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Cosmological models from String Theory setups

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To my family

"I do not know what may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

Sir Isaac Newton

"Amicus Plato - amicus Aristotle - amicus Newton - amicus Einstein - magis amica veritas"

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Resumo

Nesta tese, discutimos três modelos cosmológicos que são baseados direta ou indiretamente em ideias advindas de Teoria das Cordas. Depois de uma revisão geral de Cosmologia em Teoria das Cordas, um resumo de Cosmologia e Teoria das Cordas é apresentado, com ênfase nos conceitos fundamentais e teóricos. Então descrevemos como o acoplamento camaleônico pode potencialmente afetar as predições de inflação cósmica com campo único, com tratamento cuidadoso dos modos de perturbação cosmológica adiabáticos e de entropia. Além disso uma nova abordagem para a dualidade-T em soluções cosmológicas de supergravidade bosônica é discutida no contexto de Teoria Dupla de Campos. Por fim, propomos uma nova prescrição para o mapa holográfico em cosmologia que pode ser usado para conectar modelos fundamentais de cosmologia holográfica com outras abordagens fenomenológicas.

Palavras Chaves: Cosmologia em Teoria das Cordas; Inflação Cósmica Camaleônica; Dualidade-T em Cosmologia; Cosmologia Holográfica.

Áreas do conhecimento: Cosmologia Teórica; Teoria das Cordas; Teoria de Campos

Abstract

In this thesis we discuss three cosmological models that are based directly or indirectly on String Theory ideas. After a quick overview on String Cosmology a summary of both Cosmology and String Theory is presented, with emphasis on the fundamental and theoretical concepts. We then describe how the chameleonic coupling may potentially affect the predictions of single field cosmological inflation, with a careful treatment of adiabatic and entropy modes of cosmological perturbations. Moreover, a novel approach for T-duality of cosmological solutions of bosonic supergravity is discussed in the framework of Double Field Theory. At last, we propose a new holographic map prescription for cosmology that could be used to connect top-down setups of holographic cosmology with other phenomenological approaches.

Key Words: String Cosmology; Chameleonic Cosmological Inflation; T-duality in Cosmology; Holographic Cosmology.

Areas of Knowledge: Theoretical Cosmology; String Theory; Field Theory.

Contents

Preface

The study of Cosmology raises profound questions about the origin, fate and nature of the Universe. By studying and observing the cosmological evolution we are exploring in a unique way what the laws of Physics allow or not. On the other hand, String Theory is a proposal to solve the deepest issues in Theoretical Physics and in order to make contact with what is observed, Cosmology should have a place in its framework. If String Theory really describes or not nature only experimental and observational data can tell, and in its present form it is not clear if it does or not. Regardless of that, we do not have to wait for a possible final *String Theory* to start exploring its cosmological applications as, similarly, we have not been waiting for a full consistent theory of Quantum Gravity to use field theory in Cosmology. The personal view of the author is that any result on *String Cosmology* that we could find using what we know about String Theory at present will be of importance for the final form of the theory.

String Cosmology is an exciting field with increasing activity in recent years though it is a speculative area from the physical point of view. Putting the reality of String Theory aside, most of the models assume a lot of approximations, or even start from non-realistic scenarios from scratch. Generally this is due to the absence of knowledge about String Theory in many regimes. On theoretical grounds, the author has the opinion that we should understand the stringy effects of the simplest cases first. Otherwise, if we fail to do so, what is the hope of success in more complex situations? Also, the tools used to get around many issues in this field may potentially contribute to the understanding of String Theory itself or even be applied in other fields. These facts motivate us to study String Cosmology even though it is not close to give definite answers to the real Physics issues.

Although the main goal of this thesis is to present the author's line of research and contributions to it, it was written to also provide an introduction to Cosmology for string theorists and String Theory for cosmologists. So, the second and third chapters are reviews on selected topics on Cosmology and String Theory. Unfortunately, due to the limitations of the author, topics on both fields will be excluded from the discussion. If the reader is already familiar with both Cosmology and String Theory, she could skip chapters 2 and 3 after reading the Introduction. Chapters 4, 5 and 6 are strongly based on the following papers, that had significant contribution from the author,

- *Conformal inflation with chameleon coupling.*
  Heliudson Bernardo et al JCAP04(2019)027. [1]

- *T-dual cosmological solutions in double field theory. II.*
  Heliudson Bernardo, Robert Brandenberger, and Guilherme Franzmann, Phys. Rev. D 99, 063521 (2019). [2]

- *Holographic cosmology from "dimensional reduction" of* $\mathcal{N} = 4$ *SYM vs.* $AdS_5 \times S^5$.
  Heliudson Bernardo and Horatiu Nastase, ArXiv: 1812.07586 (in process of publication). [3].

Chapter 1

Introduction

In this thesis, we will present results on three topics within the field of String Cosmology. These are cosmological models that are based on ideas of String Theory. The purpose of this chapter is twofold. Firstly we give a very broad motivation for why one should combine String Theory and Cosmology, followed by a small overview on the subject. Then we introduce the topics in which chapters 4, 5 and 6 are based.

1.1 General motivation: A crisis in fundamental Physics

The main paradigms of fundamental Physics, General Relativity and Quantum Mechanics, are based on concepts and entities that are fundamental in the sense that they cannot be described by any further level of reductionism. In General Relativity, it is assumed that spacetime is an absolute concept that cannot be explained by some deeper structure and all that there is to know about a spacetime is encoded in its metric, which is also fundamental. For instance, gravity is a manifestation of a spacetime's property, it's curvature. On the other hand, all the energy residing in a spacetime is described in terms of quantum fields that are systems ruled by Quantum Mechanics. The question of what are quantum fields and what happens when they interact does not have answers within Quantum Field Theory, since these fields and their interactions are taken to be fundamental. In other words, physicists do not try to explain what are the fundamental elements of their theories but rather explain everything else in terms of it.

Although there is no problem in assuming fundamental entities and explaining various phenomena in terms of them, the overall concepts and rules should be compatible with each other, that is, our fundamental description of nature should not be inconsistent. Otherwise this would indicate that the fundamental theories, concepts and entities we are using are not adequate and we should look for other elements from which we could build a consistent fundamental theory.

Unfortunately, the fact that General Relativity cannot be quantized without giving non-sensical results in some extreme energy regimes, puts us in a situation where the fundamental theories that we use to describe nature are inconsistent [4–7]. This is the deepest open problem in Theoretical Physics. It is worth emphasizing that we do not have access to experimental data in the regimes where our theoretical framework is conflicting. That is, from the experimental point of view, both General Relativity and Quantum Mechanics are excellent theories and actually they agree with each other [8,9]. At the present time, the issue is theoretical and experiments cannot directly help us in finding a framework or theory that solves this crisis, a self-consistent Quantum Gravity theory.

By definition, one place where both paradigms would be important to describe nature is close to singularities. According to General Relativity, singularities of astrophysical black holes are hidden behind event horizons and so any accessible data about them are outside our reach [10, 11]. Another type of singularity is of cosmological nature and, a priori, there is no no-go mechanism that prevents us from getting information on physics close to it. Moreover, modern Cosmology was constructed by combining General Relativity and Particle Physics and the agreement between all the cosmological observations and theories is astonishing [12, 13]. The use of cosmological data to constrain theories was well explored in the last few decades and in fact there is certain experimental information that could not be obtained in other ways.

But from the pure theoretical point of view, the best physicists can do in constructing compatible fundamental theories is to try to recover what we already know about nature. In simple terms, it is to recover General Relativity from some theory that is compatible with Quantum Mechanics. Then, once we have a self-consistent theory we could move on and explore the physical implications of it. Due to the energy regime where we expect to get predictions, one is prone to expect that Cosmology would be useful to check if such a theoretical construction describes nature or not.

1.2 Cosmology and String Theory

The final goal of Cosmology is the description of the universe on the largest scales and its time evolution as a whole. After years of theories and experiments, physicists around the world contributed to what is known as Standard Cosmological Model [13–16]. In this model, the cosmological history of the universe is described in concordance with all laws already tested in laboratory and astrophysical measurements, and it predicts that all the matter in the universe consisted of a quasi-homogeneous plasma in a state of high temperature and pressure that evolved during billions years to what we observe today. Since then, the spacetime itself is expanding and such expansion can be verified through measuring the spectrum of distant galaxies, that are receding from each other following what is known as Hubble's law.

One of the most remarkable characteristics of the observable universe is the fact that on scales bigger than 300 millions light-years, the distribution of matter is nearly homogeneous and isotropic. On such scales, we can consider the constituents of this matter distribution as galaxy clusters, each containing from a few to thousands of galaxies.

In primordial times, the hot dense plasma consisted of elementary particles interacting according to the set of physical laws that we call Standard Model of Particle Physics [17, 18]. Such particles collided with each other with an interaction rate that was decreasing as the universe expanded. This resulted in the fall off of the temperature and pressure of the plasma,

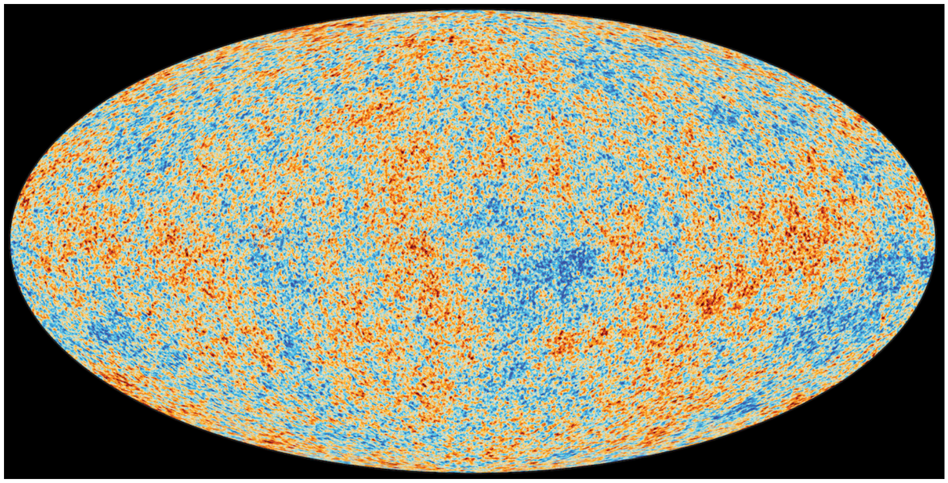


Figure 1.1: The Cosmic Microwave Background as seen from European Space Agency's Planck satellite [16]. It is a picture of the universe around 300,000 years after the moment where our cosmological models breaks. Credits: ESA/Planck Collaboration

allowing the growth of small primordial lumps on its density. Such initial irregularities got bigger under gravitational action and turned into the large scale structure we observe today, using telescopes.

Amazingly, we can witness at least part of the primordial cosmological perturbations from the cosmic background radiation (CMB). The CMB is a picture of the universe at the moment of the recombination, where the temperature of the plasma was cooler enough to allow electrons and protons to make bound states and stop scattering photons, that were then free to propagate in space. The radiation corresponds almost exactly to a blackbody with temperature of ~ 2.7K and has small anisotropies that seeded the large scales structures that we see today. Surprisingly, it seems that Quantum Mechanics plays an important role in explaining the origin of primordial inhomogeneities.

The theory that describes the gravitational collapse and the formation of structures in the universe is only successful when we assume that there exists a non-luminous form of matter that interacts very weakly with other particles besides gravitationally and that such "dark matter" dominates compared with ordinary matter, that is dubbed baryonic matter. The first evidence for dark matter came from observations of the trajectory of luminous matter in the periphery of galaxies. It was found that in order for these rotation curves to be compatible with Einstein's General Relativity theory (in its Newtonian limit), it was necessary the existence of a distribution of matter, other than the one visible, inside each galaxy. Thus, the Standard Cosmological model assumes the existence of dark matter.

At the second half of the 90's it was discovered by supernova measurements that the universe expansion is actually accelerated [19]. To describe this phase of acceleration, physicists introduced a dark energy component that enters into Einstein's equations as a cosmological constant. It was already known that a cosmological constant drives an accelerated expansion in a homogeneous and isotropic universe. The explanation of why the universe is expanding today is due to the fact that its total energy density is dominated by dark energy, though we really do not know the true nature of this component yet.

So, we have a coherent picture about the cosmic history of matter. Initially, the universe was dominated by radiation or relativistic matter, that has cooled and gave origin to a phase dominated by non-relativistic matter. Then, the matter density diminished until become smaller than the dark energy density, giving the actual accelerated era.

In fact, in the Standard Cosmology, the observable universe has flat spatial geometry and, today, its constitution is the following: $\sim 4\%$ of baryonic matter, $\sim 70\%$ of dark energy, $\sim 25\%$ of dark matter and the radiation (photons and neutrinos) percentage is negligible.

In spite of the success of the Standard Cosmological model in explaining observations, this model is commonly extended to explain why the universe has a flat geometry and how regions that appear to be causally disconnected have temperatures that are so close to each other (homogeneity). In the most popular extensions, the Universe has passed an era of accelerated expansion before being dominated by radiation. This era is called inflation, originally proposed in [20, 21]. During inflation, the space has expanded abruptly, going from microscopic to astronomical scales in a fraction of time of order 10^{-35} seconds. We do not understand completely what could generate this inflationary phase, because the energy scale involved in this period is larger than the ones tested in laboratories and particle accelerators today.

Furthermore, the most striking prediction of inflation is the origin of cosmological perturbations [21–23]. The small inhomogeneities that have grown have no explanation within the Standard Cosmological model. A natural candidate for them is the quantum fluctuations of the degree of freedom that dominates the energy density of the Universe during inflation. Remarkably, this idea is compatible with observations, and it is the most accepted explanation for the initial conditions of the perturbations. But is worth stressing that inflation *is not* the only competitive candidate mechanism that explains the primordial perturbations. There are models with periods of contraction, as matter bouncing cosmologies and Ekpyrosis that are not ruled out by the observations of cosmological fluctuations.

The fact that we need quantum theories to explain the Universe on large scales turns Cosmology into an area that corroborates fundamental particle physics theories. Although the Standard Model of Particle Physics is in accordance with all the tests done, including the cosmological tests, we do not know the true fundamental degrees of freedom and interactions at energy scales bigger than ~ 10 TeV (that corresponds to distances of order 10^{-19} m). Moreover, only three of the four fundamental interactions can be described quantum mechanically. Gravity possesses only a classical formulation, the Einstein General relativity, and is compatible with quantum mechanics only at small energies (compared with the Planck scale). Even at such small energies, we can consider it just as an effective field theory: the quantum version of General Relativity is non-renormalizable. Actually, the search for a quantum theory of gravity and its unification with the other interactions is the biggest goal of the theoretical physics.

In the last 60 years, through the work of various theoretical physicists, it was discovered that there exists a quantum theory that has various features that we expected from a unified theory. This is String Theory [24–27]. In this theory, the fundamental entities are quantum relativistic strings and their different vibrational patterns gives different quantum particles. While this theory predicts quantum gravity, it is only consistent when the spacetime has 10 dimensions, 6 more than what is observed. But since gravity is dynamical, in the phenomenological models the extra dimensions are small, each with the size of the order of Planck length 10^{-34}m and compactified in various manners, giving various different models from the 4-dimensional point of view. The goal of such models is to recover the Standard Model description, unifying all the interactions in this way. Unfortunately, since the differences in such models lie at so big scales, string theory cannot be tested in laboratories today.

So, recently, a new area has begun to be studied, to explore the best of Cosmology and String Theory: the applications of strings to cosmology or String Cosmology [28–34]. The goal is to explain the Standard Cosmological Model in the framework of String Theory. It is expected that in such explanation, we may obtain the first phenomenological implications of String Theory and the ultimate knowledge of cosmological questions. Among various applications, it is worth mentioning the description of initial conditions of the Universe, the implementation of inflation in String Theory, the use of AdS/CFT to cosmological models, brane universe models, description of the Standard Model vacuum, cosmological evolution of extra dimensions and the physical nature of the dark matter and dark energy.

1.3 Overview of String Cosmology

Most of the models in String Cosmology are attempts to embed inflation in String Theory [29]. The most convenient link with low energy physics is done by considering classes of compactifications that gives $\mathcal{N} = 1$ Supergravity in 4 dimensions (for constraints on Kähler potential and superpotential from inflation, see [35]). This procedure gives several fields in the 4 dimensional theory, organized in supersymmetric multiplets whose number depends on the details of the compactification, e.g. topology of internal manifold, fluxes and local sources. Most of them are scalar fields, called moduli fields, that should be stabilized in order for the model to be phenomenologically viable. But all fields have an explicit physical meaning, corresponding for example to positions of the local sources, volume of several different cycles of the internal manifolds and wrapping of $p-$forms around such cycles. The problem of embedding inflation in String Theory is then to find a consistent compactification with all fields stabilized in such a way that we get de Sitter space in 4 dimensions. Then, hopefully, it would be possible to deform this picture to get inflation, by adding new sources or displacing some moduli from their minima for instance.

The problem of stabilization of all moduli is a rich subject by itself and a proper discussion of it would be outside of the scope of our analysis. Suffice to say that in type IIB compactifications, it is possible to stabilize all the complex structure moduli and the dilaton moduli by using fluxes (see [36, 37] and references therein). Extra Kähler moduli are stabilized by non-perturbative effects [38,39]. These results were used by KKLT to find a supersymmetric Anti-de Sitter minimum by tuning fluxes [38]. Then, they introduced an extra positive term to the potential by adding a $\overline{D}3$-brane to "uplift" the minimum to de Sitter space, though there are several criticisms to this last step [40–43] (see also [44]). The KKLT model is the most popular model that claims to give a de Sitter vacuum in String Theory. There are other models also (see section 3.4 in [29], chapter 3 in [28] and references therein).

There are also models of inflation using branes [45–52]. Generally, in these models, the inflaton is the position of a D-brane on a warped geometry or even the separation distance of two branes. For a review on the subject, see chapter 2 of [28].

The debate about the existence of de Sitter vacua in String Theory has been recently increased due to the Swampland conjecture [44, 53, 54] (see [55] for a review). In its simpler formulation, it puts bounds on the slope of the scalar potentials coming from String Theory, in particular prohibiting a stable de Sitter space. Though it was not proven, the Swampland conjecture is based on several arguments of what we should expect from a theory of Quantum Gravity based on strings. It was applied to scalar field cosmological models [56], in particular to dark energy [57, 58]. Together with the fact that inflation is not the only way to get a spectrum of initial perturbations consistent with observations, this motivates us to look also for cosmological models that are alternatives to inflation within String Theory.

One of such alternative models is the Brandenberber-Vafa model, the String Gas Cosmology model (SGC) [59, 60], which is based on thermodynamics of strings and T-duality. By general arguments, based on the fact that there is a maximum effective physical temperature for a gas of strings, the model is free from singularities and has a natural mechanism to explain why only 3 spatial dimensions are decompactified. String Gas Cosmology predicts a blue-shifted spectrum of primordial gravitational waves, differing from usual inflation, in such a way that future observations could distinguish between the two. There are also other models based on the ideas of SGC, as the S-brane bouncing scenario [61–63]. It is worth emphasizing that SGC is the most "pure" stringy model for Cosmology, in the sense that it could not be reproduced from some effective field theory based solely on usual particles and fields. Further details of the SGC model are discussed in section 1.5 and chapter 5.

Another alternative model is Ekpyrosis [64]. Created using stringy ideas from $D = 11$ Supergravity and Horava-Witten theory [65], the ekpyrotic mechanism was rapidly detached from its origins and can now be seen as model based on scalar fields with a particular characteristic class of potentials (negative and highly steep), though it is difficult to explain the origin of the fields and its potentials [66, 67]. During the ekpyrotic phase, the universe contracts slowly but in such a way that the inhomogeneities do not increase to dominate the total energy of the Universe. A very interesting characteristic of the ekpyrotic solution is that it is an attractor solution for giving approximately scale invariant spectrum of perturbations, at the same level as inflation[1] [68].

1.4 Inflation with Chameleon Fields

Chameleon fields are scalar fields with a potential energy that depends on the energy density of the environment where it is defined (hence the name "chameleon") [69]. Such chameleonic

[1] But note that, with respect to the original proposal, at least two fields are required for generating nearly scale-invariant perturbations.

behaviour is due to a field dependent conformal factor in the metric that other sources perceive. It was first presented as a proposal to explain dark energy but with a natural screening mechanism built into it from scratch [70].

The idea that a scalar can have a "chameleon" coupling to the (non-relativistic) matter density was introduced, in part, to allow for a scalar that can be very light on cosmological scales while also "hiding" its effects from observations in the lab (on Earth), or in the Solar System [69,71]. Various laboratory searches have been initiated for such a scalar (e.g., [70,72–88]). From a theoretical perspective, this alleviates the problem of having too many a priori light scalars in string theory (moduli): if they are chameleons, they do not contradict known experiments to date. A way to embed chameleons in string theory was suggested in [89].

On the other hand, since the chameleon is a scalar, an economical ansatz is for the same field that acts as an inflaton near the Big Bang to be the chameleon. This idea was explored in [90]. In this case, however, the two regions (inflation and chameleon) are separated by a large region of vanishing potential in field space, and the inflationary era itself is not affected by the chameleon coupling. An attempt to consider what happens if we consider inflation in the presence of a chameleon or symmetron [91] coupling was considered in [92] and [93].

In chapter 4 the issue of inflation with a chameleon coupling is considered taking into account that there can be new "attractor-like" phases due to the chameleon coupling, where various forms of matter (contributions to the energy-momentum tensor) scale in the same way with the scale factor, as seen for instance in [90] at zero potential. Since we need to consider "new inflation" type of models with a plateau, a natural starting point is the system of "conformal inflation" models (see for instance [94, 95]), as analyzed in [96].

We will assume the existence of some heavy, non-relativistic matter with density ρ_m during the plateau phase (inflation), coupled to the inflaton via a chameleon coupling, $\rho_m F(\phi)$, and for the coupling the standard form $F = e^{-c\phi/M_{\rm Pl}}$. We will investigate the possibility of attractor-like behaviour due to this coupling, and see that depending on the sign of c, we can have either a prolonged period of attractor behaviour before the end inflation, or an effective inflationary potential that is different, with modified values for the CMB observables, n_s and r.

1.5 Cosmological Solutions in Double Field Theory

Target space duality [97–101] is a key symmetry of superstring theory. Qualitatively speaking, it states that physics on small compact spaces of radius R is equivalent to physics on large compact spaces of radius $1/R$ (in string units). This duality is a symmetry of the mass spectrum of free strings: to each momentum mode of energy n/R (where n is an integer) there is a winding mode of energy mR, where m is an integer. Hence, the spectrum is unchanged under the symmetry transformation $R \to 1/R$ if the winding and momentum quantum numbers m and n are interchanged. The energy of the string oscillatory modes is independent of R. This symmetry is obeyed by string interactions, and it is also supposed to hold at the non-perturbative level (see e.g. [24, 25]).

The exponential tower of string oscillatory modes leads to a maximal temperature for a gas of strings in thermal equilibrium, the Hagedorn temperature [102]. Combining these thermodynamic considerations with the T-duality symmetry leads to the proposal of *String Gas Comology* [59] (see also [103]), a nonsingular cosmological model in which the Universe loiters for a long time in a thermal state of strings just below the Hagedorn temperature, a state in which both momentum and winding modes are excited. This is the *'Hagedorn phase'*. After a phase transition in which the winding modes interact to decay into loops, the T-duality symmetry of the state is spontaneously broken, the equation of state of the matter gas changes to that of radiation, and the radiation phase of Standard Big Bang expansion can begin.

In addition to providing a nonsingular cosmology, String Gas Cosmology leads to an alternative to cosmological inflation for the origin of structure [104]: According to this picture, thermal fluctuations of strings in the Hagedorn phase lead to the observed inhomogeneities in the distribution of matter at late times. Making use of the holographic scaling of matter correlation functions in the Hagedorn phase, one obtains a scale-invariant spectrum of cosmological perturbations with a slightly red tilt, like the spectrum which simple models of inflation predict [104]. If the string scale corresponds to that of Grand Unification, then the observed amplitude of the spectrum emerges naturally. String gas cosmology also predicts a roughly scale-invariant spectrum of gravitational waves, but this time with a slightly blue tilt [105, 106], a prediction with which the scenario can be distinguished from simple inflationary models (see also [107, 108] for other distinctive predictions).

The phase transition at the end of the Hagedorn phase allows exactly three spatial dimensions to expand, the others being confined forever at the string scale by the winding and momentum modes about the extra dimension (see [109–112] for detailed discussions of this point). The dilaton can be stabilized by the addition of a gaugino condensation mechanism [113], without disrupting the stabilization of the radii of the extra dimensions. Gaugino condensation also leads to supersymmetry breaking at a high scale [114]. For detailed reviews of the String Gas Cosmology scenario see [32, 115].

However, an outstanding issue in String Gas Cosmology is to obtain a consistent description of the background space-time. Einstein gravity is clearly not applicable since it is not consistent with the basic T-duality symmetry of string theory. Dilaton gravity, as studied in *Pre-Big Bang Cosmology* [116, 117] is a promising starting point, but it also does not take into account the fact, discussed in detail in [59], that to each spatial dimension there are two position operators, the first one (x) dual to momentum, the second one ($\tilde{x}$) dual to winding. *Double Field Theory* (DFT) (see [118, 119] for original works and [120] for a detailed review) is a field theory model which is consistent both with the T-duality symmetry of string theory and the resulting doubling of the number of spatial coordinates (see also [121–123] for some early works). Hence, as a stepping stone towards understanding the dynamics of String Gas Cosmology it is of interest to study cosmological solutions of DFT.

In an initial paper [124], point particle motion in doubled space was studied, and it was argued that, when interpreted in terms of physical clocks, geodesics can be completed arbitrarily far into the past and future. In a next paper [125], the cosmological equations of dilaton gravity were studied with a matter source which has the equation of state of a gas of closed strings. Again, it was shown that the cosmological dynamics is non-singular. The full DFT equations of motion in the case of homogeneous and isotropic cosmology were then studied in [126]. The consistency of DFT with the underlying string theory leads to a constraint. In DFT, in general a stronger version of this constraint is used, namely the assumption that the fields only depend on one subset of the doubled coordinates. There are various possible frames which realize this (see also the discussion in section 3.4). In the *supergravity frame* it is assumed that the fields do not depend on the "doubled" coordinates $\tilde{x}$, while in the *winding frame* it is assumed that the fields only depend on $\tilde{x}$ and not on the x coordinates.

It was shown that for solutions with constant dilaton in the supergravity frame, the consistency of the equations demands that the equation of state of matter is that of relativistic radiation, while constant dilaton in the winding frame demands that the equation of state of matter is that of a gas of winding modes. These two solutions, however, are not T-dual. In chapter 5 solutions which are T-dual are introduced, expanding on the analysis of [126] and presenting improvements in the previous non-T-dual solutions.

1.6 Holographic Cosmology

The first indication that quantum gravity should have a holographic nature goes back to arguments about thermodynamics of black holes [127, 128]. In a modern view, holography in high energy physics deals with the description of quantum aspects of a spacetime with dimension D by using field theory defined on a flat background with dimension $D-1$. This area of research increased a lot since the explicit construction of the correspondence between String Theory on $AdS_5 \times S^5$ and $\mathcal{N}=4$ SYM by Maldacena [129] (see also [130,131]). Thenceforward, several gauge/gravity duality models were proposed and applied in many branches of theoretical physics, e.g. [132–135] (see [136] for applications in condensed matter).

The idea of a holographic cosmology has been around for a long time. The first concrete proposal of how that would look like was put forward by Maldacena in [137] (see also [138]), stating that the wave function of the Universe, as a function of spatial 3-metrics (and scalars), $\psi[h_{ij}, \phi]$ in some gravity dual background (in his specific case, proposed for some space that asymptotes to de Sitter), equals the partition function of some (3 dimensional) field theory, with sources (for the energy-momentum tensor T_{ij} and some scalar operator $\mathcal{O}$) h_{ij}, ϕ, i.e., $Z[h_{ij}, \phi] = \psi[h_{ij}, \phi]$. However, at the time, there was no concrete proposal for a gravity dual pair.

In [139], such a model was proposed, and a sort of phenomenological holographic cosmology approach was born. It was first noted that, for cosmological scale factors $a(t)$ that are both exponential (as in standard inflation, and corresponding to AdS space) or power law (as in power law inflation, and corresponding to nonconformal D-branes, for instance), a specific Wick rotation, the "domain wall/cosmology correspondence", turns the cosmology into a standard holographic space like a domain wall, that should have a field theory dual in 3 Euclidean dimensions. A holographic computation then relates the cosmological power spectrum, coming from the $\langle \delta h_{ij}(\vec{x}) \delta h_{kl}(\vec{y}) \rangle$ correlators in the bulk, with $\langle T_{ij}(\vec{x}) T_{kl}(\vec{y}) \rangle$ correlators in the boundary field theory. One can assume a regime where the field theory is perturbative, and the latter correlators can be calculated from Feynman diagrams. Then by comparing the cosmological power spectrum with CMBR data, we can find the best fit in a phenomenological class of field theories, with a "generalized conformal structure". In [140, 141] it was shown that the phenomenological fit matches the CMBR as well as the (different) standard ΛCDM with inflation, though the perturbative field theory approximation breaks down for modes with $l < 30$. But this holographic cosmology paradigm is more general than the specific class of phenomenological models: it includes standard inflationary cosmology, where the gravitational side is weakly coupled, as well as intermediate coupling field theory models, that can be treated non-perturbatively on the lattice.

Another approach to holographic cosmology was considered in [142–144], where one starts with a "top down" construction, specifically a modified version of the original $\mathcal{N} = 4$ SYM vs. string theory in $AdS_5 \times S^5$, where an FLRW cosmology with $a(t)$ replaces the Minkowski metric, and a nontrivial dilaton is introduced. On the field theory side, one has a time-dependent coupling now. The model has been used in [143, 144] to show how perturbations entering a Big Crunch exit after the Big Bang, one issue that has been very contentious in ekpyrotic and cyclic cosmologies. It was shown that the spectral index of perturbations exits unchanged, but there was no simple mechanism in [143,144] of calculating the power spectrum of fluctuations for CMBR.

Chapter 6 is a proposal for modifying the top down construction of [142–144], to fit it into the holographic paradigm of [139], for which the common concrete realization so far is a phenomenological (bottom up) approach. We will find that we can modify the general proposal of Maldacena for $Z[h_{ij}, \phi] = \psi[h_{ij}, \phi]$ to deal with this case of having both time and a radial coordinate, and then use an integration over the time coordinate, from close to zero until an arbitrary time t_0 (but not to the future of it, in this way obtaining a function of t_0), to argue that we have effectively a "dimensional reduction" over the time direction.

1.7 Notation and structure of the thesis

The metric signature used throughout the thesis is mostly plus $(-, +, \ldots, +)$ and natural units are adopted otherwise stated the contrary. In chapter 3, we tried to use the same notation as references [24] and [25] and for that reason the spacetime metric is denoted as $G_{\mu\nu}$ in that chapter rather than $g_{\mu\nu}$ as in chapter 2. The structure of the thesis is the following: in chapter 2 we review the theoretical foundations of Cosmology; in chapter 3 we discuss the main ideas of String Theory, Double Field Theory and Holography; the impact of the chameleonic coulpling to single field inflation is presented in chapter 4, for the particular case of conformal inflation; in chapter 5, T-dual cosmological solutions of DFT are presented and chapter 6 describes a novel prescription for holographic cosmology. A summary of the conclusions is presented in chapter 7 and a small digression about calculations used in chapter 6 is provided in appendix A.

Chapter 2

Basics of Cosmology

The Standard Cosmological Model used nowadays is based on two observational facts: the universe is expanding and is very homogeneous and isotropic at large scales. The former statement refers to the space itself (at large scales) and the last is called *Cosmological Principle*.

The expansion of the universe was discovered in 1929 by Edwin Hubble via measurements of the redshift of distant galaxies. Statistically, he discovered that the velocity v of the galaxies depended on its radial distance r by the relation

$$v = Hr, \tag{2.1}$$

where H was a constant, the *Hubble parameter*. This is the Hubble law for an expanding Universe. Physically, it is just the statement that each galaxy is moving away from our galaxy and from each other. More remarkably in 1998 the Universe was discovered to have an accelerated expansion [19] so the Hubble constant has not a fixed value along cosmic evolution. Today[1] the value of Hubble constant is given by

$$H_0 = 100h \frac{\mathrm{km}}{\mathrm{s} \cdot \mathrm{Mpc}}, \tag{2.2}$$

with h present value in the range $0.6 < h < 0.8$ [145].

The isotropy and homogeneity of the energy distribution of the universe is probed by sky survey observations and the measurements of the Cosmic Microwave Background Radiation (CMBR or simply CMB). On scales larger than 300 million light years, the matter distribution acts like a "cosmic fluid" made of clusters of galaxies containing a few dozens to hundreds of galaxies each. It is highly homogeneous at each point but not totally, since it has small inhomogeneities, such that visually the distribution has a "cosmic-web" structure.

The purpose of this chapter is to quickly review the basic aspects of Cosmology. We discuss standard cosmology and inflation, following biased topics that are thought to be a complete set needed to understand latter chapters. General references can be found in [14, 146, 147] and for the values of cosmological parameters see [16, 145, 148].

[1] In Cosmology one works with very large spatial and temporal scales, so when one talks about the "present" one is usually referring to a time scale small compared to billion years.

2.1 Kinematics and dynamics of an expanding Universe

From the symmetry requirements of the Cosmological Principle we get a specific form for the spacetime metric of the universe at large scales, the Friedman-Lemaître-Robertson-Walker (FLRW) metric. Using comoving coordinates, that is, coordinates of observers that are at rest with respect to the expansion, we have

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right), \tag{2.3}$$

where k can take only three values corresponding to three possible geometries of the spatial section: for $k = 1$ the space is closed (a three-sphere S^3), for $k = 0$ the space is flat (an Euclidean plane E^3) and for $k = -1$ we have an open space (a hyperbolic space H^3). The scale factor $a(t)$ is defined up to a constant rescaling that can be chosen such that $a(t_0) = 1$ and can be increasing or decreasing, describing expanding or contracting universes and its time dependence is related with the energy content of the Universe via Einstein's equations.

The Hubble law can be obtained from the FLRW metric, regardless of the functional form of the scale factor. Suppose that the coordinates are such that our galaxy is located at $r = 0$ and there is another galaxy at $r = r_g$ from which we want to check the Hubble law. At a proper time t, the proper distance to the far away galaxy is

$$d(t) = a(t) \int_0^{r_g} \frac{dr}{(1 - kr^2)^{1/2}} = \begin{cases} a(t) \sin^{-1}(r) & (\text{for} \quad k = 1); \\ a(t) r & (\text{for} \quad k = 0); \\ a(t) \sinh^{-1}(r) & (\text{for} \quad k = -1), \end{cases} \tag{2.4}$$

and in an expanding universe, the velocity of the galaxy recession as seen by us is

$$v = \dot{d}(t) = \frac{\dot{a}(t)}{a(t)} d(t), \tag{2.5}$$

which is the Hubble law with Hubble parameter given by

$$H(t) = \frac{\dot{a}(t)}{a(t)}. \tag{2.6}$$

We see that the Hubble law is a kinematic feature independent of the cosmological solution for $a(t)$.

Another kinematic feature is the redshift of distant sources due to the expansion of space. Consider two pulses of light emitted from a distant source located at $r = r_g$ at times $t = t_g$ and $t = t_g + \delta t_g$ and received at $r = 0$ at times t_0 and $t_0 + \delta t_0$, respectively. Since the comoving distance covered by the pulses is the same, we have

$$\int_{t_g}^{t_0} \frac{dt}{a(t)} = \int_{t_g + \delta t_g}^{t_0 + \delta t_0} \frac{dt}{a(t)}. \tag{2.7}$$

Assuming $a(t)$ to be constant over small time intervals δt_g and δt_0, we get

$$\frac{\delta t_g}{a(t_g)} = \frac{\delta t_0}{a(t_0)}, \tag{2.8}$$

so there is a shift z between the wavelength of the emitted and received light, λ_g and λ_0 respectively, given by

$$z = \frac{\lambda_0 - \lambda_g}{\lambda_g} = \frac{a(t_0)}{a(t_g)} - 1. \tag{2.9}$$

It is a redshift for an expanding universe and blueshift for a contracting one. This cosmological shift is due to the expansion of the space between the emitter and receiver and so is different from the Doppler shift, where the source of light and the receiver have a relative motion in the same inertial frame.

The last kinematic feature worth stressing is the existence of horizons in an expanding universe. Intuitively, from Hubble's law, two sufficiently distant points will be moving away from each other faster than the speed of light and then be causally disconnected by the expansion of the universe. To formalize this idea one defines the *cosmological event horizon*, as the maximum distance that light will be able to travel if emitted at time t:

$$d_{\mathrm{eh}}(t) = \int_t^\infty \frac{dt'}{a(t')}, \tag{2.10}$$

then by definition, at a given time, an event cannot affect any other event if their separation is greater than d_{eh}. There is also the *particle horizon* at time t, which is the distance that light could have travelled since $t = 0$:

$$d_{\mathrm{ph}}(t) = \int_0^t \frac{dt'}{a(t')}. \tag{2.11}$$

Finding some correlation between two points with comoving spatial separation greater than d_{ph} is very unlikely, as these points could never be in causal contact. In section 2.4 we will see that this situation happens to be true for different points in the CMB.

Another characteristical distance is the *Hubble horizon, distance or radius*, given by H^{-1}. The equations describing perturbations in an expanding universe are such that super-Hubble effects are negligible to the dynamics on scales well inside the Hubble radius. In fact, at such small scales and at time intervals much smaller than the Hubble distance, one could neglect the expansion and the physics would be the same as in Minkowski spacetime.

Until now we have focused on the kinematic features of an expanding universe. Turning to the dynamics, the equations ruling the evolution of $a(t)$ are given by Einstein's field equations, sourced by a perfect fluid energy-momentum tensor

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{2.12}$$

where, on the grounds of homogeneity and isotropy,

$$T_{\mu\nu} = p(t) g_{\mu\nu} + (\rho(t) + p(t))\, u_\mu u_\nu, \tag{2.13}$$

where $p(t)$ and $\rho(t)$ are the time-dependent pressure and energy densities of the fluid, respectively, and u^μ is the velocity vector field of the fluid. Then, equation (2.12) relates $a(t)$, k, $\rho(t)$ and $p(t)$. Its time-time component 00 gives the first Friedmann equation,

$$\dot{a}^2(t) + k = \frac{8\pi G}{3} \rho(t) a^2(t), \tag{2.14}$$

and each diagonal spatial components ii gives

$$2a(t)\ddot{a}(t) + \dot{a}^2(t) + k = -8\pi G p(t) a^2(t). \tag{2.15}$$

Combining these equations by derivating the first with respect to time, we get

$$\dot{\rho}(t) + 3H(t)(\rho(t) + p(t)) = 0, \tag{2.16}$$

which is exactly the continuity equation for the energy-momentum tensor, $\nabla_\mu T^\mu_\nu = 0$. It is also possible to combine equations (2.14) and (2.15) to obtain the second Friedmann equation

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho(t) + 3p(t)), \tag{2.17}$$

from which we can see that the sign of $\rho+3p$ determines whether the expansion is accelerating or decelerating.

From the first Friedmann equation (2.14), for $\rho(t) > 0$ and $k = 0$ or $k = -1$ (flat and open universes), $\dot{a}^2(t)$ will not vanish and so the universe expands forever. While for $k = 1$ (closed universe), the expansion stops at some time t_c defined by $\rho(t_c)a^2(t_c) = 3/8\pi G$, after which the universe start to contract. Defining the critical density

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{2.18}$$

we can recast the condition of open, flat and closed universes as $\rho < \rho_c$, $\rho = \rho_c$ and $\rho > \rho_c$, respectively. Measuring the total energy density in the universe today and at CMB redshift, we know that the Universe is very close to flat [145]. So, from now on we focus on the case $k = 0$. In order to find the time evolution of $a(t)$, $\rho(t)$ and $p(t)$, we need an equation of state relating energy and pressure. Assuming a constant relation, we have

$$p(t) = w\rho(t), \tag{2.19}$$

where the constant equation of state parameter w has different values depending on the nature of the cosmic fluid. Then we can find the solutions, for $w \neq -1$

$$a(t) \propto t^{\frac{2}{3(1+w)}}, \quad \rho(t) \propto a^{3(1+w)}(t) \propto t^2 \tag{2.20}$$

There are 3 simple but useful cases for w:

- *Radiation,* $w = 1/3$. In this case the energy-momentum tensor can be obtained from the electromagnetic action, which gives a traceless $T_{\mu\nu}$. This implies $w = 1/3$ for an homogeneous and isotropic distribution of photons. We then have $\rho \propto a^{-4}$ and $a(t) \propto t^{1/2}$.

- *Non-relativistic matter or dust,* $w = 0$. In this case the rest energy dominates of the kinetic energy of the components of the fluid and the pressure is negligible. The energy density scales as $\rho \propto a^{-3}$ and the scale factor evolves as $a \propto t^{2/3}$.

- *Cosmological constant or dark energy,* $w = -1$. Related to the vacuum energy density, this case corresponds to an energy momentum tensor proportional to the metric, $T_{\mu\nu} = M_{\rm Pl}^2 \Lambda g_{\mu\nu}$. The continuity equation gives a constant energy density, which implies an exponential expansion, $a \propto e^{Ht}$ with $H = \sqrt{\Lambda/3}$. Then, the FRW metric covers a patch of the de Sitter spacetime.

In Standard Cosmology the Universe is assumed to be initially in thermal equilibrium in a radiation phase. Then, after two phase transitions (hadronic and electroweak), it becomes dominated by non-relativistic matter. Finally, recent observations strongly support an accelerated phase that can be modelled by dark energy.

All the discussion in this section applies under the assumption of exact homogeneity and isotropy, an unperturbed universe. But in reality we have cosmological structures and so our Universe is not exactly unperturbed for sure. In the next section we discuss cosmological perturbations of an homogeneous and isotropic universe and their observational implications.

2.2 Cosmological Perturbations

At each fixed time, the comoving coordinates used to write (2.3) define a time slice of the spacetime that has uniform energy density and is orthogonal to comoving worldlines. Considering the $k = 0$ case it will also be flat. These slices are labelled with the time coordinate t, but there is another possible coordinate choice, the conformal time τ defined as $d\tau = a dt$. Once we perturb the background, it is impossible to find a coordinate system that preserves all these features of the unperturbed universe. We need to write the metric in a modified way such that (2.3) is retrieved for zero perturbations. There are various different choices or *gauges*. Each gauge could still have some properties of the unperturbed metric, but not all. Then, one may choose to fix the gauge and work on the perturbation or to construct gauge invariant variables and write all equations in terms of them. In the following, we will consider the gauge fixing approach.

