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UNIVERSIDADE ESTADUAL PAULISTA
"JÚLIO DE MESQUITA FILHO"

PhD Thesis IFT-T.002/19

# Aspects of Classical and Quantum integrable models


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August, 2019

Retore, Ana Lúcia
R438a Aspects of classical and quantum integrable models / Ana Lúcia Retore. – São Paulo, 2019
127 f. : il.

Tese (doutorado) - Universidade Estadual Paulista (Unesp), Instituto de Física Teórica (IFT), São Paulo
Orientador: José Francisco Gomes

1. Física matemática. 2. Teoria de campos (Física). 3. Sólitons. I. Título

Sistema de geração automática de fichas catalográficas da Unesp. Biblioteca do Instituto de Física Teórica (IFT), São Paulo. Dados fornecidos pelo autor(a).

# Acknowledgments

Primeiramente eu gostaria de agradecer meu orientador José Francisco Gomes pela orientação, tranquilidade e paciência durante todo esse tempo, e por tudo que me ensinou. Foi um grande prazer trabalhar com o senhor! Eu queria agradecer também ao Professor Zimerman, por estar sempre presente nas nossas discussões com sua sabedoria e bom humor.

I am very grateful to my co-advisor Rafael Nepomechie for all kindness and patience during supervision, for everything he taught me and for his enormous help in my career. It was a great pleasure for me to work with you, Rafael!

I would like to thank Rafael, Susana, Matt, Orlando, Nicolas and Zoltan for your kindness during my stay in US.

To all my collaborators Frank, Rafael, Nathaly, Alexis, Zimerman, and Rodrigo my sincere thank you for all you taught me during these projects.

Gostaria de agradecer aos meus amigos Nathaly, Matheus, Thiago, Rafaela, Leônidas, Geórgia, Juliana, Ernane, Víctor, Jogean, Gabriel e Bárbara por todos as brincadeiras, conversas e os inúmeros momentos bons durante estes anos. Eu gostaria de agradecer especialmente ao Luan e a Vivi por estarem sempre comigo mesmo longe. Amo vocês!

Aos professores por todo conhecimento que me passaram ao longo desses anos. Aos funcionários do IFT pelas inúmeras vezes que me ajudaram, em especial, à dona Jô.

Gostaria de agradecer aos meus pais, Angelo e Rose, minha irmã Lu e minha vó Nena por todo o amor, carinho e apoio durante todos esses anos. Amo vocês!

E finalmente eu gostaria de agradecer ao meu amor, Henrique, por todo carinho, amor, apoio e companheirismo. Obrigada por me fazer acreditar mais em mim! Você me inspira a ser melhor todos os dias! Te amo!

Eu gostaria de agradecer à CAPES e à FAPESP pelo apoio financeiro na bolsa de doutorado regular no país sob Processo número: 2015/00025-9 Fundação de Amparo à Pesquisa do Estado de São Paulo e na bolsa BEPE nos Estados Unidos sob Processo número: 2017/03072-3 Fundação de Amparo à Pesquisa do Estado de São Paulo. O suporte foi imprescindível para realização desse trabalho.

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# Abstract

Aspects of classical and quantum integrability are explored. Gauge transformations play a fundamental role in both cases.

Classical integrable hierarchies have an underlying algebraic structure which brings universality for the solutions of all the equations belonging to a hierarchy. Such universality is explored together with the gauge invariance of the zero curvature equation to systematically construct the Bäcklund transformations for the mKdV hierarchy, as well as to relate it with the KdV hierarchy. As a consequence the defect-matrix for the KdV hierarchy is obtained and a few explicit Bäcklund transformations are computed for both Type-I and Type-II. The generalization for super mKdV hierarchy is also explored.

We studied symmetries and degeneracies of families of integrable quantum open spin chains with finite length associated to affine Lie algebras $\hat{g} = A^{(2)}_{2n}$, $A^{(2)}_{2n-1}$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$ whose K-matrices depend on a discrete parameter $p$ ($p = 0, ..., n$). We show that all these transfer matrices have quantum group symmetry corresponding to removing the $p^{\text{th}}$ node of the Dynking diagram of $\hat{g}$. We also show that the transfer matrices for $C^{(1)}_n$ and $D^{(1)}_n$ also have duality symmetry and the ones for $A^{(2)}_{2n-1}$, $B^{(1)}_n$ and $D^{(1)}_n$ have $Z_2$ symmetries that map complex representations into their conjugates. Gauge transformations simplify considerably the proofs by allowing us to work in a way that only unbroken generators appear.

The spectrum of the same integrable spin chains with the addition of $D^{(2)}_{n+1}$ is then determined using analytical Bethe ansatz. We conjecture a generalization for open chains for the Bethe ansatz Reshetikhin's general formula and propose a formula relating the Dynkin labels of the Bethe states with the number of Bethe roots of each type.

**Key-words:** Integrable Hierarchies, Bäcklund transformations, quantum group symmetries, open spin chains, Bethe ansatz.

# Resumo


Aspectos de integrabilidade clássica e quântica são explorados. Transformações de gauge têm papel fundamental em ambos os casos.

Hierarquias integráveis clássicas tem uma estrutura algébrica subjacente que traz uma universalidade para as soluções de todas as equações que a compoem. Essa universalidade é explorada juntamente com a invariância da equação de curvatura nula por transformações de gauge para construir sistematicamente as transformações de Bäcklund da hierarquia mKdV, assim como para relacioná-la com a hierarquia KdV. Como uma consequência a matriz de defeito para a hierarquia KdV é obtida e alguns exemplos explícitos são calculados tanto para o Tipo-I quanto para o Tipo-II. A generalização para a hierarquia mKdV supersimétrica também é discutida.

Nós estudamos simetrias e degenerecências de famílias de cadeias de spin quânticas integráveis com comprimento finito associadas a algebras de Lie afins $\hat{g} = A^{(2)}_{2n}$, $A^{(2)}_{2n-1}$, $B^{(1)}_{n}$, $C^{(1)}_{n}$, $D^{(1)}_{n}$ cujas matrizes K dependem de um parâmetro discreto $p$ ($p = 0, ..., n$). Nós mostramos que todas essas matrizes de transferências têm simetrias de grupos quânticos correspondente a remover o nodo $p$ do diagrama de Dynkin de $\hat{g}$. Também mostramos que as matrizes de transferência para $C^{(1)}_{n}$ e $D^{(1)}_{n}$ têm também simetria de dualidade enquanto $A^{(2)}_{2n-1}$, $B^{(1)}_{n}$ e $D^{(1)}_{n}$ têm simetrias $Z_2$ que mapeiam representações complexas em seus conjugados. Transformações de gauge simplificam consideravelmente as provas pois permitem-nos trabalhar com apenas os geradores que não foram quebrados.

O espectro dessas matrizes de transferência juntamente com $D^{(2)}_{n+1}$ é então calculado usando o método do Bethe ansatz analítico. Nós conjecturamos uma generalização para cadeias de spin abertas para a fórmula de Reshetikhin e propomos uma fórmula relacionando os índices de Dynkin dos estados de Bethe com o número de raízes de Bethe de cada tipo.



**Palavras-chave:** Hierarquias Integráveis, Transformações de Bäcklund, simetrias de grupos quânticos, cadeias de spin abertas e Bethe ansatz.

# Preface

This PhD thesis is divided in two completely separated parts. The Part 1 is devoted to classical integrable models while the Part 2 is focused in quantum integrable models. Since the models we work in each part are completely different we write separated introductions and conclusions for each part.

The thesis is heavily based on the following published papers:

- R. I. Nepomechie and A. L. Retore, "The spectrum of quantum-group-invariant transfer matrices," Nucl. Phys. B **938**, 266 (2019), https://arxiv.org/pdf/1810.09048.pdf.
- R. I. Nepomechie and A. L. Retore, "Surveying the quantum group symmetries of integrable open spin chains," Nucl. Phys. B **930**, 91 (2018), https://arxiv.org/abs/1802.04864
- A. R. Aguirre, A. L. Retore, J. F. Gomes, N. I. Spano and A. H. Zimerman, "Defects in the supersymmetric mKdV hierarchy via Bäcklund transformations," JHEP **1801**, 018 (2018), https://arxiv.org/abs/1709.05568.
- J. F. Gomes, A. L. Retore and A. H. Zimerman, "Miura and generalized Bäcklund transformation for KdV hierarchy," J. Phys. A **49**, no. 50, 504003 (2016), https://arxiv.org/abs/1610.02303.
- J. F. Gomes, A. L. Retore and A. H. Zimerman, "Construction of Type-II Backlund Transformation for the mKdV Hierarchy," J. Phys. A **48**, 405203 (2015), https://arxiv.org/abs/1505.01024.

and on the conference proceedings :

- J. F. Gomes, A. L. Retore, N. I. Spano and A. H. Zimerman, "Backlund Transformation for Integrable Hierarchies: example - mKdV Hierarchy," J. Phys. Conf. Ser. **597**, no. 1, 012039 (2015), https://arxiv.org/abs/1501.00865.

but also includes some results and comments from the following works

- R. I. Nepomechie, R. A. Pimenta and A. L. Retore, "The integrable quantum group invariant $A^{(2)}_{2n-1}$ and $D^{(2)}_{n+1}$ open spin chains," Nucl. Phys. B **924**, 86 (2017), https://arxiv.org/abs/1707.09260.
- A. R. Aguirre, A. L. Retore, N. I. Spano, J. F. Gomes and A. H. Zimerman, "Recursion Operator and Bäcklund Transformation for Super mKdV Hierarchy,", https://arxiv.org/abs/1804.06463.
- N. I. Spano, A. L. Retore, J. F. Gomes, A. R. Aguirre and A. H. Zimerman, "The sinh-Gordon defect matrix generalized for n defects," https://arxiv.org/abs/1610.01856.
- A. R. Aguirre, J. F. Gomes, A. L. Retore, N. I. Spano and A. H. Zimerman, "An alternative construction for the Type-II defect matrix for the sshG," , https://arxiv.org/abs/1610.01855.

and on the following paper which was already submitted to J.Phys.A

- R. I. Nepomechie, R. A. Pimenta and A. L. Retore, "Towards the solution of an integrable $D^{(2)}_{2}$ spin chain, https://arxiv.org/abs/1905.11144

Since the two parts of the thesis are very different some notations were used in both parts meaning different things. K-matrix for example has two completely different meaning in Part 1 and Part 2. I ask the reader to keep this in mind when reading this work.

# Contents

# Part I

# Classical Integrability

# Chapter 1

# Introduction

Classical integrable models are known to have an infinite number of conserved charges that guarantees the stability of their solitons solutions. Along the history they were studied in various approaches such as inverse scattering method, the Lax method and the Zakharov-Shabat formulation [1]-[3]. These methods are systematic providing ways to construct many integrable non-linear differential equations such as the Korteweg-de Vries (KdV) equation, the non-linear Schrödinger (NLS) equation and the Sinh-Gordon (S-G) equation .

There is another method, however, which has several interesting qualities. It consists of constructing infinite towers of nonlinear integrable differential equations starting from a zero curvature equation [5, 6] and an underlying graded affine algebra. These models are called integrable hierarchies. The advantage of them is that due to their algebraic construction one can compute the features of an infinite number of differential equations in a universal way. For example, $\hat{sl}(2)$ with a principal gradation generates the so-called mKdV hierarchy. This process turns clear that both mKdV (modified KdV) equation and S-G equation are part of the same integrable hierarchy and therefore have soliton solutions with the same general structure [4].

This method involves two (1+1)-dimensional gauge potentials $A_x$ and $A_{t_N}$ called Lax pair. While the $A_x$ is the same for all the equations within a hierarchy, each $A_{t_N}$ is related to a different time evolution and therefore generates a different evolution equation. The universality of the $A_x$ in each hierarchy has a fundamental role in the computation of many physical quantities in a general way.

The solutions of these models can be obtained through the so-called Dressing method [7]-[9] which makes use of gauge transformations to create multi-soliton solutions by starting with a vacuum solution. The Dressing method provides a way to simultaneously construct the soliton solutions for all the equations of a given hierarchy.

Another interesting way to compute solutions of integrable models is called Bäcklund transformations (BT). These Bäcklund transformations, among other applications, generate an infinite sequence of soliton solutions from a non-linear superposition principle (see [10],[1],[11],[12]).

Bäcklund transformations have also been employed to describe integrable defects [13]-[17] in the sense that two solutions of an integrable model may be interpolated by a defect at certain spatial position. After the introduction of the defect, the integrability is only preserved if the two field configurations are related by a Bäcklund transformation. Under such formulation the energy and momentum have to be modified to take into account the contribution of the defect [13]. Well known (relativistic) integrable models as the sine (sinh)-Gordon, Tzitzeica [18], Lund-Regge [14] and other (non-relativistic) models as Non-Linear Schroedinger (NLS), mKdV, etc have been studied within such context [16]. Also the N=1 and N=2 supersymmetric sinh-Gordon and the super Liouville were studied using integrable defects.

There are two known types of Bäcklund transformations. The first involves only the fields of the theory and is called Type-I. In particular, it may be observed that the space component of the Type-I Bäcklund transformations for the mKdV and sinh-Gordon equations coincides for their corresponding fields [10]. Today we know that this happens because they are part of the same integrable hierarchy (the mKdV hierarchy) and that actually all the equations in this hierarchy have the same spatial part of the Bäcklund transformation.

More recently a new type of Bäcklund transformations involving auxiliary fields was shown to be compatible with the equations of motion for the sine (sinh)-Gordon and Tzitzeica models [18]. These are known as Type-II Bäcklund transformations. They are obtained by introducing two defects instead of only one and taking the limit where both defects are the same point is [21]-[27].

Historically all the methods to compute Bäcklund transformations require an individual computation for each model. When studying integrable defects, for example, for each model we want to compute the Bäcklund transformations we need to construct its Lagrangian. Although this would be in principle feasible and would bring a lot of interesting discussions, it would also be very hard. This is because the Lagrangians for differential equations with high orders could take very complicated forms.

In [19] it was shown that the Bäcklund transformations may be constructed from gauge transformation relating two field configurations of the same equation of motion. Such gauge transformation is encoded in the so-called K-matrix (or defect matrix).

The outline of Part 1 is as follows. The chapter 2 is dedicated to explain the construction of integrable hierarchies and to extend the results in [19] to all integrable equations of the mKdV hierarchy[20], [28]. This is again a consequence of the universality of the spacial Lax along the hierarchy. In the chapter 3 we construct the KdV hierarchy starting from the Lax pair of the mKdV hierarchy and using gauge transformations. As a consequence we also obtain the K-matrix of the KdV hierarchy in terms of the one for the mKdV hierarchy. We also introduce the idea that the Type-II K-matrix can be constructed as a product of two Type-I K-matrices and discuss some solutions [29]-[30]. The Chapter 4 is dedicated to the generalization for the super mKdV hierarchy[31]-[33]. In Chapter 5 we present some conclusions and further developments.

# Chapter 2

# Integrable Hierarchies and Bäcklund transformations

The outline of this chapter is as follows. The zero curvature equation is introduced in section 2.1. In section 2.2 is introduced the concept of integrable hierarchy. The construction of integrable hierarchies is discussed and quickly exemplified using the mKdV, AKNS and KdV hierarchies. The gauge invariance of the zero curvature equation is presented in section 2.3. The section 2.4 is dedicated to introduce the concept of Bäcklund transformations as well as to discuss a its construction using gauge transformations as well as integrable defects.

## 2.1 Zero Curvature equation

Consider a linear system, with coordinates $(x, t_N)$,

$$(\partial_x + A_x)\psi = 0,$$
$$(\partial_{t_N} + A_{t_N})\psi = 0, \tag{2.1}$$

in which, $A_x$ and $A_{t_N}$ are two-dimensional gauge potentials, in our case written as matrices whose elements are functions of the $n$ fields of the theory and their derivatives of any order. They are known as Lax pair.

By acting with $\partial_x$ in the second equation of (2.1) and with $\partial_{t_N}$ on the first equation and then subtracting the results we obtain

$$[\partial_x + A_x, \partial_{t_N} + A_{t_N}] = 0. \tag{2.2}$$

which is called **zero curvature equation**.

In (1+1) dimensions, we say that a model is classically integrable if it can be generated by a zero curvature representation.

The zero curvature equation plays a very important role in classical integrability since it enables the construction of the so called integrable hierarchies. An integrable hierarchy is an infinite set of integrable equations constructed from a given algebra. The differential equations belonging to the same integrable hierarchy have several properties in common such as: its equations have soliton solutions in a universal form and also its Bäcklund transformations can be constructed in a systematic way. Such universality due to their algebraic structure makes possible to understand features of an infinite number of equations at once.

Usually, for each hierarchy, $A_x$ remains the same for all its equations while each $A_{t_N}$ gives rise to a new differential equation. The order of the differential equation is directly related with $N$. For example, for the mKdV (modified Kortweg-deVries) hierarchy, the $A_{t_3}$ generates the mKdV equation

$$4\partial_{t_3} v = \partial_x^3 v - 6v^2 \partial_x v, \tag{2.3}$$

while $A_{t_{-1}}$ generates the Sinh-Gordon equation

$$\partial_{t_{-1}} v = e^{2\int_0^x v(y,t_{-1})dy} - e^{-2\int_0^x v(y,t_{-1})dy}. \tag{2.4}$$

So, the $A_{t_3}$ generates a non-linear differential equation with order 3, while $A_{t_{-1}}$ generates one with one integral, so we can say, that it has order $-1$.

Notice that the equation (2.4) becomes the usual Sinh-Gordon equation[1]

[1]Notice that for Sinh-Gordon equation $t_{-1} = z$ and $x = \bar{z}$ are the light-cone coordinates .

$$\partial_{t_{-1}}\partial_x\phi = e^{2\phi} - e^{-2\phi} \tag{2.5}$$

if we change variable $v = \partial_x\phi$.

Let us now see more in detail how such hierarchies are constructed.

## 2.2 Constructing integrable hierarchies

In order to construct an integrable hierarchy we need:

- choose an algebra;
- choose a gradation of this algebra;
- choose a semi simple element $E^{(1)}$.

Let us consider the $\hat{\mathrm{sl}}(2)$ centerless Kac-Moody algebra.

The gradation operator $Q$ decomposes the algebra in graded subspaces in the form

$$\hat{\mathcal{G}} = \oplus\mathcal{G}_m, \tag{2.6}$$

where

$$[Q, \mathcal{G}_n] = n\mathcal{G}_n, \tag{2.7}$$

with $n \in \mathbb{Z}$ and it is called degree. In addition, due to Jacobi identity we have

$$[\mathcal{G}_n, \mathcal{G}_m] \subset \mathcal{G}_{m+n}. \tag{2.8}$$

As we mentioned, in this and on the next chapter we are dealing with hierarchies related to a $\hat{\mathrm{sl}}(2)$ centerless Kac-Moody algebra. It is generated by [2] $h^{(m)} = \lambda^m h, \;\; E^{(m)}_{\pm\alpha} = \lambda^m E_{\pm\alpha}, \quad \lambda \in C, \quad m \in Z$ satisfying

$$[h^{(m)}, E^{(n)}_{\pm\alpha}] = \pm 2E^{(m+n)}_{\pm\alpha}, \quad [E^{(m)}_{\alpha}, E^{(n)}_{-\alpha}] = h^{(m+n)}. \tag{2.9}$$

For this algebra two different gradations will be considered: the **principal gradation** and the **homogeneous gradation**.

The **principal gradation** is defined by $Q_p = 2\lambda\frac{d}{d\lambda} + \frac{1}{2}h$ and generates the so called mKdV hierarchy (modified Kortweg-de-Vries hierarchy). $Q_p$ decomposes the affine $\hat{sl}(2)$ algebra into graded subspaces according to powers of the spectral parameter $\lambda$ (2.6)-(2.8), where the subspaces $\mathcal{G}_{2m}$, $\mathcal{G}_{2m+1}$ and $\mathcal{G}_{2m-1}$ contain the following generators

$$\begin{aligned} \mathcal{G}_{2m} &= \{h^{(m)} = \lambda^m h\}, \\ \mathcal{G}_{2m+1} &= \{E^{(m)}_{\alpha} = \lambda^m E_{\alpha}\}, \\ \mathcal{G}_{2m-1} &= \{E^{(m)}_{-\alpha} = \lambda^m E_{-\alpha}\} \end{aligned} \tag{2.10}$$

i.e.

$$[Q_p, \mathcal{G}_{2n+1}] = (2n+1)\,\mathcal{G}_{2m+1}, \quad [Q_p, \mathcal{G}_{2n-1}] = (2n-1)\,\mathcal{G}_{2m+1} \quad \text{and} \quad [Q_p, \mathcal{G}_{2n}] = (2n)\,\mathcal{G}_{2n}. \tag{2.11}$$

The **homogeneous gradation** is defined by $Q_h = \zeta\frac{d}{d\zeta}$ and it generates the AKNS hierarchy (Ablowitz-Kaup-Newel-Segur hierarchy) and the KdV hierarchy (Kortweg-de-Vries hierarchy). $Q_h$ decomposes the affine $\hat{sl}(2)$ algebra into graded subspaces according to powers of the spectral parameter $\zeta$[3] (2.6)-(2.8) where the subspace $\mathcal{G}_n$ contains the following generators

$$\mathcal{G}_n = \left\{h^{(n)} = \zeta^n h, E^{(n)}_{\alpha} = \zeta^n E_{\alpha}, E^{(n)}_{-\alpha} = \zeta^n E_{-\alpha}\right\}, \tag{2.12}$$

i.e.

$$[Q_h, \mathcal{G}_n] = n\,\mathcal{G}_n. \tag{2.13}$$

[2]The representation used is: $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

[3]Notice that for the principal gradation we are using $\lambda$ as spectral parameter while in the homogeneous gradation we are using $\zeta$ as spectral parameter. This is to avoid some confusion.

After choosing a gradation, the next step is to construct the Lax pair $(A_x, A_{t_N})$ of the model. Let us define $A_x = A_0 + E^{(1)}$ where $E^{(1)}$ is a semi simple element and has degree 1, while $A_0$ contains the fields of the theory and has degree 0. But how to construct $E^{(1)}$ and $A_0$? First of all, by choosing a gradation we automatically know which generators have degree 0 and 1, according to (2.10) and (2.12). We do now a new decomposition in the algebra: $\hat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$ where $\mathcal{K}$ stands for Kernel and $\mathcal{M}$ is the image. The kernel is defined as

$$\mathcal{K} = \left\{ f \in \hat{\mathcal{G}} / \left[ f, E^{(1)} \right] = 0 \right\} \tag{2.14}$$

and $\mathcal{M}$ is its complement. Notice that to choose a semi simple $E^{(1)}$ means that $\mathcal{K}$ and $\mathcal{M}$ have to satisfy

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M} \quad \text{and} \quad [\mathcal{M}, \mathcal{M}] \subset \mathcal{K}. \tag{2.15}$$

With these informations we are able to construct the spatial Lax operator $A_x$. And how about the $A_{t_N}$? In order to find $A_{t_N}$ we separate the hierarchy into two sub-hierarchies: the **positive sub-hierarchy** and the **negative sub-hierarchy**.

### 2.2.1 Positive sub-hierarchy

The positive integrable sub-hierarchy is obtained by writing $A_{t_N} = D^{(N)} + D^{(N-1)} + ... + D^{(0)}$ in such a way that the zero curvature equation decomposes (2.2) according to the graded structure as

$$\begin{aligned} [E^{(1)}, D^{(N)}] &= 0 \\ [E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0 \\ \vdots &= \vdots \\ [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_N} A_0 &= 0, \end{aligned} \tag{2.16}$$

Here $D^{(i)} \in \mathcal{G}_i$. The equations in (2.16) allows solving for $D^{(i)}, i = 0, \cdots N$ and the last equation in (2.16) yields the time evolution for fields in $A_0$. We should point out that $D^{(i)}$ are constructed systematically for each value of $N$ and so is $A_{t_N}$.

### 2.2.2 Negative sub-hierarchy

The negative integrable sub-hierarchy is obtained by writing $A_{t_{-N}} = D^{(-N)} + D^{(-N+1)} + ... + D^{(-1)}$. The zero curvature equation then is decomposed in

$$\begin{aligned} \partial_x D^{(-N)} + [A_0, D^{(-N)}] &= 0 \\ \partial_x D^{(-N+1)} + [A_0, D^{(-N+1)}] + [E^{(1)}, D^{(-N)}] &= 0 \\ \vdots &= \vdots \\ \partial_x D^{(-1)} + [A_0, D^{(-1)}] + [E^{(1)}, D^{(-2)}] &= 0 \\ [E^{(1)}, D^{(-1)}] - \partial_{t_{-N}} A_0 &= 0, \end{aligned} \tag{2.17}$$

Notice that just as in the positive case, the equation with degree zero is the only one which depends on the $t_{-N}$. So, we start by solving the equation with degree equal to $-N$ and recursively solve until the one with degree 0.

We discuss in the following the mKdV hierarchy, the AKNS hierarchy and the KdV hierarchy.

### 2.2.3 mKdV hierarchy

The informations about $E^{(1)}$ and $A_0$ for all the hierarchies we work on this and on the next chapter can be find in the Table 2.1

| Gradation | Gradation Operator | $E^{(1)}$ | $A_0$ | Hierarchy |
|---|---|---|---|---|
| principal | $Q_p = 2\lambda \frac{d}{d\lambda} + \frac{1}{2} h$ | $E_\alpha^{(0)} + E_{-\alpha}^{(1)}$ | $v(x, t_N) h$ | mKdV |
| homogeneous | $Q_h = \zeta \frac{d}{d\zeta}$ | $h^{(1)}$ | $q(x, t_N) E_\alpha + r(x, t_N) E_{-\alpha}$ | AKNS |
| homogeneous | $Q_h = \zeta \frac{d}{d\zeta}$ | $h^{(1)}$ | $-E_\alpha + J(x, t_N) E_{-\alpha}$ | KdV |

Table 2.1: In this table we put the explicit forms of $E^{(1)}$ and $A_0$ for the hierarchies considered.

Let us start with the mKdV hierarchy and therefore with the principal gradation. The semi simple element $E^{(1)}$ is defined as $E^{(1)} = E_{\alpha}^{(0)} + E_{-\alpha}^{(1)}$ while $A_0 = v(x, t_N)h$. Notice that there is not much freedom to choose $E^{(1)}$ and $A_0$ in this case, since the only objects in $\mathcal{G}_1$ are $E_{\alpha}^{(0)}$ and $E_{-\alpha}^{(1)}$ and the only object in $\mathcal{G}_0$ is $h$.

### Positive sub-hierarchy

Let us do an example in order to make things clearer. Consider $N = 3$ and therefore $A_{t_3} = D^{(3)} + D^{(2)} + D^{(1)} + D^{(0)}$ with $D^{(i)}$'s as linear combinations of the generators in $\mathcal{G}_i$. For $N = 3$, (2.10) becomes

$$\begin{aligned} \mathcal{G}_3 &= \{E_{\alpha}^{(1)}, E_{-\alpha}^{(2)}\}, \\ \mathcal{G}_2 &= \{h^{(1)}\}, \\ \mathcal{G}_1 &= \{E_{\alpha}^{(0)}, E_{-\alpha}^{(1)}\}, \\ \mathcal{G}_0 &= \{h\}. \end{aligned} \tag{2.18}$$

Consequently,

$$\begin{aligned} D^{(3)} &= a_3 E_{\alpha}^{(1)} + b_3 E_{-\alpha}^{(2)}, \\ D^{(2)} &= c_2 h^{(1)}, \\ D^{(1)} &= a_1 E_{\alpha}^{(0)} + b_1 E_{-\alpha}^{(1)}, \\ D^{(1)} &= c_0 h. \end{aligned} \tag{2.19}$$

Until this moment we do not know what are the $a_i$'s, $b_i$'s and $c_i$'s. In order to compute them we substitute the $D^{(3)}$ in the first equation in(2.16) for $N = 3$. From this equation we conclude that $D^{(3)}$ needs to be on the Kernel $\mathcal{K}$. So, $b_3 = a_3$. Now we substitute this new $D^{(3)}$ and the $D^{(2)}$ (from (2.19)) in the second equation in (2.16) and solving the resulting equations we obtain $a_3 = \text{constant} \equiv 1$ and $c_2 = v$. By continuing this process we find that the complete solution is

$$a_3 = b_3 \equiv 1, \quad b_2 = v, \quad a_1 = -\frac{1}{2}v^2 + \frac{1}{2}\partial_x v, \quad b_1 = -\frac{1}{2}v^2 - \frac{1}{2}\partial_x v \quad \text{and} \quad c_0 = \frac{1}{4}\left(\partial_x^2 v - 2v^3\right), \tag{2.20}$$

while the last equation in (2.16) is the one which has the time-derivative and therefore it gives the time-evolution. By substituting on it $D^{(0)}$ with $c_0$ given by (2.20) we obtain the mKdV equation

$$4\partial_{t_3} v - \partial_x \left(\partial_x^2 v - 2v^3\right) = 0 \qquad \text{mKdV} \tag{2.21}$$

which is the equation that names the hierarchy.

