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ROBUST $\mathscr{H}_\infty$ SWITCHED STATIC OUTPUT FEEDBACK CONTROL DESIGN FOR LINEAR SWITCHED SYSTEMS SUBJECT TO ACTUATOR SATURATION

Ilha Solteira
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"What we know is a drop, what we don’t know is an ocean." Isaac Newton

ABSTRACT

This thesis is devoted to the study of the robust $\mathscr{H}_\infty$ control problem of continuous-time switched linear systems subject to actuator saturation with polytopic uncertainties, considering an output-dependent switching law and a switched static output feedback controller. The proposed method offers new sufficient conditions based on linear matrix inequalities (LMIs) for designing the switched controllers using parameter-dependent Lyapunov functions. The method is based on a static output feedback $\mathscr{H}_\infty$ control design recently presented in the literature that avoids linear matrices equalities (LMEs) and the need to impose any constraints on output system matrices, that is, the output matrices of the system are allowed to be of non-full row rank. In order to extend those results, the actuator saturation constraint is also studied. Theoretical analyses and simulation results show that these new procedures are less conservative than recent methods available in the literature. The conditions of the proposed methods are a particular class of Bilinear Matrix Inequalities (BMIs), which contain some bilinear terms as the product of a matrix and a scalar, related to a suitable convex combination and scalars parameters to provide extra free dimensions in the solution space. The hybrid algorithm Differential Evolution-Linear Matrix Inequality (DE-LMI), is proposed for obtaining feasible solutions of this particular NP-hard problem. Examples show that the proposed methodologies reduce the design conservatism of two recent known procedures for solving the presented control problems. In particular, an example presents an implementation of the switched controllers in an Active Suspension System manufactured by Quanser®.

Keywords: Robust $\mathscr{H}_\infty$ switching control. Switched static output feedback. Linear matrix inequalities. Actuator saturation.

RESUMO

Este trabalho dedica-se ao estudo do problema de controle robusto envolvendo custo $\mathscr{H}_\infty$ para sistemas lineares chaveados no tempo contìnuo, sujeitos à saturação no atuador e com incertezas politópicas, considerando leis de chaveamento e controladores chaveados dependentes da saída da planta. Os métodos propostos oferecem novas condições baseadas em Desigualdades Matriciais Lineares (LMIs - do inglês, *Linear Matrix Inequalities*) para o projeto de controladores chaveados utilizando funções de Lyapunov dependentes de parâmetros. O método é baseado em um resultado recentemente introduzido na literatura para o projeto de controle $\mathscr{H}_\infty$ de saída o qual evita igualdades matriciais lineares (LMEs - do inglês, *Linear Matrix Equalities*) e a necessidade de impor restrições nas matrizes de saída do sistema, isto é, as matrizes de saída do sistema podem ser de posto linha incompleto. Com o objetivo de estender estes resultados, a restrição de saturação no atuador é estudada. Análises teóricas e resultados de simulações mostram que os novos procedimentos são menos conservativos quando comparados a métodos publicados recentemente na literatura. No método proposto, as condições são uma classe particular de desigualdades matriciais bilineares (BMIs - do inglês, *Bilinear Matrix Inequalities*), as quais contêm alguns termos bilineares devido à multiplicação de matrizes por escalares. Estes termos estão relacionados à combinação convexa das matrizes de chaveamento bem como a outros parâmetros escalares que proporcionam dimensões extras livres no espaço de solução. Para tanto, o algoritmo híbrido denominado DE-LMI (do inglês, *Differential Evolution-Linear Matrix Inequality*) é proposto a fim de encontrar soluções factíveis para este problema *NP-hard*. Exemplos mostram que as metodologias propostas reduzem o conservadorismo de dois procedimentos recentes presentes na literatura para resolver os problemas de controle tratados. Em particular, um exemplo apresenta a implementação do controle chaveado em um sistema de suspensão ativa fabricado pela Quanser®.

Keywords: Controle $\mathscr{H}_\infty$ chaveado e robusto . Controle chaveado estático com realimentação da saída. Desigualdades Matriciais Lineares. Saturação no atuador.

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LIST OF ABBREVIATIONS AND ACRONYMS

LMIs	Linear Matrix Inequalities
BMIs	Bilinear Matrix Inequalities
SOF	Static Output Feedback
T-S	Takagi-Sugeno
DE	Differential Evolution
DE-LMI	Differential Evolution-Linear Matrix Inequality based

LIST OF SYMBOLS

$\mathbb{R}$	Set of real numbers.
$\mathbb{R}^n$	Set of the vectors with $n \times 1$ real elements.
$\mathbb{R}^{n\times m}$	Set of the matrices with $n \times m$ real elements.
$\mathbb{K}_N$	Set of the first N positive integers $\{1,2,\ldots,r\}$.
M'	Transpose of the real matrix M.
$M > (\geq) 0$	M is a symmetric positive definite (semi-definite) matrix.
$M < (\leq) 0$	M is a symmetric negative definite (semi-definite) matrix..
I	Identity matrix.
$\|z\|$	Absolute value of a real number z.
$\|\|x\|\|$	Euclidean norm of the vector $x \in \mathbb{R}^n : \|\|x\|\| = \sqrt{x^T x}$.
$\mathrm{co}(\mathcal{M})$	Convex hull of a set $\mathcal{M}$
$\mathscr{L}_2$	Set of all finite vector $\xi(t)$ such that $\int_0^\infty \xi(t)'\xi(t)\mathrm{d}t < \infty$.
Λ_r	Set $\Lambda_r = \left\{\alpha \in \mathbb{R}^n : \alpha_j \geq 0,\ j \in \mathbb{K}_r,\ \sum_{j=1}^r \alpha_j = 1\right\}$.
A_λ	Set $A_\lambda = \sum_{j=1}^r \lambda_i A_i,\ \lambda = \begin{bmatrix}\lambda_1 & \lambda_2 & \ldots & \lambda_r\end{bmatrix}' \in \Lambda_r$.

TABLE OF CONTENTS

1 INTRODUCTION

This chapter aims to familiarise the reader with the topic of investigation, exploiting the results presented in the literature and setting the problem statement and the objectives. Additionally, the thesis outline and the notation used throughout it are presented.

1.1 BACKGROUND

The hybrid system concept arises when the dynamical systems manifest continuous and discrete behaviours. Continuous switched systems are a special case of the hybrids one, which are composed of a family of continuous subsystems where a switching rule or strategy (discrete behaviour) defines the active subsystem at each instant of time (LIBERZON, 2003).

Recently, the designing of control laws for switched systems have received lots of attention (YU; WU, 2015; ZHANG; ZHUANG; BRAATZ, 2016). The growing interest in this topic is mainly due their widespread practical applications, such as power electronics (CARDIM *et al.*, 2009; DEAECTO *et al.*, 2010), embedded systems (ZHANG; HU, 2008), road traffic control strategies (PAPAGEORGIOU *et al.*, 2003), among others. A significant result concerning the stability of switched linear systems was presented in Wicks, Peleties and DeCarlo (1994): it was demonstrated that if there exists a Hurwitz convex combination of the subsystems matrices, then there exists a state switching rule that stabilises the switched linear system. Regarding the concepts of robust stabilisation, Zhai, Lin and Antsaklis (2003) proposed a quadratic stabilisation rule for uncertain switched linear systems based on LMIs. In Lin and Antsaklis (2007) were developed two necessary and sufficient conditions for providing global stability for a class of switched linear systems with time-variant parametric uncertainties. Concerning stability and stabilisability of switched linear systems, in Lin and Antsaklis (2009) can be found a survey of available results and a proposed necessary and sufficient condition for asymptotic stabilisability.

Additionally, Daafouz, Riedinger and Iung (2002) proposed two different LMI-based conditions. In the first one, it is presented a classical method while the second incorporates slack variables in order to relax the conditions. Moreover, Ding and Yang (2009) describe more relaxed conditions through of Finsler's Lemma and piecewise quadratic Lyapunov functions for Static Output Feedback (SOF) control. For robust stabilisation of switched linear systems, when all subsystems matrices are not Hurwitz, in Yu and Wu (2015) are presented sufficient conditions, under some assumptions, for stability using the invariant subspace theory and average dwell time method. Considering some hypotheses, the authors in Yu and Zhao (2016) developed a necessary condition of stability for discrete-time switched linear systems.

With regard to the output feedback control design problem for uncertain switched linear systems, its solution is among one of the most challenging problems in literature, due to their non-convex characteristic (SADABADI; PEAUCELLE, 2016; SYRMOS *et al.*, 1997). Nevertheless, in recent years, the design of SOF controllers has been scrutinised by several authors, mainly due to the use of output feedback techniques results in simpler implementation routines for practical applications. In Peaucelle and Arzelier (2005), the authors proposed a two-step iterative algorithm focused on $\mathscr{H}_2$ optimisation. In doing so, Agulhari, Oliveira and Peres (2010) presented an extension of a previous method considering polynomial Lyapunov functions.

Concerning a performance criterion, in this case, the $\mathscr{H}_\infty$ cost, several authors have proposed LMI-based conditions considering an output feedback strategy. Crusius and Trofino (1999) presented sufficient conditions for SOF controllers, adopting linear matrices equalities and inequalities and imposing constraints on the output systems matrices. In this approach, the output systems matrices are required to be full-row rank. Aiming to overcome these drawbacks, Dong and Yang (2013) presented new LMI conditions, for cases where the output matrix is not required to be full rank. In sequence, Chang, Park and Zhou (2015) have extended the flexibility of conditions for robust SOF $\mathscr{H}_\infty$ controller design. More specifically, the developed method is applicable for uncertain systems relaxing the constraint in system matrices.

Furthermore, in Shi *et al.* (2017), the authors investigated the dynamic output feedback $\mathscr{H}_\infty$ control for a class of switched systems with mode-dependent average dwell time switching. In (WU *et al.*, 2017), it was proposed a sliding mode control (SMC) for stochastic systems via output feedback considering among others, the exogenous disturbance constraint in SMC design. An extended state observer (ESO) was used in order to reject external disturbance considering SMC for power converters (LIU *et al.*, 2017). Additionally, in Ban *et al.* (2018), it was designed a technique for robust $\mathscr{H}_\infty$ finite-time control for discrete-time polytopic uncertain switched linear systems.

The actuator saturation constraint plays an important role regarding the design of controllers due to practical applications limitations. Besides compromises the closed-loop performance, considering that the controller was not designed taking into account that constraint, if the closed-loop system is under saturation it may become an unstable system (CAO; LIN, 2003). To deal with the problem of design controllers under actuator saturation restriction Hu, Lin and Chen (2002) employs an improved condition based on convex hull representation by means of adding auxiliary matrices.

Concerning switched controllers and actuator saturation constraint, Alves *et al.* (2016) provided conditions to the design of smoothing switched controllers for uncertain nonlinear systems subject to actuator saturation. To cope with the robust $\mathscr{H}_\infty$ control design for Takagi-Sugeno (T-S) fuzzy system subject to actuator saturation, Oliveira *et al.* (2018) propose conditions to obtain switched controllers. Moreover, the bounded energy disturbance approach allowed a significant result, thus, this concept is adopted in the present work.

1.2 PROBLEM STATEMENT AND CONTRIBUTIONS

In practical applications, the state vector may not be completely available. In this situation, it is important to aim at strategies for switching based on the measured output of the plant. Nowadays, to the best of the author's knowledge, there are not available in the literature papers which consider switched SOF $\mathscr{H}_\infty$ controllers design for uncertain switched linear systems subject to actuator saturation with output-depending switching. Regarding the aforementioned researches, usually, papers on this subject consider full or reduced order output feedback controllers through of estimated state-dependent switching or state feedback.

The major contribution proposed is an exclusively output-dependent switching strategy jointly with the design of switched SOF $\mathscr{H}_\infty$ controllers to cope with the actuator saturation constraint. It is important to highlight that the proposed methodology also provides conditions to design switched output feedback $\mathscr{H}_\infty$ controllers for plants with only one dynamic subsystem. These two different situations are detailed in numerical examples.

Two different strategies to design output feedback controllers considering output dependent switching strategy for uncertain switched linear systems are presented. Firstly, it is considered that there does not exist exogenous input either control input. In sequence, the results presented in Mainardi Júnior *et al.* (2015) are relaxed and the inclusion of guaranteed cost performance and decay rate criterion is approached. Following, novel conditions are proposed based on less conservative results available in Liu and Zhang (2003), Teixeira, Assunção and Avellar (2003), Mozelli and Palhares (2011).

Besides that, novel and less conservative conditions for switching SOF $\mathscr{H}_\infty$ control of continuous-time switched linear systems are proposed. The conditions are based on a recent SOF $\mathscr{H}_\infty$ control design presented in Chang, Park and Zhou (2015) that avoids linear matrices equalities and does not impose any constraints on output systems matrices, as treated in Crusius and Trofino (1999). The results presented in Chang, Park and Zhou (2015) are relaxed considering the inclusion of switched output feedback $\mathscr{H}_\infty$ controllers jointly with an output-dependent switching strategy.

Furthermore, as a step to achieve the main result, conditions design to robust SOF $\mathscr{H}_\infty$ design for system subject to actuator saturation, based on Chang, Park and Zhou (2015), are proposed.

Finally, the main result that concerns conditions to design robust $\mathscr{H}_\infty$ switched static output feedback control design for linear switched systems subject to actuator saturation, using parameter-dependent Lyapunov function, is stated. This result takes into account the contributions of (ALVES *et al.*, 2016) and (OLIVEIRA *et al.*, 2018) concerning switched controllers under saturation, operation region and bounded energy disturbance approach.

Moreover, some bilinear terms appear in conditions of the proposed theorems. The conditions of the proposed methods are a special class of BMIs (Bilinear Matrix Inequalities), which contain some bilinear terms as the product of a matrix and a scalar, related to a suitable convex combination and two scalar parameters to provide extra free dimensions in the solution space. Currently, to the best of the authors' knowledge, there are not available solvers (deterministic methods) able to find the optimum solution for non-convex problems. Thus, the proposed design method of the output gains in order to stabilise an uncertain switched linear system is an NP-hard problem (LIN; ANTSAKLIS, 2009). Therefore, it is proposed the use of a hybrid metaheuristic technique, called DE-LMI (Differential Evolution - Linear Matrix Inequality) (STORN; PRICE, 1997) for finding quasi-optimum parameters values and/or a suitable convex combination in the design of SOF gains (SANDOU, 2013). The proposed procedure can also be used for designing robust controllers for uncertain plants subject to structural failures, considering the plant uncertainties and the structural failures as polytopic uncertainties (SILVA *et al.*, 2013).

1.3 OUTLINE AND NOTATION

- Chapter 2 presents the initial concepts involving dynamical systems, Lyapunov theory and a general definition of polytopic uncertain switched linear systems.
- Chapter 3 addresses the first and second problems statement and relaxation results for robust SOF control design for uncertain switched linear systems with an output dependent switching law introduced in Mainardi Júnior *et al.* (2015) jointly with a performance criterion and the decay rate. Based on the relaxation concepts available in Liu and Zhang (2003), Teixeira, Assunção and Avellar (2003) and Mozelli and Palhares (2011) are proposed novel conditions for stability of uncertain switched linear systems. A theoretical analysis shows if the conditions given in Mainardi Júnior *et al.* (2015) hold, then the novel proposed conditions also hold. A numerical example illustrates the flexibility obtained through of these less conservative conditions comparing feasible area and guaranteed cost obtained in the theorems proposed in this chapter.
- Chapter 4 presents the results for robust SOF $\mathscr{H}_\infty$ control developed in Chang, Park and Zhou (2015). The third problem is stated and it is developed novel and less conservative conditions for robust switching SOF $\mathscr{H}_\infty$ control of continuous time switched linear

systems. Furthermore, the results available in Chang, Park and Zhou (2015) are generalised through a switched output $\mathscr{H}_\infty$ controller. A theoretical analysis shows that these new conditions hold when the conditions presented in Chang, Park and Zhou (2015) hold. Finishing the contributions of this chapter, it is showed that the proposed methodology to design robust SOF $\mathscr{H}_\infty$ switched controllers can be directly applied to non-switched linear systems. In Example III (Chapter 8) is shown that there exist cases where the conditions from Theorem 7 are less conservative than the conditions from Theorem 2.

- Chapter 5 deals with the problem of design output-dependent $\mathscr{H}_\infty$ controllers for linear systems subject to actuator saturation. The conditions presented in Chang, Park and Zhou (2015) are exploited to obtain conditions to cope with the $\mathscr{H}_\infty$ problem considering operation region for linear systems subject to actuator saturation.

- Chapter 6 addresses the main contribution, aiming the design of switched static output feedback controllers to cope with the $\mathscr{H}_\infty$ control problem for linear switched systems subject to actuator saturation and considering operation region.

- Chapter 7 introduces the DE-LMI algorithm and briefly describes how it is applied in order to solve the proposed control problem.

- Six examples in Chapter 8 illustrate the effectiveness of the proposed methods. The first example is related to switched systems without either control input or disturbance. The following examples show, in some cases, that proposed conditions hold while the conditions present in literature do not hold. Furthermore, a practical application and an implementation example are presented.

- Finally, in Chapter 9 the conclusions and suggestions for future work are discussed.

1.3.1 Notation

The notation used in this document are described as follows. For real matrices or vectors $(')$ indicates transpose. The set composed by the first N positive integers $\{1,\ldots,N\}$ is represented by $\mathbb{K}_N$. The set of all vectors $\lambda = [\lambda_1 \; \ldots \; \lambda_N]'$ such that $\lambda_i \geq 0$, $i \in \mathbb{K}_N$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_N = 1$ is designated by Λ_N. The convex combination of a set of matrices $(A_1,\ldots,A_N)$ is denoted by $A_\lambda = \sum_{i=1}^{N} \lambda_i A_i$, where $\lambda \in \Lambda_N$. In addition, an asterisk $(*)$ will be used in matrix expressions to express the transpose of the symmetric element. Moreover, for in-line expressions, the symbol $(*)$ represents the transpose of the left side term. The notation $\mathrm{He}(M)$ refers to $M + M'$. The set of all finite $\zeta(t)$ trajectories, such that $\int_0^\infty \zeta(t)'\zeta(t)dt < \infty$ is denoted by $\mathscr{L}_2$. For simplicity of notation, $\sigma(t) = \sigma$. Abusing of the notation already defined as Λ_N, the following one denotes the convex combination of the vector $\alpha \in \mathbb{K}_r$,

$$\alpha = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_r]^T \in \Lambda_r = \left\{ \alpha \in \mathbb{R}^r : \alpha_i \geq 0, i \in \mathbb{K}_r, \sum_{i=1}^{r} \alpha_i = 1 \right\}. \tag{1}$$

2 INITIAL CONCEPTS

This chapter covers the initial concepts involving dynamical systems, Lyapunov Stability Theory and uncertain switched linear systems.

2.1 DYNAMICAL SYSTEMS

A dynamical autonomous (time-invariant) system can be described employing a set of equations as:

$$\dot{x}(t) = f(x(t)), \ \ x(t_0) \tag{2}$$

where $t \geq t_0$, $x(t_0)$ is the given initial condition, $x(t) \in \mathbb{R}^{n_x \times 1}$ represents the state vector and $f(.)$ it is an linear or non-linear function, depending on system.

If the systems (2) has an input control and is subject to a disturbance, it is possible to represent it as follows

$$\dot{x}(t) = f(x(t), u(t), w(t)), \ \ x(t_0) \tag{3}$$

where $u(t) \in \mathbb{R}^{n_u \times 1}$ is the control input and $w(t) \in \mathbb{R}^{n_w \times 1}$ is the exogenous disturbance.

2.2 LYAPUNOV STABILITY THEORY

The theory is named after Aleksandr Mikhailovich Lyapunov, who was a Russian mechanician, mathematician, and physicist. He provided two methods for stability analysis in his work *The General Problem of Motion Stability* first published in 1892. The addressed theory is also called the Lyapunov's Second Method or Direct Method of Lyapunov and it is based on the systems energy concept. Through the rate change of the system energy is possible to ascertain its stability. The Lyapunov method allows to conclude about a system stabilisation by means of a scalar function, named Lyapunov candidate function or simply Lyapunov function. The theoretical formulation of the method is given in Theorem 1.

Theorem 1. *(SLOTINE; LI, 1991) Assume that there exists a scalar function $V(x)$, with continuous first order derivatives such that:*

- *$V(x)$ is positive definite;*
- *$\dot{V}(x)$ is negative definite;*

- *$V(x) \to \infty$ as $\|x\| \to \infty$.*

then the equilibrium at the origin of the system (3) *is globally asymptotically stable.*

2.3 SWITCHED LINEAR SYSTEMS

A continuous-time uncertain switched linear system can be defined by the following state-space description:

$$\begin{cases} \dot{x}(t) &= A(\sigma,\alpha)x(t)+B(\sigma,\alpha)u(t)+E(\sigma,\alpha)w(t), \quad x(0)=x_0, \\ z(t) &= C_1(\sigma,\alpha)x(t)+D(\sigma,\alpha)u(t)+F(\sigma,\alpha)w(t), \\ y(t) &= C_2(\alpha)x(t)+H(\alpha)w(t), \end{cases} \tag{4}$$

where $x(t) \in \mathrm{I\!R}^{n_x}$ is the state vector, $y(t) \in \mathrm{I\!R}^{n_y}$ is the measured output, $z(t) \in \mathrm{I\!R}^{n_z}$ is the controlled output, $u(t) \in \mathrm{I\!R}^{n_u}$ is the control input, $w(t) \in \mathrm{I\!R}^{n_w}$ is an exogenous disturbance input with $w(t) \in \mathscr{L}_2\,[0,\infty)$ and x_0 is the initial condition. The constant vector $\alpha = [\alpha_1\ \alpha_2\ \ldots\ \alpha_r]' \in \Lambda_r$ represents the polytopic uncertainties of the plant or structural failures (SILVA *et al.*, 2013) and (CARDIM *et al.*, 2016). Consider that $\sigma(t) \in \mathrm{I\!K}_N$ is the switching strategy which selects at each instant of time an available subsystem $i \in \mathrm{I\!K}_N$. The matrices $A(\sigma,\alpha)$, $B(\sigma,\alpha)$, $E(\sigma,\alpha)$, $C_1(\sigma,\alpha)$, $D(\sigma,\alpha)$, $F(\sigma,\alpha)$, $C_2(\alpha)$ and $H(\alpha)$ are constant matrices of appropriate dimensions and can be described by convex combinations of their vertices, as below:

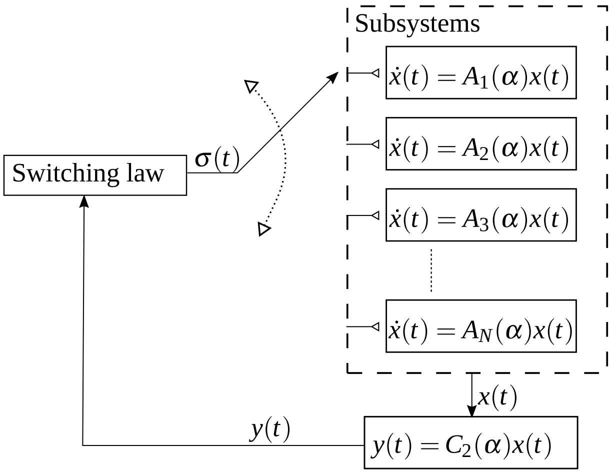
$$\Big\{ \Big[\ A(\sigma,\alpha),\ \ B(\sigma,\alpha),\ \ E(\sigma,\alpha),\ \ C_1(\sigma,\alpha),\ \ D(\sigma,\alpha),\ \ F(\sigma,\alpha),\ \ C_2(\alpha),\ \ H(\alpha)\ \Big]$$
$$= \sum_{j=1}^{r} \alpha_j \Big[\ A_{\sigma j},\ \ B_{\sigma j},\ \ E_{\sigma j},\ \ C_{1\sigma j},\ \ D_{\sigma j},\ \ F_{\sigma j},\ \ C_{2j},\ \ H_j\ \Big] \Big\} = \Delta,$$
$$\alpha_j \geq 0,\ \ j \in \mathrm{I\!K}_r,\ \ \sum_{j=1}^{r} \alpha_j = 1,\ \ \sigma \in \mathrm{I\!K}_N, \tag{5}$$

where r is the number of vertices of the polytope and N is the number of subsystems.

In Chapter 3, the switched linear system (4) and (5) is considered without control input ($u(t) = 0$, $t \geq 0$), without exogenous disturbance ($w(t) = 0, t \geq 0$) and assuming constant output matrices $C_{2j} = C$, for all $j \in \mathrm{I\!K}_r$. Therefore, since there is no control input ($u(t) = 0$), only the switching law $\sigma(t)$ is responsible to stabilise the system, selecting at each instant of time an available subsystem. Figure 1 illustrates this scenario.

In the remainder of this thesis, we assume the presence of the exogenous disturbance ($w(t)$) and that the system control input ($u(t)$) can be considered for feedback. Moreover, in Chapters 4 and 5 the gain control matrix (K_{ic}, $i \in \mathbb{K}_N$, $c \in \mathbb{K}_r$) is also switched considering the switching signal. The amount of gain control matrices to be designed depends on the number of subsystems (N) and the number of vertices of the polytope (r). We consider $N \times r$ control

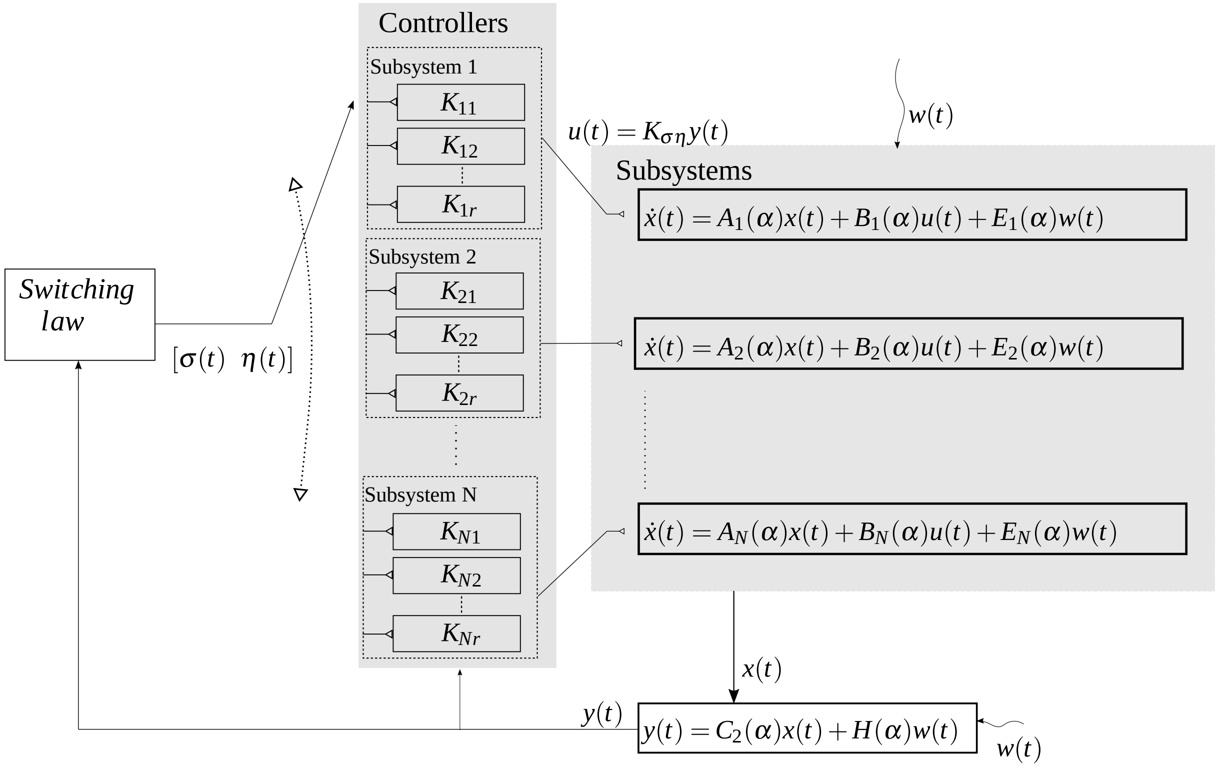
Figure 1 - Schematic of the switched system without control input ($u(t)$).



Source: Own author.

gain matrices, that is, for each subsystem jointly with a vertex a gain control matrix is designed. For instance, if the system is described by two subsystems ($N = 2$) and the uncertainty polytope with four vertices ($r = 4$), one obtains 8 controllers given following the notation K_{11}, K_{12}, K_{13}, K_{14}, K_{21}, K_{22}, K_{23} and K_{24}. Therefore, the switching strategy selects the subsystem and the control gain matrix in each instant of time, as presented in Figure 2. In order words, the proposed switching strategy assigns the index related to the subsystem and gain selection (σ) and the index of the control gain matrix selection related to the each vertex (η). Therefore, in each instant of time a control gain matrix $K_{\sigma\eta}$ is selected.

Figure 2 - Schematic of the switched linear control system.



Source: Own author.

3 OUTPUT SWITCHING STRATEGY FOR UNCERTAIN SWITCHED LINEAR SYSTEMS

In practical applications, the state vector may not be completely available. In these cases, it is important to propose switching strategies based on the measured output of the plant. Therefore, this chapter is devoted to present results concerning the output-dependent switching control.

In this chapter the plant equations are given by the switched linear system (4) and (5), but without control input ($u(t) = 0,\ t \geq 0$), without exogenous disturbance ($w(t) = 0, t \geq 0$) and assuming constant output matrices $C_{2j} = C$, for all $j \in \mathbb{K}_r$, as in (Mainardi Júnior *et al.*, 2015).

Assume that the state vector $x(t) \in \mathbb{R}^{n_x}$ is not completely available, but $y(t) \in \mathbb{R}^{n_y}$ is accessible for feedback.

Problem 1. *Determine a switching strategy* $\sigma(.) : \mathbb{R}^p \to \{1, 2, \ldots, N\}$ *such that:*

$$\sigma(t) = f(y(t)),\ \ \textit{for all}\ \ t \geq 0, \tag{6}$$

which makes the origin ($x = 0$) of the polytopic uncertain switched linear system (4) *and* (5) *a globally asymptotically stable equilibrium point, supposing that $u(t) = 0$ and $w(t) = 0$ for $t \geq 0$, and assuming constant output matrices $C_{2j} = C$, for all $j \in \mathbb{K}_r$.*

Mainardi Júnior *et al.* (2015) presented less conservative conditions for Problem 1, described in Theorem 2, than conditions available in the previous literature. Observe that, auxiliaries matrices $\hat{\mathcal{Q}}_i \in \mathbb{K}_N$ are applied to determine the switching strategy.

Theorem 2. *(Mainardi Júnior et al., 2015) If there exist $\lambda \in \Lambda_N$, matrices $X_{1_{ik}} \in \mathbb{R}^{n_x \times n_x}$, $X_{2_{ik}} \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices $Q_{0jk} \in \mathbb{R}^{n_x \times n_x}$, $\hat{\mathcal{Q}}_i \in \mathbb{R}^{n_y \times n_y}$ and symmetric positive definite matrices $P_{jk} \in \mathbb{R}^{n_x \times n_x}$, such that*

$$\begin{bmatrix} X_{1_{ik}}A_{ij} + A_{ij}'X_{1_{ik}}' + X_{1_{ij}}A_{ik} + A_{ik}'X_{1_{ij}}' & P_{jk} - X_{1_{ik}} + A_{ij}'X_{2_{ik}}' + P_{kj} - X_{1_{ij}} + A_{ik}'X_{2_{ij}}' \\ P_{jk} - X_{1_{ik}}' + X_{2_{ik}}A_{ij} + P_{kj} - X_{1_{ij}}' + X_{2_{ij}}A_{ik} & -X_{2_{ik}} - X_{2_{ik}}' - X_{2_{ij}} - X_{2_{ij}}' \end{bmatrix} < \begin{bmatrix} Q_{0jk} + Q_{0kj} + 2C'\hat{\mathcal{Q}}_iC & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

$$Q_{0jk} + C'\hat{\mathcal{Q}}_{\lambda}C < 0, \tag{8}$$

for all $i \in \mathbb{K}_N$, $j \in \mathbb{K}_r$ *and* $k \in \mathbb{K}_r$, *then the switching strategy*

$$\sigma(y) = arg \min_{i \in \mathbb{K}_N} (y'\hat{\mathcal{Q}}_i y) \tag{9}$$

makes the origin $x = 0$ *of the uncertain switched linear system* (4) *and* (5) *a globally asymptotically stable equilibrium point, supposing that* $u(t) = 0$ *and* $w(t) = 0$ *for all* $t \geq 0$, *and assuming constant output matrices* $C_{2j} = C$, *for all* $j \in \mathbb{K}_r$.

Proof. Consider that (7) and (8) are feasible. It is known that the minimum of a set of real numbers is less than or equal to an arbitrary convex combination of these numbers. Then, from (8) and (9), for $x \neq 0$ it follows that:

$$0 > x'(Q_{0jk} + C'\hat{\mathcal{Q}}_{\lambda}C)x \geq x'Q_{0jk}x + \min_{i \in \mathbb{K}_N} (y'\hat{\mathcal{Q}}_i y) = x'(Q_{0jk} + C'\hat{\mathcal{Q}}_{\sigma}C)x, \tag{10}$$

where $\lambda = [\lambda_1 \; \lambda_2 \; \ldots \; \lambda_N \;]$, $\sum_{i=1}^{N} \lambda_i = 1$ and $\lambda_i \geq 0$, for all $i \in \mathbb{K}_N$. Observe that (10) can be rewritten as:

$$x'(Q_{0jk} + C'\hat{\mathcal{Q}}_{\sigma}C)x = \\ x' \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix}' \begin{bmatrix} Q_{0jk} + C'\hat{\mathcal{Q}}_{\sigma}C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix} x < 0. \tag{11}$$

Thus, multiplying (11) by $\alpha_j \times \alpha_k$ and taking the sum from $j = 1$ to $j = r$ and $k = 1$ to $k = r$, respectively, from (7), note that:

$$\begin{aligned} 0 &> \sum_{k=1}^{r} \alpha_k \sum_{j=1}^{r} \alpha_j \, x' \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix}' \begin{bmatrix} Q_{0jk} + Q_{0kj} + 2C'\hat{\mathcal{Q}}_{\sigma}C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix} x \\ &> \sum_{k=1}^{r} \alpha_k \sum_{j=1}^{r} \alpha_j \, x' \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix}' \begin{bmatrix} X_{1\sigma k}A_{\sigma j} + A_{\sigma j}'X_{1\sigma k}' + X_{1\sigma j}A_{\sigma k} + A_{\sigma k}'X_{1\sigma j}' \\ P_{jk} - X_{1\sigma k}' + X_{2\sigma k}A_{\sigma j} + P_{kj} - X_{1\sigma j}' + X_{2\sigma j}A_{\sigma j} \end{bmatrix. \\ &\qquad \left. \begin{matrix} P_{jk} - X_{1\sigma k} + A_{\sigma j}'X_{2\sigma k}' + P_{kj} - X_{1\sigma j} + A_{\sigma k}'X_{2\sigma j}' \\ -X_{2\sigma k} - X_{2\sigma k}' - X_{2\sigma j} - X_{2\sigma j}' \end{matrix} \right] \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix} x \\ &= 2 \sum_{k=1}^{r} \alpha_k \sum_{j=1}^{r} \alpha_j \, x' \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix}' \begin{bmatrix} X_{1\sigma k}A_{\sigma j} + A_{\sigma j}'X_{1\sigma k}' & P_{jk} - X_{1\sigma k} + A_{\sigma j}'X_{2\sigma k}' \\ P_{jk} - X_{1\sigma k}' + X_{2\sigma k}A_{\sigma j} & -X_{2\sigma k} - X_{2\sigma k}' \end{bmatrix} \\ &\qquad \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix} x. \end{aligned} \tag{12}$$

Now, define $P(\alpha) = (\alpha_1\alpha_1 P_{11} + \alpha_1\alpha_2 P_{12} + \ldots + \alpha_r\alpha_r P_{rr})$, $X_1(\sigma,\alpha) = (\alpha_1 X_{1\sigma 1} + \alpha_2 X_{1\sigma 2} +$

$\ldots + \alpha_r X_{1\sigma r})$, $X_2(\sigma,\alpha) = (\alpha_1 X_{2\sigma 1} + \alpha_2 X_{2\sigma 2} + \ldots + \alpha_r X_{2\sigma r})$. Then, from (5) and (12), one has:

$$0 > x' \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix}' \begin{bmatrix} X_1(\sigma,\alpha)A(\sigma,\alpha) + A'(\sigma,\alpha)X_1'(\sigma,\alpha) & P(\alpha) - X_1(\sigma,\alpha) + A'(\sigma,\alpha)X_2'(\sigma,\alpha) \\ P(\alpha) - X_1'(\sigma,\alpha) + X_2(\sigma,\alpha)A(\sigma,\alpha) & -X_2(\sigma,\alpha) - X_2'(\sigma,\alpha) \end{bmatrix} \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix} x$$

$$= x' \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix}' \left\{ \begin{bmatrix} 0 & P(\alpha) \\ P(\alpha) & 0 \end{bmatrix} + \begin{bmatrix} X_1(\sigma,\alpha) \\ X_2(\sigma,\alpha) \end{bmatrix} \begin{bmatrix} A(\sigma,\alpha) & -I_n \end{bmatrix} + \begin{bmatrix} A'(\sigma,\alpha) \\ -I_n \end{bmatrix} \begin{bmatrix} X_1'(\sigma,\alpha) & X_2'(\sigma,\alpha) \end{bmatrix} \right\} \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix} x$$

$$= x' \begin{bmatrix} I_n & A'(\sigma,\alpha) \end{bmatrix} \begin{bmatrix} 0 & P(\alpha) \\ P(\alpha) & 0 \end{bmatrix} \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix} x. \quad (13)$$

Considering a Lyapunov function candidate $V(x) = x'P(\alpha)x$, note that from (5), $V(x) > 0$ for $x \neq 0$ and from (4), supposing that $u(t) = 0$ and $w(t) = 0$ for $t \geq 0$, and assuming constant output matrices $C_{2j} = C$, for all $j \in \mathbb{K}_r$, and (13) it follows that $\dot{V}(x) = \dot{x}(t)'P(\alpha)x(t) + x(t)'P(\alpha)\dot{x}(t) < 0$ for $x \neq 0$. The proof is concluded. □

In order to establish a performance criterion for the switched linear system (4) and (5), the following control problem introduces the guaranteed cost performance (DEAECTO *et al.*, 2010), as an extension of Problem 1. This problem was also addressed in Carniato (2016).

