mu\nu}$, we have

$$g_{\mu\nu}\Pi^{\mu\nu}_\Psi(q) = (q^2 - dq^2)\Pi_\Psi(q^2) = q^2(1-d)\Pi_\Psi(q^2). \tag{5.6}$$

That is, we have projected out the Lorentz structure of the correlator and now we just have to worry about one scalar function $\Pi_\Psi(q^2)$

$$\Pi_\Psi(q^2) = \frac{-i}{(d-1)q^2}\int \mathrm{d}^d x\, e^{iq\cdot x}\langle\Omega|T\left\{J^\mu_\Psi(x)J_{\Psi\,\mu}(0)\right\}|\Omega\rangle. \tag{5.7}$$

Using the Gell-Mann-Low Theorem, we can express the interacting vacuum in terms of the free vacuum and the interacting lagrangian given in terms of free fields, and so

$$\langle\Omega|T\left\{J^\mu_\Psi(x)J_{\Psi\,\mu}(0)\right\}|\Omega\rangle = \frac{\left\langle 0|T\left\{J^\mu_\Psi(x)J_{\Psi\,\mu}(0)e^{i\int \mathrm{d}^d y\,\mathcal{L}_{\mathrm{int}}(y)}\right\}|0\right\rangle}{\langle 0|T\left\{e^{i\int \mathrm{d}^d y\,\mathcal{L}_{\mathrm{int}}(y)}\right\}|0\rangle}. \tag{5.8}$$

The factor in the denominator is a normalization constant. It contains simply vacuum bubbles and in a full perturbative expansion, it will cancel against the bubbles which are disconnected from external legs (in a Feynman diagram language) coming from the numerator.

The first term, coming from no insertions of the interaction lagrangian, is given by:

$$\Pi^{\mathrm{I}}_{\Psi,\,\mathrm{pert.}}(q^2) = \frac{-i}{(d-1)q^2}\int \mathrm{d}^d x\, e^{iq\cdot x}\langle 0|T\left\{\bar{c}(x)\gamma^\mu c(x)\bar{c}(0)\gamma_\mu c(0)\right\}|0\rangle \tag{5.9}$$

$$= \frac{-i}{(d-1)q^2}\int \mathrm{d}^d x\, e^{iq\cdot x}\langle 0|T\left\{\bar{c}^a_i(x)\gamma^\mu_{ij}c^a_j(x)\bar{c}^b_k(0)\gamma_{\mu\,kl}c^b_l(0)\right\}|0\rangle, \tag{5.10}$$

which leads to

$$\Pi^{\mathrm{I}}_{\Psi,\,\mathrm{pert.}}(q^2) = \frac{i}{(d-1)q^2}\gamma^\mu_{ij}\gamma_{\mu\,kl}\int \mathrm{d}^d x\, e^{iq\cdot x}\,\overline{c^a_j(x)\bar{c}^b_k}(0)\overline{c^b_l(0)\bar{c}^a_i}(x), \tag{5.11}$$

since, from the Wick Theorem,

$$T\{\bar{c}_i^a(x)c_j^a(x)\bar{c}_k^b(0)c_l^b(0)\} = \langle 0|T\left\{c_j^a(x)\bar{c}_k^b(0)\right\}|0\rangle\langle 0|T\left\{\bar{c}_i^a(x)c_l^b(0)\right\}|0\rangle + \\ + \langle 0|T\left\{c_j^a(x)\bar{c}_k^b(0)\right\}|0\rangle : c_l^b(0)\bar{c}_i^a(x) : + \dots, \tag{5.12}$$

and we know that the vacuum expectation value of the time-ordered product of quark fields, $\langle 0|T\{\bar{\psi}_j^{bg}(y)\psi_i^{af}(x)\}|0\rangle$ is the same as the contraction of quark fields, which gives the quark propagator

$$\overbrace{\psi_i^{af}(x)\bar{\psi}_j^{bg}}(y) = \delta^{ab}\delta^{fg}S_{ij}^f(x-y) \tag{5.13}$$

$$= \delta^{ab}\delta^{fg}\int\frac{\mathrm{d}^d p}{(2\pi)^d}S_{ij}^f(p)e^{-ip(x-y)}; \qquad S^f(p) = \frac{i}{\not{p} - m_f + i\varepsilon}, \tag{5.14}$$

where f, g are the flavours of the quark fields (charm in the case at hand, in short c), a and b are color indices and i and j are Dirac indices. We also use that $\langle 0|0\rangle = 1$.

A better expression for the propagator in momentum space of a fermion for the calculation, obtained from Eq.5.14 by multiplying above and below by $\not{p} + m$ and simplifying, is

$$S^f(p) = \frac{i(\not{p} + m_f)}{p^2 - m_f^2 + i\varepsilon}. \tag{5.15}$$

The uncontracted terms in Eq.5.12 give rise to the condensate of charm quarks after taking the expectation value in the QCD vacuum, but the value of this condensate is negligible, as explained in Chapter 3.6, and we do not include it in our calculation.

Substituting the expressions for the propagators Eq.5.14 in Eq.5.11, we arrive at

$$\Pi^{\mathrm{I}}_{\Psi,\mathrm{pert.}}(q^2) = \frac{i}{(d-1)q^2}\gamma^{\mu}_{ij}\gamma_{\mu\,kl}\,\delta^{ab}\delta^{ab}\int \mathrm{d}^d x\, e^{iq\cdot x} \\ \int\frac{\mathrm{d}^d p}{(2\pi)^d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-ip\cdot x}e^{ik\cdot x}S^c_{jk}(p)S^c_{li}(k) \tag{5.16}$$

$$= \frac{3i}{(d-1)q^2}\gamma^{\mu}_{ij}\gamma_{\mu\,kl}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,S^c_{jk}(p)S^c_{li}(k) \\ \int \mathrm{d}^d x\, e^{ix\cdot(q-p+k)}, \tag{5.17}$$

where we simply used the fact that the color space is three-dimensional, and thus

$$\delta^{ab}\delta^{ab} = 3. \tag{5.18}$$

We can recognize that the exponential in Eq.5.17 gives rise to a δ-function, which imposes momentum conservation

$$\int \mathrm{d}^d x\, e^{ix\cdot(q-p+k)} = (2\pi)^d \delta^{(d)}(q-p+k). \tag{5.19}$$

Using this we can simplify Eq.5.17

$$\Pi^{\mathrm{I}}_{\Upsilon,\,\mathrm{pert.}}(q^2) = \frac{3i}{(d-1)q^2}\gamma^{\mu}_{ij}\gamma_{\mu\,kl}$$

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, S^c_{jk}(p) S^c_{li}(k)\,(2\pi)^d\delta^{(d)}(q-p+k) \tag{5.20}$$

$$= \frac{3i}{(d-1)q^2}\gamma^{\mu}_{ij}\gamma_{\mu\,kl}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, S^c_{jk}(p) S^c_{li}(p-q) \tag{5.21}$$

$$= \frac{3i}{(d-1)q^2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{Tr}\left[\gamma^{\mu} S^c(p)\gamma_{\mu} S^c(p-q)\right]. \tag{5.22}$$

After contracting back the Dirac indices we see that the result for the Wilson coefficient associated to the unity operator at first order perturbation theory in the OPE side of the QCD Sum Rule is basically given by the calculation of a Feynman diagram of a fermion-loop, in contrast to common perturbation theory calculations, where one finds tree-level contributions at first order. A graphical representation of the contribution from Eq.5.22 is shown in Fig.5.1.

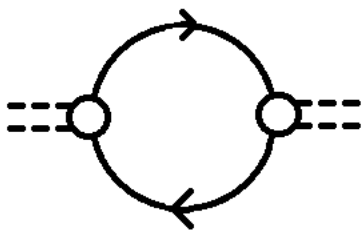

FIGURE 5.1: Graph representing the first order perturbative contribution to the OPE. The white blob means the insertion of external momentum.

The integral of Eq.5.22 has two external dimensionful parameters, which are the mass and the external momentum. For massless integrals (which have only one scale, the external momentum) or vacuum bubbles (for which there is no external momentum entering the diagram and the only scale is the internal mass) the calculation is much simpler and can usually be performed by simpler methods, like Feynman or Schwinger parametrization and solving the integrals in the parameters. When more scales come into play, we need more sophisticated methods.

We will now let this integral there and leave it to solve it in Chapter 5.2 and start the calculation of other contributions to the OPE.

5.1.2 Second Order Perturbative Contribution

In order to obtain the second-order correction to the Wilson coefficient associated to the unity operator we have to make two insertions of the interaction lagrangian of QCD, describing the interaction between quarks and gluons, into the time ordered product of the currents, Eq.5.4, according to the Gell-Mann-Low theorem, Eq. 5.8.

The relevant interaction lagrangian of QCD between quarks and gluons is

$$\mathcal{L}(x) = \frac{ig_S}{2}\bar{c}^a(x)\gamma^\alpha B^A_\alpha(x)\lambda^{A\,ab}c^b(x), \tag{5.23}$$

thus

$$\begin{aligned}
\Pi^{\mathrm{II}}_{\Psi,\,\mathrm{pert.}}(q^2) &= \frac{-i}{(d-1)q^2}\int \mathrm{d}^d x, y, z\, \frac{1}{2!}e^{iq\cdot x}\langle 0|T\left\{J^\mu(x)J_\mu(0)\mathcal{L}(y)\mathcal{L}(z)\right\}|0\rangle \\
&= \frac{-i}{2(d-1)q^2}\int \mathrm{d}^d x, y, z \left(\frac{ig_S}{2}\right)^2 e^{iq\cdot x} \\
&\quad \langle 0|T\{\bar{c}^a_i(x)\gamma^\mu_{ij}c^a_j(x)\bar{c}^b_k(0)\gamma^\mu_{\mu\,kl}c^b_l(0) \\
&\quad \bar{c}^d_m(y)\gamma^\alpha_{mn}B^A_\alpha(y)\lambda^{A\,de}c^e_n(y)\bar{c}^f_o(z)\gamma^\beta_{op}B^B_\beta(z)\lambda^{B\,fg}c^g_p(z)\}|0\rangle.
\end{aligned} \tag{5.24}$$

Again, doing the Wick contractions, where now one has more possible contractions between the charm quark fields,

$$\begin{aligned}
\Pi^{\mathrm{II}}_{\Psi,\,\mathrm{pert.}}(q^2) =& \frac{-ig_S^2}{8(d-1)q^2}\gamma^{\mu}_{ij}\gamma_{\mu\,kl}\gamma^{\alpha}_{mn}\gamma^{\beta}_{op}\lambda^{A\,de}\lambda^{B\,fg} \\
& \int \mathrm{d}^d x, y, z e^{iq\cdot x} B^A_\alpha(y) B^B_\beta(z) \\
& \Big(c^b_l(0)\bar{c}^a_i(x) c^a_j(x)\bar{c}^f_o(z) c^g_p(z)\bar{c}^d_m(y) c^e_n(y)\bar{c}^b_k(0) \\
& + c^b_l(0)\bar{c}^f_o(z) c^g_p(z)\bar{c}^d_m(y) c^e_n(y)\bar{c}^a_i(x) c^a_j(x)\bar{c}^b_k(0) \\
& + c^b_l(0)\bar{c}^d_m(y) c^e_n(y)\bar{c}^a_i(x) c^a_j(x)\bar{c}^f_o(z) c^g_p(z)\bar{c}^b_k(0) \\
& + c^b_l(0)\bar{c}^a_i(x) c^a_j(x)\bar{c}^d_m(y) c^e_n(y)\bar{c}^f_o(z) c^g_p(z)\bar{c}^b_k(0) \\
& + c^b_l(0)\bar{c}^d_m(y) c^e_n(y)\bar{c}^f_o(z) c^g_p(z)\bar{c}^a_i(x) c^a_j(x)\bar{c}^b_k(0) \\
& + c^b_l(0)\bar{c}^f_o(z) c^g_p(z)\bar{c}^a_i(x) c^a_j(x)\bar{c}^d_m(y) c^e_n(y)\bar{c}^b_k(0) \Big).
\end{aligned} \tag{5.25}$$

Expressing the quark contractions as quark propagators using the formula of Eq.5.14, and remembering that the contraction of gluon fields gives rise to the gluon propagator,

$$B^A_\mu(x) B^B_\nu(y) = \delta^{AB} D_{\mu\nu}(x-y) = \delta^{AB}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-ik\cdot(x-y)} D_{\mu\nu}(k), \tag{5.26}$$

where

$$D_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} \tag{5.27}$$

in Feynman gauge, we have:

$$\begin{aligned}
\Pi^{\mathrm{II}}_{\Psi,\,\mathrm{pert.}}(q^2) =& \frac{-ig_S^2}{4(d-1)q^2}\gamma^{\mu}_{ij}\gamma_{\mu\,kl}\gamma^{\alpha}_{mn}\gamma^{\beta}_{op}\lambda^{A\,de}\lambda^{B\,fg} \\
& \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d p_1, p_2, p_3, p_4}{((2\pi)^d)^4} \int \mathrm{d}^d x, y, z\, \delta^{AB} D_{\alpha\beta}(k) e^{-ik\cdot(y-z)} e^{iq\cdot x} \\
& (\delta^{ab}\delta^{af}\delta^{gd}\delta^{eb} S^c_{li}(p_1) S^c_{jo}(p_2) S^c_{pm}(p_3) S^c_{nk}(p_4) e^{ix\cdot p_1} e^{-i(x-z)\cdot p_2} e^{-i(z-y)\cdot p_3} e^{-iy\cdot p_4} \\
& + \delta^{bf}\delta^{dg}\delta^{ae}\delta^{ab} S^c_{lo}(p_1) S^c_{pm}(p_2) S^c_{ni}(p_3) S^c_{jk}(p_4) e^{iz\cdot p_1} e^{-i(z-y)\cdot p_2} e^{-i(y-x)\cdot p_3} e^{-ix\cdot p_4} \\
& + \delta^{bd}\delta^{ae}\delta^{af}\delta^{bg} S^c_{lm}(p_1) S^c_{ni}(p_2) S^c_{jo}(p_3) S^c_{pk}(p_4) e^{-iy\cdot p_1} e^{-i(y-x)\cdot p_2} e^{-i(x-z)\cdot p_3} e^{-iz\cdot p_4}),
\end{aligned} \tag{5.28}$$

which, upon Dirac and color indices contraction is

$$\Pi^{\mathrm{II}}_{\Psi,\mathrm{pert.}}(q^2) = \frac{-ig_S^2}{4(d-1)q^2}\mathrm{Tr}[\lambda^A\lambda^A]$$
$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d p_1, p_2, p_3, p_4}{((2\pi)^d)^4}\int \mathrm{d}^d x, y, z D_{\alpha\beta}(k)e^{-ik\cdot(y-z)}e^{iq\cdot x}$$
$$(\mathrm{Tr}[S^c(p_1)\gamma^\mu S^c(p_2)\gamma^\beta S^c(p_3)\gamma^\alpha S^c(p_4)\gamma_\mu]e^{ix\cdot p_1}e^{-i(x-z)\cdot p_2}e^{-i(z-y)\cdot p_3}e^{-iy\cdot p_4}$$
$$+\mathrm{Tr}[S^c(p_1)\gamma^\beta S^c(p_2)\gamma^\alpha S^c(p_3)\gamma^\mu S^c(p_4)\gamma_\mu]e^{iz\cdot p_1}e^{-i(z-y)\cdot p_2}e^{-i(y-x)\cdot p3}e^{-ix\cdot p_4}$$
$$+\mathrm{Tr}[S^c(p_1)\gamma^\alpha S^c(p_2)\gamma^\mu S^c(p_3)\gamma^\beta S^c(p_4)\gamma_\mu]e^{iy\cdot p_1}e^{-i(y-x)p_2}e^{-i(x-z)p_3}e^{-izp_4}). \tag{5.29}$$