The first next few explicit equations are

$$16\partial_{t_5} v - \partial_x \left(\partial_x^4 v - 10v^2(\partial_x^2 v) - 10v(\partial_x v)^2 + 6v^5\right) = 0 \tag{2.22}$$

$$\begin{aligned} 64\partial_{t_7} v &- \partial_x \left(\partial_x^6 v - 70(\partial_x v)^2(\partial_x^2 v) - 42v(\partial_x^2 v)^2 - 56v(\partial_x v)(\partial_x^3 v)\right) \\ &+ \partial_x \left(14v^2\partial_x^4 v - 140v^3(\partial_x v)^2 - 70v^4(\partial_x^2 v) + 20v^7\right) = 0 \\ &\cdots \text{etc.} \end{aligned} \tag{2.23}$$

constructed from $N = 5$ and $N = 7$, respectively.

Notice that, for positive mKdV sub-hierarchy we did not mentioned any differential equation for *even* $N$. This has a very fundamental reason. If we start with an even $N$, let us say $N = 2n$, $D^{(N)}$ has to be of the form $D^{(2n)} = h^{(n)}$ (due to (2.10)) which belongs to the image $\mathcal{M}$. But the first equation in (2.16) requires $D^{(N)}$ being in the Kernel. Therefore if we try to construct a differential equation with *even* order in the *mKdV* hierarchy we just find that all the coefficients are equal to zero. Notice that the equations (2.21)-(2.23) are invariant under $v \to -v$.

**Negative sub-hierarchy**

An interesting fact is that if we do the same procedure for the negative sub-hierarchy, for $v = \partial_x \phi$ with $N = 1$ we will obtain the Sinh-Gordon equation

$$\partial_{t_{-1}} \partial_x \phi = 2\sinh(2\phi) \tag{2.24}$$

and for $N = 3$ and $N = 5$ we have

$$\begin{aligned}
\partial_{t_{-3}} \partial_x \phi &= 4\, e^{-2\phi} d^{-1} \left(e^{2\phi} d^{-1} (\sinh 2\phi)\right) + 4\, e^{2\phi} d^{-1} \left(e^{-2\phi} d^{-1} (\sinh 2\phi)\right) \\
\partial_{t_{-5}} \partial_x \phi &= 8\, e^{-2\phi} d^{-1} \left(e^{2\phi} d^{-1} \left(e^{-2\phi} d^{-1} \left(e^{2\phi} d^{-1} (\sinh 2\phi)\right) + e^{2\phi} d^{-1} \left(e^{-2\phi} d^{-1} (\sinh 2\phi)\right)\right)\right) + \\
&\quad 8\, e^{+2\phi} d^{-1} \left(e^{-2\phi} d^{-1} \left(e^{-2\phi} d^{-1} \left(e^{2\phi} d^{-1} (\sinh 2\phi)\right) + e^{2\phi} d^{-1} \left(e^{-2\phi} d^{-1} (\sinh 2\phi)\right)\right)\right) \\
&\ddots
\end{aligned} \tag{2.25}$$

where $d^{-1} f = \int^x f(y) dy$. And for $N = 2$ and $N = 4$ we have

$$\begin{aligned}
\partial_{t_{-2}} \partial_x \phi &= 4e^{2\phi} d^{-1} \left(e^{-2\phi}\right) + 4e^{-2\phi} d^{-1} \left(e^{2\phi}\right), \\
\partial_{t_{-4}} \partial_x \phi &= 4e^{2\phi} d^{-1} \left(e^{-2\phi} \left(e^{2\phi} d^{-1} \left(e^{-2\phi}\right) + e^{-2\phi} d^{-1} \left(e^{2\phi}\right)\right)\right) + \\
&\quad 4e^{-2\phi} d^{-1} \left(e^{+2\phi} \left(e^{2\phi} d^{-1} \left(e^{-2\phi}\right) + e^{-2\phi} d^{-1} \left(e^{2\phi}\right)\right)\right)
\end{aligned} \tag{2.26}$$

Notice, that opposite to what happens in the positive sub-hierarchy we do not have any restriction for *even* $N$. For the negative sub-hierarchy decomposition (2.17) the first equation does not require that $D^{(N)}$ commutes with the $E^{(1)}$ and therefore does not create any difficulty.

However, the equations constructed from even $N$ do have some properties that are different from the ones with odd $N$. The first property is that while the equations for odd $N$ are invariant under $\phi \to -\phi$ (by inspection on equations (2.24)-(2.25))the ones for even $N$ are not (see (2.26)). The second property is that on contrary of odd $N$ (2.24)-(2.25), the equations for even $N$ (2.26) do not have the vacuum $\phi = 0$ as a solution.

### 2.2.4 AKNS hierarchy

Following the same procedure, but now considering the homogeneous gradation and $E^{(1)}$ and $A_0$ as in the Table 2.1 we obtain for $N = 2$

$$\begin{aligned}
\partial_{t_2} q &= -\frac{\alpha}{2} \partial_x^2 q + \gamma \partial_x q + \alpha q^2 r \\
\partial_{t_2} r &= \frac{\alpha}{2} \partial_x^2 r + \gamma \partial_x r - \alpha r^2 q
\end{aligned} \tag{2.27}$$

where $\alpha$ and $\gamma$ are constants. Notice that for $q = \psi$ and $r = \psi^*$ we obtain the nonlinear Schrödinger equation.

Now for $N = 3$ we obtain

$$\begin{aligned}
\partial_{t_3} q &= \frac{\alpha}{4} \partial_x^3 q - \frac{3}{2} \alpha r q \partial_x q - \frac{1}{2} \gamma \partial_x^2 q + \gamma q^2 r \\
\partial_{t_3} r &= \frac{\alpha}{4} \partial_x^3 r - \frac{3}{2} \alpha q r \partial_x r + \frac{1}{2} \gamma \partial_x^2 r - \gamma r^2 q.
\end{aligned} \tag{2.28}$$

The above equations are just examples. One could continue obtaining higher and higher degree differential equations by assuming larger values of $N$.

### 2.2.5 KdV hierarchy

By making $q = -1$ and $r = J$ in the AKNS Lax and doing again the computations for $N = 3$ we obtain the KdV equation

$$4\partial_{t_3} J = \partial_x^3 J + 6J \partial_x J, \tag{2.29}$$

Notice that by doing $\gamma = 0$, $q = -1$ and $r = J$ directly on equation (2.28) we obtain the equation (2.29).

Similarly for $A_{t_5}$ and for $A_{t_7}$, (details are given in the appendix A), we find the Sawada-Kotera equation [12]

$$16\partial_{t_5} J = \partial_x^5 J + 20\partial_x J\partial_x^2 J + 10J\partial_x^3 J + 30J^2\partial_x J, \tag{2.30}$$

and

$$\begin{aligned} 64\partial_{t_7} J = \partial_x^7 J - 70\partial_x^2 J\partial_x^3 J + 42\partial_x J\partial_x^4 J + 70(\partial_x J)^3 + +14J\partial_x^5 J \\ +280J\partial_x J\partial_x^2 J + 70J^2\partial_x^3 J + 140J^3\partial_x J, \end{aligned} \tag{2.31}$$

respectively. Higher flows (time evolutions) can be systematically constructed for generic $N$ from the same formalism.

## 2.3 Gauge-invariance of the zero curvature equation

If we do a gauge transformation on the Lax pair as follows

$$A_x' = KA_xK^{-1} - \partial_x KK^{-1}, \tag{2.32}$$

$$A_{t_N}' = KA_{t_N}K^{-1} - \partial_{t_N} KK^{-1} \tag{2.33}$$

the zero curvature equation (2.2) remains invariant.

The gauge invariance of the zero curvature equation plays a very important in finding universal features of integrable hierarchies. It allows the systematic construction of universal soliton solutions for the hierarchies through the so called Dressing Method [7]-[9]. Also, it allows the generation of Bäcklund transformations for entire hierarchies in a systematic way as we will see in next section and next two chapters.

## 2.4 Bäcklund transformations

### 2.4.1 Basics

**Bäcklund transformations (BT)** [10, 2, 1, 3] consists of a system of equations which relates two solutions of the same nonlinear differential equation [4]. Bäcklund transformations have at least one order in $x$ less than the original equation which makes them, in most of the cases, a lot easier to solve. Also, they depend on at least one new parameter (which is not present on the original equation) and is called Bäcklund parameter.

Usually nonlinear differential equations can be very hard to solve. What makes BT so important is that they work as generating functions of new solutions given that you already know some solutions (at least one). One simple example is that if you know that the vacuum is a solution, you can substitute it in the BT and then solve the system to find a 1-soliton solution.

Let us see the Bäcklund transformations of the Sinh-Gordon equation as an example. The equations

$$\begin{aligned} \partial_{t_{-1}}(\phi_1 + \phi_2) &= -\frac{4}{\beta}\sinh(\phi_1 - \phi_2) \\ \partial_x(\phi_1 - \phi_2) &= -\beta\sinh(\phi_1 + \phi_2) \end{aligned} \tag{2.34}$$

are the BT for the Sinh-Gordon equation (2.24). The $\beta$ is the Bäcklund parameter, while $\phi_1$ and $\phi_2$ are two solutions of the S-G equation. Notice that while the S-G equation has two derivatives, one in $x$ and one in $t_{-1}$, its Bäcklund transformations have or one derivative in $t_{-1}$ or one derivative in $x$. If we act with $\partial_x$ on the first equation on (2.34) and use the second one, the $\beta$ disappears and we obtain two S-G equations, one for $\phi_1$ and one for $\phi_2$.

[4] There are also BT which can relate a solution of a differential equation with a solution of another differential equation. Those, however, will not be considered in this thesis.

The solutions of the S-G equation for vacuum, 1-soliton and 2-soliton solutions are given by [5][6].

$$\phi_{0-sol} = 0$$
$$\phi_{1-sol} = \ln\left(\frac{1+R_1\rho_1}{1-R_1\rho_1}\right)$$
$$\phi_{2-sol} = \ln\left(\frac{1+\delta\left(\rho_1-\rho_2\right)+\rho_1\rho_2}{1-\delta\left(\rho_1-\rho_2\right)+\rho_1\rho_2}\right) \tag{2.35}$$

where $\rho_i = \text{Exp}\left(2k_i x + 2k_i^{-1}t_{-1}\right)$, $R_1$ is a constant and $\delta = \frac{k_1+k_2}{k_1-k_2}$.

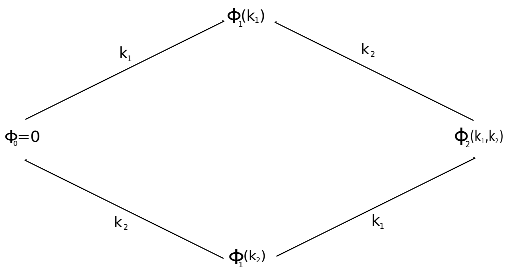


Figure 2.1: Permutability theorem for solutions of vacuum, one-soliton and two-solitons.

The solutions were introduced now in order to present a very important theorem: the **permutability theorem**. It says that if we start with a vacuum solution $\phi_{0-sol}$ and using the BT go to 1-soliton solution $\phi_{1-sol}(k_1)$ with $\beta = f(k_1)$ and then to a $\phi_{2-sol}(k_1,k_2)$ with $\beta = f(k_2)$ we will obtain a result which is completely equivalent to the one obtained by going from $\phi_{0-sol}$ to 1-soliton solution $\phi_{1-sol}(k_2)$ with $\beta = f(k_2)$ and then to a $\phi_{2-sol}(k_1,k_2)$ with $\beta = f(k_1)$ as is represented in the figure 2.1.

Another possible solution is to start with a 1-soliton and obtain another 1-soliton with a phase difference. If we start with a $\phi_{(1-sol)} = \ln\left(\frac{1+R_1\rho_1}{1-R_1\rho_1}\right)$ we could obtain $\phi_{(1-sol)} = \ln\left(\frac{1+R_2\rho_1}{1-R_2\rho_1}\right)$ with $R_2 = \left(\frac{2k_1+\beta}{2k_1-\beta}\right)R_1$.

The Bäcklund transformations presented in this section are called Type-I Bäcklund transformations. They depend only on the fields of the theory $\phi_1$ and $\phi_2$ and have one Bäcklund parameter only (called $\beta$). There is another type which was introduced by Corrigan et al [18] in the context of defects which depends on the fields of the theory but also on an auxiliary field which they called $\Lambda$ and depends on two Bäcklund parameters.

### 2.4.2 Gauge transformations and the Bäcklund transformations for mKdV hierarchy

Consider a gauge transformation as in (2.33) but now make $\left(A'_x, A'_{t_N}\right) \equiv \left(A_x(\phi_2), A_{t_N}(\phi_2)\right)$ and $\left(A_x, A_{t_N}\right) \equiv \left(A_x(\phi_1), A_{t_N}(\phi_1)\right)$ in such a way that (2.33) becomes

$$\partial_x K = KA_x(\phi_1) - A_x(\phi_2)K \tag{2.36}$$
$$\partial_{t_N} K = KA_{t_N}(\phi_1) - A_{t_N}(\phi_2)K \tag{2.37}$$

i.e. the $K$ matrix connects two field configurations $\phi_1$ and $\phi_2$.

But how to obtain the $K$? Remember that $A_x(\phi)$ is the same for the whole hierarchy. So if we can find $K$ using only the spacial part of the Lax we automatically have a $K$ which is valid for the entire hierarchy. Notice that both Type-I and Type-II Bäcklund transformations for S-G equation were well known. What we did was to write $K$ as a polynomial function on the spectral parameter, substitute the $A_x(\phi_i)$ (which we had from the hierarchy construction) on the first equation of (2.37) and solve for the coefficients[7] in order to find the spacial part of the Type-I BT and the spacial part of Type-II BT.

By doing this we found

$$K_{Type-I} = \begin{bmatrix} 1 & \frac{-\beta}{2\lambda}e^{-p} \\ -\frac{\beta}{2}e^{p} & 1 \end{bmatrix} \quad \text{and} \quad K_{Type-II} = \begin{bmatrix} 1-\frac{1}{\sigma^2\lambda}e^{q} & \frac{-\beta}{2\sigma\lambda}e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right) \\ -\frac{2}{\sigma}e^{p-\Lambda} & 1-\frac{1}{\sigma^2\lambda}e^{-q}. \end{bmatrix} \tag{2.38}$$

Using those transformations, we explicitly constructed the Bäcklund transformations for several equations of the mKdV hierarchy, including many in the negative sub-hierarchy (including with even $N$). In order to

[5]The solution of any other equation with *odd* order in the mKdV hierarchy is the same as (2.35) by just making $-1 \to N$ in the definition of $\rho_i$.

[6]The equations for *even* $N$ will be very similar, except that vacuum is $\phi$ = constant nonzero and that we have to add a constant in the 1-soliton and 2-soliton solutions.

[7]In order to see the complete calculation see the appendix B of [my first proceedings]

see some examples please see [20, 28]. But since the K matrix is general we could construct the Bäcklund transformations for any equation in the mKdV hierarchy.

We would like to highlight that the $K$ matrix for Type-II was already developed before by [19]. Our contribution in this part was to notice that since the $K$ could be constructed using only the spacial part of the Lax pair we could use this $K$ to obtain any Bäcklund transformation in the hierarchy.

### 2.4.3 Integrable defects

There are also applications of Bäcklund transformations to describe integrable defects [13, 16, 18] in the sense that one solution when hits a defect becomes another solution through Bäcklund transformations. So, it is possible to compute Bäcklund transformations using Lagrangian defects. It is considered a line and introduced a defect in $x = 0$. In each side of the defect we have a theory described by $\phi_1$ and $\phi_2$ respectively. In order to preserve the integrability after the introduction of the defect the $\phi_1$ and $\phi_2$ have to satisfy a Type-I Bäcklund transformation at the defect point. Historically the way the Type-II defect was introduced was by considering two defects, one at a point $x_1$ and one at a point $x_2$, and considering a field $\phi_1$ in $(-\infty, x_1)$, a field $\Lambda$ in $(x_1, x_2)$ and a field $\phi_2$ in $(x_2, \infty)$. Then they took the limit of $x_2 \to x_1$ and obtained a Bäcklund transformation depending on two parameters and on the fields $\phi_1$, $\phi_2$ and $\Lambda$.

We thought that maybe something similar could be done using $K$ matrices (defect matrices). In chapter 3 one of the things we show is that the Type-II $K$ matrix can be obtained by the product of two Type-I $K$ matrices, i.e. $K_{Type-II}(\phi_1, \phi_2) = K_{Type-I}(\phi_1, \phi_0)K_{Type-I}(\phi_0, \phi_2)$ [29]. In the proceedings [30] we discuss the generalization for $n$-defects. And in [31] we use this idea to reconstruct the $K$ matrix for the super Sinh-Gordon equation.

Another thing we show in the next chapter is related with Miura transformations. It is well known that one can relate the mKdV and the KdV equations through Miura transformations. What we show is that this can be generalized in order to connect the whole mKdV hierarchy with the whole KdV hierarchy [29], by using a sequence of two gauge transformations. This allows us also to construct the $K$ matrix for the KdV hierarchy using the known one for the mKdV hierarchy, and therefore to have the Bäcklund transformations of all the equations of the KdV hierarchy.

The chapter 4 will be dedicated to construct explicitly the Bäcklund transformations for the super mKdV hierarchy.

# Chapter 3

# Miura and Generalized Bäcklund transformations for KdV hierarchy

This chapter is divided in three main parts. The section 3.1 is dedicated to construct the KdV hierarchy through gauge transformations applied on the Lax pair of the mKdV hierarchy. In the section 3.2 we present the construction of the $K$ matrix for the KdV hierarchy using the one for the mKdV hierarchy and the results obtained in section 3.1. In this same section, we show some solutions of the Type-I Bäcklund transformations constructed in this section. The section 3.3 is then dedicated to the construction of the Type-II K-matrix for the KdV hierarchy. Actually, we start by showing that the mKdV Type-II K-matrix can be obtained by the product of two Type-I K-matrices. And then we proceed to construct the one for the KdV hierarchy. Again, some solutions are discussed.

## 3.1 The Algebraic Formalism for KdV Hierarchy

Following the algebraic formalism described in the chapter 2 we recall that the nonlinear equations of the mKdV hierarchy can be derived from the zero curvature representation(2.2) underlined by an affine $\hat{sl}(2)$ centerless Kac-Moody algebra and using a principal gradation. With this in mind we discuss in this section the construction of the KdV using a different approach from the one discussed in the section 2.2.5. The gauge transformations will be again a powerful tool for the process. The procedure will consist in construct the KdV hierarchy starting from a mKdV hierarchy.

Consider now the *global* gauge transformation generated by

$$g_1 = \begin{pmatrix} \zeta & 1 \\ \zeta & -1 \end{pmatrix}, \quad \zeta^2 = \lambda \tag{3.1}$$

which transforms $A^{princ}_{x,mKdV} = E^{(1)} + v(x,t_N)h = \begin{pmatrix} v & 1 \\ \lambda & -v \end{pmatrix}$, into

$$A^{hom}_{x,mKdV} = g_1 \left(A^{princ}_{x,mKdV}\right) g_1^{-1} = g_1 \left(E^{(1)} + v(x,t_N)h\right) g_1^{-1} = \begin{pmatrix} \zeta & v \\ v & -\zeta \end{pmatrix}, \tag{3.2}$$

i.e., transforms the principal into homogeneous gradation, $Q_h = \zeta \frac{d}{d\zeta}$ .

A subsequent *local* Miura-gauge transformation [39], [36]

$$g_2(v,\epsilon) = \begin{pmatrix} 1 & \epsilon \\ -\epsilon v & -v + 2\epsilon\zeta \end{pmatrix}, \tag{3.3}$$

transforms $A_{x,mKdV} \to A_{x,KdV}$. i.e.,

$$A_{x,KdV} = g_2(v,\epsilon) A^{hom}_{x,mKdV} g_2^{-1}(v,\epsilon) - \partial_x g_2(v,\epsilon) g_2^{-1}(v,\epsilon) = \begin{pmatrix} \zeta & -1 \\ J & -\zeta \end{pmatrix} \tag{3.4}$$

and realizes the Miura transformation,

$$J = \epsilon\partial_x v - v^2, \quad \epsilon^2 = 1. \tag{3.5}$$

We should emphasize that for each solution $v(x,t_N)$ of the evolution equations for the mKdV hierarchy, the Miura transformation (3.5) generates two towers of solutions, $J_\epsilon(x,t_N), \quad \epsilon=\pm 1$, of the KdV hierarchy [36]. The zero curvature under the homogeneous gradation

$$[\partial_x + A_{x,KdV}, \quad \partial_{t_N} + A_{t_N,KdV}] = 0, \tag{3.6}$$

with $A_{t_N,KdV} = \tilde{\mathcal{D}}^{(N)} + \tilde{\mathcal{D}}^{(N-1)} + \cdots + \tilde{\mathcal{D}}^{(0)}, \quad \tilde{\mathcal{D}}^{(j)} \in \tilde{\mathcal{G}}^j$ yields the KdV hierarchy equations of motion. For instance

$$A_{t_3,KdV} = \begin{bmatrix} \zeta^3 + \frac{1}{2}\zeta J + \frac{1}{4}\partial_x J & -\zeta^2 - \frac{1}{2}J \\ \zeta^2 J + \frac{1}{2}\zeta\partial_x J + \frac{1}{4}\partial_x^2 J + \frac{1}{2}J^2 & -\zeta^3 - \frac{1}{2}\zeta J - \frac{1}{4}\partial_x J \end{bmatrix} \tag{3.7}$$

yields the KdV equation

$$4\partial_{t_3} J - \partial_x^3 J - 6J\partial_x J = (\epsilon\partial_x - 2v)\,[4\partial_{t_3} v - \partial_x(\partial_x^2 v - 2v^3)] = 0, \tag{3.8}$$

Similarly from $A_{t_5}$ and for $A_{t_7}$, given in the appendix, we find the Sawada-Kotera equation [12]

$$\begin{aligned} & 16\partial_{t_5} J - \partial_x^5 J - 20\partial_x J\partial_x^2 J - 10J\partial_x^3 J - 30J^2\partial_x J \\ = \;& (\epsilon\partial_x - 2v)\,[16\partial_{t_5} v - \partial_x(\partial_x^4 v - 10v^2\partial_x^2 v - 10v(\partial_x v)^2 + 6v^5)] = 0, \end{aligned} \tag{3.9}$$

and [1]

$$\begin{aligned} & 64\partial_{t_7} J - \partial_x^7 J - 70\partial_x^2 J\partial_x^3 J - 42\partial_x J\partial_x^4 J - 70(\partial_x J)^3 - 14J\partial_x^5 J \\ - \;& 280J\partial_x J\partial_x^2 J - 70J^2\partial_x^3 J - 140J^3\partial_x J \\ = \;& (\epsilon\partial_x - 2v)\,(64\partial_{t_7} v - \partial_x(\partial_x^6 v - 70(\partial_x v)^2\partial_x^2 v - 42v(\partial_x^2 v)^2 - 56v\partial_x v\partial_x^3 v \\ - \;& 14v^2\partial_x^4 v + 140v^3(\partial_x v)^2 + 70v^4\partial_x^2 v - 20v^7)) = 0 \end{aligned} \tag{3.10}$$

respectively. Eqns.( 3.8-3.10) are displayed as explicit examples as illustration of the formalism. Higher flows (time evolutions) can be systematically constructed for generic $N$ from the same formalism.

## 3.2 Bäcklund Transformation

### 3.2.1 mKdV

In this section we start by noticing that the zero curvature representation (2.2) and (3.6) are invariant under gauge transformations of the type

$$A_\mu(\phi, \partial_x\phi, \cdots) \to \tilde{A}_\mu = K^{-1}A_\mu K + K^{-1}\partial_\mu K, \tag{3.11}$$

where $A_\mu$ stands for either $A_{t_N}$ or $A_x$.

The key ingredient of this section is to consider two field configurations $\phi_1$ and $\phi_2$ embedded in $A_\mu(\phi_1)$ and $A_\mu(\phi_2)$ satisfying the zero curvature representation and assume that they are related by a Bäcklund-gauge transformation generated by $K(\phi_1,\phi_2)$ preserving the equations of motion (e.g,, zero curvature (2.2) or (3.6) ) , i.e.,

$$K(\phi_1,\phi_2)A_\mu(\phi_1) = A_\mu(\phi_2)K(\phi_1,\phi_2) + \partial_\mu K(\phi_1,\phi_2). \tag{3.12}$$

If we now consider the Lax operator $L = \partial_x + A_x$ for mKdV case within the principal gradation,

$$A_{x,mKdV} = E^{(1)} + A_0 = \begin{bmatrix} \partial_x\phi(x,t_N) & 1 \\ \lambda & -\partial_x\phi(x,t_N) \end{bmatrix} \tag{3.13}$$

is *common to all members* of the hierarchy defined by (2.2). We find that

$$K(\phi_1,\phi_2)A_{x,mKdV}(\phi_1) = A_{x,mKdV}(\phi_2)K(\phi_1,\phi_2) + \partial_x K(\phi_1,\phi_2), \tag{3.14}$$

where the Bäcklund-gauge generator $K(\phi_1,\phi_2)$ is given by [20], [28]

$$K(\phi_1,\phi_2) = \begin{bmatrix} 1 & -\frac{\beta}{2\lambda}e^{-(\phi_1+\phi_2)} \\ -\frac{\beta}{2}e^{(\phi_1+\phi_2)} & 1 \end{bmatrix} \tag{3.15}$$

[1] In general, we find $KdV(J) = (\epsilon\partial_x - 2v)mKdV(v)$.

and $\beta$ is the Bäcklund parameter. Eqn. (3.14) is satisfied provided

$$\partial_x (\phi_1 - \phi_2) = -\beta \sinh (\phi_1 + \phi_2) . \tag{3.16}$$

For the sinh-Gordon (s-g) model, the equations of motion $\partial_t \partial_x \phi_a = 2 \sinh 2\phi_a, \ a = 1, 2$ are satisfied if we further introduce the time component of the Bäcklund transformation,

$$\partial_t (\phi_1 + \phi_2) = \frac{4}{\beta} \sinh (\phi_2 - \phi_1) . \tag{3.17}$$

Eqn. (3.17) is compatible with (3.12) for $A_\mu = A_{t_N}$ with

$$A_{t,s-g} = \left[ \begin{array}{cc} 0 & \lambda^{-1} e^{-2\phi} \\ e^{2\phi} & 0 \end{array} \right] . \tag{3.18}$$

For higher graded time evolutions the time component of the Bäcklund transformation can be derived from the appropriated time component of the two dimensional gauge potential. Several explicit examples within the positive and negative graded mKdV sub-hierarchies were discussed in [20]. We now give a general argument that the Bäcklund Transformation derived from the gauge transformation (3.14) for arbitrary $N$ provides equations compatible with the eqn. of motion. Consider the zero curvature representation for certain field configuration, namely $\phi_1$, i.e.,

$$[\partial_x + A_x(\phi_1), \partial_{t_N} + A_{t_N}(\phi_1)] = 0. \tag{3.19}$$

Under the gauge transformation,

$$\begin{aligned} & K(\phi_1, \phi_2)[\partial_x + A_x(\phi_1), \partial_{t_N} + A_{t_N}(\phi_1)] K(\phi_1, \phi_2)^{-1} \\ & = [K(\partial_x + A_x(\phi_1))K^{-1}, K(\partial_{t_N} + A_{t_N}(\phi_1))K^{-1}] \\ & = [\partial_x + A_x(\phi_2), \partial_{t_N} + A_{t_N}(\phi_2)] = 0. \end{aligned} \tag{3.20}$$

where the last equality comes from our assumption (3.14).