The perturbation of the metric corresponds to 10 degrees of freedom that can be decomposed as scalars, vectors and tensors with respect to 3-rotations. In fact, consider the most general form of the perturbed metric in conformal time,

$$ds^2 = a^2(\tau)\left\{-(1+2\phi)d\tau^2 - 2S_i d\tau dx^i + [(1+2\psi)\delta_{ij} + E_{ij}]\, dx^i dx^j\right\}. \tag{2.21}$$

The spacetime functions $\phi(x^\mu)$, $\psi(x^\mu)$ are two of the 4 scalar degrees of freedom. The other two, $B(x^\mu)$ and $E(x^\mu)$, lie within the decomposition of $S_i(x^\mu)$ and $E_{ij}(x^\mu)$:

$$\begin{aligned} S_i &= \partial_i B + B_i, \\ E_{ij} &= 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E + 2\partial_{(i}E_{j)} + h_{ij}, \end{aligned} \tag{2.22}$$

where B_i, E_i and h_{ij} are divergenceless, with h_{ij} traceless, giving 4 vector and 2 tensor degrees of freedom.

Note that by perturbing the metric, we are also perturbing the velocity vector field of any fluid on the spacetime. Indeed from $u^\mu u_\nu = -1$, we get $2u_\mu \delta u^\mu = -\delta g_{\mu\nu} u^\mu u^\nu$. To first order in perturbations and in (τ, x^i) coordinates, the perturbed 4-velocity is

$$u^\mu(x^\mu) = a^{-1}(\tau)\left(1 - \phi(x^\mu), v^i(x^\mu)\right), \tag{2.23}$$

where the coordinate 3-velocity $v^i = v_i$ is first order in perturbation. Then using (2.13), the perturbed energy momentum tensor components in (τ, x^i) coordinates are

$$\begin{aligned} T_0{}^0 &= -(\rho + \delta\rho), \\ T_i{}^0 &= (\rho + p)(v_i - B_i), \\ T_0{}^i &= -(\rho + p)v^i, \\ T_j{}^i &= (p + \delta p)\delta_j^i + \Pi_j^i, \end{aligned} \tag{2.24}$$

where the traceless Π_j^i is an anisotropic stress perturbation.

At linear order, the Einstein's equations does not mix scalar, vector and tensor perturbations. We will not discuss vector perturbations, since they decay quickly in our expanding Universe and do not play a big role in large scale structure formation. Tensor perturbations may be interpreted as gravitational waves and they are discussed in section 2.4, in the context of inflation. In the rest of this thesis we shall focus on scalar perturbations, as they have more impact on the CMB anisotropies.

The perturbations of the metric and energy-momentum tensor are not invariant under coordinate transformations,

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x^\nu). \tag{2.25}$$

The infinitesimal parameters ξ^μ have 4 degrees of freedom, that give rise to "fake" metric fluctuations. Decomposing $\xi^i = \partial^i \xi + \xi_\perp^i$ with $\partial_i \xi_\perp^i = 0$, the components ξ^0 and ξ correspond to scalar fluctuations, while $\xi_\perp^i$ is related to vector mode perturbations. As tensor fluctuations are gauge invariant, coordinate transformations cannot produce fake tensor perturbations.

Using the expression for the transformation of the components of any rank-2 tensor $A_{\mu\nu}$,

$$\tilde{A}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} A_{\alpha\beta}(x), \tag{2.26}$$

it is possible to find the transformations for scalar mode perturbations in the metric,

$$\phi \to \tilde{\phi} = \phi - \xi^{0\prime} - aH\xi^0, \tag{2.27}$$

$$\psi \to \tilde{\psi} = \psi - aH\xi^0 - \frac{1}{3}\nabla^2\xi, \tag{2.28}$$

$$B \to \tilde{B} = B + \xi^0 - \xi', \tag{2.29}$$

$$E \to \tilde{E} = E - \xi^0, \tag{2.30}$$

and in the energy-momentum tensor,

$$\delta\rho \to \tilde{\delta\rho} = \delta\rho - \xi^0 \rho', \tag{2.31}$$

$$\delta p \to \tilde{\delta p} = \delta p - \xi^0 p', \tag{2.32}$$

$$v \to \tilde{v} = v + \xi', \tag{2.33}$$

$$\Pi \to \tilde{\Pi} = \Pi, \tag{2.34}$$

where Π and v are the scalar parts of Π_{ij} and v^i, respectively. The prime in the equations above denotes derivative with respect to conformal time.

There are special combinations of the metric and energy-momentum tensor perturbations that are gauge invariant. For the metric, we have the *Bardeen variables*,

$$\Phi = \phi + B' - E'' + aH(B - E'), \tag{2.35}$$

$$\Psi = -\psi - aH(B - E') + \frac{1}{3}\nabla^2 E, \tag{2.36}$$

and for the energy-momentum tensor, we have the comoving density perturbation,

$$\Delta = \frac{\delta\rho}{\rho} + \frac{\rho'}{\rho}(v + B). \tag{2.37}$$

Using coordinate transformations, we can set two of the 8 scalar perturbations in the metric and in the cosmic fluid to zero. Each choice defines a different gauge[2]. The most common gauges in cosmology are the following:

- *Newtonian or longitudinal gauge:* $B = 0 = E$. In this gauge, the time slices are isotropic and orthogonal to the wordlines of comoving observers. In terms of Bardeen variables, $\Phi = \phi$ and $\Psi = -\psi$, and the form of the metric

  $$ds^2 = a^2(\tau)\left[-(1 + 2\Phi)d\tau^2 + (1 - 2\Psi)\delta_{ij}dx^i dx^j\right], \tag{2.38}$$

  is similar to the one valid in regions of spacetime containing weak gravitational fields (hence the gauge's name), the difference being that for the latter metric $\Phi = \Psi$. We will see that this is exactly the case for zero anisotropic stress.

- *Spatially-flat or uniform curvature gauge:* $\psi = 0 = E$. As the name indicates, in this gauge the constant time hypersurfaces are flat. This choice simplifies curvature perturbation computations.

- *Synchronous gauge:* $\phi = 0 = B$. In this case the wordlines of constant x^i are geodesics and the time slices are orthongonal to them. Since there is no time lapse, the clocks of observers within a Hubble radius are synchronized, giving the name for this gauge.

One also defines the *uniform density gauge*, with $\delta\rho = 0$, and *comoving gauge*, with $v = 0$, though they do not fix the gauge redundancy entirely since only the choice of ξ^0 is used. There are different versions of these gauges for different uses of the ξ choice (most commonly, $B = 0$ is assumed).

Now we discuss the dynamical equations for perturbations. They are obtained from the conservation of the energy momentum tensor and Einstein's equations at first order in perturbation theory (which are not totally independent due to the Bianchi identity). To get

[2]Some gauge choices do not fix all the gauge invariance, as there could still be residual coordinate transformations that preserves the gauge fixing. For us, it is a important fact that for the longitudinal gauge, if ϕ and ψ vanishes at infinity, the gauge is completely fixed.

gauge invariant equations, we work in longitudinal gauge and then use $\Phi = \phi$ and $\Psi = -\psi$ to write the final equations in terms of gauge invariant metric perturbations.[3]

From the 0-component of the conservation equation $\nabla_\mu T^\mu_\nu = 0$, we get the continuity equation for the perturbations,

$$\delta' + (1+w)\left(\partial_i v^i - 3\Phi'\right) - 3aHw\left(\delta - \frac{\delta p}{p}\right) = 0, \tag{2.39}$$

where $\delta \equiv \delta\rho/\rho$. The divergence of the perturbed Euler equation, coming from the i-components, gives

$$(\partial_i v^i)' + aH(1-3w)\partial_i v^i + \frac{w'}{1+w}\partial_i v^i + \frac{1}{\rho+p}\nabla^2\delta p - \frac{2}{3}\frac{w}{1+w}\nabla^2\Pi = 0. \tag{2.40}$$

The 00-component of Einstein equations gives,

$$\nabla^2\Psi - 3aH(\Psi' + aH\Phi) = 4\pi G a^2\rho\delta, \tag{2.41}$$

while the scalar part of the $0i$-component yields

$$\Psi' + aH\Phi = -4\pi G a^2(\rho+p)v. \tag{2.42}$$

The ij-component of Einstein's equations have a trace and trace-free scalar parts. From the former we obtain

$$\Psi'' + \frac{1}{3}\nabla^2(\Phi - \Psi) + \left(2(aH)' + (aH)^2\right)\Phi + aH(\Phi' + 2\Psi') = 4\pi G a^2\delta p, \tag{2.43}$$

and from the latter we get

$$\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)(\Psi - \Phi) = 8\pi G a^2\Pi_{ij}. \tag{2.44}$$

In the absence of anisotropic stress tensor, Π_{ij}, equation (2.44) gives $\Phi = \Psi$, and we can write all the equations in terms of the gravitational potential Φ. At first order, we can assume that to be the case.

Combining equations (2.41) and (2.42) we get the gauge invariant version of the Poisson equation,

$$\nabla^2\Phi = 4\pi G a^2\rho\Delta, \tag{2.45}$$

from which we have that the gauge invariant function Δ defined in (2.37) can be interpreted as the overdensity of energy in comoving gauge ($v = 0 = B$).

The gauge invariant comoving curvature perturbation, $\mathcal{R}(x^\mu)$, defined as

$$\mathcal{R} = \psi - \frac{1}{3}\nabla^2 E + aH(B + v), \tag{2.46}$$

[3] Another possibility is to find the gauge transformation for the perturbation in Einstein and energy-momentum tensor and construct combinations of these perturbations to get gauge invariant equations, as done in [149]

is the curvature perturbation in the slices of constant τ in the comoving gauge. Indeed, using the perturbed metric (2.21), the intrisic curvature of the constant time hypersurfaces is

$$R_{(3)} = -\frac{4}{a^2}\nabla^2\left(\psi - \frac{1}{3}\nabla^2 E\right), \tag{2.47}$$

and we see that $\mathcal{R}$ reduces to the scalar curvature perturbation

$$\psi - \frac{1}{3}\nabla^2 E, \tag{2.48}$$

in the comoving gauge. There is also the uniform density curvature perturbation, $\zeta(x^\mu)$, defined as

$$\zeta = \psi - \frac{1}{3}\nabla^2 E - aH\frac{\delta\rho}{\rho'}, \tag{2.49}$$

that is also gauge invariant and reduces to the scalar curvature perturbation for $\delta\rho = 0$. It is related with $\mathcal{R}$ by

$$\mathcal{R} = \zeta - \frac{2}{9(1+w)}\frac{\nabla^2\Phi}{(aH)^2}, \tag{2.50}$$

as one can show using equation (2.45). So, on super-Hubble scales, we have $\mathcal{R} = \zeta$.

A crucial property of the comoving and uniform density perturbations is the fact that it does not evolve outside the Hubble radius under some assumptions on the energy-momentum tensor perturbations. For vanishing anisotropic stress, we have

$$\frac{3}{2}(aH)^2(1+w)\mathcal{R}' = -4\pi G a^2 aH\left(\delta p - \frac{p'}{\rho'}\delta\rho\right) + aH\frac{p'}{\rho'}\nabla^2\Phi, \tag{2.51}$$

and since the last term is negligible for superhorizon modes, a barotropic equation of state,

$$p = p(\rho), \tag{2.52}$$

implies that $\mathcal{R}$ and ζ are frozen outside the Hubble radius. The condition (2.52) is an adiabatic condition, that implies that the perturbations in the pressure can be written as perturbations in the energy density. In presence of different fluid components ρ_i with equation of state w_i, from the adiabatic condition for energy densities

$$\rho_i = \rho_i(\rho), \tag{2.53}$$

where ρ is the sum of all ρ_i, we have that the individual perturbations $\delta\rho_i$ are zero in the uniform density energy gauge, on which $\delta\rho = 0$. For other slicings, $\delta\rho_i$ are obtained by a common local shift in time,

$$\delta\rho_i = -\rho_i'\delta\tau, \tag{2.54}$$

that together with the continuity equation (and in absence of energy transfer between the fluid components), gives

$$\frac{\delta_i}{1+w_i} = \frac{\delta_j}{1+w_j}. \tag{2.55}$$

In cosmological evolution, the fluid components that will be relevant correspond to matter δ_m and radiation δ_r. Then (2.55) gives $\delta_r = 4\delta_m/3$. So, all adiabatic perturbations are related and can be written in terms of the total energy density perturbation $\delta\rho$. There are also isocurvature or entropy perturbations, that correspond to uncorrelated perturbations between different components, but all observations are consistent with purely adiabatic perturbations.

The fact that the gauge invariant curvature perturbations are conserved outside the horizon is a very important result. It is used to explain how microphysics may generate primordial perturbations, as if we have a mechanism to make the perturbation modes to get out of the Hubble horizon and then get back inside it, then the initial conditions for the CMB perturbations have a casual cause. All proposals for explaining structure formation explore this idea, the most promising and famous being inflation, to be discussed latter in this chapter.

2.3 The Cosmic Microwave Background Radiation

2.3.1 CMB Anisotropies

The Cosmic Microwave Background Radiation (CMBR or simply CMB) is the radiation permeating all observable universe and corresponds to photons travelling freely since recombination, a moment in time when the electrons and protons first combined to form H atoms and Thomson scattering ($e + \gamma \to e + \gamma$) was not enough to keep the baryons and photons in equilibrium. The CMB has an approximate thermal blackbody spectrum with temperature $\bar{T} \simeq 2.7$K.

At the level of background cosmology, the CMB defines a frame where it is isotropic, due to homogeneity of the Universe at the time of recombination. When we observe CMB today, the motion of the Solar System with respect to this frame produces a dipolar effect in the observed temperature.

The relation between the momentum of an observed photon coming from direction $\hat{n}$ in the sky, $p_o(\hat{n})$, and its momentum p in the CMB rest frame is given by

$$p_o(\hat{n}) = \frac{p}{\gamma(v)(1 - \hat{n} \cdot \vec{v})} \approx p(1 + \hat{n} \cdot \vec{v}), \tag{2.56}$$

where $\vec{v}$ is the Solar System velocity and $\gamma(v)$ is its relativistic gamma factor. Since CMB has a blackbody spectrum and $T \propto 1/a$, we can write this change in momenta as a change in the observed temperature of the CMB:

$$\frac{\delta T(\hat{n})}{\bar{T}} \equiv \frac{T_o(\hat{n}) - \bar{T}}{\bar{T}} = \frac{p_o(\hat{n}) - p}{p} = \hat{n} \cdot \vec{v} = v \cos\theta. \tag{2.57}$$

From CMB observations we find $v \approx 368$km/s.

To find how the photons are affected by cosmological perturbations let us consider their geodesics in a perturbed background (in longitudinal gauge),

$$ds^2 = a^2(\tau)[-(1 + 2\Phi)d\tau^2 + (1 - 2\Psi)\delta_{ij}dx^i dx^j]. \tag{2.58}$$

From the 0th component of the geodesic equation, we get

$$\frac{d}{d\tau}\ln(ap) = -\frac{d\Phi}{d\tau} + \frac{\partial}{\partial\tau}(\Phi + \Psi), \tag{2.59}$$

where p is the photon momentum. Integrating this equation along a line-of-sight, gives

$$\ln(ap)|_0 = \ln(ap)|_* + (\Phi_* - \Phi_0) + \int_{\tau_*}^{\tau_0} d\tau \partial_\tau(\Phi + \Psi), \tag{2.60}$$

where we assume that all photons were emitted at a fixed time τ_*, i.e., an instantaneous recombination. We call τ_* the moment of last scattering.

To relate this result with temperature anisotropies, we use

$$ap \propto a\bar{T}\left(1 + \frac{\delta T}{\bar{T}}\right) \tag{2.61}$$

to get

$$\left.\frac{\delta T}{\bar{T}}\right|_0 = \left.\frac{\delta T}{\bar{T}}\right|_* + (\Phi_* - \Phi_0) + \int_{\tau*}^{\tau_0} d\tau \partial_\tau(\Phi + \Psi) \tag{2.62}$$

The term Φ_0 only affects the monopole perturbation, and so is unobservable and we'll drop it from the equation. Since $\rho_\gamma \propto \bar{T}^4 \implies \delta\rho_\gamma \propto 4\bar{T}^3\delta T = 4\rho_\gamma \delta T/\bar{T}$, we have

$$\left.\frac{\delta T}{\bar{T}}\right|_0 = \left(\frac{1}{4}\delta_\gamma + \Phi\right)_* + \int_{\tau_*}^{\tau_0} d\tau \partial_\tau(\Phi + \Psi). \tag{2.63}$$

The last term is called Integrated Sachs-Wolfe (ISW) term and it vanishes during matter domination, when $\dot{\Phi} \approx \dot{\Psi} = 0$. The combination $1/4\delta_\gamma + \Phi$ is called Sachs-Wolfe term (SW).

Including the motion of electrons at the surface of last scattering, leads to an extra term for $\delta T/\bar{T}$:

$$\frac{\delta T}{\bar{T}}(\hat{n}) = \left(\frac{1}{4}\delta_\gamma + \Phi + \hat{n}\cdot\vec{v}_e\right)_* + \int_{\tau_*}^{\tau_0} d\tau(\dot{\Phi} + \dot{\Psi}) \tag{2.64}$$

For adiabatic initial conditions, we can relate all the fluid perturbations with the metric ones. Working in longitudinal gauge, we have

$$\zeta = -\Psi - aH\frac{\delta\rho}{\rho'} = -\Psi - \frac{\delta}{3(1+w)}, \tag{2.65}$$

and so, at super Hubble scales we get

$$\zeta = -\Psi - \frac{2}{3}\frac{\Phi + (aH)^{-1}\dot{\Phi}}{1+w}, \tag{2.66}$$

and since ζ is constant, the non-decaying solution for the gravitational potential before horizon entry is (for $\Pi = 0$) is

$$\Phi = \Psi = -\frac{3+3w}{5+3w}\zeta. \tag{2.67}$$

Together with equation (2.65), this implies that, during radiation domination,

$$\delta = -2\Phi = \frac{4}{3}\zeta. \tag{2.68}$$

This result will be used as initial condition for the oscillations in the baryon-photon fluid.

Also, at super-Hubble scales the SW term is proportional to Φ_*, since at τ_* we have matter domination and so $\delta_\gamma = -4\delta_m/3 = -8\Phi/3$, giving $(\delta_\gamma/4 + \Phi)_* = \Phi_*/3$. Thus, an overdense region ($\Phi < 0$) appears as a cold spot in the sky today.

2.3.2 CMB Power Spectrum

The anisotropies on the CMB temperature are of the order $\delta T/\bar{T} \sim 10^{-5}$. The primordial perturbations in the early universe are imprinted in the observed angular statistic of these fluctuations. In this section we briefly comment on the CMB power spectrum.

Assuming isotropic initial conditions, we have

$$\left\langle \frac{\delta T(\hat{n})}{\bar{T}} \frac{\delta T(\hat{n}')}{\bar{T}} \right\rangle = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos\theta), \tag{2.69}$$

since the correlation will only depend on the relative orientations of $\hat{n}$ and $\hat{n}'$, $\hat{n} \cdot \hat{n}' = \cos\theta$. The $P_l(\cos\theta)$ functions are the Legendre polynomials and the expansion coefficients C_l are the angular power spectrum.

The observed spectrum of CMB anisotropies results from sub-horizon evolution of perturbations in the photon density and metric, that were sourced by initial super-horizon perturbations. Such evolution corresponds to sound waves in the photon-baryon fluid.

We can write

$$C_l = \frac{4\pi}{(2l+1)^2} \int d\ln k T_l^2(k) P_\zeta(k), \tag{2.70}$$

where $P_\zeta(k)$ is the power spectrum of the uniform density curvature perturbation ζ and $T_l(k)$, called the transfer function, comes from the solution of the line-of-sight equation, as we shall see in the next subsection. Ignoring the ISW term, we have

$$\begin{aligned} T_l &= T_{SW}(k) j_l(kr_*) + T_D(k) j_l'(kr_*) \\ T_{SW} &\equiv \frac{(1/4\delta_\gamma + \Phi)_*}{\zeta(k)}, \quad T_D(k) \equiv -\frac{(v_e)_*}{\zeta(k)}. \end{aligned} \tag{2.71}$$

In equations above, j_l and j_l' are Bessel funcions and its derivatives. They act like delta functions mapping Fourier modes k to the harmonics moments $l \sim kr_*$, with $r_* = \tau_0 - \tau_*$ the distance from last scattering surface. So, for a scale invariant P_ζ,

$$C_l \sim \frac{4\pi}{(2l+1)^2} [T_{SW}^2(k) + T_D^2(k)]_{k \sim l/r_*} \tag{2.72}$$

and the cross term $T_{SW}(k) T_D(k)$ is negligible.

2.3.3 Sound waves in the photon-baryon fluid

From the continuity and Euler equations for perturbations, we get:

$$\delta_\gamma'' + \frac{R'}{1+R}\delta_\gamma' - c_s^2 \nabla^2 \delta_\gamma = 4\Psi'' + \frac{4}{3}\nabla^2\Phi + \frac{R'}{1+R}\Psi' \tag{2.73}$$

where $R \equiv 3\bar{\rho}_b / 4\bar{\rho}_\gamma$ and $c_s^2 \equiv 1/3(1+R)$. The metric potentials, Φ and Ψ are determined by Einstein's equations as in the previous section. To gain some intuition, let us obtain approximate analytic results from this equation.

At early times, during radiation domination $R \ll 1$ and so let us consider $R = 0$. Ignoring time dilation terms, we get

$$\ddot{\Theta} - c_s^2 \nabla^2 \Theta = 0, \tag{2.74}$$

where $c_s^2 \approx 1/3$ and $\Theta \equiv \delta_\gamma/4 + \Phi$. We have

$$\Theta(k, \tau) = A_k \cos(c_s k \tau) + B_k \sin(c_s k \tau), \tag{2.75}$$

with A_k and B_k fixed by initial conditions. For adiabatic initial conditions, all perturbations in the limit $\tau \to 0$ are analytic function of k^2, so[4] $B_k = 0$. The coefficient A_k is obtained by matching with super-horizon data at the last scattering surface. Thus,

$$\Theta(k, \tau_*) = \frac{\zeta(k)}{3} \cos(c_s k \tau_*) \tag{2.76}$$

This solution for the SW term already indicates the origin of the oscillations in the CMB power spectrum C_l, since $T_{SW}(k) \sim \cos(c_s k \tau_*)$.

Modes at the extrema of their oscillations, $k_n = n\pi/(c_s \tau_*)$, will produce enhanced fluctuations. These will correspond to peaks at multipoles of the fundamental scale $k_* \equiv \pi/s_*$, where $s_* = c_s \tau_* \approx \tau_*/\sqrt{3}$ is the sound horizon at recombination. The mode k_* corresponds to a characteristic angular scale:

$$\theta_* = \frac{\lambda_*}{D_A}, \quad l_* = k_* D_A \approx \frac{\tau_*}{\tau_0}, \tag{2.77}$$

where D_A is the angular diameter distance from the last scattering. In a flat universe, $D_A = \tau_0 - \tau_* \approx \tau_0$. Assuming a purely matter dominated universe after recombination, $\tau \propto a^{1/2}$, we have

$$\theta_* \approx \left(\frac{1}{1100}\right)^{1/2} \approx 2^\circ, \quad l_* \approx 200. \tag{2.78}$$

Comparing with the actual CMB temperature power spectrum in Figure 2.1, we see that the first peak is indeed around $l_* \approx 200$. Observations of θ_* are consistent with a flat universe.

Let us consider the effects of baryons in the fluid. We have $R \propto a$ increasing with time until $R \sim \mathcal{O}(1)$ at recombination. We now have:

$$\frac{d^2}{d\tau^2}(\Theta + R\Phi) - \frac{1}{3}\nabla^2(\Theta + R\Phi) = 0, \tag{2.79}$$

where we have ignored the time variation of R. We see that the equilibrium point of oscillation shifts to $\Theta_{equi} = -R\Phi$. This leads to odd and even peaks in the CMB having unequal heights, because the C_l's depends of the square of the solution.

During the radiation era, the gravitational potential Φ decays inside horizon. So, for a mode that enters the sound horizon during radiation domination, the gravitational potential decays after horizon crossing and drives the acoustic amplitude higher.

[4] This is basically due to the fact that super-horizon modes enter the Hubble radius with vanishing velocity.

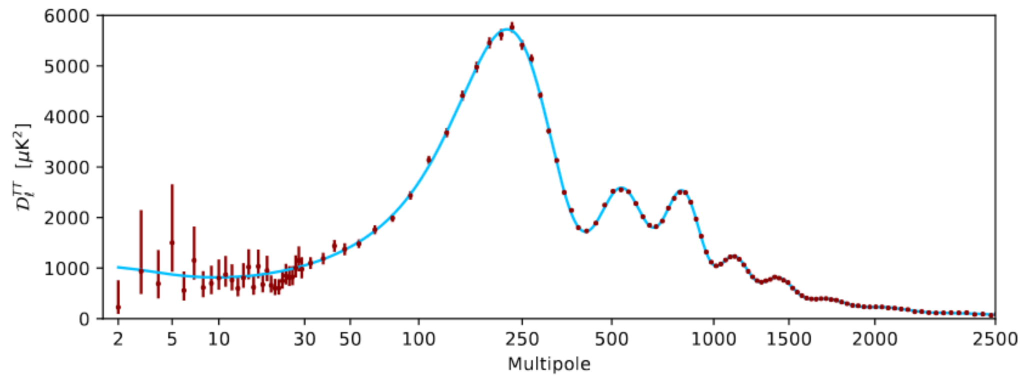


Figure 2.1: The cosmic microwave background radiation temperature power spectrum [16]. The combination $l(l+1)C_l/2\pi$ is plotted in the vertical axis.

There is another effect we must consider: the photon-baryon fluid is not a perfect fluid. The coupling between photons and electrons is not perfect and the photons have a finite mean free path

$$\lambda_C = \frac{1}{n_e \sigma_T a}, \tag{2.80}$$

where n_e is the electron density and σ_T is the Thomson cross section. This leads to a damping of small-scale fluctuations, known as Silk damping. The random walk of the photons through baryons will mix cold and hot regions and so fluctuations will be erased below the diffusion length,

$$\lambda_D \equiv \sqrt{N}\lambda_C = \sqrt{\tau/\lambda_C}\lambda_C = \sqrt{\tau\lambda_C}. \tag{2.81}$$

This effectively generates viscosity in the fluid:

$$\ddot{\Theta} + \mu c_s^2 k^2 \dot{\Theta} + c_s^2 k^2 \Theta = 0, \quad \mu \equiv \lambda_C \left(\frac{16}{15} + \frac{R^2}{1+R} \right), \tag{2.82}$$

and so there will be exponential suppression for the modes with $k > k_D \equiv 2\pi/\lambda_D$. Indeed, the photon transfer function receives the following correction:

$$T(k) \to D(k)T(k), \quad \text{where} \quad D(k) = e^{-k^2/k_D^2}. \tag{2.83}$$

The approach taken on this subsection was a simplified description of what is actually done, the real world is more complex. In practice, there are codes to calculate the transfer function by solving a Boltzmann equation for the radiation distribution functions (for details, see chapter 11 in [14]). In the following section, we will see how inflation provides the initial condition for CMB anisotropies, $\zeta(k)$.

2.4 Cosmological Inflation

2.4.1 The flatness and horizon problems

Despite its successes, the Big Bang cosmology does not explain why the Universe is so close to the flat geometry and what is the origin of the primordial inhomogeneity of the cosmic fluid, that stayed imprinted in the cosmic microwave background as observed on its anisotropy. There is still another problem related to causality due to the existence of horizons. Let us focus on the flatness problem first.

One can write the Friedmann equations in terms of the energy density parameter Ω, defined as the ratio of ρ by the critical density (2.18),

$$\Omega = \frac{\rho}{\rho_c}. \tag{2.84}$$

Then we have

$$\frac{d\Omega}{d\ln a} = (1+3w)\Omega\,(\Omega-1) \tag{2.85}$$

Although $\Omega = 1$ for flat geometry, in general Ω is not constant. For matter or radiation we have $(1+3w) > 0$ which leads to the conclusion that flat space is an unstable fixed point: if Ω begins smaller than 1 then it evolves for even smaller values; if Ω begins greater than 1 then it tends to increase even more. That is, we have

$$\frac{d|\Omega-1|}{d\ln a} > 0, \quad \text{for} \quad (1+3w) > 0. \tag{2.86}$$

So, having an Universe close to the flat geometry today is a highly fine-tuned state. To give a quantitative idea, from the CMB measurements we know that at present time, $|\Omega_0 - 1| < 0.02$, at least. So, taking $\Omega_0 = 1 \pm 0.05$ implies that at the recombination we have $\Omega_{\rm rec} = 1 \pm 0.0004$, and at the time of primordial nucleosynthesis, $\Omega_{\rm nuc} = 1 \pm 10^{-12}$. This fine-tuned situation is called the *flatness problem.*

Moving to the issue of causal structure of an expanding universe, let us work with conformal time τ. Using such coordinates, the FLRW metric is conformally related to the Minkowski metric and since a conformal factor does not affect the condition for having null geodesics, we can analyze causal structure as in Minkowski spacetime. Note that, by the definition of the particle horizon (2.11), we have

$$d_{\rm ph}(t) = \int_0^t \frac{dt'}{a(t')} = \int_0^{\tau(t)} d\tau' = \tau(t), \tag{2.87}$$

and thus the comoving particle horizon size is the age of the universe in conformal time. The expanding Universe has finite age, and therefore unlike the Minkowski spacetime we cannot prolong the light cone of an observer to infinite past: the light cone "stops" at $\tau = 0$. In this way, two points on the CMB at 180° apart in the sky will have past light cones that do not overlap. They "stop" at the spatially infinite surface at $\tau = 0$. Thus, these points are causally disconnected and there is no reason why such points reach the thermal equilibrium observed.

In quantitative terms, the particle horizon size at the time of the recombination $t_{\rm rec}$ is much smaller than the comoving distance that the radiation travels after decoupling:

$$\int_0^{t_{\rm rec}} \frac{dt}{a(t)} \ll \int_{t_{\rm rec}}^{t_0} \frac{dt}{a(t)}. \tag{2.88}$$

Therefore, there is no way to explain why the temperature of CMB is so isotropic within the standard cosmological evolution. This is the *horizon problem*.

Let us again use the first Friedmann equation, but now in the form

$$|\Omega - 1| = \frac{|k|}{a^2 H^2}. \tag{2.89}$$

In the Standard Cosmological model aH is always decreasing, and then Ω evolves away from flatness. But if we require that

$$\frac{d}{dt}\left(\frac{H^{-1}}{a}\right) < 0, \tag{2.90}$$

which means that the Hubble length H^{-1} in comoving coordinates decreases with time, then Ω evolves *towards* flatness. This condition is equivalent to $\ddot{a} > 0$, which is the definition of *inflation*, an primordial epoch of accelerated expansion. From equation (2.85) inflation is achieved if $(1 + 3w) < 0$,

$$\frac{d|\Omega - 1|}{d \ln a} < 0, \quad \text{for} \quad (1 + 3w) < 0. \tag{2.91}$$

and we see that this result implies inflation by looking at (2.17). Inflation solves the flatness problem automatically, but any model has to describe enough inflation in order not to lose the almost flat character after the end of inflation.

From (2.90) it is possible to conclude that inflation solves also the horizon problem. Since the comoving Hubble length is shrinking during inflation, distances that can be seen before inflation are much larger than distances that can be seen after inflation. This is true for scales compared to the horizon too, that is, distances that are smaller than the horizon before inflation are "red-shifted" to scales larger than the horizon after inflation. In other words, inflation is a mechanism to turn sub-horizon scales into super-horizon ones.

As an example let us consider the de Sitter case, for which we have $p = -\rho$ and

$$\frac{d \ln \Omega}{d \ln a} = 2(1 - \Omega). \tag{2.92}$$

In this case, the conformal time is

$$d\tau = \frac{dt}{a(t)} = \exp\left(-Ht\right) dt, \tag{2.93}$$

and so,

$$\tau = -\frac{1}{aH}. \tag{2.94}$$

Then, during inflation, the conformal time is negative and in this case the value $\tau = 0$ represents the transition of inflationary expansion to the radiation dominated era. The two opposite points in the sky cited before can now share a causal past in the negative domain of the conformal time.

2.4.2 Inflation from scalar fields: The slow-roll inflation

The de Sitter example of last section is an explicit picture of inflation. But the physics responsible for accelerated expansion at early times cannot be Einstein's cosmological constant, because it dominates over matter and radiation contributions at late times. We need a transition between the inflation era and the radiation dominated phase, i.e., the "vacuum" energy that drives inflation must be time dependent. For this purpose, inflation is implemented with fields, most simply with scalar fields. We are interested in models with single real scalar field, that in this case is called *the inflaton*.

Thus, it is natural to begin with the action for the inflaton minimally coupled to gravity,

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{2.95}$$

where $V(\phi)$ is the potential energy of the scalar field. Assuming flat geometry, the equation of motion for ϕ in a FLRW background is

$$\ddot{\phi} + 3H\dot{\phi} - \nabla^2\phi + \frac{\delta V}{\delta\phi} = 0 \tag{2.96}$$

We are interested in an homogeneous field for which $\nabla\phi = 0$. The equation of motion simplifies to

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{2.97}$$

where the prime represents derivative with respect to the field. The energy-momentum tensor of the scalar field is

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right), \tag{2.98}$$

and for the homogeneous case it has the same form as the energy-momentum tensor for a perfect fluid, with energy and pressure densities given by

$$\begin{aligned}\rho &= \frac{1}{2}\dot{\phi}^2 + V(\phi),\\ p &= \frac{1}{2}\dot{\phi}^2 - V(\phi).\end{aligned} \tag{2.99}$$

Then we see that the de Sitter limit $p \simeq -\rho$ is the regime in which the potential energy dominates the kinetic energy, $\dot{\phi}^2 \ll V(\phi)$. In this case the universe expands almost exponentially. The implementation with scalar fields and the last approximation frequently leads to a quasi-de Sitter universe. For this reason, it is convenient to define the number of e-folds N as

$$dN \equiv -Hdt, \tag{2.100}$$

note that N decreases as time (and also the scale factor) increases. Inserting (2.99) into the Friedmann equations (2.14) and (2.15) gives

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{2.101}$$

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + 3p) \equiv H^2(1-\epsilon), \tag{2.102}$$

where we have defined the dimensionless parameter

$$\epsilon = \frac{3}{2}\left(\frac{p}{\rho}+1\right) = 4\pi G\left(\frac{\dot{\phi}}{H}\right)^2 = -\frac{\dot{H}}{H^2}. \tag{2.103}$$

In the de Sitter limit, $\epsilon \to 0$, the potential energy dominates the kinetic energy, and we have

$$H^2 \simeq \frac{8\pi G}{3}V(\phi). \tag{2.104}$$

We will assume that the frictional term in the equation (2.97) dominates the "acceleration" term $\ddot{\phi}$:

$$\ddot{\phi} \ll 3H\dot{\phi} \tag{2.105}$$

and then the equation of motion for the scalar field is, approximately,

$$3H\dot{\phi} + V'(\phi) \simeq 0. \tag{2.106}$$

The assumption (2.105) can be written in terms of another dimensionless parameter $|\eta| \ll 1$, where we defined

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{2.107}$$

Using equations (2.104) and (2.105) we can write the parameters ϵ and η approximately as

$$\begin{aligned} \epsilon &= 4\pi G\left(\frac{\dot{\phi}}{H}\right)^2 \simeq \frac{1}{16\pi G}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \\ \eta &= -\frac{\ddot{\phi}}{H\dot{\phi}} \simeq \frac{1}{8\pi G}\left(\frac{V''(\phi)}{V(\phi)}\right). \end{aligned} \tag{2.108}$$

The approximations (2.104) and (2.105) are called the *slow-roll* approximation and they are valid as long as the *slow-roll parameters* parameters ϵ and η, satisfy

$$\epsilon \ll 1, \quad |\eta| \ll 1. \tag{2.109}$$

The first of these conditions tells us that the field has a very flat potential, $V'(\phi) \ll V(\phi)$ and the second slow roll condition informs that the curvature $V''(\phi)$ of the potential must be small.

Using the definition of ϵ we can write the number of e-folds of inflation as an integral in the field space:

$$N_* = -\int H dt = -\int \frac{H}{\dot{\phi}} d\phi = 2\sqrt{\pi G}\int \frac{d\phi}{\sqrt{\epsilon}} \simeq 8\pi G \int_{\phi_e}^{\phi_{N_*}} \frac{V(\phi)}{V'(\phi)} d\phi, \tag{2.110}$$

where ϕ_e is the field value at the end of inflation, that is obtained by breaking the slow-roll approximation, $\epsilon(\phi_e) = 1$, and ϕ_{N_*} is just an adjusted value that leads to a certain total e-folding number N_*. The number of e-folds of inflation needed to solve the horizon and flatness problem comes from the condition of having present cosmological scales inside the horizon before inflation. It turns out that we need $N_* \gtrsim 60$.

We now can put all this together and construct a qualitative picture of slow-roll inflation. At very early times the energy density of the Universe is dominated by the potential energy of a scalar field, the inflaton. The potential is nearly constant and inflation stands as long as the inflaton rolls down this region slowly, while the space expands almost exponentially. Inflation ends when the field enters a steeper and more curved region and the slow-roll conditions are not satisfied anymore. To obtain the standard Big Bang cosmology, the energy of the inflaton must decay into the Standard Model particles in a process that is called *reheating*. The theory of *reheating* is far from complete, as it depends on the extension of the Standard Model to high energies. This process is model dependent, the inflaton being or not a fundamental field. It is remarkable that only the potential energy of the inflaton is important for the physcis of slow roll inflation. Thus, the potential specifies the model.

This single view of the dynamics of inflation driven by scalar fields is an effective representation of some underlying theory that is not yet established. That is one reason why trying to get inflaton potentials from String Theory has dominated the work of the String Cosmology community in these past years.

In the next subsection, we discuss how inflation generates the primordial density perturbation, $\mathcal{R}$ or ζ. Though inflation is not the only model that explains the origin of ζ, it is the most popular one, since it solves other puzzles (horizon and flatness problems) at the same time.

2.4.3 Primordial curvature perturbation from inflation

In addition to its success in solving the flatness and horizon problems, it is remarkable that the primordial density perturbation of the cosmic fluid can be obtained from inflation. This is done by considering the quantum nature of the inflaton. In essence, the primordial inhomogeneity comes from *quantum fluctuations* of the inflaton. It is worthwhile to remember that it is this very primordial inhomogeneity that leads to the formation of the large structures of our Universe. It is this density perturbation that leads to the initial conditions for the inhomogeneities of cosmic radiation background, see e.g. (2.76). Indeed, from observations of the CMB at large scales, one can construct the *power spectrum* of this perturbation on the cosmic fluid.

We begin by solving the Klein-Gordon equation for a free scalar field φ, that is not the inflaton and does not affect the scale factor. It is just an arbitrary scalar field in a fixed background. The equation of motion is (2.97), but with $V = 0$ and in conformal time coordinate, we have

$$\varphi'' + 2\left(\frac{a'}{a}\right)\varphi' - \nabla^2\varphi = 0, \tag{2.111}$$

where now the prime indicates derivative with respect to the conformal time, $' = d/d\tau$.

Consider its Fourier expansion in comoving wave vectors $\boldsymbol{k}$:

$$\varphi(\tau, \boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left(\varphi_{\boldsymbol{k}}(\tau) b_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \bar{\varphi}_{\boldsymbol{k}}(\tau) \bar{b}_{\boldsymbol{k}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \right). \tag{2.112}$$

Then, the equation of motion for the mode $\varphi_{\boldsymbol{k}}$ is

$$\varphi''_{\boldsymbol{k}} + 2\left(\frac{a'}{a}\right)\varphi'_{\boldsymbol{k}} + k^2 \varphi_{\boldsymbol{k}} = 0. \tag{2.113}$$

Using the field redefinition $u_k \equiv a(\tau)\varphi_{\boldsymbol{k}}(\tau)$ gives

$$u''_k + \left(k^2 - \frac{a''}{a}\right) u_k = 0. \tag{2.114}$$

We shall analyze this equation in two regimes:

- *Long wavelength limit*, $k \ll a''/a$. In this limit, we have

$$a'' u_k = a u''_k \tag{2.115}$$

  with solution $u_k \propto a$ which implies that $\varphi_k = const.$. Then, the field modes φ_k are not dynamical, but have a non-zero amplitude. This is an example of *mode freezing* with the mode asymptoting to a constant;

- *Short wavelength limit*, $k \gg a''/a$. In this case, equation (2.114) reduces to the Klein-Gordon equation in Minkowski space, but in conformal coordinates,

$$u''_k + k^2 u_k = 0, \tag{2.116}$$

  that has the solution

$$u_k(\tau) = \frac{1}{\sqrt{2k}} \left(A_k e^{-ik\tau} + B_k e^{ik\tau} \right). \tag{2.117}$$

  This can be identified with the exact Minkowskian solution in the ultraviolet limit.

Hence, all we need to do is set the boundary conditions on field perturbations in the ultraviolet limit (short wavelength limit). But in such limit we have a Minkowskian description of the fields, equation (2.116), and we know how to quantize fields in Minkowski spacetime. In fact, one of the conditions comes from quantization and the other comes from the vacuum selection, which is the same as in the Minkowski case, the Bunch-Davies vacuum. Together these conditions specify A_k and B_k.

Under canonical quantization, the coefficients $b_{\boldsymbol{k}}$ and $\bar{b}_{\boldsymbol{k}}$ of the expansion (2.112) are promoted to annihilation and creation operators with the commutation relation

$$\left[\hat{b}_{\boldsymbol{k}}, \hat{b}^\dagger_{\boldsymbol{k}'}\right] = \delta^3(\boldsymbol{k} - \boldsymbol{k}') \tag{2.118}$$

The quantum field $\hat{\varphi}(\tau, \boldsymbol{x})$ is written in terms of these operators and has a canonical conjugate momentum

$$\hat{\Pi}(\tau, \boldsymbol{x}) = a^2(t) \frac{\partial \hat{\varphi}}{\partial \tau}. \tag{2.119}$$

The commutation relation

$$\left[\hat{\varphi}(\tau,\boldsymbol{x}),\hat{\Pi}(\tau,\boldsymbol{x'})\right]=i\delta^3(\boldsymbol{x}-\boldsymbol{x'}),\tag{2.120}$$

leads to

$$u_k\frac{\partial\bar{u}_k}{\partial\tau}-\bar{u}_k\frac{\partial u_k}{\partial\tau}=i,\tag{2.121}$$

which gives, for the ultraviolet mode solution (2.117),

$$|A_k|^2-|B_k|^2=1.\tag{2.122}$$

The second condition on these constants comes from the vacuum selection. We choose the vacuum which gives the usual Minkowski vacuum in the ultraviolet limit: $A_k=1$, $B_k=0\rightarrow u_k(\tau)\propto e^{-ik\tau}$.