Now, the exponentials again give rise to δ-functions, but they are different for the different lines in Eq.5.29. For the first two lines, the δs are the following

$$\int \mathrm{d}^d x\, e^{ix\cdot(q+p_1-p_2)} = (2\pi)^d\delta^{(d)}(q+p_1-p_2) \tag{5.30}$$
$$\int \mathrm{d}^d y\, e^{iy\cdot(p_3-p_4-k)} = (2\pi)^d\delta^{(d)}(p_3-p_4-k) \tag{5.31}$$
$$\int \mathrm{d}^d z\, e^{iz\cdot(p_2-p_3+k)} = (2\pi)^d\delta^{(d)}(p_2-p_3+k), \tag{5.32}$$

whereas for the last one, they are

$$\int \mathrm{d}^d x\, e^{ix\cdot(q+p_2-p_3)} = (2\pi)^d\delta^{(d)}(q+p_2-p_3) \tag{5.33}$$
$$\int \mathrm{d}^d y\, e^{iy\cdot(p_1-p_2-k)} = (2\pi)^d\delta^{(d)}(p_1-p_2-k) \tag{5.34}$$
$$\int \mathrm{d}^d z\, e^{iz\cdot(p_3-p_4+k)} = (2\pi)^d\delta^{(d)}(p_3-p_4+k). \tag{5.35}$$

The color factor in front is

$$\mathrm{Tr}[\lambda^A\lambda^A] = 4\mathrm{Tr}[t^At^A] = 4\frac{1}{2}\delta^{AA} = 4\frac{8}{2} = 16, \tag{5.36}$$

and putting this all together, we have

$$\Pi^{\mathrm{II}}_{\Psi,\mathrm{pert.}}(q^2) = \frac{-4ig_S^2}{(d-1)q^2}$$
$$\Bigg(2\int \frac{\mathrm{d}^d k, p}{((2\pi)^2)^d} D_{\alpha\beta}(k)\mathrm{Tr}[S^c(p)\gamma^\beta S^c(p+k)\gamma^\alpha S^c(p)\gamma_\mu S^c(p-q)\gamma^\mu]$$
$$+\int \frac{\mathrm{d}^d p_1, p_2}{((2\pi)^2)^d} D_{\alpha\beta}(p_1-p_2)\mathrm{Tr}[S^c(p_1)\gamma^\alpha S^c(p_2)\gamma^\mu S^c(p_2+q)\gamma^\beta S^c(p_1+q)\gamma_\mu]\Bigg). \tag{5.37}$$

This is the α_S correction to the sum rule and these are the most challenging integrals and the main subject of this work. To find a method for solving them, which furthermore could be easily generalized to more complicated cases, is the aim of this dissertation. These two loop integrals will be solved in a systematic way in Chapter 5.3, using the same method than the one for one loop integrals. The graphs corresponding to the integrals of this radiative correction are displayed in Fig.5.2 and Fig.5.3, where the gluon lines are depicted as curly lines.

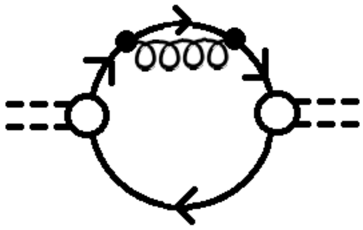

FIGURE 5.2: Second order perturbative graph corresponding to the term in the first line of Eq.5.37.

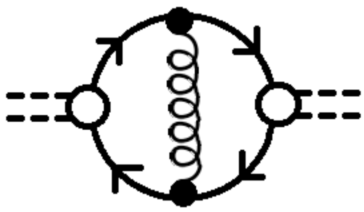

FIGURE 5.3: Second order perturbative graph corresponding to the term in the second line of Eq.5.37.

5.1.3 Gluon Condensate Contribution

The next relevant contribution to the sum rule is the gluon condensate. In order to obtain it, we start with Eq.5.24, use the Wick theorem, but rather than taking the fully contracted contribution, we take the expression where the gluon fields are not contracted. From the Gell-Mann-Low theorem one would get $\langle 0|B^A_\alpha(y)B^B_\beta(z)|0\rangle$, and this matrix element vanishes. However, as we are interested in calculating the coefficient of the OPE, which is state independent as explained just after Eq.3.1, we can substitute the sandwiching of the non-interacting vacuum by a sandwiching between the true vacuum,

$\langle 0|B^A_\alpha(y)B^B_\beta(z)|0\rangle \to \langle\Omega|B^A_\alpha(y)B^B_\beta(z)|\Omega\rangle$, to get the non-vanishing contribution which includes the non-perturbative effects (Hilger, 2012).

$$
\begin{aligned}
&\Pi_{\Psi,\langle G^2\rangle}(q^2) = \frac{-ig_S^2}{4(d-1)q^2}\mathrm{Tr}[\lambda^A\lambda^B] \\
&\quad\int \frac{\mathrm{d}^d p_1, p_2, p_3, p_4}{((2\pi)^d)^4} \int \mathrm{d}^d x, y, z\, e^{iq\cdot x}\, \langle\Omega|B^A_\alpha(y)B^B_\beta(z)|\Omega\rangle \\
&\quad\big(2\mathrm{Tr}[S^c(p_1)\gamma^\mu S^c(p_2)\gamma^\beta S^c(p_3)\gamma^\alpha S^c(p_4)\gamma_\mu]e^{ix\cdot p_1}e^{-i(x-z)\cdot p_2}e^{-i(z-y)\cdot p_3}e^{-iyp_4} \\
&\quad + \mathrm{Tr}[S^c(p_1)\gamma^\alpha S^c(p_2)\gamma^\mu S^c(p_3)\gamma^\beta S^c(p_4)\gamma_\mu]e^{iy\cdot p_1}e^{-i(y-x)\cdot p_2}e^{-i(x-z)\cdot p_3}e^{-iz\cdot p_4}\big).
\end{aligned}
\tag{5.38}
$$

The color factor in front gives

$$\mathrm{Tr}[\lambda^A\lambda^B] = 4\mathrm{Tr}[t^At^B] = 2\delta^{AB}, \tag{5.39}$$

which makes the adjoint index of the gluon fields the same.

We can now expand the gluon fields to first order in their positions using the fixed-point gauge expression, Eq.A.10, to first order in the position. We can calculate this using any gauge we like, since in the end we intend to express the expectation value of the gluon fields in terms of a gauge-invariant quantity, the gluon-condensate. The result is

$$
\begin{aligned}
\langle\Omega|B^A_\alpha(y)B^A_\beta(z)|\Omega\rangle &= \frac{1}{4}y^\sigma z^\rho\langle\Omega|G^A_{\alpha\sigma}(0)G^A_{\beta\rho}(0)|\Omega\rangle + \dots \\
&= \frac{1}{4d(d-1)}y^\sigma z^\rho(g_{\sigma\rho}g_{\alpha\beta} - g_{\sigma\beta}g_{\rho\alpha})\langle\Omega|G^A_{\mu\nu}(0)G^{A\,\mu\nu}(0)|\Omega\rangle + \dots,
\end{aligned}
\tag{5.40}
$$

where between the first and second line, we projected out the Lorentz structure of the expectation value using the fact that $G^A_{\mu\nu}$ is antisymmetric in exchange of μ by ν and the parity and time-reversal invariance of the QCD vacuum, which forbids terms like $\langle\varepsilon^{\alpha\beta\gamma\delta}G^A_{\alpha\beta}G^A_{\gamma\delta}\rangle$. This is the point where we take the short-distance expansion. We are assuming that in Eq.5.40 the most important contributions are coming from the region $y \to 0$ and $z \to 0$. The higher order terms, coming from the Taylor expansion in Eq.A.10, are associated to condensates of higher mass dimension and the assumption that the OPE converges has been used to neglect these terms.

The final expression is written as

$$\langle\Omega|B^A_\alpha(y)B^A_\beta(z)|\Omega\rangle = \frac{1}{4d(d-1)}y^\sigma z^\rho(g_{\sigma\rho}g_{\alpha\beta} - g_{\sigma\beta}g_{\alpha\rho})\langle G^2\rangle, \tag{5.41}$$

which we can then plug into Eq.5.38,

$$\Pi_{\Psi,\langle G^2\rangle}(q^2) = \frac{-ig_S^2}{2(d-1)q^2}\frac{(g_{\sigma\rho}g_{\alpha\beta}-g_{\sigma\beta}g_{\alpha\rho})}{4d(d-1)}\langle G^2\rangle \int \mathrm{d}^d y, z\, y^\sigma z^\rho$$
$$\Bigg(2\int \frac{\mathrm{d}^d p_1, p_3, p_4}{((2\pi)^d)^3}\mathrm{Tr}[S^c(p_1)\gamma^\mu S^c(p_1+q)\gamma^\beta S^c(p_3)\gamma^\alpha S^c(p_4)\gamma_\mu]e^{iy\cdot(p_3-p_4)}e^{iz\cdot(p_1+q-p_3)}$$
$$+\int \frac{\mathrm{d}^d p_1, p_2, p_4}{((2\pi)^d)^3}\mathrm{Tr}[S^c(p_1)\gamma^\alpha S^c(p_2)\gamma^\mu S^c(p_2+q)\gamma^\beta S^c(p_4)]e^{iy\cdot(p_1-p_2)}e^{iz\cdot(p_2+q-p_4)}\Bigg). \tag{5.42}$$

We recognize the derivative of the integrand as in

$$f'(p) = \int dx\, ix\, e^{ix\cdot p} f(x) = \int \frac{dx\, dk}{2\pi}\, ix\, e^{ix\cdot(p-k)} f(k) = f'(k)|_{k=p}, \tag{5.43}$$

and thus

$$\Pi_{\Psi,\langle G^2\rangle}(q^2) = \frac{-ig_S^2}{2(d-1)q^2}\frac{(g_{\sigma\rho}g_{\alpha\beta}-g_{\sigma\beta}g_{\alpha\rho})}{4d(d-1)}\langle G^2\rangle$$
$$\Bigg(2\int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\Big(\frac{\partial}{\partial p_1^\rho}\frac{\partial}{\partial p_4^\sigma}\mathrm{Tr}[S^c(p_1)\gamma^\mu S^c(p_1+q)\gamma^\beta S^c(p_3)\gamma^\alpha S^c(p_4)\gamma_\mu]\Big)_{p_3=p_4=p_1+q}$$
$$+\int \frac{\mathrm{d}^d p_2}{(2\pi)^d}\Big(\frac{\partial}{\partial p_1^\sigma}\frac{\partial}{\partial p_4^\rho}\mathrm{Tr}[S^c(p_1)\gamma^\alpha S^c(p_2)\gamma^\mu S^c(p_2+q)\gamma^\beta S^c(p_4)]\Big)_{p_1=p_2,\,p_4=p_2+q}\Bigg). \tag{5.44}$$

This is a one loop integral, and like the first order perturbative contribution, it is solved in Chapter 5.2 below. The graphical depiction of these integrals is shown in Fig.5.4 and Fig.5.5. The small red circles in the two gluon lines represent the condensates.

FIGURE 5.4: Gluon condensate graph corresponding to the term in the first line of Eq.5.44.

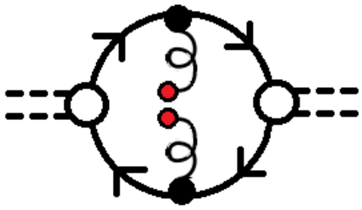

FIGURE 5.5: Gluon condensate graph corresponding to the term in the second line of Eq.5.44.

5.2 Solving the First Order Perturbative and Gluon Condensate Integrals

5.2.1 Decomposing in Terms of Master Integrals

We start with the expression from Eq.5.22 and solve the trace in Dirac space in d spacetime dimensions

$$\Pi^{\mathrm{I}}_{\Upsilon,\,\mathrm{pert.}}(q^2) = \frac{3i}{(d-1)q^2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{Tr}\left[\gamma^\mu S^c(p)\gamma_\mu S^c(p-q)\right] \tag{5.45}$$

$$= \frac{-12i}{(d-1)q^2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{\left(dm^2-(d-2)p^2-(d-2)(p\cdot q)\right)}{(p^2-m^2)\left((p+q)^2-m^2\right)}, \tag{5.46}$$

where m is the mass of the charm quark. The $i\epsilon$ in the denominators are implied and we have to take care when doing Wick rotations, so that we do not cross the poles, but we will not indicate them.