The gauge transformation of the first entry in the zero curvature representation implies the x-component of the Bäcklund transformation (3.16). Since the zero curvature (3.19) and (3.20) implies that both $\phi_1$ and $\phi_2$ satisfy the same equation of motion, the gauge transformation (3.14) for $A_\mu = A_{t_N}$ of the second entry in (3.20) generates the time component of BT which, by construction has to be consistent with the equations of motion with respect to time $t_N$.

### 3.2.2 KdV

In order to extend the same philosophy to the KdV hierarchy recall the fact that the two dimensional gauge potential $A_{x,KdV}$ can be obtained by Miura-gauge transformation from the homogeneous mKdV gauge potentials $A^{hom}_{mKdV}$ as in (3.4), i.e.,

$$A_{x,KdV}(J) = g_2(v, \epsilon) g_1 \left( A_{x,mKdV}(v) \right) g_1^{-1} g_2^{-1}(v, \epsilon) - \partial_x g_2(v, \epsilon) g_2^{-1}(v, \epsilon), \tag{3.21}$$

where $v = \partial_x \phi(x, t_N)$ By assuming (3.12) for the KdV hierarchy, i.e.,

$$\tilde{K}(J_1, J_2) A_{x,KdV}(J_1) = A_{x,KdV}(J_2) \tilde{K}(J_1, J_2) + \partial_x \tilde{K}(J_1, J_2). \tag{3.22}$$

the Bäcklund-gauge transformation for the KdV hierarchy $\tilde{K}(J_1, J_2)$ constructed in terms of $K(\phi_1, \phi_2)$ can be written as

$$\tilde{K} = g_2(v_2, \epsilon_2) \left( g_1 K(\phi_1, \phi_2) g_1^{-1} \right) g_2(v_1, \epsilon_1)^{-1}. \tag{3.23}$$

At this stage we should recall that for each solution of the mKdV hierarchy $v$, the Miura transformation (3.5) generates two solutions, $J_{\epsilon_i} = \epsilon_i \partial_x v_i - v_i^2, \quad \epsilon = \pm 1$ satisfying the associated equation of motion of the KdV hierarchy. This is precisely why we assume $\epsilon_1$ and $\epsilon_2$ in eqn. (3.23) independent. In terms of mKdV variables

$v_i = \partial_x \phi_i$, $\tilde{K}$ is given for the particular case where $\epsilon_1 = -\epsilon_2 \equiv \epsilon$ and denote $\tilde{K}(J_1, J_2) = \tilde{K}(\epsilon_1 = -\epsilon_2 \equiv \epsilon)$ [2],

$$\begin{aligned}
\tilde{K}(J_1,J_2)_{11} &= 1 - \frac{v_1 \epsilon}{\zeta} - \frac{\beta}{4\zeta}(1-\epsilon)e^{-p} - \frac{\beta}{4\zeta}(1+\epsilon)e^{p} \\
\tilde{K}(J_1,J_2)_{12} &= -\frac{1}{\zeta} \\
\tilde{K}(J_1,J_2)_{21} &= \frac{\beta}{4\zeta}\left(-v_1(1+\epsilon) + v_2(1-\epsilon)\right)e^{-p} \\
&\quad + \frac{\beta}{4\zeta}\left(v_1(1-\epsilon) - v_2(1+\epsilon)\right)e^{p} - \frac{v_1 v_2}{\zeta} \\
\tilde{K}(J_1,J_2)_{22} &= -1 - \frac{v_2 \epsilon}{\zeta} - \frac{\beta}{4\zeta}(1+\epsilon)e^{-p} - \frac{\beta}{4\zeta}(1-\epsilon)e^{p}
\end{aligned} \tag{3.24}$$

where $p = \phi_1 + \phi_2$. Substituting in eqn. (3.22) we find the following equations:

- Matrix element 11:

$$\zeta^{-1} \quad : \quad J_1 - \epsilon \partial_x v_1 - \frac{1}{2}\beta v_1 (e^{p} - e^{-p}) + v_1 v_2 = 0 \tag{3.25}$$

- Matrix element 12:

$$\zeta^{-1} \quad : \quad v_1 - v_2 + \frac{\beta}{2}(e^{p} - e^{-p}) = 0 \tag{3.26}$$

- Matrix element 21:

$$\begin{aligned}
\zeta^{0} : \quad & J_1 + J_2 + \frac{\beta v_1}{2}(1+\epsilon)e^{-p} - \frac{\beta v_1}{2}(1-\epsilon)e^{p} \\
& - \frac{\beta v_2}{2}(1-\epsilon)e^{-p} + \frac{\beta v_2}{2}(1+\epsilon)e^{p} + 2 v_1 v_2 = 0
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\zeta^{-1} : \quad & \epsilon(J_1 v_2 - J_2 v_1) - v_1 \partial_x v_2 - v_2 \partial_x v1 \\
& - \frac{\epsilon\beta}{2} v_1 v_2 (e^{p} - e^{-p}) = 0
\end{aligned} \tag{3.28}$$

- Matrix element 22:

$$\zeta^{-1} \quad - J_2 - \epsilon \partial_x v_2 - \frac{\beta v_2}{2}(e^{p} - e^{-p}) = 0. \tag{3.29}$$

Using the mixed Miura transformation, i.e., $\epsilon_2 = -\epsilon_1 \equiv \epsilon$,

$$J_1 = \epsilon \partial_x v_1 - v_1^2, \qquad J_2 = -\epsilon \partial_x v_2 - v_2^2 \tag{3.30}$$

together with the mKdV Bäcklund transformation (3.16)

$$v_1 - v_2 = -\frac{\beta}{2}(e^{p} - e^{-p}), \tag{3.31}$$

we find that eqns. (3.25), (3.26), (3.28) and (3.29) are identically satisfied. Defining the new variable $Q$ and taking into account the Bäcklund eqn. (3.31) we find the following equality

$$\frac{1}{2}Q \;=\; \epsilon v_1 + \frac{\beta}{4}(1-\epsilon)e^{-p} + \frac{\beta}{4}(1+\epsilon)e^{p} \tag{3.32}$$

$$\;=\; \epsilon v_2 + \frac{\beta}{4}(1+\epsilon)e^{-p} + \frac{\beta}{4}(1-\epsilon)e^{p}. \tag{3.33}$$

[2] Notice that $\tilde{K}$ is given in terms of mKdV variables $v_1, v_2$ and we need to rewrite it in terms of KdV variables $J_1, J_2$. This requires solving Riccati eqn. $v = v(J)$ (3.5)

Eliminating $v_1$ and $v_2$ from eqns (3.32) and (3.33) we find

$$v_1 = \frac{\epsilon}{2}Q - \frac{\beta}{4}(\epsilon-1)e^{-p} - \frac{\beta}{4}(\epsilon+1)e^{p} \tag{3.34}$$

$$v_2 = \frac{\epsilon}{2}Q - \frac{\beta}{4}(\epsilon+1)e^{-p} - \frac{\beta}{4}(\epsilon-1)e^{p} \tag{3.35}$$

and henceforth

$$\frac{\beta}{4}\left(-v_1(1+\epsilon)+v_2(1-\epsilon)\right)e^{-p} + \frac{\beta}{4}\left(v_1(1-\epsilon)-v_2(1+\epsilon)\right)e^{p} - v_1v_2 = \frac{\beta^2}{4} - \frac{Q^2}{4}. \tag{3.36}$$

Eqn. (3.27) then becomes

$$J_1 + J_2 = \frac{\beta^2}{2} - \frac{Q^2}{2}. \tag{3.37}$$

From (3.32) and (3.33) we find that

$$Q = \epsilon(v_1+v_2) + \frac{\beta}{2}(e^{p}+e^{-p}) \tag{3.38}$$

Acting with $\partial_x$ in (3.38) and using (3.30) and (3.31),

$$\begin{aligned}\partial_x Q &= \epsilon\partial_x(v_1+v_2) + \frac{\beta}{2}(v_1+v_2)(e^{p}-e^{-p}) \\ &= \epsilon\partial_x(v_1+v_2) - (v_1-v_2)(v_1+v_2) \\ &= J_1 - J_2 \\ &= \partial_x\left(\omega_1-\omega_2\right)\end{aligned} \tag{3.39}$$

where we have used $J_i \equiv \partial_x w_i, i=1,2$. It therefore follows that

$$Q = w_1 - w_2 \tag{3.40}$$

and the Bäcklund transformation for the spatial component of the KdV equation becomes,

$$J_1 + J_2 = \partial_x P = \frac{\beta^2}{2} - \frac{(w_1-w_2)^2}{2}, \qquad P = w_1 + w_2. \tag{3.41}$$

which is in agreement with the Bäcklund transformation proposed in [11] and with [40].

In the new variable $Q$ defined in (3.32) and (3.33) we rewrite the gauge-Bäcklund transformation $\tilde{K}(J_1,J_2)$ in (3.24) as

$$\tilde{K}(J_1,J_2,\beta) = -\frac{1}{\zeta}\begin{pmatrix} -\zeta+\frac{1}{2}Q & 1 \\ \frac{-\beta^2}{4}+\frac{1}{4}Q^2 & \zeta+\frac{1}{2}Q\end{pmatrix}, \tag{3.42}$$

Other cases with $\epsilon_1 = \epsilon_2 = \pm 1$ lead to trivial Bäcklund transformations in the sense that (3.22) for $\tilde{K}(\pm1,\pm1)$ is trivially satisfied for mKdV Bäcklund and Miura transformations (3.16) and (3.5). There is no new equation relating the two KdV fields $J_1$ and $J_2$. From now on we shall only consider $\tilde{K}(+1,-1) \equiv \tilde{K}$ given in (3.42) and Miura transformation given by (3.30)

We now discuss the extension of the Bäcklund transformation to the time component of the KdV hierarchy. Notice that in the zero curvature representation the spatial component of the two dimensional gauge potential $A_x$ is the same for all flows and therefore universal among the different evolution equations. They differ from the time component $A_{t_N}$ written according to the algebraic graded structure and parametrized by the integer $N$.

$$A_{t_N,KdV} = \tilde{\mathcal{D}}^{(N)} + \tilde{\mathcal{D}}^{(N-1)} + \cdots + \tilde{\mathcal{D}}^{(0)}, \quad \tilde{\mathcal{D}}^{(j)} \in \tilde{\mathcal{G}}^{j}. \tag{3.43}$$

The Bäcklund-gauge transformation (3.42) acting on the potentials $A_{t_3,KdV}$, $A_{t_5,KdV}$ and $A_{t_7,KdV}$ given by eqns. (A.2)- (A.4) of the appendix leads to the following Bäcklund equations respectively

$$4\partial_{t_3} P = -Q\partial_x^2 Q + \frac{1}{2}\left((\partial_x Q)^2 + 3(\partial_x P)^2\right) \tag{3.44}$$

$$\begin{aligned} 16\partial_{t_5} P &= -Q\partial_x^4 Q + \partial_x Q\partial_x^3 Q + 5\partial_x P\partial_x^3 P \\ &+ \frac{1}{2}\left(5(\partial_x^2 P)^2 - (\partial_x^2 Q)^2\right) + \frac{5}{2}\partial_x P\left((\partial_x P)^2 + 3(\partial_x Q)^2\right) \end{aligned} \tag{3.45}$$

$$\begin{aligned} 64\partial_{t_7} P &= -Q\partial_x^6 Q + \partial_x Q\partial_x^5 Q + 7\partial_x P\partial_x^5 P - \partial_x^2 Q\partial_x^4 Q + 14\partial_x^2 P\partial_x^4 P \\ &+ 35\partial_x Q\partial_x^2 Q\partial_x^2 P + 35\partial_x P\partial_x Q\partial_x^3 Q + \frac{21}{2}(\partial_x^3 P)^2 + \frac{1}{2}(\partial_x^3 Q)^2 \\ &+ \frac{35}{2}\partial_x P\left((\partial_x^2 P)^2 + (\partial_x^2 Q)^2\right) + \frac{35}{2}\partial_x^3 P\left((\partial_x P)^2 + (\partial_x Q)^2\right) \\ &+ \frac{105}{4}(\partial_x P)^2(\partial_x Q)^2 + \frac{35}{8}(\partial_x P)^4 + \frac{35}{8}(\partial_x Q)^2, \end{aligned} \tag{3.46}$$

where $\partial_x P = J_1 + J_2$. Equations (3.41) and (3.44) coincide with the Bäcklund transformation proposed in [11] for the KdV equation. Equations (3.41) and (3.45) correspond to those derived for the Sawada-Kotera equation in [40] [3]. In the appendix we have checked the consistency between the spatial, (3.41) and time components (3.44) -(3.46) of the Bäcklund transformations for $N = 3, 5$ and 7. By direct calculation, using software Mathematica, we indeed recover the evolution equations (3.8)-(3.10). We would like to point out that our method is systematic and provides the Bäcklund transformations for arbitrary time evolution in terms of its time component 2-d gauge potential $A_{t_N,KdV}$ in terms of graded subspaces $\tilde{D}^{(i)}, i = 0, \cdots N$. The examples given above for $t_3, t_5$ and $t_7$ just illustrate the potential of the formalism.

### 3.2.3 Examples

- Vacuum - One soliton solution

  Consider $\phi_1 = 0$ and $\phi_2 = ln(\frac{1+\rho}{1-\rho})$, $\quad \rho = e^{2kx+2k^N t_N}, N = 3, 5, 7$ two solutions of the mKdV hierarchy. The mixed Miura transformation yields

$$J_+^1 = \partial_x^2\phi_1 - (\partial_x\phi_1)^2 = 0, \qquad J_-^2 = -\partial_x^2\phi_2 - (\partial_x\phi_2)^2 \tag{3.47}$$

  Integrating to obtain $J = \partial_x w$ we find

$$w_1 = 0, \qquad w_2 = -\frac{4k}{1+\rho} + 2k \tag{3.48}$$

  Type-I Bäcklund transformation $\partial_x(w_1 + w_2) = \frac{\beta^2}{2} - \frac{1}{2}(w_1 - w_2)^2$ is satisfied by (3.48) for $\beta = \pm 2k$

- Scattering of two One-soliton Solutions

  Consider the one-soliton of the mKdV hierarchy given by

$$\phi_i = ln\left(\frac{1 + R_i\rho}{1 - R_i\rho}\right), i = 1, 2 \qquad \rho = e^{2kx+2k^N t_N}, \quad N = 3, 5, 7\cdots \tag{3.49}$$

  Miura transformation generates two one-soliton solutions of the KdV hierarchy, namely

$$J_+^1 = \partial_x^2\phi_1 - (\partial_x\phi_1)^2; \tag{3.50}$$
$$J_-^2 = -\partial_x^2\phi_2 - (\partial_x\phi_2)^2; \tag{3.51}$$

  leading to

$$w_1 = -\frac{4k}{1 + R_1\rho} + 2k, \qquad w_2 = -\frac{4k}{1 - R_2\rho} + 2k \tag{3.52}$$

  The Type-I Bäcklund transformation is satisfied for $R_1 = R_2$. Notice that although $R_1 = R_2$, $J_1$ and $J_2$ correspond to different solutions due to opposite $\epsilon$-sings in the Miura transformation.

[3] Notice that there are typos in eqn. (45.11) of ref. [40]

- One-Soliton into Two-Soliton Solution

  Taking $\phi_1$ given by the one-soliton solution (3.49) and $\phi_2$ by

$$\phi_2 = ln\left(\frac{1+\delta(\rho_1-\rho_2)-\rho_1\rho_2}{1-\delta(\rho_1-\rho_2)-\rho_1\rho_2}\right), \quad \delta = \frac{k_1+k_2}{k_1-k_2} \tag{3.53}$$

  leading to

$$w_2 = -\frac{2(k_1^2-k_2^2)(1+\rho_1)(1+\rho_2)}{k_1-k_2-(k_1+k_2)(\rho_1-\rho_2)-(k_1-k_2)\rho_1\rho_2} \tag{3.54}$$

  where $\rho_i = e^{2k_i x+2k_i^N t_N}, i=1,2$ satisfy the Type-I Bäcklund transformation for $\beta = \pm 2k_2$.

All these verifications were made in the software Mathematica.

## 3.3 Fusing and Type-II Bäcklund Transformation

In this section we shall consider the composition of two gauge-Bäcklund transformations leading to the Type-II Bäcklund transformation. Let us consider a situation in which we start with a Bäcklund relation transforming solution $v_1$ into another solution $v_0$. A second subsequent Bäcklund relation transforms $v_0$ into $v_2$. Such algebraic relation for the mKdV hierarchy is described by

$$K^{II}(v_1,v_0,v_2) = K(v_2,v_0)K(v_0,v_1) \tag{3.55}$$

where $K(v_i,v_j)$ is given in (3.15) with $\beta=\beta_{ij}$. It leads to

$$K^{II}(v_1,v_0,v_2) = \begin{bmatrix} 1+\frac{\beta_{10}\beta_{02}}{4\lambda}e^{q} & \frac{e^{-\phi_0}}{2\lambda}(\beta_{01}e^{-\phi_1}+\beta_{02}e^{-\phi_2}) \\ -\frac{1}{2}e^{\phi_0}(\beta_{01}e^{\phi_1}+\beta_{02}e^{\phi_2}) & 1+\frac{\beta_{10}\beta_{02}}{4\lambda}e^{-q} \end{bmatrix} \tag{3.56}$$

where $q=\phi_1-\phi_2$ and $\sigma^2 = -\frac{4}{\beta_{10}\beta_{02}}$. Inserting the following identity

$$(\beta_{01}e^{\phi_1}+\beta_{02}e^{\phi_2})(\beta_{01}e^{-\phi_1}+\beta_{02}e^{-\phi_2}) = \beta_{01}\beta_{02}(\eta+e^{q}+e^{-q}) \tag{3.57}$$

where $\eta = \frac{\beta_{10}^2+\beta_{02}^2}{\beta_{10}\beta_{02}}$. Defining $\Lambda = -\phi_0 - \ln(\beta_{02}e^{-\phi_1}+\beta_{01}e^{-\phi_2}) - \ln\frac{\sigma}{4}$, eqn. (3.56) becomes

$$K^{II}(v_1,v_0,v_2) = \begin{bmatrix} 1-\frac{1}{\sigma^2\lambda}e^{q} & \frac{e^{\Lambda-p}}{2\lambda\sigma}(e^{q}+e^{-q}+\eta) \\ -\frac{2}{\sigma}e^{p-\Lambda} & 1-\frac{1}{\lambda\sigma^2}e^{-q} \end{bmatrix}. \tag{3.58}$$

Eqn. (3.14) with $K^{II}(v_1,v_0,v_2)$ leads to the following Bäcklund equations

$$\partial_x q = -\frac{1}{2\sigma}e^{\Lambda-p}(e^{q}+e^{-q}+\eta) - \frac{2}{\sigma}e^{p-\Lambda} \tag{3.59}$$

$$\partial_x \Lambda = \frac{1}{2\sigma}e^{\Lambda-p}(e^{q}-e^{-q}). \tag{3.60}$$

Eqns. (3.59) and (3.60) coincide with the $x$-component of the Type-II Bäcklund transformation proposed for the sine-gordon model in [18]. Considering now the time component of the 2-d gauge potential for $t=t_3$ , (i.e., for the mKdV equation),

$$\begin{aligned} A_{t_3} &= \lambda E_\alpha + \lambda^2 E_{-\alpha} + v\lambda h + \frac{1}{2}(\partial_x v - v^2)E_\alpha - \frac{1}{2}(\partial_x v + v^2)\lambda E_{-\alpha} \\ &+ \frac{1}{4}(\partial_x^2 v - 2v^3)h, \end{aligned} \tag{3.61}$$

we find from eqn. (3.12),

$$\begin{aligned} 16\sigma^3\partial_{t_3} q &= e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right)\left[2\sigma^2(\partial_x^2 p+\partial_x^2 q)+\sigma^2\left(\partial_x p+\partial_x q\right)^2 - 8e^{q}\right] + \\ &+ 4e^{p-\Lambda}\left[-2\sigma^2(\partial_x^2 p+\partial_x^2 q)+\sigma^2\left(\partial_x p+\partial_x q\right)^2 - 8e^{-q}\right] + \\ &+ 16\sigma\partial_x p\left(e^{q}+e^{-q}+\eta\right) \end{aligned} \tag{3.62}$$

together with

$$4\sigma\partial_{t_3}\Lambda = (v_1^2+\partial_x v_1)e^{\Lambda-q-p} - (v_2^2+\partial_x v_2)e^{\Lambda+q-p}, \tag{3.63}$$

which is compatible with equations of motion for the mKdV model. These Type-II Bäcklund equations (3.59)-(3.63) coincide with those derived in detail in ref. [20] where $x \to x_+, t \to x_-$ and was extended to all positive higher graded equation within the mKdV hierarchy [4]. In the case of the KdV hierarchy

$$\tilde{K}^{typeII}(J_1,J_0,J_2) = \tilde{K}(J_2,J_0,\beta_{02})\tilde{K}(J_0,J_1,\beta_{01}) \tag{3.64}$$

where

$$\tilde{K}(J_j,J_i,\beta_{ij}) = -\frac{1}{\zeta}\begin{bmatrix} -\zeta+\frac{1}{2}Q_{ij} & 1 \\ -\frac{\beta_{ij}^2}{4}+\frac{1}{4}Q_{ij}^2 & \zeta+\frac{1}{2}Q_{ij}\end{bmatrix}.$$

Such transformation can be interpreted as an extended Bäcklund transformation dubbed Type-II Bäcklund transformation (see [18]). Explicitly we find directly from (3.64)

$$\begin{aligned}
[\tilde{K}^{II}(J_1,J_0,J_2)]_{11} &= 1-\frac{1}{2\zeta}Q-\frac{(\beta_++\beta_-)}{2\zeta^2}+\frac{Q}{8\zeta^2}(Q+P-2\Omega)\\
[\tilde{K}^{II}(J_1,J_0,J_2)]_{12} &= \frac{1}{2\zeta^2}Q\\
[\tilde{K}^{II}(J_1,J_0,J_2)]_{22} &= 1+\frac{1}{2\zeta}Q-\frac{(\beta_+-\beta_-)}{2\zeta^2}+\frac{Q}{8\zeta^2}(Q-P+2\Omega)\\
[\tilde{K}^{II}(J_1,J_0,J_2)]_{21} &= -\frac{\beta_+}{4\zeta^2}Q+\frac{\beta_-}{4\zeta^2}(P-2\Omega)+\frac{Q}{8\zeta^2}(-\Omega^2+\Omega P-\frac{P^2}{4}+\frac{Q^2}{4})\\
&- \frac{\beta_-}{\zeta}+\frac{Q}{4\zeta}(P-2\Omega)
\end{aligned} \tag{3.65}$$

where $Q = Q_{10}+Q_{02} = w_1-w_2$, $P = Q_{10}-Q_{02}+2\Omega = w_1+w_2$, $\Omega = w_0$, $4\beta_\pm = \beta_{01}^2 \pm \beta_{02}^2$ and $Q_{ij} = w_i - w_j$.

Acting with $\tilde{K}^{typeII}(J_1,J_0,J_2)$ in (3.22) we find the Bäcklund transformation for the KdV equation, i.e.,

$$\begin{aligned}
\partial_x Q &= 2\beta_- - \frac{1}{2}PQ+\Omega Q,\\
\partial_x(2\Omega+P) &= 2\beta_+ - \frac{1}{4}P^2-\frac{1}{4}Q^2-\Omega^2+\Omega P.
\end{aligned} \tag{3.66}$$

Similarly for the time component gauge potential (A.2) we find

$$\begin{aligned}
\partial_{t_3}Q &= \frac{1}{2}\partial_x P\partial_x Q+\frac{1}{2}\partial_x\Omega\partial_x Q+\frac{1}{4}Q\partial_x^2\Omega+\frac{1}{4}\Omega\partial_x^2 Q-\frac{P}{8}\partial_x^2 Q-\frac{Q}{8}\partial_x^2 P\\
\partial_{t_3}(2\Omega+P) &= \frac{1}{4}(\partial_x P)^2+\frac{1}{4}(\partial_x Q)^2+\frac{1}{2}\partial_x P\partial_x\Omega+(\partial_x\Omega)^2-\frac{P}{8}\partial_x^2 P-\frac{Q}{8}\partial_x^2 Q\\
&+ \frac{1}{4}P\partial_x^2\Omega+\frac{1}{4}\Omega\partial_x^2 P-\frac{1}{2}\Omega\partial_x^2\Omega.
\end{aligned} \tag{3.67}$$

Equations (3.66) and (3.67) are compatible and lead to the eqns. of motion (3.8).

Alternatively in terms of the mKdV Bäcklund transformation (3.23), eqn. (3.64) can be obtained by gauge-Miura transformation, i.e.,

$$\tilde{K}^{TypeII}(J_1,J_0,J_2) = g_2(v_2,\epsilon_2)g_1\left(K(\phi_2,\phi_0)\mathbb{I}K(\phi_0,\phi_1)\right)g_1^{-1}g_2(v_1,\epsilon_1)^{-1}$$

where we may introduce the identity element, $\mathbb{I} = g_1^{-1}g_2(v_0,\epsilon_0)^{-1}g_2(v_0,\epsilon_0)g_1$ depending upon an arbitrary $\epsilon$-sign, say, $\epsilon_0$. As argued when establishing (3.23), we are considering transitions with opposite $\epsilon$-signs such that $\epsilon_1 = -\epsilon_0 = \epsilon$ and $\epsilon_0 = -\epsilon_2 = -\epsilon$. It therefore follows that

$$\begin{aligned}
\tilde{K}^{TypeII}(J_1,J_0,J_2) &= g_2(v_2,\epsilon)g_1\left[K(\phi_2,\phi_0)K(\phi_0,\phi_1)\right]g_1^{-1}g_2(v_1,\epsilon)^{-1}\\
&= g_2(v_2,\epsilon)g_1K^{II}(\phi_2,\phi_1)g_1^{-1}g_2(v_1,\epsilon)^{-1}
\end{aligned} \tag{3.68}$$

[4] Observe that the Type-II Bäcklund transformation via gauge transformation was constructed in [19] where a solution presented there was chosen to reproduce the Bäcklund transformation proposed in [18]. Here we choose a gauge transformation solution of [19] that reproduces [28].