Problem 2. *Determine a switching strategy* (6) *which makes the origin $x = 0$ of the controlled polytopic uncertain switched linear system* (4) *and* (5), *supposing that $u(t) = 0$ and $w(t) = 0$ for all $t \geq 0$, and assuming constant output matrices $C_{2j} = C$, for all $j \in \mathbb{K}_r$, a globally asymptotically stable equilibrium point and the guaranteed cost*

$$J = \int_0^\infty z(t)'z(t)dt = \int_0^\infty x(t)'C_1(\sigma,\alpha)'C_1(\sigma,\alpha)x(t)dt < \mu x_0' I_n x_0 \quad (14)$$

holds for a given scalar $\mu > 0$ and all initial conditions, $x_0 = x(0) \neq 0$.

A solution of Problem 2, that is an extension of the results presented in (Mainardi Júnior *et al.*, 2015) for cases where is required a guaranteed cost performance given in (14) and related to the controlled output $z(t) = C_1(\sigma,\alpha)x(t)$ defined in (4), is proposed in Theorem 3. Furthermore, a specification of the decay rate is also considered in the problem. The decay rate is defined in (BOYD *et al.*, 1994) as the largest positive real constant (χ), such that:

$$\lim_{t\to\infty} e^{\chi t} \|x(t)\| = 0. \quad (15)$$

Moreover, if $P(\alpha) = P(\alpha)' > 0$ is a constant matrix and $V(x(t)) = x(t)'P(\alpha)x(t)$ is a Lyapunov function for a given system, then the condition $\dot{V}(x(t)) \leq -2\chi V(x(t))$ assures that the decay rate is greater than or equal to χ.

Theorem 3. *Consider that there exist $\lambda \in \Lambda_N$, a scalar $\mu > 0$, matrices $X_{1_{ik}} \in \mathbb{R}^{n_x \times n_x}$, $X_{2_{ik}} \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices $Q_{0jk} \in \mathbb{R}^{n_x \times n_x}$, $\hat{\mathcal{Q}}_i \in \mathbb{R}^{n_y \times n_y}$ and symmetric positive definite matrices $P_{jk} \in \mathbb{R}^{n_x \times n_x}$, such that*

$$P_{jk} - \mu I_n < 0, \tag{16}$$

$$\begin{bmatrix} X_{1_{ik}}A_{ij} + A'_{ij}X'_{1_{ik}} + X_{1_{ij}}A_{ik} + A'_{ik}X'_{1_{ij}} + 2C'_{1ij}C_{1ik} + 2\chi(P_{jk} + P_{kj}) & P_{jk} - X_{1_{ik}} + A'_{ij}X'_{2_{ik}} + P_{kj} - X_{1_{ij}} + A'_{ik}X'_{2_{ij}} \\ P_{jk} - X'_{1_{ik}} + X_{2_{ik}}A_{ij} + P_{kj} - X'_{1_{ij}} + X_{2_{ij}}A_{ik} & -X_{2_{ik}} - X'_{2_{ik}} - X_{2_{ij}} - X'_{2_{ij}} \end{bmatrix} < \begin{bmatrix} Q_{0_{jk}} + Q_{0_{kj}} + 2C'\hat{\mathcal{Q}}_i C & 0 \\ 0 & 0 \end{bmatrix}, \tag{17}$$

$$Q_{0_{jk}} + C'\hat{\mathcal{Q}}_\lambda C < 0. \tag{18}$$

Then, the switching strategy (9) *makes the origin $x = 0$ of the uncertain switched linear system* (4) *and* (5), *supposing that $u(t) = 0$ and $w(t) = 0$ for $t \geq 0$, and assuming constant output matrices $C_{2j} = C$, for all $j \in \mathbb{K}_r$, a globally asymptotically stable equilibrium point, the decay rate is greater than or equal to χ and the guaranteed cost* (14) *holds for all initial conditions, $x_0 = x(0) \neq 0$.*

Proof. Considering that $C'_{1ij}C_{1ij} \geq 0$ and $2\chi(P_{jk} + P_{kj}) > 0$, then if (17) and (18) hold, then (7) and (8) also hold. Therefore, from Theorem 2, the conditions (17) and (18) assure that the equilibrium point $x = 0$ of the controlled system (4), (5), is globally asymptotically stable. The proof of the Theorem 2 was performed considering a Lyapunov function $V(x(t)) = x(t)'P(\alpha)x(t)$, where $P(\alpha) = \sum_{j=1}^r \sum_{k=1}^r \alpha_j\alpha_k P_{jk}$ and $P_{jk} = P'_{jk} > 0$, for all $j,k \in \mathbb{K}_r$. Now, following the same steps used in the proof of Theorem 2, applied to the conditions (17) and (18), supposing that $C_{1ij} \neq 0$, from (4), one obtains for $x(t) \neq 0$:

$$\begin{aligned} 0 &> x(t)' \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix}' \begin{bmatrix} C_1(\sigma,\alpha)'C_1(\sigma,\alpha) + 2\chi P(\alpha) & P(\alpha) \\ P(\alpha) & 0 \end{bmatrix} \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix} x(t), \\ 0 &> \dot{V}(x(t)) + 2\chi V(x(t)) + z(t)'z(t), \end{aligned} \tag{19}$$

where $V(x(t)) = x(t)'P(\alpha)x(t)$ was defined above. Therefore, from (19), $\dot{V}(x(t)) < -2\chi V(x(t))$. Thus, the decay rate is greater than or equal to χ. From (5), note that $V(x(t)) > 0$ for $x(t) \neq 0$. Observe that, from (4) and (19), $\dot{V}(x(t)) < 0$ for $x(t) \neq 0$ and thus $x(\infty) = 0$. Now, integrating (19) from zero to infinity, considering $x_0 = x(0) \neq 0$, knowing that $V(x(\infty)) = 0$, from (16) and remembering that from (5) $(\alpha_1+\alpha_2+\ldots+\alpha_r)^2=1$, it follows that:

$$J=\int_0^{\infty} z(t)'z(t)dt < x_0'P(\alpha)x_0 < \mu x_0' I_n x_0. \tag{20}$$

The proof is concluded. □

Now, in order to relax the feasibility of the LMIs from Theorem 3, based on the results presented in Liu and Zhang (2003), Teixeira, Assunção and Avellar (2003), Souza *et al.* (2014), Deaecto, Geromel and Daafouz (2011) and inspired on the Finsler's Lemma (QIU; FENG; YANG, 2008; MOZELLI; PALHARES, 2011), less conservative conditions are proposed in Theorem 4.

Theorem 4. *Consider that there exist* $\lambda \in \Lambda_N$, *a scalar* $\mu>0$, *symmetric matrices* $Q_{0_{jk}} \in \mathbb{R}^{n_x\times n_x}$, $\hat{Q}_i \in \mathbb{R}^{n_y\times n_y}$ *and matrices* $\xi_{ijk}=\xi_{ikj}' \in \mathbb{R}^{2n_x\times 2n_x}$, $\phi_{jk}=\phi_{kj}' \in \mathbb{R}^{n_x\times n_x}$, $w_{jk}=w_{kj}' \in \mathbb{R}^{n_x\times n_x}$, $P_{jk}=P_{kj}' \in \mathbb{R}^{n_x\times n_x}$, $X_{1_{ik}} \in \mathbb{R}^{n_x\times n_x}$, $X_{2_{ik}} \in \mathbb{R}^{n_x\times n_x}$, *such that*

$$\psi_{ikk}-\gamma_{ikk} < \xi_{ikk}, \tag{21}$$

$$\psi_{ijk}-\gamma_{ijk}+(*) < \xi_{ijk}+\xi_{ijk}',\ k\neq j, \tag{22}$$

$$\theta_{kk} < \phi_{kk}, \tag{23}$$

$$\theta_{jk}+\theta_{jk}' < \phi_{jk}+\phi_{jk}',\ k\neq j, \tag{24}$$

$$P_{kk} > w_{kk} \tag{25}$$

$$P_{jk}+P_{jk}' > w_{jk}+w_{jk}',\ k\neq j, \tag{26}$$

$$P_{kk}-\mu I_n < 0, \tag{27}$$

$$\frac{1}{2}\times\left(P_{jk}+P_{jk}'\right)-\mu I_n < 0,\ \ k\neq j, \tag{28}$$

$$W^{\star}>0, \tag{29}$$

$$\Phi^{\star} < 0, \tag{30}$$

$$\Xi_i^{\star} < 0, \tag{31}$$

for all $i\in\mathbb{K}_N$, $j,k\in\mathbb{K}_r$, *where,*

$$W^{\star}=\begin{bmatrix} w_{11} & \ldots & w_{1r}\\ \vdots & \ddots & \vdots\\ w_{r1} & \ldots & w_{rr}\end{bmatrix}, \tag{32}$$

$$\Phi^{\star}=\begin{bmatrix} \phi_{11} & \ldots & \phi_{1r}\\ \vdots & \ddots & \vdots\\ \phi_{r1} & \ldots & \phi_{rr}\end{bmatrix}, \tag{33}$$

$$\Xi_i^\star = \begin{bmatrix} \xi_{i11} & \dots & \xi_{i1r} \\ \vdots & \ddots & \vdots \\ \xi_{ir1} & \dots & \xi_{irr} \end{bmatrix}, \tag{34}$$

$$\psi_{ijk} = \begin{bmatrix} \nu_{ijk} & \vartheta_{ijk} \\ (*) & -X_{2_{ik}} - X'_{2_{ik}} - X_{2_{ij}} - X'_{2_{ij}} \end{bmatrix}, \tag{35}$$

$$\gamma_{ijk} = \begin{bmatrix} Q_{0_{jk}} + Q_{0_{kj}} + 2C'\hat{\mathcal{Q}}_i C & 0 \\ 0 & 0 \end{bmatrix}, \tag{36}$$

$$\theta_{jk} = Q_{0_{jk}} + C'\hat{\mathcal{Q}}_\lambda C, \tag{37}$$

$\nu_{ijk} = X_{1_{ik}} A_{ij} + A'_{ij} X'_{1_{ik}} + X_{1_{ij}} A_{ik} + A'_{ik} X'_{1_{ij}} + 2C'_{1ij} C_{1ik} + 2\chi(P_{jk} + P_{kj})$, $\vartheta_{ijk} = P_{jk} - X_{1_{ik}} + A'_{ij} X'_{2_{ik}} + P_{kj} - X_{1_{ij}} + A'_{ik} X'_{2_{ij}}$, $\hat{\mathcal{Q}}_\lambda = \lambda_1 \hat{\mathcal{Q}}_1 + \lambda_2 \hat{\mathcal{Q}}_2 + \ldots + \lambda_N \hat{\mathcal{Q}}_N$, $\lambda = [\lambda_1\ \lambda_2 \ldots \lambda_N]$, $\sum_{i=1}^N \lambda_i = 1$, *and* $\lambda_i \geq 0$ *for all* $i \in \mathbb{K}_N$. *Then, the switching strategy* (9) *makes the origin* $x = 0$ *of the uncertain switched linear system* (4) *and* (5), *supposing that* $u(t) = 0$ *and* $w(t) = 0$ *for* $t \geq 0$, *and assuming constant output matrices* $C_{2j} = C$, *for all* $j \in \mathbb{K}_r$, *a globally asymptotically stable equilibrium point, the decay rate is greater than or equal to* χ *and the guaranteed cost* (14) *holds for all initial conditions,* $x_0 = x(0) \neq 0$.

Proof. Take into account the following definitions:

$$X = \begin{bmatrix} X_1(\sigma,\alpha) \\ X_2(\sigma,\alpha) \end{bmatrix}, \quad T = \begin{bmatrix} I_n \\ A(\sigma,\alpha) \end{bmatrix},$$

$$\sum = \sum_{j=1}^r \alpha_j \sum_{k=1}^r \alpha_k, \quad \Delta(\sigma,\alpha) = \begin{bmatrix} \alpha_1 T' & \alpha_2 T' & \dots & \alpha_r T' \end{bmatrix}'. \tag{38}$$

Afterwards, consider that (21)-(31) are feasible. Then, from (21), (22), (33)-(38), it follows that, for $x = x(t) \neq 0$:

$$0 > x'\Delta'(\sigma,\alpha)\Xi_i^\star \Delta(\sigma,\alpha)x = x' \sum T'\xi_{ijk} T x > x' \sum T' \left(\psi_{ijk} - \gamma_{ijk}\right) T x. \tag{39}$$

From (35)-(37), note that (39) can be rewritten as:

$$0 > x' \sum T' \left(\begin{bmatrix} \nu_{ijk} & \vartheta_{ijk} \\ \vartheta'_{ijk} & -X_{2_{ik}} - X'_{2_{ik}} - X_{2_{ij}} - X'_{2_{ij}} \end{bmatrix} \right. \\ \left. - \begin{bmatrix} Q_{0_{jk}} + Q_{0_{kj}} + 2C'\hat{\mathcal{Q}}_i C & 0 \\ 0 & 0 \end{bmatrix} \right) T x. \tag{40}$$

Observe that, $\sum \left(X_{1_{ik}} A_{ij} + X_{1_{ij}} A_{ik}\right) = 2 \sum X_{1_{ik}} A_{ij}$, $\sum \left(P_{jk} + P_{kj}\right) = 2 \sum P_{jk}$, $\sum \left(X_{1_{ik}} + X_{1_{ij}}\right) = 2 \sum X_{1_{ik}}$ and $\sum \left(Q_{0_{jk}} + Q_{0_{kj}}\right) = 2 \sum Q_{0_{jk}}$. Therefore, verify that (40) can also be described as follows:

$$0 > 2x' \sum T' \left(\begin{bmatrix} X_{1_{ik}} A_{ij} + (*) + C'_{1ij} C_{1ik} + 2\chi P_{jk} & (*) \\ P_{jk} - X'_{1_{ik}} + X_{2_{ik}} A_{ij} & -X_{2_{ik}} - X'_{2_{ik}} \end{bmatrix} - \begin{bmatrix} Q_{0_{jk}} + C' \hat{\mathcal{Q}}_i C & 0 \\ 0 & 0 \end{bmatrix} \right) Tx. \quad (41)$$

Then, define $X_1(\sigma,\alpha) = \alpha_1 X_{1_{\sigma 1}} + \alpha_2 X_{1_{\sigma 2}} + \ldots + \alpha_r X_{1_{\sigma r}}$, $X_2(\sigma,\alpha) = \alpha_1 X_{2_{\sigma 1}} + \alpha_2 X_{2_{\sigma 2}} + \ldots + \alpha_r X_{2_{\sigma r}}$, $Q_0(\alpha) = \alpha_1\alpha_1 Q_{0_{11}} + \alpha_1\alpha_2 Q_{0_{12}} + \alpha_2\alpha_1 Q_{0_{21}} + \ldots + \alpha_r\alpha_r Q_{0_{rr}}$, $\hat{\mathcal{Q}}_i(\alpha) = \alpha_1\alpha_1 Q_{i_{11}} + \alpha_1\alpha_2 Q_{i_{12}} + \alpha_2\alpha_1 Q_{i_{21}} + \ldots + \alpha_r\alpha_r Q_{i_{rr}}$, $C_1(\sigma,\alpha) = \alpha_1 C_{1\sigma 1} + \alpha_2 C_{1\sigma 2} + \ldots + \alpha_r C_{1\sigma r}$ and $P(\alpha) = \alpha_1\alpha_1 P_{11} + \alpha_1\alpha_2 P_{12} + \alpha_2\alpha_1 P_{21} + \ldots + \alpha_r\alpha_r P_{rr}$. Hence, from (5) and (41), considering (38) and $i = \sigma$ one obtains:

$$\begin{aligned} 0 > x'T' & \left(\begin{bmatrix} X_1(\sigma,\alpha)A(\sigma,\alpha) + (*) + C_1(\sigma,\alpha)'C_1(\sigma,\alpha) + 2\chi P(\alpha) & (*) \\ P(\alpha) - X_1'(\sigma,\alpha) + X_2(\sigma,\alpha)A(\sigma,\alpha) & -X_2(\sigma,\alpha) - X_2'(\sigma,\alpha) \end{bmatrix} - \begin{bmatrix} Q_0(\alpha) + C'\hat{\mathcal{Q}}_\sigma C & 0 \\ 0 & 0 \end{bmatrix} \right) Tx \\ = x'T' & \left(\left\{ \begin{bmatrix} C_1(\sigma,\alpha)'C_1(\sigma,\alpha) + 2\chi P(\alpha) & P(\alpha) \\ P(\alpha) & 0 \end{bmatrix} + X \begin{bmatrix} A'(\sigma,\alpha) \\ -I_n \end{bmatrix}' + \begin{bmatrix} A'(\sigma,\alpha) \\ -I_n \end{bmatrix} X' \right\} - \begin{bmatrix} Q_0(\alpha) + C'\hat{\mathcal{Q}}_\sigma C & 0 \\ 0 & 0 \end{bmatrix} \right) Tx \\ = x'T' & \left(\begin{bmatrix} C_1(\sigma,\alpha)'C_1(\sigma,\alpha) + 2\chi P(\alpha) & P(\alpha) \\ P(\alpha) & 0 \end{bmatrix} - \begin{bmatrix} Q_0(\alpha) + C'\hat{\mathcal{Q}}_\sigma C & 0 \\ 0 & 0 \end{bmatrix} \right) Tx. \end{aligned} \quad (42)$$

Now, considering that the minimum of a set of real numbers is less than or equal to an arbitrary convex combination of these numbers, observe that, from (4), (5), (9), (37), (38) and defining $V(x(t)) = x'(t)P(\alpha)x(t)$, (42) becomes:

$$\begin{aligned} 0 &> \dot{V}(x) + 2\chi V(x) + z'z - x'\left(Q_0(\alpha)\right)x - \min_{i \in \mathbb{K}_N} (y'\hat{\mathcal{Q}}_i y) \\ &\geq \dot{V}(x) + 2\chi V(x) + z'z - x' \sum (Q_{0_{jk}} + C'\hat{\mathcal{Q}}_\lambda C)x \\ &= \dot{V}(x) + 2\chi V(x) + z'z - x' \sum \theta_{jk} x. \end{aligned} \quad (43)$$

Consequently, considering a Lyapunov function candidate $V(x(t)) = x(t)'P(\alpha)x(t)$, where $P(\alpha)$ was defined above, from (23), (24), (37), note that (43) can be represented as:

$$\begin{aligned} 0 &> \dot{V}(x) + 2\chi V(x) + z'z - \sum_{j=1}^{r} \alpha_j \sum_{k=1}^{r} \alpha_k x' \theta_{jk} x, \\ &> \dot{V}(x) + 2\chi V(x) + z'z - \sum_{j=1}^{r} \alpha_j \sum_{k=1}^{r} \alpha_k x' \phi_{jk} x. \end{aligned} \quad (44)$$

Now, from (30) and (33), it follows that:

$$\dot{V}(x) + 2\chi V(x) + z'z < x' \sum_{j=1}^{r} \alpha_j \sum_{k=1}^{r} \alpha_k \, \phi_{jk} x,$$

$$\dot{V}(x) + 2\chi V(x) + z'z < x' \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \cdots \\ \alpha_r I \end{bmatrix}' \Phi^\star \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \cdots \\ \alpha_r I \end{bmatrix} x < 0. \quad (45)$$

Moreover, from (5), (25), (26), (29) and (32), observe that for $x(t) \neq 0$:

$$\begin{aligned} V(x(t)) &= x(t)' \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \cdots \\ \alpha_r I \end{bmatrix}' \begin{bmatrix} P_{11} & \cdots & P_{1r} \\ \vdots & \ddots & \vdots \\ P_{r1} & \cdots & P_{rr} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \cdots \\ \alpha_r I \end{bmatrix} x(t), \\ &> x(t)' \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \cdots \\ \alpha_r I \end{bmatrix}' \begin{bmatrix} w_{11} & \cdots & w_{1r} \\ \vdots & \ddots & \vdots \\ w_{r1} & \cdots & w_{rr} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \cdots \\ \alpha_r I \end{bmatrix} x(t), \\ &= x(t)' \sum_{j=1}^{r} \alpha_j \sum_{k=1}^{r} \alpha_k W^\star x(t) > 0. \end{aligned} \quad (46)$$

Furthermore, from (46), note that $V(x(t)) > 0$, for $x(t) \neq 0$. Now, considering that $P_{jk} = P'_{kj}$, from (5), (27) and (28) it follows that $P(\alpha) - \mu I_n < 0$. Now, from (45), $z(t)'z(t) \leq -\dot{V}(x(t))$, and integrating both sides from zero to infinity, considering $x_0 = x(0) \neq 0$, knowing that, from (45) and (46), the equilibrium point $(x = 0)$ of the uncertain switched system (4), (5), and (9) is globally asymptotically stable, then $V(x(\infty)) = 0$ and one obtains (14). Therefore, the proof is concluded. □

The following theorem shows that if the conditions given in Theorem 3 hold, then the conditions given in Theorem 4 also hold. In other words, the conditions proposed in Theorem 4 yields at least the same results as the conditions in Theorem 3. A comparative study regarding the feasibility of Theorems 3 and 4 will be presented in Example 3.1.

Theorem 5. *If the conditions given in Theorem 3 hold, then the conditions given in Theorem 4 also hold.*

Proof. Considering the definitions (35), (36) and the above conditions, $\psi_{ijk} = \psi'_{ijk}$, $\gamma_{ijk} = \gamma'_{ijk}$, then the (17) is equivalent to $\psi_{ijk} - \gamma_{ijk} < 0$, for all $i \in \mathbb{K}_N$ and $k, j \in \mathbb{K}_r$. If (17) holds, then, there exist $\xi_{ikk} = \psi_{ikk} - \gamma_{ikk} + \varepsilon I$, where $\varepsilon > 0$, is sufficiently small such that, $\xi_{ikk} < 0$, for all $i \in \mathbb{K}_N$ and $k \in \mathbb{K}_r$, and $\xi_{ijk} = 0$ for $k \neq j$, $i \in \mathbb{K}_N$, $j,k \in \mathbb{K}_r$, such that, (21) and (22) hold. Furthermore, in this case, the condition (31) also holds, because it can be rewritten as showed in (34). Note that in Theorem 3, $P_{jk} = P'_{jk} > 0$, but it is not necessary that $P_{jk} = P'_{kj}$, for $j \neq k$. However, observe that one can also rewrite $P(\alpha)$ as $P(\alpha) = \sum_{j=1}^{r} \alpha_j^2 P_{jj} + \frac{1}{2}\sum_{j\neq k}^{r} \alpha_j \alpha_k \left(P_{jk} + P_{kj}\right) = \sum_{j=1}^{r} \alpha_j^2 P_{jj} + \sum_{j\neq k}^{r} \alpha_j \alpha_k P_{N_{jk}}$, where $P_{N_{jk}} = P'_{N_{kj}} = \frac{1}{2} \times \left(P_{jk} + P_{kj}\right)$, because from Theorem 3 $P_{jk} = P'_{jk}$ for all $j,k \in \mathbb{K}_r$. Furthermore, in the LMI (17) given in Theorem 3, note that P_{jk} appears added to P_{kj} and then, $P_{jk} + P_{kj} = P_{N_{jk}} + P_{N_{kj}}$. From Theorem 3 it also follows that $P_{jk} > 0$ for all $j,k \in \mathbb{K}_r$. Then, note that, without loss of generality, one can consider in Theorem 3 that $P_{jk} = P'_{kj}$. Therefore, there exist $w_{kk} = P_{kk} - \varepsilon I$, where $\varepsilon > 0$ is sufficiently small such that, $w_{kk} > 0$ for all $k \in \mathbb{K}_r$, and $w_{jk} = 0$ for $j \neq k$, such that the conditions (25) and (26) hold. In this situation, the condition (29) also holds, because it can be rewritten as described (32). Now, from definition (37), the condition (18) is equivalent to $\theta_{jk} < 0$, $j,k \in \mathbb{K}_r$. Hence, when (18) holds, observe that (23) and (24) also hold for $\phi_{kk} = \theta_{kk} + \varepsilon I$, where $\varepsilon > 0$ is sufficiently small such that, $\phi_{kk} < 0$, for all $k \in \mathbb{K}_r$, and $\phi_{jk} = 0$, for $j \neq k$ and $k,j \in \mathbb{K}_r$. In this case, the condition (30) also holds, because it can be rewritten as presented in (33). Finally, when (16) holds for some $\mu > 0$, the conditions (27) and (28) also hold. The proof is concluded. $\square$

Remark 1. Note that the terms $\hat{\mathcal{Q}}_\lambda = \sum_{i=1}^{N} \lambda_i \hat{\mathcal{Q}}_i$ presented in the conditions (8), (18) and (37) of the previous theorems can be seen as BMIs since $\hat{\mathcal{Q}}_i$ and λ_i are variables to be found. However, if the parameters λ_i are set to be known, those conditions are LMIs that can be solved. Further, an example will exploit the optimisation of the parameters λ_i in order to achieve suboptimal values for the guaranteed cost.

3.1 EXAMPLE I - SWITCHED LINEAR SYSTEM WITHOUT INPUT CONTROL AND DISTURBANCE

In order to compare the potentiality of the proposed theorems, is presented in this section a numerical example. The example shows a comparison between the feasible region obtained through the conditions of Theorems 3 and 4. Furthermore, it is presented a comparative study regarding the guaranteed cost (14).

This example was borrowed from (Mainardi Júnior *et al.*, 2015). Consider the uncertain system (4) and (5), supposing that $u(t) = 0$ and $w(t) = 0$ for $t \geq 0$, and assuming constant output matrices $C_{2j} = C$, for all $j \in \mathbb{K}_r$, with $r = 2$, $N = 3$, $i \in \{1,2,3\}$, $j \in \{1,2\}$ and the following matrices given below:

$$A_{11} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -5 & 0 \\ 2 & 0 & -2 \end{bmatrix}, A_{12} = \begin{bmatrix} h_1 & 1 & 0 \\ 3 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}, A_{21} = \begin{bmatrix} -5 & -3 & 1 \\ -3 & -2 & 0 \\ 0 & 2 & -2 \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} -5 & -6 & 1 \\ -3 & h_2 & 0 \\ 0 & 2 & -2 \end{bmatrix},\ A_{31} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 3 & 0 & -3 \end{bmatrix},\ A_{32} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -2 & 0 \\ 3 & h_3 & -3 \end{bmatrix},$$

$$C_{21} = C_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{47}$$

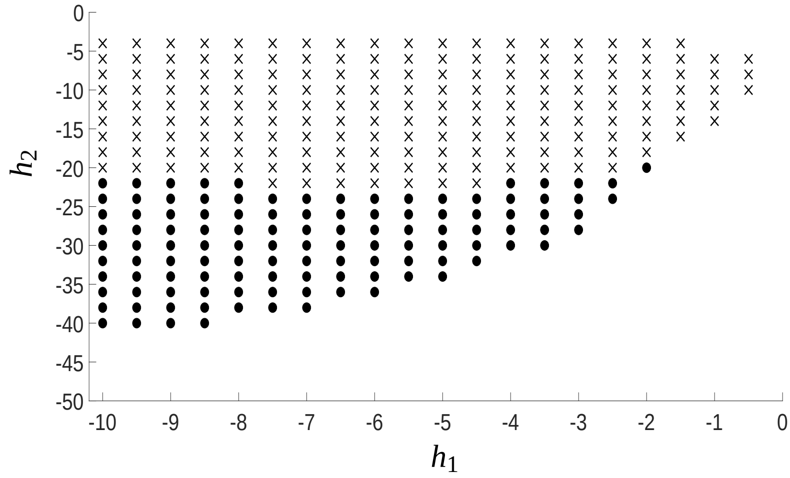
A comparative study on feasibility analysis and guaranteed cost for some pairs of h_1 and h_2, where $h_1 \in [-10,\ 0]$, $h_2 \in [-50,\ 0]$ and $h_3 = -(h1+h2)/3$ is performed in this example. Increments of 0.5 and 2 regarding the variables h_1 and h_2 were considered, respectively. Note that the matrices A_{12}, A_{22} and A_{32} depend on the parameters h_1, h_2 and h_3, respectively. Observe that the matrix A_{12} is Hurwitz (i.e., it has all eigenvalues with negative real parts) for all $h_1 \in [-10,\ -2]$ and it is not Hurwitz for all $h_1 \in [-1.5,\ 0]$. The matrix A_{22} is Hurwitz for all $h_2 \in [-50,\ -4]$ and it is not Hurwitz for all $h_2 \in [-2,\ 0]$. Moreover, verify that, the matrix A_{32} is Hurwitz for all $h_3 \in [0,\ 18]$ and it is not Hurwitz for all $h_2 \in [18.5,\ 20]$. Furthermore, note that the matrices A_{11} and A_{31} are not Hurwitz and the matrix A_{21} is Hurwitz. It was considered $\lambda_1 = 0.4$, $\lambda_2 = 0.3$, and $\lambda_3 = 0.3$. In this example the condition regarding the guaranteed cost (14) is removed, adopting $C_{111} = C_{112} = C_{121} = C_{122} = C_{131} = C_{132} = 0$. Thus, note that the conditions from Theorem 3 of this thesis are equivalent to the conditions presented on Theorem 5 in Mainardi Júnior *et al.* (2015). Figure 3 shows a comparison between the feasible regions obtained through of the conditions of Theorems 3 and 4. Note that, the proposed methodology (Theorem 4) presents a greater feasible region than that obtained with the conditions given in (Theorem 5) (Mainardi Júnior *et al.*, 2015). This fact and the result presented in Theorem 5 show that the conditions proposed in Theorem 4 are less conservative than that presented in (Mainardi Júnior *et al.*, 2015).

Furthermore the potentiality of proposed theorems are compared establishing the guaranteed cost as performance criterion, considering $x_0 = x(0) = [-0.25\ 0.5\ -0.75]'$, $C_{111} = C_{112} = C_{121} = C_{122} = C_{131} = C_{132} = C_{21} = C_{22}$, in (47) for $h_1 \in [-10,\ -7]$ and $h_2 \in [-30,\ -12]$. Observe that, from Figure 4, the less conservative conditions proposed in Theorem 4 reduce the guaranteed cost when compared to the conditions of Theorem 3. Additionally, note that, when $h_2 \le -17$, the conditions of Theorem 3 are unfeasible.

3.2 CHAPTER CONCLUSION

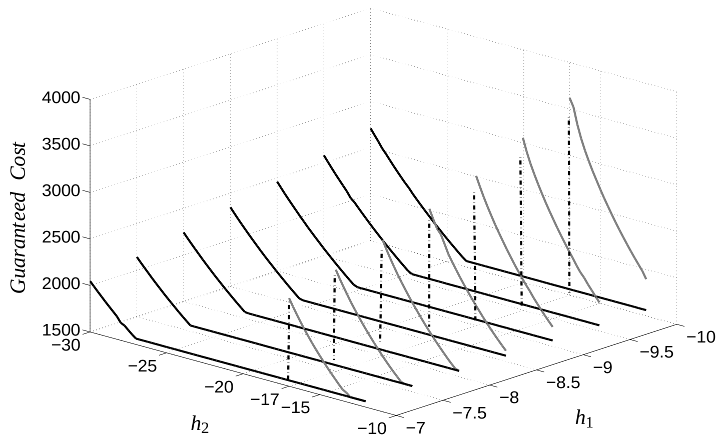
In this Chapter, considering an output-dependent switching strategy introduced in Mainardi Júnior *et al.* (2015), more relaxed conditions were proposed to deal with uncertain switched linear systems. Moreover, the performance criteria (guaranteed cost and decay rate) were addressed. Theorem 5 showed that if the conditions proposed in Mainardi Júnior *et al.* (2015) hold, then the proposed conditions (Theorem 4) also hold. Furthermore, through Example 3.1,

Figure 3 - Feasible regions obtained with Theorem 3 and Theorem 4 without the guaranteed cost specification, where the region obtained with Theorem 3 is illustrated by $(\times)$ and the region obtained for Theorem 4 is illustrated by $(\times)$ and (•).



Source: Author's own results.

Figure 4 - Guaranteed cost obtained from conditions of Theorem 3 (gray) and 4 (black).



Source: Author's own results.

it is observed that the proposed conditions yield a larger feasible region and a lower guaranteed cost value compared with the conditions proposed in Mainardi Júnior *et al.* (2015).

4 ROBUST SWITCHING STATIC OUTPUT FEEDBACK $\mathscr{H}_\infty$ CONTROL OF CONTINOUS TIME SWITCHED LINEAR SYSTEMS

This chapter is devoted to coping with the problem of designing robust SOF $\mathscr{H}_\infty$ switching controllers for switched linear uncertain systems.

4.1 PREVIOUS RESULTS PRESENTED IN THE LITERATURE

The output feedback controller for the uncertain switched system (4) and (5) is given by

$$u(t) = \mathcal{K}y(t), \tag{48}$$

where $\mathcal{K} \in \mathbb{R}^{n_u \times n_y}$ represents a constant feedback gain matrix. Consequently, the closed-loop system results in

$$\begin{cases} \dot{x}(t) &= \widetilde{A}(\sigma,\alpha)x(t) + \widetilde{E}(\sigma,\alpha)w(t), \quad x(0) = x_0 \\ z(t) &= \widetilde{C_1}(\sigma,\alpha)x(t) + \widetilde{F}(\sigma,\alpha)w(t), \\ y(t) &= C_2(\alpha)x(t) + H(\alpha)w(t), \end{cases} \tag{49}$$

where,

$$\begin{aligned} \widetilde{A}(\sigma,\alpha) &= A(\sigma,\alpha) + B(\sigma,\alpha)\mathcal{K}C_2(\alpha), \\ \widetilde{E}(\sigma,\alpha) &= E(\sigma,\alpha) + B(\sigma,\alpha)\mathcal{K}H(\alpha), \\ \widetilde{C_1}(\sigma,\alpha) &= C_1(\sigma,\alpha) + D(\sigma,\alpha)\mathcal{K}C_2(\alpha), \\ \widetilde{F}(\sigma,\alpha) &= F(\sigma,\alpha) + D(\sigma,\alpha)\mathcal{K}H(\alpha). \end{aligned} \tag{50}$$

Initially, the approach presented by Chang, Park and Zhou (2015) is introduced. These results are important to develop the contribution of this chapter. The following lemma was presented in Chang, Park and Zhou (2015) and employed to establish the main result of the work.

Lemma 1. *(CHANG; PARK; ZHOU, 2015) For matrices $\mathscr{T}$, $\mathscr{P}$, $\mathscr{U}$ and $\mathscr{A}$ with appropriate dimension and a scalar β, the following statements are equivalent:*

(i) : $\begin{bmatrix} \mathscr{T} & * \\ \beta\mathscr{P}' + \mathscr{U}\mathscr{A} & -\beta\mathscr{U} - \beta\mathscr{U}' \end{bmatrix} < 0,$

(ii) : $\mathscr{T} < 0, \;\; \mathscr{T} + \mathscr{A}'\mathscr{P}' + \mathscr{A}\mathscr{P} < 0.$

Theorem 6 provides LMI-based conditions for designing a SOF $\mathscr{H}_\infty$ controller. It is important to stress that this recent design method is less conservative since it does not impose any constraint on system matrices and avoids linear matrices equalities.

Theorem 6. *(CHANG; PARK; ZHOU, 2015) Given a scalar $\gamma > 0$, for known scalar parameters β and ρ, if there exist matrices $V \in \mathbb{R}^{n_u \times n_y}$, $U \in \mathbb{R}^{n_y \times n_y}$, and $X_j > 0 \in \mathbb{R}^{n_x \times n_x}$, $j = 1, 2, \ldots, r$ satisfying the following LMIs:*

$$\Omega_{jj} < 0, \tag{51}$$

$$\Omega_{jk} + \Omega_{kj} < 0, \quad j < k, \tag{52}$$

for all $j,\ k \in \mathbb{K}_r$, with:

$$\Omega_{jk} = \begin{bmatrix} He(A_j X_k + B_j V \mathscr{F}^k) & * & * & * \\ E_j' + H_k' V' B_j' & -\gamma^2 I & * & * \\ C_{1j} X_k + \rho \mathscr{F}_0' V' B_j' + D_j' V \mathscr{F}^k & F_j + D_j V H_k & -I + He(\rho D_j V \mathscr{F}_0) & * \\ \beta V' B_j' + C_{2j} X_k - U \mathscr{F}^k & H_j - U H_k & \beta V' D_j' - \rho U \mathscr{F}_0 & -\beta U - \beta U' \end{bmatrix}, \tag{53}$$

$$\mathscr{F}^k = \begin{cases} (C_2 C_2')^{-1} C_2, & C_2(\alpha) \text{ is fixed, } C_2(\alpha) = C_2 \text{ and } C_2 \text{ is of full rank}, \\ C_2, & C_2(\alpha) \text{ is fixed, } C_2(\alpha) = C_2 \text{ and } C_2 \text{ is of non-full rank}, \\ C_{2k}, & C_2(\alpha) \text{ is non-fixed}, \end{cases} \tag{54}$$

$$\mathscr{F}_0 = \begin{cases} I_{n_y}, & n_y = n_z, \\ \begin{bmatrix} I_{n_y} & 0_{n_y \times (n_z - n_y)} \end{bmatrix} & n_y < n_z, \\ \begin{bmatrix} I_{n_z} \\ 0_{(n_y - n_z) \times n_z} \end{bmatrix} & n_y > n_z, \end{cases} \tag{55}$$

then the system (4) *and* (5)*, for $N = 1$, is asymptotically stable with the $\mathscr{H}_\infty$ performance γ and the controller gain matrix in* (48) *is given by*

$$\mathcal{K} = K = V U^{-1}. \tag{56}$$

Proof. See Chang, Park and Zhou (2015). □

Remark 2. In Theorem 6 if ρ and β are set to be known the conditions became LMI which, when feasible, can be easily solved. As mentioned in (CHANG; PARK; ZHOU, 2015), these parameters are not necessary but they provide extra free dimensions for the design problem. Therefore it is possible to use numerical optimisation to search suboptimal values for β and ρ

to reduce the $\mathscr{H}_\infty$ bound. In (CHANG; PARK; ZHOU, 2015) the function fminsearch of the Matlab optimisation toolbox (GAHINET *et al.*, 1994) was used to obtain a locally convergent solution.