Now we can apply this analysis to perturbations of the inflaton field. We split the inflaton as the background value plus fluctuation parts

$$\phi(t,x)=\phi(t)+\varphi(t,x),\tag{2.123}$$

where in the general case of a perturbed background we will assume a flat slicing in order to define φ. We need to take into account the evolution of the background spacetime that is encoded in $a(\tau)$. A good approximation is to consider the slow parameter ϵ as constant, but in general this is not the case. For an arbitrary equation of state parameter, we get

$$\tau=-\left(\frac{1}{aH}\right)\left(\frac{1}{1-\epsilon}\right),\tag{2.124}$$

and from (2.14) and (2.17) with flat geometry, we have

$$\frac{a''}{a}=a^2H^2(2-\epsilon).\tag{2.125}$$

Then equation (2.114) is

$$u_k''+\left[k^2-a^2H^2(2-\epsilon)\right]u_k=0,\tag{2.126}$$

and using (2.124) to write aH in terms of the conformal time, we have

$$\tau^2(1-\epsilon)^2u_k''+\left[(k\tau)^2(1-\epsilon)^2-(2-\epsilon)\right]u_k=0.\tag{2.127}$$

This is a Bessel equation and its solutions depends on the first and second Bessel functions, J_ν and Y_ν,

$$u_k\propto\sqrt{-k\tau}\left(J_\nu(-k\tau)\pm iY_\nu(-k\tau)\right),\tag{2.128}$$

where

$$\nu=\frac{3-\epsilon}{2(1-\epsilon)}.\tag{2.129}$$

For the de Sitter case, $\epsilon=0$, the Bessel index is $\nu=3/2$ and the solution is

$$u_k\propto\left(\frac{k\tau-i}{k\tau}\right)e^{\pm ik\tau}.\tag{2.130}$$

The short wavelength limit is $k\tau \to \infty$. This can be seen from the equation (2.124),

$$(-k\tau)(1-\epsilon) = \frac{k}{aH}, \tag{2.131}$$

since the quantity (k/aH) represents the wavenumber k in units of the comoving Hubble size $r_H = (aH)^{-1}$, then the limit $k\tau \to 0$ corresponds to $(k/aH) \ll 1$, the long wavelength limit. Taking the short wavelength limit of the de Sitter solution and selecting the Bunch-Davies vacuum, $u_k \propto e^{ik\tau}$, with the normalization fixed by the canonical quantization, we have

$$u_k = \frac{1}{\sqrt{2k}} e^{-ik\tau}. \tag{2.132}$$

Thus, the exact solution to the mode function of the inflaton in the de Sitter case is

$$u_k = \frac{1}{\sqrt{2k}} \left(\frac{k\tau - i}{k\tau} \right) e^{-ik\tau}. \tag{2.133}$$

This solution is valid for all wavelengths, we have used the short limit just to find the proper normalization. Let us confirm the long wavelength limit, $-k\tau \longrightarrow 0$:

$$u_k \longrightarrow \frac{1}{\sqrt{2k}} \left(\frac{-i}{k\tau} \right) = \frac{-i}{\sqrt{2k}} \left(\frac{aH}{k} \right) \propto a. \tag{2.134}$$

Comparing with the solution of (2.115), we see the consistency with the vacuum choice. Therefore, the field amplitude φ_k is, in the long wavelength limit,

$$|\varphi_k| = \left| \frac{u_k}{a} \right| = \frac{H}{\sqrt{2k^3}} = const. \tag{2.135}$$

The amplitude of the quantum fluctuations can be obtained from the two-point correlation function of the quantum field $\hat{\varphi}$,

$$\langle 0|\hat{\varphi}(\tau, \boldsymbol{x})\hat{\varphi}(\tau, \boldsymbol{x'})|0\rangle = \int \frac{d^3k}{(2\pi)^3} |\varphi|^2 e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})} \equiv \int \frac{dk}{k} P_\varphi(k) e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x'})}, \tag{2.136}$$

where we have defined the *power spectrum* for the quantum field fluctuations $P_\varphi(k)$ as

$$P_\varphi(k) \equiv \left(\frac{k^3}{2\pi^2} \right) \left| \frac{u_k}{a} \right|^2 = \left(\frac{H}{2\pi} \right)^2. \tag{2.137}$$

Note that this power spectrum is scale independent, as H is constant. For a more general model, the spacetime is *not exactly* de Sitter and the power spectrum of the field fluctuations will only be approximately scale invariant. Since the initial density perturbations for CMB will be given by super-horizon modes, it is conventional to evaluate the power spectrum at $aH = k$, i.e., at *horizon crossing*.

As mentioned before, the primordial density perturbation, that is responsible for structure formation, is generated by the fluctuations of the inflaton. Using the general relativistic perturbation theory reviewed in section 2.2, it is possible to show that the power spectrum of the primordial curvature perturbations ζ is related to the power spectrum of the inflaton fluctuations $P_\varphi(k)$. Indeed, using equation (2.49), on the flat slice where the inflaton fluctuations were defined, we have

$$\zeta = -aH\frac{\varphi}{\phi'}, \tag{2.138}$$

and so the power spectrum of the density curvature perturbation ζ is

$$P_\zeta = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2, \tag{2.139}$$

where the right hand side is evaluated at the horizon exit, $k = aH$. Using the slow-roll approximation (2.109), we have

$$P_\zeta \simeq \frac{(8\pi G)^2 V}{24\pi^2\epsilon}. \tag{2.140}$$

The spectral index n_s is a parameter that measures the departure from scale invariance,

$$n_s - 1 \equiv \frac{d\ln P_\zeta}{d\ln k}, \tag{2.141}$$

that is, the power spectrum can be parametrized as

$$P_\zeta = A_\zeta \left(\frac{k}{k_0}\right)^{(n_s-1)}, \tag{2.142}$$

where k_0 is some reference scale and A_ζ is an amplitude. For $n_s = 1$ we have scale invariance. It is possible to use the slow roll conditions to find

$$n_s - 1 = -6\epsilon + 2\eta. \tag{2.143}$$

Since in most models $\eta \ll \epsilon$, the spectral index is smaller than 1. Thus P_ζ receives a bigger contribution from modes with $k < 1$, and hence we say that the spectrum has a red tilt.

The tensor perturbations h_{ij} are related to the production of gravitational waves. Since the transverse and longitudinal polarization states of the gravitational waves evolves as independent scalars, we can use the power spectrum of the inflaton fluctuations to calculate the power spectrum of the tensor perturbations as the sum of the 2 correlated functions for the separate polarizations,

$$P_h = 2 \times (8\pi G)\left(\frac{H}{2\pi}\right)^2, \tag{2.144}$$

where the factor $(8\pi G)$ comes from the Einstein-Hilbert action normalization. This power spectrum has a spectral index $n_t = -2\epsilon$, and so it is also red-tilted. Then we have a consistent condition between the amplitudes of the power spectra P_h and P_ζ,

$$r = \frac{A_h}{A_\zeta} \simeq 16\epsilon. \tag{2.145}$$

As inflation is not the only mechanism to generate the scalar and tensor primordial perturbations, their measurement alone are not a proof of inflation. Unfortunately, we have not detected primordial tensor perturbations yet, but it would be a very important check of inflation as the consistency condition for r could then be verified. It could be however that the tensor perturbations spectrum is not red-tilted or even if it is, it does not satisfy (2.145). In this case we will have to look for alternatives models for generating the primordial perturbations.

Therefore, for each model specified by the inflaton potential, we have three independent parameters to calculate: the amplitude of the scalar power spectra A_ζ, the scalar spectral index n_s and the tensor spectral index n_t. Since tensor perturbation were not measured yet, we only have observational values for A_ζ and n_s and a constraint for r. From Planck 2018 cosmological parameter results [148], we have

$$\begin{aligned} n_s &= 0.9649 \pm 0.0042, \\ r &< 0.11, \\ A_\zeta &\simeq 2.01 \times 10^{-9}. \end{aligned} \tag{2.146}$$

Chapter 3

Basics of String Theory

The purpose of this chapter is to quickly review the foundations of String Theory. We will state several results without proof, but special attention will be devoted on the assumptions they stand on. Historically, String Theory was created as an attempt to explain phenomenological results of hadronic physics in the late 60's, that led to several amplitudes with nice analytical properties. But this goal was never completed and Quantum Chromodynamics turned up to be the right theory ruling strong interactions. Then in early 70's, a new interpretation of the energy scale of the stringy process was taken and String Theory was first recognized as a potential theory of quantum gravity. After that, research in String Theory was conducted with the hope of getting a final theory that would describe all the interactions of nature, a Theory of Everything.

In the following sections, we show how to get the string spectra and how to find low energy actions describing the behaviour of the massless fields that appear in the various kinds of superstrings. Type II theories have a special treatment, as solution to them are used to define the AdS/CFT correspondence that is used in chapter 6. General references for the present chapter are [24–27]. Double Field Theory was initially introduced in [119, 150] (for reviews see [120, 151]) and the discussion presented in section 3.4 was inspired by [152]. For further details on AdS/CFT see [153] and references therein.

3.1 Relativistic (or massless) quantum bosonic strings

All modern discussions on the subject begin with a simplified kind of string, the bosonic string. This string is the simplest possible relativistic string and after quantization, its modes describes quanta of bosonic fields. Similar to the relativistic classical point particle case, one finds the equations of motion for a relativistic string from an action that has the physical interpretation of the area swept out by the string in Minkowski spacetime, the area of the string's worldsheet,

$$S = -\frac{T}{2}\int_\Sigma dA = -\frac{T}{2}\int_\Sigma d^2\sigma\sqrt{-h}, \tag{3.1}$$

where $\sigma^a = (\tau, \sigma)$ are intrinsic coordinates of the two-dimensional worldsheet Σ and h is the determinant of its induced metric, given by

$$h_{ab} = \frac{\partial X^\mu}{\partial \sigma^a} \frac{X^\nu}{\partial \sigma^b} \eta_{\mu\nu}, \tag{3.2}$$

where $X^\mu(\sigma)$ are the embedding functions, describing how worldsheet coordinates are mapped to spacetime. Although relativistic strings do not have a "mass density" term in the action, they do have a tension T. Equation (3.1) is called the Nambu-Goto action. One can also write an action for the bosonic string by introducing an independent worldsheet metric, $\gamma_{ab}(\sigma)$,

$$S_P[X, \gamma] = -\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{3.3}$$

This is the Polyakov action and it is equivalent to the Nambu-Goto action after using the equation of motion for γ_{ab}. For historical reasons, the string tension is written in terms of α', that sets the units used in string theory (it has dimension of length squared). The Polyakov action has three important symmetries:

- D-dimensional Poincaré invariance,

$$X^\mu(\sigma) \to X'^\mu(\sigma) = \Lambda^\mu_\nu X^\nu(\sigma) + a^\mu, \quad \gamma_{ab}(\sigma) \to \gamma'_{ab}(\sigma) = \gamma_{ab}(\sigma); \tag{3.4}$$

- 2-dimensional diffeomorphism invariance (diff. or reparameterization invariance), $\sigma \to \sigma'(\sigma)$ and

$$X^\mu(\sigma) \to X'^\mu(\sigma') = X^\mu(\sigma), \quad \gamma_{ab}(\sigma) \to \gamma'_{ab}(\sigma') = \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b} \gamma_{cd}(\sigma); \tag{3.5}$$

- 2-dimensional Weyl invariance,

$$X^\mu(\sigma) \to X'^\mu(\sigma) = X^\mu(\sigma), \quad \gamma_{ab}(\sigma) \to \gamma'_{ab}(\sigma) = e^{2w(\sigma)} \gamma_{ab}(\sigma). \tag{3.6}$$

The latter has no analog in the Nambu-Goto form.

Since varying the action with respect to the metric gives the energy-momentum tensor T_{ab} of the worldsheet theory, the equation of motion of for γ_{ab} is

$$T_{ab} = -\frac{1}{\alpha'} \left(\partial_a X^\mu \partial_b X_\mu - \frac{1}{2} \gamma_{ab} \gamma^{cd} \partial_c X^\mu \partial_d X_\mu \right) = 0. \tag{3.7}$$

Due to the Weyl invariance, we also have $T^a_{\ a} = 0$ off-shell. It is possible to use reparameterization and Weyl invariances to set the worldsheet metric to a flat metric, a gauge choice called conformal gauge. But then one must impose the equation of motion for γ_{ab} as a constraint. In conformal gauge, the equation of motion for $X^\mu(\sigma)$ is

$$\eta^{ab} \partial_a \partial_b X^\mu(\sigma) = 0, \tag{3.8}$$

i.e., the wave equation in two dimensions. We should also impose equation (3.7) (for $\gamma = \eta$) as a constraint, that gives the Virasoro constraints,

$$\partial_a X^\mu \partial_b X_\mu = \frac{1}{2} \eta_{ab} \eta^{cd} \partial_c X^\mu \partial_d X_\mu. \tag{3.9}$$

We also need boundary conditions for X^μ. If we impose periodic boundary conditions, we get closed strings. If the string has endpoints (it is an open string), we have a surface term in the variation of the Polyakov action that vanishes under the condition

$$\delta X^\mu \partial_\sigma X_\mu|_{\text{ endpoints}} = 0. \tag{3.10}$$

For $\partial_\sigma X_\mu|_{\text{endpoints}} = 0$ we have Neumann boundary conditions that together with the Virasoro constraints implies that the string endpoints move at the speed of light. The choice $\delta X_\mu|_{\text{endpoints}}$ means that the endpoints of the strings are fixed in some timelike hypersurface. In this case we have a Dirichlet boundary condition and Poincaré invariance is broken. The "solution" for this issue is to assume that there are dynamical fundamental objects, called D-branes, that absorb the momentum flowing from the string endpoints. More generally, we can impose Neumann boundary conditions in p spatial directions and Dirichlet conditions in the remaining $D-1-p$ directions, giving a Dp-brane. We are not going to discuss D-branes in much detail (a good introduction is [154]).

Solving the equation of motion for open strings with $\sigma \in [0, \pi]$ and for Neumann directions,

$$X^\mu(\tau, \sigma) = x^\mu + (2\alpha')p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos(n\sigma), \tag{3.11}$$

while for Dirichlet directions, we have

$$X^\mu(\tau, \sigma) = x^\mu + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{-in\tau}\sin(n\sigma). \tag{3.12}$$

Note that the string center of mass momentum p^μ vanishes for Dirichlet directions. For closed strings, with $\sigma \in [0, 2\pi)$, we get

$$X^\mu(\tau, \sigma) = x^\mu + (2\alpha')p^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left(\frac{\alpha_n^\mu}{n}e^{-in(\tau+\sigma)} + \frac{\tilde{\alpha}_n^\mu}{n}e^{-in(\tau-\sigma)}\right), \tag{3.13}$$

where x^μ is constant for all the solutions. Due to the reality of X^μ, the oscillation modes should satisfy $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$, $(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu$.

Even after going to the conformal gauge there is still a combination of Weyl and diffeomorphism transformations that preserves such gauge. This gives rise to a residual gauge symmetry, the conformal symmetry. One can use this residual redundancy to go to a gauge where spacetime coordinates are fixed, commonly the light-cone gauge, and then quantize the string to obtain its spectrum. One may also choose to quantize the string covariantly and then the generators of the conformal transformations should have null action on the states of the string. This is the quantum version of imposing the constraints in the classical solution[1].

[1] In a more systematic treatment, the conformal symmetry will give rise to Faddeev-Poppov ghosts that also contributes to the two dimensional worldsheet quantum theory. Then, in the path integral formalism, one can use the BRST symmetry coming from the residual redundancy to find the spectrum of the string.

In fact, the residual conformal invariance play a big role in String Theory in general. Conformal transformations in two dimensions are very special as they are infinite dimensional. In Euclidean complex coordinates,

$$w = \sigma + i\tau_E, \quad \bar{w} = \sigma - i\tau_E \tag{3.14}$$

where τ_E is the Euclidean worldsheet time, the conformal transformations are given by

$$w \to f(w), \quad \bar{w} \to f^*(\bar{w}) \tag{3.15}$$

where $f(w)$ is a holomorphic function. The worldsheet positions $X^\mu(w, \bar{w})$ are scalars under such transformations. Indeed the conformal gauge fixed Euclidean Polyakov action in complex coordinates,

$$S_P = \frac{1}{2\pi\alpha'} \int d^2w \partial_w X^\mu \bar{\partial}_{\bar{w}} X_\mu, \tag{3.16}$$

is invariant under conformal transformations. It turns out that conformal symmetry tools and complex analysis are combined in a very useful way in String Theory.

We now would like to quantize the bosonic string. Special care with the residual redundancy should be taken. We will first canonically quantize the X^μ and then impose the Virasoro constraints. The first step produces a non-positive definite Hilbert space, an issue to be fixed by imposing the constraints. We will focus on closed strings first. Then, in conformal gauge, the worldsheet Σ is an infinite cylinder corresponding to a closed string propagating in spacetime. The conformal transformation,

$$w \to z = e^{-iw}, \quad \bar{w} \to \bar{z} = e^{i\bar{w}}, \tag{3.17}$$

maps the cylinder to the complex plane z, with e^{τ_E} parametrizing the radial position (radial time) and σ playing the role of the angular displacement. Working with $(z, \bar{z})$ coordinates allows us to Laurent expand holomorphic and antiholomorphic functions.

Since X^μ are worldsheet scalars, we can write the mode expansion (3.13) in z complex coordinates directly as

$$X^\mu(z, \bar{z}) = x^\mu - i\frac{\alpha'}{2} p^\mu \ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{1}{m} \left(\alpha_m^\mu z^{-m} + \tilde{\alpha}_m^\mu \bar{z}^{-m} \right), \tag{3.18}$$

and then

$$\alpha_m^\mu = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial_z X^\mu(z), \quad \tilde{\alpha}_m^\mu = -\sqrt{\frac{2}{\alpha'}} \oint \frac{d\bar{z}}{2\pi} \bar{z}^m \bar{\partial} X^\mu(\bar{z}), \tag{3.19}$$

with $\alpha_m^0 = \tilde{\alpha}_m^0 = \sqrt{\alpha'/2} p^\mu$. From the operator product expasion (OPE) of the X^μ's (or simply from the canonical commutation relations), we get

$$[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu} \delta_{m+n,0} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu], \tag{3.20}$$

and we also have

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}. \tag{3.21}$$

From these commutation relations, we have that the spectrum is given by starting with a state $|0,0;k\rangle$ that has momentum k^μ and is annihilated by all of the lowering modes, α_n^μ for $n>0$, and acting in all possible ways with the raising modes α_n^μ for $n<0$. The same is true for $\tilde{\alpha}_m^\mu$ operators, that act on the second entry of the state. We see that before imposing the constraints, the Hilbert space has negative norm states coming from the $\mu=0=\nu$ commutation relations.

The generators of conformal transformations are the off-shell components of the energy-momentum tensor. In complex coordinates, they are given by

$$T_{zz}(z) \equiv T_B(z) = -\frac{1}{\alpha'} : \partial X^\mu \partial X_\mu :, \quad T_{\bar{z}\bar{z}} \equiv \tilde{T}_B(\bar{z}) = -\frac{1}{\alpha'} : \bar{\partial} X^\mu \bar{\partial} X^\mu :, \tag{3.22}$$

where $T_{z\bar{z}}=0$ due to the symmetry of T_{ab} and $(\bar{T}(\bar{z}))T(z)$ is a (anti) holomorphic function due to two-dimensional diff. invariance. The :: means that the operators are normal ordered such that the contact divergence coming from $X^\mu(z,\bar{z})X^\nu(z',\bar{z}')$,

$$X^\mu(z,\bar{z})X^\nu(0,0) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z\bar{z}), \tag{3.23}$$

is absent[2]. The Laurent expansion gives,

$$T(z) = \sum_m \frac{L_m}{z^{m+2}}, \quad \tilde{T}(\bar{z}) = \sum_m \frac{\tilde{L}_m}{\bar{z}^{m+2}}, \tag{3.24}$$

and the OPE

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) \tag{3.25}$$

implies the Virasoro algebra,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \tag{3.26}$$

where the central charge is given by the number of spacetime dimensions, $c=D$. The situation is analog for $\tilde{T}(\bar{z})$, where it is possible to find another copy of the Virasoro algebra for $\tilde{L}_m$.

In CFT language, the OPE (3.25) implies that $T(z)$ is a quasi-primary operator with conformal weight 2. Would $c=0$, then $T(z)$ would be a primary operator. This explains why there is a -2 in the power of z inside the $T(z)$ Laurent expansion: it cancels the z dependence in the conformal transformation of $T(z)$ such that the modes L_m are the same as the Fourier modes coming from expansion in $(w,\bar{w})$ coordinates. In general, Laurent expansions on the z-plane of primary or quasi-primary operators with conformal weight h will have a contribution $-h$ to the power of z.

By consistency with the Virasoro algebra, we should impose

$$(L_n - a\delta_{n,0})|\psi\rangle = 0 = (\tilde{L}_n - a\delta_{n,0})|\psi\rangle, \quad \text{for} \quad n \geq 0 \tag{3.27}$$

[2]The symbol $\sim$ in the expression for OPEs means equality up to non-singular terms.

in any state $|\psi\rangle$. The action of L_n for $n < 0$ will also be null on amplitudes, since $L_n^\dagger = L_{-n}$, due to reality of the energy-momentum tensor. The constant a is included to handle possible ordering issues. States satisfying the previous condition are called physical states.

Not all states satisfying (3.27) are in the Hilbert space of the quantum string. If on top of satisfying (3.27) some state $|\phi\rangle$ is orthogonal to all physical states, then $|\psi\rangle + |\phi\rangle$ (with $|\psi\rangle$ physical) is also physical. Such state $|\phi\rangle$ is called null. Moreover, as $|\phi\rangle$ decouples from any inner products, we have an equivalence relation

$$|\psi\rangle \cong |\psi\rangle + |\phi\rangle, \tag{3.28}$$

and so the real Hilbert space should be the set of equivalence classes of the relation between physical states and physical plus null states. Orthogonal states have the form

$$|\phi\rangle = \sum_{n=1}^{\infty} L_{-n}^{m} |\phi_n\rangle, \tag{3.29}$$

for any $|\phi_n\rangle$, and so the equivalence relation (3.28) is the same as imposing

$$L_n|\phi\rangle \cong 0, \quad \text{for} \quad n < 0 \tag{3.30}$$

on orthogonal states. To find the spectrum, one applies the constraints (3.27) and (3.30) paying attention to the existence and structure of the null states at each level of string excitation.

The final spectrum depends on a and D. For $a = 1$, $D = 26$ the spectrum is free from negative norm states and we have the critical bosonic string. For $a > 1$ we do not get rid of all negative norm states and for $a < 1$ the spectrum is positive definite but there are extra states as compared with the critical case. Though not inconsistent, there is no known way to include interaction in this last case, that is called non-critical strings. Also non-critical string theory does not respect full D-dimensional Lorentz invariance. We will consider only critical string theory in this thesis.

The L_0 and $\tilde{L}_0$ constraints gives the mass-shell condition,

$$M^2 = -p^2 = \frac{4}{\alpha'}(N-1) = \frac{4}{\alpha'}(\tilde{N}-1), \tag{3.31}$$

where the number operators,

$$N = \sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n\mu}, \quad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_{n\mu}, \tag{3.32}$$

have integer eigenvalues that define the level of the string states. Note that we should have the level matching condition $L_0 = \tilde{L}_0$ for closed strings, which implies that $N = \tilde{N}$, i.e., the number of α raising operators acting on any states should be the same as the number of $\tilde{\alpha}$ raising operators.

At lowest level ($N = 0 = \tilde{N}$), we get a scalar particle $|0, 0; k\rangle$ with negative mass squared, $M^2 = -4/\alpha'$. This tachyon is an artifact of the bosonic string, and it is absent in superstring theory. The next level, with $N = 1 = \tilde{N}$, corresponds to massless states $e_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,0;k\rangle$, where the polarization tensor $e_{\mu\nu}$ satisfies

$$k^{\mu}e_{\mu\nu} = 0 = k^{\nu}e_{\mu\nu}, \quad e_{\mu\nu} \sim e_{\mu\nu} + a_{\mu}k_{\nu} + b_{\nu}k_{\mu}, \quad \text{with} \quad a_{\mu}k^{\mu} = 0 = b_{\mu}k^{\mu}. \tag{3.33}$$

Then, starting with the D^2 polarizations, we can use the equivalence condition to fix $2(D-1)$ of them and the transverse conditions to fix other $(2D-2)$ polarizations. So, we get $(D-2)^2$ massless states that can be decomposed in irreducible representations of the rotation group $SO(D-2)$, giving a traceless symmetric tensor, an antisymmetric tensor and a scalar. Therefore, the states at the first level of the closed string corresponds to quanta of the metric $g_{\mu\nu}(x)$, an antisymmetric rank-2 tensor field called Kalb-Ramond field $B_{\mu\nu}(x)$ and a scalar field, the dilaton $\phi(x)$.

For open strings with Neumann boundary conditions, the worldsheet is an infinite strip with $\sigma \in [0,\pi]$ and the conformal map $w \to z = -e^{-iw}$ turns the infinite strip into the upper-half z-plane. The boundary conditions is now $\partial_z X^{\mu} = \bar{\partial}_{\bar{z}} X^{\mu}$ on the real axis (Im(z)) and the mode expansion in complex coordinates depends on one set of modes α^{μ}_m only:

$$X^{\mu}(z,\bar{z}) = x^{\mu} - i\alpha' p^{\mu}\ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{\alpha^{\mu}_m}{m}\left(z^{-m} + \bar{z}^{-m}\right). \tag{3.34}$$

Similarly to the closed case, the $X^{\mu}X^{\nu}$ OPE gives

$$[\alpha^{\mu}_m, \alpha^{\nu}_n] = m\eta^{\mu\nu}\delta_{m+n,0}, \tag{3.35}$$

and we have usual commutation relations for x^{μ} and p^{μ}. Thus, before imposing the constraints, the states are built from $|0,k\rangle$, that is annihilated by α^{μ}_n for $n > 0$, by acting with the raising operators α^{μ}_n for $n < 0$.

The Neumann boundary condition implies that the energy-momentum tensor components are related, $T_{zz} = T_{\bar{z}\bar{z}}$, along Im$(z) = 0$. Before doing the Laurent expansion, we use the doubling trick to define the energy momentum tensor on the lower half-z-plane,

$$T(z) \equiv \bar{T}(\bar{z}') \quad \text{for} \quad \text{Im}(z) < 0, \tag{3.36}$$

with $z' = \bar{z}$. This ensures that the $T(z)$ is holomorphic in the entire complex z-plane. Thus, from the expansion of $T(z)$ in (3.24) there will be only one set of Virasoro generators L_m, that satisfies the Virasoro algebra (3.26).

Since there are no $\tilde{L}_m$, the analysis of the spectrum is simpler. The L_0 constraint gives the mass-shell condition

$$M^2 = \frac{1}{\alpha'}(N-1), \tag{3.37}$$

where the number operator N is again given by (3.32). So, at the lowest level ($N = 0$) we also have a tachyon with mass $M^2 = -1/\alpha'$. At next level (N=1), there are D states $e_{\mu}\alpha^{\mu}_{-1}|0,k\rangle$. The other physical conditions imply that the polarization vector e_{μ} should satisfy

$$k^{\mu}e_{\mu} = 0, \quad e_{\mu} \sim e_{\mu} + \lambda k_{\mu}, \tag{3.38}$$

for any arbitrary consistent parameter λ. Thus we have $(D-2)$ massless states that form a vector of $SO(D-2)$. This corresponds to the quanta of a spacetime vector field $A^\mu(x)$, as in Maxwell's theory of electromagnetism, i.e., a photon.

For both open and closed cases, at higher levels of excitation, we get a tower of massive states with increasing spins and masses quantized in units of $1/\alpha'$. At every level, all states fall into irreducible representations of the Lorentz group.

From the discussion in this section, we see that we can obtain the quanta of various spacetime fields from quantum excitations of a relativistic string. More remarkably, it is possible to find the interaction of these spacetime fields from string interactions, a topic that is going to be to be very little discussed in this thesis. But the bosonic strings have some unpleasant features: they do not describe quanta of fermionic spacetime fields and have a tachyon in the spectrum. Though the latter issue is thought to be due to a wrong choice of vacuum, the former is truly an obstacle for string phenomenology. In the next section, we explain how introducing supersymmetry gives fermions in spacetime and how to get a string theory without tachyons.

3.2 Relativistic quantum superstrings

In the previous section, it was shown that the mass-shell condition and string spectrum come from the constraint algebra. If one wants to modify the spectrum to get fermions, one could think of enlarging the constraint algebra by introducing extra symmetries and so extra fields. Since supersymmetry relates bosons to fermions, it is a natural step to consider the supersymmetric version of the Polyakov action S_p, giving the Ramond-Neveu-Schwarz superstring (RNS). Since S_P is based on maps from the worldsheet to spacetime, one can also consider a supersymmetric generalization by changing the maps to be onto the superspace, that gives the Green-Schwarz (GS) superstring. At the end, RNS and GS formalisms are equivalent but they have different formal aspects: the GS string has spacetime supersymmetry manifested and RNS has manifest worldsheet supersymmetry. This difference is crucial in quantizing the superstring, as there is no easy way to quantize the GS covariantly. The formalism that allows us to covariantly quantize the superstring with manifest spacetime supersymmetry is called pure spinor formalism, giving the Berkovits superstring [155]. In the following we will consider the RNS superstring.

In conformal gauge, the supersymmetric Polyakov action is, using complex coordinates,

$$S_{SP} = \frac{1}{4\pi}\int d^2w \left(\frac{2}{\alpha'}\partial_w X^\mu \bar{\partial}_{\bar{w}} X_\mu + \psi^\mu \bar{\partial}_{\bar{w}}\psi_\mu + \tilde{\psi}^\mu \partial_w \tilde{\psi}_\mu\right), \tag{3.39}$$

with ψ^μ and $\tilde{\psi}^\mu$ are anticommuting Gransmann worldsheet functions. The equation of motion for them is

$$\partial_{\bar{w}}\psi^\mu = 0 = \partial_w \tilde{\psi}^\mu, \tag{3.40}$$

which implies that ψ^μ is holomorphic and $\tilde{\psi}^\mu$ is antiholomorphic.

For closed superstrings, there are two possible periodic boundary conditions for ψ^μ and for $\tilde{\psi}^\mu$: the Ramond condition, in which the anticommuting fields are periodic under $w \to w+2\pi$, and the Neveu-Schwarz condition in which they are antiperiodic. The choice of periodicities defines the Ramond (R) or Neveu-Schwarz (NS) sectors. Thus we have

$$\psi^\mu(w+2\pi) = e^{2\pi i\nu}\psi^\mu(w), \quad \tilde{\psi}^\mu(\tilde{w}+2\pi) = e^{-2\pi i\tilde{\nu}}\tilde{\psi}^\mu(\tilde{w}), \tag{3.41}$$

with $\tilde{\nu}, \nu = 0$ for the R sector and $\tilde{\nu}, \nu = 1/2$ for the NS sector. Going to the z-plane, the solutions may be Laurent expanded as

$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{\psi^\mu_r}{z^{r+1/2}}, \quad \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\tilde{\nu}} \frac{\tilde{\psi}^\mu_r}{\bar{z}^{r+1/2}}, \tag{3.42}$$

where for R sector the sums are over integers and for NS sector the sums are semi-integers.

For open strings with Neumann boundary conditions, we have

$$\psi^\mu = e^{i\pi\nu}\tilde{\psi}^\mu, \quad \text{for} \quad \mathrm{Re}(w) = 0, \tag{3.43}$$

and

$$\psi^\mu = e^{i\pi\tilde{\nu}}\tilde{\psi}^\mu, \quad \text{for} \quad \mathrm{Re}(w) = \pi. \tag{3.44}$$

By a field redefinition, we can set $\tilde{\nu} = 0$. In $(z, \bar{z})$ coordinates, the doubling trick is used to define $\psi^\mu(z)$ in the entire z-plane,

$$\psi^\mu(z) = \tilde{\psi}^\mu(\bar{z}'), \quad \text{for} \quad \mathrm{Im}(z) < 0, \tag{3.45}$$

with $z' = \bar{z}$. Then, the boundary condition $\tilde{\nu} = 0$ is a consistent condition and (3.44) is a condition on the periodicity of the extended holomorphic $\psi^\mu(z)$: it is periodic for $\nu = 0$ and antiperiodic for $\nu = 1/2$. This is one copy of what was found in the closed string case, and so the Laurent expansion for ψ^μ is simply the first equation in (3.42).

From the OPE of ψ's and $\tilde{\psi}$'s (or from canonical commutation relations), we have

$$\{\psi^\mu_r, \psi^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} = \{\tilde{\psi}^\mu_r, \tilde{\psi}^\nu_s\} \tag{3.46}$$

and the commutation for the α's and $\tilde{\alpha}$'s modes are the same as in previous section. The difference between NS and R sector lies in the different possible modes in the anticommutation relations above.

Postponing the discussion about residual symmetry, let us investigate the spectrum of a single set of NS or R sectors, corresponding to an open string (or one "side" of closed string). The closed string spectrum will then be defined as tensor products of the NS and R spectra.

The NS sector does not contain the $r = 0$ mode, the ground state is defined by $\psi^\mu_r|0\rangle_{\mathrm{NS}} = 0$ for $r > 0$ and the states are obtained by acting once (since the raising operators are anticommuting) with $r < 0$ modes. In the R sector, the ground state is degenerated since $\{\psi^\mu_0, \psi^\nu_0\} = \eta^{\mu\nu}$, which is (after identifying $\Gamma^\mu = \sqrt{2}\psi^\mu_0$) the Dirac gamma matrix algebra (a Clifford algebra). In fact, defining the ground states to be those annihilated by the $r > 0$ modes, the ψ^μ_0 operator will take the ground states into ground states.

Therefore, the ground states of the R sector forms a representation of the Clifford algebra and other states are created by acting with the $r < 0$ modes. In $D = 2k + 2$ such representation is labelled by $k + 1$ parameters $s_0, s_1, \ldots, s_k$, each one being $+1/2$ or $-1/2$, giving a representation of dimension 2^{k+1}. The critical dimension for superstrings is $D = 10$, that gives $2^{4+1} = 32$ dimensional Dirac representation,

$$|s_0, s_1, s_2, s_3, s_4\rangle_{\mathrm{R}} \equiv |\mathbf{s}\rangle_{\mathrm{R}}, \quad s_a = \pm\frac{1}{2}. \tag{3.47}$$

These states are eigenstates of the Lorentz generators

$$S_a = i^{\delta_{a,0}}\Sigma^{2a,2a+1}, \quad \Sigma^{\mu\nu} = -\frac{i}{2}\sum_{r\in\mathbb{Z}+\nu}\left[\psi^\mu_r,\, \psi^\nu_{-r}\right]. \tag{3.48}$$

In terms of irreducible Weyl representations we can reduce the Dirac representation as $\mathbf{32} = \mathbf{16} + \mathbf{16}'$. Regarding the chirality operator $\Gamma = 2^5 S_0S_1S_2S_3S_4$, states in $\mathbf{16}$ have $+1$ eigenvalue while states in $\mathbf{16}'$ have -1 eigenvalue. The operator $e^{\pi i F}$, where

$$F = \sum_{a=0}^{4} S_a, \tag{3.49}$$

can also be used to identify each Weyl representation a state belongs to. It commutes with the entire ψ^μ and so a raising operator changes F by one, which implies that F counts the fermionic "nature" of the states.

Notice that not only the ground state of the R sector is a spacetime fermion, but all states have half-integer spin as well, since the raising operators are spacetime vectors. The same reasoning implies that the NS states will have integer spin, as the NS ground state is a non-degenerated Lorentz singlet.

But we also need to impose the constraints. The residual conformal symmetry is now enlarged to a superconformal symmetry, whose infinitesimal action on the fields is

$$\delta X^\mu(w,\bar{w}) = \sqrt{\frac{\alpha'}{2}}\epsilon\left(\eta(w)\psi^\mu(w) + \eta^*(\bar{w})\tilde{\psi}^\mu(\bar{w})\right), \tag{3.50}$$

$$\delta\psi^\mu(w) = -\sqrt{\frac{\alpha'}{2}}\epsilon\eta(w)\partial_w X^\mu(w), \tag{3.51}$$

$$\delta\tilde{\psi}(\bar{w}) = \sqrt{\frac{\alpha'}{2}}\epsilon\eta^*(\bar{w})\bar{\partial}_{\bar{w}} X^\mu(\bar{w}), \tag{3.52}$$

where $\eta(w)$ is a Grassmann parameter. These transformations are generated by the worldsheet supercurrents

$$T_F(w) = i\sqrt{\frac{2}{\alpha'}}\psi^\mu(w)\partial_w X_\mu(w), \quad \tilde{T}_F = i\sqrt{\frac{2}{\alpha'}}\tilde{\psi}^\mu(\bar{w})\tilde{\partial}_{\bar{w}} X_\mu(\bar{w}). \tag{3.53}$$

The anticommuting fields contributes to the energy-momentum tensor,

$$T_B = -\frac{1}{\alpha'} : \partial_w X^\mu \partial_w X_\mu : -\frac{1}{2} : \psi^\mu \partial_w \psi_\mu :, \quad \tilde{T}_B = -\frac{1}{\alpha'} : \tilde{\partial}_w X^\mu \tilde{\partial}_{\bar{w}} X_\mu : -\frac{1}{2} : \tilde{\psi}^\mu \bar{\partial}_{\bar{w}} \tilde{\psi}_\mu :, \tag{3.54}$$

that generates conformal transformations, under which $\psi^\mu(z)$ and $\tilde{\psi}^\mu(\bar{z})$ are primary operators with conformal weight $1/2$. The normal ordering handles the divergence

$$\psi^\mu(z)\psi^\nu(0) \sim \frac{\eta^{\mu\nu}}{z}, \quad \tilde{\psi}^\mu(\bar{z})\tilde{\psi}^\nu(0) \sim \frac{\eta^{\mu\nu}}{z}. \tag{3.55}$$

The commutator of two superconformal transformations gives a conformal transformation. So, the superconformal and conformal transformations forms a closed algebra, as can be seen from the OPEs of the generators,

$$T_B(z)T_B(0) \sim \frac{3D}{4z^4} + \frac{2}{z^2}T_B(0) + \frac{1}{z}\partial T_B(0), \tag{3.56}$$

$$T_B(z)T_F(0) \sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0), \tag{3.57}$$

$$T_F(z)T_F(0) \sim \frac{D}{z^3} + \frac{2}{z}T_B(0), \tag{3.58}$$

and so the central charge is now $c = 3D/2$. There are analog forms for OPE's with $\tilde{T}_F$. Together with the mode expansions

$$T_F(z) = \sum_{r\in\mathbb{Z}+\nu} \frac{G_r}{z^{r+3/2}}, \quad \tilde{T}_F(\bar{z}) = \sum_{r\in\mathbb{Z}+\nu} \frac{\tilde{G}_r}{\bar{z}^{r+3/2}}, \tag{3.59}$$

the previous OPE's (or simply the canonical commutation relations) gives the super-Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-m)\delta_{m+n,0}, \tag{3.60}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0}, \tag{3.61}$$

$$[L_m, G_r] = \frac{m-2r}{2}G_{m+r}, \tag{3.62}$$

with another copy for $\tilde{L}_m$, $\tilde{G}_r$.

Similarly to the previous section, we should impose

$$(L_n - a\delta_{n,0})|\psi\rangle = 0, \quad n \geq 0, \tag{3.63}$$

$$G_r|\psi\rangle = 0, \quad r \geq 0 \tag{3.64}$$

and the equivalence relations

$$L_n|\phi\rangle \cong 0, \quad n < 0, \tag{3.65}$$

$$G_r|\phi\rangle \cong 0, \quad r < 0. \tag{3.66}$$

The critical dimension turns out to be $D = 10$ and the constant a depends on the sector, being 0 and 1/2 for R and NS sectors, respectively.

For open strings, we have four possible total sectors, corresponding to R or NS and the value of $\exp(\pi i F)$: NS_+, NS_-, R_+ and R_-. The mass-shell condition coming from L_0 constraint gives

$$M^2 = \frac{1}{\alpha'}(N - \nu), \tag{3.67}$$

where the number operator N now includes contributions from the worldsheet fermions,

$$N = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha^\mu_{-n}\alpha_{\mu n} : + \frac{1}{2}\sum_{r\in\mathbb{Z}+\nu} r : \psi^\mu_{-r}\psi_{\mu r} : + \nu. \tag{3.68}$$

The NS ground state, $|0; k\rangle_{\rm NS}$ is a tachyon with mass $M^2 = -1/(2\alpha')$ and has fermion number $\exp(\pi i F) = -1$. The first excited states are $e_\mu \psi^\mu_{-1/2}|0; k\rangle_{\rm NS}$ with zero mass and[3] $\exp(\pi i F) = +1$. The polarization vector e^μ satifies equation (3.38), hence at first level we get a massless vector field that is in $\mathbf{8_v}$ vector representation of the SO(8) that preserves the null-like momentum.

In the R sector, the lowest level states are

$$|u; k\rangle_{\rm R} = \sum_{\mathbf{s}} |\mathbf{s}; k\rangle_{\rm R} u_{\mathbf{s}}, \tag{3.69}$$

and the mass-shell condition implies that they are massless, as $\nu = 0$ in R sector. The G_0 constraint gives the Dirac equation in momentum space,

$$k_\mu \Gamma^\mu_{\mathbf{s}'\mathbf{s}} u_{\mathbf{s}} = 0, \tag{3.70}$$

and so we have a $D = 10$ spacetime Dirac spinor. In terms of the SO(8) representations, the Dirac equation leaves an spinorial $\mathbf{8_s}$ and $\mathbf{8_s}'$, with $\exp(\pi i F) = +1$ and $\exp(\pi i F) = -1$, repectively.

For closed strings, we just take the the tensorial product of the open sectors above. The mass-shell condition is

$$M^2 = \frac{4}{\alpha'}(N - \nu) = \frac{4}{\alpha'}(\tilde{N} - \tilde{\nu}), \tag{3.71}$$

and the level matching condition $N = \tilde{N}$ prevents pairing the sector NS_- with any other but itself. At lowest level we get a tachyon with mass $M^2 = -2/\alpha'$. At the first excited level, there are several states in each possible total sectors, built from the tensor product of the massless states in the NS_+, NS_-, R_+ and R_- total sectors. Products of the form $\mathrm{NS} \otimes \mathrm{NS}$ and $\mathrm{R} \otimes \mathrm{R}$ contains spacetime bosons and $\mathrm{NS} \otimes \mathrm{R}$ or $\mathrm{R} \otimes \mathrm{NS}$ contains spacetime fermions.

To give an example, for the sector $(\mathrm{NS}_+ \otimes \mathrm{NS}_+)$ we have a tensor decomposition of the form

$$\mathbf{8_v} \times \mathbf{8_v} = [0] + [2] + (2), \tag{3.72}$$

where $[p]$ and (p) denotes rank-p antisymmetric and traceless symmetric tensor representations, respectively. So, we get 10 dimensional dilaton $\phi(x)$, metric $g_{\mu\nu}(x)$ and antisymmetric tensor $B_{\mu\nu}(x)$ in this total sector.

The total sectors can be truncated and then combined to eliminate the tachyon and get an consistent string theory. Consistency with OPEs of vertex operators and modular invariance of one-loop torus amplitude impose some restrictions on which states could be projected out. These are the Gliozzi-Scherk-Olive (GSO) projections [156]. For closed strings, keeping only sectors with $\exp(\pi i F) = +1 = \exp(\pi i \tilde{F})$ gives the *type IIB theory*,

$$\text{IIB:} \quad (\mathrm{NS}_+ \otimes \mathrm{NS}_+) \oplus (\mathrm{NS}_+ \otimes \mathrm{R}_+) \oplus (\mathrm{R}_+ \otimes \mathrm{NS}_+) \oplus (\mathrm{R}_+ \otimes \mathrm{R}_+), \tag{3.73}$$

while taking a GSO projections with $\exp(\pi i \tilde{F}) = -1$ for the R sector gives the *type IIA theory*,

$$\text{IIA:} \quad (\mathrm{NS}_+ \otimes \mathrm{NS}_+) \oplus (\mathrm{NS}_+ \otimes \mathrm{R}_-) \oplus (\mathrm{R}_+ \otimes \mathrm{NS}_+) \oplus (\mathrm{R}_+ \otimes \mathrm{R}_-). \tag{3.74}$$

[3] To find these fermion number values, one should consider the contribution of the vacuum states of the ghosts (see [25])

Since the fermions are in the NS-R, R-NS sectors, the GSO projections are such that the type IIA and IIB theories has non-chiral and chiral spectra, respectively.