The equation (3.68) yields

$$
\begin{aligned}
\left[\tilde{K}^{TypeII}\left(J_1,J_0,J_2\right)\right]_{11} &= 1+\frac{(1+\epsilon)}{4\sigma\zeta}e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right)-\frac{(1-\epsilon)}{\sigma\zeta}e^{p-\Lambda}\\
&-\frac{(1-\epsilon)}{2\sigma^2\zeta^2}e^{-q}-\frac{(1+\epsilon)}{2\sigma^2\zeta^2}e^{q}-\frac{(1-\epsilon)}{\sigma\zeta^2}v_1e^{p-\Lambda}\\
&-\frac{(1+\epsilon)}{4\sigma\zeta^2}v_1e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right),\\
\left[\tilde{K}^{TypeII}\left(J_1,J_0,J_2\right)\right]_{12} &= \frac{(1-\epsilon)}{\sigma\zeta^2}e^{p-\Lambda}-\frac{(1+\epsilon)}{4\sigma\zeta^2}e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right),
\end{aligned}
$$

$$
\begin{aligned}
\left[\tilde{K}^{TypeII}\left(J_1,J_0,J_2\right)\right]_{21} &= -\frac{\epsilon}{\sigma^2\zeta}\left(e^{q}-e^{-q}\right)-\frac{(1-\epsilon)}{\sigma\zeta}(v_1+v_2)e^{p-\Lambda}\\
&-\frac{(1+\epsilon)}{4\sigma\zeta}\left(v_1+v_2\right)e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right)\\
&+\frac{(v_1+v_2)}{2\sigma^2\zeta^2}\left(e^{q}-e^{-q}\right)-\frac{\epsilon(v_1-v_2)}{2\sigma^2\zeta^2}\left(e^{q}+e^{-q}\right)\\
&+\frac{(1+\epsilon)}{4\sigma\zeta^2}v_1v_2e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right)-\frac{(1-\epsilon)}{\sigma\zeta^2}v_1v_2e^{p-\Lambda},\\
\left[\tilde{K}^{TypeII}\left(J_1,J_0,J_2\right)\right]_{22} &= 1-\frac{(1+\epsilon)}{4\sigma\zeta}e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right)+\frac{(1-\epsilon)}{\sigma\zeta}e^{p-\Lambda}\\
&-\frac{(1+\epsilon)}{2\sigma^2\zeta^2}e^{-q}-\frac{(1-\epsilon)}{2\sigma^2\zeta^2}e^{q}+\frac{(1-\epsilon)}{\sigma\zeta^2}v_2e^{p-\Lambda}\\
&+\frac{(1+\epsilon)}{4\sigma\zeta^2}v_2e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right).
\end{aligned}
\tag{3.69}
$$

Comparing the matrix elements of (3.64) with (3.68) we find the following relations between the mKdV and KdV variables:

- matrix element 11

$$
\zeta^{-1}: \quad -\frac{1}{2}Q=\frac{(1+\epsilon)}{4\sigma}e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right)-\frac{(1-\epsilon)}{\sigma}e^{p-\Lambda} \tag{3.70}
$$

$$
\begin{aligned}
\zeta^{-2}: \quad -\frac{(\beta_++\beta_-)}{2}+\frac{Q}{8}(Q+P-2\Omega) &= -\frac{(1-\epsilon)}{2\sigma^2}e^{-q}-\frac{(1+\epsilon)}{2\sigma^2\zeta^2}e^{q}\\
-\frac{(1-\epsilon)}{\sigma\zeta^2}v_1e^{p-\Lambda}&-\frac{(1+\epsilon)}{4\sigma\zeta^2}v_1e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right)
\end{aligned}
\tag{3.71}
$$

- matrix element 21:

$$
\begin{aligned}
\zeta^{-1}: \quad -\beta_-+\frac{Q}{4}(P-2\Omega) &= -\frac{\epsilon}{\sigma^2}\left(e^{q}-e^{-q}\right)-\frac{(1-\epsilon)}{\sigma}(v_1+v_2)e^{p-\Lambda}\\
&-\frac{(1+\epsilon)}{4\sigma\zeta}\left(v_1+v_2\right)e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right)
\end{aligned}
\tag{3.72}
$$

$$
\begin{aligned}
\zeta^{-2}: \quad -\frac{\beta_+}{4}Q+\frac{\beta_-}{4}(P-2\Omega)+\frac{Q}{8}(-\Omega^2+\Omega P-\frac{P^2}{4}+\frac{Q^2}{4}) &=\\
+\frac{(v_1+v_2)}{2\sigma^2}\left(e^{q}-e^{-q}\right)-\frac{\epsilon(v_1-v_2)}{2\sigma^2}\left(e^{q}+e^{-q}\right)&\\
+\frac{(1+\epsilon)}{4\sigma}v_1v_2e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right)-\frac{(1-\epsilon)}{\sigma}v_1v_2e^{p-\Lambda}&
\end{aligned}
\tag{3.73}
$$

- matrix element 22:

$$
\begin{aligned}
\zeta^{-2}: \quad -\frac{(\beta_+-\beta_-)}{2}+\frac{Q}{8}(Q-P+2\Omega) &= -\frac{(1+\epsilon)}{2\sigma^2}e^{-q}-\frac{(1-\epsilon)}{2\sigma^2}e^{q}\\
+\frac{(1-\epsilon)}{\sigma}v_2e^{p-\Lambda}&+\frac{(1+\epsilon)}{4\sigma}v_2e^{\Lambda-p}\left(e^{q}+e^{-q}+\eta\right).
\end{aligned}
\tag{3.74}
$$

The element 12 and the element 22 with $\zeta^{-1}$ gives us the same result of (3.70). Eliminating the mKdV variables $p, q$ and $\Lambda$ we recover the Type-II Bäcklund transformation for the KdV hierarchy (3.66) as shown in the Appendix B.

### 3.3.1 Examples and Solutions

- Vacuum - 1-soliton - Vacuum

  The first example is to consider vacuum to 1-soliton and back to vacuum again given by the following configuration,

$$w_1 = 0, \qquad w_0 = \Omega = -\frac{4k}{1+\rho(x,t_N)} + 2k, \quad w_2 = 0 \tag{3.75}$$

  with $\rho(x,t_N) = e^{2kx+2k^N t_N}$. It is straightforward to check that eqns. (3.66) and (3.67) are satisfied for $\beta_- = 0$ and $\beta_+ = 2k^2$.

- 1-soliton - 2soliton - 1-soliton

  Consider now a configuration of 1-soliton transforming into a 2-solitons solution and back to 1-soliton. It is described by

$$w_i = -\frac{4k_i}{1+\rho_i(x,t_N)} + 2k_i, \quad i = 1,2 \qquad \rho_i = e^{2k_i x + 2k_i^N t_N}, \tag{3.76}$$

$$\Omega = w_0 = -\frac{2(k_1^2 - k_2^2)(1+\rho_1)(1+\rho_2)}{k_1 - k_2 - (k_1+k_2)(\rho_1 - \rho_2) - (k_1 - k_2)\rho_1\rho_2} \tag{3.77}$$

  Eqns. (3.66) and (3.67) are satisfied for $\beta_- = k_2^2 - k_1^2$ and $\beta_+ = k_1^2 + k_2^2$.

- Vacuum - 1-soliton - 2-soliton

  Consider the solution of eqn. (3.66) and (3.67)

$$w_1 = 0, w_0 = \Omega = -\frac{4k_1}{1+\rho_1(x,t_N)} + 2k_1, \tag{3.78}$$

  and

$$w_2 = -\frac{2(k_1^2 - k_2^2)(1+\rho_1)(1+\rho_2)}{k_1 - k_2 - (k_1+k_2)(\rho_1 - \rho_2) - (k_1 - k_2)\rho_1\rho_2}, \tag{3.79}$$

  where $\rho_i = e^{2k_i x + 2k_i^N t_N}$ Eqns. (3.66) and (3.67) are satisfied for $\beta_- = k_1^2 - k_2^2$ and $\beta_+ = k_1^2 + k_2^2$.

# Chapter 4

# Super mKdV hierarchy and its Bäcklund transformations

This chapter is divided in two sections. The section 4.1 is dedicated to a review of the construction of the super mKdV hierarchy. In section 4.2 we obtain the Bäcklund transformations for the super mKdV hierarchy as a generalization of the discussion presented in the previous chapters. The explicit examples of $N = 3$ and $N = 5$ are computed.

## 4.1 The supersymmetric mKdV hierarchy

In this section we present a brief review of the systematic construction of the supersymmetric mKdV hierarchy based on the affine Kac-Moody superalgebra $\widehat{\mathcal{G}} = \widehat{sl}(2,1)$ [35]. The structure explained in the first chapter of the thesis and used in the second one will be again crucial here (with some adaptations to include the supersymmetric generators).

Let us start by considering the super Lie algebra $sl(2,1)$, which has four bosonic generators $\{h_1, h_2, E_{\pm\alpha_1}\}$, and four fermionic generators $\{E_{\pm\alpha_2}, E_{\pm(\alpha_1+\alpha_2)}\}$, where $\alpha_1$ is bosonic simple root and $\alpha_2$, $\alpha_1 + \alpha_2$ are fermionic simple roots. The affine $\widehat{sl}(2,1)$ structure is introduced by extending each generator $T_a \in sl(2,1)$ to $T_a^{(n)}$, where $d$ is defined by $\left[d, T_a^{(n)}\right] = nT_a^{(n)}$. The hierarchy is further specified by introducing a decomposition of the $\widehat{sl}(2,1)$ superalgebra through the definition of a constant grade one element $E^{(1)}$, where

$$E^{(2n+1)} = h_1^{(n+1/2)} + 2h_2^{(n+1/2)} - E_{\alpha_1}^{(n)} - E_{-\alpha_1}^{(n+1)}, \tag{4.1}$$

and the so called principal grading operator

$$Q_p = 2d + \frac{1}{2}h_1^{(0)}. \tag{4.2}$$

The grading operator $Q_p$ and the constant element $E^{(1)}$ decompose the affine superalgebra $\widehat{\mathcal{G}} = \oplus\widehat{\mathcal{G}}_m = \mathcal{K} \oplus \mathcal{M}$, where $m$ is the degree of the subspace $\widehat{\mathcal{G}}_m$ according to $Q_p$, $\mathcal{K} = \left\{x \in \widehat{\mathcal{G}} / [x, E^{(1)}] = 0\right\}$ is the kernel of $E^{(1)}$, and $\mathcal{M}$ its complement, in the following way

$$\begin{aligned}
\hat{\mathcal{G}}_{2n+1} &= \left\{K_1^{(2n+1)},\, K_2^{(2n+1)},\, M_1^{(2n+1)}\right\}, \\
\hat{\mathcal{G}}_{2n} &= \left\{M_2^{(2n)}\right\}, \\
\hat{\mathcal{G}}_{2n+\frac{1}{2}} &= \left\{F_2^{(2n+\frac{1}{2})},\, G_1^{(2n+\frac{1}{2})}\right\}, \\
\hat{\mathcal{G}}_{2n+\frac{3}{2}} &= \left\{F_1^{(2n+\frac{3}{2})},\, G_2^{(2n+\frac{3}{2})}\right\},
\end{aligned} \tag{4.3}$$

where the generators $F_i, G_i, K_i$, and $M_i$ are defined as linear combinations of the $\widehat{sl}(2,1)$ generators [35]. The representation of these generators is given in Appendix D.

Now the construction of the integrable hierarchy is based on the zero curvature condition (2.2). In general, $A_x$ is defined as $A_x = E^{(1)} + A_0 + A_{1/2}$ where $A_0 + A_{1/2} \in \mathcal{M}$, i.e.,

$$A_0 = uM_2^{(0)}, \qquad A_{1/2} = \sqrt{i}\bar{\psi}\, G_1^{(1/2)}. \tag{4.4}$$

Here, $u$ and $\bar{\psi}$ are the corresponding fields of the integrable hierarchy. Now, we will assume that $A_{t_N} = D^{(N)} + D^{(N-1/2)} + ... + D^{(1/2)} + D^{(0)}$ for the positive hierarchy, where $D^{(m)}$ has grade $m$. Then, the equation to be solved reads

$$\left[\partial_x + E^{(1)} + A_0 + A_{1/2}, \partial_{t_N} + D^{(N)} + D^{(N-1/2)} + ... + D^{(1/2)} + D^{(0)}\right] = 0. \tag{4.5}$$

The solving method consists on splitting the above equation grade by grade, which leads us to the following set of relations,

$$\begin{aligned}
(N+1): \quad & \left[E^{(1)}, D^{(N)}\right] = 0, \\
(N+1/2): \quad & \left[E^{(1)}, D^{(N-1/2)}\right] + \left[A_{1/2}, D^{(N)}\right] = 0, \\
(N): \quad & \partial_x D^N + \left[A_0, D^{(N)}\right] + \left[E^{(1)}, D^{(N-1)}\right] + \left[A_{1/2}, D^{(N-1/2)}\right] = 0, \\
& \vdots \\
(1): \quad & \partial_x D^{(1)} + \left[A_0, D^{(1)}\right] + \left[E^{(1)}, D^{(0)}\right] + \left[A_{1/2}, D^{(1/2)}\right] = 0, \\
(1/2): \quad & \partial_x D^{(1/2)} + \left[A_0, D^{(1/2)}\right] + \left[A_{1/2}, D^{(0)}\right] - \partial_{t_N} A_{1/2} = 0, \\
(0): \quad & \partial_x D^{(0)} + \left[A_0, D^{(0)}\right] - \partial_{t_N} A_0 = 0.
\end{aligned} \tag{4.6}$$

Note that, the image part of the zero and the one-half grade components of (4.6) yields the time evolution for the fields introduced in eq. (4.4). Now, it is possible to expand each term $D^{(m)}$ by using the generators in eq. (4.3), as follows

$$\begin{aligned}
D^{(2n+1)} &= \tilde{a}_{2n+1} K_1^{(2n+1)} + \tilde{b}_{2n+1} K_2^{(2n+1)} + \tilde{c}_{2n+1} M_1^{(2n+1)}, \\
D^{(2n)} &= \tilde{a}_{2n} M_2^{(2n)}, \\
D^{(2n+\frac{1}{2})} &= \tilde{a}_{2n+\frac{1}{2}} F_2^{(2n+\frac{1}{2})} + \tilde{b}_{2n+\frac{1}{2}} G_1^{(2n+\frac{1}{2})}, \\
D^{(2n+\frac{3}{2})} &= \tilde{a}_{2n+\frac{3}{2}} F_1^{(2n+\frac{3}{2})} + \tilde{b}_{2n+\frac{3}{2}} G_2^{(2n+\frac{3}{2})},
\end{aligned} \tag{4.7}$$

where the $\tilde{a}_m, \tilde{b}_m$, and $\tilde{c}_m$ are functionals of the fields $u$ and $\bar{\psi}$. Substituting this parametrization in eq. (4.6), one can solve recursively for all $D^{(m)}, m = 0, \cdots N$. Notice that the Lax component $A_x$ does not depend on the index $N$ and will be the same for the entire hierarchy. It takes the following form (see for instance [26, 34])

$$A_x = \left(\begin{array}{cc|c}
\lambda^{1/2} - \partial_x\phi & -1 & \sqrt{i}\,\bar{\psi} \\
-\lambda & \lambda^{1/2} + \partial_x\phi & \sqrt{i}\,\lambda^{1/2}\,\bar{\psi} \\
\hline
\sqrt{i}\,\lambda^{1/2}\,\bar{\psi} & \sqrt{i}\,\bar{\psi} & 2\,\lambda^{1/2}
\end{array}\right), \tag{4.8}$$

where we have redefined $u = -\partial_x\phi$. This parametrization establishes the explicit relationship between the relativistic (sinh-Gordon) and non-relativistic (mKdV) field variables.

In what follows we will apply the procedure and consider explicit solutions of the integrable hierarchy equations (4.6) for the simplest members. For the $N = 3$ member, we find that the solution for the Lax component $A_{t_3} = D^{(3)} + D^{(5/2)} + D^{(2)} + D^{(3/2)} + D^{(1)} + D^{(1/2)} + D^{(0)}$, is given by

$$A_{t_3} = \left(\begin{array}{cc|c}
a_0 + \lambda^{1/2}a_{1/2} - \lambda\partial_x\phi + \lambda^{3/2} & a_+ - \lambda & \mu_+ + \lambda^{1/2}\nu_+ + \lambda\sqrt{i}\bar{\psi} \\
-\lambda a_- - \lambda^2 & -a_0 + \lambda^{1/2}a_{1/2} + \lambda\partial_x\phi + \lambda^{3/2} & \lambda^{1/2}\mu_- + \lambda\nu_- + \lambda^{3/2}\sqrt{i}\bar{\psi} \\
\hline
\lambda^{1/2}\mu_- - \lambda\nu_- + \lambda^{3/2}\sqrt{i}\bar{\psi} & \mu_+ - \lambda^{1/2}\nu_+ + \lambda\sqrt{i}\bar{\psi} & 2\lambda^{1/2}a_{1/2} + 2\lambda^{3/2}
\end{array}\right), \tag{4.9}$$

where

$$\begin{aligned}
a_0 &= -\frac{1}{4}\left(\partial_x^3\phi - 2(\partial_x\phi)^3 + 3i\partial_x\phi\bar{\psi}\partial_x\bar{\psi}\right), \quad a_{1/2} = -\frac{i}{2}\bar{\psi}\partial_x\bar{\psi}, \quad a_\pm = \frac{1}{2}\left(\partial_x^2\phi \pm (\partial_x\phi)^2 \mp i\bar{\psi}\partial_x\bar{\psi}\right), \\
\mu_\pm &= \frac{\sqrt{i}}{4}\left(\partial_x^2\bar{\psi} \pm \partial_x\phi\partial_x\bar{\psi} \mp \bar{\psi}\partial_x^2\phi - 2\bar{\psi}(\partial_x\phi)^2\right), \qquad \nu_\pm = \frac{\sqrt{i}}{2}\left(\partial_x\bar{\psi} \pm \bar{\psi}\partial_x\phi\right).
\end{aligned} \tag{4.10}$$

The equations of motion, which correspond to the zero and one-half grade components of (4.6), are in this case the $\mathcal{N}=1$ supersymmetric mKdV equations, namely

$$4\partial_{t_3} u = \partial_x^3 u - 6u^2\partial_x u + 3i\bar{\psi}\partial_x\left(u\partial_x\bar{\psi}\right), \tag{4.11}$$

$$4\partial_{t_3}\bar{\psi} = \partial_x^3\bar{\psi} - 3u\partial_x\left(u\bar{\psi}\right). \tag{4.12}$$

Now, for the $N=5$ member, the solution for the Lax component $A_{t_5} = D^{(5)} + D^{(9/2)} + \cdots + D^{(0)}$ is given explicitly in appendix E. In this case, we find the following equations of motion,

$$\begin{aligned}16\partial_{t_5} u = {} & \partial_x^5 u - 10(\partial_x u)^3 - 40u(\partial_x u)(\partial_x^2 u) - 10u^2(\partial_x^3 u) + 30u^4(\partial_x u) + 5i\partial_x\bar{\psi}\partial_x(u\partial_x^2\bar{\psi}) \\ & + 5i\bar{\psi}\partial_x(u\partial_x^3\bar{\psi} - 4u^3\partial_x\bar{\psi} + \partial_x u\partial_x^2\bar{\psi} + \partial_x^2 u\partial_x\bar{\psi}),\end{aligned} \tag{4.13}$$

$$16\partial_{t_5}\bar{\psi} = \partial_x^5\bar{\psi} - 5u\partial_x(u\partial_x^2\bar{\psi} + 2\partial_x u\partial_x\bar{\psi} + \partial_x^2 u\bar{\psi}) + 10u^2\partial_x(u^2\bar{\psi}) - 10(\partial_x u)\partial_x(\partial_x u\bar{\psi}). \tag{4.14}$$

It is worth pointing out that the negative integrable hierarchy can be also constructed by considering the following zero curvature condition,

$$\left[\partial_x + E^{(1)} + A_0 + A_{1/2}, \partial_{t_{-M}} + D^{(-M)} + D^{(-M+1/2)} + \cdots + D^{(-1)} + D^{(-1/2)}\right] = 0. \tag{4.15}$$

The solutions are in general non-local, however, for the simplest case of $N=-M=-1$, we find that the Lax component $A_{t_{-1}} = D^{(-1)} + D^{(-1/2)}$, corresponds to the $\mathcal{N}=1$ sshG equation [27, 35], i.e

$$A_{t_{-1}} = \left(\begin{array}{cc|c} \lambda^{-1/2} & -\lambda^{-1}e^{2\phi} & -\lambda^{1/2}\sqrt{i}\,\psi e^{\phi} \\ -e^{-2\phi} & \lambda^{-1/2} & -\sqrt{i}\,\psi e^{-\phi} \\ \hline \sqrt{i}\,\psi e^{-\phi} & \lambda^{1/2}\sqrt{i}\,\psi e^{\phi} & 2\lambda^{-1/2}\end{array}\right). \tag{4.16}$$

In this case, the fields $\phi$, $\bar{\psi}$ and $\psi$ satisfy

$$\begin{aligned}\partial_{t_{-1}}\partial_x\phi &= 2\sinh 2\phi + 2\bar{\psi}\psi\sinh\phi, \\ \partial_{t_{-1}}\bar{\psi} &= 2\psi\cosh\phi, \\ \partial_x\psi &= 2\bar{\psi}\cosh\phi,\end{aligned} \tag{4.17}$$

the equations of motion of the $\mathcal{N}=1$ sshG model in the light-cone coordinates $(x, t_{-1})$. We note that the equations of motion for all members of the hierarchy are invariant under the following supersymmetric transformations,

$$\delta\phi = \sqrt{i}\bar{\epsilon}\,\bar{\psi}, \qquad \delta\bar{\psi} = \frac{1}{\sqrt{i}}\bar{\epsilon}\,\partial_x\phi, \tag{4.18}$$

where $\bar{\epsilon}$ is a Grassmannian parameter.

## 4.2 Super-Bäcklund transformations

In this section we derive a general method to generate the super-Bäcklund transformations (sBT) for all members of the hierarchy. We will use the defect matrix associated to the hierarchy in order to derive the sBT in components. The key ingredient is the gauge invariance of the zero curvature representation generated by the defect matrix which, in turn is again assumed to relate two field configurations.

As explicit examples we construct the super-Bäcklund transformation for the first two flows, namely, $N=3$ (smKdV) equation, and for the $N=5$ super-equation. One nice check will be to put the fermioninc fields to zero and recover the “classical” Bäcklund transformations.

Based upon the fact that the spatial Lax operator is common to all members of the mKdV hierarchy, it has been shown recently in the previous chapters that the spatial component of the Bäcklund transformation, and consequently the associated defect matrix, are also common and henceforth universal within the entire hierarchy. Here, we will extend these results to the supersymmetric mKdV hierarchy starting from the defect matrix already derived for the ($N=-1$ member), the super sinh-Gordon equation. The so-called type-I defect matrix can be written as follows [26],

$$K = \left(\begin{array}{cc|c} \lambda^{1/2} & -\frac{2}{\omega^2}e^{\phi_+}\lambda^{-1/2} & -\frac{2\sqrt{i}}{\omega}e^{\frac{\phi_+}{2}}f_1 \\ -\frac{2}{\omega^2}e^{-\phi_+}\lambda^{1/2} & \lambda^{1/2} & -\frac{2\sqrt{i}}{\omega}e^{-\frac{\phi_+}{2}}f_1\lambda^{1/2} \\ \hline \frac{2\sqrt{i}}{\omega}e^{-\frac{\phi_+}{2}}f_1\lambda^{1/2} & \frac{2\sqrt{i}}{\omega}e^{\frac{\phi_+}{2}}f_1 & \frac{2}{\omega^2}+\lambda^{1/2} \end{array}\right) \tag{4.19}$$

where $\phi_\pm = \phi_1 \pm \phi_2$, $\omega$ represents the Bäcklund parameter, and $f_1$ is an auxiliary fermionic field. The defect matrix $K$, connecting two different configurations $\phi_1$ and $\phi_2$, satisfies the following gauge equation,

$$\partial_x K = K A_x(\phi_1, \bar{\psi}_1) - A_x(\phi_2, \bar{\psi}_2)K. \tag{4.20}$$

Now, by substituting (4.8) and (4.19) in the (4.20), we get

$$\partial_x \phi_- = \frac{4}{\omega^2}\sinh(\phi_+) - \frac{2i}{\omega}\sinh(\frac{\phi_+}{2})f_1\bar{\psi}_+, \tag{4.21}$$

$$\bar{\psi}_- = \frac{4}{\omega}\cosh\left(\frac{\phi_+}{2}\right)f_1, \tag{4.22}$$

$$\partial_x f_1 = \frac{1}{\omega}\cosh(\frac{\phi_+}{2})\bar{\psi}_+. \tag{4.23}$$

the spatial part of the Bäcklund transformations, where we have denoted $\bar{\psi}_\pm = \bar{\psi}_1 \pm \bar{\psi}_2$. To derive the time component of the transformation, we consider the corresponding temporal part of the Lax pair $A_{t_n}$.

### 4.2.1 N=3

For the smKdV equation ($N = 3$), the second gauge condition reads,

$$\partial_{t_3} K = K A_{t_3}(\phi_1, \bar{\psi}_1) - A_{t_3}(\phi_2, \bar{\psi}_2)K. \tag{4.24}$$

By substituting (4.9) and (4.19) in the above equation, we obtain

$$\begin{aligned} 4\partial_{t_3}\phi_- =& \frac{i}{\omega}\left[\partial_x^2\phi_+\cosh\left(\frac{\phi_+}{2}\right) - (\partial_x\phi_+)^2\sinh\left(\frac{\phi_+}{2}\right)\right]\bar{\psi}_+ f_1 \\ &- \frac{i}{\omega}\left[\partial_x\phi_+\cosh\left(\frac{\phi_+}{2}\right)\partial_x\bar{\psi}_+ - 2\sinh\left(\frac{\phi_+}{2}\right)\partial_x^2\bar{\psi}_+\right]f_1 \\ &+ \frac{2}{\omega^2}\left[2(\partial_x^2\phi_+)\cosh\phi_+ - (\partial_x\phi_+)^2\sinh\phi_+ + i\bar{\psi}_+(\partial_x\bar{\psi}_+)\sinh\phi_+\right] \\ &- \frac{96i}{\omega^5}\left[\sinh\left(\frac{\phi_+}{2}\right) + 4\sinh^3\left(\frac{\phi_+}{2}\right) + 3\sinh^5\left(\frac{\phi_+}{2}\right)\right]\bar{\psi}_+ f_1 \\ &- \frac{32}{\omega^6}\sinh^3\phi_+, \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} 4\partial_{t_3} f_1 =& \frac{1}{2\omega}\cosh\left(\frac{\phi_+}{2}\right)\left[2\partial_x^2\bar{\psi}_+ - \bar{\psi}_+(\partial_x\phi_+)^2\right] + \frac{1}{2\omega}\sinh\left(\frac{\phi_+}{2}\right)\left[\bar{\psi}_+\partial_x^2\phi_+ - (\partial_x\phi_+)(\partial_x\bar{\psi}_+)\right] \\ &- \frac{12}{\omega^4}\sinh\phi_+\cosh^2\left(\frac{\phi_+}{2}\right)(\partial_x\phi_+)f_1 + \frac{12}{\omega^5}\sinh^2\phi_+\cosh\left(\frac{\phi_+}{2}\right)\bar{\psi}_+. \end{aligned} \tag{4.26}$$

Equations (4.21)–(4.23), and (4.25) and (4.26) correspond to the super-Bäcklund transformations for the smKdV in components. It can easily verified that they are consistent by cross-differentiating any of them.