4.2 NOVEL CONDITIONS TO DESIGN STATIC OUTPUT-FEEDBACK $\mathscr{H}_\infty$ SWITCHING CONTROLLERS

In order to extend the conditions of Theorem 6 for a class of switched systems and also aiming to design static output-feedback $\mathscr{H}_\infty$ switching controllers for linear uncertain systems, the following problem is stated.

Problem 3. *Find a function $f(.) : \mathbb{R}^{n_y} \to [\ \{1,2,\ldots,N\},\ \ \{1,2,\ldots,r\}\ ]$ and gains $K_{ic} \in \mathbb{R}^{n_u \times n_y}$, $i \in \mathbb{K}_N$, $c \in \mathbb{K}_r$, such that the switching strategy*

$$\varphi(t) = f(y(t)) = \begin{bmatrix} \sigma(y) & \eta(y) \end{bmatrix}, \ \ \textit{for all} \ \ t \geq 0, \tag{57}$$

and the control input (48)*, with $\mathcal{K} = K_\varphi$, make the origin $x = 0$ of the controlled polytopic uncertain switched linear system* (4) *and* (5) *a globally asymptotically stable equilibrium point such that the controlled system satisfies,*

$$\int_0^\infty z(t)'z(t)dt \leq \gamma^2 \int_0^\infty w(t)'w(t)dt. \tag{58}$$

Therefore, the $\mathscr{H}_\infty$ norm of the aforementioned system is less than γ (DONG; YANG, 2013) for $x(0) = 0$ and $w(t) \in \mathscr{L}_2[0,\infty]$.

To deal with Problem 3, Theorem 7 provides conditions for designing static output $\mathscr{H}_\infty$ switching controllers for the switched linear uncertain system (4)-(5), based on the results presented in (Mainardi Júnior *et al.*, 2015) (Theorem 2), (CHANG; PARK; ZHOU, 2015) (Theorem 6) and (SOUZA *et al.*, 2014).

Theorem 7. *Consider that for known scalar parameters β and ρ there exist $\lambda \in \Lambda$, a scalar $\gamma > 0$, matrices $V_{ic} \in \mathbb{R}^{n_u \times n_y}$, $U_{ic} \in \mathbb{R}^{n_y \times n_y}$, $Q_{0j} \in \mathbb{R}^{n_x \times n_x}$, $Q_{1j} \in \mathbb{R}^{n_x \times n_w}$, $Q_{2j} \in \mathbb{R}^{n_w \times n_w}$, and symmetric matrices $X_j > 0 \in \mathbb{R}^{n_x \times n_x}$, $\hat{\mathcal{Q}}_{ic} \in \mathbb{R}^{n_y \times n_y}$, such that*

$$W_{ijjc} < 0, \tag{59}$$

$$W_{ijkc} + W_{ikjc} < 0, \quad j < k \tag{60}$$

$$\Theta_{\lambda jjj} < 0, \tag{61}$$

$$\Theta_{\lambda jjk} + \Theta_{\lambda jkj} + \Theta_{\lambda kjj} < 0, \quad j \neq k \tag{62}$$

$$\Theta_{\lambda jkq} + \Theta_{\lambda jqk} + \Theta_{\lambda kjq} + \Theta_{\lambda kqj} + \Theta_{\lambda qjk} + \Theta_{\lambda qkj} < 0, \quad j < k,\ k < q. \tag{63}$$

for all $i \in \mathbb{K}_N$, $j,c,k,q \in \mathbb{K}_r$, where

$$W_{ijkc} = \begin{bmatrix} He(A_{ij}X_k + B_{ij}V_{ic}\mathscr{F}^k) - Q_{0j} - C_{2j}'\hat{\mathcal{Q}}_{ic}C_{2k} & * & * & * \\ E_{ij}' + H_k'V_{ic}'B_{ij}' - Q_{1j}' - H_j'\hat{\mathcal{Q}}_{ic}C_{2k} & -\gamma^2 I - Q_{2j} - H_j'\hat{\mathcal{Q}}_{ic}H_k & * & * \\ C_{1ij}X_k + \rho\mathscr{F}_0'V_{ic}'B_{ij}' + D_{ij}'V_{ic}\mathscr{F}^k & F_{ij} + D_{ij}V_{ic}H_k & -I + He(\rho D_{ij}V_{ic}\mathscr{F}_0) & * \\ \beta V_{ic}'B_{ij}' + C_{2j}X_k - U_{ic}\mathscr{F}^k & H_j - U_{ic}H_k & \beta V_{ic}'D_{ij}' - \rho U_{ic}\mathscr{F}_0 & -\beta U_{ic} - \beta U_{ic}' \end{bmatrix} \tag{64}$$

$$\Theta_{\lambda jkq} = \begin{bmatrix} Q_{1j} + C_{2j}'\hat{\mathcal{Q}}_{\lambda k}C_{2q} & Q_{1j} + C_{2j}'\hat{\mathcal{Q}}_{\lambda k}H_q \\ Q_{1j}' + H_j'\hat{\mathcal{Q}}_{\lambda k}C_{2q} & Q_{2j} + H_j'\hat{\mathcal{Q}}_{\lambda k}H_q \end{bmatrix}, \quad \hat{\mathcal{Q}}_\lambda(\alpha) = \sum_{i=1}^{N}\sum_{k=1}^{r}\alpha_k\lambda_i\hat{\mathcal{Q}}_{ik} \tag{65}$$

with $\mathscr{F}^k$ and $\mathscr{F}_0$ given in (54) *and* (55)*, respectively.*

Thus the switching strategy,

$$\varphi = \begin{bmatrix} \sigma & \eta \end{bmatrix} = arg \min_{\substack{i\in\mathbb{K}_N \\ c\in\mathbb{K}_r}} (y'\hat{\mathcal{Q}}_{ic}y) \tag{66}$$

and the control input (48)*, with $\mathcal{K} = K_\phi = K_{\sigma\eta}$, where the controller gains are given by*

$$K_{ic} = V_{ic}U_{ic}^{-1}, \quad i \in \mathbb{K}_N, c \in \mathbb{K}_r \tag{67}$$

make the closed-loop system (4), (5), (48), (66) *and* (67) *asymptotically stable with the $\mathscr{H}_\infty$ performance γ.*

Proof. From (5), (59), (60), (64) and considering $i = \sigma, c = \eta$ one obtains:

$$\sum_{j=1}^{r}\sum_{k=1}^{r}\alpha_j\alpha_k W_{\sigma jk\eta} = \sum_{j=1}^{r}\alpha^2 W_{\sigma jj\eta} + \sum_{j=1}^{r}\sum_{j<k}^{r}\alpha_j\alpha_k(W_{\sigma jk\eta} + W_{\sigma kj\eta})$$

$$= \begin{bmatrix} \mathrm{He}\big(A(\sigma,\alpha)X(\alpha) + B(\sigma,\alpha)V_{\sigma\eta}\mathscr{F}(\alpha)\big) - Q_0(\alpha) - C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & * & * & * \\ E(\sigma,\alpha)' + H(\alpha)'V_{\sigma\eta}'B(\sigma,\alpha)' - Q_1(\alpha)' - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & -\gamma^2 I - Q_2(\alpha) - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) & * & * \\ C_1(\sigma,\alpha)X(\alpha) + \rho\mathscr{F}_0'V_{\sigma\eta}'B(\sigma,\alpha)' + D(\sigma,\alpha)'V_{\sigma\eta}\mathscr{F}(\alpha) & F(\sigma,\alpha) + D(\sigma,\alpha)V_{\sigma\eta}H(\alpha) & -I + \mathrm{He}(\rho D(\sigma,\alpha)V_{\sigma\eta}\mathscr{F}_0) & * \\ \beta V_{\sigma\eta}'B(\sigma,\alpha)' + C_2(\alpha)X(\alpha) - U_{\sigma\eta}\mathscr{F}(\alpha) & H(\alpha) - U_{\sigma\eta}H(\alpha) & \beta V_{\sigma\eta}'D(\sigma,\alpha)' - \rho U_{\sigma\eta}\mathscr{F}_0 & -\beta U_{\sigma\eta} - \beta U_{\sigma\eta}' \end{bmatrix} < 0 \tag{68}$$

Since the inequalities (59) and (60) are feasible, it implies from (64) that U_{ic} are no singular, for all $i \in \mathbb{K}_N$ and $r \in \mathbb{K}_r$. From Lemma 1 with $\mathscr{A} = U_{\sigma\eta}^{-1}\begin{bmatrix} C_2(\alpha)X(\alpha) - U_{\sigma\eta}\mathscr{F}(\alpha) & H(\alpha) - U_{\sigma\eta}H(\alpha) & -\rho U_{\sigma\eta}\mathscr{F}_0 \end{bmatrix}$, $\mathscr{P}' = \begin{bmatrix} V_{\sigma\eta}' B(\sigma,\alpha)' & 0 & V_{\sigma\eta}' D(\sigma,\alpha)' \end{bmatrix}$ and

$$\mathscr{T} = \begin{bmatrix} \mathrm{He}\big(A(\sigma,\alpha)X(\alpha) + B(\sigma,\alpha)V_{\sigma\eta}\mathscr{F}(\alpha)\big) - Q_0(\alpha) - C_2(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_2(\alpha) & * & * \\ E(\sigma,\alpha)' + H(\alpha)'V_{\sigma\eta}'B(\sigma,\alpha)' - Q_1(\alpha)' - H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_2(\alpha) & -\gamma^2 I - Q_2(\alpha) - H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}H(\alpha) & * \\ C_1(\sigma,\alpha)X(\alpha) + \rho\mathscr{F}_0'V_{\sigma\eta}'B(\sigma,\alpha)' + D(\sigma,\alpha)'V_{\sigma\eta}\mathscr{F}(\alpha) & F(\sigma,\alpha) + D(\sigma,\alpha)V_{\sigma\eta}H(\alpha) & -I + \mathrm{He}(\rho D(\sigma,\alpha)V_{\sigma\eta}\mathscr{F}_0) \end{bmatrix}, \tag{69}$$

the inequality (68) results in

$$\begin{bmatrix} \mathrm{He}\big(A(\sigma,\alpha)X(\alpha) + B(\sigma,\alpha)V_{\sigma\eta}\mathscr{F}(\alpha)\big) - Q_0(\alpha) - C_2(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_2(\alpha) & * & * \\ E(\sigma,\alpha)' + H(\alpha)'V_{\sigma\eta}'B(\sigma,\alpha)' - Q_1(\alpha)' - H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_{2\sigma} & -\gamma^2 I - Q_2(\alpha) - H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}H(\alpha) & * \\ C_1(\sigma,\alpha)X(\alpha) + \rho\mathscr{F}_0'V_{\sigma\eta}'B(\sigma,\alpha)' + D(\sigma,\alpha)'V_{\sigma\eta}\mathscr{F}(\alpha) & F(\sigma,\alpha) + D(\sigma,\alpha)V_{\sigma\eta}H(\alpha) & -I + \mathrm{He}(\rho D(\sigma,\alpha)V_{\sigma\eta}\mathscr{F}_0) \end{bmatrix}$$
$$+\mathrm{He}\left(\begin{bmatrix} B(\sigma,\alpha)V_{\sigma\eta} \\ 0 \\ D(\sigma,\alpha)V_{\sigma\eta}\end{bmatrix} U_{\sigma\eta}^{-1} \begin{bmatrix} X(\alpha)C_2(\alpha)' - \mathscr{F}(\alpha)'U_{\sigma\eta}' \\ H(\alpha)' - H(\alpha)'U_{\sigma\eta}' \\ -\rho\mathscr{F}_0'U_{\sigma\eta}' \end{bmatrix}'\right) < 0. \tag{70}$$

Rewriting (70),

$$\begin{bmatrix} \mathrm{He}\big(A(\sigma,\alpha)X(\alpha)\big) & * & * \\ E(\sigma,\alpha)' & -\gamma^2 I & * \\ C_1(\sigma,\alpha)X(\alpha) & F(\sigma,\alpha) & -I \end{bmatrix}$$
$$+\mathrm{He}\left(\begin{bmatrix} B(\sigma,\alpha)V_{\sigma\eta} \\ 0 \\ D(\sigma,\alpha)V_{\sigma\eta}\end{bmatrix} U_{\sigma\eta}^{-1} \begin{bmatrix} \mathscr{F}(\alpha)'U_{\sigma\eta}' \\ H(\alpha)'U_{\sigma\eta}' \\ \rho\mathscr{F}_0'U_{\sigma\eta}' \end{bmatrix}'\right)$$
$$+\mathrm{He}\left(\begin{bmatrix} B(\sigma,\alpha)V_{\sigma\eta} \\ 0 \\ D(\sigma,\alpha)V_{\sigma\eta}\end{bmatrix} U_{\sigma\eta}^{-1} \begin{bmatrix} X(\alpha)C_2(\alpha)' - \mathscr{F}(\alpha)'U_{\sigma\eta}' \\ H(\alpha)' - H(\alpha)'U_{\sigma\eta}' \\ -\rho\mathscr{F}_0'U_{\sigma\eta}' \end{bmatrix}'\right)$$
$$-\begin{bmatrix} Q_0(\alpha) + C_2(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_2(\alpha) & * & * \\ Q_1(\alpha)' + H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_{2\sigma} & Q_2(\alpha) + H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}H(\alpha) & * \\ 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} \mathrm{He}\big(A(\sigma,\alpha)X(\alpha)\big) & * & * \\ E(\sigma,\alpha)' & -\gamma^2 I & * \\ C_1(\sigma,\alpha)X(\alpha) & F(\sigma,\alpha) & -I \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} B(\sigma,\alpha)V_{\sigma\eta} \\ 0 \\ D(\sigma,\alpha)V_{\sigma\eta} \end{bmatrix} U_{\sigma\eta}^{-1} \begin{bmatrix} X(\alpha)C_2(\alpha)' \\ H(\alpha)' \\ 0 \end{bmatrix}' \right) \\
&- \begin{bmatrix} Q_0(\alpha)+C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & * & * \\ Q_1(\alpha)'+H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_{2\sigma} & Q_2(\alpha)+H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) & * \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{He}\big(A(\sigma,\alpha)X(\alpha)+B(\sigma,\alpha)K_{\sigma\eta}C_2(\alpha)X(\alpha)\big) & * & * \\ E(\sigma,\alpha)'+H(\alpha)'K_{\sigma\eta}'B(\sigma,\alpha)' & -\gamma^2 I & * \\ \big(C_1(\sigma,\alpha)+D(\sigma,\alpha)K_{\sigma\eta}C_2(\alpha)\big)X(\alpha) & F(\sigma,\alpha)+D(\sigma,\alpha)K_{\sigma\eta}H(\alpha) & -I \end{bmatrix} \\
&- \begin{bmatrix} Q_0(\alpha)+C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & * & \\ Q_1(\alpha)'+H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & Q_2(\alpha)+H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) & * \\ 0 & 0 & 0 \end{bmatrix} < 0. \qquad (71)
\end{aligned}
$$

Using the Schur complement in (71) and considering (50), one obtains

$$
\begin{aligned}
&\begin{bmatrix} \mathrm{He}\big(\widetilde{A}(\sigma,\alpha)X(\alpha)\big)+X(\alpha)\widetilde{C}_1(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha)X(\alpha) & * \\ \widetilde{E}(\sigma,\alpha)'+\widetilde{F}(\sigma,\alpha)'\widetilde{C_1}(\sigma,\alpha)X(\alpha) & \widetilde{F}(\sigma,\alpha)'\widetilde{F}(\sigma,\alpha)-\gamma^2 I \end{bmatrix} \\
&- \begin{bmatrix} Q_0(\alpha)+C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & * \\ Q_1(\alpha)'+H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & Q_2(\alpha)+H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) \end{bmatrix} < 0. \qquad (72)
\end{aligned}
$$

Pre and post-multiplying (72) in both sides by $\begin{bmatrix} x(t)'w(t)' \end{bmatrix}$ and its transpose

$$
\begin{aligned}
&\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \left(\begin{bmatrix} \mathrm{He}\big(\widetilde{A}(\sigma,\alpha)X(\alpha)\big)+X(\alpha)\widetilde{C}_1(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha)X(\alpha) & * \\ \widetilde{E}(\sigma,\alpha)'+\widetilde{F}(\sigma,\alpha)'\widetilde{C_1}(\sigma,\alpha)X(\alpha) & \widetilde{F}(\sigma,\alpha)'\widetilde{F}(\sigma,\alpha)-\gamma^2 I \end{bmatrix} \right. \\
&\left. - \begin{bmatrix} Q_0(\alpha)+C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & * \\ Q_1(\alpha)'+H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & Q_2(\alpha)+H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\
&= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \left(\begin{bmatrix} \mathrm{He}\big(\widetilde{A}(\sigma,\alpha)X(\alpha)\big)+X(\alpha)\widetilde{C}_1(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha)X(\alpha) & * \\ \widetilde{E}(\sigma,\alpha)'+\widetilde{F}(\sigma,\alpha)'\widetilde{C_1}(\sigma,\alpha)X(\alpha) & \widetilde{F}(\sigma,\alpha)'\widetilde{F}(\sigma,\alpha)-\gamma^2 I \end{bmatrix} \right. \\
&\left. - \begin{bmatrix} Q_0(\alpha) & * \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} - \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & * \\ H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \qquad (73)
\end{aligned}
$$

Observe that,

$$
\begin{aligned}
&\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & * \\ H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\
&= \begin{bmatrix} x(t)' & w(t)' \end{bmatrix} \begin{bmatrix} C_2(\alpha)' \\ H(\alpha)' \end{bmatrix} \hat{\mathcal{Q}}_{\sigma\eta} \begin{bmatrix} C_2(\alpha) & H(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}
\end{aligned}
$$

$$= \big(x(t)'C_2(\alpha)' + w(t)'H(\alpha)'\big)\hat{\mathcal{Q}}_{\sigma\eta}\big(C_2(\alpha)x(t) + H(\alpha)w(t)\big) = y(t)'\hat{\mathcal{Q}}_{\sigma\eta}y(t). \tag{74}$$

From (72)-(74), and considering $x(t) \neq 0$, note that

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \left(\begin{bmatrix} \mathrm{He}\big(\widetilde{A}(\sigma,\alpha)X(\alpha)\big) + X(\alpha)\widetilde{C}_1(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha)X(\alpha) & * \\ \widetilde{E}(\sigma,\alpha)' + \widetilde{F}(\sigma,\alpha)'\widetilde{C_1}(\sigma,\alpha)X(\alpha) & \widetilde{F}(\sigma,\alpha)'\widetilde{F}(\sigma,\alpha) - \gamma^2 I \end{bmatrix} \right.$$
$$\left. - \begin{bmatrix} Q_0(\alpha) & * \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} - y(t)'\hat{\mathcal{Q}}_{\sigma\eta}y(t) < 0. \tag{75}$$

Now, from (61)-(63) one has

$$\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{q=1}^{r}\alpha_j\alpha_k\alpha_q\Theta_{\lambda jkq} = \sum_{j=1}^{r}\alpha_j^3\Theta_{\lambda jjj} + \sum_{j=1}^{r}\sum_{j\neq k}^{r}\alpha_j^2\alpha_k\left(\Theta_{\lambda jjk} + \Theta_{\lambda jkj} + \Theta_{\lambda jjk}\right)$$
$$+ \sum_{j=1}^{r}\sum_{j<k}^{r}\sum_{k<q}^{r}\alpha_j\alpha_k\alpha_q\left(\Theta_{\lambda jkq} + \Theta_{\lambda jqk} + \Theta_{\lambda kjq} + \Theta_{\lambda kqj} + \Theta_{\lambda qjk} + \Theta_{\lambda qkj}\right) \tag{76}$$
$$= \begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} + \begin{bmatrix} C_2(\alpha)'\hat{\mathcal{Q}}_\lambda(\alpha)C_2(\alpha) & C_2(\alpha)'\hat{\mathcal{Q}}_\lambda(\alpha)H(\alpha) \\ H(\alpha)'\hat{\mathcal{Q}}_\lambda(\alpha)C_2(\alpha) & H(\alpha)'\hat{\mathcal{Q}}_\lambda(\alpha)H(\alpha) \end{bmatrix} < 0. \tag{77}$$

Pre and post-multiplying (77) in both sides by $\begin{bmatrix} x(t)'w(t)' \end{bmatrix}$ and its transpose result in

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \left(\begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} + \begin{bmatrix} C_2(\alpha)'\hat{\mathcal{Q}}_\lambda(\alpha)C_2(\alpha) & C_2(\alpha)'\hat{\mathcal{Q}}_\lambda(\alpha)H(\alpha) \\ H(\alpha)'\hat{\mathcal{Q}}_\lambda(\alpha)C_2(\alpha) & H(\alpha)'\hat{\mathcal{Q}}_\lambda(\alpha)H(\alpha) \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} =$$
$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + y(t)'\hat{\mathcal{Q}}_\lambda(\alpha)y(t) < 0. \tag{78}$$

Now, knowing that the minimum of a set of real numbers is less than or equal to an arbitrary convex combination of these numbers it is possible to rewrite (75), considering the definition of $\hat{\mathcal{Q}}_\lambda(\alpha)$ in (65):

$$0 > -\min_{\substack{i\in\mathbb{K}_N \\ c\in\mathbb{K}_r}}(y'\hat{\mathcal{Q}}_{ic}y) + \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \left(- \begin{bmatrix} Q_0(\alpha) & * \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} \mathrm{He}\big(\widetilde{A}(\sigma,\alpha)X(\alpha)\big) + X(\alpha)\widetilde{C}_1(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha)X(\alpha) & * \\ \widetilde{E}(\sigma,\alpha)' + \widetilde{F}(\sigma,\alpha)'\widetilde{C_1}(\sigma,\alpha)X(\alpha) & \widetilde{F}(\sigma,\alpha)'\widetilde{F}(\sigma,\alpha) - \gamma^2 I \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$
$$\geq -y(t)'\hat{\mathcal{Q}}_\lambda(\alpha)y(t) - \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} Q_0(\alpha) & * \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} +$$
$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} \mathrm{He}\big(\widetilde{A}(\sigma,\alpha)X(\alpha)\big) + X(\alpha)\widetilde{C}_1(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha)X(\alpha) & * \\ \widetilde{E}(\sigma,\alpha)' + \widetilde{F}(\sigma,\alpha)'\widetilde{C_1}(\sigma,\alpha)X(\alpha) & \widetilde{F}(\sigma,\alpha)'\widetilde{F}(\sigma,\alpha) - \gamma^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \tag{79}$$

Therefore, from (79),

$$\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} \mathrm{He}\big(\widetilde{A}(\sigma,\alpha)X(\alpha)\big) + X(\alpha)\widetilde{C}_1(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha)X(\alpha) & * \\ \widetilde{E}(\sigma,\alpha)' + \widetilde{F}(\sigma,\alpha)'\widetilde{C_1}(\sigma,\alpha)X(\alpha) & \widetilde{F}(\sigma,\alpha)'\widetilde{F}(\sigma,\alpha) - \gamma^2 I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$
$$\leq y(t)'\hat{\mathcal{Q}}_\lambda(\alpha)y(t) + \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} Q_0(\alpha) & * \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \tag{80}$$

Considering (78) and (80)

$$\begin{bmatrix} \mathrm{He}\big(\widetilde{A}(\sigma,\alpha)X(\alpha)\big) + X(\alpha)\widetilde{C}_1(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha)X(\alpha) & * \\ \widetilde{E}(\sigma,\alpha)' + \widetilde{F}(\sigma,\alpha)'\widetilde{C_1}(\sigma,\alpha)X(\alpha) & \widetilde{F}(\sigma,\alpha)'\widetilde{F}(\sigma,\alpha) - \gamma^2 I \end{bmatrix} < 0. \tag{81}$$

Considering a Lyapunov function candidate $V(x(t)) = x(t)P(\alpha)x(t)$, defining $X(\alpha) = P^{-1}(\alpha)$ and pre- and post-multiplying both sides of (81) with $diag\Big\{P(\alpha), I\Big\}$ and its transpose and then pre- and post-multiplying the result in both sides with $\begin{bmatrix} x(t)' & w(t)' \end{bmatrix}$, one has

$$\begin{bmatrix} x(t)' \\ w(t)' \end{bmatrix} \begin{bmatrix} \mathrm{He}\big(\widetilde{A}'(\sigma,\alpha)P(\alpha)\big) + \widetilde{C}_1(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha) & * \\ \widetilde{E}(\sigma,\alpha)'P(\alpha) + \widetilde{F}(\sigma,\alpha)'\widetilde{C}_1(\sigma,\alpha) & \widetilde{F}(\sigma,\alpha)'\widetilde{F}(\sigma,\alpha) - \gamma^2 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$
$$= \dot{V}(t) + z(t)'z(t) - \gamma^2 w(t)'w(t) < 0. \tag{82}$$

Integrating (82) from zero to infinity

$$\int_0^\infty \dot{V}(t)dt + \int_0^\infty z(t)'z(t)dt - \gamma^2 \int_0^\infty w(t)'w(t)dt$$
$$= V(\infty) - V(x(0)) + \int_0^\infty z(t)'z(t)dt - \gamma^2 \int_0^\infty w(t)'w(t)dt < 0. \tag{83}$$

Considering $x(0) = 0$, then $V(x(0)) = 0$ and $V(\infty) \geq 0$. Therefore, from (83) one has

$$\int_0^\infty z(t)'z(t)dt \leq \gamma^2 \int_0^\infty w(t)'w(t)dt. \tag{84}$$

Thus, (58) holds and the $\mathscr{H}_\infty$ performance is fulfilled. Note that from (82) if $w(t) = 0$ one obtains $\dot{V}(t) < 0$ for all $x(t) \neq 0$. Hence, the proof is concluded. □

Remark 3. As mentioned in Remark 2, to obtain LMI conditions in Theorem 6 the parameters β and ρ are set to be known. In Theorem 7, besides β and ρ, the parameters λ_i, $i \in \mathbb{K}_N$ need to be known to have LMI conditions. It is important to highlight that the parameters λ_i, unlike β and ρ, are necessary for the conditions. As mentioned in Remark 1, the same discussion concerning λ_i can be drawn for Theorems 2, 3 and 4 since the convex combination of the parameters λ_i is needed. With the purpose of optimising the $\mathscr{H}_\infty$ norm finding the values of β, ρ and λ_i, $i \in \mathbb{K}_N$, the DE-LMI algorithm will be introduced later.

Theorem 8. *If the conditions given in Theorem 6 hold, then the conditions given in Theorem 7 also hold.*

Proof. For Theorem 7 assume the particular case where $N = 1$ and the controller gains are fixed ($\mathcal{K} = K$) which yield fixed values for $\hat{\mathcal{Q}}_{ic} = \hat{\mathcal{Q}}$, $U_{ic} = U$, $V_{ic} = V$, $\Theta_{\lambda jkq} = \Theta_j$ and $W_{ijkc} = W_{jk}$. Therefore, assuming $\hat{\mathcal{Q}} = 0$, it is possible to rewrite (64) as

$$W_{jk} = \underbrace{\begin{bmatrix} \mathrm{He}(A_jX_k + B_jV\mathscr{F}^k) & * & * & * \\ E_j' + H_k'V'B_j' & -\gamma^2 I & * & * \\ C_{1j}X_k + \rho\mathscr{F}_0'V'B_j' + D_j'V\mathscr{F}^k & F_j + D_jVH_k & -I + \mathrm{He}(\rho D_jV\mathscr{F}_0) & * \\ \beta V'B_j' + C_{2j}X_k - U\mathscr{F}^k & H_j - UH_k & \beta V'D_j' - \rho U\mathscr{F}_0 & -\beta U - \beta U' \end{bmatrix}}_{\omega_{jk}}$$

$$+ \underbrace{\begin{bmatrix} -Q_{0j} & * & * & * \\ -Q_{1j}' & -Q_{2j} & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathscr{Q}_j}, \tag{85}$$

and (65) as

$$\Theta_j = \begin{bmatrix} Q_{0j} & * \\ Q_{1j}' & Q_{2j} \end{bmatrix}. \tag{86}$$

Note that for this particular case the term ω_{jk} defined in (85) is equal to Ω_{jk} given in (53) of Theorem 6. Considering that the conditions of Theorem 6 hold, then from (51) and (52) one has that $\omega_{jj} = \Omega_{jj} < 0$ and $\omega_{jk} + \omega_{kj} = \Omega_{jk} + \Omega_{jk} < 0$. Thus, there exists sufficient small parameters $\varepsilon > 0$ and $\tau > 0$ such that $\omega_{jj} + \varepsilon I < 0$ and $\omega_{jk} + \omega_{kj} + \tau I < 0$. Therefore, from (85), for the particular case where $Q_{0j} = -\varepsilon I$, $Q_{2j} = -\varepsilon I$ and $Q_{1j} = 0$, considering (64), then (59) and (60) hold. Hence (61), (62) and (63) also hold, since $Q_{0j} = -\varepsilon I$, $Q_{2j} = -\varepsilon I$ and $Q_{1j} = 0$, because $\Theta_{\lambda jkq} = \Theta_j$ and from (86), $\Theta_j = -\varepsilon I < 0$. The proof is concluded. □

Corollary 1. *Theorem 7 also suits for designing switched output feedback $\mathscr{H}_\infty$ controllers for continuous-time uncertain linear systems. These systems can be considered a particular case of the switched systems with $N = 1$.*

In Example III (Chapter 8) is shown that there exist cases where the conditions from Theorem 7 are less conservative than the conditions from Theorem 2.

4.3 CHAPTER CONCLUSION

This chapter covered novel conditions to design switched SOF feedback controllers for switched linear systems. Firstly, a recent result introduced in Chang, Park and Zhou (2015) was presented. This result is interesting when considering the SOF $\mathscr{H}_\infty$ control design, since it overcomes the LMEs and do not impose any constraint to the output matrices of the system (CRUSIUS; TROFINO, 1999). Following, it is stated the problem of finding an output-dependent switching law, jointly with switched controllers, to make the origin of controlled polytopic uncertain swtiched linear system a globally asymptotically stable equilibrium point fulfilling the $\mathscr{H}_\infty$ performance. To deal with this problem, Theorem 7 was introduced aiming to design switched controllers.

Theorem 8 enforced that if the conditions given in the conditions proposed by Chang, Park and Zhou (2015) hold, then the proposed conditions to design switched controllers presented in Theorem 7 also hold. By means of the examples to be presented in Chapter 8 it is observed that the proposed conditions yield a larger feasible region and a lower guaranteed cost value compared with the conditions proposed in Chang, Park and Zhou (2015). Finally, Corollary 1 states that the conditions proposed in Theorem 7 also suits for design controllers for uncertain linear system, that is, a particular case of switched systems with $N = 1$.

5 ROBUST STATIC OUTPUT FEEDBACK $\mathscr{H}_\infty$ CONTROL OF CONTINUOUS TIME LINEAR SYSTEM SUBJECT TO ACTUATOR SATURATION

In this chapter the problem of dealing with input control constraint, specifically with systems subject to actuator saturation, is introduced. It is important to observe that this chapter does not consider switched systems or switching control. The main contribution is to extend the results presented in Chang, Park and Zhou (2015) for continuous-time uncertain system that are subject to actuator saturation.

5.1 CONTINUOUS-TIME UNCERTAIN LINEAR SYSTEM SUBJECT TO ACTUATOR SATURATION

A continuous-time uncertain linear system subject to actuator saturation can be defined by the following state space description:

$$\begin{cases} \dot{x}(t) = A(\alpha)x(t) + B(\alpha)\mathrm{sat}(u(t)) + E(\alpha)w(t), \quad x(0) = x_0 \\ z(t) = C_1(\alpha)x(t) + D(\alpha)\mathrm{sat}(u(t)) + F(\alpha)w(t) \\ y(t) = C_2(\alpha)x(t) + H(\alpha)w(t) \end{cases} \tag{87}$$

with $x(t) \in \mathbb{R}^{n_x}$, $w(t) \in \mathbb{R}^{n_w}$, $y(t) \in \mathbb{R}^{n_y}$, $z(t) \in \mathbb{R}^{n_z}$ and the definitions already given in Section 2.3.

In this system description $\mathrm{sat} : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is a vector saturation function defined as

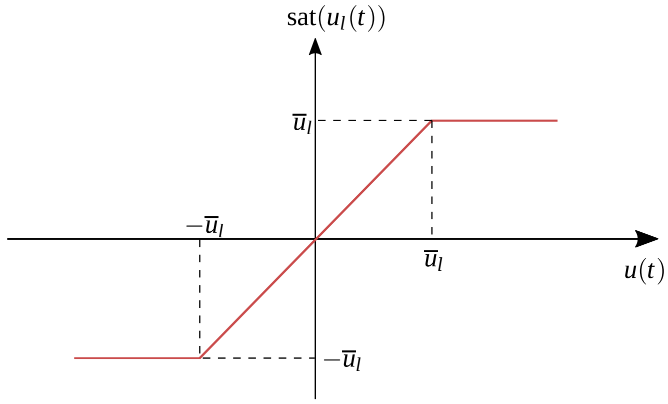
$$\mathrm{sat}(u(t)) = \Big[\mathrm{sat}(u_1(t)) \quad \mathrm{sat}(u_2(t)) \quad \ldots \mathrm{sat}(u_{n_u}(t))\Big]',$$

$$\mathrm{sat}(u_l(t)) = \mathrm{sgn}(u_l)\min\{\overline{u}_l, |u_l|\} = \begin{cases} -\overline{u}_l, & \text{if} \quad u_l(t) < -\overline{u}_l \\ u_l(t), & \text{if} \quad |u_l(t)| \leq \overline{u}_l \\ \overline{u}_l, & \text{if} \quad u_l(t) > \overline{u}_l \end{cases}, \quad l \in \mathbb{K}_{n_u} \tag{88}$$

where $\overline{u}_l$, $l \in \mathbb{K}_{n_u}$, are known positive constants that stands for the actuators saturation limit value. Figure 5 illustrates $\mathrm{sat}(u(t))$. For easier notation, in the text sat is also used to denote a scalar saturation function.

Considering the static output feedback $\mathrm{sat}(u(t)) = \mathrm{sat}(Ky(t))$, it is possible to obtain from (87) the closed loop system. It is important to highlight that due to the saturation, the closed-loop system is nonlinear. It means that, even assuming $w(t) = 0$ and considering the matrices combinations that contains the K Hurwitz $[A(\alpha) + B(\alpha)KC_2(\alpha)]$ for example, depending on the initial condition the closed-loop system can diverge from the origin. In other words, there exist initial conditions x_0 such that the trajectories converge to the origin (i. e. $x(t) \to 0$ when

Figure 5 - Illustration of the saturation function $\text{sat}(u_l(t))$.



Source: Own author.

$t \rightarrow \infty$) and, on the other hand, there exist initial conditions that the trajectories will diverge from the origin (i. e. $x(t) \rightarrow \infty$ when $t \rightarrow \infty$). A simple way to interpret this statement is to considerer that there exist a region $\mathscr{R}_s$ in which all $x_0 \in \mathscr{R}_s$ will converge to the origin. In this sense, it turns to be interesting to know the region (or domain) of attraction [$\mathscr{R}_A$], of the system with the nonlinear behaviour, defined as

$$\mathscr{R}_A \triangleq \left\{ x_0 \in \mathbb{R}^{n_x} : \lim_{t \rightarrow \infty} \aleph(t, x_0) = 0 \right\} \tag{89}$$

where $\aleph(t, x_0)$ represents the trajectory of the system. It means that the set $\mathscr{R}_A$ is composed by all the initial conditions x_0 that begin inside $\mathscr{R}_A$ and converge to the origin.

However, the determination of the set $\mathscr{R}_A$ is not an easy task. Therefore, generally, an estimation of the region of attraction for the system is used instead. An estimation of the $\mathscr{R}_A$ involves a positive invariant set. In a positive invariant set all trajectories beginning from within it will remain in it.

Since in this work the quadratic Lyapunov function is considered, it can be applied to estimate the region of attraction. In doing so, let $X(\alpha) > 0 \in \mathbb{R}^{n_x \times n_x}$ be a positive definite matrix, where $\alpha \in \Lambda_r$ defined in (1), and the Lyapunov function

$$V(x(t)) = x(t)'P(\alpha^*)x(t) = x(t)'X(\alpha)^{-1}x(t), \;\; X(\alpha) = \sum_{j=1}^{r} \alpha_j X_j, \; P(\alpha^*) = X(\alpha)^{-1}. \tag{90}$$

Note that in this thesis a matrix $X(\alpha)$ denotes a convex combination of matrices as presented in (90). In order to clarify the notation, in this work, the symbol α^*, with for instance a matrix $P(\alpha^*)$, will denote a matrix that depends on the vector α, but can not be represented by a convex combination with α as $X(\alpha)$ in (90). In other words, $P(\alpha)$ means that the matrix depends on α values but the computation of $P(\alpha^*) = (\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_j X_j)^{-1}$ may involve multiplication and sum of the terms $\alpha_j \in \Lambda_r$.