In the next step, we write the two terms in the denominator as

$$\mathrm{D}_1 = p^2 - m^2; \quad \mathrm{D}_2 = (p+q)^2 - m^2. \tag{5.47}$$

With this we rewrite the products appearing in the numerator as combinations of D_1 and D_2:

$$p^2 = \mathrm{D}_1 + m^2; \quad p \cdot q = \frac{\mathrm{D}_2 - \mathrm{D}_1 - q^2}{2}. \tag{5.48}$$

By doing this we get some cancellations between the D_n in the numerator and denominator

$$\Pi^{I}_{\Psi,\mathrm{pert.}}(q^2) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{6i(d-2)}{(1-d)\mathrm{D}_1\mathrm{D}_2} + \frac{24im^2}{(1-d)q^2\mathrm{D}_1\mathrm{D}_2} + \frac{6i(2-d)}{(1-d)q^2\mathrm{D}_1} + \frac{6i(2-d)}{(1-d)q^2\mathrm{D}_2}. \tag{5.49}$$

Next, we define scalar integrals

$$J(\text{oneloop}, n_1, n_2; q^2) \equiv \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{\mathrm{D}_1^{n_1}\mathrm{D}_2^{n_2}}, \tag{5.50}$$

and rewrite the expression for $\Pi^{\mathrm{I}}_{\Psi,\mathrm{pert.}}(q^2)$ as

$$\Pi^{\mathrm{I}}_{\Psi,\mathrm{pert.}}(q^2) = \left(\frac{6i(d-2)}{(1-d)} + \frac{24im^2}{(1-d)q^2}\right) J(\text{oneloop}, 1, 1; q^2) \tag{5.51}$$

$$+ \frac{6i(2-d)}{(1-d)q^2} J(\text{oneloop}, 1, 0) + \frac{6i(2-d)}{(1-d)q^2} J(\text{oneloop}, 0, 1; q^2). \tag{5.52}$$

We now rewrite the expressions in terms of the dimensional regularization parameter

$$d \equiv 4 - 2\varepsilon. \tag{5.53}$$

It is useful to define scalar integrals which are adimensional. In order to do this, we choose to do a change of variables $p^\mu = mp'^\mu$ and $q^\mu = mq'^\mu$ and take out from the integral factors of m. This amounts to measuring everything with dimensions of mass in units of the fermion mass. It leads to

$$\begin{aligned} J(\text{oneloop}, n_1, n_2; q^2) &= \int \frac{\mathrm{d}^{4-2\varepsilon} p}{(2\pi)^{4-2\varepsilon}} \frac{1}{(p^2 - m^2)^{n_1}((p+q)^2 - m^2)^{n_2}} \\ &= (m^{4-2\varepsilon-2(n_1+n_2)}) \int \frac{\mathrm{d}^{4-2\varepsilon} p'}{(2\pi)^{4-2\varepsilon}} \frac{1}{(p'^2 - 1)^{n_1}((p'+q')^2 - 1)^{n_2}} \\ &= \frac{(m^{4-2\varepsilon-2(n_1+n_2)})}{(2\pi)^{4-2\varepsilon}} j(\text{oneloop}, n_1, n_2; q^2). \end{aligned} \tag{5.54}$$

Where we also took out the factor $1/(2\pi)^{4-2\varepsilon}$ from the definition of $j(\text{oneloop}, n_1, n_2; q^2)$ simply by convenience.

At this point we can notice that the integrals $j(\text{oneloop}, 0, 1; q^2)$ and $j(\text{oneloop}, 1, 0; q^2)$ are the same, as can be seen through a change of variables $p \to p + q$.

The result is

$$\Pi^{\text{I}}_{\Upsilon,\text{pert.}}(q^2) = \frac{12im^{-2\varepsilon}}{(2\pi)^{4-2\varepsilon}q^2(2\varepsilon - 3)}\Big(\left(2m^2 - q^2\varepsilon + q^2\right) j(\text{oneloop}, 1, 1; q^2) + 2m^2(\varepsilon - 1) j(\text{oneloop}, 0, 1; q^2)\Big). \quad (5.55)$$

The integrals $j(\text{oneloop}, 0, 1; q^2)$ and $j(\text{oneloop}, 1, 1; q^2)$, the scalar tadpole and the scalar bubble integral, respectively, form a basis for one-loop integrals with one external momentum, labelled q here, in the sense that any integral in the family $j(\text{oneloop}, n_1, n_2; q^2)$ can be decomposed into this two integrals with the help of IBP-relations. For the first order perturbative contribution to the OPE this was not needed, because the powers of the factors in the denominators to start with were already 1 and with the cancellations against the numerator we got some to be 0. However for the gluon condensate, we will get other powers in the denominators, as explained below, and thus, one will need this relation to reduce the relevant integrals to this basis. This basis is not unique, and one can choose integrals which have desirable properties, such as being convergent in 4 spacetime dimensions or making the differential equations which they obey simpler. Henn, 2014 chooses the basis as $j(\text{oneloop}, 3, 0; q^2)$ and $j(\text{oneloop}, 2, 1; q^2)$, for example. Changing the basis is straightforward: one can simply decompose one basis in terms of the other through the same IBP relations, obtaining in this way a change of basis matrix. We have not investigated if there is a scalar product associated to this space.

Like we just said, we can decompose the expression for the gluon condensate term of the OPE, Eq.5.44, into the basis $j(\text{oneloop}, 1, 1; q^2)$ and $j(\text{oneloop}, 1, 0; q^2)$. After taking the derivatives, doing the contractions and cancelling terms in the numerator against the denominator, we obtain an expression for the integrals as a linear combination of elements of the family $j(\text{oneloop}, n_1, n_2; q^2)$, such as $j(\text{oneloop}, 2, 1; q^2)$ and $j(\text{oneloop}, 0, 3; q^2)$. The explicit expression is quite lengthy, so we will not show it here. These integrals can be then further decomposed into $j(\text{oneloop}, 1, 1; q^2)$ and $j(\text{oneloop}, 1, 0; q^2)$

by using the IBP relations, such as

$$j(\text{oneloop},0,3;q^2) = \frac{1}{2}(\varepsilon-1)\varepsilon j(\text{oneloop},0,1;q^2) \tag{5.56}$$

and

$$\begin{aligned} j(\text{oneloop},2,1;q^2) = & \frac{m^2(\varepsilon-1)\left(8m^4\varepsilon-4m^4-8m^2q^2\varepsilon+q^4\varepsilon\right)}{2q^2\left(4m^2-q^2\right)^2} j(\text{oneloop},0,1;q^2) \\ & + \frac{m^4(2\varepsilon-1)\left(2m^2+q^2\varepsilon\right)}{q^2\left(4m^2-q^2\right)^2} j(\text{oneloop},1,1;q^2). \end{aligned} \tag{5.57}$$

After this process of reduction to the basis, we get

$$\begin{aligned} \Pi_{\Psi,\langle G^2\rangle}(q^2) = & \frac{2ig_S^2\langle G^2\rangle m^{-2\varepsilon}}{3(2\pi)^{4-2\varepsilon}q^4(\varepsilon-2)(2\varepsilon-3)\left(q^2-4m^2\right)^3} \\ & \Bigg((2\varepsilon-1)\Big(72m^6-4m^4q^2(8\varepsilon^2-5\varepsilon+9) \\ & +2m^2q^4\varepsilon(8\varepsilon^2+5\varepsilon-13)-3q^6\varepsilon(\varepsilon^2-1)\Big)j(\text{oneloop},1,1;q^2) \\ & +2(\varepsilon-1)\Big(36m^6(2\varepsilon-1)+2m^4q^2\left(-16\varepsilon^3+24\varepsilon^2-26\varepsilon+9\right) \\ & +m^2q^4\varepsilon\left(6\varepsilon^2-21\varepsilon+25\right)+q^6\varepsilon(2\varepsilon-3)\Big)j(\text{oneloop},0,1;q^2)\Bigg). \end{aligned} \tag{5.58}$$

Now, solving the tadpole integral and the massive bubble integral, we will have the answers to the first order perturbative contribution and to the gluon condensate.

The graphs corresponding to the $j(\text{oneloop},1,1)$ and $j(\text{oneloop},1,0)$ integrals, are presented in Fig.5.6 and Fig.5.7 respectively.

5.2.2 The Tadpole Integral

The simplest equation to be solved is the scalar tadpole integral. It arises after the simplifications of the numerator against the denominator, when we get simply one factor in the denominator, as seen in the previous Chapter. It is a constant with respect to q^2, since no momentum flows through it and thus we will not solve it through differential equations.

FIGURE 5.6: Graph depicting the bubble integral, $j(\text{oneloop}, 1, 1)$.

FIGURE 5.7: Graph depicting the tadpole integral, $j(\text{oneloop}, 1, 0)$.

To solve it, following Grozin, 2005, we first take some factors out of the integral for convenience

$$j(\text{oneloop}, 1, 0; q^2) = \int \mathrm{d}^{4-2\varepsilon} p \frac{1}{D} \equiv -i\pi^{2-\varepsilon} V(\varepsilon); \quad D = p^2 - 1, \tag{5.59}$$

and now the aim to find $V(\varepsilon)$.

Next we do a Wick Rotation, making $p_0 = i\boldsymbol{p}_0$ and $p^2 = -\boldsymbol{p}^2$. With this, 5.59 becomes

$$\int \frac{\mathrm{d}^{4-2\varepsilon} \boldsymbol{p}}{\boldsymbol{p}^2 + 1} = \pi^{2-\varepsilon} V(\varepsilon). \tag{5.60}$$

Using the Schwinger (or α) parametrization,

$$\frac{1}{a} = \int_0^{\infty} e^{-a\alpha} d\alpha, \tag{5.61}$$

we can write $V(\varepsilon)$ as

$$V(\varepsilon) = \pi^{\varepsilon-2} \int e^{-\alpha(\boldsymbol{p}^2+1)} d\alpha \, \mathrm{d}^{4-2\varepsilon} \boldsymbol{p}. \tag{5.62}$$

The $4 - 2\varepsilon$ dimensional integral is then a product of gaussian integrals, which can be easily solved

$$\int e^{-\alpha \boldsymbol{p}^2} \mathrm{d}^{4-2\varepsilon} \boldsymbol{p} = \left(\int_{-\infty}^{+\infty} e^{-\alpha \boldsymbol{p}_1^2} \mathrm{d} \boldsymbol{p}_1 \right)^{4-2\varepsilon} = \left(\frac{\pi}{\alpha} \right)^{2-\varepsilon}. \tag{5.63}$$

Now our $V(\varepsilon)$ is

$$V(\varepsilon) = \int_0^{\infty} e^{-\alpha} \alpha^{\varepsilon-2} \, d\alpha, \tag{5.64}$$

and we can recognize the integral representation of Euler Γ function, as given in Eq.5

$$V(\varepsilon) = \Gamma(\varepsilon - 1). \tag{5.65}$$

The answer to the tadpole integral is thus

$$j(\text{oneloop}, 1, 0; q^2) = -i\pi^{2-\varepsilon}\Gamma(\varepsilon - 1). \tag{5.66}$$

The result can be expanded in a Laurent series in ε

$$j(\text{oneloop}, 1, 0; q^2) = \frac{i\pi^2}{\varepsilon} - i\pi^2 \left(\gamma_E - 1 + \log(\pi)\right) + \mathcal{O}\left(\varepsilon^1\right), \tag{5.67}$$

where γ_E is the Euler gamma constant (Eq. 10).

5.2.3 Massless Bubble Integral

Although the integrals we are interested in have massive terms in their denominators, the massless bubble integral will be needed in the solution of the massive bubble integral, $j(\text{oneloop}, 1, 1; q^2)$, in order to fix the constants of integration in the result obtained by means of its differential equation.

The solution is also simple, since the only scale appearing in the problem is the external momentum entering the diagram, and the process of solving it is analogous to the tadpole integral.

We start again defining $G(\varepsilon)$, taking out factors for convenience

$$\int \frac{\mathrm{d}^{4-2\varepsilon} p}{\mathrm{D}_1 \mathrm{D}_2} \equiv i\pi^{2-\varepsilon}(-q^2)^{-\varepsilon} G(\varepsilon); \quad \mathrm{D}_1 = (p+q)^2,\, \mathrm{D}_2 = p^2, \tag{5.68}$$

where the $(-q^2)^{-\varepsilon}$ is carrying the whole dimension of the integral, since it is the only dimensionful quantity that it can depend on. The coefficient $-\varepsilon$ is obtained simply by dimensional analysis. We will take the values of $q^2 < 0$ since in this region, before the threshold of production of two real particles, the function is analytic. Afterwards, one may extend it to the whole complex plane and obtain the discontinuity around the real axis for $q^2 > 0$.

By doing the Wick rotation in q and p and using the Schwinger parametrization, paralleling Chapter 5.2.2, where we now will have a parameter for each denominator, α_1 and α_2:

$$G(\varepsilon) = \pi^{\varepsilon-2} \int e^{-\alpha_1 (\boldsymbol{p}+\boldsymbol{q})^2 - \alpha_2 \boldsymbol{p}^2} \mathrm{d}\alpha_1, \alpha_2\, \mathrm{d}^{4-2\varepsilon} \boldsymbol{p}. \tag{5.69}$$

By doing the momentum shift in order to complete squares in the exponent,

$$\boldsymbol{p}' = \boldsymbol{p} + \frac{\alpha_1}{\alpha_1+\alpha_2}\boldsymbol{q}, \tag{5.70}$$

the momentum integration now is simply a $(4-2\varepsilon)$-fold gaussian integral, whose solution is elementary

$$G(\varepsilon) = \pi^{\varepsilon-2}\int e^{-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}}\int e^{-(\alpha_1+\alpha_2)\boldsymbol{p}^2}\mathrm{d}^{4-2\varepsilon}\boldsymbol{p}\,\mathrm{d}\alpha_1,\alpha_2 \tag{5.71}$$

$$= \int e^{-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}}(\alpha_1+\alpha_2)^{\varepsilon-2}\,d\alpha_1,\alpha_2. \tag{5.72}$$

Changing variables to η and x, by $\alpha_1 = \eta x$ $\alpha_2 = \eta(1-x)$, where x runs from 0 to 1 and η from 0 to ∞, we can recognize the definition of the Euler Γ function, Eq.5 in the η integral:

$$G(\varepsilon) = \int_0^1\int_0^\infty e^{-\eta x(1-x)}\eta^{-\varepsilon-1}\,\mathrm{d}x\,\mathrm{d}\eta \tag{5.73}$$

$$= \Gamma(\varepsilon)\int_0^1 x^{2-\varepsilon}(1-x)^{2-\varepsilon}dx. \tag{5.74}$$

In the last equation we recognize an integral representation of the Euler Beta function, Eq.6, which can be expressed in terms of Γ functions via 7. The final result is

$$G(\varepsilon) = \frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}. \tag{5.75}$$

Thus the answer is

$$\int\frac{\mathrm{d}^{4-2\varepsilon}p}{(p+q)^2p^2} = i\pi^{2-\varepsilon}(-q^2)^{-\varepsilon}\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}, \tag{5.76}$$

and as a Laurent series in ε, it is given by

$$\begin{aligned}\int \frac{\mathrm{d}^{4-2\varepsilon}p}{(p+q)^2p^2} = \frac{i\pi^2}{\varepsilon} - i\pi^2\left(\log\left(-q^2\right)+\gamma-2+\log(\pi)\right) - \\ \frac{i\pi^2\varepsilon}{12}(-6\log\left(-q^2\right)\left(\log\left(-q^2\right)+2(\gamma-2+\log(\pi))\right) \\ +\pi^2-6\left(\gamma_E^2+\log^2(\pi)+2\gamma_E(\log(\pi)-2)\right)+24(\log(\pi)-2)) \\ -\frac{i\pi^2\varepsilon^2}{12}(2\log^3\left(-q^2\right)+6(\gamma_E-2+\log(\pi))\log^2\left(-q^2\right) \\ +6(2\gamma_E-4+\log(\pi))\log(\pi)\log\left(-q^2\right) \\ +\left(6\gamma_E(\gamma_E-4)-\pi^2+48\right)\left(\log\left(-q^2\right)+\log(\pi)\right)+28\zeta(3) \\ +2\pi^2+\gamma_E\left(2\gamma_E(\gamma_E-6)-\pi^2+48\right)+2\log^3(\pi) \\ +6\gamma\log^2(\pi)-12\left(8+\log^2(\pi)\right))+\mathcal{O}\left(\varepsilon^3\right).\end{aligned} \tag{5.77}$$

5.2.4 Massive Bubble Integral with Differential Equations

Although one could solve the massive bubble integral similarly to what was done in Chapter 5.2.3, or as in Hooft and Veltman, 1973, using Schwinger parameters, we choose to derive it through the use of a differential equation, since this will introduce the method in a simple case (Henn, 2014).