For the open string case, as string interactions between open strings can generate closed strings, we should have a theory with open and closed strings. To get a consistent interacting theory we should have an unoriented theory that has GSO projected open sector ($\mathrm{NS}_+ \otimes \mathrm{R}_+$), and that together with the unoriented truncation of type IIB gives the *type I open and closed unoriented theory*. Adding Chan-Paton factors in the open sector gives a gauge group that should be $SO(32)$ in order to cancel one-loop anomalies in gauge and spacetime symmetries.

There are also *heterotic strings* that are closed strings with one "side" consisting of the same constraint and worldsheet fields of bosonic string and the other "side " being as in type IIB theory. In the fermionic construction of heterotic strings, we should have 32 worldsheet fermions that are "internal" (they do not carry a spacetime vector index):

$$S_h = \frac{1}{4\pi} \int d^2z \left(\frac{2}{\alpha'} \partial_z X^\mu \bar{\partial}_{\bar{z}} X_\mu + \lambda^a \bar{\partial}_{\bar{z}} \lambda^a + \tilde{\psi}^\mu \partial_z \tilde{\psi}_\mu \right). \tag{3.75}$$

The worldsheet theory seems to be the same as in S_{SP}, but as $\lambda(z)^a$ have internal indices, the constraints will be very different than before. The global $SO(32)$ worldsheet symmetry will give rise to local gauge spacetime symmetries, specified by the boundary conditions of $\lambda^a(z)$ and GSO projections. Cancelation of one-loop gauge anomalies will fix the gauge symmetry to have the gauge group $SO(32)$ or $E_8 \times E_8$. So there are two possible consistent heterotic strings: the $SO(32)$ *and* $E_8 \times E_8$ *heterotic theories*.

Summarizing, there are 5 kinds of consistent superstrings: the type I, the two type II and the two heterotic string theories. They are all connected by dualities, as quickly reviewed in next section, and so the modern interpretation is that they are all different approaches for the same underlying, not totally known, theory.

The GSO projections has the remarkable property of keeping the spectrum supersymmetric at each mass level, so it connects the worldsheet supersymmetry to spacetime supersymmetry. The massless spectrum of each superstring is given in table 3.1. The type II theories have two spinor-vector gravitini in the NS-R sectors, with opposite chirality for type IIA and same chiralities for type IIB, so these theories have $\mathcal{N} = 2$ spacetime supersymmetry (hence their name). The unoriented truncation kills one of type IIB gravitini in the type I theory, which so has $\mathcal{N} = 1$ supersymmetry. The heterotic strings also have $\mathcal{N} = 1$ supersymmetry, as their massless spectrum includes the type I supergravity multiplet.

We will be interested in the low energy spacetime action for the massless spectrum as in making contact with the real world we expect the massless fields to be the first non-trivial contribution for a possible string description of quantum fields. So, in the next section we describe how we can obtain actions for the spacetime fields whose quanta we got by quantizing the superstrings.

String Theory	Massless states (in irreps. of SO(8))	
	bosonic	fermionic
Type IIA	$[0]+[1]+[2]+[3]+(2)$	$\mathbf{8_s}+\mathbf{8_s'}+\mathbf{56}+\mathbf{56'}$
Type IIB	$[0]^2+[2]^2+[4]_++(2)$	$\mathbf{8_s'}^2+\mathbf{56}^2$
Type I SO(32)	$[0]+[2]+(2)+(\mathbf{8_v})_{\mathrm{SO}(32)}$	$\mathbf{8_s'}+\mathbf{56}+(\mathbf{8_s})_{\mathrm{SO}(32)}$
Heterotic SO(32)	$[0]+[2]+(2)+(\mathbf{8_v})_{\mathrm{SO}(32)}$	$\mathbf{8_s'}+\mathbf{56}+(\mathbf{8_s})_{\mathrm{SO}(32)}$
Heterotic $\mathrm{E}_8\times\tilde{\mathrm{E}}_8$	$[0]+[2]+(2)+(\mathbf{8_v})_{\mathrm{E}_8}+(\mathbf{8_v})_{\tilde{\mathrm{E}}_8}$	$\mathbf{8_s'}+\mathbf{56}+(\mathbf{8_s})_{\mathrm{E}_8}+(\mathbf{8_s})_{\tilde{\mathrm{E}}_8}$

Table 3.1: The massless spectrum of various superstrings. In case gauge symmetry is present, most of the states are gauge singlets but there are states in the adjoint of the gauge group, as indicated as a subscript with the group's name. The type I and heterotic SO(32) spectra start to be different at further mass levels. The dimensions of the representations for bosonic and fermionic states are consistent with supersymmetry. In fact, all the spectra are spacetime supersymmetric.

3.3 Low energy Supergravities and String Duality Web

3.3.1 The bosonic string case

There are various consistent ways to get low energy actions in String Theory that are all connected and equivalent. At the linear level in spacetime fields, one could obtain their equations of motion by requiring Weyl invariance of the vertex operators related to massless states, or to infer an action from stringy amplitudes between the massless states. In the following, we will discuss how the action for bosonic string spacetime fields can be obtained by requiring the cancellation of an anomaly.

First let us emphasize that "low energy" means the limit $\alpha' \to 0$. In String Theory, there is only one dimensionful parameter, α' (with mass dimension -2), that is basically the inverse of the string tension. Putting α' close to zero implies that we are neglecting the heavy high spin excitations of the string, as its tension is very big. Also, one can think of having an effective theory for characteristic length scales very long compared to the string length, $\sqrt{\alpha'}$. So, in the effective field theory approach, only the massless modes would contribute to the spacetime action at low energies.

Such intuition can be confirmed by looking at the worldsheet action in curved backgrounds. Let us consider the bosonic string in a spacetime with a non-trivial background of massless fields

$$S_{CP} = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{\gamma}\left\{\left[\gamma^{ab}G_{\mu\nu}(X)+\epsilon^{ab}B_{\mu\nu}(X)\right]\partial_a X^\mu\partial_b X^\nu - \alpha'\mathcal{R}^{(2)}\Phi(X)\right\}, \quad (3.76)$$

where $\mathcal{R}^{(2)}$ is the worldsheet Ricci scalar. For $\Phi(X)=\phi_0$ constant, the last term is $\phi_0\chi$, where χ is the Euler characteristic of the 2-dimensional manifold Σ. From the 2-dimensional point of view, this theory is an interacting field theory with infinite dimensional set of couplings given by the functions $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$. Expanding around some classical solution $X^\mu(\sigma)=x_0^\mu$, would give a series of derivative interaction terms for $\sqrt{\alpha'}Y^\mu(\sigma)=X^\mu(\sigma)-x_0^\mu$

with couplings being the derivatives of the background fields. From the metric term, this would imply that the real dimensionless coupling of the 2-dimensional theory is $\sqrt{\alpha'}/R_c$, where R_c is characteristic radius of curvature of the spacetime, coming from the derivative expansion of the metric. So, for $\sqrt{\alpha'}/R_c \ll 1$, $\alpha' \ll 1$, we can use perturbation theory in the worldsheet theory and neglect the internal structure of the string.

Turning to the quantum theory, not all backgrounds are consistent with Weyl invariance. Working with the fully covariant Polyakov action S_P (before gauge fixing to conformal gauge) in a trivial background, under an infinitesimal Weyl transformation (3.6), we have

$$\delta_W \left(\int [dX d\gamma] e^{-S_P[X,\gamma]} \right) = -\frac{1}{2\pi} \int d^2\sigma \sqrt{\gamma}\, \delta w(\sigma) \langle T^a_{\ a}(\sigma) \rangle, \tag{3.77}$$

with[4]

$$T^a_{\ a} = -\frac{c_t}{12} \mathcal{R}^{(2)}, \tag{3.78}$$

and $c_t = D - 26$ is the central charge of the full covariant worldsheet theory. The difference between c_t and the central charge c in the previous sections is the fact that the former includes contributions from the Faddeev-Popov ghosts related to the symmetries used to go to the conformal gauge. As the Weyl symmetry is a local symmetry, the total central charge c_t should be zero to cancel the trace anomaly, fixing the number of spacetime dimensions $D = 26$. So, not all flat backgrounds can consistently support quantum strings.

Now, for the Polyakov action in curved backgrounds, we get

$$T^a_{\ a} = -\frac{1}{2\alpha'} \beta^G_{\mu\nu} \gamma^{ab} \partial_a X^\mu \partial^b X^\nu - \frac{i}{2\alpha'} \beta^B_{\mu\nu} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \beta^\Phi \mathcal{R}^{(2)}, \tag{3.79}$$

where, explicitly showing all terms up to two spacetime derivatives,

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\omega} H_\nu^{\ \lambda\omega} + \mathcal{O}(\alpha'^2), \tag{3.80}$$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2} \nabla^\omega H_{\omega\mu\nu} + \alpha' H_{\omega\mu\nu} \nabla^\omega \Phi + \mathcal{O}(\alpha'^2), \tag{3.81}$$

$$\beta^\Phi = \frac{D-26}{6} - \frac{\alpha'}{2} G^{\mu\nu} \nabla_\mu \nabla_\nu \Phi + \alpha' \nabla_\omega \Phi \nabla^\omega \Phi - \frac{\alpha'}{24} H_{\mu\lambda\nu} H^{\mu\nu\lambda} + \mathcal{O}(\alpha'^2), \tag{3.82}$$

where $H_{\mu\nu\lambda} = 6\partial_{(\mu} B_{\nu\lambda)}$ is the field stregnth of the Kalb-Ramond field and $R_{\mu\nu}$ and all the covariant derivatives ∇_μ are constructed from the spacetime metric $G_{\mu\nu}$. Thus, consistent backgrounds for String Theory should satisfy $\beta^{G,\,B}_{\mu\nu} = 0 = \beta^\Phi$, which gives dynamical equations for the background fields! In another point of view, these conditions come from the fact that equations (3.80), (3.81) and (3.82) are one-loop beta functionals of the functional couplings, related to the renormalization of UV divergences in the worldsheet theory (hence the β notations). So, the worldsheet theory will be Weyl-invariant if there is no RG flow of the functional couplings. The connection between Weyl-invariance and finiteness of the 2-dimensional theory is very profound and stands at higher α' (loop) orders.

[4] There is also a possible constant term contribution for the trace but it can be canceled by including a counter-term in the action.

The vanishing of all beta functionals can be recasted as the set of equations of motion of the spacetime action

$$S = \frac{1}{2\kappa_0^2}\int d^D x\sqrt{-G}e^{-2\Phi}\left\{R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{2}{3\alpha'}(D-26) + \mathcal{O}(\alpha')\right\}. \tag{3.83}$$

Note that Φ has a wrong sign in its kinetic term and that κ_0 is has no physical meaning, as it can be absorbed by redefining Φ. In fact, making the rescaling

$$\tilde{G}_{\mu\nu}(X) = e^{-2\tilde{\Phi}/(D-2)}G_{\mu\nu}(X), \tag{3.84}$$

with $\tilde{\Phi} = \Phi - \Phi_0$, we get

$$S = \frac{1}{2\kappa_D^2}\int d^D x\sqrt{-\tilde{G}}\left\{\tilde{R} - \frac{1}{12}e^{-\frac{8}{D-2}\tilde{\Phi}}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{4}{D-2}\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi}\right.$$
$$\left. - \frac{2}{3\alpha'}(D-26)e^{\frac{4}{D-2}\tilde{\Phi}} + \mathcal{O}(\alpha')\right\}, \tag{3.85}$$

where now the Einstein-Hilbert term is properly normalized with $\kappa_D = \kappa_0 e^{\Phi_0}$ related to the D dimensional Newton's constant as $\kappa_D^2 = 8\pi G_D$ and $\tilde{\Phi}$ has a correct sign in its kinetic term. Φ_0 is to be thought as the vev of Φ in such a way that $\langle\tilde{\Phi}\rangle = 0$. All the indices are raised with the $\tilde{G}_{\mu\nu}$ metric that is referred to as the Einstein metric, while $G_{\mu\nu}$ is called the string metric. We see that bosonic string theory includes Einstein's gravity coupled with a 2-form and scalar fields, at the lowest level in α'. There are stringy corrections to the Einstein's equations, coming from α' corrections to the previous beta functionals (the $\mathcal{O}(\alpha')$ terms in the action).

3.3.2 The superstring cases

All possible consistent backgrounds for bosonic strings should come from solutions to the theory (3.83). In the superstring cases, the low energy actions are various types of supergravities in 10 dimensions. The spacetime supersymmetry manifested in the string spectrum is very useful to find the low energy actions. For completeness, in the following discussion we list the spectra and bosonic part of the actions, in accordance with table 3.1. For further details see [25] and references therein.

Notice that all 5 superstrings listed in the last section have the dilaton Φ, metric $G_{\mu\nu}$ and Kalb-Ramond form $B_{\mu\nu}$ in the massless spectrum. These are the $[0] + [2] + (2)$ irreps. of SO(8) in table 3.1, that comes from the NS-NS sector. The R-R sector will give rise to antisymmetric fields (in $[p]$ irreps.). In p-form language, antisymmetric tensors $A_{\mu_1\ldots\mu_p}$ in $[p]$ have an action

$$S[A_p] = -\frac{1}{2}\int d^D x\sqrt{-G}|F_{p+1}|^2 = -\frac{1}{2p!}\int d^D x\sqrt{-G}F_{\mu_1\ldots\mu_{p+1}}F^{\mu_1\cdots\mu_{p+1}}, \tag{3.86}$$

where $F_{p+1} = dA_p$ is the field strenght of A_p and we are using the notation

$$X_p = \frac{1}{p!}X_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \tag{3.87}$$

for any tensor tensor field $X_{\mu_1\ldots\mu_p}$ in $[p]$. The action (3.86) is invariant under gauge transformations, that in p-form language is $\delta A_p = d\lambda_{p-1}$, and implies the equation of motion

$$d * F_{p+1} = 0, \tag{3.88}$$

where the Hodge star operator $*$ is defined as

$$* X_{\mu_1\ldots\mu_{d-p}} = \frac{1}{p!}\epsilon_{\mu_1\ldots\mu_{d-p}}{}^{\nu_1\ldots\nu_p} A_{\nu_1\ldots\nu_p}. \tag{3.89}$$

For type IIA theory, the bosonic sector contains the usual dilaton Φ, metric $G_{\mu\nu}$ and Kalb-Ramond form B_2 with field strength H_3. The R-R p-forms are C_1 and C_3, with field strengths F_2 and F_4. The action is

$$\begin{aligned} S_{\mathrm{IIA}} = &\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\ &- \frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(|F_2|^2 + |\tilde{F}_4|^2\right) - \frac{1}{4\kappa_{10}^2}\int B_2\wedge F_4\wedge F_4, \end{aligned} \tag{3.90}$$

where $\tilde{F}_4 = dC_3 - C_1\wedge H_3$. Despite the explicit dependence on the B_2, the last topological *Chern-Simons* term is gauge invariant as a consequence of the Bianchi identity for F_4.

In the type IIB case, on top of the NS-NS usual fields, we have the R-R p-forms C_0, C_2 and C_4, with field strengths F_1, F_3 and F_5, the last satisfying the self-dual condition, $F_5 = *F_5$. In terms of

$$\tilde{F}_3 = F_3 - C_0\wedge H_3, \quad \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3, \tag{3.91}$$

the action is

$$\begin{aligned} S_{\mathrm{IIB}} = &\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}|H_3|^2\right) \\ &- \frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) - \frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3, \end{aligned} \tag{3.92}$$

together with the self-dual condition, that should be imposed as a constraint on the solutions. For both type II strings, we have $2\kappa_{10}^2 = (2\pi)^7\alpha'^4$ by consistency with string amplitudes.

As compared with the type IIB case, the unoriented projection of type I superstring kills off B_2, C_0 and C_4. There is also an A_1 gauge field in the adjoint of SO(32) (so we take the trace in the corresponding representation). The action is

$$S_{\mathrm{I}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{\kappa_{10}^2}{g^2}\frac{e^{\Phi}}{30}\mathrm{Tr}_{\mathrm{adj}}\left(|F_2|^2\right) - \frac{1}{2}e^{2\Phi}|\tilde{F}_3|^2\right), \tag{3.93}$$

where

$$\tilde{F}_3 = dC_2 - \frac{\kappa_{10}^2}{g^2}w_3, \quad w_3 \equiv \mathrm{Tr}_{\mathrm{v}}\left(A_1\wedge dA_1 - i\frac{2}{3}A_1\wedge A_1\wedge A_1\right). \tag{3.94}$$

The dilaton dependence in the gauge action comes from the open-stringy origin of the gauge group.

Finally, the low energy action for heterotic strings is very similar to the type I case:

$$S_{\text{Heterotic}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{\kappa_{10}^2}{30g^2}\text{Tr}_{\text{adj}}\left(|F_2|^2\right) - \frac{1}{2}|\tilde{H}_3|^2\right), \quad (3.95)$$

where the $\tilde{H}_3$ has a similar definition to the first equation in (3.94) with B_2 instead of C_2 in the first term.

3.3.3 M-Theory, $D = 11$ Supergravity and String Dualities

In String Theory, the strength of string interactions is determined by the dilaton vev $\langle\Phi\rangle$. This comes from the fact that string interactions are topological in nature, so the basic interaction vertex of strings consists of one string splitting into two or vice-versa. From the last term in the action (3.76), thinking of an amplitude coming from a path integral, a constant value for the dilaton measures the impact of the Euler characteristic (topology of the worldsheet) to a given process. So, there is an expansion on the topology of the wordsheet for any given amplitude, that for concreteness can be thought as a genus expansion. Each topology will give a contribution, that is weighted by the exponential of the dilaton. So, the string coupling is defined as $g_s = e^{\langle\Phi\rangle}$.

We have been describing the strings in a perturbative approach, with only interactions of massless states taken into account. One could ask what happens with the spectrum and with the theory in general if one consider the strong coupling limit of String Theory. Starting with the type IIA, the answer is surprising: its strong coupling limit is an eleven dimensional theory, called M-theory, not totally understood.

The low energy limit of M-theory is eleven dimensional supergravity. So, we should expect some connection between IIA supergravity and $D = 11$ supergravity. In fact, the dimensional reduction of 11-dimensional supegravity on a circle gives the IIA supergravity. In eleven dimensions, there is only one possible supersymmetry charge (one vector-spinor gravitino), in the spin $\mathbf{128_s}$ irrep. of the SO(9) little group. The bosonic sector contains a metric G_{MN} and a 3-form field A_{MNP}, denoted A_3 in form language with field strength F_4 (the indices M, N run from 0 to 10). The bosonic part of the 11-dimensional action is

$$S_{11} = \frac{1}{2\kappa_{11}}\int d^{11}x\sqrt{-G}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa_{11}}\int A_3 \wedge F_4 \wedge F_4. \quad (3.96)$$

Consider the 10th direction to be a circle of radius R. Under the Kaluza-Klein Ansatz

$$ds^2 = G_{MN}^{(11)}dx^M dx^N = e^{-2\Phi/3}G_{\mu\nu}(x^\mu)dx^\mu dx^\nu + e^{4\Phi/3}(dx^{10} + (C_1)_\mu(x^\mu)dx^\mu)^2, \quad (3.97)$$

the action S_{11} reduces to S_{IIA} of the last subsection, with C_1 coming from the $G_{\mu 10}^{(11)}$ as indicated above, $B_{\mu\nu} = A_{\mu\nu 10}$ and $(C_3)_{\mu\nu\sigma} = A_{\mu\nu\sigma}$. From the supergravity actions, the gravitational couplings are related by $\kappa_{11}^2 = 2\pi R\kappa_{10}^2 e^{2\langle\Phi\rangle}$. A glance at the metric reduction Ansatz reveals that

$$\left(\frac{R}{l_{\text{P}}^{(11)}}\right)^2 = e^{4\langle\Phi\rangle/3} = g_s^{4/3}, \quad (3.98)$$

where $l_{\rm P}^{(11)} = (2\kappa_{11}^2)^{1/9}(2\pi)^{-8/9}$ is the 11-dimensional Planck length. Thus, we have $R = g_s\sqrt{\alpha'}$, i.e., the radius R is just the string coupling in stringy units! At small string coupling, the compact direction is unperceived and the spacetime is effectively 10-dimensional, giving the IIA supergravity. But at strong coupling, the 10th direction decompactifies and we get $D = 11$ supergravity. So, at low energies, there is an equivalence between type IIA theory and M-theory compactified in a circle.

This equivalence is deeper than just a low energy match of the perturbative regimes. Even the non-perturbative spectrum of the theories (and bound states of them) are mapped to each other. At high energies, little is know about M-theory, and the best definition we have is that it is given by the strong coupling limit of type IIA.

In a very similar way, if we compactify M-theory on the orbifold $S^1/\mathbb{Z}_2$, we get the strong coupling limit of the heterotic $\rm E_8 \times E_8$ theory. At the two fixed points of the orbifold there are 9-branes, each one carrying a copy of an non-abelian $\rm E_8$ fields. Again, the eleventh dimension is unseen in perturbation theory because of an expansion around $R = 0$ for the extra dimension.

The equivalences highlighted above are just two examples of "dualities" between all 5 types of superstring theories. In the mid 1990, it was shown that all superstrings are just different ways to look at the same and unique theory [157]. We list other equivalences for completeness: T-duality, that in its simplest version is just a duality between compactification on a S^1 with radius R and on a S^1 with radius α'/R, takes both type II theories to each other and relates the SO(32) and $\rm E_8 \times E_8$ heterotic theories; S-duality, that is in its simplest version is $g_s \to 1/g_s$, takes type IIB to itself and turns SO(32) heterotic theory into type I theory.

The modern view is that there is only one *String Theory*, with different perturbative limits that gives the 5 types of superstrings. All 5 limits are connected by a "web of dualities" described in the previous paragraph.

3.4 T-duality and Double Field Theory

3.4.1 Toroidal Compactification of Bosonic Strings

From a phenomenological point of view, there should be some reason why we explicitly observe only 3 large spatial dimensions, if String Theory has something to do with reality at all. An obvious possibility is that the extra dimensions are very tiny, i.e., compactified in spaces with a small characteristic length scale as compared with the ordinary spatial directions. This idea was explored before in physics, for instance in the Kaluza-Klein-like theories.

The simplest compactification is a toroidal one. So, let us consider a spacetime with one dimension being a circle S^1_R with radius R. Take it to be the 25th direction in closed bosonic string theory. Since the string can now wind the compact direction w times, the boundary condition in such direction is now

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi w R \tag{3.99}$$

with $w \in \mathbb{Z}$. The boundary conditions on the other directions are periodic as before. The solution to the wave equation (3.8) is

$$X^\mu(\tau,\sigma) = x^\mu + \alpha' p^\mu \tau + w^\mu R\sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(\alpha_n^\mu e^{-in\sigma} + \tilde{\alpha}_n^\mu e^{in\sigma} \right), \tag{3.100}$$

where

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu), \quad w^\mu = \frac{1}{R}\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu), \tag{3.101}$$

with $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\alpha'/2}p^\mu$ for $\mu \neq 25$. If the string winds around the circle we have $\alpha_0^{25} \neq \tilde{\alpha}_0^{25}$, and so the only non-vanishing component of w^μ is $w^{25} = w$.

In $(z, \bar{z})$ coordinates, we have

$$X^\mu(z,\bar{z}) = X_L^\mu(z) + X_R^\mu(\bar{z}), \tag{3.102}$$

where

$$X_L^\mu(z) = x_L - i\frac{\alpha'}{2} p_L^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} z^{-n}, \tag{3.103}$$

$$X_R^\mu(z) = x_R - i\frac{\alpha'}{2} p_R^\mu \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}_n^\mu}{n} \bar{z}^{-n}, \tag{3.104}$$

with $x_L^\mu + x_R^\mu = x^\mu$ and

$$p_L^\mu = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu = p^\mu + \frac{w^\mu R}{\alpha'}, \quad p_R^\mu = \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0^\mu = p^\mu - \frac{w^\mu R}{\alpha'}. \tag{3.105}$$

The quantization is practically the same as in section 3.1. The main differences are that the momentum in the compact direction is quantized in units of R,

$$p^{25} = \frac{n}{R}, \tag{3.106}$$

that the level matching conditions between N and $\tilde{N}$ is now

$$N - \tilde{N} = -\frac{\alpha'}{4}\left((p_L^{25})^2 - (p_R^{25})^2 \right) = -nw, \tag{3.107}$$

and that the spectrum is

$$M_{24}^2 = \frac{1}{2}\left((p_L^{25})^2 + (p_R^{25})^2 \right) + \frac{2}{\alpha'}(N + \tilde{N} - 2) = \left(\frac{n}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2). \tag{3.108}$$

Notice that such spectrum is from the $D = 25$ point of view, so $M_{24}^2 = -p^\mu p_\mu$ does not include $\mu = 25$ (the n^2 contribution comes from the Virasoro constraint modes, that depends on all directions). Also, as in sections 3.1 and 3.2, the constraints are such that the time-like and longitudinal oscillations are not physical and we can count the number of states on a given level by considering just the transverse oscillators, $\mu = 2, \ldots, 25$.

Let us focus on the massless states, for the reasons mentioned before. For a general value of R, we should have $n = 0 = w$ and $N = 1 = \tilde{N}$, giving states of the form $\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,0;k\rangle$. Then we have 24^2 states that separates in multiplets of SO(23) little group. These are the 25-dimensional graviton, dilaton and rank-2 antisymmetric tensor fields, two 25-vector fields and a scalar. The vector field states are of the form

$$(\alpha^{\mu}_{-1}\tilde{\alpha}^{25}_{-1} + \alpha^{25}_{-1}\tilde{\alpha}^{\mu}_{-1})|0,0;k\rangle, \quad (\alpha^{\mu}_{-1}\tilde{\alpha}^{25}_{-1} - \alpha^{25}_{-1}\tilde{\alpha}^{\mu}_{-1})|0,0;k\rangle. \tag{3.109}$$

The first is nothing but a 25-vector that comes from the $G_{\mu 25}$ metric components. It was already expected from Kaluza-Klein compactification. The second one is purely stringy, coming from the so $B_{\mu 25}$ components of the rank-2 antisymmetric field. The scalar is of the form $\alpha^{25}\tilde{\alpha}^{25}|0,0;k\rangle$ and related with the G_{2525} metric component. In fact, we get a $U(1) \times U(1)$ gauge theory with n (w) being the charge of the $G_{\mu 25}$ ($B_{\mu 25}$) vector field.

At the special radius $R = \sqrt{\alpha'}$, there are four extra massless scalars (KK modes of the tachyon) and four extra massless vector bosons for $n = \pm 1 = w$, $(N, \tilde{N}) = (0,1)$ and for $n = \pm = -w$, $(N, \tilde{N}) = (1,0)$. For this specific radius the $U(1) \times U(1)$ gauge symmetry is enhanced to $SU(2) \times SU(2)$.

Let us go back to the mass-shell condition (3.108). Notice that if we exchange

$$n \leftrightarrow w \quad \textit{and} \quad R \leftrightarrow \tilde{R} \equiv \alpha'/R, \tag{3.110}$$

then the mass spectrum remains the same! In fact, such exchanges are a full symmetry of the quantum theory being also respected by string interactions. This is *T-duality*: the compactification on a circle with a given radius is equivalent to another with inverse radius (in stringy units).

Notice that the limit $R \to 0$ is equivalent to $\tilde{R} \to \infty$ and by T-duality the compact direction should not simply disappear. Indeed, starting from the theory with radius R and rewriting it in terms of

$$\tilde{X}^{25}(z,\bar{z}) = X^{25}_L(z) - X^{25}_R(\bar{z}), \tag{3.111}$$

that has same OPEs as $X^{25}(z,\bar{z})$, we get the theory with radius $\tilde{R}$, as changing $X \to \tilde{X}$ implies $p^{25}_L \to p^{25}_L$ and $p^{25}_R \to -p^{25}_R$. So, as $R \to 0$, $\tilde{X}^{25}(z,\bar{z})$ decompactifies. The center-of-mass position of $\tilde{X}^{25}$ is $\tilde{x}^{25} = x_L - x_R$. So, at a finite radius we have two position operators for the compact direction x^{25} and $\tilde{x}^{25}$. This idea was explored in the String Gas Cosmology model, as we will discuss in chapter 5.

Note that the radius $R_* = \sqrt{\alpha'}$ is self-dual under (3.110) (this is exactly the radius where we have the symmetry enhancement described before). Thus the real physical range of the modulus R can be chosen to be the half-line $R \geq \sqrt{\alpha'}$ and hence there is an effectively minimum distance scale for toroidal compacfications of perturbative string theory, given by the string length $\sqrt{\alpha'}$.

As compactifications at different radii are interpreted as different vacua for a single theory, in which R is just a modulus, T-duality is a symmetry that relates different states (vacua) of the same theory. The dilaton, that take part on specifying the vacuum, also transforms under T-duality,

$$e^{\tilde{\Phi}} = \frac{\sqrt{\alpha'}}{R} e^{\Phi}. \tag{3.112}$$

In fact, a more general background with an isometric direction, say $\mu = k$, transforms as

$$\tilde{G}_{kk} = \frac{1}{G_{kk}}, \quad \tilde{G}_{k\mu} = \frac{B_{k\mu}}{G_{kk}}, \quad \tilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{k\mu}G_{k\nu} - B_{k\mu}B_{k\nu}}{G_{kk}}, \tag{3.113}$$

$$\tilde{B}_{k\mu} = \frac{G_{k\mu}}{G_{kk}}, \quad \tilde{B}_{\mu\nu} = B_{\mu\nu} + \frac{G_{k\mu}B_{0\nu} - B_{k\mu}G_{0\nu}}{G_{kk}}, \tag{3.114}$$

$$\tilde{\Phi} = \Phi - \frac{1}{2}\ln G_{kk}. \tag{3.115}$$

These are called *Buscher rules* and can be found by a change of variables in the string path integral [158]. T-duality is an astonishing purely stringy symmetry. It does not only relates different string backgrounds but it seems that it could also be used to link different observers on a given background (see [159] for a study on that direction).

The circle compactification described above can be easily generalized to toroidal compactifications. Consider the spacetime to have k periodic directions such, i.e., it has a compact torus piece T^k. As we saw in section 3.3, String Theory puts constraints in the structure of spactime geometry (metric), and so it should constrain the possible allowed tori. We will consider generic backgrounds with constant metric and Kalb-Ramond fields in the compact directions, G_{mn}, B_{mn}, and flat metric and vanishing antisymmetric fields in the non-compact ones. The solution in the compact directions, with latin indices, is as in (3.100), but now the canonical momentum will receive contributions from $B_{\mu\nu}$:

$$p_m = \frac{n_m}{R} + B_{mn}\frac{w^n R}{\alpha'}, \tag{3.116}$$

and its left and right parts will again differ as in (3.105),

$$p_L^m = p^m + \frac{w^m R}{\alpha'}, \quad p_R^m = p^m - \frac{w^m R}{\alpha'}. \tag{3.117}$$

The mass-shell and level matching conditions are now

$$M_{26-k}^2 = \frac{1}{2}G_{mn}\left(p_L^m p_L^n + p_R^m p_R^n\right) + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{3.118}$$

$$N - \tilde{N} = -\frac{\alpha'}{4}G_{mn}\left(p_L^m p_L^n - p_R^m p_R^n\right) = -G_{mn}n^m w^n. \tag{3.119}$$

Notice that now R is just a reference scale, as the real radius of each compact direction and any other information about T^k is captured in G_{mn}. The discrete momenta (p_{Lm}, p_{Rm}) forms a lattice $\Lambda^{(k,k)}$ in the $\mathbb{R}^{2k}$ space. Any element of such a lattice can be written as $\lambda_m = \lambda_1 n_m + \lambda_2 w_m$, where the basis (λ_1, λ_2) of the lattice depends on the background fields G_{mn} and B_{mn}. Consistency with OPE of vertex operators and modular invariance of the string partition function implies that, for any element (k_{Lm}, k_{Rm}) of the lattice,

$$k_{Lm}k_L^m - k_{Rm}k_R^m \in 2\mathbb{Z}, \tag{3.120}$$

and so $\Lambda^{(k,k)}$ is a Lorentzian even lattice with (k,k) signature. Also from modular invariance, we have the self-dual condition

$$\Lambda^{(k,k)} = \Lambda^{*(k,k)}, \tag{3.121}$$

where $\Lambda^{*(k,k)}$ is the dual of $\Lambda^{(k,k)}$ lattice (that is, elements of the dual lattice have interger Lorentzian product with elements of $\Lambda^{(k,k)}$). T-dual inequivalent compactifications or toroidal backgrounds are parametrized by the coset space

$$\frac{\mathrm{O}(k,k,\mathbb{R})}{\mathrm{O}_L(k,\mathbb{R})\times\mathrm{O}_R(k,\mathbb{R})\times\mathrm{O}(k,k,\mathbb{Z})}. \tag{3.122}$$

This result comes from the fact that conditions (3.120) and (3.121) are invariant under Lorentz boosts of the $(k+k)$ momentum space and so $\mathrm{O}(k,k,\mathbb{R})$ transformations relates even self-dual lattices. But since the mass-shell condition (3.118) (and also also OPEs) is symmetric only under the action of the maximal subgroup $\mathrm{O}_L(k,\mathbb{R})\times\mathrm{O}_R(k,\mathbb{R})$, acting on the left and right parts of the momenta, we need to discount such transformations from $\mathrm{O}(k,k,\mathbb{R})$. The group factor $\mathrm{O}(k,k,\mathbb{Z})$ is the group of T-duality symmetry, that is enlarged as compared with the circle compactification case (3.110). It consists of discrete permutations of the individual lattice points (relabelling the basis vectors of the lattice), taking an initial lattice into itself. Note that this is a symmetry with constant background fields, that in turn determines the basis of the lattice.

To discuss the form of $\mathrm{O}(k,k,\mathbb{Z})$ transformations, let us rewrite the mass-shell and level matching conditions in a manifest $\mathrm{O}(k,k,\mathbb{R})$ way,

$$M^2_{26-k}=\frac{2}{\alpha'}(N+\tilde{N}-2)+\mathcal{P}^M\mathcal{H}_{MN}\mathcal{P}^N, \tag{3.123}$$

$$N-\tilde{N}=-\frac{\alpha'}{2}\mathcal{P}^M\eta_{MN}\mathcal{P}^N, \tag{3.124}$$

where the generalized momentum $\mathcal{P}^M$ and metric $\mathcal{H}_{MN}$ are defined as

$$\mathcal{P}^M=\begin{pmatrix}\frac{R}{\alpha'}w_m\\ \frac{1}{R}n^m\end{pmatrix},\quad \mathcal{H}_{MN}=\begin{pmatrix}G^{mn} & -G^{mp}B_{pn}\\ B_{mq}G^{qn} & G_{mn}-B_{mp}G^{pq}B_{qn}\end{pmatrix}, \tag{3.125}$$

and all indices can be raised or lowered by the $\mathrm{O}(k,k)$ invariant metric η_{MN} and its inverse

$$\eta_{MN}=\begin{pmatrix}0 & \delta^m{}_n\\ \delta_m{}^n & 0\end{pmatrix},\quad \eta^{MN}=\begin{pmatrix}0 & \delta_m{}^n\\ \delta^m{}_n & 0\end{pmatrix},\quad \eta_{MP}\eta^{PN}=\delta^N_M. \tag{3.126}$$

Notice that η_{MN} has signature (k,k). Any element $h\in\mathrm{O}(k,k,\mathbb{R})$ is a $2k\times 2k$ matrix with real entries that satisfies

$$h_M{}^P\eta_{PQ}h_N{}^Q=\eta_{MN}. \tag{3.127}$$

It is possible to show that

$$\mathcal{H}^{MP}\eta_{PQ}\mathcal{H}^{QN}=\eta^{MN}, \tag{3.128}$$

which together with the fact that $\mathcal{H}_{MN}$ is symmetric, it implies that the generalized metric also satisfies (3.127) and so it is a $\mathrm{O}(k,k,\mathbb{R})$ element. For the T-duality group, any transformation within $\mathrm{O}(k,k,\mathbb{Z})$ can be written as a composition of three operations [120]:

- *Diffeomorphisms*: It corresponds to a change of basis such that the combination $E_{mn} = G_{mn} + B_{mn}$ transforms as $E'_{mn} = A^p{}_m E_{pq} A^q{}_n$, where $A_m^{\ n} \in \mathrm{GL}(k, \mathbb{Z})$. As an $\mathrm{O}(k, k, \mathbb{Z})$ element,

$$h_M^{\ N} = \begin{pmatrix} A^m{}_n & 0 \\ 0 & A_m^{\ n} \end{pmatrix}; \tag{3.129}$$

- *Factorized T-dualities*: It is the generalizations of the exchange $R \leftrightarrow \tilde{R}$. For a direction k,

$$(h^{(k)})_M^{\ N} = \begin{pmatrix} \delta^m{}_n - t^m{}_n & t^{mn} \\ t_{mn} & \delta_m^{\ n} - t_m^{\ n} \end{pmatrix}, \tag{3.130}$$

where $t_{mn} = \delta_{mk}\delta_{nk}$ is a diagonal matrix with 1 in the kth position of the diagonal;

- *B-Shifts*: The element

$$h_M^{\ N} = \begin{pmatrix} \delta^m{}_n & 0 \\ b_{mn} & \delta_m^{\ n} \end{pmatrix}, \tag{3.131}$$

with $b_{mn} = -b_{mn}$ generates integer shifts in the components of the Kalb-Ramond field B_{mn}.

These three generating transformations iterates to give the full T-duality group of symmetries of the toroidal compactification of bosonic strings for an arbitrary number of compact dimensions $k < D$. Under a $h \in \mathrm{O}(k, k, \mathbb{Z})$ transformation, we have

$$\mathcal{P}^M \to h^M_{\ N} \mathcal{P}^N, \quad \mathcal{H}_{MN} \to h_M^{\ P} \mathcal{H}_{PQ} h_N^{\ Q}. \tag{3.132}$$

If h is a factorized T-duality in the kth compact direction, the transformation of the generalized metric gives the Buscher rules (3.113), (3.114), (3.115).

3.4.2 Double Field Theory (DFT)

The $\mathrm{O}(k, k)$ structure stressed in the previous subsection is made manifest in *Double Field Theory* (DFT). DFT not only makes T-duality manifest, but it also has an intrinsic $\mathrm{O}(D, D, \mathbb{R})$ invariance even on non-toroidal backgrounds, with T-duality appearing as a manifest symmetry on isometric backgrounds. In fact, in DFT the group of T-duality is the subgroup $\mathrm{O}(k, k, \mathbb{Z})$ of the $\mathrm{O}(D, D, \mathbb{R})$ group. We will simply write $\mathrm{O}(\cdot\,,\cdot)$, as the difference in considering continuous or discrete groups is irrelevant on discussing DFT at the classical level [120].

Let us motivate the DFT action from a very physical perspective. Consider the Kaluza-Klein theory that combines $(D-1)$-dimensional metric, vector and scalar fields in a D-dimensional metric. The action for the $(D-1) = d$ dimensional fields is

$$S_d = \frac{1}{2\kappa_d^2} \int d^d x \sqrt{-G^{(d)}} e^{\sigma} \left(R^{(d)} - \frac{e^{2\sigma}}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{3.133}$$

and has local symmetries parametrized by D functions: a d-vector ξ^μ that generates local diffeomorphisms, and a scalar λ that generates the gauge symmetry of the vector field. The gauge transformation is simply $\delta A_\mu = \partial_\mu \lambda$. The local diffs. acts on tensors as Lie derivatives,

$$\delta A^\mu = \mathcal{L}_\xi A^\mu = \xi^\rho \partial_\rho A^\mu - A^\rho \partial_\rho \xi^\mu \tag{3.134}$$

$$\delta G^{(d)}_{\mu\nu} = \mathcal{L}_\xi G^{(d)}_{\mu\nu} = \xi^\rho \partial_\rho G^{(d)}_{\mu\nu} + G^{(d)}_{\mu\rho} \partial_\nu \xi^\rho + G^{(d)}_{\nu\rho} \partial_\mu \xi^\rho. \tag{3.135}$$

The Kaluza-Klein "uplift" of (3.133) is made by combining the parameters λ, ξ into a D-vector $\xi^M = (\xi^\mu, \lambda)$, such that the diffeomorphisms on some D-dimensional metric reduces to the symmetries of the d-dimensional theory. In order to get such *geometrization* of all the local symmetry of the lower dimensional theory, the D-dimensional metric should have the form

$$ds^2 = G^{(D)}_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma} \left(dx^{D-1} + A_\mu dx^\mu \right)^2, \tag{3.136}$$

where the $(D-1)$th direction is periodic. Note that $G^{(D)}_{MN}$ should be independent of x^{D-1}, as is the D-vector ξ^M of local symmetry parameters, otherwise the D-dimensional diffs. fails to reproduce the gauge transformation for A^μ. As the D-dimensional theory contains a metric and is diff. invariant, we can immediately write

$$S_D = \frac{1}{2k_D^2} \int d^D x \sqrt{-G^{(D)}} R^{(D)}. \tag{3.137}$$

This action yields S_d under dimensional reduction with Kaluza-Klein Ansatz (3.136).

The mechanism of encoding the local symmetries of a lower dimensional theory into the diffeomorphisms of a higher dimensional theory described above was already touched in subsection 3.3.3. In the Kaluza-Klein reduction of $D = 11$ supergravity to $D = 10$ IIA supergravity, the $D = 10$ diffs. and the gauge symmetry of the C_1 RR form field were combined into $D = 11$ diffs. of the metric (3.97). In such an example, not all local 10-dimensional symmetries were geometrized, as the gauge transformation of other p-forms B_2 and C_3 are not included into $D = 11$ diffeomorphisms. In fact, as the R-R spectrum differs for each supergravity, while the NS-NS content is universal, it is interesting to consider the geometrization of the symmetries of the latter, that is the subject of the rest of this subsection. We will see that it is possible to start implementing this idea, but due to some important differences between the gauge symmetries of 1 and 2-forms, we get a much more evolved structure.

For generality, consider the bosonic action,

$$S[G, B_2, \Phi] = \int d^D x \sqrt{-G} e^{-2\Phi} \left(R + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} |H_3|^2 \right), \tag{3.138}$$

that is included in all supergravity actions and in the low energy action for bosonic strings (with proper dimension). It has D-dimensional diff. invariance and gauge symmetry of the Kalb-Rammond field as local symmetries:

$$\delta G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu}, \quad \delta B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu}, \quad \delta\Phi = \mathcal{L}_\xi \Phi, \tag{3.139}$$

$$\delta B_2 = d\lambda_1 \to \delta B_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu. \tag{3.140}$$

So, we have $2D$ symmetry parameters, ξ^μ and λ_μ. Following the steps for geometrizing these symmetries, we combine the gauge parameters in a generalized $2D$-vector $\xi^M = (\lambda_\nu, \xi^\mu)$.