Taking into account (90), let $\mathscr{E}(P(\alpha^*),\delta)$ be the ellipsoidal set, based on the Lyapunov function, defined as:

$$\mathscr{E}(P(\alpha^*),\delta) \triangleq \left\{x(t) \in \mathbb{R}^{n_x} : x(t)'P(\alpha^*)x(t) \leq \delta, \delta > 0\right\}. \tag{91}$$

The set represented in (91) is a contractively invariant set if

$$\dot{V}(x(t)) = 2x(t)'P(\alpha^*)f(.) < 0, \tag{92}$$

for all $x(t) \in \mathscr{E}(P(\alpha^*),\delta)\backslash\{0\}$. If a set is contractively invariant, it is inside the domain of attraction (CAO; LIN, 2003; SAIFIA *et al.*, 2012). The ellipsoidal set, based on the Lyapunov energy function, presents the propriety of contractivity with respect to the closed-loop system, since the proprieties $V(x(t)) > 0$ and $\dot{V}(x(t)) < 0$ hold. Note that, if a set is contractive relatively to a system, then this set is also positively invariant. Therefore, all trajectories beginning from within the ellipsoidal set $\mathscr{E}(P(\alpha^*),\delta)$ will remain in it and converge to zero (origin). Hence, it is possible to consider the set (91) as an estimative for the region of attraction set (89) for the closed-loop system, once $\mathscr{E}(P(\alpha^*),\delta) \subseteq \mathscr{R}_A$.

The following Lemma and remarks are important to establish an equivalent condition for $V(x(t)) = x(t)'X(\alpha)^{-1}x(t)$, in order to compute it.

Lemma 2. *Consider the matrix $X(\alpha)$ given in* (90)*. Then, for all matrices $\mathscr{V} > 0$, and α defined in* (1)*, the following conditions are equivalent:*

(i) :

$$x(t)'X(\alpha)^{-1}x(t) < x(t)'\mathscr{V}x(t),\ \alpha \in \Lambda_r,\ \ \forall x(t) \neq 0 \in \mathbb{R}^{n_x}; \tag{93}$$

(ii) :

$$x(t)'X_j^{-1}x(t) < x(t)'\mathscr{V}x(t),\ j \in \mathbb{K}_r,\ \ \forall x(t) \neq 0 \in \mathbb{R}^{n_x}. \tag{94}$$

Proof. From (90) and (93) note that applying the Schur complement, (93) is equivalent to

$$x(t)'\left[X(\alpha)^{-1} - \mathscr{V}\right]x(t) < 0,\ \ \forall x(t) \neq 0 \in \mathbb{R}^{n_x} \leftrightarrow \tag{95}$$

$$-X(\alpha)^{-1} + \mathscr{V} > 0 \leftrightarrow \begin{bmatrix} X(\alpha) & I \\ I & \mathscr{V} \end{bmatrix} > 0 \leftrightarrow \sum_{j=1}^{r} \alpha_j \begin{bmatrix} X_j & I \\ I & \mathscr{V} \end{bmatrix} > 0. \tag{96}$$

The necessary and sufficient condition for (96), for all $\alpha \in \Lambda_r$ defined in (90) is

$$\begin{bmatrix} X_j & I \\ I & \mathscr{V} \end{bmatrix} > 0,\ j \in \mathbb{K}_r, \tag{97}$$

since that, for a given $\alpha_i = 1$, $i \in \mathbb{K}_r$, one has $\alpha_j = 0$ for all $j \neq i \in \mathbb{K}_r$, and thus (97) is necessary for (96). On the other hand, if (97) is feasible, multiplying (97) by α_j and taking the sum from $j = 1$ until $j = r$ one has (96) feasible. Finally, applying the Schur complement in (97), note that (97) results in

$$\mathscr{V} - X_j^{-1} > 0 \leftrightarrow X_j^{-1} < \mathscr{V} \leftrightarrow x(t)'X_j^{-1}x(t) < x(t)\mathscr{V}x(t),\ j \in \mathbb{K}_r\ \ \forall x(t) \neq 0 \in \mathbb{R}^{n_x}, \tag{98}$$

that is, equivalent to the condition (94). Thus, the proof is concluded. □

Remark 4. Note that, if $x(t) = 0$, then $x(t)'X(\alpha)^{-1}x(t) < \delta$, state in (93) and (94), holds for all $\delta > 0$. Now, considering that Lemma 2 covers all $\mathscr{V}$ matrices, then, in particular, for all $x(t) \neq 0$, it is possible to define $\mathscr{V} = \frac{\delta I}{x(t)'x(t)}$, $\delta > 0 \in \mathbb{R}$. Thus, $x(t)'\mathscr{V}x(t) = \frac{x(t)'\delta x(t)}{x(t)'x(t)} = \delta$. Therefore, from Lemma 2 one has that $x(t)'X_j^{-1}x(t) < \delta$, for all $j \in \mathbb{K}_r$, is equivalent to $x(t)'X(\alpha)^{-1}x(t) < \delta$, for all $\alpha \in \Lambda_r$ (1).

Remark 5. Note that, to compute $V(x(t)) = x(t)'X(\alpha)^{-1}x(t)$, given (90) and (91), it is required to know the values of $\alpha \in \Lambda_r$ (1). Since those are unknown values that describe the uncertainties, it is not possible to compute directly $V(x(t))$. In doing so, taking into account Lemma 2 and Remark 4, a method to obtain a region which encompass all possible $\alpha \in \Lambda_r$ involves to considerer the intersection of the ellipsoidal sets provided by the functions $V_j(x(t)) = x(t)'X_j^{-1}x(t)$, $j \in \mathbb{K}_r$, i.e. $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \delta) = \mathscr{E}(X(\alpha)^{-1}, \delta)$. Hence, the representation $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \delta)$ is equivalent to $\mathscr{E}(X(\alpha)^{-1}, \delta)$ for all $\alpha \in \Lambda_r$. Note that, for a specific value of $\breve{\alpha} \in \Lambda_r$ one has that $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \delta) \subseteq \mathscr{E}(X(\breve{\alpha})^{-1}, \delta)$.

5.2 CONVEX HULL REPRESENTATION OF SATURATED CONTROLLERS

Let $\mathcal{D}$ be the set of $\mathbb{R}^{n_u \times n_u}$ diagonal matrices whose diagonal elements are either 1 or 0 (CAO; LIN, 2003).

$$\mathcal{D} \triangleq \left\{ \mathscr{D} \in \mathbb{R}^{n_u \times n_u} : d_{ii} = 0 \ \text{ or } 1 \text{ and } d_{ij} = 0, \forall i \neq j \right\} \tag{99}$$

As an example, if $n_u = 2$

$$\mathcal{D} = \left\{ \mathscr{D}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathscr{D}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathscr{D}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \mathscr{D}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \tag{100}$$

where $d_{ij}, \forall (i,j) \in \mathbb{K}_{n_u} \times \mathbb{K}_{n_u}$, are the elements of $\mathscr{D}$. There exist 2^{n_u} matrices in the $\mathcal{D}$ set, labelled as $\mathscr{D}_s$, $\forall s \in \mathbb{K}_{2^{n_u}}$. Let $\mathscr{D}_s^-$ denotes the element of $\mathcal{D}$ set associated with $\mathscr{D}_s$, such that $\mathscr{D}_s^- = I - \mathscr{D}_s$. For simplicity, the following notation is introduced (OLIVEIRA, 2017):

$$\mathscr{D}(\vartheta) = \sum_{s=1}^{2^{n_u}} \vartheta_s \mathscr{D}_s,\ \mathscr{D}^-(\vartheta) = \sum_{s=1}^{2^{n_u}} \vartheta_s \mathscr{D}_s^-,\ \text{ with }\ \vartheta_s \geq 0 \in \mathbb{K}_{n_u}, \sum_{s=1}^{2^{n_u}} \vartheta_s = 1. \tag{101}$$

Given the matrix $\mathcal{G}(\alpha^*) = \begin{bmatrix} G_1(\alpha^*)' & G_2(\alpha^*)' & \ldots & G_{n_u}(\alpha^*)' \end{bmatrix}' \in \mathbb{R}^{n_u \times n_x}$, and a known vector $\bar{u} = \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \ldots & \bar{u}_{n_u} \end{bmatrix}' \in \mathbb{R}^{n_u}$, $\bar{u}_l > 0$, $l \in \mathbb{K}_{n_u}$, let the polyhedral set $\mathscr{L}(\mathcal{G}(\alpha^*))$ be defined following the structure presented in Cao and Lin (2003):

$$\mathscr{L}(\mathcal{G}(\alpha^*)) \triangleq \{x(t) \in \mathbb{R}^{n_x} : |G_l(\alpha^*)x(t)| \leq \bar{u}_l,\ l \in \mathbb{K}_{n_u}\}. \tag{102}$$

Taking into account the definition of the $\mathcal{D}$, the $\mathcal{G}(\alpha^*)$ matrix and the controller gain vector K one has a set $\{\mathscr{D}_s K + \mathscr{D}_s^- \mathcal{G}(\alpha^*),\ \ s \in \mathbb{K}_{2^{n_u}}\}$ that is composed by some corresponding rows of K and the rest by rows of the auxiliary matrix $\mathcal{G}(\alpha^*)$ (HU; LIN; CHEN, 2002). Note that, as stated in (CAO; LIN, 2003), for any $x(t) \in \mathscr{L}(\mathcal{G}(\alpha^*))$

$$\mathrm{sat}(Ky(t)) \in \mathrm{co}\big\{\mathscr{D}_s K + \mathscr{D}_s^- \mathcal{G}(\alpha^*),\ \ s \in \mathbb{K}_{2^{n_u}}\big\}, \tag{103}$$

with $\mathrm{co}(\mathscr{M})$ representing the convex hull of a set $\mathscr{M}$. It is important to stress that (103) can be used only if $x(t) \in \mathscr{L}(\mathcal{G}(\alpha^*))$. Therefore, it is necessary to guarantee that the trajectories of the closed-loop system will remain within the set $\mathscr{L}(\mathcal{G}(\alpha^*))$. In Section 5.5 it will be shown an additional condition such that $\mathscr{E}(P(\alpha^*),\delta) \subset \mathscr{L}(\mathcal{G}(\alpha^*))$ holds. Knowing that $\mathscr{E}(P(\alpha^*),\delta)$ is contractively invariant with respect to the closed-loop system, and $\mathscr{E}(P(\alpha^*),\delta) \subset \mathscr{L}(\mathcal{G}(\alpha^*))$ and based on the notation given in (101), from (87) $\mathrm{sat}(u(t))$ can be represented as

$$\begin{aligned}
\mathrm{sat}(u(t)) = \mathrm{sat}(Ky(t)) &= \sum_{s=1}^{2^{n_u}} \vartheta_s[\mathscr{D}_s(Ky(t)) + \mathscr{D}_s^- \mathcal{G}(\alpha^*)x(t)] \\
&= \mathscr{D}(\vartheta)(Ky(t)) + \mathscr{D}^-(\vartheta)\mathcal{G}(\alpha^*)x(t) \\
&= \mathscr{D}(\vartheta)\big(KC_2(\alpha)x(t) + KH(\alpha)w(t)\big) + \mathscr{D}^-(\vartheta)\mathcal{G}(\alpha^*)x(t).
\end{aligned} \tag{104}$$

Considering the representation above, the notation given in (101), then substituting (104) in (87), one has

$$\begin{cases}
\dot{x}(t) &= \widetilde{A}(\alpha,\vartheta)x(t) + \widetilde{E}(\alpha,\vartheta)w(t) \\
z(t) &= \widetilde{C_1}(\alpha,\vartheta)x(t) + \widetilde{F}(\alpha,\vartheta)w(t) \\
y(t) &= C_2(\alpha)x(t) + H(\alpha)w(t)
\end{cases} \tag{105}$$

where,

$$\begin{aligned}
\widetilde{A}(\alpha,\vartheta) &= A(\alpha) + B(\alpha)[\mathscr{D}(\vartheta)KC_2(\alpha) + \mathscr{D}^-(\vartheta)\mathcal{G}(\alpha^*)] \\
\widetilde{E}(\alpha,\vartheta) &= E(\alpha) + B(\alpha)D(\vartheta)KH(\alpha) \\
\widetilde{C_1}(\alpha,\vartheta) &= C_1(\alpha) + D(\alpha)[\mathscr{D}(\vartheta)KC_2(\alpha) + \mathscr{D}^-(\vartheta)\mathcal{G}(\alpha^*)] \\
\widetilde{F}(\alpha,\vartheta) &= F(\alpha) + D(\alpha)\mathscr{D}(\vartheta)KH(\alpha).
\end{aligned} \tag{106}$$

5.3 OPERATION REGION

In case of nonlinear systems (ALVES *et al.*, 2016), represented by a T-S fuzzy model, in which the nonlinear functions of the model usually must have a bounded supremum and a bounded infinimum, an operation region should be defined to guarantee the exactly representation of the system. Further, it is possible to define an operation region, even for a linear system, to consider the inherent bound and restriction of the actual plant. Let the polyhedral set for the operation region be defined as

$$\mathcal{X}(\mathcal{N}_h) \triangleq \left\{x(t) \in \mathbb{R}^{n_x} : |\mathcal{N}_{(h)}x(t)| \leq \phi_h, \forall h \in \mathbb{K}_{n_h}\right\}, \tag{107}$$

where $\mathcal{N} = [\mathcal{N}'_{(1)} \quad \mathcal{N}'_{(2)} \quad \ldots \mathcal{N}'_{(n_h)}]' \in \mathbb{R}^{n_h \times n_x}$ and $\phi = [\phi_1 \quad \phi_2 \quad \ldots \quad \phi_{n_h}]' \in \mathbb{R}^{n_h}$ are known. Observe that n_h may be, for instance, the number of state variables which needs to be considered bounded for the controller design. Clearly, to ensure that the system trajectory will not deviate from the limitation values (ϕ), previously established by the designer, it is required that the contratively invariant set $\mathscr{E}(P(\alpha^*), \delta)$ to be inside the set $\mathcal{X}(\mathcal{N}_h)$. For this purpose in Section 5.5 it will be presented a condition such that $\mathscr{E}(P(\alpha^*), \delta) \subset \mathcal{X}(\mathcal{N}_h)$ is ensured.

5.4 ON ENLARGING OF THE CONTRACTIVELY INVARIANT SET (ESTIMATIVE OF THE DOMAIN OF ATTRACTION)

We may choose the “largest” ellipsoid to obtain the least conservative estimate of the domain of attraction (HU; LIN; CHEN, 2002).

Let $\mathscr{X} \in \mathbb{R}^n$ be a prescribed bounded convex set containing the origin. The region $\mathscr{X}$ is defined as a polyhedral set:

$$\mathscr{X} = \varpi \mathrm{co}\{w_1, \ldots, w_{n_L}\}, \varpi > 0 \tag{108}$$

where n_L is the number of vertices of the convex polyhedron.

Thus, if the constraint $\mathscr{X} \subset \mathscr{E}(P(\alpha^*), \delta)$ is enforced, it is possible to maximize the variable ϖ in order to obtain a less conservative estimation of the domain of attraction. In other words, if the polyhedral set is $\mathscr{X}$ is inside of the contractive set $\mathscr{E}(P(\alpha^*), \delta)$ and by the maximization of the ϖ the set $\mathscr{X}$ is "enlarged", consequently the set $\mathscr{E}(P(\alpha^*), \delta)$ is "enlarged" as well, since $\mathscr{X} \subset \mathscr{E}(P(\alpha^*), \delta)$ hold. Remember that $\mathscr{E}(P(\alpha^*), \delta) \subseteq \mathscr{R}_A$, with $\mathscr{R}_A$ defined in (89). That is, the ellipsoidal set is an estimation of the domain of attraction.

The condition where the constraint $\mathscr{X} \subset \mathscr{E}(P(\alpha^*), \delta)$ is ensured, will be presented in Section 5.5, as well as an illustration of the sets constraints.

5.5 SETS CONSTRAINTS

As stated in the last subsections some sets constraints are required to cope with a closed-loop system subject to actuator saturation and limited by an operation region considering the enlargement of the contractively invariant set. To this end, the following constraints are required to be ensured:

(i) $\mathscr{E}(P(\alpha^*),\delta) \subset \mathscr{L}(\mathcal{G}(\alpha^*))$: The set $\mathscr{E}(P(\alpha^*),\delta)$ is contractive invariant with respect to the closed-loop system, thus, all trajectories beginning within it, will remain inside the set and converge to zero. Therefore, if $\mathscr{E}(P(\alpha^*),\delta) \subset \mathscr{L}(\mathcal{G}(\alpha^*))$ it is ensured that the trajectories will remain in $\mathscr{L}(\mathcal{G}(\alpha^*))$, then the saturation can be represented by $\mathrm{sat}(u(t)) = \mathscr{D}(\vartheta)(Ky(t)) + \mathscr{D}^-(\vartheta)\mathcal{G}(\alpha^*)x(t)$.

(ii) $\mathscr{E}(P(\alpha^*),\delta) \subset \mathcal{X}(\mathcal{N}_h)$: Since the trajectories will remain in $\mathscr{E}(P(\alpha^*),\delta)$ and it is inside $\mathcal{X}(\mathcal{N}_h)$, the state variables will not deviate from the limitation values specified by the designer.

(iii) $\mathscr{X} \subset \mathscr{E}(P(\alpha^*),\delta)$: If the set $\mathscr{X}$ is kept inside $\mathscr{E}(P(\alpha^*),\delta)$, thus aiming to "enlarge" the ellipsoidal set one may "maximize" the set $\mathscr{X}$, related to the scalar ϖ.

It is important to highlight that the constraint (i) is required in the design in order to represent the saturation as described in the Section 5.2. However, the constraints (ii) and (iii) are optionals, which should be employed by the designer if it is needed to consider an operation region and "enlarge" the ellipsoidal set.

Figures 6 and 7 illustrate the sets aforementioned and also examples of trajectories, in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The arrow indicates the direction of the trajectory over time. The following three lemmas are devoted to ensure the constraint presented in (ii), (iii) and (iii), respectively.

Lemma 3. *Define,* $\mathcal{Z}(\alpha) = \sum_{j=1}^{r} \alpha_j \mathcal{Z}_j$, $\zeta_l(\alpha) = \sum_{j=1}^{r} \alpha_j \zeta_{j(l)}$, $j \in \mathbb{K}_r$

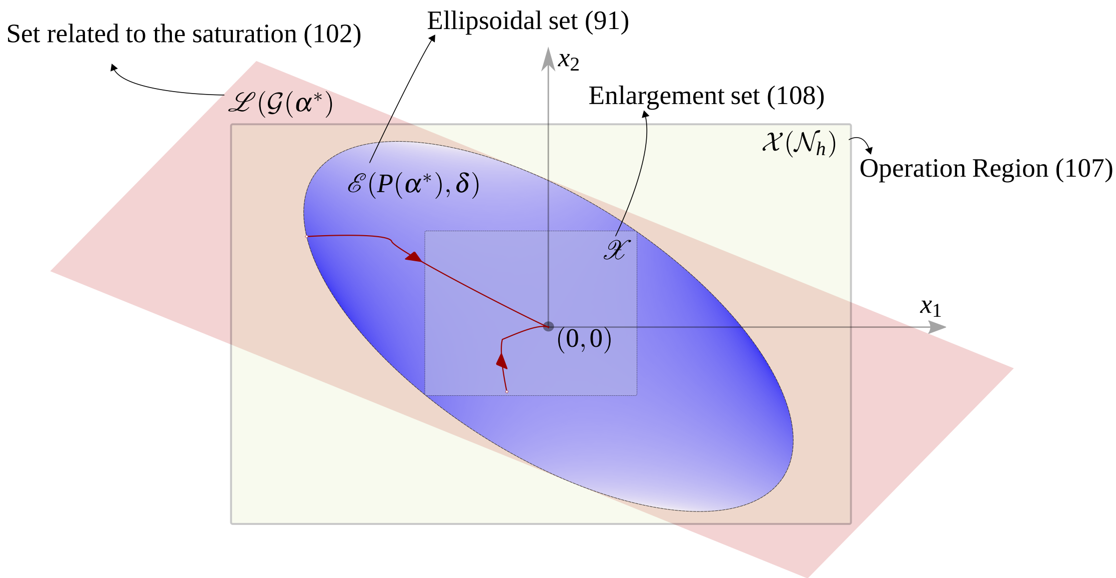
$$\begin{aligned} \mathcal{Z}_j &= \begin{bmatrix} \zeta'_{j(1)} & \zeta'_{j(2)} & \dots & \zeta'_{j(n_u)} \end{bmatrix}' \in \mathbb{R}^{n_u \times n_x}, \quad j \in \mathbb{K}_r \\ \mathcal{Z}(\alpha) &= \begin{bmatrix} \zeta_{(1)}(\alpha)' & \zeta_{(2)}(\alpha)' & \dots & \zeta_{(n_u)}(\alpha)' \end{bmatrix}', \quad l \in \mathbb{K}_{n_u} \end{aligned} \tag{109}$$

Then, given the sets $\mathscr{E}(P(\alpha^*),\delta)$ *in* (90) *and* (91) *and* $\mathscr{L}(\mathcal{G}(\alpha^*))$ *in* (102)*, the constraint* $\mathscr{E}(P(\alpha^*),\delta) \subset \mathscr{L}(\mathcal{G}(\alpha^*))$ *is satisfied if the condition*

$$\begin{bmatrix} \overline{u}_l^2 \delta^{-1} & \zeta_{j(l)} \\ \zeta'_{j(l)} & X_j \end{bmatrix} \geq 0, \tag{110}$$

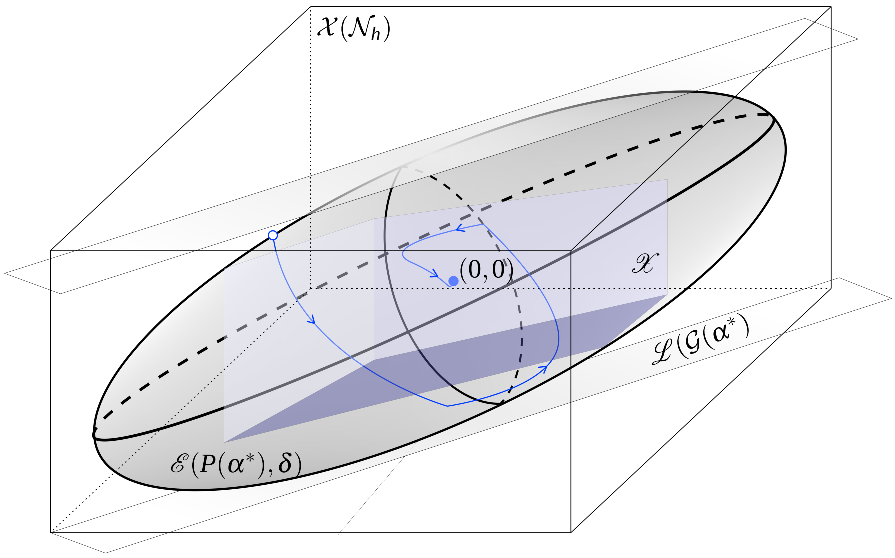
holds for all $j \in \mathbb{K}_r$ *and* $l \in \mathbb{K}_{2^{n_u}}$*, with* $P(\alpha^*) = X(\alpha)^{-1}$ *and* $\mathcal{G}(\alpha^*) = \mathcal{Z}(\alpha)P(\alpha^*)$.

Figure 6 - Representation in $\mathbb{R}^2$ of the sets $\mathscr{E}(P(\alpha^*),\delta)$, $\mathscr{L}(\mathcal{G}(\alpha^*))$, $\mathcal{X}(\mathcal{N}_h)$, $\mathscr{X}$ and state trajectories.



Source: Own author.

Figure 7 - Representation in $\mathbb{R}^3$ of the sets $\mathscr{E}(P(\alpha^*),\delta)$, $\mathscr{L}(\mathcal{G}(\alpha^*))$, $\mathcal{X}(\mathcal{N}_h)$, $\mathscr{X}$ and state trajectories.



Source: Own author.

Proof. Firstly, multiplying (110) by α_j, where $\alpha = [\alpha_1 \ \ \alpha_2 \ \ \dots \ \ \alpha_r]' \in \Lambda_r$ and taking the sum from $j = 1$ to $j = r$, from (90) and (109) one obtains

$$\begin{bmatrix} \overline{u}_l^2\delta^{-1} & \zeta_l(\alpha) \\ \zeta_l(\alpha)' & X(\alpha) \end{bmatrix} \geq 0. \tag{111}$$

Pre- and post multiplying (111) by diag$\{1, P(\alpha^*)\}$, where $P'(\alpha^*) = P(\alpha^*) = X(\alpha)^{-1}$, one has:

$$\begin{bmatrix} \overline{u}_l^2\delta^{-1} & \zeta_l(\alpha)P(\alpha^*) \\ P(\alpha^*)\zeta_l(\alpha)' & P(\alpha^*)X(\alpha)P(\alpha^*) \end{bmatrix} = \begin{bmatrix} \overline{u}_l^2\delta^{-1} & \zeta_l(\alpha)P(\alpha^*) \\ P(\alpha^*)\zeta_l(\alpha)' & P(\alpha^*) \end{bmatrix} \geq 0. \tag{112}$$

Applying the Schur complement based on the element $P(\alpha^*)$ in (112), it follows that:

$$P(\alpha^*) - P(\alpha^*)\zeta_l(\alpha)'\overline{u}_l^{-2}\delta\zeta_l(\alpha)P(\alpha^*) \geq 0. \tag{113}$$

Defining $G_l(\alpha^*) = \zeta_l(\alpha)X(\alpha)^{-1} = \zeta_l(\alpha)P(\alpha^*)$, therefore from (113)

$$P(\alpha^*) \geq \overline{u}_l^{-2}\delta G_l(\alpha^*)'G_l(\alpha^*). \tag{114}$$

Pre- and post multiplying (114) by $x(t)'$ and its transpose, for $x(t) \neq 0$,

$$x(t)'P(\alpha^*)x(t) \geq \overline{u}_l^{-2}\delta x(t)'G_l(\alpha^*)'G_l(\alpha^*)x(t). \tag{115}$$

If $x(t) \in \mathscr{E}(P(\alpha^*), \delta)$ it implies that $x(t)'P(\alpha^*)x(t) \leq \delta$. Therefore, from (115), it follows,

$$\delta \geq x(t)'P(\alpha^*)x(t) \geq \overline{u}_l^{-2}\delta x(t)'G_l(\alpha^*)'G_l(\alpha^*)x(t) \tag{116}$$

and then

$$\overline{u}_l^2 \geq x(t)'G_l(\alpha^*)'G_l(\alpha^*)x(t) = |G_l(\alpha^*)x(t)|^2 \tag{117}$$

Hence, $x(t) \in \mathscr{L}(\mathcal{G}(\alpha^*))$ and consequently $\mathscr{E}(P(\alpha^*), \delta) \subset \mathscr{L}(\mathcal{G}(\alpha^*))$. $\square$

Lemma 4. *(ALVES, 2017) The condition $\mathscr{E}(P(\alpha^*), \delta) \subset \mathcal{X}(\mathcal{N}_h)$ is ensured if the following conditions hold*

$$\begin{bmatrix} \frac{\phi_h^2}{\delta} & \mathcal{N}_h X_j \\ X_j\mathcal{N}_h' & X_j \end{bmatrix} \geq 0, \tag{118}$$

for all $n_h \in \mathbb{K}_{n_h}$, $j \in \mathbb{K}_r$, with $P(\alpha^*) = X(\alpha)^{-1}$.

Proof. Firstly, multiplying (118) by α_j, $\alpha_j > 0$ and taking the sum from $j = 1$ to $j = r$, from (90) and (118) one obtains

$$\begin{bmatrix} \frac{\phi_h^2}{\delta} & \mathcal{N}_h X(\alpha) \\ X(\alpha)\mathcal{N}_h' & X(\alpha) \end{bmatrix} \geq 0. \tag{119}$$

Pre and post multiplying (119) by diag$\{1, P(\alpha^*)\}$, with $P(\alpha^*) = X(\alpha)^{-1}$, then

$$\begin{bmatrix} \frac{\phi_h^2}{\delta} & \mathcal{N}_h \\ \mathcal{N}_h' & P(\alpha^*) \end{bmatrix} \geq 0. \tag{120}$$

Then, applying the Schur complement in (120), one has

$$\begin{aligned} P(\alpha^*) - \mathcal{N}_h' \phi^{-2} \delta \mathcal{N}_h &\geq 0, \\ P(\alpha^*) &\geq \mathcal{N}_h' \phi^{-2} \delta \mathcal{N}_h. \end{aligned} \tag{121}$$

Pre and post multiplying (121) by $x(t)'$ and its transpose it follows that

$$x(t)' P(\alpha^*) x(t) \geq x(t)' \mathcal{N}_h' \phi^{-2} \delta \mathcal{N}_h x(t). \tag{122}$$

Considering $x(t) \in \mathscr{E}(P(\alpha^*), \delta)$, from (91) one has $x(t)' P(\alpha^*) x(t) \leq \delta$, then

$$\begin{aligned} \delta &\geq x(t)' P(\alpha^*) x(t) \geq x(t)' \mathcal{N}_h' \phi^{-2} \delta \mathcal{N}_h x(t) \\ \phi^2 &\geq x(t)' \mathcal{N}_h' \mathcal{N}_h x(t). \end{aligned} \tag{123}$$

Therefore, $x(t) \in \mathcal{X}(\mathcal{N}_h)$, consequently $\mathscr{E}(P(\alpha^*), \delta) \subset \mathcal{X}(\mathcal{N}_h)$. □

Lemma 5. *The constraint $\mathscr{X} \subset \mathscr{E}(P(\alpha^*), \delta)$ is ensured if*

$$\begin{bmatrix} \delta & \varpi w_g \\ w_g' \varpi & X_j \end{bmatrix} \geq 0 \tag{124}$$

is satisfied $\forall g \in \mathbb{K}_{n_L}$, where n_L is the number of vertices of the convex polyhedron.

Proof. Firstly multiplying (124) by $\sum_{j=1}^{r} \alpha_j \times \sum_{g=1}^{n_L} \theta_g$, considering (90) and $\theta_g \geq 0, \sum_{g=1}^{n_L} \theta_g = 1,\ \ g \in n_L$ one has

$$\begin{bmatrix} \delta & \varpi w(\theta) \\ w(\theta)' \varpi & X(\alpha) \end{bmatrix} \geq 0. \tag{125}$$

Applying the Schur complement in (125), then

$$\delta - \varpi w(\theta)' P(\alpha^*) \varpi w(\theta) \geq 0. \tag{126}$$

Therefore, it follows that $\varpi w(\theta)' P(\alpha^*) \varpi w(\theta) \leq \delta$ and consequently $\varpi w(\theta) \in \mathscr{E}(P(\alpha^*), \delta)$, for any $\varpi w(\theta) \in \mathscr{X}$. Therefore, $\mathscr{X} \subset \mathscr{E}(P(\alpha^*), \delta)$. □

5.6 $\mathscr{H}_\infty$ PROBLEM CONSIDERING OPERATION REGION AND LINEAR SYSTEMS SUBJECT TO SATURATION

In order to deal with the $\mathscr{H}_\infty$ control problem for systems under constraints, as example of the actuator saturation and region of operation, it is assigned a limiting value of the disturbance $w(t)$ energy instead of considering $w(t) \in \mathscr{L}_2$ as presented in the previous sections. In doing so, consider the energy-bounded disturbance $w(t) \in \mathscr{W}$ and a positive constant ε, such that

$$\mathscr{W} \triangleq \left\{ w(t) \in \mathbb{R}^{n_w} : \int_0^\infty w(t)' w(t) dt \leq \varepsilon \right\}. \tag{127}$$

The following problem is based in Oliveira *et al.* (2018) results, inspired by Lee *et al.* (2015).

Problem 4. *Given the energy-bounded disturbance $w(t) \in \mathscr{W}$ in* (127)*, a slack variable $\varphi > 0$, a constant $\varepsilon_0 \geq 0$ as well as a constant $\gamma > 0$, determine an output-dependent control law that satisfies the following three statements (OLIVEIRA et al., 2018):*

(i) for $w(t) = 0, t \geq 0$, the uncertain linear system (105), (106) *and* (5) *is locally asymptotically stable for all $x(0) \in \mathscr{E}\left(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon\right)$ and the ellipsoid $\mathscr{E}\left(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon\right)$ is an invariant subset of the domain of attraction (i.e., if $x(0)$ belongs to this set, then $x(t)$, $t > 0$, will also stay in this set).*

(ii) for $w(t) \neq 0$, if $x(0) \in \mathscr{E}\left(P(\alpha^*), \varepsilon_0\right)$, then $x(t) \in \mathscr{E}\left(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon\right)$, for all $t \geq 0$.

(iii) for $w(t) \neq 0$, if $x(0) = 0$, then the linear system (105), (106) *and* (5) *has an $\mathscr{H}_\infty$ guaranteed cost $\gamma = \sqrt{\mu} > 0$, such that*

$$\int_0^\infty z(t)' z(t) \mathrm{d}t \leq \gamma^2 \int_0^\infty w(t)' w(t) \mathrm{d}t \tag{128}$$

and $x(t) \in \mathscr{E}\left(P(\alpha^*), \varphi^{-1}\varepsilon\right)$, for all $t \geq 0$.

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Remark 6. (SAIFIA *et al.*, 2019) Note that, the lower value of $\gamma = \sqrt{\mu}$, the better attenuation of disturbance effect ($w(t)$) in the controlled output ($z(t)$). Although, a lower attenuation yields smaller estimation of the region of attraction. To achieve a good rejection of the disturbance with the compromise of a larger region of attraction, it possible to minimize the value of $\mu - \varpi$.

In order to deal with the Problem 4 the Theorem 9, based on Theorem 6 (CHANG; PARK; ZHOU, 2015), is proposed.

Theorem 9. *Consider the system described in* (105), (106) *and* (5) *subject to actuator saturation, a energy bounded disturbance and an operation region $\mathcal{X}(\mathcal{N}_h)$* (107)*, where $\overline{u} \in \mathbb{R}^{n_u}$,*

$\mathcal{N} \in R^{n_h \times n_x}$, $\phi \in \mathbb{K}_{n_h}$, $\varepsilon_0 \geq 0$, $\varepsilon > 0$ and φ are known. For also known scalar parameters ρ, β and $\gamma > 0$ suppose that there exist symmetric matrices $X_j > 0 \in \mathbb{R}^{n_x \times n_x}$, matrices $\mathcal{Z}_j = \begin{bmatrix} \zeta'_{j(1)} & \zeta'_{j(2)} & \dots & \zeta'_{j(n_u)} \end{bmatrix}' \in \mathbb{R}^{n_u \times n_x}$, $V \in \mathbb{R}^{n_u \times n_y}$ and $U \in R^{n_y \times n_y}$ satisfying,

$$\Phi_{jj} < 0 \tag{129}$$

$$\Phi_{jk} + \Phi_{kj} < 0, \quad j < k \tag{130}$$

$$\Phi_{jk} = \begin{bmatrix} \mathrm{He}(A_j X_k + B_j[\mathscr{D}_s V \mathscr{F}_k + \mathscr{D}_s^- \mathcal{Z}_k]) & (*) & (*) & (*) \\ E'_j + H'_k V' \mathscr{D}'_s B'_j & -\varphi^{-1} I & (*) & (*) \\ C_{1j} X_k + \rho \mathscr{F}'_0 V' \mathscr{D}'_s B'_j + D_j[\mathscr{D}_s V \mathscr{F}_k + \mathscr{D}_s^- \mathcal{Z}_k] & F_j + D_j \mathscr{D}_s V H_k & -\varphi\gamma^2 I + He(\rho D_j \mathscr{D}_s V \mathscr{F}_0) & (*) \\ C_{2j} X_k - U \mathscr{F}_k + \beta V' \mathscr{D}'_s B'_j & H_j - U H_k & \beta V' \mathscr{D}'_s D'_j - \rho U \mathscr{F}_0 & -\beta U - \beta U' \end{bmatrix} < 0 \tag{131}$$

$$\begin{bmatrix} \frac{\bar{u}_l^2}{(\varepsilon_0 + \varphi^{-1}\varepsilon)} & \zeta_{j(l)} \\ \zeta'_{j(l)} & X_j \end{bmatrix} \geq 0 \tag{132}$$

$$\begin{bmatrix} \frac{\phi_h^2}{(\varepsilon_0 + \varphi^{-1}\varepsilon)} & \mathcal{N}_h X_j \\ X_j \mathcal{N}'_h & X_j \end{bmatrix} \geq 0 \tag{133}$$

with $\mathscr{F}_j$ (54) and $\mathscr{F}_0$ (55) for all j and $k \in \mathbb{K}_r$, $l \in \mathbb{K}_{n_u}$, $h \in \mathbb{K}_{n_L}$, $s \in \mathbb{K}_{2^{n_u}}$, $\mathscr{D} \in \mathcal{D}$, $\mathscr{D}_s^- = I - \mathscr{D}_s$. Then, the control gain $K = VU^{-1}$ with $X(\alpha)^{-1} = P(\alpha^*)$ satisfies the statements of Problem 4. Moreover, $x(t) \in \mathcal{X}(\mathcal{N}_h)$(operation region) for all $t \geq 0$ and the sets constraints $\mathscr{E}(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \mathscr{L}(\mathcal{G}(\alpha^*))$ and $\mathscr{E}(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \mathcal{X}(\mathcal{N}_h)$ hold.