In order to find out the differential equation obeyed by the $j(\text{oneloop}, 1, 1; q'^2)$ integral, we take its derivative. We could take the derivative of the expression Eq.5.54 with respect to q'^2, using the fact that

$$q'_\mu \frac{df(q'^2)}{dq'_\mu} = q'_\mu \frac{df(q'^2)}{dq'^2}\frac{dq'^2}{dq'_\mu} = 2q'^2\frac{df(q'^2)}{dq'^2}, \tag{5.78}$$

which implies that

$$\frac{df(q'^2)}{dq'^2} = \frac{q'_\mu}{2q'^2}\frac{df(q'^2)}{dq'_\mu}. \tag{5.79}$$

But it is even more convenient to define a new variable x and take the derivative with respect to it

$$q'^2 = \frac{-(1-x)^2}{x}. \tag{5.80}$$

This transformation is useful because it solves square roots present in the dependence of q'^2.

The transformation back from x to q'^2 is given by

$$x = \frac{\sqrt{4-q'^2}-\sqrt{-q'^2}}{\sqrt{4-q'^2}+\sqrt{-q'^2}}. \tag{5.81}$$

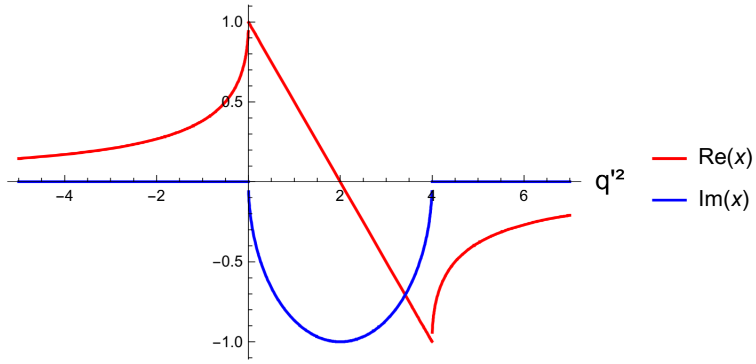

FIGURE 5.8: Relation between the x and q'^2 variables.

In this derivation, we also work for $q^2 < 0$, below the threshold of production. In this region, x is purely real and moreover positive. The graph of x as a function of q'^2 is presented in Fig.5.8.

By making the $j(\text{oneloop}, 1, 1; q'^2)$ a function of x, and taking the derivative with respect to x, the result is

$$\begin{aligned}\partial_x j(\text{oneloop}, 1, 1; x) = \frac{x+1}{2(x-1)x^2}\Big((x-1)^2 j(\text{oneloop}, 2, 1; x) \\ - x j(\text{oneloop}, 1, 1; x) + x j(\text{oneloop}, 2, 0; x)\Big).\end{aligned} \tag{5.82}$$

We notice that the derivative is expressed by a linear combination of integrals in the same family $j(\text{oneloop}, n_1, n_2; x)$. These integrals can then be decomposed in the basis $j(\text{oneloop}, 1, 0; x)$ and $j(\text{oneloop}, 1, 1; x)$ with the use of IBP-relations. After that, the reexpressed result is:

$$\begin{aligned}\partial_x j(\text{oneloop}, 1, 1; x) = \frac{-1}{x\,(x^2-1)}\Big(\Big((x^2+1)\varepsilon - 2x(\varepsilon-1)\Big) j(\text{oneloop}, 1, 1; x) \\ + 2x(\varepsilon-1) j(\text{oneloop}, 0, 1; x)\Big),\end{aligned} \tag{5.83}$$

that is, we establish a relation between the derivative of a function and itself and another function, $j(\text{oneloop}, 1, 0; x)$ (which in this case is actually a constant in x, the tadpole given by Eq.5.66). This is the differential equation we were seeking.

Although we are not indicating, it is implicit that these integrals are also functions of the dimensional regularization parameter ε. The next step is to do a Laurent series of $j(\text{oneloop}, 1, 1; x)$ around $\varepsilon = 0$. Since these integrals will be divergent, we need to start the series at negative powers. In the end, we intend to take the limit $\varepsilon \to 0$, so, one would expect us to need the result only up to order ε^0. However, as we will see, this result will be used for the two-loop calculation and there, it will be multiplied by divergent ε^{-2} terms, which makes it necessary to know the answer of this integral up to order ε^2, so that in the end we have the full result up to order ε^0. Having this in mind, we write

$$j(\text{oneloop}, 1, 1; x) = \sum_{i=-1}^{2} j_i(\text{o.l.}, 1, 1; x)\varepsilon^i + \mathcal{O}(\varepsilon^3). \tag{5.84}$$

Plugging this and also the Laurent expansion of the tadpole integral, Eq.5.67, into the differential equation, Eq.5.83. We get

$$\begin{aligned}
&\partial_x \left(\frac{j_{-1}(\text{o.l.}, 1, 1; x)}{\varepsilon} + j_0(\text{o.l.}, 1, 1; x) + j_1(\text{o.l.}, 1, 1; x)\varepsilon + j_2(\text{o.l.}, 1, 1; x)\varepsilon^2 \right) = \\
&+ \frac{2i\left(\pi^2 + i j_{-1}(\text{o.l.}, 1, 1; x)\right)}{(x^2-1)\,\varepsilon} \\
&- \frac{(x-1)^2 j_{-1}(\text{o.l.}, 1, 1; x) + 2x\left(j_0(\text{o.l.}, 1, 1; x) + i\pi^2(\gamma + \log(\pi))\right)}{x\,(x^2-1)} \\
&+ \frac{\varepsilon}{6x\,(x^2-1)} \Big(-6(x-1)^2 j_0(\text{o.l.}, 1, 1; x) - 12x j_1(\text{o.l.}, 1, 1; x) \\
&+ i\pi^2 x \left(6\gamma_E^2 + \pi^2 + 6\log^2(\pi) + 12\gamma_E \log(\pi)\right) \Big) + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{5.85}$$

Organizing order by order, we see that we need the -1-th result to solve the 0-th order equation, and so on. The -1-th equation has no dependence on higher orders in ε,

$$\partial_x j_{-1}(\text{o.l.}, 1, 1; x) = \frac{2i\left(\pi^2 + i j_{-1}(\text{o.l.}, 1, 1; x)\right)}{(x^2-1)}, \tag{5.86}$$

so we can solve it up to an integration constant

$$j_{-1}(\text{o.l.}, 1, 1; x) = \frac{c_1 x + c_1 + 2i\pi^2}{1-x}. \tag{5.87}$$

The constant has to be fixed from the high $q'^2 \to -\infty$ limit. This corresponds to the mass of the internal propagators vanishing. Eq.5.77 provides the answer for the massless bubble diagram.

We have to take into account that in the large $-q^2$ region, which corresponds to small positive x, as can be seen from Eq.5.81, x is

$$q'^2 = \frac{q^2}{m^2} \approx -\frac{1}{x} \quad \Rightarrow \quad x \approx -\frac{m^2}{q^2}, \tag{5.88}$$

from Eq.5.80[1]. To compare results for the massive bubble coming from the differential equation in the massless limit to the massless bubble answer, Eq.5.77, so that one can fix the coefficients of the differential equation general answer, one should not forget to multiply the mass factors factored out in Eq.5.54 before taking this limit.

Doing this, we find that

$$c_{-1} = -i\pi^2. \tag{5.89}$$

The solution for the coefficient of ε^{-1} is then:

$$j_{-1}(\text{o.l.}, 1, 1; x) = i\pi^2. \tag{5.90}$$

The equation for order ε^0 can now be solved plugging in the result for $j_{-1}(\text{o.l.}, 1, 1; x)$.

$$\begin{aligned} \partial_x j_0(\text{o.l.}, 1, 1; x) = &-\frac{1}{x\,(x^2-1)}\Big((x-1)^2 j_{-1}(\text{o.l.}, 1, 1; x) \\ &+2x\left(j_0(\text{o.l.}, 1, 1; x) + i\pi^2(\gamma_E + \log(\pi))\right)\Big). \end{aligned} \tag{5.91}$$

Its solution is, in general,

$$j_0(\text{o.l.}, 1, 1; x) = \frac{c_0(-(x+1)) - i\pi^2(x+1)\log(x) + 2i\pi^2(\gamma - 2 + \log(\pi))}{x-1}. \tag{5.92}$$

Fixing again the coefficient from the solution at $q'^2 \to -\infty$ limit, we have

$$c_0 = i\pi^2(\gamma_E - 2 + \log(\pi)), \tag{5.93}$$

which leads to

$$j_0(\text{o.l.}, 1, 1; x) = -\frac{i\pi^2((x-1)(\gamma_E - 2 + \log(\pi)) + (x+1)\log(x))}{x-1}. \tag{5.94}$$

[1]The expansion of x for large $-q^2$ has a nice curiosity: the coefficients of the higher powers are the Catalan numbers, which usually appear in combinatorics problems.

Repeating the same procedure for the ε^1 equation we have,

$$\partial_x j_1(\text{o.l.},1,1;x) = \frac{1}{6x\left(x^2-1\right)}\Big(-6(x-1)^2 j_0(\text{o.l.},1,1;x) - 12x j_1(\text{o.l.},1,1;x) \\ + i\pi^2 x\left(6\gamma_E^2+\pi^2+6\log^2(\pi)+12\gamma_E\log(\pi)\right)\Big). \tag{5.95}$$

The general answer is

$$j_1(\text{o.l.},1,1;x) = -\frac{i}{6(x-1)}\Big(-6ic_1(x+1) - 12\pi^2(x+1)\text{Li}_2(-x) \\ + 3\pi^2(x+1)\log(x)(\log(x)-2(2\log(x+1)+\gamma_E-2+\log(\pi))) \\ + \pi^4+6\pi^2\left(\gamma_E^2+8+2\gamma_E(\log(\pi)-2)+(\log(\pi)-4)\log(\pi)\right)\Big), \tag{5.96}$$

where $\text{Li}_m(z)$ denotes the polylogarithm, Eq.9. The coefficient is, from the massless limit,

$$c_1 = -\frac{1}{4}i\pi^2\left(2\gamma_E^2+\pi^2+16+4\gamma_E(\log(\pi)-2)+2(\log(\pi)-4)\log(\pi)\right), \tag{5.97}$$

and finally:

$$j_1(\text{o.l.},1,1;x) = \frac{i\pi^2}{12(x-1)}\Big(24(x+1)\text{Li}_2(-x)+6\gamma_E^2(x-1)+3\pi^2 x \\ + 48x + 6(x-1)\log^2(\pi) - 6(x+1)\log^2(x) \\ - 12\gamma_E(2x-2+\log(\pi)) + 12(\gamma_E-2)x\log(\pi x) \\ + 12\log(x)(x\log(\pi)+2(x+1)\log(x+1) \\ + \gamma_E - 2 + \log(\pi)) + \pi^2 - 48 + 24\log(\pi)\Big). \tag{5.98}$$

The same goes for the order ε^2 coefficient, and we get

$$\partial_x j_2(\text{o.l.},1,1;x) = -\frac{1}{6x\left(x^2-1\right)}\Big(6(x-1)^2 j_1(\text{o.l.},1,1;x) \\ + 12x j_2(\text{o.l.},1,1;x) \\ + i\pi^2 x(2\gamma_E^3+\gamma_E\pi^2+2\log^3(\pi)+6\gamma_E\log^2(\pi) \\ + 6\gamma_E^2\log(\pi)+\pi^2\log(\pi)-2\psi^{(2)}(1))\Big), \tag{5.99}$$

where $\psi^{(m)}(z)$ is the polygamma function of order m, Eq.8.