#### Limit to bosonic case

Notice also that by setting all the fermions to zero we recover the bosonic case, i.e., the Bäcklund transformation of the mKdV [28],

$$\partial_x\phi_- = \frac{4}{\omega^2}\sinh\phi_+, \tag{4.27}$$

$$4\partial_{t_3}\phi_- = \frac{4}{\omega^2}\partial_x^2\phi_+\cosh\phi_+ - \frac{2}{\omega^2}(\partial_x\phi_+)^2\sinh\phi_+ - \frac{32}{\omega^6}\sinh^3\phi_+. \tag{4.28}$$

### 4.2.2 N=5

Now, to derive the temporal part of the super-Bäcklund transformation for the $N = 5$ member of the hierarchy we consider the corresponding Lax operator $A_{t_5}$. The gauge condition reads,

$$\partial_{t_5} K = K A_{t_5}(\phi_1, \bar{\psi}_1) - A_{t_5}(\phi_2, \bar{\psi}_2) K. \tag{4.29}$$

By solving this condition for $A_{t_5}$ given in Appendix E, we obtain

$$\begin{aligned} 16\partial_{t_5}\phi_- \;=\; & -\frac{i}{\omega}\left[c_0\,\bar{\psi}_+ + c_1\,\partial_x\bar{\psi}_+ + c_2\,\partial_x^2\bar{\psi}_+ + c_3\,\partial_x^3\bar{\psi}_+ + c_4\,\partial_x^4\bar{\psi}_+\right] f_1 \\ & + \frac{1}{\omega^2}\left[c_5 + ic_6\,\bar{\psi}_+\partial_x\bar{\psi}_+ + ic_7\,\bar{\psi}_+\partial_x^2\bar{\psi}_+ + ic_8\big(\bar{\psi}_+\partial_x^3\bar{\psi}_+ - (\partial_x\bar{\psi}_+)(\partial_x^2\bar{\psi}_+)\big)\right] \\ & - \frac{i}{\omega^5}\left[c_9\,\bar{\psi}_+ + c_{10}\,\partial_x\bar{\psi}_+ + c_{11}\,\partial_x^2\bar{\psi}_+\right] f_1 + \frac{1}{\omega^6}\left[c_{12} + i\,c_{13}\,\bar{\psi}_+\partial_x\bar{\psi}_+\right] \\ & + \frac{i}{\omega^9}\,c_{14} f_1\bar{\psi}_+ + \frac{c_{15}}{\omega^{10}}, \end{aligned} \tag{4.30}$$

$$\begin{aligned} 16\partial_{t_5} f_1 \;=\; & \frac{1}{\omega}\left[d_0\,\bar{\psi}_+ + d_1\,\partial_x\bar{\psi}_+ + d_2\,\partial_x^2\bar{\psi}_+ + d_3\,\partial_x^3\bar{\psi}_+ + d_4\,\partial_x^4\bar{\psi}_+\right] \\ & + \frac{1}{\omega^4}\left[d_6 + i\,d_5\,\bar{\psi}_+\partial_x\bar{\psi}_+\right] f_1 + \frac{1}{\omega^5}\left[d_7\,\bar{\psi}_+ + d_8\,\partial_x\bar{\psi}_+ + d_9\,\partial_x^2\bar{\psi}_+\right] \\ & + \frac{d_{10}}{\omega^8} f_1 + \frac{d_{11}}{\omega^9}\,\bar{\psi}_+, \end{aligned} \tag{4.31}$$

where $c_i$, $i = 0, .., 15$ and $d_j$, $j = 0, .., 11$ are functions depending on $\phi_+$ and its derivatives, and their explicit forms are given by (F.1)-(F.16) and (F.17)-(F.28) from Appendix F, respectively. The equations (4.21)–(4.23), and (4.30) and (4.31) correspond to the super-Bäcklund transformations for the $N = 5$ super equation. Cross differentiating (4.30) and (4.31) with respect of $x$ we recover the equations of motion (4.13) and (4.14) after using eqns. (4.21)–(4.23).

In this chapter we reviewed the construction of the super mKdV hierarchy using the graded super algebra $\hat{sl}(2,1)$ and the principal gradation. We then used the universality of the K-matrix construction along the hierarchy to explicitly construct the Bäcklund transformations for the super $N = 3$ and super $N = 5$ equations.

Although we are presenting it here,we also computed the Bäcklund transformations for those equations using the super fields formalism. The super charges were also computed, giving particular attention to the changes on the conserved charges due to the introduction of the defect [31, 33].

# Chapter 5

# Conclusions and further developments

In the Part I of the thesis we explored the algebraic structure underlying integrable hierarchies in order to find universal features of their equations. The gauge invariance of the zero curvature equation allowed the construction of a defect-gauge matrix connecting two different field configurations of the same integrable model and hence generating its Bäcklund transformations. In Chapter 2 the main result is the fact that the construction of the defect matrix depends only on the $A_x$ and therefore is universal within the hierarchy. This means that one can systematically construct the Bäcklund transformations for any equation in the mKdV hierarchy [20], [28].

The main result of the Chapter 3 is the extension of such construction to the KdV hierarchy by proposing a Miura-gauge transformation denoted by the product $g_2 g_1$ given in (3.1) and (3.3) mapping the mKdV into the KdV hierarchy (see (3.4)) [29]. A subtle point is that such Miura mapping allows a sign ambiguity such that each solution of the mKdV hierarchy defines two solutions for its KdV counterpart. The Bäcklund-gauge transformation for the KdV hierarchy is constructed by Miura-gauge transforming the Bäcklund transformation of the mKdV system as shown in (3.23). An interesting fact is that the Bäcklund transformation for the KdV hierarchy is solved by mixed Miura solutions generated by the mKdV Bäcklund solutions. A few simple explicit examples illustrate our conjecture. A more general evidence of the mixed Miura solutions is shown to agree with the Bäcklund transformation proposed in [11] for the first two KdV flows.

The composition law of two subsequent Bäcklund-gauge transformations leading to Type-II Bäcklund transformation (see (3.55)) introduced in [18] in the context of sine-Gordon and Tzitzeica models was extended to the KdV hierarchy. We have showed that the ideia of fusing to defects in the Lagrangian formalism can be translated within our construction, to the direct fusion of two KdV Bäcklund-gauge transformations in (3.64) and alternatively, the Miura transformation of mKdV Type-II Bäcklund transformation as shown in (3.68). These two approaches generate relations between the mKdV and KdV variables which were shown in the Appendix B, to be consistent. In [30] we discuss the generalization of this idea for $n$ defects for the Sinh-Gordon equation.

Another interesting point is that we explored only the positive part of the mKdV hierarchy, by using it to construct the positive part of KdV hierarchy and corresponding Bäcklund transformations. It would be very interesting to understand which kind of result it would be obtained for the negative part of the KdV hierarchy by starting with the negative part of the mKdV hierarchy, both in a the sense of hierarchy itself as well as for its Bäcklund transformations.

In the Chapter 4 we have studied the presence of a Type-I integrable defects in the $\widehat{sl}(2,1)$ supersymmetric integrable hierarchy through super Bäcklund transformations. What we computed in principle would be called Type-I defect in the literature. However, let us call the attention to an important property appearing in the corresponding supersymmetric extensions. It turns to be that what we are calling type-I integrable defect for the supersymmetric mKdV hierarchy contains intrinsically an auxiliary fermionic field necessary to describe defect conditions for the fermionic fields. In that sense, this kind of defect should be treated as a "partial" Type-II defect, i.e. there is only one auxiliary fermionic time dependent quantity defined on the defect, but not a bosonic auxiliary field. A genuine Type-II defect will then contain one bosonic and two fermionic auxiliary fields as it is the case of the super sinh-Gordon model. Then, by using the universality argument, the type-II super Bäcklund transformation for the smKdV can be obtained either directly from the type-II defect matrix for the super sinh-Gordon previously obtained in [27], or by applying the fusing procedure of two partial type I defect matrices. We discuss this in [31]. The latter procedure can be achieved by performing two type-I Bäcklund transformations frozen at different points and then taking the limit when both points coincide. The auxiliary fields will then appear after an appropriate reparameterization of the "squeezed" fields valued only at the defect point. We expect that solutions for the auxiliary fields will be same as those for the super Sinh-Gordon equation due to the universality of the spatial component of the

Lax within the hierarchy, as it is in the bosonic case [20, 28]. We expect to return to these issues in future investigations.

One natural continuation for this work would be to construct the super KdV hierarchy and its K-matrix through gauge transformations starting with the super mKdV hierarchy as a generalization for what we did in the bosonic case.

An interesting result we did not discuss here but it is worth to mention is that in [32] we also showed the construction of the super integrable hierarchy and of the Bäcklund transformations by recursion relations as a generalization of [36].

Finally we should mention that the idea of an universality of the Bäcklund-gauge transformation is most probably valid for other hierarchies such as the AKNS and higher rank Toda theories. It would be interesting to see how such examples can be worked out technically.

Also, for an $\hat{sl}(3)$ for example we could construct a Toda hierarchy if we assume the principal gradation. A good question is if there exist something analog to the Miura-gauge transformations which would lead to the correspondent hierarchy for the homogeneous gradation.

One extra question would be how to translate the discussion present in this Part I to the classical r-matrix formalism.

It should be interesting to develop the concept of integrable hierarchies for discrete cases and investigate whether the arguments involving Bäcklund-gauge transformation employed in this thesis can be extended. The relation between the integrable discrete mKdV [37],[38] and its Miura transformation to discrete KdV equations should be understood under the algebraic formalism.

# Part II

# Quantum Integrability

# Chapter 6

# Introduction

The Part II of this thesis is focused in quantum integrable spin chains. More specifically we are interested in compute the spectrum of a certain class of finite length spin chains and explain the high degeneracies present in their spectra.

Quantum spin chains have many interesting applications in several different areas of physics including but not restricted to condensed matter [41], statistical mechanics [42, 43] and AdS/CFT [44] with possible applications in black-holes [45]. Recently, relations between integrable spin chains and a four dimensional Chern-Simons theory were also found [46]-[48].

To have good ways to compute the spectra of such systems is therefore very important. However, this is not usually easy since they are interacting systems with many particles.

For those quantum spin chains which are integrable, several techniques have been developed, such as quantum inverse scattering method (QISM), algebraic Bethe ansatz, analytical Bethe ansatz, nested algebraic Bethe ansatz, separation of variables, etc [49]-[53].

In the context of open integrable quantum spin chains the most important objects are the $R$-matrix and the $K$-matrices. The $R$-matrix encodes the bulk information and satisfies the so-called Yang-Baxter equation (YBE)[50, 51]. The $K$-matrices, $K^L$ and $K^R$, on the other hand contain the information about the left and right boundaries, respectively. The $K$-matrices satisfy an equation called Boundary Yang-Baxter equation (BYBE) or reflection equation [52, 53]. By having $R$, $K^L$ and $K^R$ we are able to construct the so-called transfer matrix which is the generating function of an infinite number of conserved quantities. For a review see [54, 55].

Three important types of R-matrices are: rational, trigonometric and elliptic. Spin chains constructed from rational R-matrices have classical (Lie group) symmetries which are in fact, the same symmetries of the R-matrix. This is not the case for trigonometric and elliptic R-matrices, whose transfer matrices do not have the same symmetries of the R-matrices. Several explicit R-matrices were computed by [56]-[58]. For each R-matrix the BYBE has to be solved in order to find the corresponding $K$-matrices. Several solutions for the reflection equation were find by [59]-[62].

In this work we focus on anisotropic spin chains. They are constructed from trigonometric R-matrices, which are themselves associated to affine Lie algebras. Some of these spin chains have quantum group (QG) symmetries that help to explain the degeneracies and multiplicities of their spectrum. The first example to be solved was the XXZ integrable spin chain which was proved to have QG symmetry $U_q(sl(2))$ [63, 64]. Since then many other examples with higher ranks have been investigated, see e.g. [65]-[87].

In this work we construct finite length integrable quantum spin chains using the R-matrices for the affine Lie algebras algebras $\hat{g} = \{A^{(2)}_{2n}, A^{(2)}_{2n-1}, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n\}$ [56]-[58]. Since we want to describe the spin chains which have more symmetry we use diagonal $K$-matrices [59]-[62]. Those K-matrices depend on a discrete parameter $p$ which runs from 0 to $n$. As we prove [88] in chapter 7 those spin chains have quantum group symmetry corresponding to removing the $p^{\text{th}}$ node from the Dynkin diagram of $\hat{g}$.

In the work of Nepomechie and Mezincescu [67] for R-matrices associated with $A^{(1)}_1$ and $A^{(2)}_2$ they noticed that the asymptotic monodromy matrix could be expressed in terms of only the coproducts of the generators. This is not immediately true for the $\hat{g}$ models above. A fundamental step is to do a gauge transformation on the R-matrix and K-matrices in such a way that they still satisfy YBE and BYBE but the new asymptotic monodromy would depend only on the unbroken generators, i.e. the ones corresponding to the nodes on lhs and rhs of the $p^{\text{th}}$ node.

In the Table 6.1 are summarized the QG symmetries each spin chain has.

It is important to notice that some cases were already well understood. Our main contribution is for the cases where $0 < p < n$ which from our knowledge are new. The cases for $p = 0$ were know already from a long

| $\hat{g}$ | QG symmetry | Representation at each site |
|---|---|---|
| $A_{2n}^{(2)}$ | $U_q(B_{n-p}) \otimes U_q(C_p)$ | $(2(n-p)+1,1) \oplus (1,2p)$ |
| $A_{2n-1}^{(2)}$ | $U_q(C_{n-p}) \otimes U_q(D_p) \quad (p \neq 1)$ | $(2(n-p),1) \oplus (1,2p)$ |
| $B_n^{(1)}$ | $U_q(B_{n-p}) \otimes U_q(D_p) \quad (n>1, p \neq 1)$ | $(2(n-p)+1,1) \oplus (1,2p)$ |
| $C_n^{(1)}$ | $U_q(C_{n-p}) \otimes U_q(C_p)$ | $(2(n-p),1) \oplus (1,2p)$ |
| $D_n^{(1)}$ | $U_q(D_{n-p}) \otimes U_q(D_p) \quad (n>1, p \neq 1, n-1)$ | $(2(n-p),1) \oplus (1,2p)$ |

Table 6.1: QG symmetries of the open-chain transfer matrix, where $p = 0, 1, \ldots, n$.

time [65]-[69]. The case of $p = n$ and $\hat{g} = A_{2n}^{(2)}$ was computed more recently in [78, 86] while for $\hat{g} = A_{2n-1}^{(2)}$ was computed by us in [87].

In addition to the symmetries described in Table 6.1, the cases $C_n^{(1)}$ and $D_n^{(1)}$ have also a duality symmetry $p \to n-p$. Such symmetry is a consequence of the Dynkin diagram be invariant under reflection. When $p = \frac{n}{2}$ we also have a self-duality symmetry. In addition to that, when $p = \frac{n}{2}$ and the parameter of the $K$-matrix $\gamma_0 = -1$ we also have what we called bonus symmetry. The bonus symmetry makes representations as $2(a,a)$ degenerate to $2a^2$.

The cases for $A_{2n-1}^{(2)}, B_n^{(1)}$ and $D^{(1)}$ have also $Z_2$ symmetries transforming complex representations into their conjugates.

The duality symmetry, self-duality symmetries, bonus symmetry and the $Z_2$ symmetries are all explicitly constructed in the Chapter 7 and are used to explain the degeneracies and multiplicities of the spectrum.

Recently the case of $D_{n+1}^{(2)}$ was also studied [89] and showed to have QG symmetry $U_q(B_{n-p}) \otimes U_q(B_p)$, duality $p \to n-p$, self duality and bonus symmetry in the same situations as the ones described above.

The process of directly diagonalize the transfer matrix can be computationally very difficult. The dimension of the matrices increases very fast with the number of sites in such a way that this process quickly becomes impossible to continue. An alternative is to use the method of analytical Bethe ansatz to obtain the eigenvalues. Again the cases for $p = 0$ were already been considered several years ago [68, 69, 84, 85, 87]. And some of the cases for $p = n$ [84, 86, 87].

When talking about Bethe ansatz, on Chapter 8, in addition to the cases described in Table 6.1, we also consider $D_{n+1}^{(2)}$ . For closed spin chains there is a general formula by Reshetikhin for Bethe ansatz for all algebras. In this work we conjecture a generalization of such formula for open spin chains. We also construct formulas for the Dynkin labels of the Bethe states in terms of the number of Bethe roots of each type.

In chapter 7 we prove that the spin chains have the QG symmetries presented in Table 6.1, the dualities and $Z_2$ are explicitly constructed and proved. In the process to prove these symmetries several interesting properties for the $R$-matrices were found and proved. Several examples are given in order to illustrate the symmetries. In Chapter 8 we compute the Bethe ansatz and give explicit formulas for all the algebras mentioned, as well as a conjecture for a general formula. The relation between Dynkin labels of the Bethe states and the number of Bethe roots of each type is also presented. A more detailed outline of each chapter is presented at the beginning of them.

# Chapter 7

# Surveying the quantum group symmetries of integrable open spin chains

The outline of this chapter is as follows. The transfer matrix is introduced in Sec. 7.1. The QG symmetry of the transfer matrix is proved in Sec. 7.2. The duality symmetry of the transfer matrix (for the cases $C_n^{(1)}$ and $D_n^{(1)}$), and the action of duality on the QG generators, are worked out in Sec. 7.3. The additional $Z_2$ symmetries of the transfer matrix (for the cases $A_{2n-1}^{(2)}$, $B_n^{(1)}$ and $D_n^{(1)}$), and the action of these symmetries on the QG generators, are worked out in Sec. 7.4. These symmetries are used in Sec. 7.5 to explain the degeneracies in the spectrum of the transfer matrix for generic values of the anisotropy parameter $\eta$. The R-matrices are recalled in Appendix G, details about the QG generators are presented in Appendix H, and the Hamiltonian is noted in Appendix I. Proofs of several lemmas are outlined in Appendix J.

## 7.1 Basics

We consider an integrable open quantum spin chain with a vector space $\mathcal{V} = \mathbb{C}^d$ at each of its $N$ sites, where

$$d = \begin{cases} 2n+1 & \text{for } A_{2n}^{(2)}\,, B_n^{(1)} \\ 2n & \text{for } A_{2n-1}^{(2)}\,, C_n^{(1)}\,, D_n^{(1)} \end{cases}\,, \qquad n = 1, 2, \ldots\,. \tag{7.1}$$

The Hilbert space ("quantum" space) of the spin chain is therefore $\mathcal{V}^{\otimes N}$.

### 7.1.1 R-matrix

The bulk interactions of the spin chain are encoded in the R-matrix $R(u)$, which maps $\mathcal{V} \otimes \mathcal{V}$ to itself, and satisfies the Yang-Baxter equation (YBE) on $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$

$$R_{12}(u-v)\, R_{13}(u)\, R_{23}(v) = R_{23}(v)\, R_{13}(u)\, R_{12}(u-v)\,. \tag{7.2}$$

We use the standard notations $R_{12} = R \otimes \mathbb{I}\,, R_{23} = \mathbb{I} \otimes R\,, R_{13} = \mathcal{P}_{23} R_{12} \mathcal{P}_{23} = \mathcal{P}_{12} R_{23} \mathcal{P}_{12}$, where $\mathbb{I}$ is the identity matrix on $\mathcal{V}$, and $\mathcal{P}$ is the permutation matrix on $\mathcal{V} \otimes \mathcal{V}$

$$\mathcal{P} = \sum_{i,j=1}^{d} e_{ij} \otimes e_{ji}\,, \tag{7.3}$$

where $e_{ij}$ are the $d \times d$ elementary matrices with elements $(e_{ij})_{\alpha\beta} = \delta_{i,\alpha}\delta_{j,\beta}$.

We consider here the anisotropic R-matrices (with anisotropy parameter $\eta$) corresponding to the following affine Lie algebras [1]

$$\hat{g} = \{A_{2n}^{(2)}\,, A_{2n-1}^{(2)}\,, B_n^{(1)}\,, C_n^{(1)}\,, D_n^{(1)}\}\,. \tag{7.4}$$

[1] We do not consider here the case $A_n^{(1)}$, which does not have crossing symmetry; it has been studied in a similar context in [71, 72, 77].

These R-matrices, which are given by Jimbo [56] (except for $A^{(2)}_{2n-1}$, in which case we consider instead Kuniba's R-matrix [58]), are in the homogeneous picture (gauge).[2] These R-matrices, which can be found in Appendix G, all have the following additional properties: $PT$ symmetry

$$R_{21}(u) \equiv \mathcal{P}_{12}\, R_{12}(u)\, \mathcal{P}_{12} = R_{12}^{t_1 t_2}(u)\,, \tag{7.5}$$

unitarity

$$R_{12}(u)\ R_{21}(-u) = \zeta(u)\, \mathbb{I} \otimes \mathbb{I}\,, \tag{7.6}$$

where $\zeta(u)$ is given by

$$\zeta(u) = \xi(u)\, \xi(-u)\,, \qquad \xi(u) = -2\, \delta_1\, \sinh(\frac{1}{2}(u+4\eta))\sinh(\frac{1}{2}(u+\rho))\,, \tag{7.7}$$

where $\delta_1$ is given by

$$\delta_1 = \begin{cases} i & \text{for } A^{(2)}_{2n}\,, A^{(2)}_{2n-1} \\ 1 & \text{for } B^{(1)}_n\,, C^{(1)}_n\,, D^{(1)}_n\,, \end{cases} \tag{7.8}$$

and crossing symmetry

$$R_{12}(u) = V_1\, R_{12}^{t_2}(-u-\rho)\, V_1 = V_2^{t_2}\, R_{12}^{t_1}(-u-\rho)\, V_2^{t_2}\,, \tag{7.9}$$

where the crossing parameter $\rho$ is given by

$$\rho = \begin{cases} -2\kappa\eta - i\pi & \text{for } A^{(2)}_{2n}\,, A^{(2)}_{2n-1} \\ -2\kappa\eta & \text{for } B^{(1)}_n\,, C^{(1)}_n\,, D^{(1)}_n \end{cases}\,, \tag{7.10}$$

with $\kappa$ defined in (G.4). The crossing matrix $V$ is an antidiagonal matrix given by

$$V = \delta_2 \sum_{\alpha=1}^{d} \epsilon_\alpha e^{(\bar{\alpha}-\bar{\alpha'})\eta} e_{\alpha\alpha'}\,, \qquad V^2 = \mathbb{I}\,, \tag{7.11}$$

where $\delta_2$ is given by

$$\delta_2 = \begin{cases} 1 & \text{for } A^{(2)}_{2n}\,, B^{(1)}_n\,, D^{(1)}_n \\ i & \text{for } A^{(2)}_{2n-1}\,, C^{(1)}_n \end{cases}\,,$$

and the other notations are defined in (G.5)-(G.7). The corresponding matrix $M$ is defined by

$$M = V^t\, V\,, \tag{7.12}$$

and it is given by the diagonal matrix

$$M = \delta_2^2 \sum_{\alpha=1}^{d} e^{4(\frac{d+1}{2}-\bar{\alpha})\eta} e_{\alpha\alpha}\,. \tag{7.13}$$

### 7.1.2 K-matrices

The boundary interactions are encoded in the right and left K-matrices, denoted here by $K^R(u)$ and $K^L(u)$, respectively, which map $\mathcal{V}$ to itself.[3] We choose $K^R(u)$ to be the diagonal $d \times d$ matrix

$$K^R(u) = K^R(u,p) = \text{diag}\,(\underbrace{e^{-u},\ldots,e^{-u}}_{p}, \underbrace{\frac{\gamma e^u+1}{\gamma+e^u},\ldots,\frac{\gamma e^u+1}{\gamma+e^u}}_{d-2p}, \underbrace{e^u,\ldots,e^u}_{p})\,, \tag{7.14}$$

where $p = 0, 1, \ldots, n$, and

$$\gamma = \begin{cases} \gamma_0 e^{(4p-2)\eta+\frac{1}{2}\rho} & \text{for } A^{(2)}_{2n-1}\,, B^{(1)}_n\,, D^{(1)}_n \\ \gamma_0 e^{(4p+2)\eta+\frac{1}{2}\rho} & \text{for } A^{(2)}_{2n}\,, C^{(1)}_n \end{cases}\,, \qquad \gamma_0 = \pm 1\,, \tag{7.15}$$

[2] Bazhanov's R-matrices [57] are equivalent, but are instead in the principal picture.

[3] Following Sklyanin [52], the right and left K-matrices are usually denoted instead by $K^-(u)$ and $K^+(u)$, respectively. However, we adopt a different notation here in order to avoid confusion with the $\pm$ used in subsequent sections to denote the limits $u \to \pm\infty$.

where $\rho$ is the crossing parameter (7.10). Unless otherwise noted, all the results in this chapter hold for both values ($\pm 1$) of the parameter $\gamma_0$. As observed in [62] (see also [59, 60, 61]), the matrices (7.14) are solutions of the boundary Yang-Baxter equation (BYBE) on $\mathcal{V} \otimes \mathcal{V}$ [52, 92, 93]

$$R_{12}(u-v)\, K^R_1(u)\; R_{21}(u+v)\, K^R_2(v) = K^R_2(v)\, R_{12}(u+v)\, K^R_1(u)\, R_{21}(u-v)\,. \tag{7.16}$$

For $p = 0$, we see that $K^R(u,p)$ in (7.14) is proportional to the identity matrix,

$$K^R(u,0) \propto \mathbb{I}\,, \tag{7.17}$$

which is the solution noted in [65]. We emphasize that the solution (7.14) depends on the bulk anisotropy parameter $\eta$ and the discrete boundary parameters $p$ and $\gamma_0$, but does not have any continuous boundary parameters.

For the left K-matrix, we take

$$K^L(u) = K^L(u,p) = K^R(-u-\rho, p)\, M\,, \tag{7.18}$$

where $M$ is given by (7.12), which is a solution of the corresponding BYBE [52, 53]

$$\begin{aligned} &R_{12}(-u+v)\, K^{L\,t_1}_1(u)\, M_1^{-1}\, R_{21}(-u-v-2\rho)\, M_1\, K^{L\,t_2}_2(v) \\ &\quad = K^{L\,t_2}_2(v)\, M_1\, R_{12}(-u-v-2\rho)\, M_1^{-1}\, K^{L\,t_1}_1(u)\, R_{21}(-u+v)\,. \end{aligned} \tag{7.19}$$

### 7.1.3 Transfer matrix

The open-chain transfer matrix, which maps the quantum space $\mathcal{V}^{\otimes N}$ to itself, is given by[52]

$$t(u,p) = \mathrm{tr}_a\, K^L_a(u,p)\, T_a(u)\, K^R_a(u,p)\, \widehat{T}_a(u)\,, \tag{7.20}$$

where the single-row monodromy matrices are defined by

$$\begin{aligned} T_a(u) &= R_{aN}(u)\; R_{aN-1}(u)\;\; \cdots R_{a1}(u)\,, \\ \widehat{T}_a(u) &= R_{1a}(u)\;\; \cdots R_{N-1a}(u)\; R_{Na}(u)\,, \end{aligned} \tag{7.21}$$

and the trace in (7.20) is over the "auxiliary" space, which is denoted by $a$. The transfer matrix is engineered to satisfy the fundamental commutativity property

$$[t(u,p)\,, t(v,p)] = 0 \text{ for all } u\,, v\,, \tag{7.22}$$

which is the hallmark of integrability. The transfer matrix contains the Hamiltonian ($\sim t'(0,p)$, see Appendix I) and higher local conserved quantities.The transfer matrix is also crossing invariant

$$t(u,p) = t(-u-\rho,p)\,, \tag{7.23}$$

where the crossing parameter $\rho$ is given by equation (7.10).

## 7.2 Quantum group symmetry

We now proceed to show that the transfer matrix (7.20) has QG symmetry, in accordance with the second column in Table 6.1.

A key step of our argument is to use a gauge transformation to bring the right K-matrix "as close as possible" to the identity matrix. By transforming to this "unitary" gauge, the asymptotic (single-row) monodromy matrix becomes expressed in terms of only the unbroken symmetry generators, which then allows us to bring the powerful QISM machinery to bear on the problem. To this end, we set (see e.g. [56])

$$\tilde{R}_{12}(u,p) = B_1(u,p)\, R_{12}(u)\, B_1(-u,p) = B_2(-u,p)\, R_{12}(u)\, B_2(u,p)\,, \tag{7.24}$$

and [53]

$$\begin{aligned} \tilde{K}^R(u,p) &= B(u,p)\, K^R(u,p)\, B(u,p)\,, \\ \tilde{K}^L(u,p) &= B(-u,p)\, K^L(u,p)\, B(-u,p)\,, \end{aligned} \tag{7.25}$$

where $B(u,p)$ is a diagonal matrix that maps $\mathcal{V}$ to itself, which we choose as follows

$$B(u,p) = \text{diag}\,\big(\underbrace{e^{\frac{u}{2}},\ldots,e^{\frac{u}{2}}}_{p},\underbrace{1,\ldots,1}_{d-2p},\underbrace{e^{-\frac{u}{2}},\ldots,e^{-\frac{u}{2}}}_{p}\big)\,. \tag{7.26}$$

Indeed, this gauge transformation brings $K^R(u,p)$ (7.14) to a form with mostly 1's on the diagonal

$$\tilde{K}^R(u,p) = \text{diag}\,\big(\underbrace{1,\ldots,1}_{p},\underbrace{\frac{\gamma e^u+1}{\gamma+e^u},\ldots,\frac{\gamma e^u+1}{\gamma+e^u}}_{d-2p},\underbrace{1,\ldots,1}_{p}\big)\,. \tag{7.27}$$

For $p=n$, we see that $\tilde{K}^R(u,n)$ is exactly equal to $\mathbb{I}$ if $d=2n$ (i.e., for $A^{(2)}_{2n-1}$, $C^{(1)}_n$ and $D^{(1)}_n$); and $\tilde{K}^R(u,n)$ differs from $\mathbb{I}$ only in the middle matrix element if $d=2n+1$ (i.e., for $A^{(2)}_{2n}$ and $B^{(1)}_n$).