Thus, if there is any hope to succeed in the geometrization mechanism, we must to introduce D extra coordinates to spacetime which gives the so called *double space* with $2D$ coordinates $X^M = (\tilde{x}_\mu, x^\mu)$.

The next natural step is to define a generalized metric on which $2D$-dimensional diffs. reduce to (3.139) and (3.140) under the assumption that all fields are independent of the dual coordinates $\tilde{x}_\mu$. In DFT, the *generalized metric* is

$$\mathcal{H}_{MN}(X) = \begin{pmatrix} G^{\mu\nu} & -G^{\mu\rho}B_{\rho\nu} \\ B_{\mu\sigma}G^{\sigma\nu} & G_{\mu\nu} - B_{\mu\rho}G^{\rho\sigma}B_{\sigma\nu} \end{pmatrix}, \tag{3.141}$$

That has the same structure as in (3.125), but now all fields depends on the double space (we will include the dilaton afterwards). Unfortunately, the gauge transformation of a D-dimensional 2-form field is not contained in $2D$-diffeomorphisms. So, we need a *generalized Lie derivative* $\hat{\mathcal{L}}$, such that the generalized diffs. generated by $\hat{\mathcal{L}}$ give the desired D-dimensional local transformations. This indicates that we need to depart from Riemannian geometry if we want to uplift a 2-form gauge symmetry to higher dimensions.

The generalized $2D$ diffeomorphisms of the generalized metric are generated by

$$\hat{\mathcal{L}}_\xi \mathcal{H}_{MN} = \mathcal{L}_\xi \mathcal{H}_{MN} + Y^R{}_M{}^P{}_Q \mathcal{H}_{RN} \partial_P \xi^Q + Y^R{}_N{}^P{}_Q \mathcal{H}_{MR} \partial_P \xi^Q, \tag{3.142}$$

with

$$Y^M{}_P{}^N{}_Q = -\eta^{MN}\eta_{PQ} \tag{3.143}$$

measuring the departure from Riemannian geometry and η_{MN} is a $2D \times 2D$ matrix as in (3.126). Together with the condition

$$\partial_{\tilde{x}_\mu}(\,\cdot\,) = 0, \tag{3.144}$$

when acting on any field, the generalized diffeomorphisms (3.142) reduces to (3.139) and (3.140). Condition (3.144) is the analog of the independence of the periodic dimension x^{D-1} in the Kaluza-Klein example.

A remarkable feature of the generalized Lie derivative is the fact that it preservers the $\mathrm{O}(D,D)$ metric and the delta symbol δ_M^N,

$$\hat{\mathcal{L}}_\xi \eta_{MN} = 0 = \hat{\mathcal{L}}_\xi \delta_M^N, \tag{3.145}$$

for arbitrary ξ^M. Thus the $\mathrm{O}(D,D)$ structure is preserved by generalized diffeomorphisms. In particular, the $\mathrm{O}(D,D)$ condition

$$\mathcal{H}^{MP}\eta_{PQ}\mathcal{H}^{QN} = \eta_{MN} \tag{3.146}$$

is preserved. So, generalized diffs. and the $\mathrm{O}(D,D)$ transformations

$$X^M \to X'^M = h^M_N X^N, \quad \mathcal{H}_{MN}(X) \to \mathcal{H}'_{MN}(X') = h_M^P \mathcal{H}_{PQ}(X) h_N^Q, \tag{3.147}$$

are compatible.

So far the only drastic difference with respect to the Kaluza-Klein case was the introduction of generalized diffeomorphisms. But this description is not totally consistent, since the commutator of two generalized Lie derivative is not a generalized Lie derivative: the algebra of generalized diffeomorphisms does not close. In fact, we have the following structure when acting on an arbitrary generalized tensor,

$$\left[\hat{\mathcal{L}}_{\xi_1},\, \hat{\mathcal{L}}_{\xi_2}\right] = \hat{\mathcal{L}}_{[\xi_1,\,\xi_2]_{\mathrm{C}}} + (\,\cdots\,), \tag{3.148}$$

where $[\cdot,\, \cdot]_C$, defined as

$$[\xi_1, \xi_2]^M_{\mathrm{C}} = \xi_1^N \partial_N \xi_2^M - \frac{1}{2}\xi_1^N \partial^M \xi_{2N} - (1 \leftrightarrow 2), \tag{3.149}$$

is called the C-bracket. The three-dots in equation (3.148) are terms that vanish under the *strong constraint*

$$\eta^{MN}\partial_M(\partial_N(\,\cdot\,)) = 0 \tag{3.150}$$

when acting on any double-space field. Thus we only have a consistent theory if we impose (3.150) and so DFT is a constrained theory, another novel feature as compared with the Kaluza-Klein case. In components, the strong constraint is just $\partial_\mu \partial^{\tilde{\mu}}(\cdot) = 0$, hence imposing all fields to be independent of dual coordinates (3.144) solves it automatically. This corresponds to a choice of *section* in the double-space and any $\mathrm{O}(D, D)$ rotation of such a choice also solves the strong constraint. For that reason, the strong constraint is also reffered to as the *section condition*.

Finally, let us write the DFT action. It should be invariant under generalized $2D$ diffeomorphisms and $\mathrm{O}(D, D)$ transformations and should reduces to (3.138) by imposing (3.150). The action that attend all these criteria is [150]

$$S_{\mathrm{DFT}} = \int d^D x d^D \tilde{x} e^{-2d} \mathcal{R}, \tag{3.151}$$

with

$$\begin{aligned}\mathcal{R} \equiv \Big(& 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ &+\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}\Big),\end{aligned} \tag{3.152}$$

and the generalized dilaton d is defined as

$$e^{-2d} = \sqrt{-G}e^{-2\Phi}. \tag{3.153}$$

The left-hand side of this definition transforms as a generalized scalar density,

$$\hat{\mathcal{L}}_\xi(e^{-2d}) = \partial_M(\xi^M e^{-d}), \tag{3.154}$$

while $\mathcal{R}$ is a generalized scalar. Such transformation laws guarantees that S_{DFT} is gauge invariant. Note that we could also impose $\partial_\mu(\cdot) = 0$ as the section condition. This would give the D-dimensional NS-NS supergravity action again, but defined over the dual space with $\tilde{x}^\mu$ coordinates. Even all fields depending of some $\mathrm{O}(D, D)$ invariant combination of x^μ and $\tilde{x}^\mu$ do solve the strong constraint.

3.4.3 T-duality in DFT

Note that the $\mathrm{O}(D,D)$ symmetry of DFT is independent of T-duality. The double space over which the DFT action (3.151) is defined may have periodic directions or not. The $\mathrm{O}(D,D)$ of DFT is a general symmetry of String Theory backgrounds that is not manifest in the supergravity actions: in (3.138) each term is a scalar but they are not invariant under $\mathrm{O}(D,D)$; in (3.151) all terms are $\mathrm{O}(D,D)$ invariant independently, but they are not generalized scalars [160].

The first novel feature of DFT on a toroidal background is the relaxation of the strong constraint. One can expand the DFT fields in modes of the generalized momenta in the compact directions. Then, when derivatives hit these fields, we get combinations $\mathcal{P}^M\mathcal{P}_M$. As DFT deals with fields at massless string level, equation (3.124) implies that the product of momenta in the periodic direction vanishes. Thus, we get a weak version of the strong constraint that acts on any field ϕ as

$$\eta^{MN}\partial_M\partial_N\phi = 0. \tag{3.155}$$

This is the *weak constraint* and differs from (3.150) by the fact that the latter is valid even for products of fields. Imposing (3.144) also solves the weak constraint. Equation (3.155) is a consequence of the level matching condition for DFT fields on toroidal backgrounds.

In DFT, T-duality is related with the several choices of section conditions. If we consider a toroidal background with k isometric directions, with all fields independent of those directions, then the strong constraint will not define a k-dimensional section on the $(\tilde{x}_m, x^m)$ subspace. So there will be an ambiguity in defining the D-dimensional spacetime where the supergravity fields live in, for we could choose (x^μ, x^m) or the $\mathrm{O}(k,k)$ rotated $(x^\mu, \tilde{x}_m)$. This would correspond to two different identifications of the D-dimensional spacetime metric as different components of the generalized metric would be selected out to compose $G_{\mu\nu}$. These two choices would be related by T-duality, that is the $\mathrm{O}(k,k)$ rotation that connects the two sections. The T-duality of the dilaton can be found using the fact that the generalized dilaton d is a $\mathrm{O}(D,D)$ scalar. That is how T-duality is made manifest in Double Field Theory.

As an example, consider the cosmological Ansatz

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \tag{3.156}$$

$$\Phi(t,x) = \Phi(t),\ B_{\mu\nu} = 0. \tag{3.157}$$

The generalized metric (3.141) and dilaton (3.153) are then

$$\mathcal{H}_{MN}dX^M dX^N = -d\tilde{t}^2 + a^{-2}(\tilde{t},t)\delta_{ij}d\tilde{x}^i d\tilde{x}^j - dt^2 + a^2(\tilde{t},t)\delta_{ij}dx^i dx^j, \tag{3.158}$$

$$d(\tilde{t},t) = a^{D-1}(\tilde{t},t)e^{-2\Phi(\tilde{t},t)} \tag{3.159}$$

and so if we use choose the section such that $\tilde{\partial}^\mu(\cdot) = 0$, we get (3.156) as the spacetime metric. On the other hand, if we impose the strong constraint $\partial_\mu(\cdot) = 0$, the spacetime metric would be

$$ds^2 = \tilde{G}_{\mu\nu}d\tilde{x}^\nu d\tilde{x}^\mu = -d\tilde{t}^2 + a^{-2}(\tilde{t})\delta_{ij}d\tilde{x}^i d\tilde{x}^j. \tag{3.160}$$

Thus metrics (3.156) and (3.160) are related by T-duality and we find the scale-factor T-duality transformation

$$a(t) \leftrightarrow \frac{1}{a(\tilde{t})}, \tag{3.161}$$

while for the dilaton, we get

$$\Phi(t) \leftrightarrow \Phi(\tilde{t}) - (D-1)\ln a(\tilde{t}). \tag{3.162}$$

These T-duality transformation were known previously from Pre-Big-Bang cosmology [116, 117]. In particular the duality relation (3.161) is known as *scale factor duality* [161]. [5]

The previous discussion explains how T-duality is encoded in the $\mathrm{O}(D,D)$ symmetry of DFT. It is clear that such global symmetry is present even when the double-space does not have isometries and so DFT *is not* simply a theory that turns T-duality manifest.

3.5 The AdS$_D$/CFT$_{D-1}$ Correspondence

The AdS/CFT correspondence states that String Theory defined on AdS background is the same physical system as a certain gauge theory. On top of having the same symmetries, there are maps between fields in one side of the correspondence and operators on the other; the gravity and field theory are just two formulations of the same underlying physics. The most concrete example we have is the correspondence of type IIB String Theory to $\mathcal{N}=4$ supersymmetric Yang-Mills theory, that was proposed by Maldacena in [129]. In certain limits, it is possible to do explicit checks, but a general proof of the correspondence is still elusive, though there are hints of a possible path to it [162, 163].

In this section, we briefly review AdS/CFT focusing on the results instead of calculations (detailed references can be found in [164]). We begin by commenting on the ingredients of each side of the correspondence, after which we blend them together explaining how observables in one side are related with observables of the other. We consider $D > 2$ as both CFTs in two dimensions and two dimensional AdS spaces are special and deserve a different treatment.

3.5.1 AdS$_D$ Spacetime

The D-dimensional Anti-de Sitter spacetime of radius R_{AdS}, AdS$_D$, can be defined as the subspace (hyperboloid) of the $D+1$ flat space with $(2, D)$ signature,

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^{D-1})^2 - (dx^D)^2, \tag{3.163}$$

$$-R_{\mathrm{AdS}}^2 = -(x^0)^2 + (x^1)^2 + \cdots + (x^{D-1})^2 - (x^D)^2, \tag{3.164}$$

from which we see that it is a homogeneous and isotropic spacetime with $\mathrm{SO}(2, D-1)$ isometry. The constraint can be solved by

$$x^0 = R_{\mathrm{AdS}}\cosh\rho\cos\tau, \quad x^D = R_{\mathrm{AdS}}\cosh\rho\sin\tau, \quad x^i = R_{\mathrm{AdS}}\Omega^i\sinh\rho, \tag{3.165}$$

[5] Such duality holds even for uncompactified directions, but then it is a relation between two background solutions, while for the T-duality case we have effectively the same physical solution but expressed in dual variables.

where Ω_i $(i = 1, \ldots, D-1)$ are the coordinates of a $D-2$ dimensional unit sphere, S^{D-2}, i.e. $\Omega_i^2 = 1$. The metric is now

$$ds^2 = \mathrm{R}^2_{AdS}\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2_{D-2}\right), \tag{3.166}$$

and if we take

$$0 \leq \rho < \infty, \quad 0 \leq \tau < 2\pi, \tag{3.167}$$

the (τ, ρ, Ω_i) coordinates covers the hyperboloid once. Note that τ is a closed timelike direction and to get a causal spacetime we simply unwrap such direction taking $-\infty < \tau < \infty$ to be a global time without any identification. It is this simply connected space that we consider to be Anti-de Sitter. The coordinates (τ, ρ, Ω_i) are called *global coordinates*.

To study the causal structure of the AdS_D spacetime, let us introduce the coordinate θ by $\tan\theta = \sinh\rho$, with $0 \leq \theta \leq \pi/2$. The line element is now

$$ds^2 = \frac{R^2_{AdS}}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega^2_{D-2}\right), \tag{3.168}$$

that is conformally related with a FLRW metric with $k = 1$ (for $D = 4$, see (2.3) and (2.4)), that is called Einstein static universe. But in the FLRW case, the coordinate θ take values in $[0, \pi)$ and so we get one "half" of the Einstein universe. As the conformal factor does not change how light propagates, the causal structures of the D-dimensional Anti-de Sitter and "half" of Einstein static universe are the same. The spacelike surfaces $\tau =$ const. are $D-1$ hemispheres and at the equator $\theta = \pi/2$ we woud have a boundary with the topology of S^{D-2}. Including this boundary to the space gives a *conformal compactification* of AdS_D spacetime and any other spacetime that has a conformal compactification to a region with this same boundary structure is said to be *asymptotically Anti-de Sitter*. Notice that the boundary S^{D-2} is the same for the conformally compactified $D-1$ dimensional Minkowski spacetime, and so the D-dimensional Anti-de Sitter space has the same conformal boundary as the $(D-1)$-dimensional Minkowski spacetime.

From (3.168), we see that the global coordinates leave the maximal subgroup $\mathrm{SO}(2) \times \mathrm{SO}(D-1)$ of $\mathrm{SO}(2, D-1)$ manifest. There is another set of coordinates, that covers half of the total AdS space, in which the Poincaré group $\mathrm{ISO}(1, D-2)$ of a certain (t, y^i) subspace becomes manifest

$$ds^2 = R^2_{\mathrm{AdS}} u^2\left(-dt^2 + \delta_{ij} dy^i dy^j + \frac{du^2}{u^4}\right), \tag{3.169}$$

these are *Poincaré coordinates* and they do not cover the entire space but only a half of it, with $-\infty \leq t, y^i \leq \infty$ and $0 \leq u \leq \infty$, that is called the *Poincaré patch*. In terms of $z = 1/u$, we have

$$ds^2 = \frac{R^2_{\mathrm{AdS}}}{z^2}\left(-dt^2 + \delta_{ij} dy^i dy^j + dz^2\right). \tag{3.170}$$

At $z = 0$ we are at the conformal boundary of AdS and for $z \to \infty$ we have a Killing horizon.

To conclude this quick discussion of the AdS properties, we state another important feature of the Anti-de Sitter spacetime: The AdS_D spacetime is a solution for D-dimensional

Einstein equations with a negative cosmological constant. The Ricci scalar is proportional to the inverse of the AdS radius,

$$R^{(D)} = -\frac{D(D-1)}{R_{\mathrm{AdS}}^2}, \tag{3.171}$$

that is related with the cosmological constant as

$$\Lambda = -\frac{(D-1)(D-2)}{16\pi G_D R_{\mathrm{AdS}}^2}. \tag{3.172}$$

3.5.2 The $\mathcal{N}=4$ SYM as a CFT

The conformal group in $D-1$ dimensions is the group of spacetime transformations such that the metric is preserved up to a arbitrary scale factor. It has $D(D+1)/2$ generators that corresponds to spacetime rotations ($\Lambda_{\mu\nu} = -\Lambda_{\nu\mu}$), spacetime translations (P^μ), scale transformations $x^\mu \to \lambda x^\mu$ (with generator D) and special conformal transformations (generated by K^μ), that are combinations of rotations, translations and inversions ($x^\mu \to x^\mu/x^2$). The connected components form an algebra that is isomorphic to the algebra of $\mathrm{SO}(2, D-1)$. So, the conformal group in $D-1$ dimensions is isomorphic to the group of isometries of AdS_D. This is a hint that D-dimensional field theories with conformal symmetry are related with the $D+1$ dimensional AdS space.

Conformal Field Theories (CFTs) are field theories in which fields span representations of the conformal group. For a CFT, there is no notion of scales, as the theory is the same for any rescaling of scales. This is true even for quantum CFTs, which have vanishing beta function for all couplings. Thus, there is no non-ambiguous way to define asymptotic states and so S-matrices for CFTs, and the objects of physical interest are correlation functions of operators. In a CFT, interesting operators are eigenfunctions of the dilatation operator D, with eigenvalue given by their scaling (or conformal) dimension, Δ, that classically is just the mass-dimension of the operator. From the conformal group algebra, the generator of special conformal transformation K^μ lowers the conformal dimension of a operator, while the momentum P^μ raises it. In an unitary CFT, there is a lower bound on the conformal dimension of the fields and so for each representation there should be fields that are annihilated by K^μ. These are primary operators (or fields) and for each of them we have a representation of the conformal group, that is constructed by acting with the momentum operator P^μ on primary fields and that is labelled by the spin and scaling dimensions (which are the Casimirs of the conformal group).

Conformal symmetry severely constrain correlation functions of a CFT. In particular, the two-point function of two primary operators $\mathcal{O}_1(x)$, $\mathcal{O}_2(x)$ with different conformal dimensions Δ_1, Δ_2 is completely fixed to be

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(x')\rangle = \frac{c_{12}\delta_{\Delta_1\Delta_2}}{|x-x'|^{2\Delta_1}}, \tag{3.173}$$

and so vanishes for operators with different Δ. The constant c_{12} can be absorbed by redefining the operators. Moreover, the three-point function is fixed by the conformal dimension of the operators up to a constant c_{123} and the full set of these constants defines the dynamics of the theory.

The $D = 4$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with SU(N) gauge groups is a example of a CFT, with conformal invariance at classsical and quantum levels. It has a SU(N) gauge group and a SU(4) R-symmetry, that rotates the fields. The field content is a gauge field A^μ, 6 real scalars Φ^{IJ} organized in the [2] of SU(4) $\simeq$ SO(6) and 4 Majorana fermions Ψ^I in the **4** of SU(4). All the fields carry a index of the adjoint of the SU(N) gauge group. They have scaling dimensions $\Delta_A = 1 = \Delta_\Phi$, $\Delta_\Psi = 3/2$. The action is

$$S_{\mathcal{N}=4\,\mathrm{SYM}} = \int d^4x \mathrm{Tr}\left[-\frac{1}{4g_{\mathrm{YM}}^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2} D_\mu \Phi_{IJ} D^\mu \Phi^{IJ} - i\bar{\Psi}_I \not{D} \Psi^I \right.$$
$$\left. - g_{\mathrm{YM}} \bar{\Psi}^I [\Phi_{IJ}, \Psi^J] - \frac{g_{\mathrm{YM}}^2}{4} [\Phi_{IJ}, \Phi_{KL}][\Phi^{IJ}, \Phi^{KL}] \right], \tag{3.174}$$

and it is supersymmetric though we will not discuss supersymmetry here (see [164] for details). But a simple check is to count the on-shell degrees of freedom: there are 8 bosonic, 6 from the scalars and 2 from the vector field, and 8 fermionic coming from the 4 Majorana fermions (that has 2 on-shell degrees of freedom). We also see the consistency with the number of supersymmetries, as the $\mathcal{N} = 4$ supersymmetric parameters can be organized in the fundamental of the R-symmetry group.

In principle the conformal dimension of operators receive quantum corrections, and so their quantum values are different than the classical ones. However the symmetries of the $\mathcal{N} = 4$ SYM are not just the conformal group, but actually the superconformal group SU$(2, 2|4)$. Representations of this enlarged group are constructed from chiral primary operators that are not only annihilated by K^μ but also by some combinations of the supersymmetry generators. It turns out that the scaling dimensions of the chiral operators are protected from quantum corrections, and so we can keep using their classical values even after quantization. This fact is very useful in checking the holographic dictionary, that we discuss next.

3.5.3 Holographic Map and GKPW Construction

It was already mentioned that the isometries of AdS space and the symmetries of CFTs are related. Consider the original AdS$_5$ $\times$ S^5/$\mathcal{N} = 4$ SYM correspondence. This case was constructed explicitly from a system of N D-branes in String Theory, in the limit of type IIB supergravity. The AdS space appears by considering the near horizon limit of the curved background sourced by the stack of D-branes, on which the gauge theory is defined. Focusing on bosonic fieds, on the CFT side we have the SO(2, 4) of conformal group and SO(6) $\simeq$ SU(4) of the R-symmetry. This matches exactly with the isometries in the gravity side: there is a SO(2, 4) from the AdS$_5$ piece and a SO(6) from the five-sphere S^5. The correspondence also relates the dimensionless couplings on each side: for the CFT, there is the gauge coupling $g_{\mathrm{YM}}^2 = 4\pi g_s$ and the rank of gauge group N, while on the gravity side we have the AdS radius $R_{\mathrm{AdS}}/\sqrt{\alpha'} = (4\pi g_s N)^{1/4}$ (in stringy units) that is also the radius of the five sphere. As discussed in the section 3.3, to trust in a supergravity limit we should have small curvature as compared with the string length. But from (3.171), the curvature scale for AdS is $1/R^2_{\text{AdS}}$ and so we should have

$$\alpha' R^{(D)} \ll 1 \implies \frac{\alpha'}{R^2_{\text{AdS}}} \ll 1 \implies 4\pi g_s N \gg 1, \tag{3.175}$$

and thus the gauge theory should be in the limit $\lambda \equiv g^2_{\text{YM}} N \gg 1$, where λ is the *t'Hooft coupling*. But there could be g_s quantum corrections that are due to loop of string in spacetime and so we should also have $g_s \ll 1 \implies g_{\text{YM}} \ll 1$. Thus, for consistency with (3.175), we need $N \to \infty$, that gives the t'Hooft limit of a gauge theory, where N is very large and g^2_{YM} is very small but the t'Hooft coupling λ is fixed. It can be shown that it is λ that governs the perturbative expansion of a SU(N) gauge theory in the large N limit and not simply g_{YM}. Therefore the correspondence is also a duality, as the limit of small string corrections for the supergravity side corresponds to the large t'Hooft coupling limit (non-perturbative limit) in the gauge theory side. Though the correspondence is not totally proved, there are several hints that the opposite limit is also true. The AdS/CFT conjecture is that the correspondence is true for any values of g_s and N, that is, that String Theory defined on the $\text{AdS}_5 \times \text{S}^5$ background *is* $\mathcal{N} = 4$ supersymmetric Yang-Mills gauge theory.

The map between the sides of the correspondence is as follows. First, the field theory is defined at the conformal boundary of the AdS space and operators with certain conformal dimension $\Delta_{\mathcal{O}}$ will be sources for fields with mass $m(\Delta_{\mathcal{O}})$ in the AdS space. Consider a gauge invariant composite operator $\mathcal{O}$ made from the fields in the gauge theory. It will belong to some representation I_n of the global R-symmetry SO(6) and will source a S^5 Kaluza-Klein mode $\phi_n^{I_n}$ of some field defined in the $\text{AdS}_5 \times \text{S}^5$ space

$$\phi(x, \Omega) = \sum_{I_n} \sum_n \phi_n^{I_n}(x) Y_n^{I_n}(\Omega), \tag{3.176}$$

where $Y_n^{I_n}(\Omega)$ are the spherical harmonics of S^5 in the representation I_n of the sphere symmetry group SO(6). The eigenvalue of the sphere laplacian are the levels n in the expansion. For scalar fields, the relation between their mass and the conformal dimension of the operator will be related by

$$\Delta = \frac{D-1}{2} + \sqrt{\frac{(D-1)^2}{4} + m^2 R^2_{\text{AdS}}}, \tag{3.177}$$

where $D - 1 \equiv d = 4$ is the dimension of the CFT. This formula gets modified for other spins and types of fields [164]. As further examples, a global symmetry current J_μ in the field theory is mapped to a gauge field A_μ in gravity side, and the energy momentum tensor $T_{\mu\nu}$ in the field theory side sources the metric $G_{\mu\nu}$ in the gravity side.

For relating observables, we use the Witten or Gubser-Klebanov-Polyakov (GKPW) prescription [130, 131]. It is a relation between the bulk and boundary partition functions. For the gravity side, the partition function for the quantum theory of string fields (collectively denoted by φ^i) on the AdS background depends on the value of the fields φ^i_0 on the conformal boundary that defines the Cauchy problem in the curved spacetime,

$$Z^{\text{AdS}}_{\text{String Theory}}[\varphi_0] = \int_{\varphi_0} [d\varphi] e^{i S_{\text{AdS}}[\varphi]}. \tag{3.178}$$

For the CFT side, the partition function for the quantum theory of CFT fields χ^a with a possible source term for operators $\mathcal{O}^i$ is

$$Z_{\mathcal{O}}^{\text{CFT}}[J] = \int [d\chi] e^{iS_{CFT}[\chi]+i\int d^4x J(x)\mathcal{O}(x)}. \tag{3.179}$$

The prescription is to equate these two partition functions, identifying the source of $\mathcal{O}^i$ with the boundary value φ_0^i,

$$Z_{\mathcal{O}}^{\text{CFT}}[\varphi_0] = Z_{\text{String Theory}}^{\text{AdS}}[\varphi_0]. \tag{3.180}$$

So, the string partition function on a AdS background is the generating function for correlators of $\mathcal{N} = 4$ SYM operators. If we take the limit $\alpha' \to 0$, $g_s \to 0$, the saddle point evaluation of the path integral selects the classical solution, that is just classical supergravity. In this limit, the GKPW prescription gives

$$Z_{\mathcal{O}}^{\text{CFT}}[\varphi_0] = e^{iS_{\text{SuGra}}[\tilde{\varphi}[\varphi_0]]}, \tag{3.181}$$

where $S_{\text{SuGra}}[\tilde{\varphi}[\varphi_0]]$ is just the classical action for fields in the AdS background evaluated on the classical solution $\tilde{\varphi}^i$ which has the value φ_0^i at the boundary. So, we can use this prescription to calculate correlation functions of CFT operators using the on-shell value of actions for fields defined on AdS spacetime! In the following we give some comments on the example of a scalar field.

Consider a scalar field on the AdS$_D$ spacetime in Poincaré coordinates (3.170),

$$S = \int d^D x \sqrt{-G} \left(-\frac{1}{2} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right). \tag{3.182}$$

The near boundary $z \to 0$ solutions has two independent parts

$$\phi(t, y, z) = \phi_1 z^{\Delta}(1 + \mathcal{O}(z^2)) + \phi_2 z^{d-\Delta}(1 + \mathcal{O}(z^2)), \tag{3.183}$$

where Δ is the same as in (3.177), with $D - 1 = d$.

The parts of the solution with ϕ_1 and ϕ_2 are said to be normalizable and nonrenormalizable, respectively, as the first has finite modulus with respect to the Klein-Gordon inner product, while the second has not. Tachyonic scalar fields on AdS spactimes can still be stable, if their mass satisfy the Breitenlohner-Freedman (BF) bound,

$$m^2 \geq -\frac{(D-1)^2}{4R_{\text{AdS}}^2}, \tag{3.184}$$

that is the condition for Δ be real. In holography, the normalizable modes are vacuum expectation values of the dual operators, while the non-renormalizable modes are sources for them, deforming the boundary theory (see [153]). Also, as mentioned before, there is a lower bound for the scaling dimension Δ for unitary CFTs. As we are considering a scalar field in the bulk, the dual CFT operator will also be a scalar, and the lower bound is then $\Delta \geq (d-2)/2$. The BF bound guarantees that this is the case, and so stability of the bulk solution is related to unitarity of the boundary CFT.

Due to a singular behaviour close to the boundary, the source is actually

$$\phi_0(t,y) = \lim_{z\to 0} z^{\Delta-d}\phi(t,y,z). \tag{3.185}$$

The proper way to regulate this "singular behaviour" is to do a *holographic renormalization* of the theory (see [165] and references therein). This is due to the fact that the boundary limit is actually a UV limit for the CFT and a IR limit for the gravity theory. In fact, if we would calculate the on-shell supergravity action we would get divergences coming from the near boundary limit, that corresponds to UV divergences of the boundary field theory. So, the holographic direction z is actually parameterizing the energy scale of the theories and motion on this direction corresponds to an RG flow of the CFT.

From the GKPW prescription, in the supergravity limit, we have

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \frac{1}{i^n}\frac{\delta^n \ln Z^{\mathrm{CFT}}_{\mathcal{O}}[\phi_0]}{\delta\phi_0(x_1)\cdots\delta\phi_0(x_n)}\bigg|_{\phi_0=0} = \frac{1}{i^{n-1}}\frac{\delta^n S_{\mathrm{SuGra}}[\phi_0]}{\delta\phi_0(x_1)\cdots\delta\phi_0(x_n)}\bigg|_{\phi_0=0}. \tag{3.186}$$

For the scalar field case, we get

$$\langle \mathcal{O}(x)\rangle \propto \phi_1, \quad \langle \mathcal{O}(x)\mathcal{O}(x')\rangle \propto \frac{1}{|x-x'|^{2\Delta}}, \tag{3.187}$$

that after comparison with (3.173), confirms that the conformal dimension of the dual operator is related with the power in the solution of the bulk field.

Notice that the example of a scalar field was done in D dimensions and without any mention about the compact five-sphere. In the case of $\mathrm{AdS}_5\times \mathrm{S}^5$, the field $\phi(t,x,z)$ should be thought as the zero Kaluza-Klein mode in the expansion in spherical harmonics, the $n=0$ term in (3.176) that does not transform under S^5 rotation. Then, it will correspond to a $\mathcal{N}=4$ SYM composite operator that is invariant under the SO(6) R-symmetry. A possible candidate is, for instance,

$$\mathcal{O} = \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}], \tag{3.188}$$

and if we start considering the KK modes of the bulk scalar field, we will get the dual operators

$$\mathcal{O}_n = \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}\Phi^{I_1}\cdots\Phi^{I_n}]. \tag{3.189}$$

In fact, the Kaluza-Klein modes of the supergravity fields in $\mathrm{AdS}_5\times \mathrm{S}^5$ correspond to chiral primary fields in the $\mathcal{N}=4$ SYM conformal theory, with conformal dimensions protect against quantum corrections. The matching between the masses of the KK modes in the bulk and the conformal dimension of the chiral spectrum of the CFT is one of the outstanding checks of the duality [164].

The AdS/CFT correspondence can be generalized for other cases, and have been applied in several fields of Physics as condensed matter, nuclear physics, quantum information theory and cosmology. In fact, departing from conformal symmetry, large N limit and supersymmetry, gives more general *gauge/gravity* correspondences. In all cases we get a holographic duality with the gauge side being perturbative and the gravity side being strongly coupled or vice-versa, with a holographic direction parametrizing energy scales. Up to date, the AdS/CFT is the only tool we have to understand strongly coupled gravitational physics.

Chapter 4

Conformal Inflation with Chameleon fields

4.1 Motivation: String Moduli and Chameleon screening mechanism

As already touched in the introduction, a recurrent issue on trying to construct phenomenological suitable string compactifications is the existence of several scalar fields that are not automatically stabilized, the *moduli* of the compactification. These fields are light scalars that govern the details of the compactification. Having mechanisms that creates potential for them is an important step in building a stringy model for cosmology, as massless degrees of freedom give rise to energy densities that do impact on cosmological evolution, and so they change the results of single field inflation for instance.

If the dark energy that dominates our Universe today is not totally due to a cosmological constant, the next simplest approach is to consider it to be a scalar field and there are several setups of this sort, called *quintessence models*. In fact, the current constraints on the equation of state of dark energy still allows for it to be described by quintessence fields. The problem is that light scalar fields with sufficiently large coupling with matter generates a non-observed long-range 5th interaction. The Chameleon mechanism is a way to explain how scalar fields may play a role in cosmological scales and on the other hand hide its effects on very small (terrestrial or Solar system-sized) scales [69].

The chameleonic scalar field $\varphi(x)$ is defined by its coupling with matter fields $\chi(x)$, that is given through the metric. That is, the action for a chameleon field in a fixed curved spacetime is (see [166] for a introduction to chameleonic cosmology)

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\phi)\right] + S_m[\chi, \tilde{g}_{\mu\nu}], \tag{4.1}$$

where[1]

$$\tilde{g}_{\mu\nu} = F^2(\varphi) g_{\mu\nu}, \quad F(\varphi) = e^{-c\varphi/M_{\rm Pl}}, \tag{4.2}$$

and so the equation of motion for the chameleon field is

$$\Box\varphi = \frac{dV}{d\varphi} - \tilde{g}^{\mu\nu}\tilde{T}_{\mu\nu}F^4(\varphi)\frac{d\ln F(\varphi)}{d\varphi}, \quad \tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{\tilde{g}}}\frac{\delta S_m}{\delta\tilde{g}^{\mu\nu}} \tag{4.3}$$

With cosmological application in mind, assuming a perfect fluid Jordan-frame energy-momentum tensor, we write $\tilde{T}_\mu{}^\mu = -(1-3w)\tilde{\rho}$. In terms of the Einstein-frame energy density ρ_m, that is conserved in Einstein-frame, we have

$$\Box\varphi = \frac{dV}{d\varphi} + (1-3w)\rho_m F^{(1-3w)}(\varphi)\frac{d\ln F}{d\varphi} \equiv \frac{dV_{\rm eff}}{d\varphi}, \tag{4.4}$$

where we see that the chameleon perceives an effective potential $V_{\rm eff}$ that includes contributions from the energy density

$$V_{\rm eff}(\varphi) = V(\varphi) + \rho_m F^{(1-3w)}(\varphi), \tag{4.5}$$

For radiation, $w = 1/3$, there is no change in the potential and so the chameleon does not couple to it. The more interesting case that we will discuss in this chapter is non-relativistic matter, $w = 0$. From the form of the effective potential we have that even if $V(\varphi)$ has no minima, the chameleon could still be stabilized at regions with high matter density, and hence its name: it "mimics" the environment by being light in small density regions (cosmological scales) and massive at regions with large matter density (small scales). This is the screening mechanism of chameleonic models.

In [89], the chameleon mechanism was embedded into a string compactification by adding some extra contributions to the KKLT scenario in the low energy $\mathcal{N} = 1$, $D = 4$ supergravity. In the model, the chameleon was the overall volume modulus of the compact manifold. It was shown that one can tune the chameleon coupling and other parameters to be consistent with all laboratory tests and observations.

In this chapter, we will take a more broad approach by simply looking for inflationary solutions with chameleon coupling between the inflaton and non-relativistic matter, without regarding the UV completion of the model. So, it is a pure inflationary model, with the novelty of chameleon-like coupling, but that could or could not be related with string theory. Of course the hope is that if the chameleon screening mechanism happens to be found in any other compactification/model in String Theory, we will have an explicit solution that shows how this mechanism affects inflation, at least for some classes of potentials.

4.2 Conformal inflation and set-up

4.2.1 Conformal inflation coupled to energy density

Conformal inflation is a perturbation of an exactly "conformally invariant" model, that doesn't contain a fundamental scale, even the Planck scale; all the scales appear from gauge

[1]The exponential form of the chameleon coupling function $F(\varphi)$ is motivated, for instance, by Weyl transformations used in $F(R)$ theories to go to the Einstein frame.

fixing and minimizing the potential[2]. The action is

$$S=\frac{1}{2}\int d^4x\sqrt{-g}\left[\partial_\mu\chi\partial^\mu\chi-\partial_\mu\phi\partial^\mu\phi+\frac{\chi^2-\phi^2}{6}R-\frac{\lambda}{18}(\phi^2-\chi^2)^2\right]. \tag{4.6}$$

Notice that the coupling of the scalars to the Einstein term has the conformal value, and that the potential is quartic, such as not to have any dimensionful parameters in the action.

Then we have a local "Weyl" type symmetry, acting on the fields by

$$g_{\mu\nu}\to e^{-2\sigma(x)}g_{\mu\nu};\quad \chi\to e^{\sigma(x)}\chi,\qquad \phi\to e^{\sigma(x)}\phi. \tag{4.7}$$

We also have an $SO(1,1)$ symmetry rotating (ϕ,χ) (note that we have only the combination $\chi^2-\phi^2$ and the corresponding kinetic term), which acts as a Lorentz type symmetry (originally, in [168–170], the model was motivated by 2-time physics: a covariant $(4,2)$-dimensional form led to it, with the $SO(1,1)$ being a remnant of the $SO(4,2)$ Lorentz invariance). Alternatively, the $SO(1,1)$ symmetry can also be obtained from a model with $SO(4,2)$ conformal invariance in 3+1 dimensions, as motivated in [171–173].

The field χ has the wrong sign kinetic term, so it would seem it is a ghost. However, the local "Weyl" symmetry above allows one to use a gauge choice to set it to zero, therefore the ghost is not physical. Choosing a gauge for the local "Weyl" symmetry also introduces a scale, the Planck scale, that will appear in front of the Einstein action. Yet, since the theory has only a single scale, its value is simply a definition of units, defining for instance what "one meter" is (or what is 10^{19}GeV), and physics is independent of the value we attribute to this scale, since it is a gauge choice.

The gauge we will be mostly interested in is the Einstein gauge, defined by $\chi^2-\phi^2=6M_{\rm Pl}^2$. We solve this constraint in terms of a *canonically normalized field* φ by

$$\chi=\sqrt{6}M_{\rm Pl}\cosh\frac{\varphi}{\sqrt{6}M_{\rm Pl}};\quad \phi=\sqrt{6}M_{\rm Pl}\sinh\frac{\varphi}{\sqrt{6}M_{\rm Pl}}. \tag{4.8}$$

In terms of φ, we obtain the simple Einstein plus canonical scalar action, with a cosmological constant,

$$S=\int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\lambda M_{\rm Pl}^4\right]. \tag{4.9}$$

To obtain an inflationary model from the one above, we must deform the theory so that the cosmological constant gets modified into a potential with a plateau. We do so by keeping the local "Weyl" symmetry, and deforming the $SO(1,1)$ symmetry so that it is only approximately valid, at large field values. Imposing the "Weyl" invariance, we must have a potential of the type $f(\phi/\chi)\phi^4$, since both ϕ/χ and $\sqrt{-g}\phi^4$ are locally "Weyl" invariant. Imposing moreover that the potential reduces to the $(\phi^2-\chi^2)^2$ form at large field values, we finally obtain the most general form

$$V=\lambda\left[\tilde{f}(\phi/\chi)\phi^2-\tilde{g}(\phi/\chi)\chi^2\right]^2, \tag{4.10}$$

or in another parametrization

$$V=\lambda f(\phi/\chi)\left[\phi^2-g(\phi/\chi)\chi^2\right]^2. \tag{4.11}$$

[2] See [167] for a generalization of this approach in a Weyl geometry set up.

In the last form, in order to have the $SO(1,1)$ symmetry at large field values, since in the Einstein gauge $\phi/\chi = \tanh\varphi/\sqrt{6}M_{\rm Pl}$, which goes to 1 at large φ, we must impose $g(1)=1$.

For simplicity, we will consider only cases with $f(x)=1$. Moreover, as in [96], with a simple polynomial form for $g(x)$,

$$g(x) = \omega^2 + (1-\omega^2)x^n \ , \tag{4.12}$$

where $\omega = 246{\rm GeV}/\sqrt{6}M_{\rm Pl}$ and $n>2$, we can interpolate between the conformal inflation plateau and a Higgs potential at small field values, $V \simeq \left[\varphi^2 - 6\omega^2 M_{\rm Pl}^2\right]^2$, so it presents a simple set-up, with the same scalar playing the role of inflaton and Higgs. In this case, at $\varphi\to\infty$, using that $\omega^2 \ll 1$, we obtain the potential

$$V \simeq 9(n-2)^2\lambda M_{\rm Pl}^4\left[1 - 2n e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}}\right]. \tag{4.13}$$

This exponentially-corrected plateau behaviour is the generic one for any function $g(x)$ that is well behaved near $x=1$, where we have the large-field (large φ) expansion.

One easily finds the scalar spectral index and the tensor to scalar ratio, n_s and r, in terms of the number of e-folds N_e. In this case

$$\begin{aligned} 1-n_s &\simeq \frac{2}{N_e} \\ r &\simeq 3(n_s-1)^2 \simeq \frac{12}{N_e^2}\ , \end{aligned} \tag{4.14}$$

as in the Starobinsky model. Thus the simplest model of conformal inflation effectively gives the Starobinsky model. More generally, the Starobinsky model result above is found from the general potential at $\varphi\to\infty$

$$V \simeq A\left[1 - B e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}}\right]. \tag{4.15}$$

If one replaces $\sqrt{2/3}$ in the exponent by a general factor a, n_s is unchanged, but r is changed to $8/(a^2N_e^2)$.

In the class of *conformal inflation* models, as we saw, any generic, well behaved at $x=1$, function $g(x)$ will give the simple Starobinsky model at large field φ. Yet, with a slightly more unusual function $g(x)$, which can be defined implicitly by

$$\begin{aligned} V(\varphi) &= 36\lambda M_{\rm Pl}^4 \sinh^4\frac{\varphi}{\sqrt{6}M_{\rm Pl}}\left[1 - \frac{g\left(\tanh\frac{\varphi}{\sqrt{6}M_{\rm Pl}}\right)}{\tanh^2\frac{\varphi}{\sqrt{6}M_{\rm Pl}}}\right]^2 \\ &\to \frac{9}{4}\lambda M_{\rm Pl}^4\left(1 - 4e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}}\right)e^{2\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}}\left[1 - \frac{g\left(1-2e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}}\right)}{1-4e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}}}\right]^2\ , \end{aligned} \tag{4.16}$$

we can also obtain the potential

$$V(\varphi) \simeq A\left[1 - B\left(\frac{\varphi}{M_{\rm Pl}}\right)^{-p}\right] , \tag{4.17}$$

at least approximately at large φ. Specifically, we need

$$g\left(1-2e^{-\sqrt{\frac{2}{3}}x}\right) \simeq 1-4e^{-\sqrt{\frac{2}{3}}x}+e^{-2\sqrt{\frac{2}{3}}x}\left(1-\frac{B}{2}x^{-p}\right) , \quad x \gg 1 , \tag{4.18}$$

or equivalently, near $y \simeq 1$,

$$g(y) \simeq y^2 + \left(\frac{1-y}{2}\right)^2\left(1-\frac{B}{2}\left[-\sqrt{\frac{3}{2}}\ln\frac{1-y}{2}\right]^{-p}\right). \tag{4.19}$$

The above potential is interesting because it combines the necessary inflationary plateau (the constant term) with the inverse power law potential which was the first model for a chameleon.