The solution is

$$
\begin{aligned}
j_2(\text{o.l.}, 1, 1; x) = \frac{(x+1)}{12(1-x)} \Bigg(& 12c_2 + \frac{i\pi^2}{x+1}\Big(24\gamma(x+1)\mathrm{Li}_2(-x) \\
& - 48(x+1)\mathrm{Li}_2(-x) - 24(x+1)\mathrm{Li}_3(-x) - 48(x+1)\mathrm{Li}_3(x+1) \\
& + 24(x+1)\log(\pi)\mathrm{Li}_2(-x) + 48(x+1)\mathrm{Li}_2(-x)\log(x+1) \\
& + 48(x+1)\mathrm{Li}_2(x+1)\log(x+1) + 2(x+1)\log^3(x) \\
& + 6(x+1)\log^2(\pi)\log(x) - 6(x+1)\log(\pi)\log^2(x) \\
& - 6\gamma_E(x+1)\log^2(x) + 12(x+1)\log^2(x) \\
& + 24(x+1)\log(-x)\log^2(x+1) + 24(x+1)\log(x)\log^2(x+1) \\
& - 12(x+1)\log^2(x)\log(x+1) + 12\gamma_E(x+1)\log(\pi)\log(x) \\
& - 24(x+1)\log(\pi)\log(x) + 24(x+1)\log(\pi)\log(x)\log(x+1) \\
& + 6\gamma_E^2(x+1)\log(x) - \pi^2(x+1)\log(x) - 24\gamma_E(x+1)\log(x) \\
& + 48(x+1)\log(x) + 4\pi^2(x+1)\log(x+1) \\
& + 24\gamma_E(x+1)\log(x)\log(x+1) - 48(x+1)\log(x)\log(x+1) \\
& - 4\gamma_E^3 + 24\gamma_E^2 - 2\gamma_E\pi^2 + 4\pi^2 - 96\gamma_E + 192 - 4\log^3(\pi) \\
& - 12\gamma_E\log^2(\pi) + 24\log^2(\pi) - 12\gamma_E^2\log(\pi) - 2\pi^2\log(\pi) \\
& + 48\gamma_E\log(\pi) - 96\log(\pi) + 4\psi^{(2)}(1)\Big)\Bigg),
\end{aligned} \tag{5.100}
$$

the free coefficient is fixed to be

$$
\begin{aligned}
c_2 = \frac{1}{12} i\pi^2 \Big(& 28\zeta(3) + 2\gamma_E^3 - 96 + 6\gamma_E^2(\log(\pi) - 2) + 3\pi^2(\log(\pi) - 2) \\
& + 2\log(\pi)(24 + (\log(\pi) - 6)\log(\pi)) \\
& + 3\gamma_E\left(\pi^2 + 16 + 2(\log(\pi) - 4)\log(\pi)\right)\Big)
\end{aligned} \tag{5.101}
$$

and the final answer is

$$
\begin{aligned}
j_2(\text{o.l.},1,1;x) = \frac{i\pi^2(x+1)}{12(1-x)}\Bigg(&\frac{1}{x+1}\Big(24\gamma_E(x+1)\mathrm{Li}_2(-x)-48(x+1)\mathrm{Li}_2(-x)\\
&-24(x+1)\mathrm{Li}_3(-x)-48(x+1)\mathrm{Li}_3(x+1)+24(x+1)\log(\pi)\mathrm{Li}_2(-x)\\
&+48(x+1)\mathrm{Li}_2(-x)\log(x+1)+48(x+1)\mathrm{Li}_2(x+1)\log(x+1)\\
&+2(x+1)\log^3(x)+6(x+1)\log^2(\pi)\log(x)-6(x+1)\log(\pi)\log^2(x)\\
&-6\gamma_E(x+1)\log^2(x)+12(x+1)\log^2(x)\\
&+24(x+1)\log(-x)\log^2(x+1)+24(x+1)\log(x)\log^2(x+1)\\
&-12(x+1)\log^2(x)\log(x+1)+12\gamma_E(x+1)\log(\pi)\log(x)\\
&-24(x+1)\log(\pi)\log(x)+24(x+1)\log(\pi)\log(x)\log(x+1)\\
&+6\gamma_E^2(x+1)\log(x)-\pi^2(x+1)\log(x)\\
&-24\gamma_E(x+1)\log(x)+48(x+1)\log(x)+4\pi^2(x+1)\log(x+1)\\
&+24\gamma_E(x+1)\log(x)\log(x+1)-48(x+1)\log(x)\log(x+1)\\
&-4\gamma_E^3+24\gamma_E^2-2\gamma_E\pi^2+4\pi^2-96\gamma_E+192-4\log^3(\pi)-12\gamma\log^2(\pi)\\
&+24\log^2(\pi)-12\gamma_E^2\log(\pi)-2\pi^2\log(\pi)+48\gamma_E\log(\pi)\\
&-96\log(\pi)+4\psi^{(2)}(1)\Big)+28\zeta(3)+2\gamma_E^3-96+6\gamma_E^2(\log(\pi)-2)\\
&+3\pi^2(\log(\pi)-2)+2\log(\pi)(24+(\log(\pi)-6)\log(\pi))\\
&+3\gamma_E\left(\pi^2+16+2(\log(\pi)-4)\log(\pi)\right)\Bigg). \qquad (5.102)
\end{aligned}
$$

5.2.5 Answer for the First Order and Gluon-Condensate Integrals

Now, with the answers for the scalar tadpole and scalar massive bubble integral at hand, we can simply use Eq.5.55 and Eq.5.58 to obtain the answers for the first order perturbative contribution and the gluon condensate contribution to the OPE, respectively.

Since in Eq.5.55 and in Eq.5.58 there are no poles in ε in the coefficients of the basis expansion, the answer for the scalar massive bubble integral is needed only to order ε^0, as commented just before Eq.5.84.

The answer for the scalar tadpole is given by Eq.5.67, and the answer for the scalar bubble integral is, from Eq.5.94 and Eq.5.90,

$$j(\text{oneloop},1,1;x) = \frac{i\pi}{\varepsilon} - \frac{i\pi^2((x-1)(\gamma_E - 2 + \log(\pi)) + (x+1)\log(x))}{x-1} + \mathcal{O}(\varepsilon). \tag{5.103}$$

Eq.5.103 and Eq.5.67 both agree with Hooft and Veltman, 1973; Ellis and Zanderighi, 2008.

Substituting these expressions for the scalar tadpole and scalar bubble integral into Eq.5.55, and translating back from the x variable to the momentum variable $q^2 = q'^2 m^2$, via the transformation from Eq.5.81, we get for first order perturbative contribution

$$\begin{aligned}\Pi^{\mathrm{I}}_{\Psi,\text{pert.}}(q^2) &= \frac{1}{4\pi^2\varepsilon} \\ &- \frac{1}{12\pi^2 q^2}\left(3\sqrt{1-\frac{4m^2}{q^2}}\left(2m^2+q^2\right)\log\left(\frac{2m^2 - q^2\left(1+\sqrt{\frac{q^2-4m^2}{q^2}}\right)}{2m^2}\right)\right. \\ &\left. - 12m^2 + 3q^2\log(m^2) + q^2(3\gamma_E - 5(1+\log(2)) + 3\log(\pi))\right) \\ &+ \mathcal{O}\left(\varepsilon^1\right).\end{aligned} \tag{5.104}$$

For $s = q^2 > 4m^2$ the expression from Eq.5.104 presents a branch cut due to argument of the logarithm becoming negative. We can then calculate the discontinuity across this branch cut and the answer is well known (Matheus, 2003; Schwinger, 1998; Källén and Sabry, 1955)

$$\mathrm{Im}\Pi^{\mathrm{I}}_{\Psi,\text{pert.}}(s) = \frac{\sqrt{1-\frac{4m^2}{s}}\,(2m^2+s)}{4\pi s}\Theta(s-4m^2). \tag{5.105}$$

Substituting the answers for the tadpole and scalar bubble integral into Eq.5.58, we find for the gluon condensate contribution

$$\Pi_{\Psi,\langle G^2\rangle}(q^2) = \frac{g_S^2\langle G^2\rangle}{48\pi^2 q^4\left(q^2-4m^2\right)^3}$$
$$\Bigg(48m^6 - 28m^4q^2 + 8m^2q^4 - q^6$$
$$+ 12m^4\left(q^2-2m^2\right)\sqrt{1-\frac{4m^2}{q^2}}\log\left(\frac{2m^2-q^2\left(1+\sqrt{\frac{q^2-4m^2}{q^2}}\right)}{2m^2}\right)\Bigg)$$
$$+\mathcal{O}\left(\varepsilon^1\right). \tag{5.106}$$

Eq.5.106 agrees with Colangelo and Khodjamirian, 2001, which quotes it as

$$\Pi_{\Psi,\langle G^2\rangle}(q^2) = \frac{\langle \alpha_S G^a_{\mu\nu}G^{a\mu\nu}\rangle}{48\pi q^4} f(a), \tag{5.107}$$

where $a = 1 - 4m^2/q^2$ and

$$f(a) = \frac{3(a+1)(a-1)^2}{2a^{5/2}}\log\left(\frac{\sqrt{a}+1}{\sqrt{a}-1}\right) - \frac{3a^2-2a+3}{a^2}. \tag{5.108}$$

For a massless fermion, the result is

$$\lim_{m\to 0}\Pi_{\Psi,\langle G^2\rangle}(q^2) = \frac{-1}{12q^4}\left\langle\frac{\alpha_S G^2}{\pi}\right\rangle. \tag{5.109}$$

5.3 Decomposing the Second Order Perturbative Integral into Master Integrals

For the second order perturbative integral, the process is similar to the one for the first order perturbative and gluon condensate integrals, only more involved. We start from Eq.5.37 and calculate the traces in d spacetime dimensions. Here we quote again Eq.5.37:

$$\Pi^{\mathrm{II}}_{\Psi,\mathrm{pert.}}(q^2) = \frac{-4ig_S^2}{(d-1)q^2}$$
$$\left(2\int \frac{\mathrm{d}^dk,p}{((2\pi)^2)^d} D_{\alpha\beta}(k)\mathrm{Tr}[S^c(p)\gamma^\beta S^c(p+k)\gamma^\alpha S^c(p)\gamma_\mu S^c(p-q)\gamma^\mu]\right.$$
$$\left. + \int \frac{\mathrm{d}^dk,p}{((2\pi)^2)^d} D_{\alpha\beta}(k-p)\mathrm{Tr}[S^c(p)\gamma^\alpha S^c(k)\gamma^\mu S^c(k+q)\gamma^\beta S^c(p+q)\gamma_\mu]\right). \tag{5.110}$$

We calculate separately the contributions from the first and second terms in Eq.5.110, although, as we will see, we will be able to decompose them into the same set of master integrals. We will call the first term twoloopA and the second term twoloopB.

Associated with the twoloopA term, we define the family

$$J(\text{twoloopA}, n_1, n_2, n_3, n_4, n_5, n_6; q^2) \equiv$$
$$\int \frac{\mathrm{d}^dp}{(2\pi)^d}\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(k\cdot q)^{-n_6}}{\mathrm{D}_1^{n_1}\mathrm{D}_2^{n_2}\mathrm{D}_3^{n_3}\mathrm{D}_4^{n_4}\mathrm{D}_5^{n_5}}. \tag{5.111}$$

with $\mathrm{D}_1 = p^2 - m^2$, $\mathrm{D}_2 = (p+k)^2 - m^2$, $\mathrm{D}_3 = p^2 - m^2$, $\mathrm{D}_4 = (p-q)^2 - m^2$ and $\mathrm{D}_5 = k^2$. We can see that D_1 and D_3 are actually the same and we could eliminate one of them, but we will keep them both, but pass the exponent of D_3 to D_1 whenever it arises, keeping the exponent of D_3 to be always zero, for better organization.

The reason we needed to define this family including a numerator term $k\cdot q$ is because this product is irreducible, which means that it cannot be written as a combination of the terms in the denominator. This is not a problem, as long as n_6 is negative, as in our case (Smirnov, 2012). Through the IBP relations they will be reduced to master integrals with only denominators.

After calculating the trace of the integrand and substituting the products of momenta by combinations of the D_n, we find the numerator of integrand

$$\text{Num. of twoloopA} = -4\Big(2(d-2)^2 k\cdot q\,\mathrm{D}_1$$
$$+4m^2(-(d-2)(\mathrm{D}_2+\mathrm{D}_4-\mathrm{D}_5)+(d-2)s+2\mathrm{D}_1)$$
$$+(d-2)^2\left(\mathrm{D}_1^2-(\mathrm{D}_2-\mathrm{D}_5)(s-\mathrm{D}_4)\right)+16m^4\Big) \tag{5.112}$$

and we can cancel terms of the numerator against the denominator and change $d \to 4 - 2\varepsilon$. The result is

$$
\begin{aligned}
\Pi^{\mathrm{II},a}_{\Psi,\mathrm{pert}}(q^2) = \frac{-4ig_S^2}{(3-2\varepsilon)q^2}\Bigg(
&16\Big(-2m^2 J(\text{twoloopA},2,1,0,1,1,0;q^2)\left(2m^2 - s\varepsilon + s\right) \\
&+ (\varepsilon-1)J(\text{twoloopA},2,0,0,1,1,0;q^2)\left(s(\varepsilon-1) - 2m^2\right) \\
&+ (\varepsilon-1)J(\text{twoloopA},2,1,0,1,0,0;q^2)\left(2m^2 - s\varepsilon + s\right) \\
&- 2m^2(\varepsilon-1)J(\text{twoloopA},2,1,0,0,1,0;q^2) \\
&- 2m^2 J(\text{twoloopA},1,1,0,1,1,0,q^2) \\
&- (\varepsilon-1)^2 J(\text{twoloopA},0,1,0,1,1,0;q^2) \\
&- 2(\varepsilon-1)^2 J(\text{twoloopA},1,1,0,1,1,-1;q^2) \\
&- (\varepsilon-1)^2 J(\text{twoloopA},2,0,0,0,1,0;q^2) \\
&+ (\varepsilon-1)^2 J(\text{twoloopA},2,1,0,0,0,0;q^2)\Big)\Bigg). \qquad (5.113)
\end{aligned}
$$

We can define adimensional $j(\text{twoloopA}, n_1, n_2, n_3, n_4, n_5, n_6; q^2)$ integrals taking out masses, factors of $(2\pi)^d$, like we have done in the one loop case in Eq.5.54, and, in order to get rid of the Euler gamma constants γ_E , $i\pi^2$ and $\log(\pi)$ appearing in the calculation, we can take out a factor of $e^{-2\gamma_E\varepsilon}(i\pi^{2-\varepsilon})^2$

$$
\begin{aligned}
J(\text{twoloopA}, n_1, n_2, n_3, n_4, n_5, n_6; q^2) &\equiv (m)^{2(4-2\varepsilon)-2(n_1+n_2+n_3+n_4+n_5+n_6)} \\
&\frac{e^{-2\gamma_E\varepsilon}(i\pi^{2-\varepsilon})^2}{(2\pi)^{2(4-2\varepsilon)}} j(\text{twoloopA}, n_1, n_2, n_3, n_4, n_5, n_6; q^2). \qquad (5.114)
\end{aligned}
$$

We can then use the IBP relations for this family and reduce them to a set of master integrals.