The matrix $B(u,p)$ satisfies

$$B(u,p)\,B(v,p) = B(u+v,p)\,, \qquad B(0,p)=\mathbb{I}\,, \tag{7.28}$$

as well as

$$[B_1(u,p)\,B_2(u,p)\,,R_{12}(v)] = 0\,. \tag{7.29}$$

With the help of these properties, it can be shown that the gauge-transformed R-matrix and K-matrices continue to satisfy their respective Yang-Baxter equations. The crossing symmetry (7.9) is also maintained, with [53]

$$\tilde{V}(p) = V\,B(\rho,p) = B(-\rho,p)\,V\,, \tag{7.30}$$

and

$$\tilde{M}(p) = \tilde{V}^t(p)\,\tilde{V}(p) = B(\rho,p)\,M\,B(\rho,p)\,. \tag{7.31}$$

The transfer matrix (7.20) remains invariant under these transformations [53]

$$t(u,p) = \text{tr}_a\,\tilde{K}^L_a(u,p)\,\tilde{T}_a(u,p)\,\tilde{K}^R_a(u,p)\,\hat{\tilde{T}}_a(u,p)\,, \tag{7.32}$$

where

$$\begin{aligned}\tilde{T}_a(u,p) &= \tilde{R}_{aN}(u,p)\;\tilde{R}_{aN-1}(u,p)\;\cdots\tilde{R}_{a1}(u,p)\,,\\ \hat{\tilde{T}}_a(u,p) &= \tilde{R}_{1a}(u,p)\;\cdots\tilde{R}_{N-1a}(u,p)\;\tilde{R}_{Na}(u,p)\,.\end{aligned} \tag{7.33}$$

As already remarked in the Introduction, prior to any gauge transformation, the R-matrix has the property that $\check{R}(u) = \mathcal{P}R(u)$ commutes with the coproducts of generators of the "left" quantum group $U_q(g^{(l)})$ in Table 6.1 with $p=0$, i.e.[4]

$$p=0: \qquad \left[\check{R}(u)\,,\Delta(H^{(l)}_j(0))\right] = 0 = \left[\check{R}(u)\,,\Delta(E^{\pm\,(l)}_j(0))\right]\,, \qquad j=1,\ldots,n\,. \tag{7.34}$$

In contrast, the gauge-transformed R-matrix given by (7.24) and (7.26) with $p=n$ has the property that $\check{\tilde{R}}(u,n) = \mathcal{P}\tilde{R}(u,n)$ commutes with the coproducts of generators of the "right" quantum group $U_q(g^{(r)})$ in Table 6.1 with $p=n$, i.e.

$$p=n: \qquad \left[\check{\tilde{R}}(u,n)\,,\Delta(H^{(r)}_j(n))\right] = 0 = \left[\check{\tilde{R}}(u,n)\,,\Delta(E^{\pm\,(r)}_j(n))\right]\,, \qquad j=1,\ldots,n\,. \tag{7.35}$$

We now use such gauge transformations to prove the QG invariance of the open-chain transfer matrix $t(u,p)$ for any integer $p\in[0,n]$.

Let us denote by $\tilde{R}^{\pm}(p)$ the asymptotic limits of the gauge-transformed R-matrix $\tilde{R}(u,p)$ (7.24)

$$\tilde{R}^{\pm}(p) = \lim_{u\to\pm\infty} e^{\mp u}\tilde{R}(u,p)\,, \tag{7.36}$$

and we similarly denote by $\tilde{T}^{\pm}_a(p)$ the asymptotic limits of the gauge-transformed monodromy matrix $\tilde{T}_a(u,p)$ (7.33)

$$\tilde{T}^{\pm}_a(p) = \tilde{R}^{\pm}_{aN}(p)\;\tilde{R}^{\pm}_{aN-1}(p)\cdots\tilde{R}^{\pm}_{a1}(p)\ \ . \tag{7.37}$$

Let us further denote by $\tilde{T}^{\pm}_{i,j}(p)$ $(1\le i,j\le d)$ the matrix elements of $\tilde{T}^{\pm}_a(p)$ in the auxiliary space, which are operators on the quantum space $\mathcal{V}^{\otimes N}$.

[4]Further details about the generators, coproducts, etc. can be found in Appendix H.

We show in Appendix H that the operators $\tilde{T}^{\pm}_{i,j}(p)$ can be expressed in terms of (the quantum enveloping algebra of) the unbroken $\hat{g}$ generators, i.e. the generators of the quantum groups in the second column of Table 6.1. Hence, in order to demonstrate the QG symmetry of the transfer matrix, it suffices to show that

$$\left[\tilde{T}^{\pm}_{i,j}(p)\,, t(u,p)\right] = 0 \qquad i,j = 1,2,\ldots,d\,. \tag{7.38}$$

To this end, following [95] (see also [64, 66]), we first establish several lemmas.

**Lemma 1.**

$$\left[\tilde{R}^{\pm}_{12}(p)\,, \tilde{K}^{R}_{2}(u,p)\right] = 0\,. \tag{7.39}$$

A proof is outlined in Secs. J.4 and J.5.

**Lemma 2.**

$$\left[\tilde{R}^{\pm}_{12}(p)\,, \tilde{M}_1(p)\,\tilde{K}^{L}_{2}(u,p)\right] = 0\,. \tag{7.40}$$

*Proof.* We observe that

$$\tilde{K}^{L}(u,p) = \tilde{K}^{R}(-u-\rho,p)\,\tilde{M}(p) = \tilde{M}(p)\,\tilde{K}^{R}(-u-\rho,p)\,, \tag{7.41}$$

as follows from (7.18), (7.31) and (7.25). Hence,

$$\begin{aligned}\tilde{R}^{\pm}_{12}(p)\,\tilde{M}_1(p)\tilde{K}^{L}_{2}(u,p) &= \tilde{R}^{\pm}_{12}(p)\,\tilde{M}_1(p)\tilde{M}_2(p)\tilde{K}^{R}_{2}(-u-\rho,p)\\ &= \tilde{M}_1(p)\tilde{M}_2(p)\tilde{R}^{\pm}_{12}(p)\,\tilde{K}^{R}_{2}(-u-\rho,p)\\ &= \tilde{M}_1(p)\tilde{M}_2(p)\tilde{K}^{R}_{2}(-u-\rho,p)\,\tilde{R}^{\pm}_{12}(p)\\ &= \tilde{M}_1(p)\tilde{K}^{L}_{2}(u,p)\,\tilde{R}^{\pm}_{12}(p)\,,\end{aligned} \tag{7.42}$$

where the first and last equalities follow from (7.41); the second equality is a consequence of the fact [53]

$$\left[R_{12}(u)\,, M_1 M_2\right] = 0\,; \tag{7.43}$$

and the third equality follows from Lemma 1 (7.39). □

**Lemma 3.**

$$\left[\tilde{R}^{\pm}_{12}(p)\,\tilde{T}^{\pm}_{1}(p)\,, \tilde{T}_2(u,p)\,\tilde{K}^{R}_{2}(u,p)\,\hat{\tilde{T}}_2(u,p)\right] = 0\,. \tag{7.44}$$

*Proof.* We recall the gauge-transformed fundamental relation

$$\tilde{R}_{12}(u_1-u_2,p)\,\tilde{T}_1(u_1,p)\,\tilde{T}_2(u_2,p) = \tilde{T}_2(u_2,p)\,\tilde{T}_1(u_1,p)\,\tilde{R}_{12}(u_1-u_2,p)\,. \tag{7.45}$$

Taking asymptotic limits of $u_1$ yields

$$\tilde{R}^{\pm}_{12}(p)\,\tilde{T}^{\pm}_{1}(p)\,\tilde{T}_2(u,p) = \tilde{T}_2(u,p)\,\tilde{T}^{\pm}_{1}(p)\,\tilde{R}^{\pm}_{12}(p)\,, \tag{7.46}$$

which further implies

$$\tilde{T}^{-1}_{2}(u,p)\,\tilde{R}^{\pm}_{12}(p)\,\tilde{T}^{\pm}_{1}(p) = \tilde{T}^{\pm}_{1}(p)\,\tilde{R}^{\pm}_{12}(p)\,\tilde{T}^{-1}_{2}(u,p)\,. \tag{7.47}$$

Therefore,

$$\begin{aligned}&\tilde{R}^{\pm}_{12}(p)\,\tilde{T}^{\pm}_{1}(p)\,\tilde{T}_2(u,p)\,\tilde{K}^{R}_{2}(u,p)\,\tilde{T}^{-1}_{2}(-u,p)\\ &\quad= \tilde{T}_2(u,p)\,\tilde{T}^{\pm}_{1}(p)\,\tilde{R}^{\pm}_{12}(p)\,\tilde{K}^{R}_{2}(u,p)\,\tilde{T}^{-1}_{2}(-u,p)\\ &\quad= \tilde{T}_2(u,p)\,\tilde{K}^{R}_{2}(u,p)\,\tilde{T}^{\pm}_{1}(p)\,\tilde{R}^{\pm}_{12}(p)\,\tilde{T}^{-1}_{2}(-u,p)\\ &\quad= \tilde{T}_2(u,p)\,\tilde{K}^{R}_{2}(u,p)\,\tilde{T}^{-1}_{2}(-u,p)\,\tilde{R}^{\pm}_{12}(p)\,\tilde{T}^{\pm}_{1}(p)\,,\end{aligned} \tag{7.48}$$

where the first equality follows from (7.46), the second equality follows from Lemma 1 (7.39), and the third equality follows from (7.47). We have therefore demonstrated the commutativity property

$$\left[\tilde{R}^{\pm}_{12}(p)\,\tilde{T}^{\pm}_{1}(p)\,, \tilde{T}_2(u,p)\,\tilde{K}^{R}_{2}(u,p)\,\tilde{T}^{-1}_{2}(-u,p)\right] = 0\,. \tag{7.49}$$

Finally, we see from (7.33) that

$$\begin{aligned}\tilde{T}^{-1}_{a}(u,p) &= \tilde{R}^{-1}_{a1}(u,p)\cdots\tilde{R}^{-1}_{aN}(u,p)\\ &\propto \tilde{R}_{1a}(-u,p)\cdots\tilde{R}_{Na}(-u,p) = \hat{\tilde{T}}_a(-u,p)\,,\end{aligned} \tag{7.50}$$

where the second line follows from unitarity (7.6). Substituting into (7.49) we obtain the desired result (7.44). □

**Lemma 4.**

$$\tilde{M}_1^{-1}(p)\,\left((\tilde{R}_{12}^{\pm}(p))^{-1}\right)^{t_2}\,\tilde{M}_1(p)\,\tilde{R}_{12}^{\pm\,t_2}(p) = \mathbb{I}^{\otimes 2}\,. \tag{7.51}$$

*Proof.* We write the gauge-transformed unitarity condition (7.6) as

$$\tilde{R}_{12}(u,p)\,\tilde{R}_{12}^{t_1 t_2}(-u,p) = \zeta(u)\,\mathbb{I}^{\otimes 2}\,, \tag{7.52}$$

and then use crossing symmetry (7.9) to obtain

$$\tilde{V}_1(p)\,\tilde{R}_{12}^{t_2}(-u-\rho,p)\,\tilde{V}_1(p)\,\tilde{V}_1^{t_1}(p)\,\tilde{R}_{12}^{t_1}(u-\rho,p)\,\tilde{V}_1^{t_1}(p) = \zeta(u)\,\mathbb{I}^{\otimes 2}\,, \tag{7.53}$$

where $\tilde{V}(p)$ is given by (7.30). By taking asymptotic limits of (7.53) and noting that $\tilde{V}(p)^2 = \mathbb{I}$, we obtain

$$\tilde{R}_{12}^{\pm\,t_2}(p)\,\tilde{M}_1^{-1}(p)\,\tilde{R}_{12}^{\mp\,t_1}(p)\,\tilde{M}_1(p) = \chi\,\mathbb{I}^{\otimes 2}\,, \tag{7.54}$$

where $\chi$ is given by

$$\chi = \lim_{u\to\pm\infty} e^{\mp 2u}\,\zeta(u) = \frac{1}{4}\delta_1{}^2\,. \tag{7.55}$$

Moreover, from (7.52) we obtain

$$\tilde{R}_{12}^{\pm}(p)\,\tilde{R}_{12}^{\mp\,t_1 t_2}(p) = \chi\,\mathbb{I}^{\otimes 2}\,, \tag{7.56}$$

which implies that

$$\tilde{R}_{12}^{\mp\,t_1 t_2}(p) = \chi\,(\tilde{R}_{12}^{\pm}(p))^{-1}\,, \quad \text{or} \quad \tilde{R}_{12}^{\mp\,t_1}(p) = \chi\left((\tilde{R}_{12}^{\pm}(p))^{-1}\right)^{t_2}\,. \tag{7.57}$$

Substituting into (7.54), we obtain

$$\tilde{R}_{12}^{\pm\,t_2}(p)\,\tilde{M}_1^{-1}(p)\,\left((\tilde{R}_{12}^{\pm}(p))^{-1}\right)^{t_2}\,\tilde{M}_1(p) = \mathbb{I}^{\otimes 2}\,, \tag{7.58}$$

which can be rearranged to give the desired result (7.51). □

We are finally ready to prove the main result (7.38), which is equivalent to the following

**Proposition 1.**

$$\left[\tilde{T}_1^{\pm}(p)\,, t(u,p)\right] = 0\,. \tag{7.59}$$

*Proof.* Recalling that the transfer matrix remains invariant under gauge transformations (7.32), we obtain

$$\begin{aligned}
&\tilde{T}_1^{\pm}(p)\,t(u,p)\\
&\quad= \mathrm{tr}_2\left\{\tilde{T}_1^{\pm}(p)\,\tilde{K}_2^L(u,p)\,\tilde{T}_2(u,p)\,\tilde{K}_2^R(u,p)\,\hat{\tilde{T}}_2(u,p)\right\}\\
&\quad= \mathrm{tr}_2\left\{\tilde{M}_1^{-1}(p)\,\tilde{M}_1(p)\,\tilde{K}_2^L(u,p)\,(\tilde{R}_{12}^{\pm}(p))^{-1}\,\tilde{R}_{12}^{\pm}(p)\,\tilde{T}_1^{\pm}(p)\,\tilde{T}_2(u,p)\,\tilde{K}_2^R(u,p)\,\hat{\tilde{T}}_2(u,p)\right\}\\
&\quad= \mathrm{tr}_2\left\{\tilde{M}_1^{-1}(p)\,(\tilde{R}_{12}^{\pm}(p))^{-1}\,\tilde{M}_1(p)\,\tilde{K}_2^L(u,p)\,\tilde{T}_2(u,p)\,\tilde{K}_2^R(u,p)\,\hat{\tilde{T}}_2(u,p)\,\tilde{R}_{12}^{\pm}(p)\,\tilde{T}_1^{\pm}(p)\right\}\\
&\quad= \ldots
\end{aligned} \tag{7.60}$$

In passing to the third equality, we have used Lemma 2 (7.40) and Lemma 3 (7.44). Then

$$\begin{aligned}
\ldots &= \mathrm{tr}_2\left\{\tilde{M}_1^{-1}(p)\,(\tilde{R}_{12}^{\pm}(p))^{-1}\,\tilde{M}_1(p)\,\tilde{K}_2^L(u,p)\,\tilde{T}_2(u,p)\,\tilde{K}_2^R(u,p)\,\hat{\tilde{T}}_2(u,p)\,\tilde{R}_{12}^{\pm}(p)\right\}\tilde{T}_1^{\pm}(p)\\
&= \mathrm{tr}_2\left\{A_{12}\,Q_2\,\tilde{R}_{12}^{\pm}(p)\right\}\tilde{T}_1^{\pm}(p)\\
&= \mathrm{tr}_2\left\{A_{12}^{t_2}\,\tilde{R}_{12}^{\pm\,t_2}(p)\,Q_2^{t_2}\right\}\tilde{T}_1^{\pm}(p) = \ldots
\end{aligned} \tag{7.61}$$

In passing to the second line we have made the identifications $A_{12} = \tilde{M}_1^{-1}(p)\,(\tilde{R}_{12}^{\pm}(p))^{-1}\,\tilde{M}_1(p)$ and $Q_2 = \tilde{K}_2^L(u,p)\,\tilde{T}_2(u,p)\,\tilde{K}_2^R(u,p)\,\hat{\tilde{T}}_2(u,p)$. Finally, we obtain

$$\begin{aligned}
\ldots &= \mathrm{tr}_2\left\{\tilde{M}_1^{-1}(p)\,\left((\tilde{R}_{12}^{\pm}(p))^{-1}\right)^{t_2}\,\tilde{M}_1(p)\,\tilde{R}_{12}^{\pm\,t_2}(p)\,Q_2^{t_2}\right\}\tilde{T}_1^{\pm}(p)\\
&= \mathrm{tr}_2\left\{Q_2^{t_2}\right\}\tilde{T}_1^{\pm}(p)\\
&= t(u,p)\,\tilde{T}_1^{\pm}(p)\,.
\end{aligned} \tag{7.62}$$

In passing to the second line we have used Lemma 4 (7.51); and we have used (7.32) again to pass to the third line. □

## 7.3 Duality symmetry

We now show that the transfer matrix $t(u,p)$ (7.20) for the cases $C_n^{(1)}$ and $D_n^{(1)}$ has a $p \leftrightarrow n-p$ "duality" symmetry. In order to prove the general result (8.12), we need the following lemma:

**Lemma 5.** *The R-matrices for both $C_n^{(1)}$ and $D_n^{(1)}$ obey*

$$\begin{aligned} U_1\, R_{12}(u)\, U_1 &= W_2^t(u)\, R_{12}(u)\, W_2^t(u)\,, \\ U_2\, R_{12}(u)\, U_2 &= W_1(u)\, R_{12}(u)\, W_1(u)\,, \end{aligned} \tag{7.63}$$

*where $U$ and $W(u)$ are the following $(2n)\times(2n)$ matrices*

$$\begin{aligned} U &= \left(\begin{array}{c|c} 0 & \mathbb{I}_{n\times n} \\ \hline \mathbb{I}_{n\times n} & 0 \end{array}\right)_{2n\times 2n}, & U^2 &= \mathbb{I}\,, \\ W(u) &= \left(\begin{array}{c|c} 0 & e^{-\frac{u}{2}}\mathbb{I}_{n\times n} \\ \hline e^{\frac{u}{2}}\mathbb{I}_{n\times n} & 0 \end{array}\right)_{2n\times 2n}, & W(u)^2 &= \mathbb{I}\,. \end{aligned} \tag{7.64}$$

*Furthermore, the K-matrices (7.14) and (7.18) obey*

$$\begin{aligned} W(u)\, K^R(u,p)\, W^t(u) &= f^R(u,p)\, K^R(u,n-p) \\ W^t(u)\, K^L(u,p)\, W(u) &= f^L(u,p)\, K^L(u,n-p)\,, \end{aligned} \tag{7.65}$$

*where $f^R(u,p)$ and $f^L(u,p)$ are scalar functions given by*

$$\begin{aligned} f^R(u,p) &= \frac{\gamma_0\, e^u + e^{(2n-4p)\eta}}{\gamma_0 + e^{u+(2n-4p)\eta}}\,, \\ f^L(u,p) &= \frac{\gamma_0\, e^u + e^{(4p+2n\pm4)\eta}}{\gamma_0\, e^{(4n\pm4)\eta} + e^{u+(4p-2n)\eta}} \quad \text{with} \left\{ \begin{array}{ll} + & \text{for } C_n^{(1)} \\ - & \text{for } D_n^{(1)} \end{array} \right., \end{aligned} \tag{7.66}$$

*where $\gamma_0 = \pm 1$ is a parameter appearing in the K-matrix, see (7.15).*

A proof of (7.63) is outlined in Sec. J.3.

The main duality result is given by the following proposition:

**Proposition 2.** *For the cases $C_n^{(1)}$ and $D_n^{(1)}$, the transfer matrix has the duality symmetry*

$$\mathcal{U}\, t(u,p)\, \mathcal{U} = f(u,p)\, t(u,n-p)\,, \tag{7.67}$$

*where $\mathcal{U}$ is the quantum-space operator*

$$\mathcal{U} = U_1 \ldots U_N\,, \qquad \mathcal{U}^2 = \mathbb{I}^{\otimes N}\,, \tag{7.68}$$

*and the scalar factor $f(u,p)$ is given by*

$$f(u,p) = f^L(u,p)\, f^R(u,p)\,. \tag{7.69}$$

*Proof.* We see from (7.63) that the monodromy matrices (7.21) transform as follows

$$\begin{aligned} \mathcal{U}\, T_a(u)\, \mathcal{U} &= W_a(u)\, T_a(u)\, W_a(u)\,, \\ \mathcal{U}\, \widehat{T}_a(u)\, \mathcal{U} &= W_a^t(u)\, \widehat{T}_a(u)\, W_a^t(u)\,. \end{aligned} \tag{7.70}$$

Evaluating $\mathcal{U}\, t(u,p)\, \mathcal{U}$ using the definition (7.20) of the transfer matrix together with (7.70) and (7.65), we arrive at the desired result (8.12). □

A similar duality symmetry was noted for the case $A_{n-1}^{(1)}$ in [77].

As a consequence of the duality symmetry (8.12), for each eigenvalue $\Lambda(u,p)$ of $t(u,p)$, there is a corresponding eigenvalue $\Lambda(u,n-p)$ of $t(u,n-p)$ such that

$$\Lambda(u,p) = f(u,p)\, \Lambda(u,n-p)\,. \tag{7.71}$$

### 7.3.1 Action of duality on the QG generators

For the case $C_n^{(1)}$, the transfer matrix $t(u,p)$ has the symmetry $U_q(C_{n-p}) \otimes U_q(C_p)$ (see again Table 6.1), while $t(u,n-p)$ (its image under the duality transformation (8.12)) has the symmetry $U_q(C_p) \otimes U_q(C_{n-p})$. Under a duality transformation, the generators of the "left" symmetry factor of $t(u,p)$ (namely, $U_q(C_{n-p})$) are mapped to the generators of the "right" symmetry factor of $t(u,n-p)$ (which is also $U_q(C_{n-p})$). Similarly, the generators of the "right" symmetry factor of $t(u,p)$ (namely, $U_q(C_p)$) are mapped to the generators of the "left" symmetry factor of $t(u,n-p)$ (which is also $U_q(C_p)$). The case $D_n^{(1)}$ is identical, except with $D$'s replacing the $C$'s. In other words,

$$\begin{aligned} U\, H_i^{(l)}(p)\, U &= H_i^{(r)}(n-p)\,, \qquad & U\, E_i^{\pm\,(l)}(p)\, U &= E_i^{\pm\,(r)}(n-p)\,, \qquad & i &= 1,2,\ldots,n-p\,, \\ U\, H_i^{(r)}(p)\, U &= H_i^{(l)}(n-p)\,, \qquad & U\, E_i^{\pm\,(r)}(p)\, U &= E_i^{\pm\,(l)}(n-p)\,, \qquad & i &= 1,2,\ldots,p\,. \end{aligned} \tag{7.72}$$

and similarly for the coproducts. In order to obtain the general result (7.79), we need a few lemmas.

**Lemma 6.**

$$W^t(u) = B(u,n-p)\, U\, B(-u,p)\,, \tag{7.73}$$

*where $U$ and $W(u)$ are given by (7.64).*

*Proof.* We evaluate the RHS by writing all three matrices in terms of $n \times n$ blocks:

$$\begin{aligned} RHS &= \left(\begin{array}{c|c} \begin{array}{cc} e^{\frac{u}{2}}\mathbb{I}_{(n-p)\times(n-p)} & \\ & \mathbb{I}_{p\times p} \end{array} & \\ \hline & \begin{array}{cc} \mathbb{I}_{p\times p} & \\ & e^{-\frac{u}{2}}\mathbb{I}_{(n-p)\times(n-p)} \end{array} \end{array}\right) \left(\begin{array}{c|c} & \mathbb{I}_{n\times n} \\ \hline \mathbb{I}_{n\times n} & \end{array}\right) B(-u,p) \\ &= \left(\begin{array}{c|c} & \begin{array}{cc} e^{\frac{u}{2}}\mathbb{I}_{(n-p)\times(n-p)} & \\ & \mathbb{I}_{p\times p} \end{array} \\ \hline \begin{array}{cc} \mathbb{I}_{p\times p} & \\ & e^{-\frac{u}{2}}\mathbb{I}_{(n-p)\times(n-p)} \end{array} & \end{array}\right) \left(\begin{array}{c|c} \begin{array}{cc} e^{-\frac{u}{2}}\mathbb{I}_{p\times p} & \\ & \mathbb{I}_{(n-p)\times(n-p)} \end{array} & \\ \hline & \begin{array}{cc} \mathbb{I}_{(n-p)\times(n-p)} & \\ & e^{\frac{u}{2}}\mathbb{I}_{p\times p} \end{array} \end{array}\right) \\ &= \left(\begin{array}{c|c} & e^{\frac{u}{2}}\mathbb{I}_{n\times n} \\ \hline e^{-\frac{u}{2}}\mathbb{I}_{n\times n} & \end{array}\right) = LHS\,. \end{aligned} \tag{7.74}$$

□

**Lemma 7.** *The gauge-transformed R-matrices for $C_n^{(1)}$ and $D_n^{(1)}$ obey*

$$U_1\, \tilde{R}_{12}(u,p)\, U_1 = U_2\, \tilde{R}_{12}(u,n-p)\, U_2\,. \tag{7.75}$$

*Proof.* Recalling the definition of the gauge-transformed R-matrix (7.24), we see that

$$\begin{aligned} U_1\, \tilde{R}_{12}(u,p)\, U_1 &= U_1\, B_2(-u,p)\, R_{12}(u)\, B_2(u,p)\, U_1 \\ &= B_2(-u,p)\, U_1\, R_{12}(u)\, U_1\, B_2(u,p) \\ &= B_2(-u,p)\, W_2^t(u)\, R_{12}(u)\, W_2^t(u)\, B_2(u,p) \\ &= U_2\, B_2(-u,n-p)\, R_{12}(u)\, B_2(u,n-p)\, U_2 \\ &= U_2\, \tilde{R}_{12}(u,n-p)\, U_2\,. \end{aligned} \tag{7.76}$$

In passing to the third line, we have used the result (7.63); in passing to the fourth line, we use

$$B(-u,p)\, W^t(u) = U\, B(-u,n-p)\,, \qquad W^t(u)\, B(u,p) = B(u,n-p)\, U\,, \tag{7.77}$$

which follow from (7.73); and we pass to the last line using again the definition of the gauge-transformed R-matrix. □

**Lemma 8.** *The gauge-transformed monodromy matrices for $C_n^{(1)}$ and $D_n^{(1)}$ transform under duality as*

$$\mathcal{U}\, \tilde{T}_a(u,p)\, \mathcal{U} = U_a\, \tilde{T}_a(u,n-p)\, U_a\,, \tag{7.78}$$

*where $\mathcal{U}$ is given by (7.68).*

*Proof.* This result follows immediately from the definition of $\tilde{T}_a(u,p)$ (7.33) and the result (7.75). □

Finally, taking asymptotic limits of the result (7.78), we obtain the sought-after result:

**Proposition 3.** *For the cases $C_n^{(1)}$ and $D_n^{(1)}$, the asymptotic gauge-transformed monodromy matrices $\tilde{T}_a^{\pm}(p)$ transform under duality as*

$$\mathcal{U}\,\tilde{T}_a^{\pm}(p)\,\mathcal{U} = U_a\,\tilde{T}_a^{\pm}(n-p)\,U_a\,. \tag{7.79}$$

From the result (7.79), we can read off the transformation properties of the coproducts of the QG generators under duality, thereby generalizing (7.72).