Proof. Consider the Lyapunov function candidate (90). Firstly, from Lemma 3 and Lemma 4, note that (133) and (132) ensure $\mathscr{E}(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \mathscr{L}(\mathcal{G}(\alpha^*))$ and $\mathscr{E}(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \mathcal{X}(\mathcal{N}_h)$, respectively. From (5), (129), (130) and (131), one has

$$\sum_{j=1}^{r}\sum_{k=1}^{r} \alpha_j \alpha_k \Phi_{jk} = \sum_{j=1}^{r} \alpha_j^2 \Phi_{jj} + \sum_{j=1}^{r}\sum_{j<k}^{r} \alpha_j \alpha_k \left(\Phi_{jk} + \Phi_{kj}\right)$$

$$= \begin{bmatrix} \mathrm{He}(A(\alpha)X(\alpha) + B(\alpha)[\mathscr{D}_s V \mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}(\alpha)]) & (*) \\ E(\alpha)' + H(\alpha)' V' \mathscr{D}'_s B(\alpha)' & -\varphi^{-1} I \\ C_1(\alpha)X(\alpha) + \rho \mathscr{F}'_0 V' \mathscr{D}'_s B(\alpha)' + D(\alpha)[\mathscr{D}_s V \mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}(\alpha)] & F(\alpha) + D(\alpha) \mathscr{D}_s V H(\alpha) \\ C_2(\alpha)X(\alpha) - U\mathscr{F}(\alpha) + \beta V' \mathscr{D}'_s B(\alpha)' & H(\alpha) - UH(\alpha) \end{bmatrix}$$

$$\left.\begin{matrix} (*) & (*) \\ (*) & (*) \\ -\varphi\gamma^2 I + \mathrm{He}(\rho D(\alpha)\mathscr{D}_s V \mathscr{F}_0) & (*) \\ \beta V'\mathscr{D}_s' D(\alpha)' - \rho U \mathscr{F}_0 & -\beta U - \beta U' \end{matrix}\right] < 0 \quad (134)$$

Applying Lemma 1 in (134) with $\mathscr{T} + \mathscr{P}\mathscr{A} + \mathscr{A}'\mathscr{P}'$, note that

$$\mathscr{T} = \begin{bmatrix} \mathrm{He}(A(\alpha)X(\alpha) + B(\alpha)[\mathscr{D}_s V \mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}(\alpha)]) & (*) & (*) \\ E(\alpha)' + H(\alpha)'V'\mathscr{D}_s' B(\alpha)' & -\varphi^{-1} I & (*) \\ C_1(\alpha)X(\alpha) + \rho\mathscr{F}_0' V' \mathscr{D}_s' B(\alpha)' + D(\alpha)[\mathscr{D}_s V \mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}(\alpha)] & F(\alpha) + D(\alpha)\mathscr{D}_s V H(\alpha) & -\varphi\gamma^2 I + \mathrm{He}(\rho D(\alpha)\mathscr{D}_s V \mathscr{F}_0) \end{bmatrix} < 0, \mathscr{U} = U,$$

$$\mathscr{P} = \begin{bmatrix} B(\alpha)\mathscr{D}_s V \\ 0 \\ D(\alpha)\mathscr{D}_s V \end{bmatrix}, \mathscr{A} = U^{-1}\begin{bmatrix} C_2(\alpha)X(\alpha) - U\mathscr{F}(\alpha) & H(\alpha) - UH(\alpha) & -\rho U \mathscr{F}_0 \end{bmatrix}. \quad (135)$$

Performing the variable changing $K = VU^{-1}$, observe that

$$\mathscr{T} + \mathscr{P}\mathscr{A} + \mathscr{A}'\mathscr{P}' = \mathscr{T} + \begin{bmatrix} \begin{pmatrix} B(\alpha)\mathscr{D}_s K C_2(\alpha)X(\alpha) - \\ B(\alpha)\mathscr{D}_s V\mathscr{F}(\alpha) \end{pmatrix} & \begin{pmatrix} B(\alpha)\mathscr{D}_s K H(\alpha) \\ -B(\alpha)\mathscr{D}_s V H(\alpha) \end{pmatrix} & -\rho B(\alpha)\mathscr{D}_s V \mathscr{F}_0 \\ 0 & 0 & 0 \\ \begin{pmatrix} D(\alpha)\mathscr{D}_s K C_2(\alpha)X(\alpha) \\ -D(\alpha)\mathscr{D}_s V\mathscr{F}(\alpha) \end{pmatrix} & \begin{pmatrix} D(\alpha)\mathscr{D}_s K H(\alpha) \\ -D(\alpha)\mathscr{D}_s V H(\alpha) \end{pmatrix} & -\rho D(\alpha)\mathscr{D}_s V \mathscr{F}_0 \end{bmatrix}$$
$$+ \begin{bmatrix} \begin{pmatrix} X(\alpha)'C_2(\alpha)'K'\mathscr{D}_s' B(\alpha)' \\ -\mathscr{F}(\alpha)'V'\mathscr{D}_s' B(\alpha)' \end{pmatrix} & 0 & \begin{pmatrix} X(\alpha)'C_2(\alpha)'K'\mathscr{D}_s' D(\alpha)' \\ -\mathscr{F}(\alpha)'V'\mathscr{D}_s' D(\alpha)' \end{pmatrix} \\ H(\alpha)'K'\mathscr{D}_s' B(\alpha)' - H(\alpha)'V'\mathscr{D}_s' B(\alpha)' & 0 & H(\alpha)'K'\mathscr{D}_s' D(\alpha)' - H(\alpha)V'\mathscr{D}_s' D(\alpha)' \\ -\rho\mathscr{F}_0' V'\mathscr{D}_s' B(\alpha)' & 0 & -\rho\mathscr{F}_0' V'\mathscr{D}_s' D(\alpha)' \end{bmatrix} < 0. \quad (136)$$

Rewriting (136), it follows that

$$\left[\begin{matrix} \begin{pmatrix} \mathrm{He}(A(\alpha)X(\alpha) + B(\alpha)[\mathscr{D}_s V\mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}(\alpha)] \\ +B(\alpha)\mathscr{D}_s[-V\mathscr{F}(\alpha) + KC_2(\alpha)X(\alpha)]) \end{pmatrix} \\ E(\alpha)' + H(\alpha)'V'\mathscr{D}_s' B(\alpha)' + H(\alpha)'K'\mathscr{D}_s' B(\alpha)' - H(\alpha)V'\mathscr{D}_s' B(\alpha)' \\ \begin{pmatrix} C_1(\alpha)X(\alpha) + \rho\mathscr{F}_0' V'\mathscr{D}_s' B(\alpha)' + D(\alpha)[\mathscr{D}_s V\mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}(\alpha)] + \\ D(\alpha)\mathscr{D}_s[KC_2(\alpha)X(\alpha) - V\mathscr{F}(\alpha)] - \rho\mathscr{F}_0' V'\mathscr{D}_s' B(\alpha)' \end{pmatrix} \end{matrix}\right.$$

$$\begin{matrix} (*) & (*) \\ -\varphi^{-1}I & (*) \\ \begin{pmatrix} F(\alpha)+D(\alpha)\mathscr{D}_sVH(\alpha)+D(\alpha)\mathscr{D}_sKH(\alpha) \\ -D(\alpha)\mathscr{D}_sVH(\alpha) \end{pmatrix} & \begin{pmatrix} -\varphi\gamma^2I+\mathrm{He}(\rho D(\alpha)\mathscr{D}_sV\mathscr{F}_0) \\ -\mathrm{He}(\rho D(\alpha)\mathscr{D}_sV\mathscr{F}_0) \end{pmatrix} \end{matrix}\Bigg] < 0. \tag{137}$$

Performing the operations, from (137) one has

$$\begin{bmatrix} \begin{pmatrix} \mathrm{He}(A(\alpha)X(\alpha)+B(\alpha)[\mathscr{D}_sKC_2(\alpha)X(\alpha) \\ +\mathscr{D}_s^-\mathcal{Z}(\alpha)]) \end{pmatrix} & (*) & (*) \\ E(\alpha)'+H(\alpha)'K'\mathscr{D}_s'B(\alpha)' & -\varphi^{-1}I & (*) \\ \begin{pmatrix} C_1(\alpha)X(\alpha)+D(\alpha)[\mathscr{D}_sKC_2(\alpha)X(\alpha) \\ +\mathscr{D}_s^-\mathcal{Z}(\alpha)] \end{pmatrix} & F(\alpha)+D(\alpha)\mathscr{D}_sKH(\alpha) & -\varphi\gamma^2I \end{bmatrix} < 0. \tag{138}$$

Pre- and post-multiplying (138) by $\mathrm{diag}\{P(\alpha^*),I,I\}$ in both sides, considering $P(\alpha^*) = X(\alpha)^{-1}$, one obtains:

$$\begin{bmatrix} \begin{pmatrix} \mathrm{He}(P(\alpha^*)A(\alpha)+P(\alpha^*)B(\alpha)[\mathscr{D}_sKC_2(\alpha) \\ +\mathscr{D}_s^-\mathcal{Z}(\alpha)P(\alpha^*)]) \end{pmatrix} & (*) & (*) \\ (E(\alpha)'+H(\alpha)'K'\mathscr{D}_s'B(\alpha)')P(\alpha^*) & -\varphi^{-1}I & (*) \\ C_1(\alpha)+D(\alpha)[\mathscr{D}_sKC_2(\alpha)+\mathscr{D}_s^-\mathcal{Z}(\alpha)P(\alpha^*)] & F(\alpha)+D(\alpha)\mathscr{D}_sKH(\alpha) & -\varphi\gamma^2I \end{bmatrix} < 0. \tag{139}$$

Considering the variable changing $\mathcal{G}(\alpha^*) = \mathcal{Z}(\alpha)X(\alpha)^{-1} = \mathcal{Z}(\alpha)P(\alpha^*)$, then

$$\begin{bmatrix} \begin{pmatrix} \mathrm{He}(P(\alpha^*)A(\alpha)+P(\alpha^*)B(\alpha)[\mathscr{D}_sKC_2(\alpha) \\ +\mathscr{D}_s^-\mathcal{G}(\alpha^*)]) \end{pmatrix} & (*) & (*) \\ (E(\alpha)'+H(\alpha)'K'\mathscr{D}_s'B(\alpha)')P(\alpha^*) & -\varphi^{-1}I & (*) \\ C_1(\alpha)+D(\alpha)[\mathscr{D}_sKC_2(\alpha)+\mathscr{D}_s^-\mathcal{G}(\alpha^*)] & F(\alpha)+D(\alpha)\mathscr{D}_sKH(\alpha) & -\varphi\gamma^2I \end{bmatrix} < 0. \tag{140}$$

Multiplying the result by ϑ_s, $\vartheta_s \geq 0$, $\sum_{s=1}^{2^{nu}} \vartheta_s = 1$ and following the notations (101) and (106), then

$$\begin{bmatrix} \mathrm{He}\big(P(\alpha^*)\widetilde{A}(\alpha,\vartheta)\big) & (*) & (*) \\ \widetilde{E}(\alpha,\vartheta)'P(\alpha^*) & -\varphi^{-1}I & (*) \\ \widetilde{C}_1(\alpha,\vartheta) & \widetilde{F}(\alpha,\vartheta) & -\varphi\gamma^2I \end{bmatrix} < 0. \tag{141}$$

Applying the Schur complement, note that

$$\begin{bmatrix} \mathrm{He}\big(P(\alpha^*)\widetilde{A}(\alpha,\vartheta)\big) & (*) \\ \widetilde{E}(\alpha,\vartheta)'P(\alpha^*) & -\varphi^{-1}I \end{bmatrix} + \begin{bmatrix} \widetilde{C_1}(\alpha,\vartheta)' \\ \widetilde{F}(\alpha,\vartheta)' \end{bmatrix} \varphi^{-1}\gamma^{-2}I \begin{bmatrix} \widetilde{C_1}(\alpha,\vartheta) & \widetilde{F}(\alpha,\vartheta) \end{bmatrix} < 0$$

$$\begin{bmatrix} \mathrm{He}\big(P(\alpha^*)\widetilde{A}(\alpha,\vartheta)\big)+\varphi^{-1}\gamma^{-2}\widetilde{C_1}(\alpha,\vartheta)'\widetilde{C_1}(\alpha,\vartheta) & (*) \\ \widetilde{E}(\alpha,\vartheta)'P(\alpha^*)+\varphi^{-1}\gamma^{-2}\widetilde{F}(\alpha,\vartheta)'\widetilde{C_1}(\alpha,\vartheta) & \varphi^{-1}\gamma^{-2}\widetilde{F}(\alpha,\vartheta)'\widetilde{F}(\alpha,\vartheta)-\varphi^{-1}I \end{bmatrix} < 0. \tag{142}$$

Considering a Lyapunov function candidate $V(x(t)) = x(t)'P(\alpha^*)x(t)$ and the system (105), one has

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{x}(t)'P(\alpha^*)x(t)+x(t)'P(\alpha^*)\dot{x}(t)\\
&= (x(t)'\widetilde{A}(\alpha,\vartheta)'+w(t)'\widetilde{E}(\alpha,\vartheta)')P(\alpha^*)x(t)+x(t)'P(\alpha^*)(\widetilde{A}(\alpha,\vartheta)x(t)+\widetilde{E}(\alpha,\vartheta)w(t))\\
&= x(t)'\widetilde{A}(\alpha,\vartheta)'P(\alpha^*)x(t)+w(t)'\widetilde{E}(\alpha,\vartheta)'P(\alpha^*)x(t)+\\
&\qquad x(t)'P(\alpha^*)\widetilde{A}(\alpha,\vartheta)x(t)+x(t)'P(\alpha^*)\widetilde{E}(\alpha,\vartheta)w(t)\\
&= x(t)'\big(A(\alpha)'+[\mathscr{D}(\vartheta)KC_2(\alpha)+\mathscr{D}^-(\vartheta)\mathcal{G}(\alpha^*)]'B(\alpha)'\big)P(\alpha^*)x(t)+x(t)'P(\alpha^*)\big(A(\alpha)\\
&+B(\alpha)[\mathscr{D}(\vartheta)KC_2(\alpha)+\mathscr{D}^-(\vartheta)\mathcal{G}(\alpha^*)]\big)x(t)+w(t)'\big(E(\alpha)'\\
&+H(\alpha)'K'\mathscr{D}_s'B(\alpha)'\big)P(\alpha^*)x(t)+x(t)'P(\alpha^*)\big(E(\alpha)+B(\alpha)\mathscr{D}_sKH(\alpha)\big)w(t)\\
&= x(t)'\big(A(\alpha)'P(\alpha^*)+P(\alpha^*)A(\alpha)\big)x(t)+x(t)'P(\alpha^*)\big(B(\alpha)[\mathscr{D}(\vartheta)KC_2(\alpha)x(t)\\
&+\mathscr{D}^-(\vartheta)\mathcal{G}(\alpha^*)x(t)+\mathscr{D}(\vartheta)KH(\alpha)w(t)]+E(\alpha)w(t)\big)+x(t)'\big([\mathscr{D}(\vartheta)KC_2(\alpha)\\
&+\mathscr{D}^-(\vartheta)\mathcal{G}(\alpha^*)+\mathscr{D}(\vartheta)KH(\alpha)w(t)]'B(\alpha)'w(t)'+E(\alpha)'\big)P(\alpha^*)x(t).
\end{aligned} \tag{143}$$

Admitting that $x(t)\in\mathscr{L}(\mathcal{G}(\alpha^*))$ defined in (102), and the saturation in (143) represented as in (104) then, it is possible to rewrite (143)

$$\begin{aligned}
\dot{V}(x(t)) &= x(t)'[A(\alpha)'P(\alpha^*)+P(\alpha^*)A(\alpha)]x(t)+\mathrm{sat}(Ky(t))'B(\alpha)'P(\alpha^*)x(t)\\
&+x(t)'P(\alpha^*)B(\alpha)\mathrm{sat}(Ky(t))+w(t)'E(\alpha)'P(\alpha^*)x(t)+x(t)'P(\alpha^*)E(\alpha)w(t)<0.
\end{aligned} \tag{144}$$

Bearing in mind the following multiplications,

$$\begin{aligned}
z(t)'z(t) &= \big(x(t)'\widetilde{C_1}(\alpha,\vartheta)'+w(t)'\widetilde{F}(\alpha,\vartheta)'\big)\big(\widetilde{C_1}(\alpha,\vartheta)x(t)+\widetilde{F}(\alpha,\vartheta)w(t)\big)\\
&= x(t)'\widetilde{C_1}(\alpha,\vartheta)'\widetilde{C_1}(\alpha,\vartheta)x(t)+x(t)'\widetilde{C_1}(\alpha,\vartheta)'\widetilde{F}(\alpha,\vartheta)w(t)\\
&\qquad +w(t)'\widetilde{F}(\alpha,\vartheta)'\widetilde{C_1}(\alpha,\vartheta)x(t)+w(t)'\widetilde{F}(\alpha,\vartheta)'\widetilde{F}(\alpha,\vartheta)w(t),
\end{aligned} \tag{145}$$

then, pre and post-multiplying (142) in both sides by $\begin{bmatrix} x(t)' & w(t)' \end{bmatrix}$ and its transpose it follows that

$$\begin{aligned}
&\begin{bmatrix} x(t)' & w(t)' \end{bmatrix} \begin{bmatrix} \widetilde{A}(\alpha,\vartheta)'P(\alpha^*)+P(\alpha^*)\widetilde{A}(\alpha,\vartheta)+\varphi^{-1}\gamma^{-2}\widetilde{C_1}(\alpha,\vartheta)'\widetilde{C_1}(\alpha,\vartheta) & (*) \\ \widetilde{E}(\alpha,\vartheta)'P(\alpha^*)+\varphi^{-1}\gamma^{-2}\widetilde{F}(\alpha,\vartheta)'\widetilde{C_1}(\alpha,\vartheta) & \varphi^{-1}\gamma^{-2}\widetilde{F}(\alpha,\vartheta)'\widetilde{F}(\alpha,\vartheta)-\varphi^{-1}I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}\\
&= x(t)'\widetilde{A}(\alpha,\vartheta)'P(\alpha^*)x(t)+x(t)'P(\alpha^*)\widetilde{A}(\alpha,\vartheta)x(t)+x(t)'\varphi^{-1}\gamma^{-2}\widetilde{C_1}(\alpha,\vartheta)'\widetilde{C_1}(\alpha,\vartheta)x(t)\\
&+w(t)'\widetilde{E}(\alpha,\vartheta)'P(\alpha^*)x(t)+w(t)'\varphi^{-1}\gamma^{-2}\widetilde{F}(\alpha,\vartheta)'\widetilde{C_1}(\alpha,\vartheta)x(t)+x(t)'P(\alpha^*)\widetilde{E}(\alpha,\vartheta)w(t)
\end{aligned}$$

$$+x(t)'\varphi^{-1}\gamma^{-2}\widetilde{C_1}(\alpha,\vartheta)'\widetilde{F}(\alpha,\vartheta)w(t)+w(t)'\varphi^{-1}\gamma^{-2}\widetilde{F}(\alpha,\vartheta)\widetilde{F}(\alpha,\vartheta)w(t)'-\varphi^{-1}w(t)'w(t). \tag{146}$$

Taking into account (143), (145), and (146) we have

$$\dot{V}(x(t))+\varphi^{-1}\gamma^{-2}z(t)'z(t)-\varphi^{-1}w(t)'w(t)<0 \tag{147}$$

From now, it is possible to verify that the proposed control law satisfies the aforementioned statements.

(i) Statement 1 [$w(t)=0$]:
From (90) and (105), if $x(t)\in\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)$ and $w(t)=0$, knowing that $\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)\subset\mathscr{L}(\mathcal{G}(\alpha^*))$ and $\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)\subset\mathcal{X}(\mathcal{N}_h)$, from (147) one has

$$\dot{V}(x(t))<-\varphi^{-1}\gamma^{-2}z(t)'z(t). \tag{148}$$

It is possible to observe that (148) implies in $\dot{V}(x(t))<0$ for all $x(t)\in\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)\backslash\{0\}$, since $\varphi^{-1}\gamma^{-2}\int_0^\infty z(t)'z(t)>0$. Therefore, for $w(t)=0$, the linear system (105) is asymptotically stable for all $x(0)\in\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)$.

(ii) Statement 2 [$w(t)\in\mathscr{W}$, $w(t)\neq 0$ and $x(0)'P(\alpha^*)x(0)\leq\varepsilon_0$]:
If $x(0)\in\mathscr{E}(P(\alpha)^*,\varepsilon_0)$ then $V(x(0))\leq\varepsilon_0$ and from (127) one has $\int_0^\infty w(t)'w(t)\mathrm{d}t<\varepsilon$. Integrating (147) from zero to $t,\ t>0$, and knowing that $\varphi^{-1}\gamma^{-2}\int_0^\infty z(t)'z(t)>0$ and $\gamma^2>0$, one obtains:

$$\begin{aligned} V(x(t)) &< V(x(0))+\varphi^{-1}\int_0^t w(t)'w(t)\mathrm{d}t-\varphi^{-1}\gamma^{-2}\int_0^t z(t)'z(t)\mathrm{d}t \\ &\leq V(x(0))+\varphi^{-1}\int_0^\infty w(t)'w(t)\mathrm{d}t \\ &\leq \varepsilon_0+\varphi^{-1}\varepsilon,\ t\geq 0. \end{aligned} \tag{149}$$

Thus, $V(x(t))<\varepsilon_0+\varphi^{-1}\varepsilon$ for all $t\geq 0$. Remembering that $\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)\subset\mathcal{X}(\mathcal{N}_h)$ and $\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)\subset\mathscr{L}(H(\alpha^*))$, then it is ensured that the state trajectory started in an initial condition $x(0)\in\mathscr{E}(P(\alpha^*),\varepsilon_0)$ will remain within $\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)\backslash\partial\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)$, for all $t\geq 0$, where $\partial\mathscr{E}\left(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon\right)$ is the boundary of $\mathscr{E}\left(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon\right)$.

(iii) Statement 3 [$w(t)\in\mathscr{W}$, $w(t)\neq 0$ and $x(0)=0$]:
Integrating (147) from zero to $t,\ t>0$, having $x(0)=0$ and consequently $V(x(0))=0$

$$V(x(t))-\varphi^{-1}\int_0^t w(t)'w(t)\mathrm{d}t<-\varphi^{-1}\gamma^{-2}\int_0^t z(t)'z(t)\mathrm{d}t. \tag{150}$$

Knowing that $\varphi^{-1}\gamma^{-2}\int_0^\infty z(t)'z(t) > 0$ and $w(t) \in \mathscr{W}$, it follows that

$$V(x(t)) \leq \varphi^{-1}\int_0^\infty w(t)'w(t)\mathrm{d}t \leq \varphi^{-1}\varepsilon,\ t \geq 0. \tag{151}$$

Observe that $V(x(t)) < \varphi^{-1}\varepsilon$, $t \geq 0$. Then, $x(t) \in \mathscr{E}(P(\alpha), \varphi^{-1}\varepsilon)$, for all $t \geq 0$. As well as it is a particular case of (149) with $\varepsilon_0 = 0$.

Moreover, as $V(x(t)) \geq 0$, from (150) one has

$$\int_0^\infty z(t)'z(t)\mathrm{d}t \leq \gamma^2\int_0^\infty w(t)'w(t)\mathrm{d}t. \tag{152}$$

Thus, (128) holds and the $\mathscr{H}_\infty$ performance is fulfilled.

Since the three statements of Problem 4 are satisfied, the proof is concluded. □

Remark 7. According to Remark 2 it is possible to search suboptimal values for β and ρ to reduce the $\mathscr{H}_\infty$ bound. In Theorem 9 another slack variable φ (presented in $\mathscr{E}(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon)$) was introduced. Therefore, when considering the actuator saturation numerical optimisation allows to search suboptimal values for β, ρ and φ in order to achieve a reduced $\mathscr{H}_\infty$ bound and a better estimation of the region of attraction.

5.7 CHAPTER CONCLUSION

In this chapter the robust SOF $\mathscr{H}_\infty$ control design for linear system subject to actuator saturation, considering parameter-dependent Lyapunov function, was addressed. Firstly, to obtain an estimation of the region of attraction, the ellipsoidal set, based on the Lyapunov function, was explored. Additionally, the convex hull representation of saturated controllers introduced in (CAO; LIN, 2003) was described. Subsequently, in order to consider an operation region and a better estimation of the domain of attraction the respective sets were introduced. Moreover, to ensure that the saturation can be represented as the convex hull and that all trajectories beginning within in it will remain inside the set and converge to zero the sets inclusion were assured. Furthermore, the set constraints to consider the region of operation and the ellipsoidal set "enlargment" were introduced. To cope with the $\mathscr{H}_\infty$ control for systems subject to actuator saturation the control problem, based in Oliveira *et al.* (2018), was presented. In order to deal with the previous problem, Theorem 9 was proposed. It is important to stress that, to the best of the author's knowledge, no previous results in literature explore the SOF $\mathscr{H}_\infty$ control design for linear system subject to actuator saturation, considering parameter-dependent Lyapunov function and based on (CHANG; PARK; ZHOU, 2015) approach, which one allows the output matrices of the system to be not of full row rank and avoids LME constraints.

6 ROBUST $\mathscr{H}_\infty$ SWITCHED STATIC OUTPUT FEEDBACK CONTROL OF CONTINUOUS TIME SWITCHED LINEAR SYSTEM SUBJECT TO ACTUATOR SATURATION

This chapter introduces the design of switched static output feedback controllers to cope with the $\mathscr{H}_\infty$ control problem for linear switched systems subject to actuator saturation. In doing so, the results presented in Chapters 4 and 6 are considered.

6.1 SWITCHED LINEAR SYSTEMS SUBJECT TO ACTUATOR SATURATION

Consider the continuous-time uncertain switched linear system with actuator saturation defined by the following state-space realization:

$$\begin{cases} \dot{x}(t) = A(\sigma,\alpha)x(t) + B(\sigma,\alpha)\text{sat}(u(t)) + E(\sigma,\alpha)w(t), \quad x(0) = x_0 \\ z(t) = C_1(\sigma,\alpha)x(t) + D(\sigma,\alpha)\text{sat}(u(t)) + F(\sigma,\alpha)w(t) \\ y(t) = C_2(\alpha)x(t) + H(\alpha)w(t) \end{cases} \tag{153}$$

with $x(t) \in \mathbb{R}^{n_x}$, $w(t) \in \mathbb{R}^{n_w}$, $y(t) \in \mathbb{R}^{n_y}$, $z(t) \in \mathbb{R}^{n_z}$ and the definitions already given in Section 2.3. The function sat was defined in Section 5.1.

6.2 SWITCHED CONTROL SUBJECT TO ACTUATOR SATURATION

Chapter 4 introduced a switching strategy (66) law, by means of auxiliary symmetric matrices $\hat{\mathcal{Q}}_{ic}$, that determine the value of the indexes $\sigma \in \mathbb{K}_N$ and $\eta \in \mathbb{K}_r$ that selects the subsystem and the output feedback gain, respectively. The switched control law was defined as

$$u(t) = K_{\sigma\eta}y(t), \tag{154}$$

following equations (48) and (67).

Taking into consideration the convex hull representation of saturated controllers in Section 5.2 it is necessary to redefine the set of the auxiliary matrices considering the switched control law. For this purpose, given the $\mathcal{G}_{ic}(\alpha^*) = \begin{bmatrix} G_{ic(1)}(\alpha^*)' & G_{ic(2)}(\alpha^*)' & \dots & G_{ic(n_u)}(\alpha^*) \end{bmatrix}' \in \mathbb{R}^{n_u \times n_x}$ with $i \in \mathbb{K}_N, c \in \mathbb{K}_r$ and a known vector $\overline{u} = \begin{bmatrix} \overline{u}_1 & \overline{u}_2 & \dots & \overline{u}_{n_u} \end{bmatrix} \in \mathbb{R}^{n_u}$, $\overline{u}_l > 0$, $l \in \mathbb{K}_{n_u}$, let the polyhedral set $\mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$ be defined as in Alves (2017)

$$\mathscr{L}(\mathcal{G}_{ic}(\alpha^*)) \triangleq \left\{ x(t) \in \mathbb{R}^{n_x} : |G_{ic(l)}(\alpha^*)x(t)| \leq \overline{u}_l, \;\; i \in \mathbb{K}_N, \;\; c \in \mathbb{K}_{n_r}, \;\; l \in \mathbb{K}_{n_u} \right\}. \tag{155}$$

If $x(t) \in \mathcal{G}_{ic}, \forall i \in \mathbb{K}_N$ and $\forall c \in \mathbb{K}_r$, then $x(t) \in \mathcal{G}_{\sigma\eta}$.

Since the same Lyapunov function is used along the work, the same contractively invariant set $\mathscr{E}(P(\alpha^*), \delta)$ is given as an estimation of the domain of attraction. Consider that the condition $\mathscr{E}(P(\alpha^*), \delta) \subset \mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$ holds (to be demonstrated in Section 6.3), then the saturated switched control law $\text{sat}(u(t)) = \text{sat}(K_{\sigma\eta} y(t))$ can be represented as

$$\begin{aligned}\text{sat}(u(t)) = \text{sat}(K_{\sigma\eta} y(t)) &= \sum_{s=1}^{2^{n_u}} \vartheta_s [\mathscr{D}_s(K_{\sigma\eta} y(t)) + \mathscr{D}_s^- \mathcal{G}_{\sigma\eta}(\alpha^*) x(t)] \\ &= \mathscr{D}(\vartheta)\big(K_{\sigma\eta} C(\alpha) x(t) + K_{\sigma\eta} H(\alpha) w(t)\big) + \mathscr{D}^-(\vartheta) \mathcal{G}_{\sigma\eta}(\alpha^*) x(t). \end{aligned} \tag{156}$$

Considering the representation above and the notation in (101), replacing (156) in (153) one obtains the following closed-loop system,

$$\begin{cases} \dot{x}(t) &= \widetilde{A}(\sigma, \alpha, \vartheta) x(t) + \widetilde{E}(\sigma, \alpha, \vartheta) w(t) \\ z(t) &= \widetilde{C_1}(\sigma, \alpha, \vartheta) x(t) + \widetilde{F}(\sigma, \alpha, \vartheta) w(t) \\ y(t) &= C_2(\alpha) x(t) + H(\alpha) w(t) \end{cases} \tag{157}$$

where,

$$\begin{aligned} \widetilde{A}(\sigma, \alpha, \vartheta) &= A(\sigma, \alpha) + B(\sigma, \alpha)[\mathscr{D}(\vartheta) K_{\sigma\eta} C_2(\alpha) + \mathscr{D}^-(\vartheta) \mathcal{G}_{\sigma\eta}(\alpha^*)] \\ \widetilde{E}(\sigma, \alpha, \vartheta) &= E(\sigma, \alpha) + B(\sigma, \alpha) \mathscr{D}(\vartheta) K_{\sigma\eta} H(\alpha) \\ \widetilde{C_1}(\sigma, \alpha, \vartheta) &= C_1(\sigma, \alpha) + D(\sigma, \alpha)[\mathscr{D}(\vartheta) K_{\sigma\eta} C_2(\alpha) + \mathscr{D}^-(\vartheta) \mathcal{G}_{\sigma\eta}(\alpha^*)] \\ \widetilde{F}(\sigma, \alpha, \vartheta) &= F(\sigma, \alpha) + D(\sigma, \alpha) \mathscr{D}(\vartheta) K_{\sigma\eta} H(\alpha). \end{aligned} \tag{158}$$

6.3 SET CONSTRAINT

Taking into account the Section (5.5) and the set $\mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$ previously defined in this section, the condition $\mathscr{E}(P(\alpha^*), \delta) \subset \mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$ ensures that the trajectories will remain in $\mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$. Therefore, the following lemma is announced.

Lemma 6. *Define,* $\mathcal{Z}_{ic}(\alpha) = \sum_{j=1}^{r} \alpha_j \mathcal{Z}_{ijc}$, $\zeta_{ic(l)}(\alpha) = \sum_{j=1}^{r} \alpha_j \zeta_{ijc(l)}$, $j \in \mathbb{K}_r$,

$$\begin{aligned} \mathcal{Z}_{ijc} &= \begin{bmatrix} \zeta'_{ijc(1)} & \zeta'_{ijc(2)} & \dots & \zeta'_{ijc(n_u)} \end{bmatrix}' \in \mathbb{R}^{n_u \times n_x}, \;\; i \in \mathbb{K}_N, c \in \mathbb{K}_r, j \in \mathbb{K}_r \\ \mathcal{Z}_{ic}(\alpha) &= \begin{bmatrix} \zeta_{ic(1)}(\alpha)' & \zeta_{ic(2)}(\alpha)' & \dots & \zeta_{ic(n_u)}(\alpha)' \end{bmatrix}', \;\; i \in \mathbb{K}_N, c \in \mathbb{K}_r, l \in \mathbb{K}_{n_u} \end{aligned} \tag{159}$$

Then, given the sets $\mathscr{E}(P(\alpha^*), \delta)$ *in* (91) *and* $\mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$ *in* (155) *, the constraint*

$\mathscr{E}(P(\alpha^*),\delta) \subset \mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$ is satisfied if the conditions

$$\begin{bmatrix} \bar{u}_l^2\delta^{-1} & \zeta_{ijc(l)} \\ \zeta'_{ijc(l)} & X_j \end{bmatrix} \geq 0 \tag{160}$$

hold, with $P(\alpha^*) = X(\alpha)^{-1}$ and $\mathcal{G}_{ic}(\alpha^*) = \mathcal{Z}_{ic}(\alpha)P(\alpha^*)$, for all $i \in \mathbb{K}_N$, j and $c \in \mathbb{K}_r$ and $l \in \mathbb{K}_{2^{n_u}}$, .

Proof. The proof is similar to the proof of Lemma 3. □

Remark 8. Following Remark 5, we can also compute an estimation for the sets $\mathscr{L}(\mathcal{G}(\alpha^*))$ and $\mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$, given in (102) and (155), respectively. In order to estimate the set $\mathscr{L}(\mathcal{G}(\alpha^*))$, one can consider the intersection of the sets $\mathscr{L}(\mathcal{G}_j),\ j \in \mathbb{K}_r$, i.e. $\bigcap_{j=1}^r \mathscr{L}(\mathcal{G}_j) \subseteq \mathscr{L}(\mathcal{G}(\alpha^*))$. For the set $\mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$, the indexes $i \in \mathbb{K}_N$ and $c \in K_N$ are relative to the switching strategy (173) that changes in each instant of time. It is important to stress that the values of $\alpha_j \in \mathbb{K}_r$ are unknown and observe that the set $\mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$ meets all the combinations of $i \in \mathbb{K}_N$ and $c \in \mathbb{K}_r$.Therefore an estimation for the set $\mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$ involves the intersection of the sets $\mathscr{L}(\mathcal{G}_{icj})$ for all $j \in \mathbb{K}_r$, i. e. $\bigcap_{j=1}^r \mathscr{L}(\mathcal{G}_{icj}) \subseteq \mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$.

6.4 $\mathscr{H}_\infty$ PROBLEM CONSIDERING REGION OF OPERATION AND SWITCHED LINEAR SYSTEMS SUBJECT TO SATURATION

In order to extend the conditions of Theorem 9 for a class of switched systems and also aiming to design static output-feedback $\mathscr{H}_\infty$ switching controllers for linear uncertain systems subject to actuator saturation and energy-bounded disturbance, the following problem is stated.

Problem 5. *Given the energy-bounded disturbance $w(t) \in \mathscr{W}$ in* (127)*, a slack variable $\varphi > 0$, a constant $\varepsilon_0 \geq 0$ as well as a constant $\gamma > 0$, determine gains $K_{ic} \in \mathbb{R}^{n_u \times n_y}$, $i \in \mathbb{K}_N$, $c \in \mathbb{K}_r$, such that the switching strategy*

$$\begin{bmatrix} \sigma & \eta \end{bmatrix} = arg \min_{\substack{i \in \mathbb{K}_N \\ s \in \mathbb{K}_r}} (y' \hat{\mathcal{Q}}_{ic} y) \tag{161}$$

and the control input subject to actuator saturation

$$\text{sat}(u(t)) = \mathscr{D}(\vartheta)\big(K_{\sigma\eta}C(\alpha)x(t) + K_{\sigma\eta}H(\alpha)w(t)\big) + \mathscr{D}^-(\vartheta)\mathcal{G}_{\sigma\eta}(\alpha^*)x(t) \tag{162}$$

satisfy the following statements (OLIVEIRA et al., 2018):

(i) *for $w(t) = 0, t \geq 0$, the uncertain linear system* (157), (158) *and* (5) *is locally asymptotically stable for all $x(0) \in \mathscr{E}\left(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon\right)$ and the ellipsoid $\mathscr{E}\left(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon\right)$ is an invariant subset of the domain of attraction (i.e., if $x(0)$ belongs to this set, then $x(t)$, $t > 0$, will also stay in this set).*

(ii) *for* $w(t) \neq 0$, *if* $x(0) \in \mathscr{E}\left(P(\alpha^*), \varepsilon_0\right)$, *then* $x(t) \in \mathscr{E}\left(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon\right)$, *for all* $t \geq 0$.

(iii) *for* $w(t) \neq 0$, *if* $x(0) = 0$, *then the linear system* (157), (158) *and* (5) *has an* $\mathscr{H}_\infty$ *guaranteed cost* $\gamma = \sqrt{\mu} > 0$, *such that*

$$\int_0^\infty z(t)'z(t)\mathrm{d}t \leq \gamma^2 \int_0^\infty w(t)'w(t)\mathrm{d}t \tag{163}$$

and $x(t) \in \mathscr{E}\left(P(\alpha^*), \varphi^{-1}\varepsilon\right)$, *for all* $t \geq 0$.

In order to deal with the Problem 5, based on Theorems 7 and 9, the Theorem 10 is proposed.