$$
\begin{aligned}
\Pi^{\mathrm{II},a}_{\Psi,\,\text{pert.}}(q^2) &= \frac{-4ig_S^2 e^{-\varepsilon\gamma_E}(i\pi^{2-\varepsilon})^2}{(2\pi)^{2(4-2\varepsilon)}(3-2\varepsilon)(1-2\varepsilon)q^2\left(\frac{q^2}{m^2}-4\right)} \\
&\Bigg(-\frac{8(\varepsilon-1)^2\left(m^2\right)^{-2\varepsilon} j(\text{twoloopA},0,1,0,1,0,0;q^2)}{q^2\varepsilon} \\
&\qquad \Big(-8m^4+4m^2q^2(\varepsilon-1)\left(4\varepsilon^2-6\varepsilon+1\right)+q^4\varepsilon(3-2\varepsilon)\Big) \\
&\quad -8(\varepsilon-1)(2\varepsilon-3)\left(m^2\right)^{-2\varepsilon-1} j(\text{twoloopA},1,1,0,1,0,0;q^2) \\
&\qquad \Big(m^4(8\varepsilon-4)+2m^2q^2\left(-2\varepsilon^2+\varepsilon+1\right)+q^4(\varepsilon-1)\Big) \\
&\quad -\frac{8(2\varepsilon-1)\left(m^2\right)^{-2\varepsilon} j(\text{twoloopA},0,1,0,1,1,0;q^2)}{q^2\varepsilon} \\
&\qquad \Big(8m^4(3\varepsilon-2)+4m^2q^2\varepsilon(2\varepsilon(2\varepsilon-5)+5)-q^4(\varepsilon-1)(\varepsilon(4\varepsilon-9)+4)\Big) \\
&\quad -\frac{8\left(m^2\right)^{-2\varepsilon-1} j(\text{twoloopA},0,2,0,1,1,0;q^2)}{q^2\varepsilon} \\
&\qquad \Big(16m^6(2\varepsilon-1)+8m^4q^2\left(4(\varepsilon-2)\varepsilon^2+\varepsilon+1\right) \\
&\qquad +8m^2q^4(\varepsilon(-2(\varepsilon-3)\varepsilon-5)+1)+q^6(\varepsilon-1)\varepsilon(2\varepsilon-3)\Big)\Bigg).
\end{aligned}
\tag{5.115}
$$

The master integrals needed to calculate this diagram will be referred to as simple sunset for $j(\text{twoloopA},0,1,0,1,1,0;q^2)$, sunset with doubled propagator for $j(\text{twoloopA},0,2,0,1,1,0;q^2)$, bubble-tadpole for $j(\text{twoloopA},1,1,0,1,0,0;q^2)$ and double-tadpole for $j(\text{twoloopA},0,1,0,1,0,0;q^2)$. To understand the names, we can simply look at the explicit representation of the integrals and from this we can draw representations of them in graphs. The graphical depiction of the integrals are presented in Figs.5.9 to 5.13: Fig.5.9 presents the double-tadpole diagram, Fig.5.10 the double-bubble, which will show up later, Fig.5.11 the bubble tadpole, the simple sunset diagram is in Fig.5.12 and the sunset with doubled propagator in Fig.5.13.

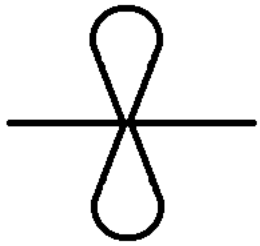

FIGURE 5.9: Master Integral $j(\text{twoloopB}, 0, 0, 1, 1, 0; q^2)$ or $j(\text{twoloopA}, 0, 1, 0, 1, 0, 0; q^2)$.

FIGURE 5.10: Master Integral $j(\text{twoloopB}, 1, 1, 1, 1, 0; q^2)$.

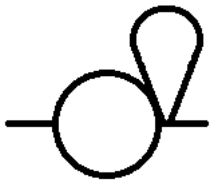

FIGURE 5.11: Master Integral $j(\text{twoloopB}, 0, 1, 1, 1, 0; q^2)$ or $j(\text{twoloopA}, 1, 1, 0, 1, 0, 0; q^2)$.

FIGURE 5.12: Master Integral $j(\text{twoloopB}, 0, 1, 1, 0, 1; q^2)$ or $j(\text{twoloopA}, 0, 1, 0, 1, 1, 0; q^2)$.

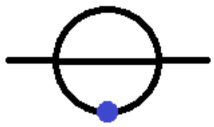

FIGURE 5.13: Master Integral $j(\text{twoloopB}, 0, 2, 1, 0, 1; q^2)$ or $j(\text{twoloopA}, 0, 2, 0, 1, 1, 0; q^2)$.

The explicit representation of the double-tadpole integral is

$$
\begin{aligned}
J(\text{twoloopA}, 0, 1, 0, 1, 0, 0; q^2) &= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(k \cdot q)^0}{\mathrm{D}_1^0 \mathrm{D}_2^1 \mathrm{D}_3^0 \mathrm{D}_4^1 \mathrm{D}_5^0} \\
&= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{((p+k)^2 - m^2)((p-q)^2 - m^2)} \\
&= \int \frac{\mathrm{d}^d p'}{(2\pi)^d} \frac{1}{p'^2 - m^2} \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \frac{1}{k'^2 - m^2},
\end{aligned}
\tag{5.116}
$$

where we made the substitution $p - q \to p'$ and $p + k \to k'$ in the last line. We can recognize that the two integrals are scalar tadpole integrals and this justifies naming it double tadpole. For the sunset integral

$$\begin{aligned} J(\text{twoloopA}, 0, 1, 0, 1, 1, 0; q^2) &= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(k \cdot q)^0}{\mathrm{D}_1^0 \mathrm{D}_2^1 \mathrm{D}_3^0 \mathrm{D}_4^1 \mathrm{D}_5^1} \\ &= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{((p+k)^2 - m^2)((p-q)^2 - m^2)(k^2)}. \end{aligned} \tag{5.117}$$

The momenta in integral in Eq.5.117 are entangled and a change of variables will not factor it into two integrals. This is a true two loop integral: the scalar sunset integral, where one of the masses of the internal propagators is 0 and the other two are m. The names for the other master integrals can be understood similarly, bubble-tadpole being a product of a scalar bubble integral and a scalar tadpole integral.

We can proceed along similar lines with the second term of Eq.5.110. Associated to it, we define the family

$$J(\text{twoloopB}, n_1, n_2, n_3, n_4, n_5; q^2) \equiv \int \frac{\mathrm{d}^d p, k}{(2\pi)^{2d}} \frac{1}{\mathrm{D}_1^{n_1} \mathrm{D}_2^{n_2} \mathrm{D}_3^{n_3} \mathrm{D}_4^{n_4} \mathrm{D}_5^{n_5}} \tag{5.118}$$

with $\mathrm{D}_1 = (q+k)^2 - m^2$, $\mathrm{D}_2 = (q+p)^2 - m^2$, $\mathrm{D}_3 = k^2 - m^2$, $\mathrm{D}_4 = p^2 - m^2$ and $\mathrm{D}_5 = (k-p)^2$. Calculating the trace, replacing the products of momenta by combinations of the D_n and cancelling terms in the numerator against the denominator, we get:

$$\begin{aligned}
\Pi^{\mathrm{II},b}_{\Psi,\,\mathrm{pert.}}(q^2) &= -4\left(2m^2+s(2-2\varepsilon)\right)J(\text{twoloopB},0,1,1,1,1;q^2)\\
&-4\left(2m^2+s(2-2\varepsilon)\right)J(\text{twoloopB},1,0,1,1,1;q^2)\\
&-4\left(2m^2+s(2-2\varepsilon)\right)J(\text{twoloopB},1,1,0,1,1;q^2)\\
&-4\left(2m^2+s(2-2\varepsilon)\right)J(\text{twoloopB},1,1,1,0,1;q^2)\\
&-2\left(s\left((4-2\varepsilon)^2-10(4-2\varepsilon)+16\right)-8m^2\varepsilon\right)J(\text{twoloopB},1,1,1,1,0;q^2)\\
&-4\left(2m^2-s\right)\left(4m^2+s(2-2\varepsilon)\right)J(\text{twoloopB},1,1,1,1,1;q^2)\\
&+4(2-2\varepsilon)J(\text{twoloopB},0,0,1,1,1;q^2)+4(2-2\varepsilon)J(\text{twoloopB},0,1,0,1,1;q^2)\\
&+2\left((4-2\varepsilon)^2-6(4-2\varepsilon)+8\right)J(\text{twoloopB},0,1,1,0,1;q^2)\\
&-4(2-2\varepsilon)J(\text{twoloopB},0,1,1,1,0;q^2)\\
&+2\left((4-2\varepsilon)^2-6(4-2\varepsilon)+8\right)J(\text{twoloopB},1,0,0,1,1;q^2)\\
&+4(2-2\varepsilon)J(\text{twoloopB},1,0,1,0,1;q^2)-4(2-2\varepsilon)J(\text{twoloopB},1,0,1,1,0;q^2)\\
&+4(2-2\varepsilon)J(\text{twoloopB},1,1,0,0,1;q^2)-4(2-2\varepsilon)J(\text{twoloopB},1,1,0,1,0;q^2)\\
&-4(2-2\varepsilon)J(\text{twoloopB},1,1,1,0,0;q^2)+4(2-2\varepsilon)J(\text{twoloopB},1,1,1,1,-1).
\end{aligned}\tag{5.119}$$

Defining reduced integrals $j(\text{twoloopB},n_1,n_2,n_3,n_4,n_5;q^2)$

$$\begin{aligned}
&J(\text{twoloopB},n_1,n_2,n_3,n_4,n_5;q^2)\equiv(m^2)^{4-2\varepsilon-(n_1+n_2+n_3+n_4+n_5)}\\
&=\frac{e^{-\varepsilon\gamma_E}(i\pi^{2-\varepsilon})^2}{(2\pi)^{2(4-2\varepsilon)}}j(\text{twoloopB},n_1,n_2,n_3,n_4,n_5;q^2),
\end{aligned}\tag{5.120}$$

and reducing to a set of master integrals using the IBP relations, we arrive at

$$
\begin{aligned}
\Pi^{\mathrm{II},b}_{\Psi,\mathrm{pert.}}(q^2) = &\frac{e^{-2\varepsilon\gamma_E}(i\pi^{2-\varepsilon})^2}{\left(\frac{q^2}{m^2}-4\right)q^2(1-2\varepsilon)\varepsilon} \\
&\Bigg(16(2\varepsilon-1)\left(m^2\right)^{-2\varepsilon} j(\mathrm{twoloopB},0,1,1,0,1;q^2) \\
&\quad \left(4m^4(3\varepsilon-2)+2m^2q^2\varepsilon\left(2\varepsilon^2-5\varepsilon+2\right)-q^4(\varepsilon-2)(\varepsilon-1)^2\right) \\
&-16q^2(\varepsilon-1)\left(m^2\right)^{-2\varepsilon-1} j(\mathrm{twoloopB},0,1,1,1,0;q^2) \\
&\quad \left(m^4\left(-8\varepsilon^2+4\varepsilon+4\right)+m^2q^2\varepsilon(3-2\varepsilon)+q^4(\varepsilon-1)\right) \\
&+4q^2(2\varepsilon-1)\left(m^2\right)^{-2\varepsilon-1} j(\mathrm{twoloopB},1,1,1,1,0;q^2) \\
&\quad \left(8m^4+4m^2q^2\varepsilon\left(-2\varepsilon^2+\varepsilon+1\right)+q^4\left(2\varepsilon^3-3\varepsilon^2+3\varepsilon-2\right)\right) \\
&-16(\varepsilon-1)^2\left(m^2\right)^{-2\varepsilon} j(\mathrm{twoloopB},0,0,1,1,0;q^2)\left(4m^4+2m^2q^2\varepsilon-q^4\right) \\
&+16\left(m^2\right)^{-2\varepsilon-1} j(\mathrm{twoloopB},0,2,1,0,1;q^2) \\
&\left(8m^6(2\varepsilon-1)-4m^4q^2\left(2\varepsilon^2-\varepsilon+1\right)+2m^2q^4\left(4\varepsilon^2-5\varepsilon+1\right)-q^6(\varepsilon-1)\right)\Bigg).
\end{aligned}
\tag{5.121}
$$

Now we notice that the master integrals appearing in Eq.5.121 are the double tadpole $j(\mathrm{twoloopB},0,0,1,1,0;q^2)$, double bubble $j(\mathrm{twoloopB},1,1,1,1,0;q^2)$, bubble-tadpole $j(\mathrm{twoloopB},0,1,1,1,0;q^2)$, sunset $j(\mathrm{twoloopB},0,1,1,0,1;q^2)$ and sunset with a doubled propagator $j(\mathrm{twoloopB},0,2,1,0,1;q^2)$, only named differently with respect to the twoloopA integrals because the initial momentum parametrization was different. The only new master integral is the double bubble. If we find the answer to these integrals, we have a solution for the the two loop diagrams we need.

5.4 Solving the Second Order Perturbative's Master Integrals

5.4.1 Double Tadpole, Double Bubble, and Bubble-Tadpole Master Integrals

The double-tadpole integral is simply the scalar tadpole integral squared. We have an answer for the tadpole for any ε, given in Eq.5.66. Squaring it, expanding in ε and taking out the factors of $e^{-2\gamma\varepsilon}(i\pi^{2-\varepsilon})^2$ as in the definition of Eq.5.114, we get

$$
\begin{aligned}
& j(\text{twoloopA}, 0, 1, 0, 1, 0, 0; q^2) = j(\text{twoloopB}, 0, 0, 1, 1, 0; q^2) = \\
& = \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \left(3 + \frac{\pi^2}{6}\right) + \frac{1}{3}\varepsilon\left(\pi^2 + 12 + \psi^{(2)}(1)\right) \\
& + \mathcal{O}\left(\varepsilon^2\right).
\end{aligned} \tag{5.122}
$$

The double bubble integral is simply the square of the bubble integral solved in Chapter 5.2.4. Now it is clear why we needed it up to order ε^2. The coefficient which multiplied this integral in Eq. 5.121 has a $\frac{1}{\varepsilon}$ pole, which means that if we want to have an answer up to order ε^0, as we do, we need the double bubble integral to order ε. However, the double bubble integral is the square of a simple bubble integral, and each bubble integral has a $\frac{1}{\varepsilon}$ pole, so that to get the answer for this integral to order ε, we need to compute the bubbles to order ε^2.