### 7.3.2 Self-duality

For $p = \frac{n}{2}$ with $n$ even, we see that the duality relation (8.12) implies that the transfer matrix is self-dual

$$\left[\mathcal{U}\,, t(u, \tfrac{n}{2})\right] = 0\,, \tag{7.80}$$

since $f(u, \frac{n}{2}) = 1$. This self-duality symmetry maps the "left" and "right" generators into each other

$$U\,H_i^{(l)}(\tfrac{n}{2})\,U = H_i^{(r)}(\tfrac{n}{2})\,, \qquad U\,E_i^{\pm\,(l)}(\tfrac{n}{2})\,U = E_i^{\pm\,(r)}(\tfrac{n}{2})\,, \qquad i = 1, 2, \ldots, \tfrac{n}{2}\,, \tag{7.81}$$

as follows from (7.72). Hence, this symmetry maps the representations $(1, \mathbf{R})$ and $(\mathbf{R}, 1)$ (i.e., with "left" and "right" singlets, respectively) into each other; and therefore these states are degenerate (i.e., have the same transfer-matrix eigenvalue). This degeneracy is discussed further in Section 7.5.

#### Bonus symmetry for $\gamma_0 = -1$

For the self-dual cases (namely, $C_n^{(1)}$ and $D_n^{(1)}$ with $p = \frac{n}{2}$ and $n$ even) with $\gamma_0 = -1$, there is an additional ("bonus") symmetry, which leads to even higher degeneracies for the transfer-matrix eigenvalues.

In order to exhibit this symmetry, it is convenient to introduce the matrix $\bar{U}$, which is similar to the duality matrix $U$ (7.64),

$$\bar{U} = \left(\begin{array}{cc|cc} & & i\mathbb{I}_{\frac{n}{2}\times\frac{n}{2}} & \\ & & & -i\mathbb{I}_{\frac{n}{2}\times\frac{n}{2}} \\ \hline -i\mathbb{I}_{\frac{n}{2}\times\frac{n}{2}} & & & \\ & i\mathbb{I}_{\frac{n}{2}\times\frac{n}{2}} & & \end{array}\right)_{2n\times 2n}\,, \qquad \bar{U}^2 = \mathbb{I}\,, \tag{7.82}$$

and which satisfies

$$\bar{U}U = -U\bar{U} = iD\,, \tag{7.83}$$

where $D$ is the diagonal matrix

$$D = \operatorname{diag}\big(\underbrace{1\,, \ldots\,, 1}_{\frac{n}{2}}, \underbrace{-1\,, \ldots\,, -1}_{n}, \underbrace{1\,, \ldots\,, 1}_{\frac{n}{2}}\big)\,. \tag{7.84}$$

Similarly to (7.75), we find that the gauge-transformed R-matrix obeys

$$\bar{U}_1\,\tilde{R}_{12}(u, \tfrac{n}{2})\,\bar{U}_1 = \bar{U}_2\,\tilde{R}_{12}(u, \tfrac{n}{2})\,\bar{U}_2\,, \tag{7.85}$$

as well as

$$D_1\,\tilde{R}_{12}(u, \tfrac{n}{2})\,D_1 = D_2\,\tilde{R}_{12}(u, \tfrac{n}{2})\,D_2\,. \tag{7.86}$$

Moreover, the gauge-transformed right K-matrix (7.27) is equal to $D$ [5]

$$\tilde{K}^R(u, \tfrac{n}{2}) = D\,. \tag{7.87}$$

It follows from the BYBE (7.16) that

$$\tilde{R}_{12}(u-v, \tfrac{n}{2})\,D_1\,\tilde{R}_{21}(u+v, \tfrac{n}{2})\,D_2 = D_2\,\tilde{R}_{12}(u+v, \tfrac{n}{2})\,D_1\,\tilde{R}_{21}(u-v, \tfrac{n}{2})\,. \tag{7.88}$$

The key result is given by the following proposition

[5] We emphasize that the result (7.87) holds only for $\gamma_0 = -1$, and we assume that $\gamma_0 = -1$ in the remainder of this subsection.

**Proposition 4.** *For the cases $C_n^{(1)}$ and $D_n^{(1)}$ with $p = \frac{n}{2}$ (n even) and $\gamma_0 = -1$, the transfer matrix has the bonus symmetry*

$$[\bar{\mathcal{U}}\,, t(u, \tfrac{n}{2})] = 0\,, \tag{7.89}$$

*where $\bar{\mathcal{U}}$ is the quantum-space operator given by* [6]

$$\bar{\mathcal{U}} = \bar{U}_1 U_2 \cdots U_N\,, \qquad \bar{\mathcal{U}}^{\,2} = \mathbb{I}^{\otimes N}\,. \tag{7.90}$$

*Proof.* We see from (7.85) that the gauge-transformed monodromy matrices (7.33) transform as follows

$$\begin{aligned}\bar{\mathcal{U}}\,\tilde{T}_a(u, \tfrac{n}{2})\bar{\mathcal{U}} &= -i\,U_a\tilde{R}_{aN}(u, \tfrac{n}{2})\,\tilde{R}_{a,N-1}(u, \tfrac{n}{2})\cdots\tilde{R}_{a2}(u, \tfrac{n}{2})\,D_a\,\tilde{R}_{a1}(u, \tfrac{n}{2})\,\bar{U}_a\,,\\ \bar{\mathcal{U}}\,\hat{\tilde{T}}_a(u, \tfrac{n}{2})\bar{\mathcal{U}} &= i\,\bar{U}_a\,\tilde{R}_{1a}(u, \tfrac{n}{2})\,D_a\,\tilde{R}_{2a}(u, \tfrac{n}{2})\,\tilde{R}_{3a}(u, \tfrac{n}{2})\cdots\tilde{R}_{Na}(u, \tfrac{n}{2})\,U_a\,,\end{aligned} \tag{7.91}$$

where we have also used (7.83). Starting from the gauge-transformed expression for the transfer matrix (7.32), and also making use of (7.87), we obtain

$$\begin{aligned}&\bar{\mathcal{U}}\,t(u, \tfrac{n}{2})\bar{\mathcal{U}}\\ &\quad= \mathrm{tr}_a\,\tilde{K}_a^L(u, \tfrac{n}{2})\,\tilde{R}_{aN}(u, \tfrac{n}{2})\cdots\tilde{R}_{a2}(u, \tfrac{n}{2})\,D_a\,\tilde{R}_{a1}(u, \tfrac{n}{2})\,D_a\,\tilde{R}_{1a}(u, \tfrac{n}{2})\,D_a\,\tilde{R}_{2a}(u, \tfrac{n}{2})\cdots\tilde{R}_{Na}(u, \tfrac{n}{2})\\ &\quad= \mathrm{tr}_a\,\tilde{K}_a^L(u, \tfrac{n}{2})\,\tilde{R}_{aN}(u, \tfrac{n}{2})\cdots\tilde{R}_{a1}(u, \tfrac{n}{2})\,D_a\,\tilde{R}_{1a}(u, \tfrac{n}{2})\cdots\tilde{R}_{Na}(u, \tfrac{n}{2})\\ &\quad= \mathrm{tr}_a\,\tilde{K}_a^L(u, \tfrac{n}{2})\,\tilde{T}_a(u, \tfrac{n}{2})\,D_a\,\hat{\tilde{T}}_a(u, \tfrac{n}{2})\\ &\quad= t(u, \tfrac{n}{2})\,,\end{aligned} \tag{7.92}$$

which implies the desired result (7.89). In passing to the first equality, we have used the cyclic property of the trace, and the fact $U_a\,\tilde{K}_a^L(u, \frac{n}{2})\,U_a = -\tilde{K}_a^L(u, \frac{n}{2})$; and to pass to the second equality, we have used the result

$$D_a\,\tilde{R}_{a1}(u, \tfrac{n}{2})\,D_a\,\tilde{R}_{1a}(u, \tfrac{n}{2})\,D_a = \tilde{R}_{a1}(u, \tfrac{n}{2})\,D_a\,\tilde{R}_{1a}(u, \tfrac{n}{2})\,, \tag{7.93}$$

which follows from (7.88). □

Recalling the definitions of $\mathcal{U}$ (7.68) and $\bar{\mathcal{U}}$ (7.90) as well as the property (7.83), it is easy to see that

$$[\mathcal{U}\,,\bar{\mathcal{U}}] = -2i\,\mathcal{D}\,, \tag{7.94}$$

where $\mathcal{D}$ is the quantum-space operator defined by

$$\mathcal{D} = D_1 = D\otimes\mathbb{I}^{\otimes(N-1)}\,, \qquad \mathcal{D}^2 = \mathbb{I}^{\otimes N}\,. \tag{7.95}$$

The fact that $\mathcal{D}$ commutes with the transfer matrix is now a simple corollary of (7.89):

**Corollary.** *For the cases $C_n^{(1)}$ and $D_n^{(1)}$ with $p = \frac{n}{2}$ (n even) and $\gamma_0 = -1$, the transfer matrix commutes with the operator $\mathcal{D}$ (7.95)*

$$[\mathcal{D}\,, t(u, \tfrac{n}{2})] = 0\,. \tag{7.96}$$

*Proof.* Using (7.94) and the Jacobi identity, we see that

$$\begin{aligned}[\mathcal{D}\,, t(u, \tfrac{n}{2})] &= \frac{i}{2}\,\big[[\mathcal{U}\,,\bar{\mathcal{U}}]\ ,t(u, \tfrac{n}{2})\big]\\ &= -\frac{i}{2}\,\big[[\bar{\mathcal{U}}\,,t(u, \tfrac{n}{2})]\ ,\mathcal{U}\big] - \frac{i}{2}\,\big[[t(u, \tfrac{n}{2})\,,\mathcal{U}]\ ,\bar{\mathcal{U}}\big]\\ &= 0\,,\end{aligned} \tag{7.97}$$

where the final equality follows from the symmetries (8.13) and (7.89). □

The symmetry (7.96) gives rise to additional degeneracies of the transfer-matrix eigenvalues. Indeed, let $|\Lambda\rangle$ be a simultaneous eigenket of the transfer matrix and of the self-duality operator $\mathcal{U}$,

$$\begin{aligned}t(u, \tfrac{n}{2})\,|\Lambda\rangle &= \Lambda(u, \tfrac{n}{2})\,|\Lambda\rangle\,,\\ \mathcal{U}\,|\Lambda\rangle &= \mu\,|\Lambda\rangle\,, \qquad \mu = \pm 1\,.\end{aligned} \tag{7.98}$$

Since $\mathcal{U}$ and $\mathcal{D}$ do not commute[7], $|\Lambda\rangle$ is not necessarily an eigenket of $\mathcal{D}$, in which case $\mathcal{D}\,|\Lambda\rangle$ is a linearly independent eigenket with the same transfer-matrix eigenvalue $\Lambda(u, \frac{n}{2})$ as $|\Lambda\rangle$. Note that $|\Lambda\rangle$ necessarily belongs to a QG representation of the form $(\mathbf{R}, \mathbf{R})$ or $(\mathbf{R}_1, \mathbf{R}_2) \oplus (\mathbf{R}_2, \mathbf{R}_1)$; hence, the bonus symmetry implies the existence of a second set of states of the form $(\mathbf{R}, \mathbf{R})$ or $(\mathbf{R}_1, \mathbf{R}_2) \oplus (\mathbf{R}_2, \mathbf{R}_1)$. In particular, the degeneracy of the corresponding transfer-matrix eigenvalue becomes *doubled* as a consequence of the bonus symmetry.

[6] Note that $\bar{\mathcal{U}}$ contains only one factor of $\bar{U}$; all the other factors are $U$.
[7] Indeed, $[\mathcal{U}\,,\mathcal{D}] = [U\,,D]\otimes U\otimes\cdots\otimes U = 2i\bar{\mathcal{U}}$, see (7.83).

## 7.4 $Z_2$ symmetries

We now show that the transfer matrix $t(u,p)$ (7.20) has a discrete "right" $Z_2$ symmetry that maps complex representations of $U_q(D_p)$ to their conjugates for the cases $A^{(2)}_{2n-1}$, $B^{(1)}_n$ and $D^{(1)}_n$; and, for the latter case, there is an additional "left" $Z_2$ symmetry that maps complex representations of $U_q(D_{n-p})$ to their conjugates. We shall see in Section 7.5 that these discrete symmetries give rise to degeneracies in the spectrum beyond those expected from QG symmetry.[8]

### 7.4.1 The "right" $Z_2$

In order to prove the main result (8.14), we need the following lemma:

**Lemma 9.** *The R-matrices for $A^{(2)}_{2n-1}$, $B^{(1)}_n$ and $D^{(1)}_n$ obey*

$$\begin{aligned} Z^{(r)}_1\, R_{12}(u)\, Z^{(r)}_1 &= Y^t_2(u)\, R_{12}(u)\, Y^t_2(u)\,, \\ Z^{(r)}_2\, R_{12}(u)\, Z^{(r)}_2 &= Y_1(u)\, R_{12}(u)\, Y_1(u)\,, \end{aligned} \tag{7.99}$$

*where $Z^{(r)}$ and $Y(u)$ are the following $d\times d$ matrices*

$$\begin{aligned} Z^{(r)} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{I}_{(d-2)\times(d-2)} & 0 \\ 1 & 0 & 0 \end{pmatrix}_{d\times d}\,, \qquad & Z^{(r)\,2} = \mathbb{I}\,, \\ Y(u) &= \begin{pmatrix} 0 & 0 & e^{-u} \\ 0 & \mathbb{I}_{(d-2)\times(d-2)} & 0 \\ e^{u} & 0 & 0 \end{pmatrix}_{d\times d}\,, \qquad & Y(u)^2 = \mathbb{I}\,. \end{aligned} \tag{7.100}$$

*Furthermore, for $p>0$, the K-matrices (7.14) and (7.18) obey*

$$\begin{aligned} Y(u)\, K^R(u,p)\, Y^t(u) &= K^R(u,p)\,, \\ Y^t(u)\, K^L(u,p)\, Y(u) &= K^L(u,p)\,. \end{aligned} \tag{7.101}$$

A proof of (7.99) is outlined in Sec. J.2.

The main result concerning the "right" $Z_2$ symmetry is contained in the following proposition:

**Proposition 5.** *For the cases $A^{(2)}_{2n-1}$, $B^{(1)}_n$ and $D^{(1)}_n$ with $p>0$, the transfer matrix has the "right" $Z_2$ symmetry*

$$\Big[\mathcal{Z}^{(r)}\,, t(u,p)\Big] = 0\,, \tag{7.102}$$

*where $\mathcal{Z}^{(r)}$ is the quantum-space operator*

$$\mathcal{Z}^{(r)} = Z^{(r)}_1 \ldots Z^{(r)}_N\,, \qquad \mathcal{Z}^{(r)\,2} = \mathbb{I}^{\otimes N}\,. \tag{7.103}$$

*Proof.* We see from (7.99) that the monodromy matrices (7.21) transform as follows

$$\begin{aligned} \mathcal{Z}^{(r)}\, T_a(u)\, \mathcal{Z}^{(r)} &= Y_a(u)\, T_a(u)\, Y_a(u)\,, \\ \mathcal{Z}^{(r)}\, \widehat{T}_a(u)\, \mathcal{Z}^{(r)} &= Y^t_a(u)\, \widehat{T}_a(u)\, Y^t_a(u)\,. \end{aligned} \tag{7.104}$$

Evaluating $\mathcal{Z}^{(r)}\, t(u,p)\, \mathcal{Z}^{(r)}$ using the definition (7.20) of the transfer matrix together with (7.104) and (7.101), we arrive at the result (8.14). □

#### Action of the "right" $Z_2$ on the QG generators

In order to determine the action of the "right" $Z_2$ on the QG generators, we use a set of lemmas that are analogous to (7.73), (7.75) and (7.78), and which have similar proofs:

**Lemma 10.**

$$Y(u) = B(-u,p)\, Z^{(r)}\, B(u,p)\,, \qquad p>0\,, \tag{7.105}$$

*where $Z^{(r)}$ and $Y(u)$ are given by (7.100).*

[8] The $Z_2$ symmetry for the case $A^{(2)}_{2n-1}$ with $p=n$ was conjectured in [87].

**Lemma 11.** *The gauge-transformed R-matrices (7.24) for $A^{(2)}_{2n-1}$, $B^{(1)}_n$ and $D^{(1)}_n$ with $p>0$ obey*

$$Z_1^{(r)}\, \tilde{R}_{12}(u,p)\, Z_1^{(r)} = Z_2^{(r)}\, \tilde{R}_{12}(u,p)\, Z_2^{(r)} \,. \tag{7.106}$$

**Lemma 12.** *The gauge-transformed monodromy matrices (7.33) for $A^{(2)}_{2n-1}$, $B^{(1)}_n$ and $D^{(1)}_n$ with $p>0$ transform under the "right" $Z_2$ as*

$$\mathcal{Z}^{(r)}\, \tilde{T}_a(u,p)\, \mathcal{Z}^{(r)} = Z_a^{(r)}\, \tilde{T}_a(u,p)\, Z_a^{(r)} \,. \tag{7.107}$$

Finally, taking asymptotic limits of the result (7.107), we obtain the sought-after result:

**Proposition 6.** *For the cases $A^{(2)}_{2n-1}$, $B^{(1)}_n$ and $D^{(1)}_n$ with $p>0$, the asymptotic gauge-transformed monodromy matrices $\tilde{T}^{\pm}_a(p)$ transform under the "right" $Z_2$ as*

$$\mathcal{Z}^{(r)}\, \tilde{T}^{\pm}_a(p)\, \mathcal{Z}^{(r)} = Z_a^{(r)}\, \tilde{T}^{\pm}_a(p)\, Z_a^{(r)} \,. \tag{7.108}$$

We can read off from this result how the coproducts of the "right" QG generators transform under this $Z_2$ symmetry. In particular, we observe that

$$\begin{aligned} Z_a^{(r)}\, H_j^{(r)}\, Z_a^{(r)} &= \begin{cases} H_j^{(r)} & \text{for } j=1,\ldots,p-1\,, \\ -H_p^{(r)} & \text{for } j=p \end{cases} \,, \\ Z_a^{(r)}\, E_j^{\pm(r)}\, Z_a^{(r)} &= \begin{cases} E_j^{\pm(r)} & \text{for } j=1,\ldots,p-2\,, \\ E_p^{\pm(r)} & \text{for } j=p-1 \end{cases} \,. \end{aligned} \tag{7.109}$$

Hence, this transformation maps complex representations of $U_q(D_p)$ to their conjugates.

### 7.4.2 The "left" $Z_2$

In order to prove the main result (8.15), we need the following lemma:

**Lemma 13.** *The R-matrix for $D^{(1)}_n$ obeys*

$$Z_1^{(l)}\, R_{12}(u)\, Z_1^{(l)} = Z_2^{(l)}\, R_{12}(u)\, Z_2^{(l)} \,, \tag{7.110}$$

*where $Z^{(l)}$ is the following $2n\times 2n$ matrix*

$$Z^{(l)} = \begin{pmatrix} \mathbb{I}_{(n-1)\times(n-1)} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \mathbb{I}_{(n-1)\times(n-1)} \end{pmatrix}_{2n\times 2n} \,, \qquad Z^{(l)\,2} = \mathbb{I}\,. \tag{7.111}$$

*Furthermore, for $p<n$, the K-matrices (7.14) and (7.18) obey*

$$\begin{aligned} Z^{(l)}\, K^R(u,p)\, Z^{(l)} &= K^R(u,p)\,, \\ Z^{(l)}\, K^L(u,p)\, Z^{(l)} &= K^L(u,p)\,. \end{aligned} \tag{7.112}$$

A proof of (7.110) is outlined in Sec. J.1.

The main result concerning the "left" $Z_2$ symmetry is contained in the following proposition:

**Proposition 7.** *For the case $D^{(1)}_n$ with $p<n$, the transfer matrix has the "left" $Z_2$ symmetry*

$$\left[\mathcal{Z}^{(l)}\,, t(u,p)\right] = 0\,, \tag{7.113}$$

*where $\mathcal{Z}^{(l)}$ is the quantum-space operator*

$$\mathcal{Z}^{(l)} = Z_1^{(l)} \ldots Z_N^{(l)}\,, \qquad \mathcal{Z}^{(l)\,2} = \mathbb{I}^{\otimes N}\,. \tag{7.114}$$

*Proof.* We see from (7.110) that the monodromy matrices (7.21) transform as follows

$$\begin{aligned} \mathcal{Z}^{(l)}\, T_a(u)\, \mathcal{Z}^{(l)} &= Z_a^{(l)}\, T_a(u)\, Z_a^{(l)}\,, \\ \mathcal{Z}^{(l)}\, \widehat{T}_a(u)\, \mathcal{Z}^{(l)} &= Z_a^{(l)}\, \widehat{T}_a(u)\, Z_a^{(l)}\,. \end{aligned} \tag{7.115}$$

Evaluating $\mathcal{Z}^{(l)}\, t(u,p)\, \mathcal{Z}^{(l)}$ using the definition (7.20) of the transfer matrix together with (7.115) and (7.112), we arrive at the result (8.15). □

**Action of the "left" $Z_2$ on the QG generators**

The gauge-transformed R-matrix (7.24) for $D_n^{(1)}$ with $p < n$ obeys

$$Z_1^{(l)}\, \tilde{R}_{12}(u,p)\, Z_1^{(l)} = Z_2^{(l)}\, \tilde{R}_{12}(u,p)\, Z_2^{(l)}\,, \tag{7.116}$$

in view of the property (7.110) and the fact that $\left[Z^{(l)}\,, B(u,p)\right] = 0$ for $p < n$. Hence, the gauge-transformed monodromy matrices (7.33) transform as follows

$$\begin{aligned} \mathcal{Z}^{(l)}\, \tilde{T}_a(u,p)\, \mathcal{Z}^{(l)} &= Z_a^{(l)}\, \tilde{T}_a(u,p)\, Z_a^{(l)}\,, \\ \mathcal{Z}^{(l)}\, \hat{\tilde{T}}_a(u,p)\, \mathcal{Z}^{(l)} &= Z_a^{(l)}\, \hat{\tilde{T}}_a(u,p)\, Z_a^{(l)}\,. \end{aligned} \tag{7.117}$$

Taking asymptotic limits of this result gives the following proposition:

**Proposition 8.** *For the case $D_n^{(1)}$ with $p < n$, the asymptotic gauge-transformed monodromy matrices $\tilde{T}_a^{\pm}(p)$ transform under the "left" $Z_2$ as*

$$\mathcal{Z}^{(l)}\, \tilde{T}_a^{\pm}(p)\, \mathcal{Z}^{(l)} = Z_a^{(l)}\, \tilde{T}_a^{\pm}(p)\, Z_a^{(l)}\,. \tag{7.118}$$

We can read off from this result how the coproducts of the "left" QG generators transform under this $Z_2$ symmetry. In particular, we observe that

$$\begin{aligned} Z_a^{(l)}\, H_j^{(l)}\, Z_a^{(l)} &= \begin{cases} H_j^{(l)} & \text{for } j = 1, \ldots, n-p-1\,, \\ -H_{n-p}^{(l)} & \text{for } j = n-p \end{cases}\,, \\ Z_a^{(l)}\, E_j^{\pm(l)}\, Z_a^{(l)} &= \begin{cases} E_j^{\pm(l)} & \text{for } j = 1, \ldots, n-p-2\,, \\ E_{n-p}^{\pm(l)} & \text{for } j = n-p-1 \end{cases}\,. \end{aligned} \tag{7.119}$$

Hence, this transformation maps complex representations of $U_q(D_{n-p})$ to their conjugates.

## 7.5 Degeneracies of the transfer matrix

The symmetries identified above can be used to understand the degeneracies in the spectrum of the transfer matrix. Most importantly, the QG symmetries of the transfer matrix (7.59), summarized in Table 6.1, are directly manifested in the degeneracies of the spectrum. Indeed, for generic values of the anisotropy parameter $\eta$, the $N$-site Hilbert space $\mathcal{V}^{\otimes N}$ can be decomposed into a direct sum of irreducible representations of the corresponding classical group, whose dimensions are generally equal to the degeneracies of the eigenvalues.

For the cases $A_{2n-1}^{(2)}$, $B_n^{(1)}$ and $D_n^{(1)}$, the transfer matrix has an additional "right" $Z_2$ symmetry (8.14) that maps complex representations of $U_q(D_p)$ to their conjugates. Moreover, for the case $D_n^{(1)}$, the transfer matrix also has a "left" $Z_2$ symmetry (8.15) that maps complex representations of $U_q(D_{n-p})$ to their conjugates. Consequently, the degeneracies of eigenvalues corresponding to complex representations are larger than expected from the decomposition of the Hilbert space.

For the cases $C_n^{(1)}$ and $D_n^{(1)}$ with $n$ even and $p = \frac{n}{2}$, the transfer matrix has a self-duality symmetry (8.13) that maps the representations $(1, \mathbf{R})$ and $(\mathbf{R}, 1)$ into each other, and therefore those states are degenerate. If $\gamma_0 = -1$, then there is a bonus symmetry (7.89), (7.96) that leads to additional degeneracies.

For the cases $C_n^{(1)}$ and $D_n^{(1)}$ with $n$ odd and $p = \frac{n\pm1}{2}$, we also observe some higher degeneracies, which presumably can also be attributed to some discrete symmetries that remain to be elucidated.

We now consider examples of each of these cases.

### 7.5.1 $A_{2n}^{(2)}$

For $A_{2n}^{(2)}$ and generic values of $\eta$, the degeneracies of the transfer matrix exactly match with the predictions from the decomposition of the Hilbert space based on the QG symmetry. That is, in contrast with the other cases considered below, we do not find any higher degeneracies. As an example, let us consider the case $n = 5$ and $N = 2$ (two sites). By direct diagonalization of the transfer matrix $t(u,p)$ for generic numerical values of $u$ and $\eta$, we find that the degeneracies are as follows:

$$\begin{aligned}
p=0: &\qquad \{1,55,65\} \\
p=1: &\qquad \{1,1,3,18,18,36,44\} \\
p=2: &\qquad \{1,1,5,10,21,27,28,28\} \\
p=3: &\qquad \{1,1,10,14,14,21,30,30\} \\
p=4: &\qquad \{1,1,3,5,24,24,27,36\} \\
p=5: &\qquad \{1,1,10,10,44,55\}\,.
\end{aligned} \tag{7.120}$$

In other words, for $p=0$, one eigenvalue is repeated 65 times, another eigenvalue is repeated 55 times, and another eigenvalue appears only once; and similarly for other values of $p$.

On the other hand, according to Table 6.1, the symmetry for $A^{(2)}_{2n}$ with $n=5$ is $U_q(B_{5-p})\otimes U_q(C_p)$, and the representation at each site is $\mathcal{V}=(11-2p,1)\oplus(1,2p)$. For generic values of $\eta$, the QG representations are the same as for the corresponding classical groups. Performing the tensor-product decompositions here and below using LieART [96], we obtain[9]

$$\begin{aligned}
p=0: B_5 &\qquad (\mathbf{11})^{\otimes 2} = \mathbf{1}\oplus\mathbf{55}\oplus\mathbf{65} \\
p=1: B_4\otimes C_1 &\qquad ((\mathbf{9},\mathbf{1})\oplus(\mathbf{1},\mathbf{2}))^{\otimes 2} = 2(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus 2(\mathbf{9},\mathbf{2})\oplus(\mathbf{36},\mathbf{1})\oplus(\mathbf{44},\mathbf{1}) \\
p=2: B_3\otimes C_2 &\qquad ((\mathbf{7},\mathbf{1})\oplus(\mathbf{1},\mathbf{4}))^{\otimes 2} = 2(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{5})\oplus 2(\mathbf{7},\mathbf{4})\oplus(\mathbf{1},\mathbf{10})\oplus(\mathbf{21},\mathbf{1})\oplus(\mathbf{27},\mathbf{1}) \\
p=3: B_2\otimes C_3 &\qquad ((\mathbf{5},\mathbf{1})\oplus(\mathbf{1},\mathbf{6}))^{\otimes 2} = 2(\mathbf{1},\mathbf{1})\oplus 2(\mathbf{5},\mathbf{6})\oplus(\mathbf{10},\mathbf{1})\oplus(\mathbf{1},\mathbf{14})\oplus(\mathbf{14},\mathbf{1})\oplus(\mathbf{1},\mathbf{21}) \\
p=4: B_1\otimes C_4 &\qquad ((\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{8}))^{\otimes 2} = 2(\mathbf{1},\mathbf{1})\oplus(\mathbf{3},\mathbf{1})\oplus(\mathbf{5},\mathbf{1})\oplus 2(\mathbf{3},\mathbf{8})\oplus(\mathbf{1},\mathbf{27})\oplus(\mathbf{1},\mathbf{36}) \\
p=5: C_5 &\qquad (\mathbf{1}\oplus\mathbf{10})^{\otimes 2} = 2(\mathbf{1})\oplus 2(\mathbf{10})\oplus\mathbf{44}\oplus\mathbf{55}\,.
\end{aligned} \tag{7.121}$$

Comparing the degeneracies (7.120) with the corresponding tensor-product decompositions (7.121), we see that they exactly match. We obtain similar results for other values of $n$ and $N$. The special cases $p=0$ and $p=n$ are discussed further in [86].