Theorem 10. *Consider the system described in* (157), (158) *and* (5) *subject to actuator saturation, a energy bounded disturbance and an operation region* $\mathcal{X}(\mathcal{N}_h)$ (107), *where* $\overline{u} \in \mathbb{R}^{n_u}$, $\mathcal{N} \in R^{n_h \times n_x}$, $\phi \in \mathbb{K}_{n_h}$, $\varepsilon_0 \geq$, $\varepsilon > 0$ *and* φ *are known. For also known scalar parameters* ρ, β *and* $\gamma > 0$ *suppose that there exist symmetric matrices* $X_j > 0 \in \mathbb{R}^{n_x \times n_x}$, *matrices* $\mathcal{Z}_{ijc} = \begin{bmatrix} \zeta'_{ijc(1)} & \zeta'_{ijc(2)} & \cdots & \zeta'_{ijc(n_u)} \end{bmatrix}' \in \mathbb{R}^{n_u \times n_x}$, $V_{ic} \in \mathbb{R}^{n_u \times n_y}$, $U_{ic} \in \mathbb{R}^{n_y \times n_y}$, $Q_{0j} \in \mathbb{R}^{n_x \times n_x}$, $Q_{1j} \in \mathbb{R}^{n_x \times n_w}$, $Q_{2j} \in \mathbb{R}^{n_w \times n_w}$ *and* $\hat{\mathcal{Q}}_{ic} \in \mathbb{R}^{n_y \times n_y}$ *satisfying,*

$$\Phi_{ijjc} < 0 \tag{164}$$

$$\Phi_{ijkc} + \Phi_{ikjc} < 0, \quad j < k \tag{165}$$

$$\Theta_{\lambda jjj} < 0 \tag{166}$$

$$\Theta_{\lambda jjk} + \Theta_{\lambda jkj} + \Theta_{\lambda kjj} < 0, \quad j \neq k \tag{167}$$

$$\Theta_{\lambda jkq} + \Theta_{\lambda jqk} + \Theta_{\lambda kjq} + \Theta_{\lambda kqj} + \Theta_{\lambda qjk} + \Theta_{\lambda qkj} < 0, \quad j < k, k < q \tag{168}$$

$$\Phi_{ijkc} = \begin{bmatrix} \mathrm{He}(A_{ij}X_k + B_{ij}[\mathscr{D}_s V_{ic}\mathscr{F}_k + \mathscr{D}_s^- \mathcal{Z}_{ikc}]) - Q_{0j} - C'_{2j}\hat{\mathcal{Q}}_{ic}C_{2k} & (*) & (*) & (*) \\ E'_{ij} + H'_k V'_{ic}\mathscr{D}'_s B'_{ij} - Q'_{1j} - H'_j\hat{\mathcal{Q}}_{ic}C_{2k} & -\varphi^{-1}I - Q_{2j} - H'_j\hat{\mathcal{Q}}_{ic}H_k & (*) & (*) \\ C_{1ij}X_k + \rho\mathscr{F}'_0 V'_{ic}\mathscr{D}'_s B'_{ij} + D_{ij}[\mathscr{D}_s V_{ic}\mathscr{F}_k + \mathscr{D}_s^- \mathcal{Z}_{ikc}] & F_{ij} + D_{ij}\mathscr{D}_s V_{ic}H_k & -\varphi\gamma^2 I + He(\rho D_{ij}\mathscr{D}_s V_{ic}\mathscr{F}_0) & (*) \\ C_{2j}X_k - U_{ic}\mathscr{F}_k + \beta V'_{ic}\mathscr{D}'_s B'_{ij} & H_j - U_{ic}H_k & \beta V'_{ic}\mathscr{D}'_s D'_{ij} - \rho U_{ic}\mathscr{F}_0 & -\beta U_{ic} - \beta U'_{ic} \end{bmatrix} < 0 \tag{169}$$

$$\Theta_{\lambda jkq} = \begin{bmatrix} Q_{0j} + C'_{2j}\hat{\mathcal{Q}}_{\lambda k}C_{2q} & Q_{1j} + C'_{2j}\hat{\mathcal{Q}}_{\lambda k}H_q \\ Q'_{1j} + H'_j\hat{\mathcal{Q}}_{\lambda k}C_{2q} & Q_{2j} + H'_j\hat{\mathcal{Q}}_{\lambda k}H_q \end{bmatrix} < 0, \quad \hat{\mathcal{Q}}_\lambda(\alpha) = \sum_{i=1}^{N}\sum_{k=1}^{r} \alpha_k \lambda_i \hat{\mathcal{Q}}_{ik} \tag{170}$$

$$\begin{bmatrix} \frac{\overline{u}_l^2}{(\varepsilon_0 + \varphi^{-1}\varepsilon)} & \zeta_{ijc(l)} \\ \zeta'_{ijc(l)} & X_j \end{bmatrix} \geq 0 \tag{171}$$

$$\begin{bmatrix} \frac{\phi_h^2}{(\varepsilon_0+\varphi^{-1}\varepsilon)} & \mathcal{N}_h X_j \\ X_j \mathcal{N}_h' & X_j \end{bmatrix} \geq 0 \tag{172}$$

with $\mathscr{F}_j$ (54) *and* $\mathscr{F}_0$ (55), *for all* c, j *and* $k \in \mathbb{K}_r$, $l \in \mathbb{K}_{n_u}$, $h \in \mathbb{K}_{n_L}$, $s \in \mathbb{K}_{2_{n_u}}$, $D \in \mathscr{D}$, $\mathscr{D}_s^- = I - \mathscr{D}_s$. *Then, the conditions* (169)-(172) *with the switching strategy*

$$\begin{bmatrix} \sigma(t) & \eta(t) \end{bmatrix} = arg \min_{\substack{i\in\mathbb{K}_N \\ c\in\mathbb{K}_r}} (y(t)'\hat{\mathcal{Q}}_{ic} y(t)) \tag{173}$$

and control gains $K_{is} = V_{is}U_{is}^{-1}$ *with* $X(\alpha)^{-1} = P(\alpha^*)$ *satisfy the statements of Problem 5. Moreover,* $x(t) \in \mathscr{X}$ *(operation region) for all* $t \geq 0$ *and the constraints* (171) *and* (172) *hold the sets constraints* $\mathscr{E}(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$ *and* $\mathscr{E}(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \mathcal{X}(\mathcal{N}_h)$.

Proof. Consider the Lyapunov function candidate (90). Firstly, from Lemma 4 and Lemma 6, note that (172) and (171) ensure $\mathscr{E}(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \mathcal{X}(\mathcal{N}_h)$ and $\mathscr{E}(P(\alpha^*), \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$, respectively. From (5), (164), (165), and (169), considering $i = \sigma$ and $c = \eta$ one has

$$\sum_{j=1}^{r}\sum_{k=1}^{r} \alpha_j\alpha_k \Phi_{\sigma jk\eta} = \sum_{j=1}^{r} \alpha_j^2 \Phi_{\sigma jj\eta} + \sum_{j=1}^{r}\sum_{j<k}^{r} \alpha_j\alpha_k \left(\Phi_{\sigma jk\eta} + \Phi_{\sigma kj\eta}\right) =$$

$$\begin{bmatrix} \mathrm{He}(A(\sigma,\alpha)X(\alpha) + B(\sigma,\alpha)[\mathscr{D}_s V_{\sigma\eta}\mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}_{\sigma\eta}(\alpha)]) - Q_0(\alpha) - C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & (*) & (*) & (*) \\ E(\sigma,\alpha)' + H(\alpha)'V_{\sigma\eta}'\mathscr{D}_s' B(\sigma,\alpha)' - Q_1(\alpha)' - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) & -\varphi^{-1}I - Q_2(\alpha) - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) & (*) & (*) \\ C_1(\sigma,\alpha)X(\alpha) + \rho\mathscr{F}_0' V_{\sigma\eta}'\mathscr{D}_s' B(\sigma,\alpha)' + D(\sigma,\alpha)[\mathscr{D}_s V_{\sigma\eta}\mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}_{\sigma\eta}(\alpha)] & F(\sigma,\alpha) + D(\sigma,\alpha)\mathscr{D}_s V_{\sigma\eta} H(\alpha) & -\varphi\gamma^2 I + \mathrm{He}(\rho D(\sigma,\alpha)\mathscr{D}_s V_{\sigma\eta}\mathscr{F}_0) & (*) \\ C_2(\alpha)X(\alpha) - U_{\sigma\eta}\mathscr{F}(\alpha) + \beta V_{\sigma\eta}'\mathscr{D}_s' B(\sigma,\alpha)' & H(\alpha) - U_{\sigma\eta}H(\alpha) & \beta V_{\sigma\eta}'\mathscr{D}_s' D(\sigma,\alpha)' - \rho U_{\sigma\eta}\mathscr{F}_0 & -\beta U_{\sigma\eta} - \beta U_{\sigma\eta}' \end{bmatrix} < 0 \tag{174}$$

Since the inequalities (164) and (165) are feasible, from (169) we have that U_{ic} are non-singular, for all $i \in \mathbb{K}_N$ and $c \in \mathbb{K}_r$. From Lemma 1, with

$$\mathscr{T} = \begin{bmatrix} \mathrm{He}(A(\sigma,\alpha)X(\alpha) + B(\sigma,\alpha)[\mathscr{D}_s V_{\sigma\eta}\mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}_{\sigma\eta}(\alpha)]) - Q_0(\alpha) - C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) \\ E(\sigma,\alpha)' + H(\alpha)'V_{\sigma\eta}'\mathscr{D}_s' B(\sigma,\alpha)' - Q_1(\alpha)' - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha) \\ C_1(\sigma,\alpha)X(\alpha) + \rho\mathscr{F}_0' V_{\sigma\eta}'\mathscr{D}_s' B(\sigma,\alpha)' + D(\sigma,\alpha)[\mathscr{D}_s V_{\sigma\eta}\mathscr{F}(\alpha) + \mathscr{D}_s^- \mathcal{Z}_{\sigma\eta}(\alpha)] \end{bmatrix}$$

$$\begin{array}{cc}(*) & (*)\\ -\varphi^{-1}I - Q_2(\alpha) - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) & (*)\\ F(\sigma,\alpha)+D(\sigma,\alpha)\mathscr{D}_sV_{\sigma\eta}H(\alpha) & -\varphi\gamma^2I + \mathrm{He}(\rho D(\sigma,\alpha)\mathscr{D}_sV_{\sigma\eta}\mathscr{F}_0)\end{array}\Bigg] < 0,$$

$$\mathscr{U} = U, \mathscr{P} = \begin{bmatrix} B(\sigma,\alpha)\mathscr{D}_sV_{\sigma\eta}\\ 0\\ D(\sigma,\alpha)\mathscr{D}_sV_{\sigma\eta}\end{bmatrix},$$

$$\mathscr{A} = U_{\sigma\eta}^{-1}\Big[C_2(\alpha)X(\alpha) - U_{\sigma\eta}\mathscr{F}(\alpha) \quad H(\alpha) - U_{\sigma\eta}H(\alpha) \quad -\rho U_{\sigma\eta}\mathscr{F}_0\Big], \tag{175}$$

and performing the operations and the variables changing $K_{\sigma\eta} = V_{\sigma\eta}U_{\sigma\eta}^{-1}$, one has

$$\left[\begin{array}{l}\mathrm{He}(A(\sigma,\alpha)X(\alpha)+B(\sigma,\alpha)[\mathscr{D}_sK_{\sigma\eta}C_2(\alpha)X(\alpha)+\mathscr{D}_s^-\mathcal{Z}_{\sigma\eta}(\alpha)]) - Q_0(\alpha) - C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha)\\ E(\sigma,\alpha)'+H(\alpha)'K_{\sigma\eta}'\mathscr{D}_s'B(\sigma,\alpha)' - Q_1(\alpha)' - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha)\\ C_1(\sigma,\alpha)X(\alpha)+D(\sigma,\alpha)[\mathscr{D}_sK_{\sigma\eta}C_2(\alpha)X(\alpha)+\mathscr{D}_s^-\mathcal{Z}_{\sigma\eta}(\alpha)]\end{array}\right.$$
$$\left.\begin{array}{cc}(*) & (*)\\ -\varphi^{-1}I - Q_2(\alpha) - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) & (*)\\ F(\sigma,\alpha)+D(\sigma,\alpha)\mathscr{D}_sK_{\sigma\eta}H(\alpha) & -\varphi\gamma^2I\end{array}\right] < 0. \tag{176}$$

Considering the variable changing $\mathcal{G}_{\sigma\eta}(\alpha^*) = \mathcal{Z}_{\sigma\eta}(\alpha)P(\alpha^*)$, with $P(\alpha^*) = X(\alpha)^{-1}$ that yields $\mathcal{Z}_{\sigma\eta}(\alpha) = \mathcal{G}_{\sigma\eta}(\alpha^*)X(\alpha)$ and rewriting (176), it follows that

$$\left[\begin{array}{l}\mathrm{He}\big((A(\sigma,\alpha)+B(\sigma,\alpha)[\mathscr{D}_sK_{\sigma\eta}C_2(\alpha)+\mathscr{D}_s^-\mathcal{G}_{\sigma\eta}(\alpha^*)])X(\alpha)\big) - Q_0(\alpha) - C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha)\\ E(\sigma,\alpha)'+H(\alpha)'K_{\sigma\eta}'\mathscr{D}_s'B(\sigma,\alpha)' - Q_1(\alpha)' - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha)\\ (C_1(\sigma,\alpha)+D(\sigma,\alpha)[\mathscr{D}_sK_{\sigma\eta}C_2(\alpha)+\mathscr{D}_s^-\mathcal{G}_{\sigma\eta}(\alpha)])X(\alpha)\end{array}\right.$$
$$\left.\begin{array}{cc}(*) & (*)\\ -\varphi^{-1}I - Q_2(\alpha) - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) & (*)\\ F(\sigma,\alpha)+D(\sigma,\alpha)\mathscr{D}_sK_{\sigma\eta}H(\alpha) & -\varphi\gamma^2I\end{array}\right] < 0. \tag{177}$$

Multiplying the result by ϑ_s, $\vartheta_s \geq 0$, $\sum_{s=1}^{2^{nu}}\vartheta_s = 1$ and following the closed-loop notation presented in (157) and (158) with $K_{\sigma\eta}$ it is possible to rewrite (177) as

$$\left[\begin{array}{l}\mathrm{He}(\widetilde{A}(\sigma,\alpha,\vartheta)X(\alpha)) - Q_0(\alpha) - C_2(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha)\\ \widetilde{E}(\sigma,\alpha,\vartheta)' - Q_1(\alpha)' - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}C_2(\alpha)\\ \widetilde{C}_1(\sigma,\alpha,\vartheta)X(\alpha)\end{array}\right.$$
$$\left.\begin{array}{cc}(*) & (*)\\ -\varphi^{-1}I - Q_2(\alpha) - H(\alpha)'\hat{\mathcal{Q}}_{\sigma\eta}H(\alpha) & (*)\\ \widetilde{F}(\sigma,\alpha,\vartheta) & -\varphi\gamma^2I\end{array}\right] < 0. \tag{178}$$

Applying the Schur complement in (178) and manipulating the result, one has

$$
\begin{bmatrix}
\begin{pmatrix} \mathrm{He}(\widetilde{A}(\sigma,\alpha,\vartheta)X(\alpha)) \\ +\varphi^{-1}\gamma^{-2}X(\alpha)\widetilde{C}_1(\sigma,\alpha,\vartheta)'\widetilde{C}_1(\sigma,\alpha,\vartheta)X(\alpha) \end{pmatrix} & (*) \\
\begin{pmatrix} \widetilde{E}(\sigma,\alpha,\vartheta)' \\ +\varphi^{-1}\gamma^{-2}\widetilde{F}(\sigma,\alpha,\vartheta)'\widetilde{C}_1(\sigma,\alpha,\vartheta)X(\alpha) \end{pmatrix} & \varphi^{-1}\gamma^{-2}\widetilde{F}(\sigma,\alpha,\vartheta)'\widetilde{F}(\sigma,\alpha,\vartheta)-\varphi^{-1}I
\end{bmatrix}
$$
$$
< \begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} + \begin{bmatrix} C_2(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_2(\alpha) & (*) \\ H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_2(\alpha) & H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}H(\alpha) \end{bmatrix}. \tag{179}
$$

Pre and post-multiplying (179) in both sides by $\begin{bmatrix} x(t)' & w(t)' \end{bmatrix}$ and its transpose, for the right side of the equation it is possible to note that

$$
\begin{aligned}
&\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \left(\begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} + \begin{bmatrix} C_2(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_2(\alpha) & (*) \\ H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}C_2(\alpha) & H(\alpha)'\hat{\mathscr{Q}}_{\sigma\eta}H(\alpha) \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\
&= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \left(\begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} + \begin{bmatrix} C_2(\alpha)' \\ H(\alpha)' \end{bmatrix} \hat{\mathscr{Q}}_{\sigma\eta} \begin{bmatrix} C_2(\alpha) & H(\alpha) \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \\
&= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + y(t)'\hat{\mathscr{Q}}_{\sigma\eta}y(t).
\end{aligned} \tag{180}
$$

From (179) and (180), and considering $x(t) \neq 0$, observe that

$$
\begin{aligned}
&\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} \mathrm{He}(\widetilde{A}(\sigma,\alpha,\vartheta)X(\alpha)) + \varphi^{-1}\gamma^{-2}X(\alpha)\widetilde{C}_1(\sigma,\alpha,\vartheta)'\widetilde{C}_1(\sigma,\alpha,\vartheta)X(\alpha) & (*) \\ \widetilde{E}(\sigma,\alpha,\vartheta)' + \varphi^{-1}\gamma^{-2}\widetilde{F}(\sigma,\alpha,\vartheta)'\widetilde{C}_1(\sigma,\alpha,\vartheta)X(\alpha) & \varphi^{-1}\gamma^{-2}\widetilde{F}(\sigma,\alpha,\vartheta)'\widetilde{F}(\sigma,\alpha,\vartheta) - \varphi^{-1}I \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\
&\qquad < \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + y(t)'\hat{\mathscr{Q}}_{\sigma\eta}y(t).
\end{aligned} \tag{181}
$$

Now, from (166)-(168) and (170)

$$
\begin{aligned}
\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{q=1}^{r} \alpha_j\alpha_k\alpha_q\Theta_{\lambda jkq} &= \sum_{j=1}^{r} \alpha_j^3\Theta_{\lambda jjj} + \sum_{j=1}^{r}\sum_{j\neq k}^{r} \alpha_j^2\alpha_k \left(\Theta_{\lambda jjk} + \Theta_{\lambda jkj} + \Theta_{\lambda jjk}\right) \\
&= + \sum_{j=1}^{r}\sum_{j<k}^{r}\sum_{k<q}^{r} \alpha_j\alpha_k\alpha_q \left(\Theta_{\lambda jkq} + \Theta_{\lambda jqk} + \Theta_{\lambda kjq} + \Theta_{\lambda kqj} + \Theta_{\lambda qjk} + \Theta_{\lambda qkj}\right) \\
&= \begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} + \begin{bmatrix} C_2(\alpha)'\hat{\mathscr{Q}}_\lambda(\alpha)H(\alpha) & C_2(\alpha)'\hat{\mathscr{Q}}_\lambda(\alpha)H(\alpha) \\ H(\alpha)'\hat{\mathscr{Q}}_\lambda(\alpha)C_2(\alpha) & H(\alpha)'\hat{\mathscr{Q}}_\lambda(\alpha)H(\alpha) \end{bmatrix} < 0.
\end{aligned} \tag{182}
$$

Pre and post-multiplying (182) in both sides by $\begin{bmatrix} x(t)' & w(t)' \end{bmatrix}$ and its transpose

$$
\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \left(\begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} + \begin{bmatrix} C_2(\alpha)'\hat{\mathscr{Q}}_\lambda(\alpha)H(\alpha) & C_2(\alpha)'\hat{\mathscr{Q}}_\lambda(\alpha)H(\alpha) \\ H(\alpha)'\hat{\mathscr{Q}}_\lambda(\alpha)C_2(\alpha) & H(\alpha)'\hat{\mathscr{Q}}_\lambda(\alpha)H(\alpha) \end{bmatrix} \right) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}
$$

$$= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}' \begin{bmatrix} Q_0(\alpha) & Q_1(\alpha) \\ Q_1(\alpha)' & Q_2(\alpha) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + y(t)'\hat{\mathscr{Q}}_\lambda(\alpha)y(t). \tag{183}$$

Now, knowing that the minimum of a set of real numbers is less than or equal to an arbitrary convex combination of these numbers ($\hat{\mathscr{Q}}_{\sigma\eta} \leq \hat{\mathscr{Q}}_\lambda(\alpha)$) it is possible to rewrite (181), considering the definition of $\hat{\mathscr{Q}}_\lambda(\alpha)$ in (170), and the equations (180), (182) and (183) one has

$$\begin{bmatrix} \mathrm{He}(\widetilde{A}(\sigma,\alpha,\vartheta)X(\alpha)) + \varphi^{-1}\gamma^{-2}X(\alpha)\widetilde{C}_1(\sigma,\alpha,\vartheta)'\widetilde{C}_1(\sigma,\alpha,\vartheta)X(\alpha) & (*) \\ \widetilde{E}(\sigma,\alpha,\vartheta)' + \varphi^{-1}\gamma^{-2}\widetilde{F}(\sigma,\alpha,\vartheta)'\widetilde{C}_1(\sigma,\alpha,\vartheta)X(\alpha) & \varphi^{-1}\gamma^{-2}\widetilde{F}(\sigma,\alpha,\vartheta)'\widetilde{F}(\sigma,\alpha,\vartheta) - \varphi^{-1}I \end{bmatrix} < 0. \tag{184}$$

From now, the proof is analogous to that presented in the Theorem 9. Observe that pre- and post-multiplying (184) by diag$\{P(\alpha^*), I\}$ in both sides, considering $P(\alpha^*) = X(\alpha)^{-1}$, one obtains an equation similar to (142), but now, instead of matrices $\widetilde{A}(\alpha,\vartheta), \widetilde{E}(\alpha,\vartheta), \widetilde{C_1}(\alpha,\vartheta)$, and $\widetilde{F}(\alpha,\vartheta)$, one considers the new matrices $\widetilde{A}(\sigma,\alpha,\vartheta), \widetilde{E}(\sigma,\alpha,\vartheta), \widetilde{C_1}(\sigma,\alpha,\vartheta)$, and $\widetilde{F}(\sigma,\alpha,\vartheta)$ of the controlled system, given in (157) and (158) . Therefore, the proof follows the same steps of the Theorem 9 to meet the statements of the Problem 5. □

Corollary 2. *Theorem 10 also suits for designing switched output feedback $\mathscr{H}_\infty$ controllers for continuous-time uncertain linear systems subject to actuator saturation. These systems can be considered a particular case of the switched systems with $N = 1$.*

Remark 9. Besides searching suboptimal values for β, ρ and φ as stated in Remark 7 in case of switched systems subject to actuator saturation the values of $\lambda_i \in \mathbb{K}_N$ are needed (Remark 3). With the purpose of achieve a reduced $\mathscr{H}_\infty$ bound and a better estimation of the region of attraction, finding the values of β, ρ, φ and λ_i, $i \in \mathbb{K}_N$, the DE-LMI algorithm is introduced in Chapter 7.

6.5 CHAPTER CONCLUSION

This chapter was focused in presenting conditions to design robust $\mathscr{H}_\infty$ switched SOF controllers for switched linear system subject to actuator saturation. Firstly, the convex hull representation presented in Chapter 5 was extended to cope with switched controllers design. Additionally, the sets inclusion to ensure that the trajectories will not deviate from the convex hull set for the saturation, considering switched controllers, was introduced. Moreover, the problem of dealing with the $\mathscr{H}_\infty$ control for switched systems subject to actuator saturation and switched controllers was introduced. In order to deal with this problem and to extend the conditions of Chapter 5 for a class of switched system, the Theorem 10 was proposed. It is important to stress that the proposed conditions also suit for designing switched SOF $\mathscr{H}_\infty$ controllers for continuous-time uncertain linear systems subject to actuator saturation, as stated in Corollary 2.

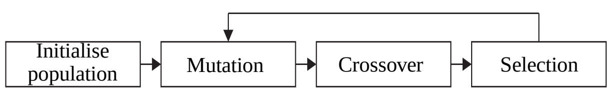
7 HYBRID DIFFERENTIAL EVOLUTION-LINEAR MATRIX INEQUALITY-BASED ALGORITHM

Note that as mentioned in Remarks 1 and 3, terms $\hat{\mathcal{Q}}_\lambda$ lead to BMIs conditions in Theorems 2, 3, 4, 7, 10. As well as stated in Remark 2, the parameters ρ and β in Theorems 6 and 7 can be found through numerical optimisation in order to reduce the $\mathscr{H}_\infty$ norm. Furthermore, as stated in Remark 7 and 9, it is possible to find the parameters β, ρ and φ in Theorems 9 and 10 aiming achieve a reduced $\mathscr{H}_\infty$ bound and a better estimation of the region of attraction. Currently, to the best of the author's knowledge, there are not available solvers (deterministic methods) in literature able to find the optimum solution for non-convex problems (SADABADI; PEAUCELLE, 2016). Thus, finding the output gains in order to stabilise an uncertain switched linear system is a NP-hard problem (LIN; ANTSAKLIS, 2009; KOUMBOULIS; TZAMTZI, 2007). Therefore, it is proposed the use of an hybrid metaheuristic technique, DE-LMI (Differential Evolution - Linear Matrix Inequality) (STORN; PRICE, 1997) addressed in Carniato (2016) for finding quasi-optimum values for the parameters ρ, β, φ and λ_i.

7.1 DIFFERENTIAL EVOLUTION

Global optimisation is considered effective in different fields of engineering, statistics, and finances models. Consequently, there are different techniques proposed in the literature to solve these problems. Differential evolution (DE) is a stochastic method based on population optimisation algorithm introduced by (STORN; PRICE, 1997). This method belongs to the class of Evolutionary Algorithms (EA), which also includes, Genetic Algorithms (GA), Evolutionary Strategies (ES) and Evolutionary Programming (EP) (PRICE; STORN; LAMPINEN, 2006). DE figured as one of the best among the competing algorithms presented at Second International Contest on Evolutionary Optimisation and over the years has attracted the attention of researchers from diverse fields of knowledge. Hence, plenty of variants of the basic DE algorithm has emerged (DAS; SUGANTHAN, 2011). DE is a real parameter algorithm and can be summarised in four stages, such as GA: initialisation of the population/parameters, mutation, crossover, and selection. Figure 8 depicts these stages.

Figure 8 - Main stages of the differential evolution algorithm.



Source: Adapted from (DAS; SUGANTHAN, 2011).

Suppose that it is needed to optimise a function with n_p real parameters. Let S_p stands for the size of population. The notation for representing the parameter vector that will be updated throughout the G generations, was adopted as following (185):

$$\left\{\vec{x}_{i,G} = \begin{bmatrix} x_{1,i,G} & x_{2,i,G} & \ldots & x_{n_p,i,G} \end{bmatrix} \mid i = 0,\ 1,\ 2\ \ldots\ S_p - 2,\ S_p - 1\right\}. \tag{185}$$

To initialise the population $(G = 0)$, in order to cover the suitable range with uniformly distributed random individuals, one has to consider the previous knowledge about the problem. For instance, if the parameters of the convex combination compose the population, knowing that $\lambda_i \leq 1$, $\lambda_i \geq 0$ and $\sum_{i=1}^{N} \lambda_i = 1$, the initial population should cover the range between 0 and 1, considering the sum of the individuals equal to 1, with uniformly distributed random values.

The mutation stage involves adding the weighted difference between two random individuals ($\vec{x}_{\beta,G}$ and $\vec{x}_{\gamma,G}$) of the population to a third one ($\vec{x}_{\kappa,G}$), as shown in (186).

$$\vec{v} = \vec{x}_{\kappa,G} + F(\vec{x}_{\beta,G} - \vec{x}_{\gamma,G}), \tag{186}$$

where the real and constant $F > 0$ factor controls the amplification of the differential variation and for each *target vector* $\vec{x}_{i,G}$ a *mutant vector* is generated (STORN; PRICE, 1997). The vector $\vec{x}_{\kappa,G}$ may be replaced by the best individual in the population at generation G ($\vec{x}_{best,G}$). In order to do that, when initialising the population the fitness function must be evaluated.

Aiming to increase the diversity, the crossover between the *mutant vector* and the *target vector* is performed to yield the *trial vector* ($\vec{u}$). The last stage (selection) involves to decide whether or not the *trial vector* should replace the *target vector*. The *trial vector* replaces the *target vector* if it yields a smaller fitness function. Otherwise, the *target vector* is held.

7.2 DIFFERENTIAL EVOLUTION-LINEAR MATRIX INEQUALITY-BASED ALGORITHM EMPLOYED TO THE PROPOSED PROBLEM

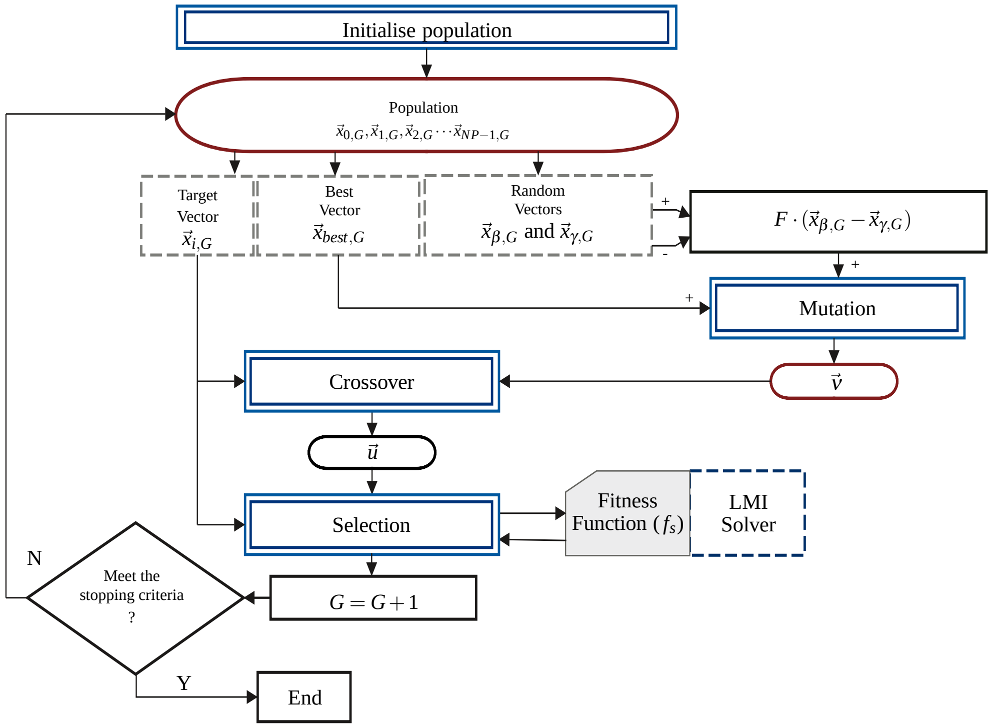
As aforementioned the proposed DE-LMI based algorithm applies the SeDuMi algorithm (STURM, 1999) or Matlab LMI toolbox (GAHINET *et al.*, 1994) for solving the LMI problems in order to provide the parameters to evaluate each individual of the population considered in the DE. Considering the DE traditional algorithm described in Section 7.1 and to illustrate the integration of DE and LMI solvers, Figure 9 depicts the algorithm.

The main goal in this work is to find the convex combination parameters (λ_i), the scalars variables β, ρ and φ when required. It is possible to build the vectors of the population with the convex combination parameters and those aforementioned scalars variables. For instance, considering Theorem 10, if $N = 2$ one has four parameters to evaluate: β, ρ, φ and λ_1. Thus, the parameter vector may assume, with $n_p = 4$, the following representation: $\vec{x} = [\ \beta\ \ \rho\ \ \varphi\ \ \lambda_1\ ]$.

The parameter λ_2 can be obtained doing $\lambda_2 = 1 - \lambda_1$, since $\sum_{i=1}^{2} \lambda_i = \lambda_1 + \lambda_2 = 1$.

The aim is to minimise the guaranteed cost for Theorems 3 and 4, minimise the $\mathscr{H}_\infty$ bound for Theorems 6 and 7, and minimise the $\mathscr{H}_\infty$ bound and maximise the estimation of the region of attraction for Theorems 9 and 10. For this purpose it is possible to apply the DE-LMI algorithm to deal with the problem, achieving quasi-optimum values for λ_i, β, ρ and φ. Note that, when the aim is just optimise the $\mathscr{H}_\infty$ bound, one can considers the fitness function being $f_s = \gamma$. On the other hand, if the actuator saturation is taken into account, in order to minimise the $\mathscr{H}_\infty$ and maximise the estimation of region of attraction the fitness function is assigned as $f_s = \gamma - \varpi$ (Remark 6). The number of generations is used as the stop condition.

Figure 9 - DE-LMI routine.



Source: Own author.

8 NUMERICAL AND PRACTICAL EXAMPLES

In this chapter, six examples are used to compare the potentiality of the theorems proposed in this work and complement the results presented in Chapter 3. The first three examples are related to the results presented in Chapter 4. The last three examples cover the Theorems proposed in Chapters 5 and 6 that are concerned with systems subject to actuator saturation.

Example II illustrates the case where all matrices of the subsystems are not Hurwitz. Comparisons regarding the guaranteed cost were performed considering a given set of λ_1, λ_2 and λ_3 and the suboptimal λ_1, λ_2 and λ_3 obtained using the proposed hybrid algorithm DE-LMI. Example II, consider that $u(t) = 0$ and $w(t) = 0$, and thus Theorems 3 and 4 are compared.

With the aim to compare the proposed technique with the results presented in (CHANG; PARK; ZHOU, 2015) for continuous-time systems, the condition from Corollary 1 is used to cover the Example III.

For the Example IV the conditions of Theorem 7 are used in order to find a feasible solution for a practical application of the method in the design of a switched robust controller for a semi-active suspension described in Cardim *et al.* (2016) and Geromel, Colaneri and Bolzern (2008) aiming to minimise the $\mathscr{H}_\infty$ cost (58).

Example V aims to draw a comparison between the Theorem 9 and Theorem 10 (Corollary 2) which proposes the strategy of switched controllers. The system of Example III is revisited considering the actuator saturation constraint.

An implementation in an active suspension system, intending to show a practical application of the switched controllers for systems subject to actuator saturation (Theorem 10 - Corollary 2) is presented in Example VI.

In order to design switched controllers for switched linear systems subject to actuator Example VII is proposed. The conditions of Theorem 10 allows designing this controller jointly with a switching strategy by means of auxiliary matrices.

Concerning the LMI solver, it was used the SeDuMi (STURM, 1999) and Matlab LMI toolbox (GAHINET *et al.*, 1994) interfaced by YALMIP in MATLAB software.

8.1 EXAMPLE II - SWITCHED LINEAR SYSTEM - GUARANTEED COST

Consider the uncertain switched linear systems represented by (4) and (5), with $r = 2$, $N = 3$, $i \in \{1,2,3\}$, $j \in \{1,2\}$ and the following matrices given below:

$$A_1(a) = \begin{bmatrix} a & 1 & 0 \\ 1 & -3 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad A_2(b) = \begin{bmatrix} -1 & -2 & 0 \\ -2 & b & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

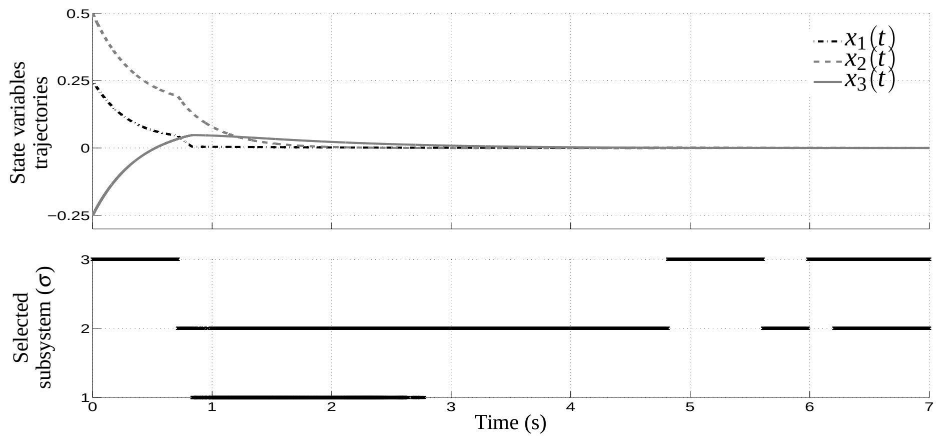
$$A_3(c) = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & c \end{bmatrix}, \quad C_{21} = C_{22} = C_{1ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{187}$$

Since there is neither control input ($u(t) = 0$) nor exogenous disturbance ($w(t) = 0$) the matrices B_{ij}, E_{ij}, D_{ij}, F_{ij} and H_j are not represented. Note that, if $a < -\frac{1}{3}$, $b < -4$, $c < -2$, the matrices $A_1(a)$, $A_2(b)$ and $A_3(c)$ are Hurwitz, otherwise, these matrices are not Hurwitz. The vertices of the polytope were obtained considering $0 \leq a \leq 1$, $-3 \leq b \leq -1.5$, $-1.5 \leq c \leq -1$. Initially, adopt $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, $\lambda_3 = 0.3$, $x_0 = x(0) = [0.25\; 0.5\; -0.25]'$ and $u(t) = 0$. From conditions of both Theorems 3 and 4, the obtained guaranteed cost (14) was 6.022. The next subsection is devoted to apply the concepts of DE-LMI algorithm in order to reduce the guaranteed cost.