The answer is, to order ε

$$
\begin{aligned}
j(\text{twoloopB}, 1, 1, 1, 1, 0; q^2) = {}& \frac{1}{\varepsilon^2} - \frac{2(-2x + (x+1)\log(x) + 2)}{(x-1)\varepsilon} \\
& + \frac{24\left(x^2-1\right)\mathrm{Li}_2(-x) + 24\left(x^2-1\right)\left(\log(x+1) - 2\right)\log(x)}{6(x-1)^2} \\
& + \frac{(x-1)\left(3\pi^2 x + 72x + \pi^2 - 72\right) + 12(x+1)\log^2(x)}{6(x-1)^2} \\
& - \frac{\varepsilon}{3(x-1)^2}\Bigg(- 12x^2\mathrm{Li}_3(-x) - 24x^2\mathrm{Li}_3(x+1) \\
& + 24\left(x^2-1\right)\mathrm{Li}_2(x+1)\log(x+1) + 12\mathrm{Li}_3(-x) + 24\mathrm{Li}_3(x+1) \\
& + 12(x+1)\mathrm{Li}_2(-x)((x+1)\log(x) + 2(x-1)(\log(x+1) - 2)) \\
& + 14x^2\zeta(3) - 6\pi^2 x^2 - 96x^2 - 2x^2\log^3(x) + 6x^2\log(x+1)\log^2(x) \\
& + 12x^2\log^2(x+1)\log(x) + 12x^2\log(-x)\log^2(x+1) + \pi^2 x^2\log(x) \\
& + 72x^2\log(x) - 48x^2\log(x+1)\log(x) + 2\pi^2 x^2\log(x+1) \\
& + 4\pi^2 x + 192x - 6x\log^3(x) - 4\log^3(x) - 24x\log^2(x) \\
& + 24x\log(x+1)\log^2(x) + 18\log(x+1)\log^2(x) - 24\log^2(x) \\
& - 12\log^2(x+1)\log(x) - 12\log(-x)\log^2(x+1) + \pi^2\log(x) \\
& + 2\pi^2 x\log(x) + 48\log(x+1)\log(x) - 72\log(x) - 2\pi^2\log(x+1) \\
& + 2x\psi^{(2)}(1) - 14\zeta(3) + 2\pi^2 - 96 - 2\psi^{(2)}(1)\Bigg) + \mathcal{O}\left(\varepsilon^2\right),
\end{aligned}
\tag{5.123}
$$

and we are using the variable x from Eq.5.81.

The bubble tadpole integral is simply the product of a tadpole and a bubble. Doing the product and expanding to order ε, we get

$$
\begin{aligned}
j(\text{twoloopA},1,1,0,1,0,0;q^2) = j(\text{twoloopB},0,1,1,1,0;q^2) = \frac{1}{\varepsilon^2}& \\
+\frac{3-\frac{(x+1)\log(x)}{x-1}}{\varepsilon}& \\
+\frac{12(x+1)\text{Li}_2(-x)+2\pi^2x+42x-3(x+1)\log^2(x)}{6(x-1)}& \\
+\frac{6(x+1)(2\log(x+1)-3)\log(x)-42}{6(x-1)}& \\
-\frac{\varepsilon}{6(x-1)}\Big(-12x\text{Li}_3(-x)-12\text{Li}_3(-x)-24x\text{Li}_3(x+1)& \\
-24\text{Li}_3(x+1)+12(x+1)\text{Li}_2(-x)(2\log(x+1)-3)& \\
+24(x+1)\text{Li}_2(x+1)\log(x+1)+14x\zeta(3)-6\pi^2x-90x& \\
+x\log^3(x)+\log^3(x)+9x\log^2(x)-6x\log(x+1)\log^2(x)& \\
-6\log(x+1)\log^2(x)+9\log^2(x)+12x\log^2(x+1)\log(x)& \\
+12\log^2(x+1)\log(x)+12x\log(-x)\log^2(x+1)& \\
+12\log(-x)\log^2(x+1)+42x\log(x)-36x\log(x+1)\log(x)& \\
-36\log(x+1)\log(x)+42\log(x)+2\pi^2\log(x+1)& \\
+2\pi^2x\log(x+1)-x\psi^{(2)}(1)+14\zeta(3)+90+3\psi^{(2)}(1)\Big)& \\
+\mathcal{O}\left(\varepsilon^2\right).&
\end{aligned}
\tag{5.124}
$$

5.4.2 Sunset Master Integrals

For the simple sunset and sunset with a doubled propagator master integrals, we will resort again to the method of differential equations. In this chapter, we will refer to these integrals as $j(\text{sunset},q^2)$ (instead of $j(\text{twoloopB},0,1,1,0,1;q^2)$ or $j(\text{twoloopA},0,1,0,1,1,0;q^2)$) and $j(\text{sunsetdp},q^2)$ (instead of $j(\text{twoloopB},0,2,1,0,1;q^2)$ or $j(\text{twoloopA},0,2,0,1,1,0;q^2)$).

Following what was done for the bubble integral in Chapter 5.2.4, we start taking the derivative of these integrals with respect to the variable x, Eq.5.80.

The differentiation will result in integrals within the same family and we can use the IBP relations to reduce them to the set of master integrals. We find that the $j(\text{sunset};q^2)$ and $j(\text{sunsetdp};q^2)$ form, along with the double tadpole integral (which will be referred to in this Chapter as $j(\text{dbtadpole};q^2)$), a closed system

$$j'(\text{sunset};x) = -\frac{(x+1)(2\varepsilon-1)j(\text{sunset};x)}{(x-1)x} - \frac{2(x+1)j(\text{sunsetdp};x)}{(x-1)x}, \tag{5.125}$$

$$j'(\text{sunsetdp};x) = \frac{(2\varepsilon-1)(3\varepsilon-2)j(\text{sunset};x)}{(x-1)(x+1)} - \frac{\left(x^2\varepsilon-10x\varepsilon+6x+\varepsilon\right)j(\text{sunsetdp};x)}{(x-1)x(x+1)} + \frac{(\varepsilon-1)^2 j(\text{dbtadpole};x)}{(x-1)(x+1)}, \tag{5.126}$$

$j(\text{dbtadpole}, q^2)$ being a constant in x, whose Laurent expansion in ε in Eq.5.122. We have a coupled system of differential equations to solve, Eq.5.125 and 5.126. However, we can produce a differential equation for $j(\text{dbtadpole}, q^2)$ alone by isolating $j(\text{sunsetdp};x)$ in Eq.5.125 and substituting it in Eq.5.126. Simplifying, we obtain

$$\frac{x\left(x^2-1\right)j''(\text{sunset};x)}{2(x+1)^2} + \frac{\left(3(x-1)^2\varepsilon+6x-2\right)j'(\text{sunset};x)}{2(x+1)^2} + \frac{(2\varepsilon-1)j(\text{sunset};x)(x((x-4)\varepsilon+2)+\varepsilon)}{2(x-1)x(x+1)} + \frac{(\varepsilon-1)^2 j(\text{dbtadpole},x)}{(x-1)(x+1)} = 0 \tag{5.127}$$

Now with a second order differential equation, we will need two boundary conditions to determine the solution completely. We will use the massless limit for the sunset diagram and the sunset vacuum bubble, which corresponds to the limit $q^2 \to 0$. The integrals in both these limits are much easier to obtain, since they have only one scale. The answer of these limiting cases of the integrals are shown in Appendix B.

Obtaining the answer now proceeds along lines similar to the ones in Chapter 5.2.4 for the bubble integral: one expands $j(\text{sunset};x)$ in a series in ε and solves the differential equation at each order. The differential equation for a specific order will depend only on the answer of previous orders. One fixes the two integration constants of each order with the corresponding coefficients from the massless and vacuum bubble integral limits. We will not present the steps in detail since this would be a little lengthy and almost no profit would be taken from it.

The answer we find, to order ε, is

$$
\begin{aligned}
j(\text{sunset};x) = {} & \frac{1}{\varepsilon^2} + \frac{x^2+10x+1}{4x\varepsilon} + \frac{1}{24(x-1)^2x}\Big(-24x\left(x^2-x+1\right)\log^2(x) \\
& -12\left(x^4-4x^3+4x-1\right)\log(x) + \left(39x^2+\left(66+4\pi^2\right)x+39\right)(x-1)^2\Big) \\
& + \frac{\varepsilon}{48(x-1)^2x} \\
& \Big(32\left(x(x(8x-7)+8)\zeta(3) - 9x((x-1)x+1)\left(2\text{Li}_3(x) - \text{Li}_3\left(x^2\right)\right)\right) \\
& -96\text{Li}_2(x)\left(x^4-4x^3+4x-2((x-1)x+1)x\log(x)-1\right) \\
& +72\text{Li}_2\left(x^2\right)\left(x^4-4x^3+4x-4((x-1)x+1)x\log(x)-1\right) \\
& +(x-1)\left(\pi^2\left(6x\left(x^2+x-5\right)+2\right)+15(x-1)(x(23x-22)+23)\right) \\
& +4\log(x)\Big(-39x\left((x-4)x^2+4\right)+36x\left((x-4)x^2+4\right)\log(x+1) \\
& \quad +12\log(1-x)\left(x^4-4x^3+4x-2((x-1)x+1)x\log(x)-1\right) \\
& \quad +6x\log(x)\left(-3x^3+6x^2+x-2((x-1)x+1)\log(x)-10\right)\Big) \\
& +24\pi^2x((x-1)x+1)+24\log(x)-216\log(x+1)+234\Big).
\end{aligned} \tag{5.128}
$$

Using now Eq.5.125 it is possible to obtain $j(\text{sunsetdp};x)$

$$
\begin{aligned}
j(\text{sunsetdp};x) = {} & \frac{1}{2\varepsilon^2} + \frac{1}{2\varepsilon} \\
& + \frac{-6\left(x^2+1\right)\log^2(x)+12\left(x^2-1\right)\log(x)+\left(\pi^2-6\right)(x-1)^2}{12(x-1)^2} \\
& - \frac{\varepsilon}{12(x-1)^2}\Big(12x^2\text{Li}_2(x) - 144x^2\text{Li}_3(-x) - 72x^2\text{Li}_3(x) \\
& + 60x^2\text{Li}_2(x)\log(x) + 12\text{Li}_2(1-x)\left(-x^2+\left(x^2+1\right)\log(x)+1\right) \\
& + 72\text{Li}_2(-x)\left(x^2+x^2\log(x)+\log(x)-1\right) - 12\text{Li}_2(x) - 144\text{Li}_3(-x) \\
& - 72\text{Li}_3(x) + 60\text{Li}_2(x)\log(x) - 32x^2\zeta(3) + 3\pi^2x^2 + 66x^2 + 6x^2\log^3(x) \\
& - 24x^2\log^2(x) + 24x^2\log(1-x)\log^2(x) - 4\pi^2x^2\log(x) - 60x^2\log(x) \\
& + 12x^2\log(1-x)\log(x) + 72x^2\log(x+1)\log(x) + 2\pi^2x - 132x \\
& + 6\log^3(x) + 12x\log^2(x) + 24\log(1-x)\log^2(x) + 24\log^2(x) \\
& - 4\pi^2\log(x) - 12\log(1-x)\log(x) - 72\log(x+1)\log(x) \\
& + 60\log(x) + 4x\psi^{(2)}(1) - 32\zeta(3) - 5\pi^2 + 66\Big).
\end{aligned} \tag{5.129}
$$

The answers to these master integrals, Eq.5.128 and 5.129 agree with Smirnov, 2012, which presents the answers up to order $\mathcal{O}(\varepsilon^0)$. The former also agrees with Bonciani, Mastrolia, and Remiddi, 2003 and the latter agrees with Czakon, Gluza, and Riemann, 2005.

5.5 Result for Second Order Perturbative Integral

Now that we have all the answers for the master integrals and the expansion of the second order perturbative contribution in terms of these integrals, we can substitute them in Eqs.5.115 and 5.121 and do the Laurent expansion to order $\mathcal{O}(\varepsilon^0)$.

$$
\begin{aligned}
\Pi^{\text{II}}_{\Psi,\text{pert.}}(q^2) = &\frac{-(i\pi^{2-\varepsilon})e^{-2\varepsilon\gamma_E}(m^2)^{2\varepsilon}}{(2\pi)^{8-4\varepsilon}} \frac{4g^2}{3(x-1)^4(x+1)} \\
&\Bigg(\frac{6(x-1)\left(x^4-26x^3+48x^2\log(x)+26x-1\right)}{\varepsilon} \\
&+\Bigg(32\mathrm{Li}_2(x)\left(x^3-3x^2-3x+1\right)\left(x^2+2\left(x^2+1\right)\log(x)-1\right) \\
&\quad -48\left(\zeta(3)+2\mathrm{Li}_3(x)+4\mathrm{Li}_3(-x)\right)\left(x^5-3x^4-2x^3-2x^2-3x+1\right) \\
&\quad +64\mathrm{Li}_2(-x)\Big(x^5-3x^4-13x^3+13x^2+3x-1 \\
&\qquad +2\left(x^5-3x^4-2x^3-2x^2-3x+1\right)\log(x)\Big) \\
&\quad +(x-1)\left(55x^4-694x^3-48\pi^2x^2+694x-55\right) \\
&\quad +8\log^2(x)\Big(x\left(-6x^4+16x^3+33x^2-27x-2\right) \\
&\qquad +2\left(x^5-3x^4-2x^3-2x^2-3x+1\right)\left(\log(1-x)+2\log(x+1)\right)\Big) \\
&\quad +4(x-1)\log(x)\Big(-3x^4+54x^3+210x^2+54x-3 \\
&\qquad +8(x+1)^2\left(x^2-4x+1\right)\log(1-x) \\
&\qquad +16\left(x^4-2x^3-15x^2-2x+1\right)\log(x+1)\Big)+\mathcal{O}(\varepsilon)\Bigg)\Bigg).
\end{aligned}
\tag{5.130}
$$

Looking at Eq.5.115 and 5.121 one could be worried that we would get a result with a divergent term $\frac{1}{\varepsilon^3}$, which is unusual for a two loop calculation. But this is not the case and this divergent term cancels. In fact, the $\frac{1}{\varepsilon^2}$ term also cancel each other and the naively anticipated divergence is also not there.

Although we have not investigated in detail why this is so, the reason is probably the gauge symmetry working behind the curtains to ameliorate the divergence. The $\frac{1}{\varepsilon^2}$ is present in the results for the contributions from Eq.5.115 and Eq.5.121 separately, but they cancel exactly when we add them both.