Consider that the canonical scalar φ is not just the inflaton as above, but is also a chameleon, coupling universally to the (non-relativistic) energy density ρ through a coupling $\rho F(\varphi)$, where

$$F(\varphi) = e^{-\frac{c}{M_{\rm Pl}}(\varphi-\varphi_0)} , \tag{4.20}$$

and where $\varphi_0 \gg M_{\rm Pl}$ is some large VEV introduced to normalize the energy density when $\varphi = \varphi_0$.

The field φ being a chameleon, means that the above coupling is relevant whenever there is a non-relativistic energy density. That is certainly true during the current matter (and Λ) dominated phase, the period for which the chameleon was invented.

However, it will also be relevant in the case that there is a very heavy non-relativistic species X during inflation and before, with a decay time scale $\tau = 1/\Gamma$ of the order of (or larger) than the time scale for inflation. Such a species will couple to the chameleon through its energy density ρ_m, for a total energy density $\rho_X = \rho_m F(\varphi)$. While the existence of such a species X might seem arbitrary, certainly at the Planck scale, we expect that there will be many massive species, so it is not a stretch to assume such a species will exist also during inflation. As for it being non-relativistic, we should remember that in slow-roll inflation also one assumes kinetic energy much smaller than the potential one. Extending this assumption to X would mean kinetic energy much smaller than the mass term (rest energy), leading to a non-relativistic species. Then moreover the natural scale for Γ is also of the order of H during inflation. In conclusion, our assumption is certainly special, but not more so than inflation itself.

4.2.2 Equations of motion and two cases

The equations of motion for the cosmology are the Friedmann equation for gravity, together with the energy conservation equation or equivalently the acceleration equation, and the Klein-Gordon (KG) equation for the scalar. We will write them as an evolution in terms of the number of e-folds, N, defined by $dN = d\ln a$, instead of as a time evolution. Doing so simplifies the analysis.

In terms of N, the non-relativistic, chameleon-coupled energy density scales as

$$\rho_X = F(\varphi)\rho_m = \frac{F(\varphi)\rho_0 a_0^3}{a^3} \propto e^{-3N} F(\varphi). \tag{4.21}$$

and radiation scales as

$$\rho_{\rm rad} = \frac{\rho_{\rm rad,0} a_0^4}{a^4} \propto e^{-4N}. \tag{4.22}$$

The KG equation for the scalar is now

$$\ddot{\varphi} + 3H\dot{\varphi} = -\frac{\rho_0 a_0^3}{a^3}\frac{dF}{d\varphi} - \frac{dV}{d\varphi} = -\frac{\rho_X}{F}\frac{dF}{d\varphi} - \frac{dV}{d\varphi}. \tag{4.23}$$

To write the Friedmann and acceleration equations for gravity, note first that the energy-momentum tensor of a homogeneous scalar with canonical kinetic term and potential V is

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}[(\partial\varphi)^2 + 2V] \, , \tag{4.24}$$

which means that energy density and pressure associated with it are

$$\rho_\varphi = \frac{\dot{\varphi}^2}{2} + V; \quad p_\varphi = \frac{\dot{\varphi}^2}{2} - V. \tag{4.25}$$

The Friedmann equation is

$$3M_{\rm Pl}^2 H^2 = \rho_{\rm tot} = \rho_X + \frac{\dot{\varphi}^2}{2} + \rho_{\rm rad} + V. \tag{4.26}$$

To simplify our notation going forward, we will denote $\frac{d}{dN} \equiv \;'$. Using

$$\varphi' = \frac{d\varphi}{dt}\frac{dt}{d\ln a} = \frac{\dot{\varphi}}{H} \, , \tag{4.27}$$

we rewrite the Friedmann equation as

$$3M_{\rm Pl}^2 H^2 = \rho_X + \frac{1}{2}H^2\varphi'^2 + \rho_{\rm rad} + V \, , \tag{4.28}$$

which gives

$$3M_{\rm Pl}^2 H^2 = \frac{\rho_X + \rho_{\rm rad} + V}{1 - \frac{\varphi'^2}{6M_{\rm Pl}^2}}. \tag{4.29}$$

The acceleration equation is

$$\frac{\ddot{a}}{a} = H\frac{dH}{dN} + H^2 = -\frac{(\rho_{\rm tot} + 3p_{\rm tot})}{6M_{\rm Pl}^2} = -\frac{1}{6M_{\rm Pl}^2}(\rho_X + 2\rho_{\rm rad} + 2\dot{\varphi}^2 - 2V) \, , \tag{4.30}$$

where as usual $\cdot = \frac{d}{dt}$. Now we can rewrite the KG equation first as

$$H^2\varphi'' + (3H^2 + HH')\varphi' = -\frac{\rho_X}{F}\frac{dF(\varphi)}{d\varphi} - \frac{dV}{d\varphi} \, , \tag{4.31}$$

and using the acceleration and the Friedmann equation to calculate the friction term (the bracket in front of φ'), we find the N-dependent form

$$H^2\varphi'' + \frac{1}{3M_{\rm Pl}^2}\left(\frac{3}{2}\rho_X + \rho_{\rm rad} + 3V\right)\varphi' = -\frac{\rho_X}{F}\frac{dF}{d\varphi} - \frac{dV}{d\varphi} = -\frac{dV_{\rm eff}}{d\varphi}\ , \tag{4.32}$$

where

$$V_{\rm eff} = V + \rho_X = V + F(\varphi)\rho_{m,0}e^{-3N}\ , \tag{4.33}$$

and the second term scales as $e^{-\frac{c(\varphi-\varphi_0)}{M_{\rm Pl}}-3N}$. Note that the coupling of the chameleon field to matter can, in principle, avoid that the chameleon-coupled matter density is redshifted away during inflation if a linear (in the number of e-folds N) solution for the scalar field exists (see also [90] for how this attractor mechanism works). In fact, if $\varphi(N) \sim -3NM_{Pl}/c$ the energy density associated to matter is a constant instead of decaying as e^{-3N}. We now investigate whether such solutions exist here.

We can define the usual ratios of the densities to the critical density $3H^2M_{\rm Pl}^2$,

$$\Omega_X \equiv \frac{\rho_X}{3H^2M_{\rm Pl}^2};\quad \Omega_{\rm rad} \equiv \frac{\rho_{\rm rad}}{3H^2M_{\rm Pl}^2};\quad \Omega_{{\rm kin},\varphi} \equiv \frac{\dot\varphi^2/2}{3H^2M_{\rm Pl}^2} = \frac{\varphi'^2}{6M_{\rm Pl}^2}\ ,\quad \Omega_V \equiv \frac{V}{3H^2M_{\rm Pl}^2}\ , \tag{4.34}$$

and then the Friedmann equation becomes just the fact that the sum of all the Ω's is one,

$$\Omega_X + \Omega_{\rm rad} + \Omega_{{\rm kin},\varphi} + \Omega_V = 1\ , \tag{4.35}$$

which implies

$$\varphi' = M_{\rm Pl}\sqrt{6(1-\Omega_X-\Omega_{\rm rad}-\Omega_V)}. \tag{4.36}$$

The nontrivial equation is the KG equation, which becomes

$$\varphi'' + \left(\frac{3}{2}\Omega_X + \Omega_{\rm rad} + 3\Omega_V\right)\varphi' = 3cM_{\rm Pl}\Omega_X - \frac{1}{H^2}\frac{dV}{d\varphi}\ , \tag{4.37}$$

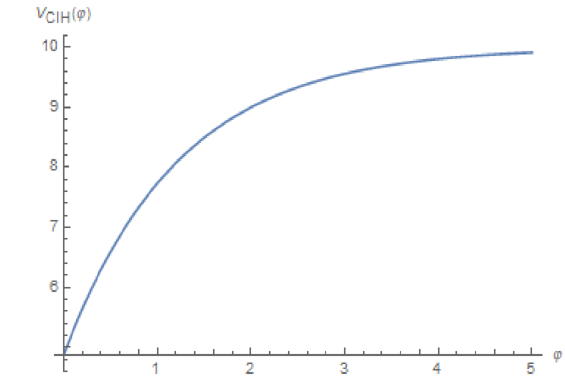
We now have to solve this equation, given some initial conditions, which are a certain $\varphi = \varphi_i$, and an initial "velocity" (derivative with respect to N) $\varphi_i' < 0$. As suggested by the $V = 0$ case considered in [90], we expect to find some attractor-like behaviour, where some of the energy density components scale in the same way[3].

We must distinguish however between the cases of $c > 0$, when the coupling factor $F(\varphi)$ increases away from the inflationary plateau, i.e. at small φ (the plateau is at large φ), and the case of $c < 0$, when $F(\varphi)$ decreases in the same direction as the potential, namely towards small φ, as one can see explicitly from Figure 4.1 and Figure 4.2.

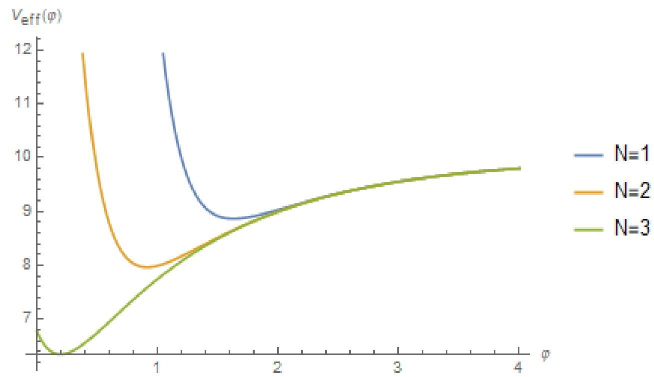
In the $c > 0$ case, we can have a local, time (or N) dependent minimum of the effective potential $V_{\rm eff}$, which allows for the possibility that ρ_X and ρ_V scale in the same way. Also the kinetic energy $\rho_{{\rm kin},\varphi}$ can *a priori* scale the same way.

In the $c < 0$ case, ρ_V is approximately constant with N on the plateau, but ρ_X will decrease as $\sim e^{|c|\frac{\varphi}{M_{\rm Pl}}-3N} \sim e^{-|c|\frac{|\varphi'|}{M_{\rm Pl}}N-3N}$, so it will be subleading. But the kinetic energy $\rho_{{\rm kin},\varphi}$ can *a priori* scale in the same way.

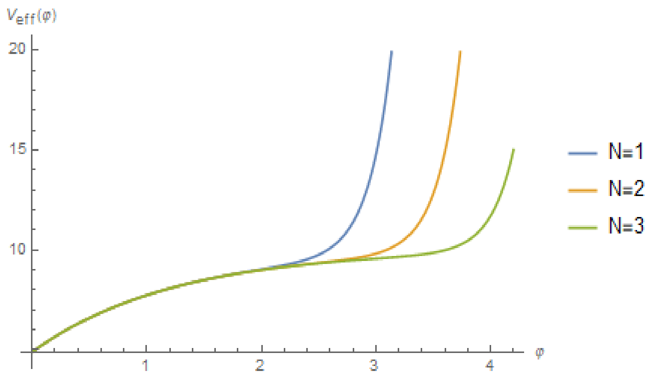
[3]That is, we expect to find solutions in which the fractional densities Ω_i tend to a fixed-point behaviour.



(a) Plateau behaviour for the "conformal inflation" model.



(b) The coupling of the matter potential to the chameleon field, for the case $c > 0$, generates an instantaneous minimum that evolves with the number of e-folds N.



(c) For the case $c < 0$ there is no instantaneous minimum.

Figure 4.1: Typical potential behaviour without (Figure 1(a)) and with chameleon coupling in the case $c > 0$ (Figure 1(b)) and $c < 0$ (Figure 1(c)) for the "conformal inflation" model.

In both cases, $\rho_{\mathrm{rad}} \sim e^{-4N}$, so it will subleading.

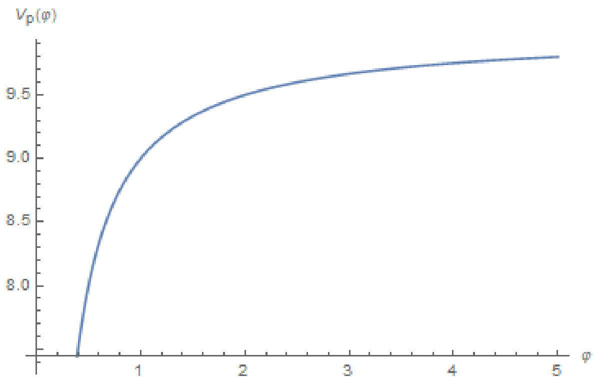
We will study the possible attractor behaviours in both cases, both analytically and numerically, in the next section.

4.3 Evolution and attractors

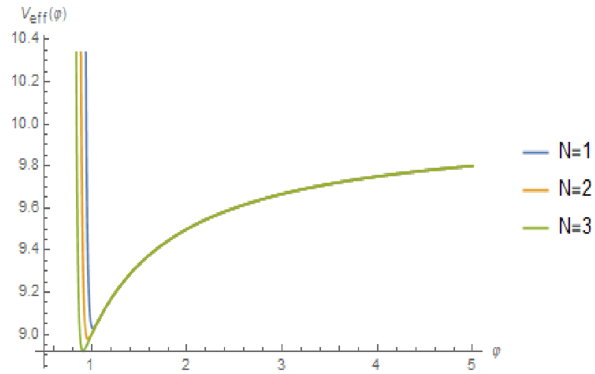
In order to find possible attractor-like behaviours, we consider an ansatz where the kinetic energy density $\rho_{\mathrm{kin},\varphi}$ is constant, so φ' is constant,

$$\varphi = \varphi_i - kNM_{\mathrm{Pl}}, \quad k > 0. \tag{4.38}$$

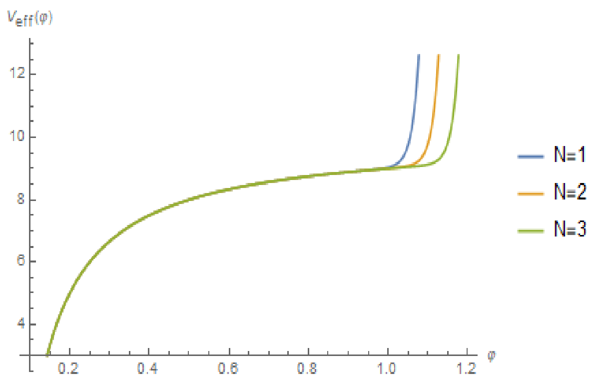
If moreover the Hubble constant H is constant, as is expected if $\rho_V = V$ is approximately constant and the other terms are either negligible or scale in the same way in (4.29), then



(a) Plateau behaviour for the power law model.



(b) Minimum of the effective potential evolving with the number of e-folds N in the power law model for the case $c > 0$.



(c) No instantaneous minimum in the case $c < 0$.

Figure 4.2: There is no minimum formed without the chameleon coupling (Figure 2(a)) or with the chameleon coupling in the case $c < 0$ (Figure 2(c)). Figure 2(b) shows the formation and evolution, with the number of e-folds N, of the instantaneous minimum in the case $c > 0$ for the power law model.

also $\Omega_{\text{kin},\varphi} = \rho_{\text{kin},\varphi}/(3H^2 M_{\rm Pl}^2)$ is constant, and so is $\Omega_V = V/(3H^2M_{\rm Pl}^2)$, which is what we would mean by an attractor-like behaviour.

Since as we saw, $\rho_{\rm rad}$ scales as e^{-4N} so it quickly becomes subleading, for an attractor-like behaviour as above, we find from the KG (4.37) and Friedmann (4.29) equations

$$\begin{aligned} 3\left(\frac{1}{2}\Omega_X + \Omega_V\right)(-k) &= 3c\Omega_X - \frac{1}{H^2}\frac{V'}{M_{\rm Pl}}, \\ \frac{k^2}{6} &= 1 - \Omega_X - \Omega_V. \end{aligned} \tag{4.39}$$

We now analyze separately the cases $c > 0$ and $c < 0$.

4.3.1 The case $c > 0$

In this case, ρ_X increases away from the plateau (at small φ), so we can have a local minimum of $V_{\rm eff}$.

Analytical results: Attractor 1

It is clear from the attractor equations (4.39) that, in order to have an attractor, and if as we said we have $H \simeq$ constant, we need to have $V' \equiv dV/d\varphi \simeq$ constant. Let us therefore assume that

$$\frac{dV}{d\varphi} \equiv \alpha \equiv \tilde{\alpha}\frac{A}{M_{\rm Pl}} \tag{4.40}$$

is approximately constant, over a relevant number of e-folds N, and is small in Planck units, i.e., $\alpha \ll M_{\rm Pl}^3$; moreover, $\tilde{\alpha} \ll 1$. Here $A = \lambda M_{\rm Pl}^4$ is the plateau value for the potential. Then we have approximately

$$\Omega_V = \frac{\rho_V}{3H^2 M_{\rm Pl}^2} \simeq \frac{A}{3H^2 M_{\rm Pl}^2} \equiv \Omega_{V,0}. \tag{4.41}$$

The definition of $\tilde{\alpha}$, together with the condition $\tilde{\alpha} \ll 1$ is done so that, in the context of inflation, the first slow-roll parameter ϵ is small,

$$\tilde{\alpha} = \sqrt{2\epsilon} \ll 1 \Rightarrow \epsilon \ll 1. \tag{4.42}$$

Then, substituting Ω_X from the second equation in (4.39) into the first, and using the definition of $\tilde{\alpha}$ and of $\Omega_V \simeq \Omega_{V,0}$, we find

$$3\left(1 - \Omega_V - \frac{k^2}{6}\right)\left(\frac{k}{2} + c\right) + 3\Omega_V k = \frac{V'}{H^2 M_{\rm Pl}} \simeq 3\tilde{\alpha}\Omega_V. \tag{4.43}$$

Consider now an attractor with Ω_V scaling in the same way as Ω_X, that is with

$$\rho_X \propto e^{-c\frac{\varphi}{M_{\rm Pl}} - 3N} \sim e^{(ck-3)N} \sim \text{constant} \Rightarrow k = \frac{3}{c}. \tag{4.44}$$

Then we obtain

$$3c\left[\left(1 - \Omega_V - \frac{3}{2c^2}\right)\left(1 + \frac{3}{2c^2}\right) + \frac{3}{c^2}\Omega_V\right] = \tilde{\alpha}\Omega_V. \tag{4.45}$$

Incidentally, note that the equations are invariant under changing simultaneously the signs of c and $\tilde{\alpha}$. Indeed, this is what happens if we redefine φ to $\tilde{\varphi} = \varphi_i - \varphi$, to have a variable that increases from 0, instead of one that decreases from a large value: then in terms of $\tilde{\varphi}$, we change both the signs of $\tilde{\alpha}$ and of c. The more general equation (4.43) is also invariant under the simultaneous change of signs of $\tilde{\alpha}, c$ and k, for the same reason.

The above is a linear equation in Ω_V, which is solved by

$$\Omega_V \simeq \frac{1 + \frac{3}{2c^2}}{1 + \frac{\tilde{\alpha}/c}{1 - \frac{3}{2c^2}}}. \tag{4.46}$$

We can now obtain some constraints on parameters. First, note that

$$\Omega_X = 1 - \Omega_V - \frac{3}{2c^2} \geq 0 \Rightarrow 1 - \Omega_V \geq \frac{3}{2c^2} \Rightarrow \Omega_V \leq 1 - \frac{3}{2c^2}. \tag{4.47}$$

Substituting the approximate value of Ω_V in (4.46), we obtain the condition

$$\frac{\tilde{\alpha}}{c} = \frac{\tilde{\alpha}/c}{1-\frac{3}{2c^2}}\left(1-\frac{3}{2c^2}\right) \geq \frac{3}{c^2} \Rightarrow \tilde{\alpha} \geq \frac{3}{c}. \tag{4.48}$$

It seems then that, since we need $\tilde{\alpha}$ to be small we could have c be very large, so that we still have $\tilde{\alpha} > 3/c$. The problem is that then, we have Ω_V very close to 1. Expanding in $\tilde{\alpha}$, we get

$$\Omega_V \simeq 1 + \frac{3}{2c^2} - \frac{\tilde{\alpha}}{c} = 1 - \frac{1}{c}\left(\tilde{\alpha} - \frac{3}{2c}\right). \tag{4.49}$$

A set of reasonable values for the parameters $\tilde{\alpha}$ and c, that do not give too small values for Ω_X and $\Omega_{\text{kin},\varphi}$, are:

$$(\sqrt{2\epsilon} =)\;\; \tilde{\alpha} \simeq \frac{1}{5}\,, \qquad c \sim 20. \tag{4.50}$$

These values satisfy the constraints above. The resulting Ω's for them are

$$\Omega_V \simeq 1 - \frac{1}{100}\,, \quad \Omega_{\text{kin},\varphi} = \frac{3}{2c^2} \simeq \frac{3}{2\cdot 400} \simeq \frac{1}{300}\,, \quad \Omega_X = 1 - \Omega_V - \Omega_{\text{kin}} \simeq \frac{2}{300}. \tag{4.51}$$

These values are very small, but measurable, so in this case we have a nontrivial attractor.

However, we need to remember that we will in fact compare with inflation in the next section, and that leaves a measurable imprint in the CMBR. The correct analysis will be done in the next section, but for now we will assume that the parameters we find for the attractor-like solution are also the parameters during inflation, which are measured in the CMBR. In this way, we will get oriented for the kind of values we need to take for $\tilde{\alpha}$ and c.

Since as we said, $\tilde{\alpha} = \sqrt{2\epsilon}$, assuming the ϵ here is the same relevant for the CMBR (which is not quite correct, as we will see next section, but we will assume this for now), we must have

$$n_s - 1 = -6\epsilon + 2\eta \simeq 0.97. \tag{4.52}$$

If $\eta \ll \epsilon$, we have $\epsilon \simeq 1/200$, so $\tilde{\alpha} = \sqrt{2\epsilon} \simeq 1/10$. Then $c > 3/\tilde{\alpha} \sim 30$. Consider the value $c \sim 40$. Then we get the Ω's

$$\Omega_V \simeq 1 - \frac{1}{400}\,, \quad \Omega_{\text{kin},\varphi} = \frac{3}{2c^2} \simeq \frac{1}{1000}\,, \quad \Omega_X \simeq \frac{1}{700}\,, \tag{4.53}$$

which are still not unreasonable.

For the generic conformal inflation (4.15), we have

$$\tilde{\alpha} = \left[\sqrt{\frac{2}{3}} B e^{-\sqrt{\frac{2}{3}}\frac{\varphi_i}{M_{\text{Pl}}}}\right] e^{\sqrt{\frac{2}{3}}\frac{(\varphi_i - \varphi)}{M_{\text{Pl}}}}. \tag{4.54}$$

From the above value $\tilde{\alpha} = 1/10$, it means that the square bracket must equal this value. In the case where $B = 2n$ (coming from a power law function $g(x)$), this is indeed consistent with $\varphi_i \gg M_{\text{Pl}}$. Then

$$\sqrt{\frac{2}{3}}\frac{\tilde{\varphi}}{M_{\text{Pl}}} = \frac{\sqrt{6}}{c} N. \tag{4.55}$$

With $c \sim 40$, we get a coefficient of about $1/15$. Then for $N \sim 5$ e-foldings, $\tilde{\alpha}$ varies by a factor of $e^{1/3} \simeq 1.4$ (a 40% change), which is still reasonable, meaning that the attractor holds for about 5 e-foldings in this case.

In the case of the inverse power law potential (4.17),

$$\tilde{\alpha} = pB\left(\frac{\varphi}{M_{\rm Pl}}\right)^{-p-1} , \tag{4.56}$$

if we want the same $\tilde{\alpha} \sim 1/10$, for $p = 2$ we obtain

$$B\left(\frac{\varphi}{M_{\rm Pl}}\right)^{-3} \sim \frac{1}{20} , \tag{4.57}$$

and now we can choose an initial value φ_i large enough so that the above quantity does not vary for many e-folds.

However, one more potential problem is that generically we have $|\eta| \gg \epsilon$, instead of $|\eta| \ll \epsilon$, as assumed until now.

For the generic conformal inflation potential (4.15), we have

$$\eta \equiv M_{\rm Pl}^2 \frac{V''}{V} = -\frac{2}{3} B e^{-\sqrt{\frac{2}{3}}\frac{\varphi}{M_{\rm Pl}}} = -\sqrt{\frac{2}{3}}\tilde{\alpha} , \tag{4.58}$$

so indeed $|\eta| \gg \epsilon = \tilde{\alpha}^2/2$.

But then the scalar tilt is

$$n_s - 1 \simeq 2\eta = -2\sqrt{\frac{2}{3}}\tilde{\alpha} \simeq \frac{3}{100} , \tag{4.59}$$

which implies that we must have

$$\tilde{\alpha} \simeq \frac{1}{50}. \tag{4.60}$$

Since $c \geq 3/\tilde{\alpha}$, we can choose for instance

$$c \sim 200. \tag{4.61}$$

Then the ratios of energy densities are

$$\Omega_V \simeq 1 - \frac{1}{50 \cdot 150} = 1 - \frac{1}{7500} , \quad \Omega_{{\rm kin},\varphi} = \frac{3}{2c^2} = \frac{3}{8 \cdot 10^4} , \quad \Omega_X = \frac{1}{7500} - \frac{3}{8 \cdot 10^4} \sim \frac{1}{10^4}. \tag{4.62}$$

In this case,

$$\frac{\varphi_i - \varphi}{M_{\rm Pl}} = \frac{3}{c} N \simeq \frac{N}{70} , \tag{4.63}$$

which means that $\tilde{\alpha}$ and η do not change significantly even over 70 e-folds, meaning that the attractor lasts at least as much.

For the inverse power law potential (4.17),

$$\eta = -p(p+1)B\left(\frac{\varphi}{M_{\rm Pl}}\right)^{-p-2} = -(p+1)\tilde{\alpha}\left(\frac{M_{\rm Pl}}{\varphi}\right) = -\frac{p+1}{p}\frac{2\epsilon}{B}\left(\frac{\varphi}{M_{\rm Pl}}\right)^{p}. \tag{4.64}$$

Then generically, we also have $|\eta| \gg \epsilon$ (though with a sufficiently small B and not too large $\varphi/M_{\rm Pl}$ we could avoid it).

But then, since $2\eta \simeq -3/100$, we get

$$\tilde{\alpha}\frac{M_{\rm Pl}}{\varphi} \sim \frac{1}{100} \ , \tag{4.65}$$

which means, for $p = 2$, that

$$B\left(\frac{M_{\rm Pl}}{\varphi}\right)^4 \sim \frac{1}{200}. \tag{4.66}$$

Then, with $B \sim 1$, we can impose $\varphi_i \sim 4M_{\rm Pl}$, and since as we saw, $\tilde{\varphi}/M_{\rm Pl} \simeq N/70$, $\tilde{\alpha}$ and η are virtually unchanged even over 100 e-folds.

It is worth noticing that we did not need to satisfy the CMBR constraints for $V_{\rm eff}$, as the perturbations during the attractor-like era considered in this section are not the same as canonical single field inflation, as we will see in the next section. But we wanted to make sure that, even if we considered the tightest constraints, we can still obtain a nontrivial attractor, so we used the CMBR constraints as if this attractor-like solution already describes inflation observables (which it does not).

Finally, consider the fact that the effective potential

$$V_{\rm eff} = V + \rho_X = V + \rho_m e^{-c\frac{(\varphi-\varphi_0)}{M_{\rm Pl}}} \tag{4.67}$$

has an instantaneous (at fixed N) minimum given by

$$\frac{dV_{\rm eff}}{d\varphi} = 0 \Rightarrow \frac{\tilde{\alpha}A}{M_{\rm Pl}} \equiv V'(\varphi) = \frac{c}{M_{\rm Pl}}\rho_{m,0}e^{-c\left(\frac{\varphi_i-\varphi_0}{M_{\rm Pl}}-kN\right)-3N}. \tag{4.68}$$

But for the attractor, with $k = 3/c$, the N dependence cancels out on the right hand side of the above equation, which means that the (late time) attractor sits (or rather, "tracks" it) at the instantaneous minimum of the effective potential, $\varphi_{\rm att}(N) \simeq \varphi_{\rm min}(N)$, meaning that there is a sort of adiabaticity[4].

Analytical results: Attractor 2

However, there is still one more attractor-like case to consider, that will also play a role in the numerics.

In the previous case, we had assumed that ρ_X scales like ρ_V, so we have a constant and nonzero Ω_X.

If we consider instead that $\Omega_X = 0$, just like $\Omega_{\rm rad} = 0$ for the attractor, we obtain another one. Then the Friedmann equation (4.29) becomes

$$k \equiv -\frac{d\varphi}{M_{\rm Pl}dN} = \sqrt{6}\sqrt{1-\Omega_V} \ , \tag{4.69}$$

[4]A similar behaviour for quintessence field coupled with the inflaton via nonrenormalizable interactions was found in [174]. Note this is a different system (two different scalars) than the chameleon coupling with matter considered here, though they have in common a time-dependent effective potential with an attractor for the scalar at the time-dependent minimum.

or equivalently

$$\Omega_V = 1 - \frac{k^2}{6}. \tag{4.70}$$

On the other hand, the generic KG equation (4.37) in the assumption of an attractor ($d^2\varphi/dN^2 = 0$) with $\Omega_X = 0$ becomes

$$3\Omega_V M_{\rm Pl} k = \frac{V_0}{H^2}\frac{\tilde{\alpha}}{M_{\rm Pl}}\ , \tag{4.71}$$

which gives

$$k\Omega_V = \frac{V_0}{3H^2 M_{\rm Pl}^2}\tilde{\alpha} \Rightarrow k = \tilde{\alpha}. \tag{4.72}$$

For this attractor-like solution, we get

$$\Omega_{{\rm kin},\varphi} = \frac{\tilde{\alpha}^2}{6}\ ,\quad \Omega_V = 1 - \frac{\tilde{\alpha}^2}{6}. \tag{4.73}$$

Then, for the maximum value of $\tilde{\alpha}$ we considered (unrestricted by the CMBR), $\tilde{\alpha} \sim 1/5$, we get

$$\Omega_{{\rm kin},\varphi} \sim \frac{1}{150}\ ,\ \ \Omega_V \sim 1 - \frac{1}{150}\ , \tag{4.74}$$

whereas for the CMBR-constrained values we obtain

$$\Omega_{{\rm kin},\varphi} \sim \frac{1}{1.5\times 10^4}\ ,\ \ \Omega_V \sim 1 - \frac{1}{1.5\times 10^4}. \tag{4.75}$$

But of course, like for any dynamical system, if there is more than one attractor, each one of them has a "basin of attraction", a well defined region in the (phase) space of initial conditions, such that if we start inside it, we reach the given attractor.

To find initial conditions compatible with the $\Omega_X \neq 0$ attractor, we consider the fact that H^2 was scaled out of the KG equation near the attractor, but was otherwise given by the Friedmann equation as

$$H^2 = \frac{\rho_V + \rho_{\rm rad} + \rho_X}{3M_{\rm Pl}^2(1-\varphi'^2/6)} = \frac{\rho_V/M_{\rm Pl}^2}{3(1-\varphi'^2/6)}(1 + \Omega_X/\Omega_V). \tag{4.76}$$

Thus, since the overall ρ_V just changes the scale of H^2, if $\Omega_X/\Omega_V \simeq \Omega_{X,att.}/\Omega_{V,att}$, which is about 10^{-4} in the $\tilde{\alpha} = 1/50$ case, and if $\phi' \simeq \phi'_{\rm att.} = 3/c$, we will be quickly driven to the $\Omega_X \neq 0$ attractor.

Otherwise, if for instance we start with $\Omega_X/\Omega_V \ll \Omega_{X,att.}/\Omega_{V,att.}$, the competition between the two terms (with Ω_X and with $dV/d\phi$) on the right hand side of (4.37) means that the other attractor will win.

Before turning to the numerical work, we note that we have defined the notion of "attractor" in a physical sense, not in a strict mathematical sense[5]. We defined it implicitly by having $d^2\varphi/dN^2 \simeq 0$ (compared to other terms) for a long enough number of e-folds (certainly

[5] We also used "attractor-like solutions" to designate these physical "attractors". We let the search for a formal proof that these solutions are attractors in mathematical sense to further investigation.

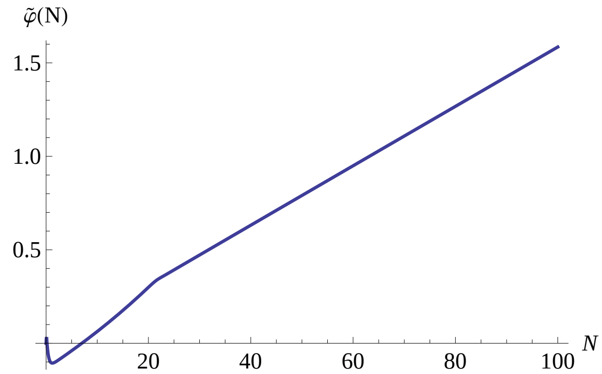


Figure 4.3: Conformal inflation attractor $\tilde{\varphi}(N) = \varphi_i - \varphi(N)$ with $\varphi_0 = \varphi_i = \sqrt{\frac{3}{2}}\ln(\sqrt{\frac{2}{3}}\frac{B}{\tilde{\alpha}})$, $\varphi_i' = -(\frac{3}{c} + 0.5)$. We have $\varphi(N) = -0.0158977N \approx -3N/c$ once the attractor is reached.

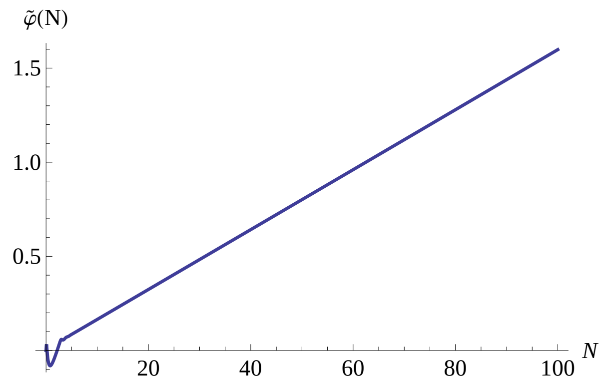


Figure 4.4: Inverse power law attractor $\tilde{\varphi}(N) = \varphi_i - \varphi(N)$ with $\varphi_0 = \varphi_i = 3$, $\varphi_i' = -(\frac{3}{c} + 0.5)$. In this case, $\varphi(N) = -0.0159053N \approx -3N/c$ after onset of the attractor.

larger than the number of e-folds needed to reach the "attractor"). Also note that we cannot be sure there are no more more attractors (other than the two above), but our extensive numerical work seems to suggest that there are not.

Numerical results: Attractor 1

We solved numerically the KG equation (4.37) for $\varphi(N)$ with H given by (4.29), ensuring a flat universe. For generic conformal inflation, we used $B = 6$ while for the inverse power law potential $B = 1$ and $p = 2$, with $A = 10^{-9}$, $\rho_{m,0} = 10^{-4}A$, $\rho_{rad,0} = 0.5A$, $\tilde{\alpha} = 1/50$ and $c = 3/\tilde{\alpha} + 40$ in both cases (we used Planckian units in all numerical simulations). We considered the attractors-like solutions (4.63) and (4.66), since they last enough to have sufficient number of e-folds of inflation. The numerical solution corresponding to them are shown in Figure 4.3 and Figure 4.4.

For generic conformal inflation, we can calculate the exact minimum of the potential at fixed N and compare with the numerical solution, to verify if the attractor follows the instantaneous minimum of the effective potential. This is shown in Figure 4.5, from which we found numerically that at late times the attractor solution sits at the minimum.

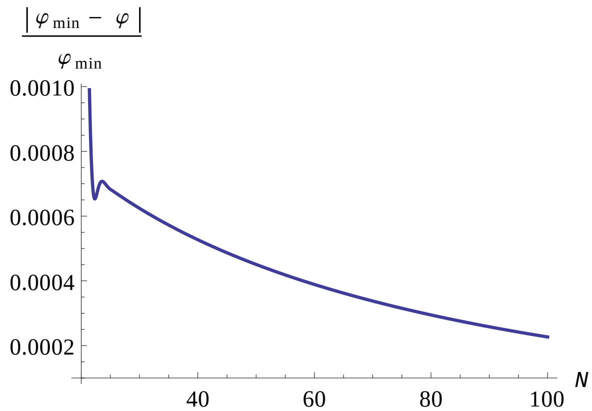


Figure 4.5: Comparison between the instantaneous minimum of effective potential and numerical (attractor) solution.

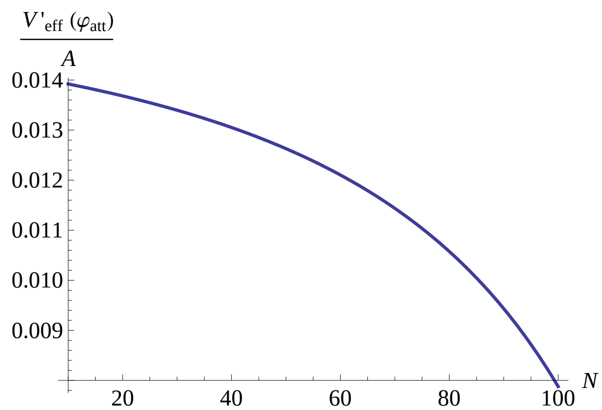


Figure 4.6: The derivative of the effective potential using the inverse power law potential is effectively zero after $N = 25$ e-folds. The numerical solution follows the minimum of the effective potential. A is the amplitude of the potential.

For the inverse power law potential, the instantaneous minimum satisfies a transcendental equation and in order to check if the attractor follows it, we computed $V'_{\text{eff}}(\varphi(N), N)$ on the numerical solution. If $\varphi_{\text{att}}(N) \simeq \varphi_{\text{min}}(N)$, we should have $V'_{\text{eff}}(\varphi_{\text{att}}(N)) \simeq 0$, after the attractor is reached. Figure 4.6 shows that this is actually the case.

Numerical results: Attractor 2

In this case, we expect the attractor-like solution to last less than attractor 1. This is so because $k = \tilde{\alpha}$ and then from $c > 3/\tilde{\alpha}$ we have that $\phi(N)$ changes faster with N. Also, since $\Omega_X \sim e^{-c\varphi-3N} \sim e^{-(3/\tilde{\alpha}+\Delta c)(-\tilde{\alpha}N)-3N} \sim e^{\tilde{\alpha}\Delta c N}$, we see that even starting with very small matter density it will increase so that numerical solutions will only approximate attractor 2.

We consider the $\tilde{\alpha} = 1/50$ case with $\rho_{X,0} = 10^{-6}A$, such that $\Omega_{X,0}/\Omega_{V,0} \ll 10^{-4}$. Figures 4.7 and 4.8 show the numerical solutions for generic conformal inflation and inverse power law potentials. In the inverse power law case $\varphi_i = \varphi_0 = 4.64$, and $\varphi'_i = -\tilde{\alpha}$ for both potentials. All other parameters are the same as attractor 1. We see that attractor 2 does not last more

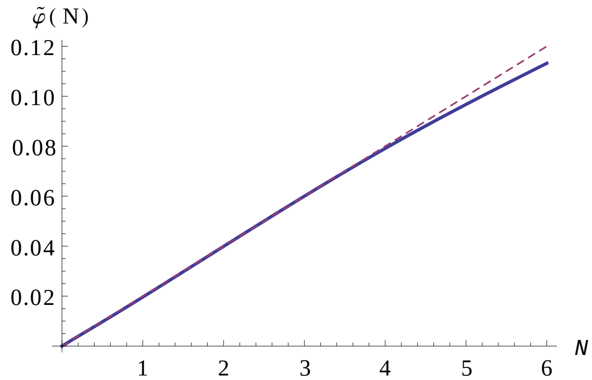


Figure 4.7: Numerical solution (thick line) for conformal inflation potential and analytical result (dashed line) for the attractor 2.

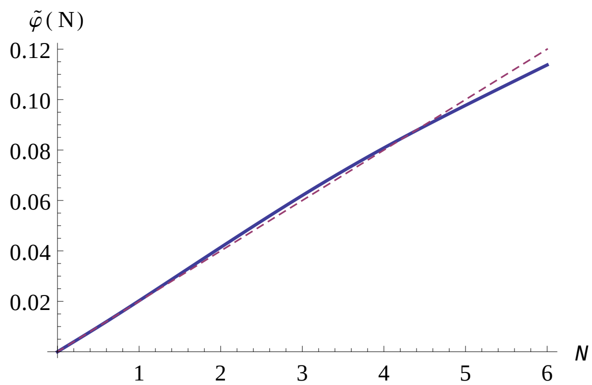


Figure 4.8: Numerical solution (thick line) for inverse power law potential and analytical result (dashed line) for the attractor 2.

than 5 e-folds.

4.3.2 The case $c < 0$

In this case, ρ_X decreases in the same direction, towards small φ, so there is no local minimum for V_{eff}.

Analytical results

In this case, the second attractor from the previous subsection ($c > 0$ case) is still valid. Indeed, we saw that this attractor-like solution corresponds to $\Omega_X = 0$, so in this case the sign of the exponent of the coupling in Ω_X is irrelevant.

Since moreover, in this case, there cannot be any nontrivial attractor with $\Omega_X \neq 0$, since $\Omega_X \sim e^{-c\frac{\varphi}{M_{\text{Pl}}}-3N} \sim e^{-|c|\frac{|\varphi'|}{M_{\text{Pl}}}N-3N}$ decays, while Ω_V stays constant, the only possible endpoint for any initial condition is the attractor with $\Omega_X = 0$. We will call this attractor a *kinetic-potential phase*.

Note that it is different from the case of inflation, in which Ω_V dominates, and the kinetic energy is generically small, and then starts increasing.

The nonzero energy densities are still given by (4.73), but this time we do not have any constraints on the value of $\tilde{\alpha}$ (like $\tilde{\alpha} \geq 3/c$ and $\Omega_V \simeq 1 + 3/(2c^2) - \tilde{\alpha}/c$ before), so we can fix it as we like. However, we still find that, for the value $\tilde{\alpha} \sim 1/50$, consistent with CMBR in the hypothesis of inflation, the attractor is maintained for over 70 e-folds.

Generic initial conditions should lead to the attractor, but we must ensure that the attractor is reached in less than the number of e-folds it persists.

An interesting case, however, is now possible: we can have a kind of *kinetic domination*, usually called *kination*. Not quite that, of course, since in fact this is an attractor with small, though constant and nonzero, Ω_V. That is, we can have $k = \tilde{\alpha} \simeq \sqrt{6}$, so that

$$\Omega_{\mathrm{kin},\varphi} = \frac{\varphi'^2}{6M_{\mathrm{Pl}}^2} \simeq 1 \tag{4.77}$$

and Ω_V very small, though nonzero, whereas Ω_{rad} and Ω_X are truly 0.