8.1.1 Finding the suboptimal parameters of convex combination

Note that, the conditions (18), (23) and (24) are BMIs, contain terms as the product of a scalar by a matrix. However, as discussed earlier, the proposed hybrid metaheuristic DE-LMI algorithm is able to find feasible solutions in order to reduce the guaranteed cost (14), obtaining values of convex combination parameters that yield suboptimal guaranteed cost value. In this case, the DE-LMI algorithm searches the values of λ_i, $i \in \mathbb{K}_N$, such that, $\sum_{i=1}^{N} \lambda_i = 1$ and $\hat{\mathcal{Q}}_\lambda = \lambda_1 \hat{\mathcal{Q}}_1 + \lambda_2 \hat{\mathcal{Q}}_2 + \ldots + \lambda_N \hat{\mathcal{Q}}_N$. From the conditions of Theorem 4, the DE-LMI algorithm stopped in the $17th$ generation due the relative tolerance stop criterion, yielding a value of 2.087 for the guaranteed cost with the solution $\lambda_1 = 0.2449$, $\lambda_2 = 0.3623$, and $\lambda_3 = 0.3928$. The result shows that the guaranteed cost was considerably reduced when compared with the value obtained disregarding optimisation. Figure 10 presents the trajectories of the state variables and the selected subsystem over time, obtained through of conditions proposed in Theorem 4. It is important to highlight that the switching strategy (9) stabilises the system described in (187), even when the state matrices are not Hurwitz. Furthermore, note that when $t > 4s$, the time responses of the state variables are close to zero. However, observe that the switching strategy continues selecting the best available subsystem considering (9).

Figure 10 - Time response of the state variables and switching selection of the controlled system (4), (5), (9) and (187).



Source: Author's own results.

8.2 EXAMPLE III - SWITCHED $\mathscr{H}_\infty$ CONTROL FOR LINEAR SYSTEMS

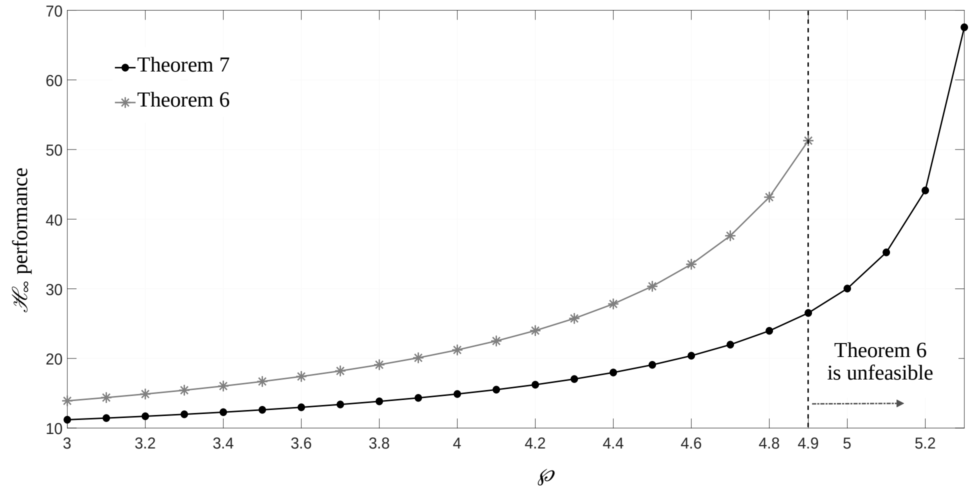
This numerical example was introduced in (CHANG; PARK; ZHOU, 2015). It is represented by (4) and (5), with $r = 2$, $N = 1$ (Corollary 1) and the following matrices:

$$
\begin{aligned}
&A_1 = \begin{bmatrix} -0.9896 & 17.41 & 96.15 \\ 0.2648 & -0.8512 & -11.39 \\ 0 & 0 & -30+\wp \end{bmatrix},\ A_2 = \begin{bmatrix} -1.702 & 50.72 & 263.5 \\ 0.2201 & -1.418 & -31.99 \\ 0 & 0 & -30+\wp \end{bmatrix},\\
&B_1 = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix},\ B_1 = \begin{bmatrix} -85.09 \\ 0 \\ 30 \end{bmatrix},\ E_1 = E_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ C_{11} = C_{12} = I_{3\times 3},\\
&D_1 = D_2 = 0,\ F_1 = F_2 = 0,\ C_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ C_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},\\
&H_1 = H_2 = 0.
\end{aligned}
\tag{188}
$$

The parameter $\wp$ was added in the matrices A_1 and A_2 to draw a comparison between Theorem 6 (CHANG; PARK; ZHOU, 2015) and Theorem 7 for the $\mathscr{H}_\infty$ performance, considering $\wp$ increments.

Figure 11 shows that Theorem 7 achieves better $\mathscr{H}_\infty$ performance when compared with Theorem 6. Furthermore, it is important to highlight that for $\wp \geq 4.9$ the conditions of Theorem 6 are unfeasible.

Figure 11 - $\mathscr{H}_\infty$ cost comparison between Theorem 6 and Theorem 7 for $\wp$ increments.



Source: Author's own results.

8.3 EXAMPLE IV - PRACTICAL APPLICATION: SEMI-ACTIVE SWITCHED SUSPENSION

In order to evaluate the proposed technique presented in the Theorem 7 in a practical application, consider the semi-active switched suspension addressed in Geromel, Colaneri and Bolzern (2008) and in Cardim *et al.* (2016). The problem consists in designing an output feedback controller that jointly with the switching strategy mitigates the passenger's discomfort. Figure 12 depicts the system. The mathematical model can be seen as the system represented in (4) and (5) with the following matrices:

$$A_1(\alpha) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k(\alpha)}{M(\alpha)} & \frac{-c_{min}}{M(\alpha)} & \frac{k(\alpha)}{M(\alpha)} & \frac{c_{min}}{M(\alpha)} \\ 0 & 0 & 0 & 1 \\ \frac{k(\alpha)}{m} & \frac{c_{min}}{m} & \frac{-(k(\alpha)+k_t)}{m} & \frac{-c_{min}}{m} \end{bmatrix},\ B_1(\alpha) = B_2(\alpha) = \begin{bmatrix} 0 \\ \frac{1}{M(\alpha)} \\ 0 \\ \frac{-1}{m} \end{bmatrix},$$

$$A_2(\alpha) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k(\alpha)}{M(\alpha)} & \frac{-c_{max}}{M(\alpha)} & \frac{k(\alpha)}{M(\alpha)} & \frac{c_{max}}{M(\alpha)} \\ 0 & 0 & 0 & 1 \\ \frac{k(\alpha)}{m} & \frac{c_{max}}{m} & \frac{-(k(\alpha)+k_t)}{m} & \frac{-c_{max}}{m} \end{bmatrix},\ E_1(\alpha) = E_2(\alpha) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_t}{m} \end{bmatrix},$$

$$C_{11}(\alpha) = \begin{bmatrix} \frac{-k}{M(\alpha)} & \frac{-c_{min}}{M(\alpha)} & \frac{k}{M(\alpha)} & \frac{c_{min}}{M(\alpha)} \end{bmatrix},\ C_{12}(\alpha) = \begin{bmatrix} \frac{-k}{M(\alpha)} & \frac{-c_{max}}{M(\alpha)} & \frac{k}{M(\alpha)} & \frac{c_{max}}{M(\alpha)} \end{bmatrix},$$

$$D_{11} = D_{12} = D_{21} = D_{22} = 0,\ F_{11} = F_{12} = F_{21} = F_{22} = 0$$

$$C_{21} = C_{22} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \; H_1 = H_2 = 0, \tag{189}$$

where the state vector is $x(t) = [\delta y_1 \; \delta \dot{y}_1 \; \delta y_2 \; \delta \dot{y}_2]'$, δy_1 and δy_2 are the variations of y_1 and y_2 around an equilibrium point. Additionally $y_1(t)$, $y_2(t)$, $\zeta(t)$ are the vertical position of the body, the unsprung mass, and the road profile, respectively. The others plant parameters are the quarter-car body mass (M), the unsprung mass (m), the stiffness of the suspension spring (k), the stiffness of the tire (k_t) and damping coefficient of the passive shock absorber (c_σ). Furthermore, it is considered a switching strategy such that the coefficient of the passive shock absorber (c_σ) can assume only two values, previously named, c_{min} and c_{max}. Moreover, it is important to stress that $z(t)$ corresponds to the body acceleration, that is, $\ddot{y}_1(t)$.

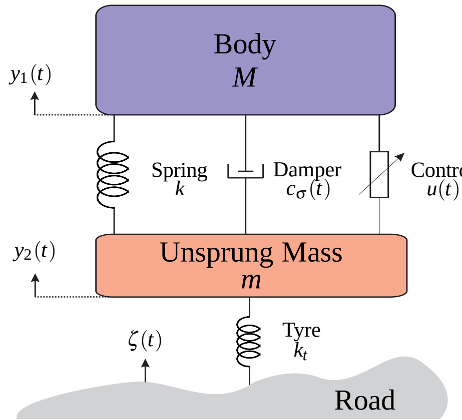
8.3.1 Case I: uncertain quarter-car body mass (CARDIM *et al.*, 2016)

For the first simulation it is assumed that the quarter-car body mass is uncertain and belongs to $M_{min} < M < M_{max}$. Thus, for this case, the system is represented by (4),(5) and (189) with $r = 2$, $N = 2$, $i \in \{1,2\}$, $j \in \{1,2\}$. The following values were adopted: $M_{min} = 350$ kg, $M_{max} = 450$ kg, $c_{min} = 3 \times 10^2$ N.s/m, $c_{max} = 3.9 \times 10^3$ N.s/m, $m = 50$ kg, $k = 2 \times 10^4$ N/m, $k_t = 2.5 \times 10^5$ N/m. Due to practical implementation issues, the output feedback gains must be bounded. In this sense, through the DE-LMI it is possible to constraint the norm of vectors gains. For this case the values were constrained according to $norm(K_{ic}) \leq 10 \times 10^3$, for all $i, c \in \mathbb{K}_2$. The DE-LMI algorithm, considering the solver *LMIlab* (GAHINET *et al.*, 1994), was applied in order to obtain the output feedback gains (K_{ic}) and the switching decision matrices ($\hat{\mathcal{Q}}_{ic}$) for the conditions proposed in Theorem 7. The obtained values are the following:

$$\begin{aligned}
&\beta = 0.1709, && \rho = 1.8066 \\
&\lambda_1 \approxeq 0, && \lambda_2 \approxeq 1 \\
&K_{11} = \begin{bmatrix} 619.8018 & -0.6575 \end{bmatrix}, && K_{21} = 1 \times 10^3 \begin{bmatrix} -7.5318 & -1.0005 \end{bmatrix}, \\
&K_{12} = \begin{bmatrix} 635.0843 & -0.0200 \end{bmatrix}, && K_{22} = 1 \times 10^3 \begin{bmatrix} -9.8440 & -1.0180 \end{bmatrix}, \\
&\hat{\mathcal{Q}}_{11} = 1 \times 10^7 \begin{bmatrix} -0.5164 & -1.7135 \\ -1.7135 & 0.0129 \end{bmatrix}, && \hat{\mathcal{Q}}_{12} = 1 \times 10^7 \begin{bmatrix} -0.5163 & -1.7138 \\ -1.7138 & 0.0261 \end{bmatrix}, \\
&\hat{\mathcal{Q}}_{21} = 1 \times 10^7 \begin{bmatrix} -0.5164 & -1.7135 \\ -1.7135 & 0.0126 \end{bmatrix}, && \hat{\mathcal{Q}}_{22} = 1 \times 10^7 \begin{bmatrix} -0.5164 & -1.7135 \\ -1.7135 & 0.0126 \end{bmatrix},
\end{aligned} \tag{190}$$

The road profile is treated in this work as the exogenous disturbance, that is $w(t) = \zeta(t)$. It is important to highlight that in (CARDIM *et al.*, 2016) the guaranteed cost was selected as the performance criterion and there is no $\mathscr{H}_\infty$ approach involved. For comparison purpose with the simulation results presented in (CARDIM *et al.*, 2016), the sa

Figure 12 - Active suspension system (quarter car).



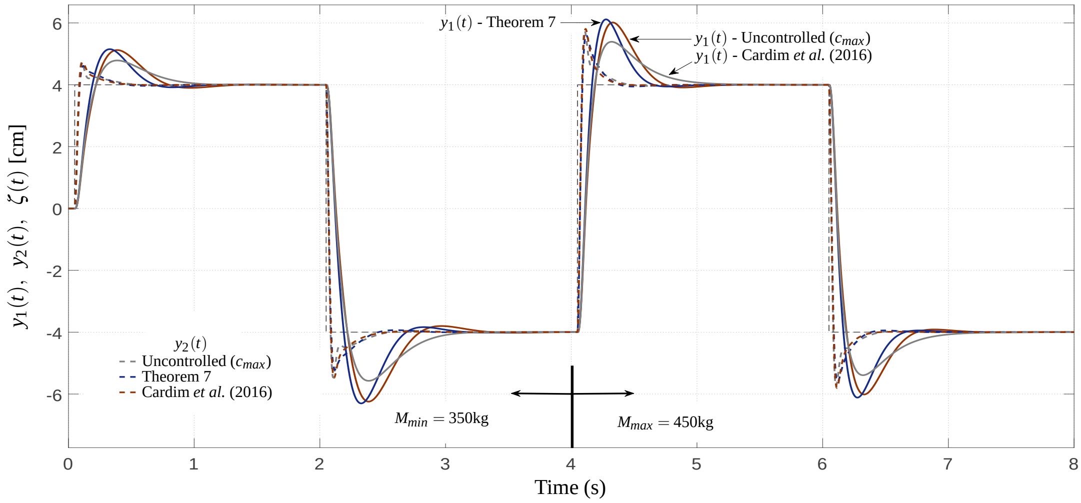
Source: Adapted from Cardim *et al.* (2016)

me road profile ($w(t) = \zeta(t)$), a square wave form with amplitude $\pm$4cm, was considered. Since the quarter-car body mass is an uncertain parameter (M), the simulation examines two different scenarios, for $M = M_{min} = 350$kg ($0 \leq t < 4$s) and $M = M_{max} = 450$kg ($t > 4s$). For the open-loop simulation, it was used the maximum value of c_σ, in this case $c_{max} = 3.9 \times 10^3$ N.s/m. Figures 13, 14 and 15 show the simulation results.

Figure 13 shows the comparison of the time response of $y_1(t)$ and $y_2(t)$ for the conditions of Theorem 7, the conditions introduced in (CARDIM *et al.*, 2016) in Section 5.2, and the uncontrolled situation with c_{max}.

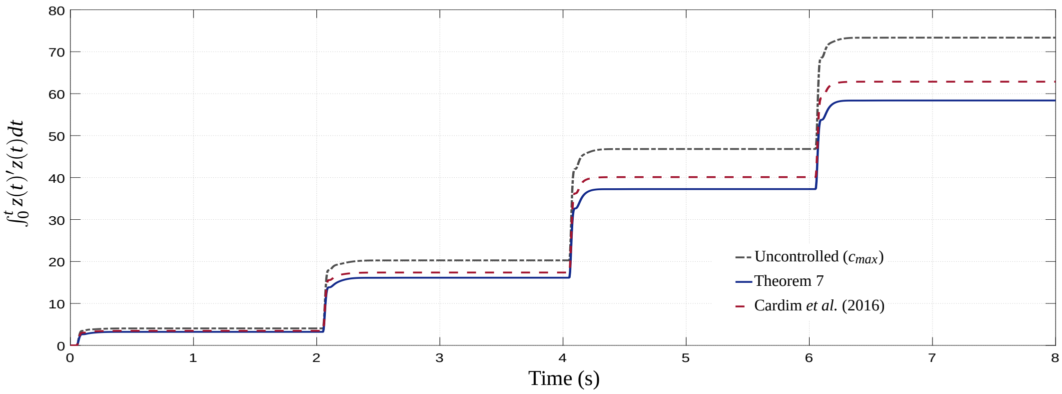
It is possible to observe in Figure 14 that the proposed Theorem 7 presents the best performance for $y_1(t)$, considering the reduction of the guaranteed cost. This fact is linked with the reduction of $\int_0^t z(t)'z(t)dt$, as depicted in Figure 14, since $z(t)$ correspond to the body acceleration $\ddot{y}_1(t)$. Theorem 7 provides about 20.4% and 7.11% of reduction in the $\int_0^t z(t)'z(t)dt$ final value, when compared to the open-loop case and the response related to the conditions proposed in (CARDIM *et al.*, 2016), respectively. It is important to stress that the proposed DE-LMI method do not ensure the global optimisation, due to non-convex characteristics related to BMIs. Therefore different values for the controllers can be found, depending on the ED initialisation parameters values (STORN; PRICE, 1997). Figure 15 shows the control input $u(t)$ time response. The switching selection was omitted in this example, as in this case for Theorem 7 the switching function returned $\sigma = 2$ and $\eta = 1$ for the simulation time, that is, the controller gain K_{21} was kept. Concerning the dynamical output feedback design proposed in (GEROMEL; COLANERI; BOLZERN, 2008), note that it can not be directly applied in this example that presents uncertainties, since in the aforementioned paper the plant was supposed known and without uncertain parameters.

Figure 13 - Comparison of the time response: $M = M_{min} = 350$kg ($0 \le t < 4$s) and $M = M_{max} = 450$kg ($t > 4s$).



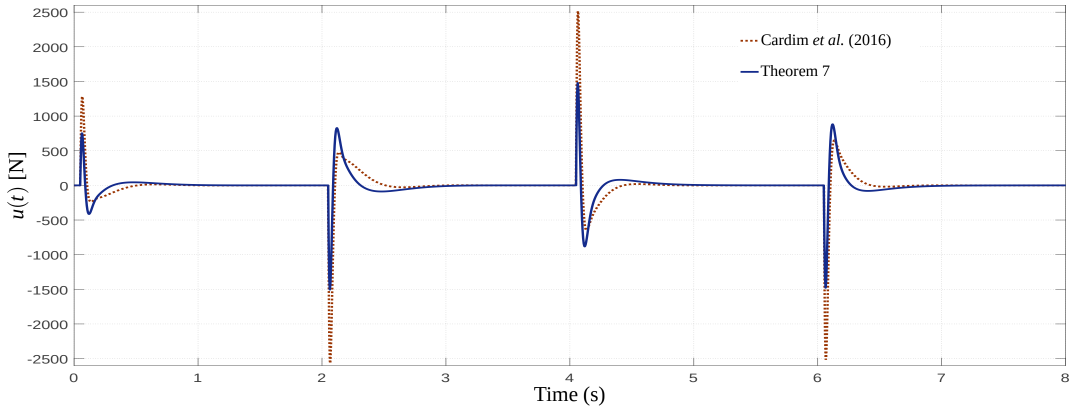
Source: Author's own results.

Figure 14 - Comparison of the guaranteed cost: $M = M_{min} = 350$kg ($0 \le t < 4$s) and $M = M_{max} = 450$kg ($t > 4s$).



Source: Author's own results.

Figure 15 - Control signal $u(t)$: $M = M_{min} = 350$kg $(0 \leq t < 4$s) and $M = M_{max} = 450$kg $(t > 4s)$.



Source: Author's own results.

8.3.2 Case II: uncertain quarter-car body mass (M) and suspension spring stiffness (k) variation

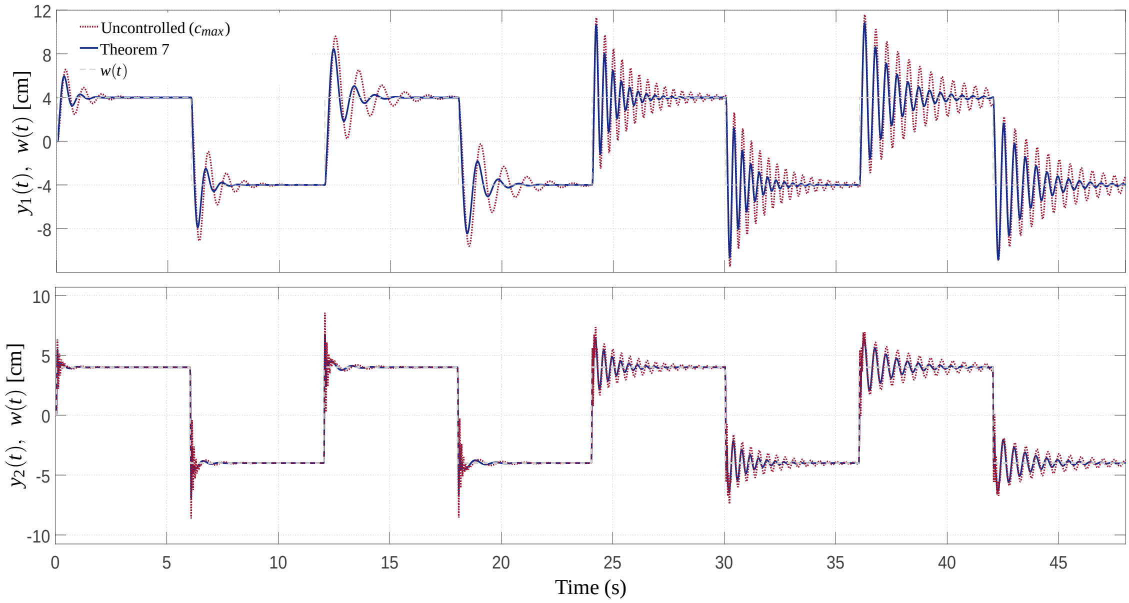
For this second case besides the quarter-car body mass, the stiffness of the suspension spring was treated as an uncertain parameter, belonging to $k_{min} < k < k_{max}$. Now, the system is represented by (4), (5) and (189) with $r = 4$, $N = 2$, $i \in \{1,2\}$, $j \in \{1,2,3,4\}$. The following values were adopted: $M_{min} = 300$ kg, $M_{max} = 500$ kg, $c_{min} = 6 \times 10^2$ N.s/m, $c_{max} = 9 \times 10^2$ N.s/m, $m = 50$ kg, $k_{min} = 2 \times 10^4$ N/m, $k_{max} = 12 \times 10^4$ N/m, $k_t = 2.5 \times 10^5$ N/m. It is important to highlight that the range of the quarter-car body mass was increased and the suspension spring stiffness was treated as another uncertainty parameter with the purpose of exploiting and to show the potential of Theorem 7. For this case the norm of the vectors gains was also constrained according to $norm(K_{ic}) \leq 10 \times 10^3$. The DE-LMI algorithm, considering the solver *LMIlab* (GAHINET *et al.*, 1994), was used for the conditions proposed in Theorem 7. The obtained values for the parameters λ_i, ρ, β, for the controller gains K_{ic} and the switching decision matrices $\hat{\mathcal{Q}}_{ic}$ are the following:

$$\begin{aligned}
&\beta = 0.2559, && \rho = -0.3026\\
&\lambda_1 = 0.15020, && \lambda_2 = 0.8498\\
&K_{11} = 1\times 10^3 \begin{bmatrix} -7.1936 & -0.7666 \end{bmatrix}, && K_{21} = 1\times 10^3 \begin{bmatrix} -9.3346 & -0.9770 \end{bmatrix},\\
&K_{12} = 1\times 10^3 \begin{bmatrix} -5.5566 & -0.7633 \end{bmatrix}, && K_{22} = 1\times 10^3 \begin{bmatrix} -7.4555 & -0.9792 \end{bmatrix},\\
&K_{13} = 1\times 10^3 \begin{bmatrix} -5.0453 & -0.7935 \end{bmatrix}, && K_{23} = 1\times 10^3 \begin{bmatrix} -7.2094 & -0.9824 \end{bmatrix},\\
&K_{14} = 1\times 10^3 \begin{bmatrix} -4.3620 & -0.7681 \end{bmatrix}, && K_{24} = 1\times 10^3 \begin{bmatrix} -6.4612 & -0.9838 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\hat{\mathcal{Q}}_{11} &= \begin{bmatrix} 30444329.2000 & 26631481.8465\\ 26631481.8465 & 5045455.02273 \end{bmatrix}, & \hat{\mathcal{Q}}_{21} &= \begin{bmatrix} 30444329.1999 & 26631481.8476\\ 26631481.8476 & 5045454.97914 \end{bmatrix},\\
\hat{\mathcal{Q}}_{12} &= \begin{bmatrix} 30444329.2000 & 26631481.8469\\ 26631481.8469 & 5045455.01505 \end{bmatrix}, & \hat{\mathcal{Q}}_{22} &= \begin{bmatrix} 30444329.1998 & 26631481.8477\\ 26631481.8477 & 5045454.97644 \end{bmatrix},\\
\hat{\mathcal{Q}}_{13} &= \begin{bmatrix} 30444329.1998 & 26631481.8478\\ 26631481.8478 & 5045455.00590 \end{bmatrix}, & \hat{\mathcal{Q}}_{23} &= \begin{bmatrix} 30444329.1998 & 26631481.8478\\ 26631481.8478 & 5045454.97463 \end{bmatrix},\\
\hat{\mathcal{Q}}_{14} &= \begin{bmatrix} 30444329.1999 & 26631481.8477\\ 26631481.8477 & 5045455.00677 \end{bmatrix}, & \hat{\mathcal{Q}}_{24} &= \begin{bmatrix} 30444329.1998 & 26631481.8479\\ 26631481.8479 & 5045454.97452 \end{bmatrix},
\end{aligned} \tag{191}$$

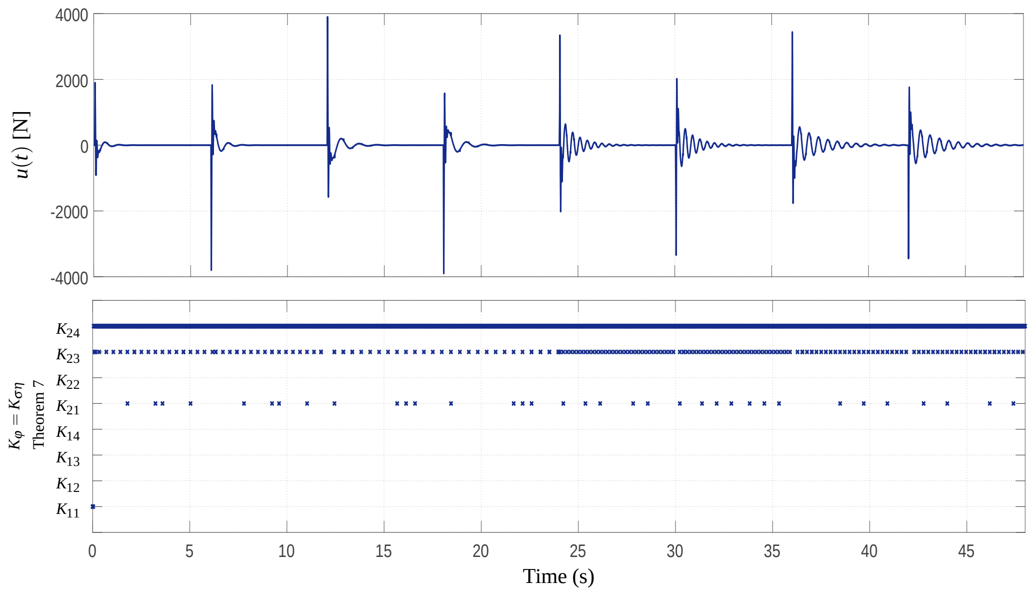
For the open-loop simulation, it was used the maximum value of c_σ, in this case $c_{max} = 9\times 10^2$ (N.s)/m. The simulation examines four different scenarios: 1. $M = M_{min} = 300$kg and $k = k_{min} = 2\times 10^4$N/m ($0 \le t < 12$s), 2. $M = M_{max} = 500$kg and $k = k_{min} = 2\times 10^4$N/m ($12 \le t < 24$s), 3. $M = M_{min} = 300$kg and $k = k_{max} = 12\times 10^4$N/m ($24 \le t < 36$s) and 4. $M = M_{max} = 300$kg and $k = k_{max} = 12\times 10^4$N/m ($t > 36$s) . Figure 16 shows the comparison of the time response of $y_1(t)$ and $y_2(t)$ for the conditions of Theorem 7 and the open-loop system. It is possible to observe that the oscillation and peak values of $y_1(t)$ and $y_2(t)$ are significantly reduced when considering the switching controllers. Finally, Figure 17 shows the control input signal ($u(t)$) and the controllers/subsystems switching.

Figure 16 - Comparison of the time response: $M = M_{min} = 300$kg and $k = k_{min} = 2 \times 10^4$ $(0 \leq t < 12\text{s})$, $M = M_{max} = 500$kg and $k = k_{min} = 2 \times 10^4$ $(12 \leq t < 24\text{s})$, $M = M_{min} = 300$kg and $k = k_{max} = 12 \times 10^4$ $(24 \leq t < 36\text{s})$, and $M = M_{max} = 500$kg and $k = k_{max} = 12 \times 10^4$ $(36 \leq t < 48\text{s})$.



Source: Author's own results.

Figure 17 - Control signal $u(t)$ and switching selection: $M = M_{min} = 300$kg and $k = k_{min} = 2 \times 10^4$ $(0 \leq t < 12\text{s})$, $M = M_{max} = 500$kg and $k = k_{min} = 2 \times 10^4$ $(12 \leq t < 24\text{s})$, $M = M_{min} = 300$kg and $k = k_{max} = 12 \times 10^4$ $(24 \leq t < 36\text{s})$, and $M = M_{max} = 500$kg and $k = k_{max} = 12 \times 10^4$ $(36 \leq t < 48\text{s})$.



Source: Author's own results.

8.4 EXAMPLE V - SWITCHED $\mathscr{H}_\infty$ OUTPUT CONTROL UNDER ACTUATOR SATURATION

In this numerical example the same matrices of the system employed in Example III (Section 8.2) are considered. However, in order to study systems subject to actuator saturation and operation region, the following constraints are addressed. For the operation region $x(t) \in \mathcal{X}(\mathcal{N}_h)$, the bounds values are given by $\phi = \begin{bmatrix} 40 & 5 & 5 \end{bmatrix}$ with $\mathcal{N} = I_3$. The disturbance is bounded according to (127), with $\varepsilon = 5$. This example consists in evaluate and a draw a comparison between Theorem 9 and Theorem 10 ($N = 1$, Corollary 2), considering $\wp$ increments. Two different cases are studied, concerning the value of $\bar{u}$ and ε_0.

The Tables 1, 2, 3 and 4 show the results obtained with the conditions of Theorem 9 and 10 (Corollary 2) for comparison. It presents the values of $\gamma = \sqrt{\mu}$, φ, f_s, $\bar{v}$, β, ρ. As defined before, γ refers to the $\mathscr{H}_\infty$ performance. The parameters φ, β, and ρ were optimised, using the DE-LMI algorithm, in order to obtain suboptimal values to reduce the $\mathscr{H}_\infty$ bound (Remark 7) and obtain a "larger" estimation of the domain of attraction. The value of $\bar{v}$ indicates the average between the volumes of the ellipsoidal sets obtained for each vertex, that is, for each value of X_j where $P(\alpha^*) = X_j^{-1}$ for $j \in \mathbb{K}_r$. Finally, f_s is the fitness function applied in the

DE-LMI algorithm, given by $f_s = \gamma - \varpi$, where ϖ is the parameter related to the ellipsoidal set "enlargement" considering $n_L = 6$ with $w_1 = [1\ 0\ 0],\ w_2 = [-1\ 0\ 0],\ w_3 = [0\ 1\ 0],\ w_4 = [0\ -1\ 0],\ w_5 = [0\ 0\ 1],\ w_6 = [0\ 0\ -1]$ (Section 5.4).

8.4.1 Case I: $\overline{u} = 0.5$ and $\varepsilon_0 = 0$

The first approach adopts a saturation level $\overline{u} = 0.5$ as the saturation value for single input ($l = 1$). The ellipsoidal set is given by $\mathscr{E}(P(\alpha^*), \varphi^{-1}\varepsilon)$, that is to say that $\varepsilon_0 = 0$ and consequently for $w(t) \neq 0$ the initial condition should be $x(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}'$. The results for the conditions of Theorem 9 and Theorem 10 are presented in Table 1 and Table 2, respectively. It is possible to observe that Theorem 10, which one employs the switched controller, achieves better performance when compared with Theorem (9). This fact is noticed by the $\mathscr{H}_\infty$ performance value and the volume of the ellipsoids ($\overline{v}$). Furthermore, for $\wp = 4.9$ the conditions of Theorem 9 are unfeasible while the conditions of Theorem 10 are feasible, allowing to obtain the controller even under the imposed constraints.

Table 1 - Results for Example V - Case I with $\overline{u} = 0.5$ and $\varepsilon_0 = 0$, considering the conditions of Theorem 9.

	γ	ϖ	f_s	φ	$\overline{v}$	β	ρ
$\wp = 4.5$	29.67	1.47	28.2	4.83	767.5	0.15304	0.14819
$\wp = 4.6$	35.31	1.4	33.91	6.19	767	0.15492	0.16374
$\wp = 4.7$	46.33	1.37	44.96	62.47	783	0.15610	0.15989
$\wp = 4.8$	91.2	1.16	90	0.448	792.4	0.15727	0.15031
$\wp = 4.9$	-	-	-	-	-	-	-

Source: Author's own results.

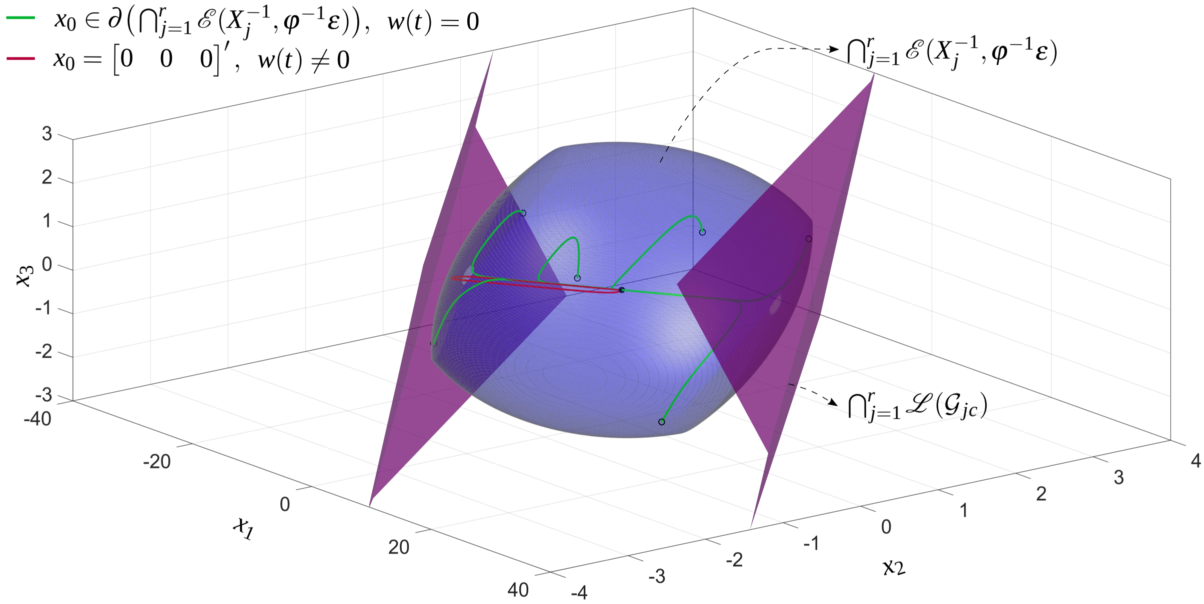
Table 2 - Results for Example V - Case I with $\overline{u} = 0.5$ and $\varepsilon_0 = 0$, considering the conditions of Theorem 10.

	γ	ϖ	f_s	φ	$\overline{v}$	β	ρ
$\wp = 4.5$	20.79	1.65	19.14	3.54	791.3	0.15684	0.15984
$\wp = 4.6$	23.15	1.62	21.53	3.62	795	0.15738	0.16212
$\wp = 4.7$	26.61	1.58	25	0.0791	795	0.15891	0.16019
$\wp = 4.8$	32.5	1.54	31	0.186	790.1	0.16056	0.16687
$\wp = 4.9$	46.2	1.49	44.7	0.894	780	0.16175	0.16434

Source: Author's own results.

Concerning $\wp = 4.9$, Figure 18 shows the set $\mathscr{E}(P(\alpha^*), \varphi^{-1}\varepsilon)$ and an estimation for the set $\mathcal{G}_c(\alpha^*)$ given by $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varphi^{-1}\varepsilon)$ and $\bigcap_{j=1}^{r} \mathcal{G}_{jc}$, respectively (refer to Remarks 5 and

Figure 18 - Example V: Case I - Representation of the system trajectories and sets $\mathscr{E}(P(\alpha^*),\varphi^{-1}\varepsilon)$ and $\mathcal{G}_c(\alpha^*)$



Source: Author's own results.

8). The green lines represent the trajectories beginning in $\partial\left(\bigcap_{j=1}^{r}\mathscr{E}(X_j^{-1},\varphi^{-1}\varepsilon)\right)$ (boundary) with $w(t)=0$ in order to illustrated the first statement given in Problem 4. The red trajectory corresponds to an initial condition $x_0=[0\ \ 0\ \ 0]'$, aiming to evaluate the third statement of the aforementioned problem. From Figure 18 note that sets constraints, introduced in Section 5.5, are fulfilled, since $\bigcap_{j=1}^{r}\mathscr{E}(X_j^{-1},\varphi^{-1}\varepsilon)\subset\mathcal{X}(\mathcal{N}_h)$ and $\bigcap_{j=1}^{r}\mathscr{E}(X_j^{-1},\varphi^{-1}\varepsilon)\subset\bigcap_{j=1}^{r}\mathcal{G}_{jc}$.

8.4.2 Case II: $\bar{u}=0.45$ and $\varepsilon_0=0.1\varphi^{-1}\varepsilon$

In Case II, in order to explore the second statement of Problem 4, it was adopted $\varepsilon_0 = 0.1\varphi^{-1}\varepsilon$. It means that the inner ellipsoid given by $\mathscr{E}(P(\alpha^*),\varepsilon_0)$ is proportional to the outer ellipsoid $\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)$. The results for the conditions of Theorem 9 and Theorem 10 are presented in Tables 3 and 4, respectively.