In the massless limit, the result becomes

$$\Pi^{\mathrm{II}}_{\Psi,\,\mathrm{pert.}}(q^2) \to \frac{g^2}{32\pi^4\varepsilon} - \frac{g^2\left(-12\log\left(-\frac{1}{s}\right) + 48\zeta(3) + 12\gamma_E - 55 - 12\log(4\pi)\right)}{192\pi^4}, \tag{5.131}$$

which agrees with Pascual and Tarrach, 1984; Grozin, 2005. In this limit, of massless fermions, the $\frac{1}{\varepsilon}$ divergence is independent of momentum. Again, we have not investigated in detail the reason why the $\frac{1}{\varepsilon}$ divergence is dependent on the external momentum in the case of arbitrary masses, but we suppose that this is due to the phenomenon of overlapping divergences which shows up first at two loop level (Peskin and Schroeder, 1995). A suitable renormalization procedure at one-loop level is expected to eliminate this divergence.

Taking the analytic continuation to the region corresponding to $q^2 > 0$, we can calculate the discontinuity. The result we find for the finite part (order ε^0) is not consistent with the one by Schwinger, 1998, which is quoted in Reinders, Rubinstein, and Yazaki, 1980. The comparison can be seen in the plot in Fig. 5.14.

For large q^2 our results become more consistent, but this regime corresponds to the mass becoming irrelevant, and so, since we used the result for massless diagrams as a boundary conditions when solving the differential equations, it is simply a fit to the known result.

Also, due to the $\log(x)$ in our $\frac{1}{\varepsilon}$ term, we get a divergent imaginary part, which was unexpected. We still do not know for sure what the problem is, but, assuming the calculations are correct, the difference may be due to the lack of renormalization in our result. It is possible that Schwinger uses an on-shell renormalization to arrive at his final result, but his calculations are quite intricate and we do not know how to see this explicitly. By calculating the counter terms and including them in the OPE terms, which we will be able to solve by using Master Integrals, we hope to find agreement with the aforementioned results. In general lines, what we think is happening is this: since we have two loops, the finite part of a loop, the $\log(x)$, is multiplying the divergent part of the other, the $\frac{1}{\varepsilon}$ pole. By doing the renormalization procedure, we will eliminate the poles of each loop and the result will be then

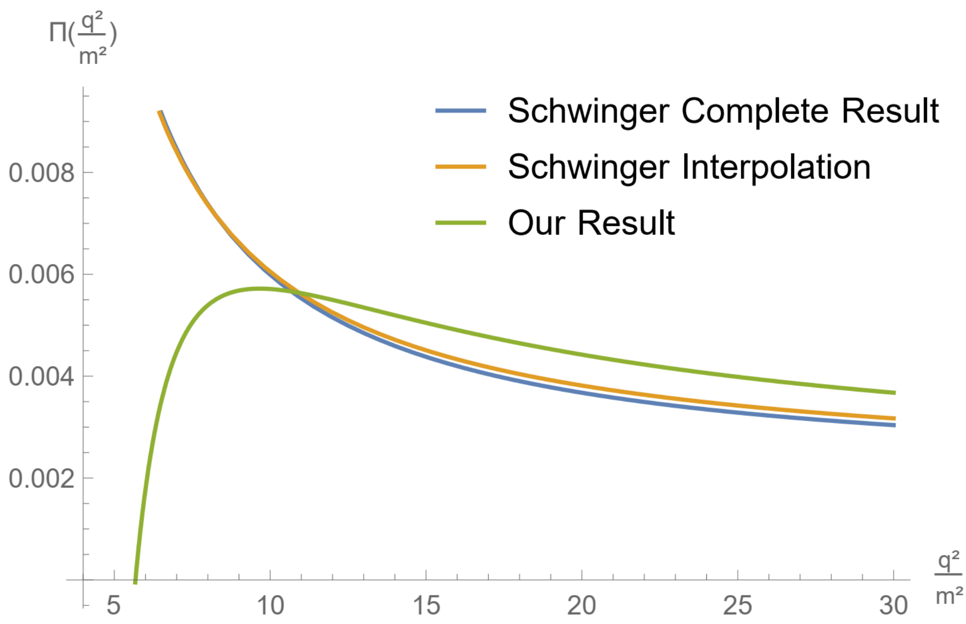

FIGURE 5.14: Graph showing the result for the discontinuity of $\Pi^{\mathrm{II}}_{\Psi,\,\mathrm{pert.}}(q^2)$ for positive q^2 in Schwinger, 1998 and its approximate interpolation, also presented there and quoted in Reinders, Rubinstein, and Yazaki, 1980, in comparison with the result we obtained.

finite automatically.

6 Conclusion

Although we were not fully successful in our attempt of calculating the two loop contribution to the OPE, we want to make the point that the reason for this is not the method applied. If our calculations are correct, we probably need only to take care of the renormalization and the disagreement with the established result should disappear. Then again, our calculations may contain mistakes, and we would have to redo them more carefully. The method of master integrals is solid and a whole community of scientists has been working on it recently. It is also possible that the Schwinger result is incorrect, although in order to be sure of this, we would need to find an independent calculation of the same quantity.

That said, we conclude that the method of master integrals is suited for calculating the coefficients of the OPE in QCD Sum Rules. It is extremely versatile and can be applied to diagrams with massive and massless particles with essentially no modification. A great number of integrals are already available in the literature, but when one finds an integral which has not been calculated before, one can take advantage of the fact that these integrals obey differential equations and solve them through these equations. Some libraries have been implemented to ease the process of obtaining the relevant set of master integrals and to reduce the integrals in which one is interested to this set, which makes this part rather straightforward. Solving the master integrals order by order can be also done in symbolic manipulation software.

Automatizing the process of solution of these integrals is, in my opinion, badly needed. QCD Sum Rules calculations, specially for massive systems, has very complicated integrals and cumbersome expressions. In order to speed up the computation of OPE coefficients and get more reliable results, we should rely more and more in symbolic manipulation software. The high chances of losing a minus sign or a numeric factor in calculating traces of gamma matrices and the time lost searching for these mistakes are good arguments for implementing things on a computer. And now that we are aware that the process of solving the integrals themselves can be carried on systematically, I think we should move in this direction.

In the future, precise results including more physical information in QCD

Sum Rules, specially in more complicated system such as the tetraquark, will become increasingly relevant.

A Fixed-Point Gauge

The so-called Schwinger or Fixed-Point gauge is defined by

$$x^\mu B_\mu(x) = 0. \tag{A.1}$$

One advantage of this gauge is that the gluon-field is easily expressed in terms of the field strength-tensor by

$$B^A_\mu(x) = \int_0^1 d\alpha\, \alpha\, G^A_{\rho\mu}(\alpha x) x^\rho. \tag{A.2}$$

To prove this we start with the expression of the derivative of $B^A_\rho y^\rho$. After a simple manipulation, we have

$$B^A_\mu(y) = \frac{\partial}{\partial y^\mu}\left(B^A_\rho y^\rho\right) - y^\rho \frac{\partial B^A_\rho(y)}{\partial y^\mu} \tag{A.3}$$

Now, from the definition of $G^A_{\mu\nu}(x)$ and the gauge condition, we have that

$$y^\mu G^A_{\mu\nu} = y^\mu \partial_\mu B^A_\nu - y^\mu \partial_\nu B^A_\mu, \tag{A.4}$$

or more explicitly

$$y^\rho \frac{\partial B^A_\rho(y)}{\partial y^\mu} = y^\rho G^A_{\mu\rho}(y) + y^\rho \frac{\partial B^A_\mu(y)}{\partial y^\rho}. \tag{A.5}$$

Substituting Eq.A.5 into in Eq.A.3, we get

$$B^A_\mu(y) + y^\rho \frac{\partial B^A_\mu(y)}{\partial y^\rho} = y^\rho G^A_{\rho\mu}(y) \tag{A.6}$$

If we now make the substitution $y = \alpha x$, we see that the last equation is a total derivative.

$$\frac{d}{d\alpha}(\alpha B^A_\mu(\alpha x)) = \alpha x^\rho G^A_{\rho\mu}(\alpha x) \tag{A.7}$$

Integrating in α from 0 to 1 we then get Eq.A.2. If we are interested in the expansion of $B^A_\mu(x)$ near $x^\mu = 0$, as is the case in the short-distance expansion needed in QCD Sum Rules, we can expand Eq.A.2.

$$B_\mu^A(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)} x^\omega x^{\omega_1} \dots x^{\omega_n} \partial_{\omega_1} \dots \partial_{\omega_n} G_{\omega\mu}^A(x)|_{x=0} \tag{A.8}$$

Due to the gauge condition and the fact that in it $B_{\omega_1}^A(0) = 0$, we notice that

$$x^{\omega_1} \partial_{\omega_1} G_{\omega\mu}^A(0) = x^{\omega_1} [D_{\omega_1}(0), G_{\omega\mu}(0)] \tag{A.9}$$

and similarly to higher order derivatives. The final expression can be obtained in terms of covariant derivatives

$$B_\mu^A(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)} x^{\omega_0} \dots x^{\omega_n} [D_{\omega_1}(0), [\dots, [D_{\omega_n}(0), F_{\omega_0\mu}(0)]]]. \tag{A.10}$$

B Sunset Integral in Massless and Vacuum Bubble Limit

The simple sunset in the massless limit is easy to solve (Grozin, 2005). To do so, one can first parametrize the momenta in such a way that the external momentum dependence is present on only one of the loops and solve the other one with Schwinger parameters as was done for the massless bubble integral. The answer will be proportional to the other loop momentum squared, raised to the power $-\varepsilon$ and the constant of proportionality will come out of the calculation as well. Then one will have something like a bubble calculation, but with the power of the propagator ε and, using again Schwinger parameters, one gets

$$\int \frac{d^{4-2\varepsilon}k, p}{(p+q)^2(k^2)(p-k)^2} = \pi^{4-2\varepsilon}(-q^2)^{1-2\varepsilon}\frac{\Gamma^3(1-\varepsilon)\Gamma(2\varepsilon-1)}{\Gamma(3-3\varepsilon)}. \tag{B.1}$$

For obtaining the result in the limit $q^2 \to 0$, one can use again Schwinger parameters and after solving the momenta integration, expressing the integrals of the parameters in terms of gamma functions and simplifying, one obtains (Smirnov, 2012)

$$\int \frac{d^{4-2\varepsilon}k, p}{(p^2-m^2)(k^2-m^2)(p-k)^2} = \pi^{4-2\varepsilon}\left(m^2\right)^{1-2\varepsilon}\frac{\Gamma(\varepsilon-1)}{1-2\varepsilon}. \tag{B.2}$$

Bibliography

Alkofer, R. and L. von Smekal (2001). In: *Phys. Rept. 353, 281*.

Bonciani, R., P. Mastrolia, and E. Remiddi (2003). In: *Nucl.Phys.B661:289-343*.

Chetyrkin, K. G. and F. V. Tkachov (1981). In: *Nucl. Phys.* B192, pp. 159–204.

Cohen, T.D. et al. (1995). In: *Progress in Particle and Nuclear Physics, 35 221-298*.

Colangelo, P. and A. Khodjamirian (2001). *QCD Sum Rules, a Modern Perspective*. In Boris Ioffe Festschrift "At the Frontier of Particle Physics / Handbook of QCD". World Scientific.

Czakon, M., J. Gluza, and T. Riemann (2005). In: *Phys.Rev. D71 073009*.

Durr, S. et al. (2008). In: *Science 322 1224-1227*.

Ellis, R.K. and G. Zanderighi (2008). In: *JHEP 0802 002*.

Gattringer, C. and C. Lang (2010). *Quantum Chromodynamics on the Lattice*. Springer.

Grozin, A. (2005). "Lectures on QED and QCD." Lectures given at Dubna International Advanced School on Theoretical Physics. arXiv:hep-ph/0508242v1.

Henn, J. (2014). *Scattering Amplitudes in Gauge Theories*. Springer.

— (2015). In: *J.Phys. A48 153001*.

Henn, J. and V.A. Smirnov (2013). In: *JHEP 1311 041*.

Hilger, T. (2012). "Medium Modifications of Mesons. Chiral Symmetry Restoration, in-medium QCD Sum Rules for D and ρ Mesons, and Bethe-Salpeter Equation". PhD thesis. Universität Dresden.

Hooft, G.'t and M.J.G. Veltman (1973). *Diagrammar*. CERN.

Ioffe, B.L. (1981). In: *Nucl.Phys. B188 317-341*.

Källén, G. and A. Sabry (1955). In: *Dan.Mat.Fys.Medd.29, 17*.

Lee, R. (2012). "Presenting LiteRed: a tool for the Loop InTEgrals REDuction [arXiv:1212.2685]".

Matheus, R. D'Elia (2003). "O Fator de Forma J - D - D em Regras de Soma da QCD". MA thesis. Universidade de São Paulo.

— (2006). "Partículas Exóticas em Regras de Soma da QCD". PhD thesis. Universidade de São Paulo.

Muta, T. (1998). *Foundations of Quantum Chromodynamics: an Introduction to Perturbative Methods in Gauge Theories*. World Scientific.

Pascual, P. and R. Tarrach (1984). *QCD: Renormalization for the Practitioner*. Springer.

Passarino, G. and M.J.G. Veltman (1979). In: *Nucl. Phys. B 160, 151*.

Patrignani, C. et al. (2016). In: *Chin. Phys. c40 10 100001*.

Peskin, M.E. and D.V. Schroeder (1995). *An introduction to Quantum Field Theory*. Addison-Wesley.

Reinders, R.J., H.R. Rubinstein, and S. Yazaki (1980). In: *Nuclear Physics B186, 109-146*.

Remiddi, Ettore (1997). In: *Nuovo Cim.* A110, pp. 1435–1452.

Schwinger, J. (1998). *Particles, Sources and Fields. Vol. 3*. Perseus Books.

Shifman, M.A., A. I. Vainshtein, and V. I. Zakharov (1979). In: *Nuclear Physics B147, 385-447*.

Shtabovenko, V., R. Mertig, and F. Orellana (2016). In: *Comput. Phys. Commun. 207, 432*.

Smirnov, V.A. (2012). *Analytic tools for Feynman integrals*. Springer.

Wang, C.Y. et al. (2017). arxiv:1708.04563.

Weinberg, S. (2005). *The Quantum Theory of Fields: Vol. 1*. Cambridge University Press.

Wilson, K. (1969). In: *Phys. Rev. 179, 1449*.