However, we still expect to find a condition on c for the existence of this attractor-like solution, since at $c = 0$ (no chameleon), we have no attractor (this is the standard inflationary case, which does not admit the kination phase). To understand this qualitatively, we rewrite the KG equation (4.37), in the case $\Omega_{\mathrm{rad}} = 0$, as

$$H^2\varphi'' + \left(\rho_V + \frac{\rho_X}{2}\right)\frac{1}{M_{\mathrm{Pl}}^2}\varphi' = c\frac{\rho_X}{M_{\mathrm{Pl}}} - \frac{dV}{d\varphi}. \tag{4.78}$$

This takes the form of a "force law", with the first term on the left hand side being the "mass times acceleration", and the second being a friction term, proportional to the velocity and opposing the acceleration. Then the term $-dV/d\varphi < 0$ is a force driving us towards smaller φ, and the first term, for $c < 0$, is another driving force, in the same direction as the potential, allowing the constant velocity motion to go on for longer. If the initial condition has a very large ρ_X, this term will dominate initially, so even if after a long time it decays to zero, its effect is felt through the settling the initial motion into the attractor. This suggests that there should be some minimum value for c, depending on the initial conditions, below which we do not get the attractor-like behaviour.

Numerical results

We considered just the generic conformal inflation potential for this case.

For $\rho_X = 0$, we are back to the usual inflationary scenario. In this case, the conformal inflation potential is a plateau for large field values and becomes steep right before becoming negative. For the parameters used in previous section we have $\epsilon \simeq 0.21$ for $\varphi = 3$, so inflation ends after around that. For typical initial conditions, with φ_i well at the plateau and initial velocity smaller than the critical value $\sqrt{6}$, the field rolls down the potential slowly.

This picture changes with $\rho_X \neq 0$, where we can have the kinetic dominated attractor. But if $|c|$ is not big enough, we found numerically that kination phase does not last much and φ starts to roll down the potential slowly until inflation ends around $\varphi \simeq 3$. But increasing the value of $|c|$ we get kination, as shown in Figure 4.9.

Table 4.1 shows what is the minimal value of $|c|$ in terms of $\rho_{X,0}$ such that the kinetic dominated attractor-like behaviour is achieved. Other parameters are the same used previously.

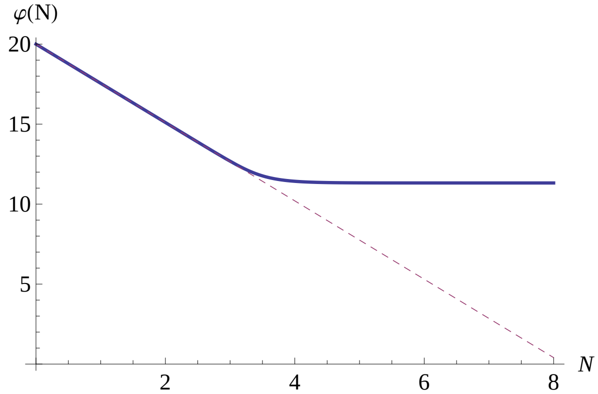


Figure 4.9: Numerical solution with $\rho_{X,0} = e^{10^{-2}}$, $\varphi_i = \varphi_0 = 20$, $\varphi_i' = -1$, $\rho_{r,0} = 0.5A$, $c = -200$ (thick line) and $c = -571$ (dashed). All other parameters are the same as before. For these initial conditions, $c_{\min} = -571$ and the dashed line corresponds to $\varphi(N) = \varphi_i - \sqrt{6}N$.

$\rho_{X,0}$	$c_{\min}$
$e^{10^{-2}}$	-571
$e^{10^{-1}}$	-186
1	-179
e^{10}	-70
e^{10^2}	-2

Table 4.1: Minimal value of c from which we get kination for specific values of ρ_X.

4.4 Modifications to inflationary era and CMBR observables

In this section, we explore the consequences for inflation of the existence of the attractor-like behaviours found in the previous section.

4.4.1 The case $c < 0$ and shortened inflation

In this case, the chameleon coupling term decreases in the same direction as the potential, and we saw that nevertheless we found a kinetic-potential attractor, with a constant "velocity" $d\varphi/dN$ and a constant Ω_V. More interestingly, we could have an *almost kination* attractor, when $\Omega_{\text{kin},\varphi} \simeq 1$. Normally, in inflation the scalar rolls down the potential slowly, and then accelerates as the slope steepens, finally ending inflation.

But in the current case, the main effect of the chameleon coupling is to start with a kinetic (almost kination) phase, and thus delay the onset of the region of inflation *per se*. Once the attractor behaviour ends, and the ρ_X component has sufficiently decayed so as to become irrelevant, we are back to usual inflation. This of course depends on the initial value for ρ_X at the start of the kinetic phase.

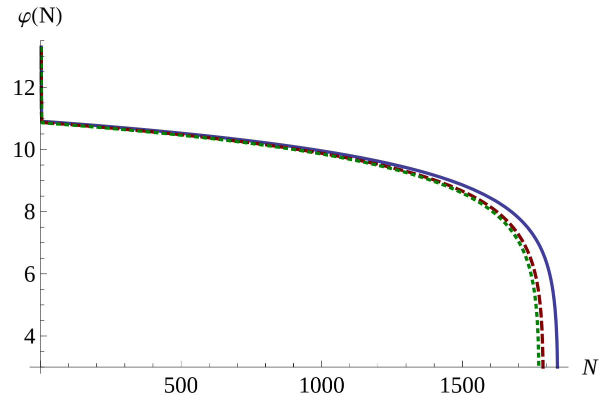


Figure 4.10: End of inflation in numerical solutions with $\rho_{X,0} = e^{10^{-2}}$ (blue thick), $\rho_{X,0} = e^{10^{-1}}$ (red dashed) and $\rho_{X,0} = e^{10}$ (green dotted). In all these cases, the kination phase ends before $N = 5$.

To compare with the pure inflation case ($c = 0$) we need to set the initial value of φ such that transition from the kinetic phase to normal inflation occurs before $\varphi \simeq 3$. But then pure inflation will remain for a large number of e-folds. Indeed, using the parameter values as in Figure 4.9, we get that inflation ends around $N = 1.7 \times 10^6$. Now, for $c \neq 0$ and varying $\rho_{X,0}$, we found numerically that inflation ends earlier than that, assuming that it ends when φ reaches 3. Figure 4.10 shows numerical solutions with $c = -50$ and different values of $\rho_{X,0}$.

4.4.2 The case $c > 0$ and modified inflation

In this case, the chameleon coupling term increases in the direction that the potential decreases, leading to the existence of an instantaneous (at fixed N) minimum of the effective potential $V_{\text{eff}} = V + \rho_X$. As we saw, the attractor 1 (with $\Omega_X \neq 0$) follows at late times this instantaneous minimum $\varphi_{\min}(N)$ of the effective potential, $\varphi_{\text{att}}(N) = \varphi_{\min}(N)$.

Unlike the $c < 0$ case then, we now have a motion that is qualitatively modified: the motion is not due to rolling down the potential itself anymore, but rather to the variation with N of the instantaneous minimum of V_{eff}, where the scalar field sits. It is also not clear how and when this behaviour will end. We will look more into it in the next subsection.

But the result is that we have a new form of inflation, a *modified inflation phase*. Then strictly speaking, we have to calculate the spectrum of perturbations in this phase ab initio. In order to do so we rewrite our model in terms of a second scalar field that has a non-minimal kinetic coupling with the chameleon field. We show that such a model leads to equations of motion that are equivalent to the ones we had studied in the previous sections. The perturbations for two scalar fields where one of them has a non-minimal kinetic coupling and some general potential were computed in [175]. We will apply their results to our scenario.

Another important consistency check for the model is whether the presence of a second scalar field will lead to an increase of the entropy fluctuations. In the following section we check that the chameleon coupling in our model actually prevents the entropy modes to grow during inflation. We are then allowed to compute the scalar tilt and tensor to scalar ratio parameters in terms of the adiabatic modes only.

Unfortunately, in trying to compare with CMBR data, we face a quandary: we saw that the attractor behaviour $\varphi_{\rm att}(N)$ seems to go on forever, and there is no way to end inflation. That is relevant, since the scale that we see the CMBR in the sky now, $k_0 \sim 10^{-3}{\rm Mpc}^{-1}$, is situated a number $N = N(k_0)$ e-folds of inflation before the *end of inflation*, related (by the known evolution of the Universe, assuming a normal Einstein-gravity radiation dominated phase after reheating until Big Bang Nucleonsynthesis) by the usual formula

$$N = \ln \frac{a_{\rm end}}{a_{k_0}} = 56 - \frac{2}{3} \ln \frac{10^{16} GeV}{\rho_*^{1/4}} - \frac{1}{3} \ln \frac{10^9 GeV}{T_R} \ , \tag{4.79}$$

where ρ_* is the energy density at the end of inflation and T_R is the reheat temperature.

Assuming some standard values for ρ_* and T_R, one gets around 60 e-folds of inflation from the scale k_0 (relevant for CMBR) exiting the horizon, when we should measure n_s and r, and the end of inflation.

Thus in order to be consistent with observations, we must find a way to end inflation.

4.4.3 Ending inflation for $c > 0$

The simplest way to end inflation in the case $c > 0$ is to remember that the heavy non-relativistic particle(s) that make up ρ_X must decay, since they are not there after inflation. That is, they must have a decay constant $\Gamma = 1/\tau$ with τ of the order of the age of the Universe at the end of inflation. Then the energy density ρ_m of this component satisfies the usual modified equation of motion

$$\dot{\rho}_m + 3H\rho_m + \Gamma \rho_m = 0. \tag{4.80}$$

This implies an extra factor in the decreases of $\rho_X = \rho_m F(\varphi)$ with N of

$$e^{-\Gamma t} = e^{-\frac{\Gamma}{H} H t} \sim a^{-\frac{\Gamma}{H}} = e^{-\frac{\Gamma}{H} N}. \tag{4.81}$$

Then, for instance for a $\Gamma \sim 0.1H$ (or $\tau \sim 10H^{-1}$), we would obtain an extra factor of $e^{-0.1N}$. After about 100 e-folds, this would give a contribution of about $e^{-10} \simeq 10^{-4.3}$.

However, the drawback of this method is that it goes on very slowly, instead of the sudden end to inflation that we usually need.

One alternative possibility would be to say that, once the slope of the potential $V(\varphi)$ starts becoming steep (so that its own ϵ or η would be of order 1, so that normal inflation would have ended), some unknown particle physics mechanism would allow ρ_m to decay *into the inflaton/chameleon particles themselves*, thus ending inflation, and allowing reheating (the conversion of the inflaton into lighter particles) to happen.

This can be implemented as an ad-hoc turning off of ρ_X at the same time normal inflation would have ended, though it is not clear how one could implement it in detail from a particle physics perspective. In the rest of this section we calculate the CMBR observables as a function of the number of e-folds of inflation.

4.4.4 Microscopic description

To proceed with the calculation of the inflationary observables, we show the consistency of our model with the one considered in [175]. In fact, the equations of motion we have assumed during this chapter have a microscopic description given by the Lagrangian:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right] + S_m[\tilde{g}_{\mu\nu} = F^2(\varphi)g_{\mu\nu}], \tag{4.82}$$

where the non-relativistic matter can be described, without loss of generality, by a scalar field,

$$S_m = \int d^4x\sqrt{-\tilde{g}}\left[-\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - U(\chi)\right]. \tag{4.83}$$

In terms of the Einstein-frame metric the above action can be rewritten as

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}F^2(\varphi)g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - \tilde{V}(\varphi,\chi)\right], \tag{4.84}$$

which has exactly the form considered in [175] with

$$F(\varphi) = e^{b(\varphi)}, \quad b(\varphi) = -c(\varphi-\varphi_0)/M_{\rm Pl}, \quad \tilde{V}(\varphi,\chi) = V(\varphi) + F^4(\varphi)U(\chi). \tag{4.85}$$

In our case we consider everything in terms of the inflaton and the energy density of a matter ($p = 0$ in the Einstein-frame) fluid. We need to find a mapping between the energy density[6] ρ_m and the second scalar field χ. In order to do so, we compute

$$\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S}{\delta\tilde{g}^{\mu\nu}} \tag{4.86}$$

and then use[7] $\rho_m = F(\varphi)^3\tilde{\rho}_m$. Assuming a homogeneous field, we have that

$$\begin{aligned}\tilde{T}_{\mu\nu} &= \partial_\mu\chi\partial_\nu\chi - \tilde{g}_{\mu\nu}\left(\frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + U\right)\\ \implies \tilde{T}_{00} &= \frac{1}{2}\dot{\chi}^2 + F^2U \implies \tilde{\rho}_m = -\tilde{T}^0_0 = \frac{1}{2}F^{-2}\dot{\chi}^2 + U.\end{aligned} \tag{4.87}$$

Then we find

$$\rho_m = \frac{1}{2}F\dot{\chi}^2 + F^3U. \tag{4.88}$$

Also, from $T_{\mu\nu} = F^2\tilde{T}_{\mu\nu}$ and

$$\tilde{T}_{ij} = \tilde{g}_{ij}F^{-2}\left(\frac{1}{2}\dot{\chi}^2 - F^2U\right) \tag{4.89}$$

we have

$$T_{ij} = pg_{ij} = F^2\tilde{p}\tilde{g}_{ij} \implies p = F^4\tilde{p} = F^2\left(\frac{1}{2}\dot{\chi}^2 - F^2U\right). \tag{4.90}$$

[6] Remember that $\rho_X = F(\varphi)\rho_m$.

[7] See [166] for a detailed proof of this relation.

Therefore, the condition for χ to describe a matter fluid, $p = 0$, implies $\frac{1}{2}\dot{\chi}^2 = F^2 U$. Using this into the expression for ρ_m, we find

$$\rho_m = F\dot{\chi}^2. \tag{4.91}$$

From the above mapping, one can rewrite the equation of motion coming from the action (4.84)

$$\begin{aligned} \ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial \tilde{V}}{\partial \varphi} &= \frac{\partial b}{\partial \varphi} e^{2b}\dot{\chi}^2, \\ \ddot{\chi} + \left(3H + 2\frac{\partial b}{\partial \varphi}\dot{\varphi}\right)\dot{\chi} + e^{-2b}\frac{\partial \tilde{V}}{\partial \chi} &= 0, \end{aligned} \tag{4.92}$$

as

$$\begin{aligned} \ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} &= -\rho_m \frac{\partial F}{\partial \varphi}, \\ \dot{\rho}_m + 3H\rho_m &= 0, \end{aligned} \tag{4.93}$$

which are exactly the equations we have used throughout this chapter. Note that to find it we had to use the matter fluid condition, since the chameleon coupling is sensitive to the equation of state of the fluid.

Adiabatic and entropy perturbations for the action in (4.84) were also computed in detail in [175]. Having showed our macroscopic description is recovered by the microscopic one given by (4.84) we now move on to investigate whether these perturbations are under control in our scenario.[8]

Adiabatic and Entropy Perturbations

Following [175], in order to proceed with calculations we first decompose the fields in its adiabatic and entropy components (see also [176–178]):

$$\begin{aligned} d\sigma &= \cos\theta d\varphi + e^b \sin\theta d\chi \\ ds &= e^b \cos\theta d\chi - \sin\theta d\varphi, \end{aligned} \tag{4.94}$$

where

$$\cos\theta = \frac{\dot{\varphi}}{\dot{\sigma}}, \quad \sin\theta = \frac{e^b\dot{\chi}}{\dot{\sigma}}, \quad \text{with} \quad \dot{\sigma} = \sqrt{\dot{\varphi}^2 + e^{2b}\dot{\chi}^2}, \tag{4.95}$$

and we write the metric perturbations in longitudinal gauge

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)(1-2\Phi)\delta_{ij}dx^i dx^j. \tag{4.96}$$

[8] One could perhaps think that entropy perturbations are avoided by being at the minimum of the effective potential, with a large effective mass, but since we have a dynamical (time-dependent) situation, and one depending on the density of matter, it is not clear, and must be checked. We will in fact find that it is a quite complicated analysis.

The curvature and entropy perturbations, ζ and δs, defined as

$$\begin{aligned} \zeta &= \Phi - \frac{H}{\dot{H}}\left(\dot{\Phi} + H\Phi\right) \\ \delta s &= -\frac{\dot{\sigma}}{2H(d\tilde{V}/ds)}\left(\dot{\zeta} - \frac{k^2}{a^2}\frac{H}{\dot{H}}\Phi\right) \end{aligned} \tag{4.97}$$

satisfy the following equations of motion

$$\begin{aligned} \ddot{\zeta} + \left(3H - 2\frac{\dot{H}}{H} + \frac{\ddot{H}}{\dot{H}}\right)\dot{\zeta} + \frac{k^2}{a^2}\zeta = \frac{H}{\dot{\sigma}}\Bigg[&\frac{d}{dt}(\dot{\theta}\delta s) - 2\left(\frac{1}{\dot{\sigma}}\frac{d\tilde{V}}{d\sigma} + \frac{\dot{H}}{H}\right)\dot{\theta}\delta s \\ &+2\frac{db}{d\varphi}h(t) + \frac{d^2b}{d\varphi^2}\dot{\sigma}^2\sin 2\theta\delta s\Bigg], \end{aligned} \tag{4.98}$$

$$\begin{aligned} \ddot{\delta s} + 3H\dot{\delta s} + \Bigg[&\frac{k^2}{a^2} + \frac{d^2\tilde{V}}{ds^2} + 3\dot{\theta}^2 + \left(\frac{db}{d\varphi}\right)^2 g(t) + \frac{db}{d\varphi}f(t) - \\ &-\frac{d^2b}{d\varphi^2}\dot{\sigma}^2 - 4\frac{1}{\dot{\sigma}^2}\left(\frac{d\tilde{V}}{ds}\right)^2\Bigg]\delta s = 2\frac{1}{H}\frac{d\tilde{V}}{ds}\dot{\zeta}, \end{aligned} \tag{4.99}$$

where

$$\begin{aligned} f(t) &= \frac{d\tilde{V}}{d\varphi}(1 + \sin^2\theta) - 4\frac{d\tilde{V}}{ds}\sin\theta \\ g(t) &= -\dot{\sigma}^2(1 + 3\sin^2\theta) \\ h(t) &= \dot{\sigma}\frac{d}{dt}(\sin\theta\delta s) - \sin\theta\left(\frac{\dot{H}}{H}\dot{\sigma} + 2\frac{d\tilde{V}}{d\sigma}\right)\delta s - 3H\dot{\sigma}\sin\theta\delta s \\ \dot{\theta} &= \dot{\sigma}\left(-\frac{1}{\dot{\sigma}^2}\frac{d\tilde{V}}{ds} - \frac{db}{d\varphi}\sin\theta\right). \end{aligned} \tag{4.100}$$

In order to apply the above results to our model we only need to write all functions in terms of N and to replace all possible χ dependence for ρ.

Perturbations in conformal inflation with chameleon coupling

From equation (4.98) and (4.99), we see that entropy perturbations will feed adiabatic modes and vice versa. Since there is no evidence for entropy modes in the CMB data, we must check if entropy perturbations are under control in our model. In order to do so, we rewrite equation (4.99) as

$$\begin{aligned} \delta\ddot{s} + 3H\delta\dot{s} + \left(\frac{k^2}{a^2} + m_{\rm eff}^2\right)\delta s &= -\frac{k^2}{a^2}\frac{d\tilde{V}}{ds}\frac{4M_{Pl}^2\Phi}{\dot{\sigma}^2}, \\ m_{\rm eff}^2 \equiv \frac{d^2\tilde{V}}{ds^2} + 3\dot{\theta}^2 + \left(\frac{db}{d\varphi}\right)^2 g(t) &+ \frac{db}{d\varphi}f(t) - \frac{d^2b}{d\varphi^2}\dot{\sigma}^2. \end{aligned} \tag{4.101}$$

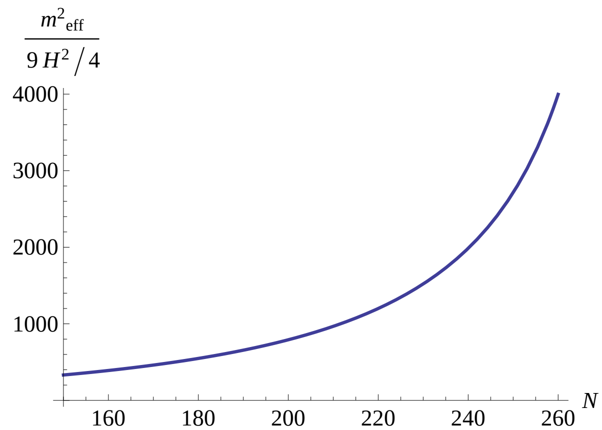


Figure 4.11: Evolution of the effective mass for the *conformal inflation model* during inflation, evaluated for the solution showed in Figure 4.3. The entropy perturbation field gets heavier due to the coupling to the chameleon field.

One can see that the entropy evolves freely at super-Hubble scales. Moreover, in order for small-scale quantum fluctuations to generate super-Hubble perturbations during inflation, the fluctuation should be light compared to the Hubble scale. In fact, the existence of super-Hubble entropy perturbation requires that

$$m_{\text{eff}}^2 < \frac{9}{4}H^2, \tag{4.102}$$

otherwise the perturbations stay at the vacuum state and there is a strong suppression at large scales.

It is straightforward to rewrite the effective mass in terms of the number of e-folds N and evaluate it in terms of the solutions we obtained in the previous sections. Figure 4.11 and 4.12 show that the effective mass is much bigger than the Hubble scale during inflation. More than that, in both cases the chameleon coupling effectively makes the entropy modes more and more massive during inflation, strongly suppressing its effects.

One can ask what happens with the sub-Hubble modes. Assuming an usual Bunch-Davies vacuum initial condition, ζ and δs initially have amplitudes of the same order. Using the time scales for the variation of the perturbations, $\dot{\zeta} \sim H\zeta$ and $\delta\dot{s} \sim m_{\text{eff}}\delta s$, one can see from (4.99) that the feeding of entropy modes by adiabatic perturbations is negligible also in small scales. Therefore, in a semi-classical point of view (where ζ and δs are classical fields with quantum initial conditions), the contributions for the power spectrum of adiabatic perturbations are of the order $\mathcal{O}(H^2/m_{\text{eff}}^2)$ and can be safely neglected in our model.

From the above discussion, we only need to solve the homogeneous equation for the adiabatic modes

$$\ddot{\zeta} + \left(3H - 2\frac{\dot{H}}{H} + \frac{\ddot{H}}{\dot{H}}\right)\dot{\zeta} + \frac{k^2}{a^2}\zeta = 0, \tag{4.103}$$

which leads to the usual relation for the power spectra of the perturbations

$$n_s - 1 = 2\eta - 4\varepsilon, \tag{4.104}$$

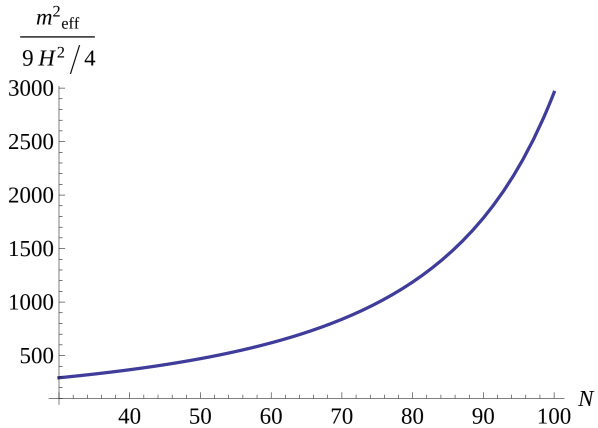


Figure 4.12: Evolution of the effective mass for the *power law model* during inflation, evaluated for the solution displayed at Figure 4.4. The entropy perturbation field also gets heavier due to the coupling to the chameleon field in this scenario.

with the ε and η written in terms of H as functions of N:

$$
\begin{aligned}
\varepsilon(N) &= -\frac{H'(N)}{H(N)} \\
\eta(N) &= \varepsilon - \frac{1}{2\varepsilon}\varepsilon'(N) \\
&= -\frac{1}{2}\left(\frac{H''(N)}{H'(N)} + \frac{H'(N)}{H(N)}\right).
\end{aligned} \tag{4.105}
$$

Also, since tensor perturbations are independent of scalar ones, we can use:

$$r = 16\varepsilon(N). \tag{4.106}$$

As we have previously discussed, in order to leave the attractor phase we need to choose a time to end inflation. We will use the time inflation would end if we did not have a chameleon coupling for both models. This will also be important to compare how the coupling with chameleon change the observables values for the conformal inflation model.

In the conformal inflation potential case, the tilt and scalar to tensor ratio are then given by

$$n_s - 1 = 0.9743, \quad r = 0.0508, \tag{4.107}$$

for 50 e-folds (everything evaluated at $N = 201$) and

$$n_s - 1 = 0.9765, \quad r = 0.0421, \tag{4.108}$$

for 60 e-folds (everything evaluated at $N = 191$).

For the inverse power law potential case, we have

$$n_s - 1 = 0.9712, \quad r = 0.0292, \tag{4.109}$$

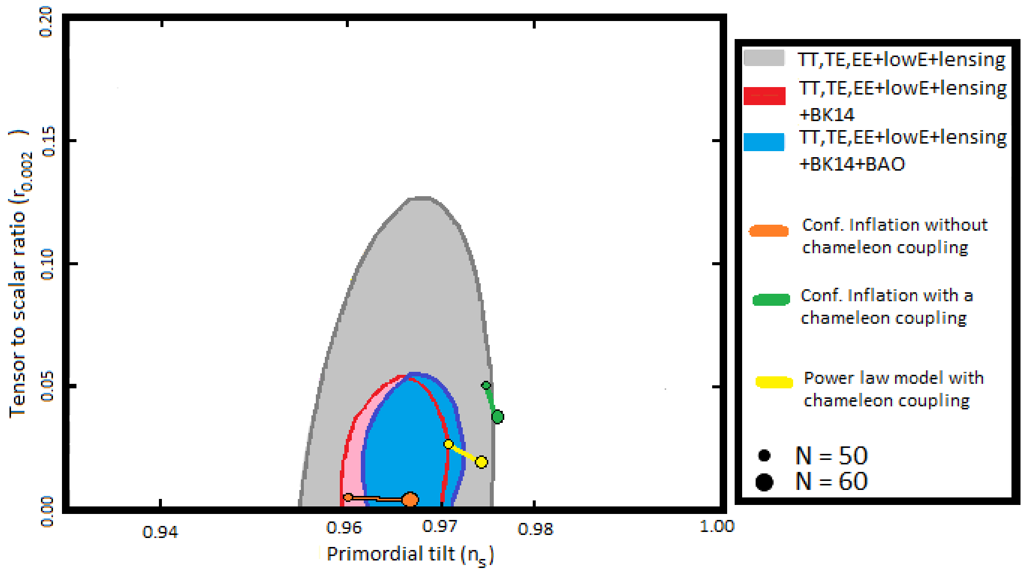


Figure 4.13: The chameleon coupling to the conformal inflation model almost takes its observables outside the allowed region by Planck satelite measurents [148]. For the power law model it makes inflation last enough e-folds and the observables lie inside the region allowed by observations.

for 50 e-folds (everything evaluated at $N = 49$) and

$$n_s - 1 = 0.9742, \quad r = 0.0229, \tag{4.110}$$

for 60 e-folds (everything evaluated at $N = 39$).

It is interesting to compare these results with the pure conformal inflation class of models, where for 50 e-folds we get $n_s = 0.9600$, while for 60 e-folds $n_s = 0.9667$ (see Figure 4.13). So, we see that the coupling of the chameleon with non-relativistic matter shifts the values of cosmological observables from the sweet spot of Planck data to almost out of the region allowed by the observations in the conformal inflation case but they still lie inside the region allowed by observations for the power law model considered here.

4.5 Final Comments

In this chapter we have investigated the possibility that the inflaton in conformal inflation models is also a chameleon, containg a coupling $\rho_m F(\varphi) = \rho_m e^{-c(\varphi-\varphi_0)}$ to the energy density ρ_m of some heavy non-relativistic particles present during inflation.

After this project was concluded, we became aware of the paper [179], that also deals with inflation coupled to nonrelativistic matter, though mostly from supergravity embeddings. Their conclusions are somewhat different, because of the differences in models, the several approximations they consider, and their specific initial conditions, but they also find that inflation is modified by the chameleon coupling. For $c = -\sqrt{2/3}$ (the Starobinsky model, the only one that overlaps with our work), we checked that their initial conditions indeed lead to similar results to theirs. However, we have numerically considered a very large class of initial conditions, values for c, and made no approximations, which allowed us to find that in the case of natural initial conditions we obtain attractor behaviour.

Chapter 5

T-dual Cosmological Solutions in Double Field Theory

5.1 Motivation: String Gas Cosmology and Stringy Symmetries

The String Gas Cosmology (SGC) model is based on stringy thermodynamics on compact spaces. For a gas of closed strings, there is a maximum temperature called *Hagedorn temperature* T_H, and if we consider the gas to be in a closed box with length scale R the temperature $T(R)$ will stay close to T_H for a range of R around the string length. As the entropy of the gas increases, the range of R for which $T(R) \sim T_H$ increases [59]. In SGC, these properties are used to postulate that the very early universe can be modeled by a gas of strings in a quasi-static phase with temperature close to T_H. There are proposals that explain the origin of cosmological perturbations by thermal fluctuations during this phase [60] and indications that the model is self-consistent by showing that most of the moduli (size and shape of the torii) are stabilized [180]. Unfortunately, this is done in the limit of constant dilaton supergravity, with the motivation of describing the late time phase dynamics of the string gas, and there is no theory that rules the full background evolution of the string gas in a consistent way. The SGC setup stands by results about one-loop thermal string partition functions (for a review on string thermodynamics and applications to cosmology, see [181]).

As compactification on a torus is one of the assumptions of SGC, one can ask what T-duality has to say about the string gas. In fact, as pointed out in the original proposal [59], there should be two position operators for each direction, one (x^μ) associated with the momentum modes around the compactified directions and other ($\tilde{x}^\mu$) related with the purely stringy winding modes. On T-duality grounds, these operators should be dual to each other. Consider the simplest case of one compactified direction. Equation (3.108) shows that the momentum mode contribution for the string energy is proportional to the inverse of the compactification scale R, while the contribution from winding modes are proportional to R (in stringy units). Thus, considering the partition of energy for the string gas, for large values of R (with respect to the string length) the momentum modes dominates the counting for the partition function (as they are energetically favorable) and for small R the winding modes are dominant. Then, the phase $R < 1$ prior to the quasi-static state around $R = 1$ would be T-dual to the one for $R > 1$, and we would have a "bouncing-like" cosmology. A momentum mode dominated gas has a radiation equation of state parameter $w = 1/(D-1)$ and a winding dominated gas has an equation of state parameter $w = -1/(D-1)$. We see that SGC evokes T-duality symmetry automatically.

The issue is that we do not have total control around $R = 1$ as the oscillatory modes contribution (that have a dust-like equation of state parameter) cannot be disregarded and all tower of string states are important. But as full String Theory is T-duality invariant, the picture described in the previous paragraph is prone to be right. Although dynamical equations for the background are unknown, we expect them to be T-duality invariant, and so a natural step is to study Double Field Theory for cosmology.

The first applications of DFT to cosmology are recent [2,124–126,182,183]. In particular, in [126] the authors coupled the double space DFT equations evaluated for a cosmological Ansatz (that were obtained previously in [182]) with a perfect fluid in a T-duality consistent way, in the same lines as [184] did in supergravity. They found cosmological solutions that were not T-dual to each other and an unpleasant feature on the dual energy density: it should be negative[1].

Even not considering SGC, there are still motivations to study cosmological solutions of DFT. If we try to construct a field theory that comes from String Theory, this QFT should implement the stringy symmetries somehow and in particular it should be T-duality invariant. Using symmetries of string theory in usual field theories is also an approach to String Cosmology. Some results in this direction were recently studied in [183] by generalizing Einstein's equations to include the all NS-NS massless sector fields and $\mathrm{O}(D, D)$ symmetry in a "stringy-like" gravity [185].

In this chapter, we show how to get solutions to DFT equations under a cosmological Ansatz that are truly T-dual. We show also that the negativity of dual energy density previously found in [126] was due to a misinterpretation of the energy density in the T-dual section, and once we write the equations in terms of the proper one, we get positive energy densities in both double-space sections.

5.2 T-Dual Frames vs. T-Dual Variables

We consider an underlying D-dimensional space-time. The fields of DFT then live in a $2D$ dimensional space with coordinates (t, x) and dual coordinates $(\tilde{t}, \tilde{x})$, where t is time and x denote the $D - 1$ spatial coordinates. In general, the *generalized metric* of DFT is made up of the $D-$dimensional space-time metric, the dilaton and an antisymmetric tensor field, all being functions of the $2D$ coordinates.

In this section (like in the rest of this chapter) we consider only homogeneous and isotropic space-times and transformations which preserve the symmetries. In this case, the basic fields

[1]The authors of [126] tried to fix that issue in the last section of the essay, but it turns out that the proposed argument was ineffective in solving the "minus sign issue" in the dual energy density.

reduce to the cosmological scale factor $a(t,\tilde{t})$ and the dilaton $\phi(t,\tilde{t})$. It is self-consistent to neglect the antisymmetric tensor field. These are the same fields which also appear in dilaton gravity.

In supergravity, the T-duality transformation of the fields can be defined as

$$a(t) \to \frac{1}{a(t)}, \quad d(t) \to d(t), \tag{5.1}$$

where $d(t)$ is the rescaled dilaton

$$d(t) = \phi(t) - \frac{D-1}{2} \ln a(t) \tag{5.2}$$

which is invariant under a T-duality transformation. In DFT this definition can be generalized to be

$$a(t,\tilde{t}) \to \frac{1}{a(\tilde{t},t)}, \quad d(t,\tilde{t}) \to d(\tilde{t},t). \tag{5.3}$$

This implies that the dilaton transforms as

$$\phi(t,\tilde{t}) \;\to\; \phi(\tilde{t},t) - (D-1)\ln a(\tilde{t},t)\,. \tag{5.4}$$

An important assumption of DFT is the need to impose a *section condition*, a condition which states that the fields only depend on a D-dimensional subset of the space-time variables. The different choices of this section condition are called *frames*, and different frames are related via T-duality transformations. The *supergravity frame* is the frame in which the fields only depend on the (t,x). The second frame which we will consider is the *winding frame* in which the fields only depend on the $(\tilde{t},\tilde{x})$ coordinates.

In this chapter we are interested in finding supergravity frame solutions

$$(\phi(t), a(t), d(t)) \tag{5.5}$$

and winding frame solutions

$$(\phi(\tilde{t}), a(\tilde{t}), d(\tilde{t})) \tag{5.6}$$

which are T-dual to each other, i.e.

$$d(\tilde{t}) \;\to\; d(t(\tilde{t})) \tag{5.7}$$

$$a(\tilde{t}) \;\to\; \frac{1}{a(t(\tilde{t}))}\,, \tag{5.8}$$

where $t(\tilde{t}) = \tilde{t}$.

5.3 Equations

Our starting point is given by the equations for DFT under a cosmological ansatz [182] (Eqs. (8) in [126]):

$$\begin{aligned} 4d'' - 4(d')^2 - (D-1)\tilde{H}^2 + 4\ddot{d} - 4\dot{d}^2 - (D-1)H^2 = 0 \\ (D-1)\tilde{H}^2 - 2d'' - (D-1)H^2 + 2\ddot{d} = 0 \\ \tilde{H}' - 2\tilde{H}d' + \dot{H} - 2H\dot{d} = 0\,, \end{aligned} \tag{5.9}$$

where the prime denotes the derivative with respect to $\tilde{t}$, and the overdot the derivative with respect to t. In addition,

$$H = \frac{\dot{a}}{a}, \quad \tilde{H} = \frac{a'}{a}. \tag{5.10}$$

These equations are invariant under T-duality, since $d(t, \tilde{t})$ is a scalar and $H \leftrightarrow -\tilde{H}$ under this transformation. Then, we couple these equations with matter in the following way [126]

$$\begin{aligned} 4d'' - 4(d')^2 - (D-1)\tilde{H}^2 + 4\ddot{d} - 4\dot{d}^2 - (D-1)H^2 = 0 \\ (D-1)\tilde{H}^2 - 2d'' - (D-1)H^2 + 2\ddot{d} = \frac{1}{2}e^{2d}E \\ \tilde{H}' - 2\tilde{H}d' + \dot{H} - 2H\dot{d} = \frac{1}{2}e^{2d}P. \end{aligned} \tag{5.11}$$

Now, these new equations are invariant under T-duality provided $E \to -E$ and $P \to -P$. But this is exactly the case since, as explained in [126], the energy and pressure in the winding frame are given by

$$\begin{aligned} E(t, \tilde{t}) &= -2\frac{\delta F}{\delta g_{tt}(t, \tilde{t})} \to -2\left(-g_{tt}^2(\tilde{t}, t)\frac{\delta F}{\delta g_{tt}(\tilde{t}, t)}\right) = -E(\tilde{t}, t), \\ P(t, \tilde{t}) &= -\frac{2}{D-1}\frac{\delta F}{\delta g_{ij}(t, \tilde{t})}g_{ij}(t, \tilde{t}) = -\frac{\delta F}{\delta \ln a(t, \tilde{t})} \to -\frac{\delta F}{\delta \ln(1/a(\tilde{t}, t))} = -P(\tilde{t}, t), \end{aligned} \tag{5.12}$$

where we used $g_{tt} = 1$ for our case and assumed that the matter action in double space F is $O(D, D)$ invariant. The invariance of Eqs. (5.11) under T-duality is a strong support for the correctness of the coupling with matter.

Solutions to Eqs. (5.11) may be found after imposing the strong condition of DFT. One may impose that all functions are $\tilde{t}$-independent or t-independent, corresponding to the supergravity (SuGra) or winding frames, respectively. In [126], solutions based on either the SuGra or winding frames were found for the case of constant dilaton $\phi(t, \tilde{t}) = \phi_0$. But notice that by (5.4) the dilaton transforms non-trivially under T-duality. Hence, the solutions found in [126] in the SuGra and winding frames, respectively, are not T-dual to each other. The fact that two solutions both with constant dilaton in the respective frames are not related by T-duality (or $O(D, D,)$ more generally) can be confirmed by noting that equations (12) in [126] obtained from (5.11) after assuming constant dilaton are not T-dual invariant. These equations were obtained by imposing

$$\begin{aligned} 2d(t, \tilde{t}) &= 2\phi_0 - (D-1)\ln a(t, \tilde{t}) \\ \Longrightarrow\ 2\dot{d} &= -(D-1)H, \quad 2d' = -(D-1)\tilde{H}, \end{aligned} \tag{5.13}$$

which is not compatible with T-duality, since $2d'$ does not transform to $2\dot{d}$ as it should.

From the point of view of a field theory with doubled coordinates, there is no problem in considering constant dilaton in the way it was considered in [126]. However, since the SuGra and winding frame solutions are not T-dual to each other, the comparison of these solutions used to motivate the correspondence $\tilde{t} \to t^{-1}$ is tenuous.

In this work, we look for equations and solutions that respect T-duality, and specifically with constant dilaton *only* in the SuGra frame or in the winding frame. We also solve an apparent inconsistency with positive energy density in the winding frame, found in [126].

5.4 T-duality preserving ansatz and equations for each frame

Starting from the supergravity frame, let us look for solutions with constant dilaton. In this case

$$\begin{aligned} 2d(t) &= 2\phi_0 - (D-1)\ln a(t) \\ \Longrightarrow\ 2\dot{d} &= -(D-1)H\,. \end{aligned} \tag{5.14}$$

We now seek solutions in the winding frame which are T-dual. By the invariance of d, $d(t) = d(\tilde{t}(t))$, we have

$$\begin{aligned} \phi_0 - \frac{D-1}{2}\ln a &= \phi(\tilde{t}) - \frac{D-1}{2}\ln a(\tilde{t}) \\ \Longrightarrow\ \phi(\tilde{t}) &= \phi_0 - \frac{D-1}{2}\ln\left(\frac{a(t(\tilde{t}))}{a(\tilde{t})}\right)\,. \end{aligned} \tag{5.15}$$

Now by the scale-factor duality which comes from the transformation of the generalized metric, $a(t(\tilde{t})) = 1/a(\tilde{t})$, and so

$$\phi(\tilde{t}) = \phi_0 + (D-1)\ln a(\tilde{t})\,, \tag{5.16}$$

and hence

$$\begin{aligned} d(\tilde{t}) &= \phi_0 + \frac{D-1}{2}\ln a(\tilde{t}) \\ \Longrightarrow\ 2d'(\tilde{t}) &= (D-1)\tilde{H}\,. \end{aligned} \tag{5.17}$$

Thus, the ansatz for the rescaled dilaton $d(t,\tilde{t})$ in the winding frame will be such that

$$2\dot{d}(t) = -(D-1)H, \quad 2d'(\tilde{t}) = (D-1)\tilde{H}, \tag{5.18}$$

which is related to the supergravity frame dilaton by T-duality. Similarly, for a constant dilaton in the winding frame we have

$$2\dot{d}(t) = (D-1)H, \quad 2d'(\tilde{t}) = -(D-1)\tilde{H}\,. \tag{5.19}$$

Equations (5.18) and (5.19) are *Ansätze* compatible with T-duality between the SuGra and winding frames.

To find the equations in each frame under these assumptions, let us consider

$$2\dot{d}(t) = \alpha(D-1)H, \quad 2d'(\tilde{t}) = \tilde{\alpha}(D-1)\tilde{H}\,, \tag{5.20}$$

which takes both cases into account: for $(\alpha,\tilde{\alpha}) = (-1,1)$ we have a constant dilaton in the SuGra frame and non-constant dilaton in the winding frame; for $(\alpha,\tilde{\alpha}) = (1,-1)$, we have constant dilaton in the winding frame and non-constant dilaton in the SuGra frame. The case $(\alpha,\tilde{\alpha}) = (-1,-1)$ corresponds to having the dilaton constant in both frames and was considered in [126]. But, as already argued, this breaks the T-duality between the frames.

Here,we are looking for solutions in each frame that are T-dual to each other, so we will not consider the case $(\alpha, \tilde{\alpha}) = (1, 1)$.

Applying the section conditions, we get equations for SuGra and winding frame,

$$
\begin{aligned}
4\ddot{d} - 4\dot{d}^2 - (D-1)H^2 &= 0 & 4d'' - 4(d')^2 - (D-1)\tilde{H}^2 &= 0 \\
-(D-1)H^2 + 2\ddot{d} &= \frac{1}{2}e^{2d}E(t) & (D-1)\tilde{H}^2 - 2d'' &= \frac{1}{2}e^{2d}E(\tilde{t}) \\
\dot{H} - 2H\dot{d} &= \frac{1}{2}e^{2d}P(t) & \tilde{H}' - 2\tilde{H}d' &= \frac{1}{2}e^{2d}P(\tilde{t})
\end{aligned}
\tag{5.21}
$$

Before solving them, notice that the energy and pressure in the winding frame are given by

$$
\tilde{E}(\tilde{t}) = -2\frac{\delta F}{\delta g_{\tilde{t}\tilde{t}}(\tilde{t})} = -2\left(-g_{tt}^2(\tilde{t})\frac{\delta F}{\delta g_{tt}(\tilde{t})}\right) = -E(\tilde{t}), \tag{5.22}
$$

$$
\begin{aligned}
\tilde{P}(\tilde{t}) &= -\frac{2}{D-1}\frac{\delta F}{\delta g_{\tilde{i}\tilde{j}}(\tilde{t})}g_{\tilde{i}\tilde{j}}(\tilde{t}) = -2\frac{\delta F}{\delta(a^{-2}(\tilde{t}))}a^{-2}(\tilde{t}) \\
&= \frac{\delta F}{\delta \ln a(\tilde{t})} = -P(\tilde{t}).
\end{aligned}
\tag{5.23}
$$

Thus, under T-duality, $E(t) \to \tilde{E}(\tilde{t})$ and $P(t) \to \tilde{P}(\tilde{t})$. This observation allows to reinterpret the minus sign appearing in the equation for $\tilde{H}^2$ in [126]. In contrast to what happens in the SuGra frame, the energy measured in the winding frame is not simply the function $E(t, \tilde{t})$ projected to $E(\tilde{t})$ upon applying the section condition, but actually the negative of it. The difference appears because the definition of $E(t, \tilde{t})$ selects the SuGra frame as a preferred frame, since $g_{\tilde{t}\tilde{t}}$ does not enter in this definition. As explained in [126], to work only with $E(t, \tilde{t})$ was a choice since the variations with respect to g_{tt} can be written as $g_{\tilde{t}\tilde{t}}$ variations. But this choice selects t as a preferred variable and so it is natural that the energy in the winding frame is different from $E(\tilde{t})$.