Table 3 - Results for Example V - Case II with $\bar{u}=0.45$ and $\varepsilon_0=0.1\varphi^{-1}\varepsilon$considering the conditions of Theorem 9.

	γ	ϖ	f_s	φ	$\bar{\nu}$	β	ρ
$\wp=4.5$	63.90	1.51	62.40	0.30	865.20	0.15491	0.15627
$\wp=4.6$	-	-	-	-	-	-	-

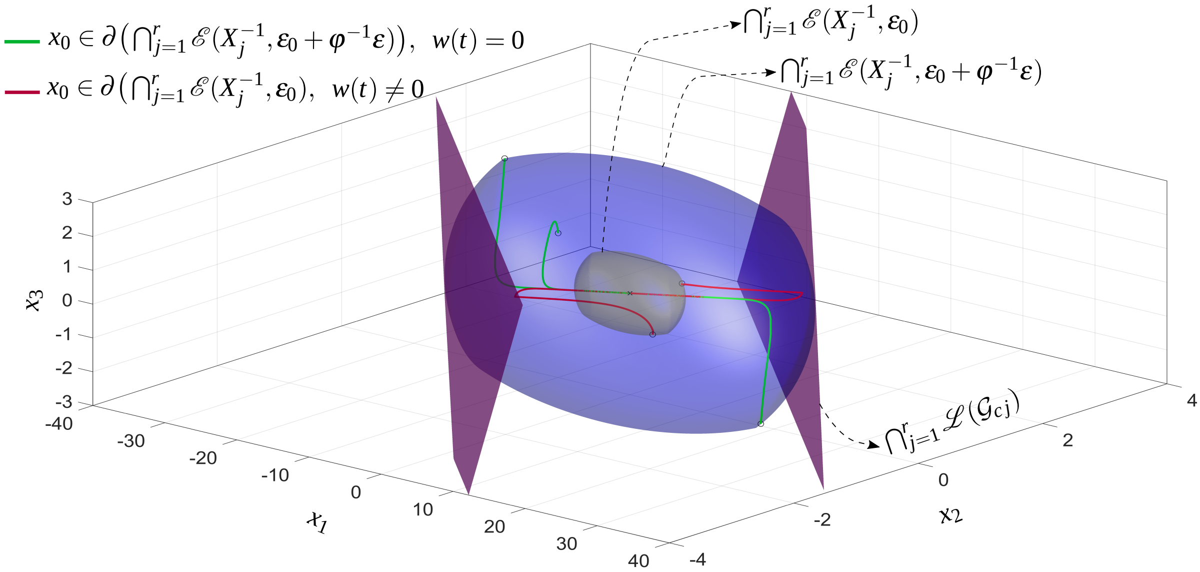
Source: Author's own results.

In Tables 3 and 4, it is possible to note that for $\wp=4.5$ the conditions of Theorem

Table 4 - Results for Example V - Case II with $\bar{u} = 0.45$ and $\varepsilon_0 = 0.1\varphi^{-1}\varepsilon$, considering the conditions of Theorem 10.

	γ	ϖ	f_s	φ	$\bar{v}$	β	ρ
$\wp = 4.5$	23.26	1.64	21.62	23.14	780.32	0.15666	0.15844
$\wp = 4.6$	26.83	1.61	25.22	23.17	782.36	0.15758	0.16106
$\wp = 4.7$	32.86	1.58	31.28	28.14	781.28	0.15935	0.16330
$\wp = 4.8$	46.89	1.52	45.37	0.75	777.78	0.16066	0.16431
$\wp = 4.9$	-	-	-	-	-	-	-

Figure 19 - Example V: Case 2 - Representation of the system trajectories and sets $\mathscr{E}(P(\alpha^*),\varepsilon_0)$, $\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)$ and $\mathcal{G}_c(\alpha^*)$



Source: Author's own results.

10 results in a better performance for the $\mathscr{H}_\infty$ cost value (γ). As long as the value of $\bar{v}$ is greater for the conditions of Theorem 9, the value of γ obtained for with conditions of Theorem 10 is considerably lower. From these results, it is important to stress that for $4.6 \leq \wp \leq 4.8$ the conditions of Theorem 9 are unfeasible, although employing the switched controllers (conditions of Theorem 10) it is possible to obtain feasible solutions. For $\wp = 4.8$, Figure 19 presents the sets $\mathscr{E}(P(\alpha^*),\varepsilon_0)$ and $\mathscr{E}(P(\alpha^*),\varepsilon_0+\varphi^{-1}\varepsilon)$ as well as an estimation for the $\mathcal{G}_c(\alpha^*)$ given by $\bigcap_{j=1}^{r}\mathscr{E}(X_j^{-1},\varepsilon_0)$, $\bigcap_{j=1}^{r}\mathscr{E}(X_j^{-1},\varepsilon_0+\varphi^{-1}\varepsilon)$ and $\bigcap_{j=1}^{r}\mathcal{G}_{jc}$, respectively (refer to Remark 5 and Remark 8). The green lines represent the trajectories beginning in $\partial\left(\bigcap_{j=1}^{r}\mathscr{E}(X_j^{-1},\varepsilon_0+\varphi^{-1}\varepsilon)\right)$ (boundary) with $w(t) = 0$ in order to illustrated the first statement given in Problem 5. The red trajectories correspond to initial conditions $x_0 \in \partial\left(\bigcap_{j=1}^{r}\mathscr{E}(X_j^{-1},\varepsilon_0)\right)$ with $w(t) \neq 0, \int_0^t w(t)'w(t) = \varepsilon = 5$, aiming to evaluate the second statement of the aforementioned problem.

Observing the curves in Figure 19 it is important to stress that, as required in the second statement of Problem 3, the trajectories, even under disturbance ($w(t) \neq 0$), beginning within $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varepsilon_0)$ remain in $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varepsilon_0 + \varphi^{-1}\varepsilon)$ for all $t \geq 0$. From Figure 19 note that sets constraints discussed in Section 5.5, are fulfilled, since $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \mathcal{X}(\mathcal{N}_h)$ and $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varepsilon_0 + \varphi^{-1}\varepsilon) \subset \bigcap_{j=1}^{r} \mathcal{G}_{jc}$.

8.5 EXAMPLE VI - PRACTICAL IMPLEMENTATION IN AN ACTIVE SUSPENSION SYSTEM

This section is devoted to the design and practical implementation of the switched output-dependent controller, considering the actuator saturation, in an Active Suspension (AS) system manufactured by Quanser®, which represents $1/4$ of a vehicle. Its schematic model is depicted in Fig. 20. The dynamic model can be represented by (87), (5), with $r = 4$, $j \in \{1, 4\}$ and

$$A(\alpha) = \begin{bmatrix} 1 & 1 & 0 & -1 \\ \frac{-k_s}{M_s(\alpha)} & \frac{-b_s}{M_s(\alpha)} & 0 & \frac{b_s}{M_s(\alpha)} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{M_{us}} & \frac{b_s}{M_{us}} & \frac{-k_{us}}{M_{us}} & \frac{(-b_s - b_{us})}{M_{us}} \end{bmatrix}, \ B(\alpha) = \begin{bmatrix} 0 \\ \frac{1}{M_s(\alpha)} \\ 0 \\ -\frac{1}{M_{us}} \end{bmatrix}, \ C_1(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

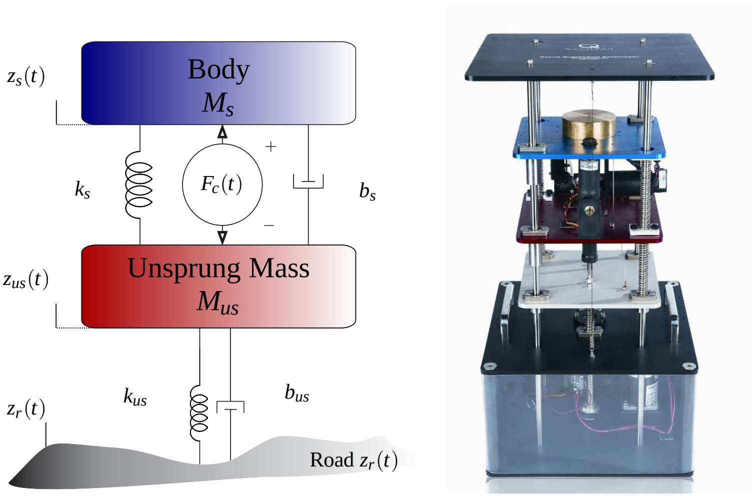
$$C_2(\alpha) = \begin{bmatrix} 0 & s_f(\alpha) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ E(\alpha) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ \frac{b_{us}}{M_{us}} \end{bmatrix}, F(\alpha) = 0, \ \ D(\alpha) = 0, \ \ H(\alpha) = 0,$$

$$x(t) = \begin{bmatrix} z_s(t) - z_{us}(t) \\ \dot{z}_s(t) \\ z_{us}(t) - z_r(t) \\ \dot{z}_{us}(t) \end{bmatrix}, \ \ w(t) = \dot{z}_r(t), \ \ u(t) = F_c(t). \tag{192}$$

The values for the system parameters are presented in Table 5. The body mass (M_s) is an uncertainty parameter since it may assume the range $1.455kg \leq M_s \leq 2.45kg$. Furthermore, the possibility of a sensor measurement fault is addressed. This fault was modelled as parametric uncertainties with $0.8 \leq s_f \leq 1.2$ instead of $s_f = 1$.

The controller design takes into account the actuator saturation and the region of operation. Due to the physics restrictions of the spring length, interval $-0.02 \leq z_u s(t) - z_r(t) \leq 0.02m$ is assumed for the state variable $z_u s(t) - z_r(t)$ (OLIVEIRA, 2017). In doing so, the operation region $x(t) \in \mathcal{X}(\mathcal{N}_h)$ is given by $\phi = 0.02$ with $\mathcal{N} = [0\ 0\ 1\ 0]$. According to the manufacturer the control input $u(t) = F_c(t)$ should be limited in the interval $-39.2\,N \leq u(t) \leq 39.2\,N$, thus, for the design $\overline{u} = 39.2\ N$ is applied. Furthermore the design consider a bounded-energy disturbance for $w(t) = \dot{z}_r(t) \in (127)$ with $\varepsilon = 0.0013$ and $\varepsilon_0 = 0$. For the ellipsoidal set "enlargement" it was considered $n_L = 8$ with $w_1 = [1\ 0\ 0\ 0]$, $w_2 = [-1\ 0\ 0\ 0]$, $w_3 = [0\ 1\ 0\ 0]$, $w_4 = [0\ -$

Figure 20 - Active suspension system (quarter car).



Source: Adapted from Oliveira *et al.* (2018)

Table 5 - Active suspension parameters

Parameter	Symbol	Value
Mass of $1/4$ of the total vehicle (kg)	M_s	$[1.455, 2.45]$
Mass of the tire set (kg)	M_{us}	1
Stiffness constant of the spring (N/m)	k_s	900
Stiffness constant of the spring (N/m)	k_{us}	2500
Damping coefficient (Ns/m)	b_s	7.5
Damping coefficient (Ns/m)	b_s	5

Source: Oliveira *et al.* (2018).

$1\ 0\ 0]$, $w_5 = [0\ 0\ 1\ 0]$, $w_6 = [0\ 0\ -1\ 0]$, $w_7 = [0\ 0\ 0\ 1]$, $w_8 = [0\ 0\ 0\ -1]$ (Section 5.4). The conditions proposed in Theorem 10 (Corollary 2) yield, with $\beta = 0.0720509$, $\rho = -0.003199$ and $\varphi = 981.2$, an $\mathscr{H}_\infty$ performance $\gamma = 0.2227$ and the parameter $\varpi = 0.0173$ related with the ellipsoidal set "enlargement". The following switched static output gains (K_c, $c \in \mathbb{K}_4$) and switching decision matrices ($\hat{\mathcal{Q}}_c$, $c \in K_4$), were obtained

$$
\begin{aligned}
K_1 &= \begin{bmatrix} -20.2743 & 5.3406 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -20.5511 & 5.5188 \end{bmatrix}, \\
K_3 &= \begin{bmatrix} -21.1370 & 5.8537 \end{bmatrix}, \quad K_4 = \begin{bmatrix} -22.0959 & 6.0093 \end{bmatrix}, \\
\hat{\mathcal{Q}}_1 &= \begin{bmatrix} 24006.49 & 3096.98 \\ 3096.98 & -23541.72 \end{bmatrix}, \quad \hat{\mathcal{Q}}_2 = \begin{bmatrix} 18592.10 & 3177.80 \\ 3177.80 & -26545.58 \end{bmatrix} \\
\hat{\mathcal{Q}}_3 &= \begin{bmatrix} 9768.79 & 1828.86 \\ 1828.86 & 26441.86 \end{bmatrix}, \quad \hat{\mathcal{Q}}_4 = \begin{bmatrix} -4259.03 & 2114.01 \\ 2114.01 & 8491.82 \end{bmatrix}
\end{aligned}
$$

The practical implementation considers for the road profile ($z_r(t)$) a sine wave signal, with amplitude 0.15 cm and the frequency varying linearly from 1 to 12 Hz for $0.5 \leq t \leq 9.5$ s. For $0 \leq t < 0.5$s and $9.5 < t \leq 10$s, $z_r(t) = 0$. That is the same road profile considered in Oliveira *et al.* (2018) . Furthermore, the minimum body mass value ($M_s = 1.445$kg) and the worst case of the fault ($s_f = 0.8$) were employed.

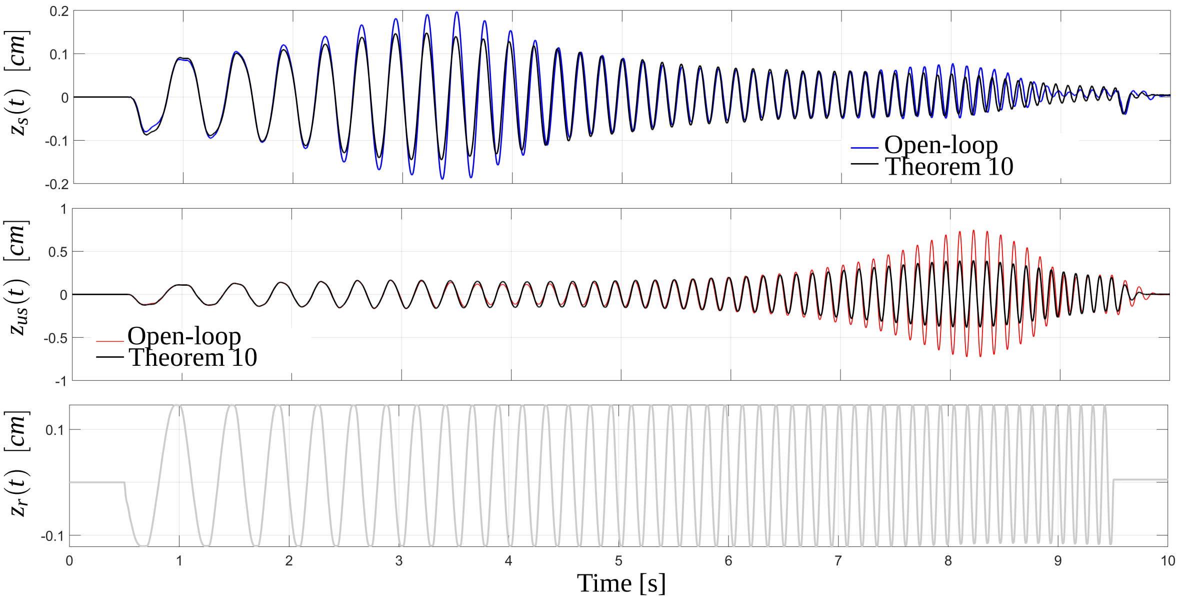
Figure 21 shows AS plates displacement (z_s and z_{us}) for the open-loop and controllers gains K_c, $c \in \mathbb{K}_4$ cases. It is possible to verify that the open-loop system is stable, since the response is bounded. Although, the system shows large amplitude oscillations, which causes mechanical stress that can compromise the components of the system and road handling. Moreover, large oscillations could cause discomfort for the driver (OLIVEIRA *et al.*, 2018). The closed-loop system provides, even under the constraint in the actuator saturation and the sensor measurement fault, a reduction in the peak value of the displacement of the plates for the switched controller, thus improving the safety and comfort for the system and driver. The red plate displacement (z_{us}) peak was reduced in about 52%.

Figure 22 presents the dynamic behaviour of the of the control input ($u(t) = F_c(t)$) and the switching signal, which selects the controller gain in each instant of time. Note that, for this test, 3 of the 4 possible gains were activated.

Figure 22 also shows the ratio $\gamma_r(t) = \sqrt{\int_0^t z(t)'z(t) / \int_0^t w(t)'w(t)}$. Additionally, it shows the $\gamma = 0.2227$ obtained for this design. It is possible to observe that the real value for the $\mathscr{H}_\infty$ performance ($\gamma_r(t)$) is always below the value of γ (bound).

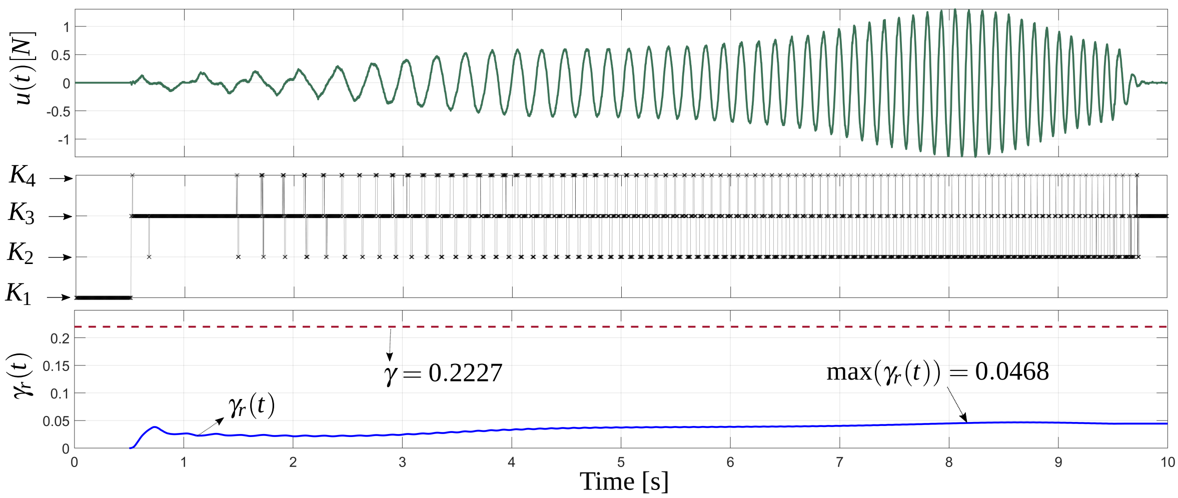
Finally, it is important to highlight that even considering actuator saturation and output sensor measurement fault the design of switched controllers ensures a performance improvement.

Figure 21 - Dynamic response of z_s [Blue plate] and z_{us} [Red plate] for the given road profile z_r [Silver plate] - Example VI.



Source: Author's own results.

Figure 22 - Dynamic response of $u(t)$, the switching selection and $\gamma_r(t)$ - Example VI



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Source: Author's own results.

8.6 EXAMPLE VII - SWITCHED LINEAR SYSTEM SUBJECT TO SATURATION

Consider the switched uncertain linear system subject to actuator saturation, described in (153) and (5), with $r = 2$, $N = 2$, $i \in 1,2$, $j \in 1,2$ and the following matrices given below

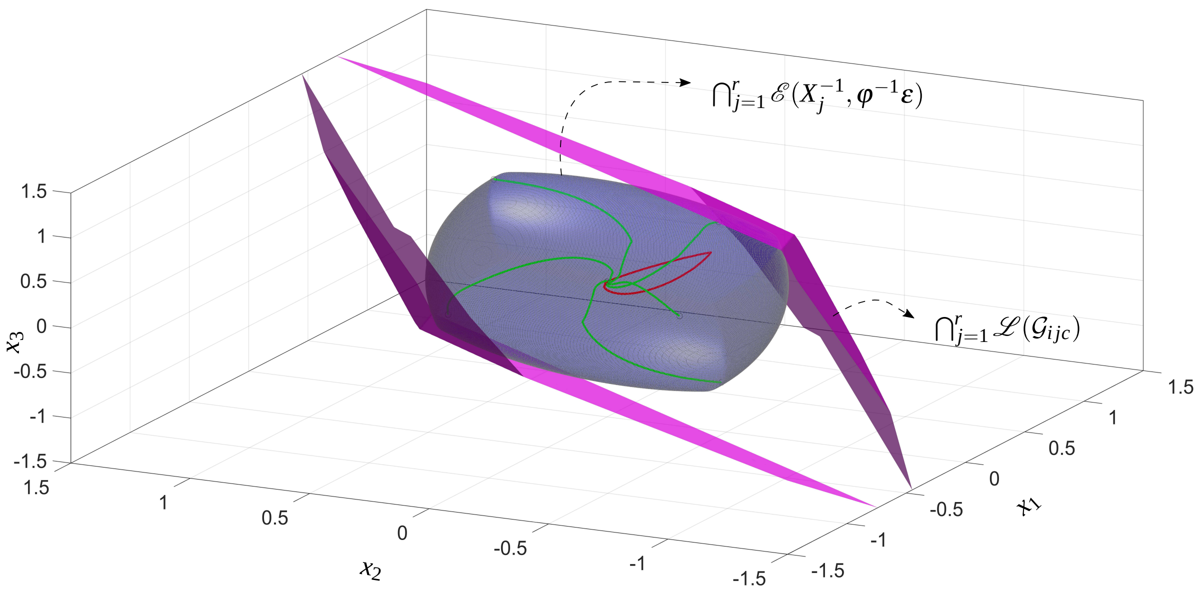
$$A_{11} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 0 \\ 0 & 2 & -1 \end{bmatrix},\ A_{12} = \begin{bmatrix} -5 & -6 & 1 \\ -3 & -3 & 0 \\ 0 & 2 & -2 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix},\ A_{22} = \begin{bmatrix} -5 & -6 & 1 \\ -3 & -2 & 0 \\ 0 & 2 & -2 \end{bmatrix},$$
$$B_{11} = B_{12} = B_{21} = B_{22} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ E_{11} = E_{12} = E_{21} = E_{22} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$
$$C_{111} = C_{112} = C_{121} = C_{122} = I_3,\ D_{11} = D_{12} = D_{21} = D_{22} = 0,\ F_{11} = F_{12} = F_{21} = F_{22} = 0,$$
$$C_{21} = C_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, H_1 = H_2 = 0. \tag{193}$$

For the operation region $x(t) \in \mathcal{X}(\mathcal{N}_h)$, the bounds values are given by $\phi = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$. The actuator saturation is adopted with $\bar{u} = 1.3$. The energy of the disturbance $w(t)$ is limited following (127), with $\varepsilon = 1.5$ and $\varepsilon_0 = 0$. Aiming maximize the volume of the ellipsoidal set $\mathscr{E}(P(\alpha^*), \varphi^{-1}\varepsilon)$, consider $n_L = 6$ with $w_1 = [1\ 0\ 0]$, $w_2 = [-1\ 0\ 0]$, $w_3 = [0\ 1\ 0]$, $w_4 = [0\ -1\ 0]$, $w_5 = [0\ 0\ 1]$, $w_6 = [0\ 0\ -1]$ (Section 5.4).

In order to obtain the switched controllers K_{ic} and the auxiliary switching matrices $\hat{\mathcal{Q}}_{ic}$, for $i \in \mathbb{K}_N$ and $c \in \mathbb{K}_r$, the conditions of Theorem 10 were employed, resulting in an $\mathscr{H}_\infty$ bound performance $\gamma = 1.7472$. For this result, the DE-LMI algorithm returned $\beta = 0.317213447$, $\rho = -0.488461476$, $\varphi = 0.6332$,$\lambda_1 = 0.8587$ and $\lambda_2 = 0.1413$ with $\varpi = 0.6984$ and the following gains and decision matrices

$$K_{11} = \begin{bmatrix} 2.0037 & 0.2738 \end{bmatrix},\ K_{12} = \begin{bmatrix} 2.1800 & 0.3624 \end{bmatrix}$$
$$K_{21} = \begin{bmatrix} 1.8196 & 0.8198 \end{bmatrix},\ K_{22} = \begin{bmatrix} 1.8461 & 0.8100 \end{bmatrix}$$
$$\hat{\mathcal{Q}}_{11} = \begin{bmatrix} 7326864.90436335 & -25793772.1917703 \\ -25793772.1917703 & 3629704.95760790 \end{bmatrix}$$
$$\hat{\mathcal{Q}}_{12} = \begin{bmatrix} 7326864.88844529 & -25793772.1634784 \\ -25793772.1634784 & 3629704.96637357 \end{bmatrix}$$
$$\hat{\mathcal{Q}}_{21} = \begin{bmatrix} 7326866.88834941 & -25793773.9057567 \\ -25793773.9057567 & 3629706.26542099 \end{bmatrix}$$

Figure 23 - Representation of the system trajectories and sets $\mathscr{E}(P(\alpha^*), \varphi^{-1}\varepsilon)$ and $\mathcal{G}_{ic}(\alpha^*)$ - Example VII.



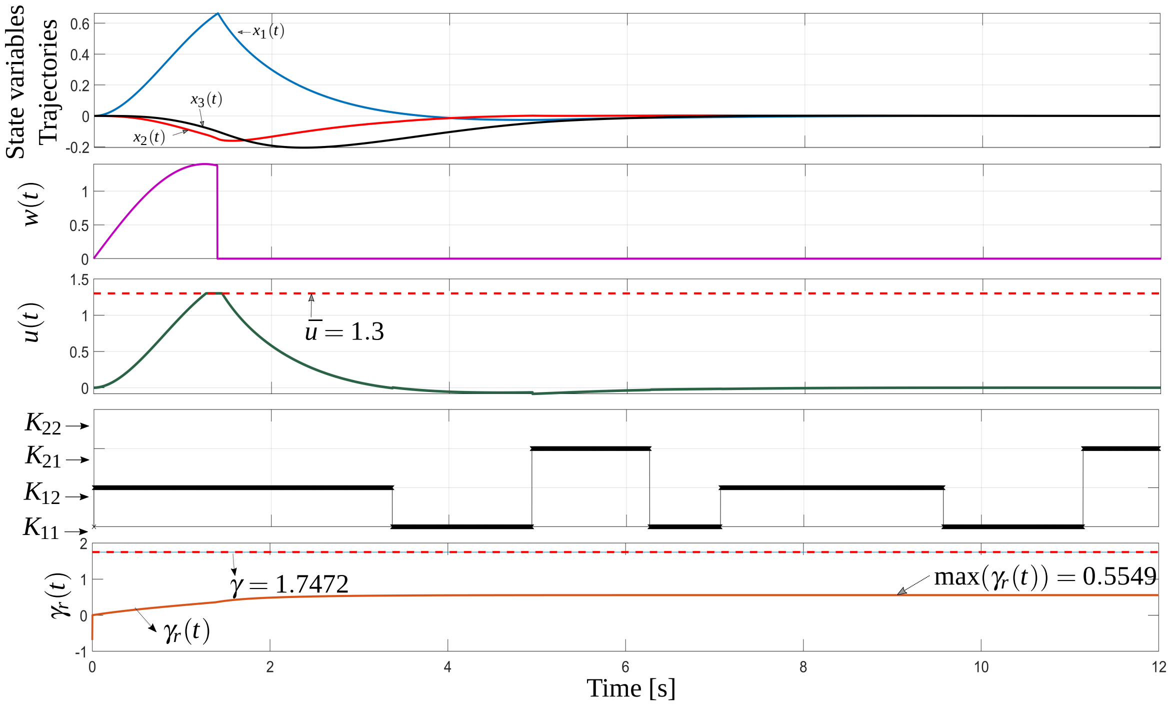
Source: Author's own results.

$$\hat{\mathcal{Q}}_{22} = \begin{bmatrix} 7326866.91551809 & -25793774.0253663 \\ -25793774.0253663 & 3629706.76490712 \end{bmatrix} \tag{194}$$

Figure 23 shows the representation of the sets $\mathscr{E}(P(\alpha^*), \varphi^{-1}\varepsilon)$ and $\mathcal{G}_{ic}(\alpha^*)$, that following the Remark 4, 5 and 8, are obtained doing $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varphi^{-1}\varepsilon)$ and $\bigcap_{j=1}^{r} \mathscr{L}(\mathcal{G}_{icj}) \subseteq \mathscr{L}(\mathcal{G}_{ic}(\alpha^*))$, respectively. Furthermore, green trajectories begin in $\partial\left(\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varphi^{-1}\varepsilon)\right)$ (boundary) with $w(t) = 0$ in order to illustrated the first statement given in Problem 5. The red trajectory corresponds to an initial condition $x_0 = [0 \;\; 0 \;\; 0]'$ with $w(t) \neq 0, \int_0^t w(t)'w(t) = \varepsilon = 1.5$, aiming to evaluate the third statement of the aforementioned problem. Note that sets constraints discussed in Section 5.5, are fulfilled, since $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varphi^{-1}\varepsilon) \subset \mathcal{X}(\mathcal{N}_h)$ and $\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varphi^{-1}\varepsilon) \subset \bigcap_{j=1}^{r} \mathcal{G}_{ijc}$.

Figure 24 depicts, for the origin as initial condition ($x_0 = [0 \; 0 \; 0]$), the state trajectory, the disturbance energy $w(t) \neq 0$ with $\int_0^t w(t)'w(t) = \varepsilon = 1.5$, the input control ($u(t)$), the switching selection and $\gamma_r(t)$. Observe that, even under the maximum disturbance, the trajectory does not leave the operation region $\mathcal{X}(\mathcal{N}_h)$. Moreover, the input signal ($u(t)$) is saturated in the time interval $1.25 \leq t \leq 1.45$. Although, as proposed in the conditions of Theorem 10, even under saturation the proposed control design ensures the system stabilization with $x(t) \in \mathcal{X}(\mathcal{N}_h)$. That is, after energy bounded disturbance is vanished, the closed-loop system backs to the origin. Note that, for this simulation, the system switches between its subsystem and controller gains. Concerning the $\mathscr{H}_\infty$ performance, observe that the real value for the ($\gamma_r(t)$) is always below the

Figure 24 - State trajectory, disturbance ($w(t) \neq 0$), control input ($u(t)$), switching selection and $\gamma_r(t)$ for $x_0 = [0\ 0\ 0]$ - Example VII.

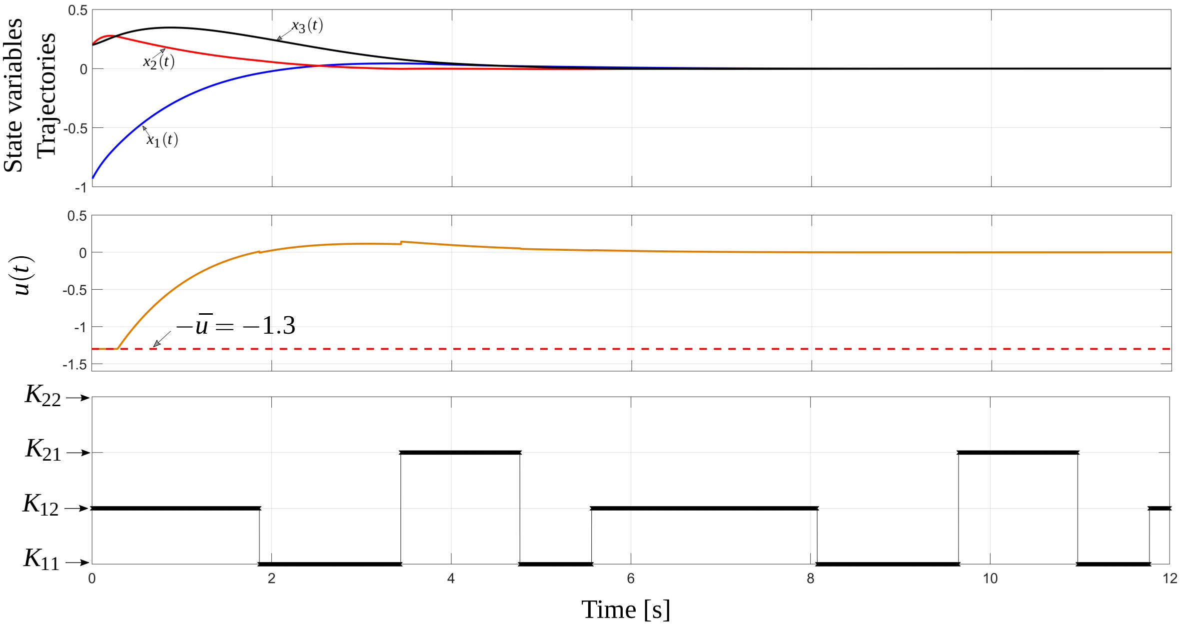


Source: Author's own results.

value of $\gamma = 1.7472$ (bound).

In order to evaluate the system stability in a disturbance free scenario, Figure 25 shows the dynamic response of the system, the input control $(u(t))$, and the switching signal considering the initial condition $x_0 = [-0.93\ 0.2034\ 0.2] \in \partial\left(\bigcap_{j=1}^{r} \mathscr{E}(X_j^{-1}, \varphi^{-1}\varepsilon)\right)$. Note that, at the beginning of simulation the input control $(u(t))$ is saturated in $u(t) = -\bar{u} = -1.3$. Note that, the controller designed considering the conditions of Theorem 10 ensures the system stability for all $x(t) \in (X_j^{-1}, \varphi^{-1}\varepsilon))$, even under actuator saturation.

Figure 25 - State trajectory, control input ($u(t)$) and switching selection for $w(t) = 0$ and $x_0 = [-0.93\ 0.2034\ 0.2]$ - Example VII.



Source: Author's own results.

9 CONCLUSIONS AND FUTURE RESEARCH

This chapter is devoted to draw the conclusions and discuss the future work perspectives.

9.1 CONCLUSIONS

Initially, in this work was presented in Theorem 4 a strategy to design an exclusive output-dependent switching strategy for controlling linear time-invariant continuous-time uncertain switched linear systems.

Theorem 5 shows that, if the known conditions of Theorem 3 hold, then the conditions proposed in Theorem 4 also hold. Furthermore, from simulations results (Examples I and II), the conditions proposed in Theorem 4 present a greater feasible region and reduce the guaranteed cost when compared with the conditions of Theorem 3. Therefore, the conditions proposed in Theorem 4 are less conservative than that presented in Theorem 3. The second control problem studied in this work was the robust switching static output feedback $\mathscr{H}_\infty$ control of continuous-time switched linear time-invariant systems. For a particular case of switched systems with only one subsystem, a proof in Theorem 8 shows that if the known conditions of Theorem 6 hold, then the conditions proposed in Theorem 7 also hold.

Additionally, from simulations results (Example III), the conditions proposed in Theorem 7 present a greater feasible region and reduce the $\mathscr{H}_\infty$ cost when compared with the conditions of Theorem 6. Therefore, the conditions proposed in Theorem 7 are less conservative than that presented in Theorem 6.

The conditions of the proposed methods are a special class of BMIs (Bilinear Matrix Inequalities), which contain some bilinear terms as the product of a matrix and a scalar, related to a suitable convex combination and two scalar parameters to provide extra free dimensions in the solution space. The hybrid algorithm DE-LMI is proposed for obtaining feasible solutions of this particular NP-hard problem.

In Example IV, it was presented a practical application on a semi-active suspension system. It was possible to observe a dynamic response improvement considering the reduction of the guaranteed cost, when compared with the results obtained considering the procedure presented in (CARDIM *et al.*, 2016). A second study regarding this problem, considering an uncertain bounded mass and a fault in the spring, confirms the effectiveness of the proposed approach.

Regarding actuator saturation and switched controllers, Examples V and VI explore the conditions of Theorem 9 and 10. These examples shows that for some cases the proposed conditions, considering switched controllers, hold while the conditions proposed in Chang, Park and Zhou (2015) does not. Moreover, it is possible to observe that the switched controllers method yields a better $\mathscr{H}_\infty$ bound (γ).

Finally, Example VII presents the robust $\mathscr{H}_\infty$ switched controllers designing, based on Theorem 10, for a switched linear system subject to actuator saturation. This examples shows that, even under actuator saturation, the systems stability is ensured for all $x(t) \in \mathcal{X}(\mathcal{N}_h)$ and the set constraints $\mathscr{E}(P(\alpha^*), \varphi^{-1}\varepsilon) \subset \mathcal{X}(\mathcal{N}_h)$ and $\mathscr{E}(P(\alpha^*), \varphi^{-1}\varepsilon) \subset \mathcal{G}_{ic}(\alpha^*)$ hold.

It is important to highlight that the proposed methods are LMI-based and consider a parameter-dependent Lyapunov function. Furthermore, based on recent result presented in the literature the design avoids linear matrices equalities and the need to impose any constraints on system matrices

9.2 FUTURE RESEARCH DIRECTIONS

As futures research directions, the following proposes are listed:

- Generalize the control design for uncertain linear discrete-time systems.
- Extend this work to cope with for uncertain nonlinear systems described by T–S fuzzy systems subject to actuator saturation.

9.3 PUBLICATIONS

- **CARNIATO, LEONARDO ATAIDE**; CARNIATO, ALEXANDRE ATAIDE; TEIXEIRA, MARCELO CARVALHO MINHOTO; CARDIM, RODRIGO; MAINARDI JUNIOR, EDSON ITALO; ASSUNÇÃO, EDVALDO . Output control of continuous-time uncertain switched linear systems via switched static output feedback. INTERNATIONAL JOURNAL OF CONTROL, v. 1, p. 1-20, 2018. https://doi.org/10.1080/00207179.2018.1495341

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