Using (5.20) in SuGra frame, we have

$$
\begin{aligned}
2\alpha\dot{H} - H^2(\alpha^2(D-1)+1) &= 0, \\
\alpha\dot{H} - H^2 &= \frac{1}{2(D-1)}e^{2d}E, \\
\dot{H} - \alpha(D-1)H^2 &= \frac{1}{2}e^{2d}P,
\end{aligned}
\tag{5.24}
$$

which implies

$$
\begin{aligned}
H^2 &= \frac{e^{2\phi_0}a^{(\alpha+1)(D-1)}}{(D-1)(\alpha^2(D-1)-1)}\rho, \\
w &= -\frac{1}{\alpha}\frac{1}{D-1}, \\
\dot{\rho} &+ (D-1)H(\rho+p) = 0\,.
\end{aligned}
\tag{5.25}
$$

Notice that ϕ_0 is the value of the dilaton in the frame where it is constant.

In winding frame we obtain

$$
\begin{aligned}
2\tilde{\alpha}\tilde{H}' - \tilde{H}^2(\tilde{\alpha}^2(D-1)+1) &= 0, \\
-\tilde{H}^2 + \tilde{\alpha}\tilde{H}' &= \frac{1}{2(D-1)} e^{2d}\tilde{E}, \\
-\tilde{H}' + \tilde{\alpha}(D-1)\tilde{H}^2 &= \frac{1}{2} e^{2d}\tilde{P},
\end{aligned}
\tag{5.26}
$$

which are equivalent to

$$
\begin{aligned}
\tilde{H}^2 &= \frac{e^{2\phi_0} a^{(\tilde{\alpha}+1)(D-1)}}{(D-1)(\tilde{\alpha}^2(D-1)-1)}\tilde{\rho}, \\
w &= \frac{1}{\tilde{\alpha}}\frac{1}{D-1}, \\
\tilde{\rho}' + (D-1)&\left[\frac{(D-1)-1/\tilde{\alpha}}{(D-1)+1/\tilde{\alpha}}\right]\tilde{H}(\tilde{\rho}+\tilde{p}) = 0\,,
\end{aligned}
\tag{5.27}
$$

where w is the equation of state parameter

$$
w \;=\; \frac{p}{\rho}\,, \tag{5.28}
$$

p and ρ being pressure and energy density, respectively.

From these equations, we conclude that the equation of state is the same in both frames regardless in which frame the dilaton is taken to be constant. For constant dilaton in the SuGra frame we obtain the equation of state of radiation, for constant dilaton in the winding frame, on the other hand, the equation of state is that of a gas of winding modes.

5.5 Solutions

Solving the equations of the previous section in the SuGra frame, we obtain

$$
\rho(t) \propto a^{-(D-1)+1/\alpha}(t), \tag{5.29}
$$

$$
a(t) \propto \left(\frac{\alpha}{2}(D-1) - \frac{1}{2\alpha}\right)^{\frac{2}{-\alpha(D-1)-1/\alpha}} t^{\frac{2}{-\alpha(D-1)-1/\alpha}}, \tag{5.30}
$$

while in the winding frame we get

$$
\tilde{\rho}(\tilde{t}) \propto a^{-(D-1)+1/\tilde{\alpha}}(\tilde{t}), \tag{5.31}
$$

$$
a(\tilde{t}) \propto \left(\frac{-\tilde{\alpha}}{2}(D-1) - \frac{1}{2\tilde{\alpha}}\right)^{\frac{2}{-\tilde{\alpha}(D-1)-1/\tilde{\alpha}}} \tilde{t}^{\frac{2}{-\tilde{\alpha}(D-1)-1/\tilde{\alpha}}}. \tag{5.32}
$$

In particular, for constant dilaton in the SuGra frame, we have

$$
\rho(t) \propto a^{-D}(t), \qquad \tilde{\rho}(\tilde{t}) \propto a^{-(D-2)}(\tilde{t}), \tag{5.33}
$$

$$
a(t) \propto t^{2/D}, \qquad a(\tilde{t}) \propto \tilde{t}^{-2/D}. \tag{5.34}
$$

We see that given a radiation equation of state in both frames, the energy density in the winding frame has the same a dependence as a fluid with winding equation of state. The reason for this is that in the winding frame the dilaton is not constant, and hence the relationship between equation of state and scale factor dependence of the energy density which we are used to from Einstein gravity changes.

For constant dilaton in the winding frame, we find

$$\rho(t) \propto a^{-(D-2)}(t), \qquad \tilde{\rho}(\tilde{t}) \propto a^{-D}(\tilde{t}), \tag{5.35}$$

$$a(t) \propto t^{-2/D}, \qquad a(\tilde{t}) \propto \tilde{t}^{2/D}, \tag{5.36}$$

which shows that a fluid with winding equation of state has time dependence of the scale factor like radiation in the winding frame.

As we can check from the above results, we found solutions in the SuGra and winding frame which are T-dual to each other. Also, the solutions exhibit a symmetry connected with T-duality: if we change t to $\tilde{t}$ in the SuGra frame solution with constant dilaton in that frame, we get the winding frame solution with constant dilaton in the winding frame, and vice-versa.

5.6 Final Comments

Since Double Field Theory is based on the same T-duality symmetry which is key to superstring theory, one could hope that Double Field Theory could provide a consistent background for superstring cosmology, and provide a good background for String Gas Cosmology. Let us consider the background space to be toroidal. In this case, as argued in [59] and explained in the beginning of the chapter, for large values of the radius R of the torus (in string units), the light degrees of freedom correspond to the momenta, and the supergravity frame is hence the one in which observers made up of light degrees of freedom measure physical quantities. In contrast, for small values of R, it is the winding modes which are light, and hence the winding frame is the frame in which observers describe the physics. In the transition region (the Hagedorn phase) the full nature of double space will be important. It is possible that the section condition becomes dynamical [2]. It would be interesting in this context to explore the connection with the recent ideas in [186–189].

[2] We thank Laurent Freidel for discussions on this point.

Chapter 6

Holographic Cosmology from dimensional reduction of $\mathcal{N} = 4$ SYM vs. $\mathbf{AdS}_5 \times \mathbf{S}^5$

6.1 Motivation: Strong Gravity in the Very early universe

The inflationary paradigm is a semi-classical approach for the physics of the very early universe. From the effective field theory point of view, the energy scale given by H should be small compared with the Planck scale, as should be the momentum modes of the cosmological perturbations. In fact, the consistent models of inflation that we have are on the verge of being outside the range of validity of the effective theory approach. This situation is called the *trans-Planckian problem* [190]. Also, inflation does not solves the problem of initial singularity. To solve these problems we should deal with the strong coupling limit of gravity.

Therefore, it is natural to try to use ideas from holography to describe the perturbations even at the strong non-perturbative regime. This is one of the goals of holographic cosmology. We can divide the approaches to holographic cosmology in three classes:

- In the theoretical extremum, we can try to construct a dS/QFT correspondence, that could or not have a string construction. A proposal for a dS/CFT correspondence was initiated in [138];
- Deform the original AdS/CFT to include cosmological solutions in the gravity side of the correspondence. As we will describe, this was done in [142, 191] for flat isotropic and homogeneous metric. In section 6.3 we show that actually there are only 2 other cosmological possibilities, corresponding to the open and closed universe cases (to the knowledge of the author, this was first recognized in [3]);
- Construct a field theory dual to cosmology from a purely phenomenological point of view. This approach was created in [139], with the field theory being fixed by consistency of some observational data.

These approaches have similarities and differences, but each one follows the basic ideas of holography.

In this chapter, we will attempt to "push" the third approach into the second class described above. We will start from the deformed AdS/CFT case [142–144] and then propose a prescription to fix the field theory for [139].

6.2 Holographic cosmology paradigm

In this section we review the holographic cosmology paradigm of [139]. One considers a cosmological FLRW model, coupled with a scalar ϕ, and having fluctuations in both,

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)[\delta_{ij} + h_{ij}(t,\vec{x})]dx^i dx^j \ , \\ \phi(t,\vec{x}) &= \phi(t) + \delta\phi(t,\vec{x})a. \end{aligned} \tag{6.1}$$

After a Wick rotation, the "domain wall/cosmology correspondence", putting $t = -iz$, but also $\bar{\kappa}^2 = -\kappa^2, \bar{q} = -iq$ (here κ is the Newton constant and q is momentum), which in field theory corresponds to $\bar{q} = -iq, \bar{N} = -iN$, we obtain the domain wall gravity dual

$$\begin{aligned} ds^2 &= +dz^2 + a^2(z)[\delta_{ij} + h_{ij}(z,\vec{x})]dx^i dx^j \ , \\ \phi(z,\vec{x}) &= \phi(z) + \delta\phi(z,\vec{x})a \ , \end{aligned} \tag{6.2}$$

The generic "domain wall" above can correspond to (asymptotically) AdS space, for (asymptotically) exponential $a(z)$, in which case expect a field theory that is conformal in the UV. Or it can correspond to some holographic dual of the type of nonconformal branes, for power law $a(z)$, in which case one expects a "generalized conformal structure": the theory has as only dimensional parameter the YM coupling g_{YM}, which appears as an overall factor in front of the action. Therefore it is of the type that we would obtain by dimensionally reducing a 4 dimensional conformal field theory. Specifically, the phenomenological class of models considered for the fit to the CMBR is a super-renormalizable theory of $SU(N)$ gauge fields A_i^a, scalars ϕ^{aM} and fermions ψ^{aL}, where a is an adjoint $SU(N)$ index and M, L are flavour indices, with action

$$\begin{aligned} S_{\text{QFT}} &= \int d^3x \, \text{Tr} \left[\frac{1}{2} F_{ij}F^{ij} + \delta_{M_1M_2} D_i \Phi^{M_1} D^i \Phi^{M_2} + 2\delta_{L_1L_2} \bar{\psi}^{L_1} \gamma^i D_i \psi^{L_2} \right. \\ &\quad \left. + \sqrt{2} g_{YM} \mu_{ML_1L_2} \Phi^M \bar{\psi}^{L_1} \psi^{L_2} + \frac{1}{6} g_{YM}^2 \lambda_{M_1 \ldots M_4} \Phi^{M_1} \ldots \Phi^{M_4} \right] \\ &= \frac{1}{g_{YM}^2} \int d^3x \, \text{Tr} \left[\frac{1}{2} F_{ij}F^{ij} + \delta_{M_1M_2} D_i \Phi^{M_1} D^i \Phi^{M_2} + 2\delta_{L_1L_2} \bar{\psi}^{L_1} \gamma^i D_i \psi^{L_2} \right. \\ &\quad \left. + \sqrt{2} \mu_{ML_1L_2} \Phi^M \bar{\psi}^{L_1} \psi^{L_2} + \frac{1}{6} \lambda_{M_1 \ldots M_4} \Phi^{M_1} \ldots \Phi^{M_4} \right]. \end{aligned} \tag{6.3}$$

Here $\lambda_{M_1 \ldots M_4}$ and $\mu_{ML_1L_2}$ are dimensionless, and only g_{YM} is dimensional, and in the second line the fields have been rescaled by g_{YM} in order to obtain g_{YM} as an overall factor, and the dimensions of the fields to be the ones in 4 dimensions. The generalized conformal structure means that the momentum dependence organizes into a dependence on the effective dimensionless coupling of the theory,

$$g_{\rm eff}^2 = \frac{g_{YM}^2 N}{q}. \tag{6.4}$$

Correlators will thus depend on $g_{\rm eff}^2$, and in perturbation theory one obtains, as usual, a combination of powers of $g_{\rm eff}^2$ and $\ln g_{\rm eff}^2$.

The CMBR power spectrum is defined in terms of the standard scalar and tensor fluctuations in momentum space $\zeta(q)$ and $\gamma_{ij}(q)$ as

$$\begin{aligned}\Delta_S^2(q) &\equiv \frac{q^3}{2\pi^3}\langle\zeta(q)\zeta(-q)\rangle \\ \Delta_T^2(q) &\equiv \frac{q^3}{2\pi^3}\langle\gamma_{ij}(q)\gamma_{ij}(-q)\rangle.\end{aligned} \tag{6.5}$$

In principle, one could relate them to the two-point functions of the energy-momentum tensors via the Maldacena relation $Z[h_{ij}] = \psi[h_{ij}]$ as follows. From general theory, the partition function is represented as the generating functional of correlators as

$$Z[h_{ij}] = \exp\left[\int \frac{1}{2} h^{ij}\langle T_{ij}T_{kl}\rangle h^{kl} + ...\right] , \tag{6.6}$$

which leads to the 2-point function of cosmological fluctuations h_{ij} as

$$\langle h_{ij}h_{kl}\rangle = \int \mathcal{D}h_{mn}|\psi[h_{pq}]|^2 h_{ij}h_{kl} \sim \frac{1}{{\rm Im}\langle T_{ij}T_{kl}\rangle} , \tag{6.7}$$

where the last equality is qualitative, and involves a nontrivial calculation. The more precise relation was found in [192], based on the formalism in [193, 194], and is reviewed in the Appendix A. Decomposing the energy-momentum tensor correlators as

$$\langle T_{ij}(\bar q)T_{kl}(-\bar q)\rangle = A(\bar q)\Pi_{ijkl} + B(\bar q)\pi_{ij}\pi_{kl} , \tag{6.8}$$

where

$$\Pi_{ijkl} = \pi_{i(k}\pi_{l)j} - \frac{1}{2}\pi_{ij}\pi_{kl} , \quad \pi_{ij} = \delta_{ij} - \frac{\bar q_i \bar q_j}{\bar q^2} \tag{6.9}$$

are the 4-index transverse traceless projection operator (Π_{ijkl}), and the 2-index transverse projection operator (π_{ij}), we obtain the power spectra

$$\begin{aligned}\Delta_S^2(q) &= -\frac{q^3}{16\pi^2{\rm Im}B(-iq)} \\ \Delta_T^2(q) &= -\frac{2q^3}{\pi^2{\rm Im}A(-iq)} ,\end{aligned} \tag{6.10}$$

where we have already performed the analytical continuation to Lorentzian signature through $\bar q = -iq$ and $\bar N = -iN$.

6.3 Top-down model from dimensional reduction of $\mathcal{N} = 4$ SYM vs. $\mathrm{AdS}_5 \times \mathrm{S}^5$

Another holographic approach was developed in [142–144], and we will present it in a way that can fit into the holographic cosmology paradigm from the previous section. We consider a 4+1 dimensional geometry that is a solution of the 10 dimensional type IIB equations of motion, with a metric ansatz

$$ds^2 = \frac{R^2}{z^2}[dz^2 + (-dT^2 + a^2(T)d\vec{x}^2)] + R^2 d\Omega_5^2 \; , \tag{6.11}$$

and with a nontrivial dilaton $\phi = \phi(T)$. More generally, for the metric ansatz

$$ds^2 = \frac{R^2}{z^2}[dz^2 + g_{\mu\nu}(x)dx^\mu dx^\nu)] + R^2 d\Omega_5^2 \; , \tag{6.12}$$

the equations of motion are

$$R_{\mu\nu}[g_{\rho\sigma}] = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi \; , \quad \partial_A(\sqrt{-G}G^{AB}\partial_B\phi) = 0 \; , \tag{6.13}$$

where G_{AB} is the 5-dimensional metric. With a flat FLRW cosmological ansatz as in (6.11), one finds the unique solution

$$a(T) \propto T^{1/3} \; , \quad e^{\phi(T)} = \left(\frac{T}{R}\right)^{2/\sqrt{3}} \; , \tag{6.14}$$

which corresponds to a "stiff matter" cosmology, with equation of state $P = w\rho$, with $w = +1$. Indeed, in general for FLRW we have $a(T) \propto T^{\frac{2}{3(1+w)}}$.

Making a transformation to conformal time t (called τ in chapter 2), we obtain

$$-dT^2 + a^2(T)d\vec{x}^2 = a^2(t)[-dt^2 + d\vec{x}^2] \Rightarrow a \sim T^{1/3} \sim t^{1/2} \; , \tag{6.15}$$

so in particular

$$e^{\phi(t)} = \left(\frac{t}{R}\right)^{\sqrt{3}} . \tag{6.16}$$

In fact, for a general homogeneous and isotropic cosmological ansatz for the metric, we have

$$R_{00}[g_{\rho\sigma}] = -3\frac{\ddot{a}}{a} \; , \quad R_{ij}[g_{\rho\sigma}] = \left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right)\delta_{ij} \; , \tag{6.17}$$

with $k = -1, 0, 1$ for open, flat and closed universes. So, to have a 10 dimensional solution with homogeneous dilaton, we should have

$$\dot{\phi}^2 = -6\frac{\ddot{a}}{a} \; , \quad \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 0 \; , \tag{6.18}$$

which in conformal time reads

$$\left(\frac{d\phi}{dt}\right)^2 = 6\left[\frac{1}{2a^4}\left(\frac{d}{dt}(a^2)\right)^2 + 2k\right] \; , \quad \frac{1}{2a^2}\frac{d^2}{dt^2}(a^2) + 2k = 0. \tag{6.19}$$

Solving these equations for $k = 0$ gives the results before. For $k = 1$ the solution is

$$a(t) \propto |\sin(2t)|^{1/2}, \quad e^{\phi(t)} \propto |\tan(t/R)|^{\sqrt{3}}, \tag{6.20}$$

and for $k = -1$ we have

$$a(t) \propto |\sinh(2t)|^{1/2}, \quad e^{\phi(t)} \propto |\tanh(t/R)|^{\sqrt{3}}. \tag{6.21}$$

We conclude that, for homogeneous dilaton, there is unique solutions for each possible spatial topology.

Note that the original G_{AB} metric was in Einstein frame, and $\phi(T)$ was the dilaton. If we make the conformal transformation by $a(T)$ we move away from the Einstein frame. Then $\phi(T) = \phi(t)$ is the dilaton, thus $e^{\phi(t)}$ is the string coupling, corresponding in the boundary field theory to the YM coupling $g^2_{YM}/(4\pi)$. In terms of the time t of Minkowski space, we have then a time-dependent SYM coupling,

$$g_{YM}(t) = g_{YM,0}\left(\frac{|t|}{R}\right)^{\sqrt{3}}. \tag{6.22}$$

The conformal transformation on the boundary is allowed, given that the boundary field theory is conformal. However, when doing that in holography, we will obtain a modification of the holographic map, that will be calculated in the next subsection.

As an aside, note that the solution

$$\begin{aligned} ds_4^2 &= |\sinh(2t)|\left[-dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_2^2\right] \\ e^{\phi(t)} &= g_s|\tanh(t/R)|^{\sqrt{3}} \end{aligned} \tag{6.23}$$

is not just conformally flat, but actually asymptotically flat. For $t \to \pm\infty$, there is a coordinate transformation that takes away the conformal factor, giving $\mathrm{AdS}_5 \times S^5$ and constant dilaton in these regimes, as analyzed in [142]. However, *close to the strong coupling gravity region* $t \sim 0$, we get still a conformal factor deviating from 1, and the solution is the same as before, $a^2(t) \propto t$ and $e^{\phi(t)} \propto t^{\sqrt{3}}$.

In order to embed the approach presented in this section into the paradigm from the last section, we need to consider how to extend it to the case when there is both a radial coordinate, and a time coordinate. For the general set-up of Maldacena, the wavefunction of the Universe $\psi[h_{ij}]$ is evolved in time with the Hamiltonian, which corresponds on the boundary to the RG flow of the correlators obtained from $Z[h_{ij}]$, as the energy scale is varied. In the framework of [139], the Wick rotation (”domain wall/cosmology correspondence”) means that time evolution is replaced by a radial ”Hamiltonian” evolution, corresponding to the same, and in line with the usual AdS/CFT construction.

The Maldacena map is based on the fact that the wavefunction of the Universe can be thought of as a path integral, integrated over time (in the past), but with the boundary condition of spatial 3-metric h_{ij} at the corresponding time t. Then it is really just a type of analytical continuation of the usual AdS/CFT map between the partition function of the field theory, with sources h_{ij}, and the partition function of the gravity or string theory (written as a path integral), with a boundary condition of h_{ij}.

But now we have both a radial direction and a time direction, and we have to decide how to generalize the set-up of Maldacena to this situation, so that maybe in a second step, we can relate it to the paradigm of [139].[1]

There are now two possible Hamiltonians in the gravitational theory: both the radial one, who gives the evolution that, via the usual AdS/CFT correspondence, corresponds to the RG flow of the boundary field theory, and the true Hamiltonian, which gives the evolution of gravity along the time direction, and should similarly correspond to a Hamiltonian evolution in time in the boundary field theory.

It seems therefore reasonable to assume that the correct prescription to use is to have, on the gravity side, a partition function with boundary condition both at time t and at radial size r, which therefore is still a wavefunction of the Universe, corresponding in field theory to a partition function integrated over time until the corresponding time t, and both be as usual functions of spatial 3-metrics h_{ij},

$$\psi[h_{ij}]_{t,r} = Z[h_{ij}]_{t,q}. \tag{6.24}$$

Here q is the energy scale corresponding holographically to the radial direction r in the bulk. The time t is arbitrary, and the path integration is assumed to be for times between $-\infty$ and t, but not in its future. In this way, both sides of the equation are functions of this time t, which are evolved with the Hamiltonian. Of course, in the context of the "top-down" model, the bulk will have a time-dependent Hamiltonian, which can be interpreted in terms of particle production. The evolution with the Hamiltonian from t to t' should be equivalent with the path integration until a later time t'.

Next, we need to understand the effect of the integration over t on both sides of the equality, and how to take it into account. On the gravity side, the integration over time gives the wavefunction of the Universe, and there is nothing we need to do with it. Since the holographic map is the same, the calculation of the correlators of metric fluctuations in (6.7) is unchanged, and we should obtain the same relation (6.10).

On the field theory side, we should do the path integral over the time direction until the time t. Because of the fact that $g_{YM}(t)$ is a positive power law, and appears in the denominator in the action,

$$e^{-S} = e^{-\int dt \frac{1}{g_{YM}^2(t)} \int d^3x \mathcal{L}_{\rm SYM}}, \tag{6.25}$$

the largest contribution to the weight e^{-S} will be from small times. But then, if at small t the fields are *positive* power laws in time (which should be the case since fields must not be singular at $t = 0$, and must be Taylor expandable), which would correspond to "massive KK modes" in a "KK" expansion in t of the fields of SYM, these would give small contributions to the path integral. The leading contribution must be from the time-independent fields, i.e., the "KK dimensionally reduced" fields. We also should split the Lorentz indices according to

[1]See [195–197] for early treatments of having both time and radial direction, though outside the holographic cosmology context we introduce here.

this dimensional reduction, finally obtaining a 3 dimensional field theory action, with coupling factor integrated over time from a time $t_X \sim t_{\rm Pl}$ of the order of the Planck scale up to the relevant t,

$$\int dt \frac{1}{g_{YM}^2(t)} \sim \frac{1}{g_{YM,0}^2}\int_{t_{\rm Pl}}^{t_X} \frac{dt}{(t/R)^{\sqrt{3}}} = \frac{R}{g_{YM,0}^2}\,(t/R)^{1-\sqrt{3}}\Big|_{t_{\rm Pl}}^{t_X} \equiv \frac{RK}{g_{YM,0}^2} \equiv \frac{1}{g_{3d}^2}. \tag{6.26}$$

Here $K = (t_X/R)^{1-\sqrt{3}} - (t_{\rm Pl}/R)^{1-\sqrt{3}}$ is very large.

But then the *effective* (dimensionles) 3 dimensional coupling is

$$g_{\rm eff}^2 \equiv \frac{g_{3d}^2 N}{\bar{q}} = \frac{g_{YM,0}^2 N}{K(R\bar{q})}. \tag{6.27}$$

Since both $g_{YM,0}^2 N \gg 1$ (from the usual holographic condition on the validity of the supergravity approximation for $\mathrm{AdS}_5 \times \mathrm{S}^5$) and $K \gg 1$, we can have even $R\bar{q} \sim 1$, and still *we can choose* the effective coupling to be perturbative, $g_{\rm eff}^2 < 1$, though that is not necessary.

In this case, we see that we obtain a specific 3 dimensional field theory with generalized conformal structure, one obtained from the dimensional reduction of $\mathcal{N} = 4$ SYM. However, in [140, 141] the best fit to the CMBR data of the *perturbative* phenomenological field theory was analyzed, and it was found that for no fermions (introducing fermions moves the fit away from the desired region), the number of adjoint scalars for a good match is of the order of 10^4, which is much larger than the one obtained from dimensionally reducing $\mathcal{N} = 4$ SYM (which is 7: 6 originally, and one from the A_0 component of the gauge field). That means that this theory does not fit the CMBR data *perturbatively*.

It could be that one needs to choose a larger coupling (so as not to have $g_{\rm eff}^2 < 1$) in order to find the fit, though to test that we would need access to lattice data. Or it could be that $\mathcal{N} = 4$ SYM is just a toy model, and we would need to apply the same methods to other top down gravity dual pairs, though we will leave that for further work. In particular, we saw that the $a(t)$ uniquely selected by the type IIB equations of motion corresponded to a "stiff matter" cosmology, with $w = 1$, which is different than, say, inflation.

6.3.1 Transformation of dilaton and operator VEV

We could ask: where do we see the dependence on the cosmological model $a(t)$? There is not much dependence in the constant K, defining g_{3d}^2, and there would be a small dependence if we took into account corrections due to non-constant field theory modes (considering "the full KK tower" of fields, instead of the dimensionally reduced ones only). Of course, the type IIB equations of motion only allow a specific $a(t)$, so it cannot be varied, but it still seems strange. Here we want to see that there is in fact one quantity that depends on it, though it should affect only correlators away from the perturbative regime.

We have already noted that a conformal rescaling on the boundary, to go from a conformally flat space to a flat space (by the $a^2(t)$ factor that takes us from a cosmological model to a simple flat space), corresponds in the bulk to a coordinate transformation.

Indeed, a conformal transformation *on the boundary* can be thought of as embedded in the set of general coordinate transformations *on the boundary* (conformal transformations are global $SO(4,2)$ transformations in $d=4$, embedded in the infinite dimensional "group" of general coordinate transformations). But by applying a conformal transformation, we just obtain a specific coordinate transformation, differing from what we have, which means that we cannot remove the conformal factor by a conformal transformation *on the boundary.*

But we can remove *any* conformal factor on the boundary by a coordinate transformation *in the bulk*, as shown in [198], eqs. 8,9,10.

Let us apply this procedure to our case. Writing $\rho = z^2$, the general coordinate transformation is expanded as

$$\begin{aligned}
\rho &= \rho' e^{-2\sigma(x')} + \sum_{k\geq 2} a_{(k)}(x')\rho'^k \\
x^i &= x'^i + \sum_{k\geq 1} a^i_{(k)}(x')\rho'^k \ ,
\end{aligned} \tag{6.28}$$

which gives

$$g'_{(0)ij} = e^{2\sigma} g_{(0)ij} \tag{6.29}$$

and higher orders, which do not interest us.

For us, we have

$$g_{(0)ij} = a^2(t)\delta_{ij} \ , \ \ g'_{(0)ij} = \delta_{ij} \ , \tag{6.30}$$

so $e^{2\sigma} = a^{-2}(t)$. That means that we only need to transform time, as

$$t = t' + \sum_{k\geq 1} a^0_{(k)}(t')\rho'^k \ , \tag{6.31}$$

but not space (since the metric is space independent). Then the formulas for the relevant coefficients are (note that we are not interested in the transformation on ρ, so we don't care about $a_{(k)}$'s)

$$\begin{aligned}
a^0_{(1)} &= \frac{1}{2}\partial^t\sigma e^{-2\sigma} \\
a^0_{(2)} &= -\frac{1}{4}e^{-4\sigma}\left(\partial_t\sigma g^{tt}_{(2)} + \frac{1}{2}\partial^t\sigma(\partial\sigma)^2 + \frac{1}{2}\Gamma^t{}_{tt}\partial^t\sigma\partial^t\sigma\right) \ \text{, where} \\
g_{(2)ij} &= \frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)}Rg_{(0)ij}\right) .
\end{aligned} \tag{6.32}$$

Here indices are raised and lowered with $g_{(0)ij} = a^2(t)\delta_{ij}$.

We consider in particular the cosmological solution of the type IIB equations of motion, which has

$$a^2(t) = t \ , \quad e^{\phi(t)} = t^{\sqrt{3}} \Rightarrow \phi = \sqrt{3}\ln t \ , \tag{6.33}$$

and solves

$$R_{ij} = \frac{1}{2}\partial_i\phi\partial_j\phi \ , \tag{6.34}$$

giving for $g_{(2)ij}$ the value

$$g_{(2)ij} = \frac{1}{2}\left(\frac{\partial_i\phi\partial_j\phi}{2} - \frac{1}{6}(\partial\phi)^2 g_{(0)ij}\right) \ , \tag{6.35}$$

or more precisely

$$g_{(2)tt} = \frac{1}{6}(\partial_t \phi)^2. \tag{6.36}$$

We also calculate the relevant Christoffel symbol,

$$\Gamma^t{}_{tt} = \frac{1}{2t}. \tag{6.37}$$

Then, after a bit of algebra, we find the coefficients

$$a^0_{(1)} = \frac{1}{4t}\,, \quad a^0_{(2)} = \frac{1}{16t^3}. \tag{6.38}$$

Substituting in the coordinate transformation of the time direction, we find

$$t = t' + \frac{1}{4t'}\rho' + \frac{1}{16t'^3}\rho'^2. \tag{6.39}$$

The scalar transformation law is $\phi'(t') = \phi(t)$, so we obtain

$$\phi'(t') = \phi(t) = \phi\left(t' + \frac{\rho'}{4t'} + \frac{\rho'^2}{16t'^3}\right) = \sqrt{3}\ln\left[t' + \frac{\rho'}{4t'} + \frac{\rho'^2}{16t'^3}\right]. \tag{6.40}$$

Expanding near the boundary at $\rho = 0$, we find

$$\phi'(t') = \sqrt{3}\left[\ln t' + \ln\left(1 + \frac{\rho'}{4t'^2} + \frac{\rho'^2}{16t'^4}\right)\right] \simeq \sqrt{3}\left[\ln t' + \frac{\rho'}{4t'^2} + \frac{\rho'^2}{32t'^4}\right]. \tag{6.41}$$

The leading term in the ρ' expansion of on-shell fields is the source on the boundary, and we see that it is unmodified in the case of $\phi(t)$. The second term in the expansion of $\phi(t)$ (with ρ') is related to the first, but the third (with ρ'^2) is related to an operator VEV on the boundary.

That means that we have, besides the source, also an operator VEV in the $\mathcal{N} = 4$ SYM with time dependent coupling. This coupling $g^2_{YM}(t)$ is unchanged, but we have obtained a nonzero VEV, of

$$\langle \mathrm{Tr}[F^2_{\mu\nu}]\rangle \propto \frac{1}{32t'^4} \neq 0. \tag{6.42}$$

This operator VEV is truly dependent on the cosmological solution $a(t)$, as we have seen, and its presence should modify nonperturbatively the SYM correlators. But in the perturbation theory we have considered, there is no modification.[2]

[2] In a CFT, a state is created by a local operator, so a correlator in a different state is equivalent with adding two more operators in the vacuum correlator. However, the perturbation theory considered here in 3 Euclidean dimensions, after the "dimensional reduction" of the time direction. From this theory's point of view, we are in a nonperturbative state: at fixed time, the VEV calculated here is constant throughout the space, and thus is not the effect of a local operator.

6.4 Final Comments

In this chapter we have extended the holographic cosmology map $Z[h_{ij}, \phi] = \psi[h_{ij}, \phi]$ of Maldacena, between the wavefunction of the Universe and the boundary partition function, to the case where there is *both* an Euclidean holographic direction, and a Minkowskian time direction, obtaining $\psi[h_{ij}, \phi]_{t,r} = Z[h_{ij}, \phi]_{t,q}$. Stressing once again, this prescription could be applied for more general top down holographic pairs, and that we could in principle start with another phenomenological holographic model as starting point.

Chapter 7

Conclusions

The field of String Cosmology is only on its first steps and the more we understand about String Theory *and* Cosmology, the more creative ways we have to construct cosmological models from String Theory or to search for observational implications of String Theory setups. In this thesis, we presented contributions to a small corner of the space of possible setups that we summarize in the following.

In chapter 4, we studied a chameleonic inflation model with the scalar moduli issue of string compactification in mind, though the results are independent of String Theory and could be seen as purely inflationary model. Depending on the sign of c in the exponent of the chameleon coupling, we have found two different behaviours. In the $c < 0$ case we could introduce a long period of almost kinetic domination, with an attractor-like behaviour with $1 - \Omega_{\mathrm{kin},\varphi} \ll 1$, that precedes the inflationary phase, and thus shortens it. In the $c > 0$ case we found that there are attractor-like behaviours possible, one of which corresponds to sitting at the instantaneous minimum $\varphi_{\min}(N)$ of the effective potential $V_{\mathrm{eff}}(\varphi, N) = V(\varphi) + \rho_X(\varphi, N)$. We have shown the modifications of the CMBR inflationary observables n_s and r after proving the equivalence between our equations of motion and the ones obtained by the microscopic description presented in [175]. We have checked that the presence of a second heavy field does not generate entropy modes during inflation. More than that, the chameleon coupling with the second scalar field strongly suppresses the growth of entropy modes during inflation by increasing its effective mass. For conformal inflation models we have found that the presence of non-relativistic matter coupled to a chameleon shifts the value of n_s and r from the sweet spot of Planck data to almost out of the region allowed region by the data. For the inverse power law case we have shown that the coupling with chameleon extends the period of inflation and the values for the observables lie in the region allowed by observations.

In chapter 5, we have constructed supergravity and winding frame solutions of the cosmological equations of Double Field Theory which are T-dual to each other. When the correct transformation of the energy and pressure is taken into account, there is no need for complexification of the scale factor as done previously in the literature [126]. The behaviour of the solutions on each frame were compatible with T-duality and with what we would expect based on the intuition from String Gas Cosmology. From the String Theory perspective, the solutions are valid only asymptotically on t and $\tilde{t}$, as close to $t = 0 = \tilde{0}$ we should consider all α' corrections to the DFT equations (something that apparently was done in [199] in vacuum). That is, we still need a better understanding of the string corrections to the setup in order to "glue" both asymptotically solutions together.

A prescription to extend the holographic map for cosmology was presented in chapter 6. We considered the case of a cosmological solution of the type IIB equations of motion with a time-dependent dilaton $\phi(t)$, where the conformal factor $a^2(t)$ relates it conformally to a flat space solution, corresponding to the usual $\mathrm{AdS}_5 \times \mathrm{S}^5$ vs. $\mathcal{N} = 4$ SYM in flat space. This is therefore a "top down" holographic cosmology, obtained by a modification of the original AdS/CFT case. We have then proposed that to integrate over the time direction as needed, we can, in the boundary partition function, "dimensionally reduce" the theory on the time direction, by considering only time-independent quantities, except for the overall coupling $g_{YM}(t)$. In so doing, we obtain the set-up of [139], just that from a top down, as opposed to bottom up, construction. While the resulting cosmology was not, perturbatively, consistent with the CMBR data, we could think of the possibility of either a non-perturbative match, where the SYM results would be obtained on the lattice, or of using the same construction for a different top down starting point. These possibilities are left for further work. We have also shown that the effect of the scale factor $a(t)$ (of the cosmology) on the correlators of SYM is to introduce a nonzero time-dependent VEV $\langle \mathrm{Tr}[F_{\mu\nu}^2] \rangle$ non-perturbatively.

Appendix A

Holographic calculation of the scalar and tensor two-point functions

In this Appendix, we review the holographic calculation in [192–194], relating $\langle \delta h_{ij} \delta h_{kl} \rangle$ correlators (experimentally derived from the CMBR) to $\langle T_{ij} T_{kl} \rangle$ correlators in the $\mathcal{N} = 4$ SYM field theory, using the radial Hamiltonian formalism.

Consider an asymptotically AdS metric in Fefferman-Graham coordinates,

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \left(g_{(0)ij} + ... + z^d g_{(d)ij} + ...\right)\right] \ , \tag{A.1}$$

The one-point function of the energy-momentum tensor in the presence of sources is then

$$\langle T_{ij}(x) \rangle = -\frac{1}{\sqrt{g_{(0)(x)}}} \frac{\delta W[g_{(0)}, ...]}{\delta g^{ij}_{(0)}(x)} \ , \tag{A.2}$$

where W, the generating functional of connected graphs, equals by the AdS/CFT prescription (minus) the on-shell action $S_{\text{on-shell}}$.

We use a radial Hamiltonian formulation for AdS gravity, with r,

$$z = e^{-r} \ , \tag{A.3}$$

acting as "time" in the "ADM parametrization"

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \hat{\gamma}_{ij} d\hat{x}^i d\hat{x}^j + 2N_i d\hat{x}^i dr + (N^2 + N_i N^i) dr^2. \tag{A.4}$$

Then the asymptotically AdS metric is

$$ds^2 = dr^2 + g_{ij}(r, x) dx^i dx^j \tag{A.5}$$

and $g_{(p)ij}$ means (as in (A.1)) the expansion of g_{ij} in $z^{2p-2} = e^{(2-2p)r}$.

Then, like in the usual ADM construction, we can always choose a gauge such that $N = 1, N_i = 0$, and the ADM parametrization becomes the same as the Fefferman-Graham expansion above, with

$$\hat{\gamma}_{ij} = g_{ij} = \frac{1}{z^2}(g_{(0)ij}(x) + \mathcal{O}(z^2)) \simeq e^{2r} g_{(0)ij}(x). \tag{A.6}$$

The resulting on-shell action is

$$S_{\rm on-shell} = -\frac{1}{8\pi G_N}\int_{r_0}^{r_\epsilon} dr \int d^d x \sqrt{\hat{\gamma}} N \left[\hat{R} + 8\pi G_N(\tilde{T}_{ij} - \mathcal{L}_m)\right] , \tag{A.7}$$

and one defines the canonically conjugate momentum to $\hat{\gamma}_{ij}$ as (at the position $r_\epsilon = 1/\epsilon$, close to the boundary at $z = 0$)

$$\pi^{ij}(r_\epsilon, x) = \frac{\delta S_{\rm on-shell}}{\delta \hat{\gamma}_{ij}(r_\epsilon, x)}. \tag{A.8}$$

We obtain

$$\begin{aligned} \partial_r &\simeq \int d^d x\, 2\hat{\gamma}_{ij}\frac{\delta}{\delta\hat{\gamma}_{ij}} + \int d^d x (\Delta_I - d)\Phi_I \frac{\delta}{\delta \Phi^I} \\ &= \delta_D(1 + \mathcal{O}(e^{-2r})) , \end{aligned} \tag{A.9}$$

where D is the dilatation operator.

Thus we can identify the radial expansion with the expansion in the eigenfunctions of the dilation operator. In particular, we could do that for the canonical momentum, which is found in the radial picture to equal

$$\pi_{ij} = \frac{\sqrt{g}}{16\pi G_N}(K_{ij} - K\hat{\gamma}_{ij}) , \tag{A.10}$$

where K_{ij} is the extrinsic curvature of the radial surface,

$$K_{ij} = \frac{1}{2}\partial_r g_{ij} \to \frac{1}{2}\delta_D g_{ij} , \tag{A.11}$$

$K = K_{ij}\hat{\gamma}^{ij}$, and expand the canonical momentum in eigenvalues of δ_D,

$$\delta_D \pi_{ij}^{(n)} = -n\pi_{ij}^{(n)}. \tag{A.12}$$

This would not be important in the unrenormalized case, but in the renormalized case, it is.

Then, identifying $S_{\rm on-shell}$ with $-W$ as before, we obtain a relation between the one-point function of the energy-momentum tensor and the canonical momentum conjugate to $\hat{\gamma}_{ij}$,

$$\langle T_{ij}\rangle = -\frac{2}{\sqrt{g}}\pi_{ij} , \tag{A.13}$$

which is valid even in the renormalized case, provided we keep the piece of engineering dimension equal to the spatial one, d (3 in the physical case), so

$$\langle T_{ij}\rangle = \left(-\frac{2}{\sqrt{g}}\pi_{ij}\right)_{(d)} = -\frac{1}{8\pi G_N}(K_{ij} - K\hat{\gamma}_{ij})_{(d)} = -\frac{1}{8\pi G_N}(K_{(d)ij} - K_{(d)}\hat{\gamma}_{ij})$$

$$
\begin{aligned}
&= -\frac{1}{16\pi G_N}(\partial_r g_{(d)ij} - \hat{\gamma}^{kl}\partial_r g_{(d)kl}\hat{\gamma}_{ij}) \\
&\simeq -\frac{d}{16\pi G_N} g_{(d)ij}. \qquad \text{(A.14)}
\end{aligned}
$$

The 2-point function is found from the variation of the one-point function in the presence of sources,

$$
\delta\langle T_{ij}(x)\rangle = -\int d^3y\sqrt{g_{(0)}}\left(\frac{1}{2}\langle T_{ij}(x)T_{kl}(y)\delta g^{kl}_{(0}(y) + \mathcal{O}(\delta\phi_I)\right) , \qquad \text{(A.15)}
$$

so that

$$
\langle T_{ij}(x)T_{kl}(y)\rangle = \frac{1}{\sqrt{g_{(0)}}}\frac{\delta}{\delta g^{(0)}_{kl}(y)}\langle T_{ij}(x)\rangle = \frac{1}{\sqrt{g_{(0)}}}\frac{\delta}{\delta g^{(0)}_{kl}(y)}\left(-\frac{2}{\sqrt{g}}\pi_{ij}\right)_{(d)} . \qquad \text{(A.16)}
$$

The right-hand side, when we take out the trivial index structure, was in a sense the definition of the linear response functions, which to linear order satisfy

$$
E = \frac{\delta\pi^{\gamma}_{q}}{\delta\gamma_q} + \text{nonlinear} , \quad \Omega = \frac{\delta\pi^{\zeta}}{\delta\zeta_q} + \text{nonlinear} , \qquad \text{(A.17)}
$$

so after decomposing, in momentum space

$$
\langle T_{ij}(q)T_{kl}(-q)\rangle = A(q)\pi_{ijkl} + B(q)\pi_{ij}\pi_{kl} , \qquad \text{(A.18)}
$$

we find

$$
A(q) = 4E_{(0)}(q) , \quad B(q) = \frac{1}{4}\Omega_{(0)}(q). \qquad \text{(A.19)}
$$

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