### 7.5.2 $A^{(2)}_{2n-1}$

For $A^{(2)}_{2n-1}$ and generic values of $\eta$, the degeneracies of the transfer matrix either match with the predictions from QG symmetry, or are larger due to the "right" $Z_2$ symmetry (8.14). As an example, let us consider the case $n=5$ and $N=2$ (two sites). By direct diagonalization of the transfer matrix $t(u,p)$ for generic numerical values of $u$ and $\eta$, we find that the degeneracies are as follows:

$$\begin{aligned}
p=0: &\qquad \{1,44,55\} \\
p=2: &\qquad \{1,1,6,9,14,21,24,24\} \\
p=3: &\qquad \{1,1,5,10,15,20,24,24\} \\
p=4: &\qquad \{1,1,3,16,16,28,35\} \\
p=5: &\qquad \{1,45,54\}\,.
\end{aligned} \tag{7.122}$$

Note that we exclude the case $p=1$.

On the other hand, according to Table 6.1, the symmetry for $A^{(2)}_{2n-1}$ with $n=5$ and $p\neq 1$ is $U_q(C_{5-p})\otimes U_q(D_p)$, and the representation at each site is $\mathcal{V}=(10-2p,1)\oplus(1,2p)$. The tensor-product decompositions are as follows:

$$\begin{aligned}
p=0: C_5 &\qquad (\mathbf{10})^{\otimes 2} = \mathbf{1}\oplus\mathbf{44}\oplus\mathbf{55} \\
p=2: C_3\otimes D_2 &\qquad ((\mathbf{6},\mathbf{1})\oplus(\mathbf{1},\mathbf{4}))^{\otimes 2} = 2(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\bar{\mathbf{3}})\oplus 2(\mathbf{6},\mathbf{4})\oplus(\mathbf{1},\mathbf{9}) \\
&\qquad\qquad \oplus(\mathbf{14},\mathbf{1})\oplus(\mathbf{21},\mathbf{1}) \\
p=3: C_2\otimes D_3 &\qquad ((\mathbf{4},\mathbf{1})\oplus(\mathbf{1},\mathbf{6}))^{\otimes 2} = 2(\mathbf{1},\mathbf{1})\oplus(\mathbf{5},\mathbf{1})\oplus 2(\mathbf{4},\mathbf{6})\oplus(\mathbf{10},\mathbf{1}) \\
&\qquad\qquad \oplus(\mathbf{1},\mathbf{15})\oplus(\mathbf{1},\mathbf{20}') \\
p=4: C_1\otimes D_4 &\qquad ((\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{8}_v))^{\otimes 2} = 2(\mathbf{1},\mathbf{1})\oplus(\mathbf{3},\mathbf{1})\oplus 2(\mathbf{2},\mathbf{8}_v)\oplus(\mathbf{1},\mathbf{28})\oplus(\mathbf{1},\mathbf{35}_v) \\
p=5: D_5 &\qquad (\mathbf{10})^{\otimes 2} = \mathbf{1}\oplus\mathbf{45}\oplus\mathbf{54}\,.
\end{aligned} \tag{7.123}$$

Comparing the degeneracies (7.122) with the corresponding tensor-product decompositions (7.123), we see that they match, except for $p=2$. For the latter case, the degeneracies are larger, due to the "right" $Z_2$ symmetry mapping complex representations of $D_p$ to their conjugates (here, the $\mathbf{3}$ and $\bar{\mathbf{3}}$). We obtain similar results for other values of $n$ and $N$. The special cases $p=0$ and $p=n$ are discussed further in [87].

[9] We recall that $A_1=B_1=C_1$, while the $D_n$ series starts with $n=2$.

### 7.5.3 $B_n^{(1)}$

For $B_n^{(1)}$ and generic values of $\eta$, the degeneracies of the transfer matrix also either match with the predictions from QG symmetry, or are larger due to the "right" $Z_2$ symmetry (8.14). As an example, let us consider the case $n = 5$ and $N = 2$ (two sites). By direct diagonalization of the transfer matrix $t(u, p)$ for generic numerical values of $u$ and $\eta$, we find that the degeneracies are as follows:

$$\begin{aligned}
p = 0:& \quad \{1, 55, 65\} \\
p = 2:& \quad \{1, 1, 6, 9, 21, 27, 28, 28\} \\
p = 3:& \quad \{1, 1, 10, 14, 15, 20, 30, 30\} \\
p = 4:& \quad \{1, 1, 3, 5, 24, 24, 28, 35\} \\
p = 5:& \quad \{1, 1, 10, 10, 45, 54\}\,.
\end{aligned} \tag{7.124}$$

Note that we again exclude the case $p = 1$.

On the other hand, according to Table 6.1, the symmetry for $B_n^{(1)}$ with $n = 5$ and $p \neq 1$ is $U_q(B_{5-p}) \otimes U_q(D_p)$, and the representation at each site is $\mathcal{V} = (11 - 2p, 1) \oplus (1, 2p)$. The tensor-product decompositions are as follows:

$$\begin{aligned}
p = 0: B_5 &\quad (\mathbf{11})^{\otimes 2} = \mathbf{1} \oplus \mathbf{55} \oplus \mathbf{65} \\
p = 2: B_3 \otimes D_2 &\quad ((\mathbf{7}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) \oplus (\mathbf{1}, \mathbf{9}) \oplus 2(\mathbf{7}, \mathbf{4}) \\
&\quad \oplus (\mathbf{21}, \mathbf{1}) \oplus (\mathbf{27}, \mathbf{1}) \\
p = 3: B_2 \otimes D_3 &\quad ((\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{6}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus 2(\mathbf{5}, \mathbf{6}) \oplus (\mathbf{10}, \mathbf{1}) \oplus (\mathbf{14}, \mathbf{1}) \\
&\quad \oplus (\mathbf{1}, \mathbf{15}) \oplus (\mathbf{1}, \mathbf{20'}) \\
p = 4: B_1 \otimes D_4 &\quad ((\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}_v))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{5}, \mathbf{1}) \oplus 2(\mathbf{3}, \mathbf{8}_v) \oplus (\mathbf{1}, \mathbf{28}) \oplus (\mathbf{1}, \mathbf{35}_v) \\
p = 5: D_5 &\quad (\mathbf{1} \oplus \mathbf{10})^{\otimes 2} = 2(\mathbf{1}) \oplus 2(\mathbf{10}) \oplus \mathbf{45} \oplus \mathbf{54}\,.
\end{aligned} \tag{7.125}$$

Comparing the degeneracies (7.124) with the corresponding tensor-product decompositions (7.125), we see that they match, except for $p = 2$. For the latter case, the degeneracies are larger, due to the "right" $Z_2$ symmetry mapping complex representations of $D_p$ to their conjugates (here, the $\mathbf{3}$ and $\bar{\mathbf{3}}$). We obtain similar results for other values of $n$ and $N$.

### 7.5.4 $C_n^{(1)}$

For $C_n^{(1)}$ and generic values of $\eta$, the degeneracies of the transfer matrix match with the predictions from QG symmetry, except when $n$ is even and $p = \frac{n}{2}$ (in which case there is a self-duality symmetry (8.13)) or when $n$ is odd and $p = \frac{n \pm 1}{2}$. Moreover, the spectrum exhibits a $p \to n - p$ duality symmetry.

#### Example 1: even $n$

As a first example, let us consider the case $n = 4$ and $N = 2$ (two sites). By direct diagonalization of the transfer matrix $t(u, p)$ for generic numerical values of $u$ and $\eta$, we find that the degeneracies are as follows:

$$\begin{aligned}
p = 0:& \quad \{1, 27, 36\} \\
p = 1:& \quad \{1, 1, 3, 12, 12, 14, 21\} \\
p = 2:& \quad \begin{cases} \{1, 1, 10, 16, 16, 20\} & \text{for } \gamma_0 = +1 \\ \{2, 10, 20, 32\} & \text{for } \gamma_0 = -1 \end{cases} \\
p = 3:& \quad \{1, 1, 3, 12, 12, 14, 21\} \\
p = 4:& \quad \{1, 27, 36\}\,.
\end{aligned} \tag{7.126}$$

The fact that the degeneracies are the same for $p$ and $n-p$ is a consequence of the duality symmetry (8.12), (7.71).

On the other hand, according to Table 6.1, the symmetry for $C_n^{(1)}$ with $n = 4$ is $U_q(C_{4-p}) \otimes U_q(C_p)$, and the representation at each site is $\mathcal{V} = (8 - 2p, 1) \oplus (1, 2p)$. The tensor-product decompositions are as follows:

$$\begin{aligned}
p = 0: C_4 &\quad (\mathbf{8})^{\otimes 2} = \mathbf{1} \oplus \mathbf{27} \oplus \mathbf{36} \\
p = 1: C_3 \otimes C_1 &\quad ((\mathbf{6}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus 2(\mathbf{6}, \mathbf{2}) \oplus (\mathbf{14}, \mathbf{1}) \oplus (\mathbf{21}, \mathbf{1}) \\
p = 2: C_2 \otimes C_2 &\quad ((\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}) \oplus 2(\mathbf{4}, \mathbf{4}) \oplus (\mathbf{10}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{10})\,.
\end{aligned} \tag{7.127}$$

There is no need to display the tensor-product decompositions for $p > 2$ due to the symmetry $p \to n - p$.

Comparing the degeneracies (7.126) with the corresponding tensor-product decompositions (7.127), we see that they match, except for $p = 2$. For the latter case, the degeneracies are larger, due to the self-duality symmetry (8.13) for even $n$ and $p = \frac{n}{2}$, which here maps $(\mathbf{1}, \mathbf{5})$ to $(\mathbf{5}, \mathbf{1})$ (resulting in a 10-fold degeneracy), and also maps $(\mathbf{1}, \mathbf{10})$ to $(\mathbf{10}, \mathbf{1})$ (resulting in a 20-fold degeneracy). If $\gamma_0 = -1$, then the bonus symmetry (7.89), (7.96) implies that the two $(\mathbf{4}, \mathbf{4})$ are degenerate (giving rise to a 32-fold degeneracy), as well as the two $(\mathbf{1}, \mathbf{1})$ (resulting in a 2-fold degeneracy).

**Example 2: odd $n$**

As a second example, let us consider the case $n = 5$ and $N = 2$ (two sites). By direct diagonalization of the transfer matrix $t(u, p)$ for generic numerical values of $u$ and $\eta$, we find that the degeneracies are as follows:

$$\begin{aligned}
p = 0 : &\quad \{1, 44, 55\} \\
p = 1 : &\quad \{1, 1, 3, 16, 16, 27, 36\} \\
p = 2 : &\quad \{1, 1, 5, 21, 34, 38\} \\
p = 3 : &\quad \{1, 1, 5, 21, 34, 38\} \\
p = 4 : &\quad \{1, 1, 3, 16, 16, 27, 36\} \\
p = 5 : &\quad \{1, 44, 55\} \,.
\end{aligned} \tag{7.128}$$

We see again that the degeneracies are the same for $p$ and $n - p$, as a consequence of the duality symmetry (8.12), (7.71).

On the other hand, according to Table 6.1, the symmetry for $C_n^{(1)}$ with $n = 5$ is $U_q(C_{5-p}) \otimes U_q(C_p)$, and the representation at each site is $\mathcal{V} = (10-2p, 1) \oplus (1, 2p)$. The tensor-product decompositions are as follows:

$$\begin{aligned}
p = 0 : C_5 &\qquad (\mathbf{10})^{\otimes 2} = \mathbf{1} \oplus \mathbf{44} \oplus \mathbf{55} \\
p = 1 : C_4 \otimes C_1 &\qquad ((\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus 2(\mathbf{8}, \mathbf{2}) \oplus (\mathbf{27}, \mathbf{1}) \oplus (\mathbf{36}, \mathbf{1}) \\
p = 2 : C_3 \otimes C_2 &\qquad ((\mathbf{6}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}) \oplus 2(\mathbf{6}, \mathbf{4}) \oplus (\mathbf{1}, \mathbf{10}) \oplus (\mathbf{14}, \mathbf{1}) \oplus (\mathbf{21}, \mathbf{1}) \,.
\end{aligned} \tag{7.129}$$

Again, there is no need to display the tensor-product decompositions for $p > 2$ due to the symmetry $p \to n-p$.

Comparing the degeneracies (7.128) with the corresponding tensor-product decompositions (7.129), we see that they match, except for $p = 2$. For the latter case, the degeneracies are larger: the $(\mathbf{1}, \mathbf{10})$ and one $(\mathbf{6}, \mathbf{4})$ are degenerate (resulting in a 34-fold degeneracy); and the $(\mathbf{14}, \mathbf{1})$ and the other $(\mathbf{6}, \mathbf{4})$ are degenerate (resulting in a 38-fold degeneracy). We expect that such degeneracies for odd $n$ and $p = \frac{n\pm1}{2}$ can be attributed to some discrete symmetries, which remain to be elucidated.

### 7.5.5 $D_n^{(1)}$

For $D_n^{(1)}$ and generic values of $\eta$, the degeneracies of the transfer matrix match with the predictions from QG symmetry, except for the following exceptions: when $n$ is even and $p = \frac{n}{2}$ (in which case there is a self-duality symmetry (8.13)); when $n$ is odd and $p = \frac{n\pm1}{2}$; and when there are additional degeneracies due to the "right" and "left" $Z_2$ symmetries (8.14), (8.15). Moreover, the spectrum exhibits a $p \to n - p$ duality symmetry.

**Example 1: even $n$**

As a first example, let us consider the case $n = 6$ and $N = 2$ (two sites). By direct diagonalization of the transfer matrix $t(u, p)$ for generic numerical values of $u$ and $\eta$, we find that the degeneracies are as follows:

$$\begin{aligned}
p = 0 : &\quad \{1, 66, 77\} \\
p = 2 : &\quad \{1, 1, 6, 9, 28, 32, 32, 35\} \\
p = 3 : &\quad \begin{cases} \{1, 1, 30, 36, 36, 40\} & \text{for } \gamma_0 = +1 \\ \{2, 30, 40, 72\} & \text{for } \gamma_0 = -1 \end{cases} \\
p = 4 : &\quad \{1, 1, 6, 9, 28, 32, 32, 35\} \\
p = 6 : &\quad \{1, 66, 77\} \,.
\end{aligned} \tag{7.130}$$

Note that we exclude the cases $p = 1$ and $p = n - 1$. The fact that the degeneracies are the same for $p$ and $n - p$ is a consequence of the duality symmetry (8.12), (7.71).

On the other hand, according to Table 6.1, the symmetry for $D_n^{(1)}$ with $n = 6$ and $p \ne 1, n-1$ is $U_q(D_{6-p}) \otimes U_q(D_p)$, and the representation at each site is $\mathcal{V} = (12-2p, 1) \oplus (1, 2p)$. The tensor-product decompositions are as follows:

$$\begin{aligned}
p = 0 : D_6 \qquad & (\mathbf{12})^{\otimes 2} = \mathbf{1} \oplus \mathbf{66} \oplus \mathbf{77} \\
p = 2 : D_4 \otimes D_2 \qquad & ((\mathbf{8}_v, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) \oplus (\mathbf{1}, \mathbf{9}) \oplus 2(\mathbf{8}_v, \mathbf{4}) \\
& \qquad \oplus (\mathbf{28}, \mathbf{1}) \oplus (\mathbf{35}_v, \mathbf{1}) \\
p = 3 : D_3 \otimes D_3 \qquad & ((\mathbf{6}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{6}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus 2(\mathbf{6}, \mathbf{6}) \oplus (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{15}) \\
& \qquad \oplus (\mathbf{20}', \mathbf{1}) \oplus (\mathbf{1}, \mathbf{20}') \,.
\end{aligned} \tag{7.131}$$

There is no need to display the tensor-product decompositions for $p > 3$ due to the symmetry $p \to n - p$.

Comparing the degeneracies (7.130) with the corresponding tensor-product decompositions (7.131), we see that they match for $p = 0$. For $p = 2$, the degeneracies are larger due to the the "right" $Z_2$ symmetry (8.14), which maps $(\mathbf{1}, \mathbf{3})$ to $(\mathbf{1}, \bar{\mathbf{3}})$, and results in a 6-fold degeneracy.

For $p = 3$, the degeneracies are larger due to the self-duality symmetry (8.13) for even $n$ and $p = \frac{n}{2}$, which maps $(\mathbf{1}, \mathbf{15})$ to $(\mathbf{15}, \mathbf{1})$ (resulting in a 30-fold degeneracy), and also maps $(\mathbf{1}, \mathbf{20}')$ to $(\mathbf{20}', \mathbf{1})$ (resulting in a 40-fold degeneracy). If $\gamma_0 = -1$, then the bonus symmetry (7.89), (7.96) implies that the two $(\mathbf{6}, \mathbf{6})$ are degenerate (giving rise to a 72-fold degeneracy), as well as the two $(\mathbf{1}, \mathbf{1})$ (resulting in a 2-fold degeneracy).

**Example 2: odd $n$**

As a second example, let us consider the case $n = 5$ and $N = 2$ (two sites). By direct diagonalization of the transfer matrix $t(u, p)$ for generic numerical values of $u$ and $\eta$, we find that the degeneracies are as follows:

$$\begin{aligned}
p = 0 : \qquad & \{1, 45, 54\} \\
p = 2 : \qquad & \{1, 1, 6, 20, 33, 39\} \\
p = 3 : \qquad & \{1, 1, 6, 20, 33, 39\} \\
p = 5 : \qquad & \{1, 45, 54\} \,.
\end{aligned} \tag{7.132}$$

We again exclude the cases $p = 1, n-1$, and observe that the degeneracies are the same for $p$ and $n - p$, as a consequence of the duality symmetry (8.12), (7.71).

On the other hand, according to Table 6.1, the symmetry for $D_n^{(1)}$ with $n = 5$ and $p \ne 1, n-1$ is $U_q(D_{5-p}) \otimes U_q(D_p)$, and the representation at each site is $\mathcal{V} = (10-2p, 1) \oplus (1, 2p)$. The tensor-product decompositions are as follows:

$$\begin{aligned}
p = 0 : D_5 \qquad & (\mathbf{10})^{\otimes 2} = \mathbf{1} \oplus \mathbf{45} \oplus \mathbf{54} \\
p = 2 : D_3 \otimes D_2 \qquad & ((\mathbf{6}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}))^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) \oplus 2(\mathbf{6}, \mathbf{4}) \oplus (\mathbf{1}, \mathbf{9}) \\
& \qquad \oplus (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{20}', \mathbf{1}) \,.
\end{aligned} \tag{7.133}$$

Again, there is no need to display the tensor-product decompositions for $p > 2$ due to the symmetry $p \to n - p$.

Comparing the degeneracies (7.132) with the corresponding tensor-product decompositions (7.133), we see that they match for $p = 0$. For $p = 2$, the 6-fold degeneracy is due to "right" $Z_2$ symmetry, which maps $(\mathbf{1}, \mathbf{3})$ to $(\mathbf{1}, \bar{\mathbf{3}})$. Moreover, the $(\mathbf{1}, \mathbf{9})$ and one $(\mathbf{6}, \mathbf{4})$ are degenerate (resulting in a 33-fold degeneracy); and the $(\mathbf{15}, \mathbf{1})$ and the other $(\mathbf{6}, \mathbf{4})$ are degenerate (resulting in a 39-fold degeneracy). We expect that such degeneracies for odd $n$ and $p = \frac{n \pm 1}{2}$ can be attributed to some discrete symmetries, which remain to be elucidated.

# Chapter 8

# The spectrum of quantum-group-invariant transfer matrices

The outline of this chapter is as follows. The key results of the Chapter 7 and [88, 89] are briefly reviewed in Sec. 8.1. Expressions for the eigenvalues of the transfer matrix and corresponding Bethe equations are obtained in Sec. 8.2. Formulas for the Dynkin labels of the Bethe states (in terms of the numbers of Bethe roots of each type) are obtained and illustrated with some examples in Sec. 8.3. We briefly study how duality transformations are implemented on the Bethe ansatz solutions in Sec. 8.4. A connection between "bonus" symmetry and singular solutions of the Bethe equations is noted in Appendix K, and some additional cases are considered in Appendix L.

## 8.1 Review of previous results

### 8.1.1 R-matrix

As in the previous chapter, we consider here the trigonometric R-matrices given by Jimbo [56] (except for $A^{(2)}_{2n-1}$, in which case we consider instead Kuniba's R-matrix [58]), corresponding to the following non-exceptional affine Lie algebras

$$\hat{g} = \left\{ A^{(2)}_{2n-1}\,, A^{(2)}_{2n}\,, B^{(1)}_{n}\,, C^{(1)}_{n}\,, D^{(1)}_{n}\,, D^{(2)}_{n+1} \right\} . \tag{8.1}$$

We use the specific expressions for the R-matrices given in Appendix G (including the one for $D^{(2)}_{n+1}$ tat had not been discussed in the previous chapter). We emphasize that we consider here exclusively generic values of $\eta$. Various useful parameters related to these R-matrices are collected in Table 8.1. In particular, $d$ is the dimension of the vector space at each site of the spin chain; hence, the R-matrix is a $d^2 \times d^2$ matrix. Also, $\delta = 0$ ($\delta = 2$) for the untwisted (twisted) cases, respectively.

| $\hat{g}$ | $A^{(2)}_{2n-1}$ | $A^{(2)}_{2n}$ | $B^{(1)}_{n}$ | $C^{(1)}_{n}$ | $D^{(1)}_{n}$ | $D^{(2)}_{n+1}$ |
|---|---|---|---|---|---|---|
| $d$ | $2n$ | $2n+1$ | $2n+1$ | $2n$ | $2n$ | $2n+2$ |
| $\kappa$ | $2n$ | $2n+1$ | $2n-1$ | $2n+2$ | $2n-2$ | $2n$ |
| $\rho$ | $-2\kappa\eta - i\pi$ | $-2\kappa\eta - i\pi$ | $-2\kappa\eta$ | $-2\kappa\eta$ | $-2\kappa\eta$ | $-\kappa\eta$ |
| $\omega$ | $\kappa+2$ | $\kappa-2$ | $\kappa+2$ | $\kappa-2$ | $\kappa+2$ | $\kappa$ |
| $\bar{\omega}$ | $\kappa-2$ | $\kappa+2$ | $\kappa-2$ | $\kappa+2$ | $\kappa-2$ | $\kappa$ |
| $\delta$ | 2 | 2 | 0 | 0 | 0 | 2 |
| $\xi$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $\xi'$ | 0 | 0 | 0 | 0 | 1 | 0 |

Table 8.1: Parameters related to the R-matrices.

### 8.1.2 K-matrices

For all the cases except $D^{(2)}_{n+1}$, we take the right K-matrices given by the expressions (7.14) and (7.15) where $p$ is a discrete parameter taking the values

$$\begin{aligned} &p = 0, ..., n && \text{for} \quad A^{(2)}_{2n}\,, C^{(1)}_{n}\,,\\ &p = 0, ..., n\,, \quad p \neq 1\,, && \text{for} \quad A^{(2)}_{2n-1}\,, B^{(1)}_{n}\,,\\ &p = 0, ..., n\,, \quad p \neq 1\,, n-1\,, && \text{for} \quad D^{(1)}_{n}\,. \end{aligned} \tag{8.2}$$

and $\gamma_0$ is another discrete parameter

$$\gamma_0 = \pm 1\,. \tag{8.3}$$

It is convenient to define the corresponding parameter

$$\varepsilon = \frac{1-\gamma_0}{2}\,, \tag{8.4}$$

which therefore can take the values $\varepsilon = 0, 1$.

Note that in (8.2) (as well as in the previous chapter) the following cases are excluded:

$$\begin{aligned} &A^{(2)}_{2n-1} && \text{with} \quad p = 1,\\ &B^{(1)}_{n} && \text{with} \quad p = 1,\\ &D^{(1)}_{n} && \text{with} \quad p = 1\,, n-1\,. \end{aligned} \tag{8.5}$$

For these cases, to which we henceforth refer as "special" cases, we take instead the following right K-matrices:

$$K^R(u, 1) = \text{diag}(e^{-2u}, \underbrace{1, ..., 1}_{d-2}, e^{2u}) \tag{8.6}$$

for $A^{(2)}_{2n-1}\,, B^{(1)}_{n}\,, D^{(1)}_{n}$ with $p = 1$; and

$$K^R(u, n-1) = \text{diag}(\underbrace{e^{-u}, ..., e^{-u}}_{n-1}, e^{u}\,, e^{-u}, \underbrace{e^{u}, ..., e^{u}}_{n-1}) \tag{8.7}$$

for $D^{(1)}_{n}$ with $p = n - 1$. We choose these K-matrices because they lead to QG symmetry, as explained in Sec. 8.1.3.

For the case $D^{(2)}_{n+1}$, the right K-matrix is given by the $d \times d$ block-diagonal matrix [89][1]

$$K^R(u,p) = \left(\begin{array}{c|c|cc|c|c} k_-(u)\mathbb{I}_{p\times p} & & & & & \\ \hline & g(u)\mathbb{I}_{(n-p)\times(n-p)} & & & & \\ \hline & & k_1(u) & k_2(u) & & \\ & & k_2(u) & k_1(u) & & \\ \hline & & & & g(u)\mathbb{I}_{(n-p)\times(n-p)} & \\ \hline & & & & & k_+(u)\mathbb{I}_{p\times p} \end{array}\right), \tag{8.8}$$

where

$$\begin{aligned} k_\pm(u) &= e^{\pm 2u}\,,\\ g(u) &= \frac{\cosh\left(u-(n-2p)\eta+\frac{i\pi}{2}\varepsilon\right)}{\cosh\left(u+(n-2p)\eta-\frac{i\pi}{2}\varepsilon\right)}\,,\\ k_1(u) &= \frac{\cosh(u)\cosh\left((n-2p)\eta+\frac{i\pi}{2}\varepsilon\right)}{\cosh\left(u+(n-2p)\eta+\frac{i\pi}{2}\varepsilon\right)}\,,\\ k_2(u) &= -\frac{\sinh(u)\sinh\left((n-2p)\eta+\frac{i\pi}{2}\varepsilon\right)}{\cosh\left(u+(n-2p)\eta+\frac{i\pi}{2}\varepsilon\right)}\,. \end{aligned} \tag{8.9}$$

and $\varepsilon$ is, again, a discrete parameter that can take the values $\varepsilon = 0, 1$.

[1] The $D^{(2)}_{n+1}$ K-matrices $K^R(u, n)$ (i.e. with $p = n$) with $\varepsilon = 0$ and $\varepsilon = 1$ are proportional to the $D^{(2)}_{n+1}$ K-matrices $K^-(u)$ in [87] for the cases I and II, respectively; explicitly, $K^-_{[87]\,\mathrm{I,II}}(u) = -2e^{2u+n\eta}\cosh\left(u-n\eta+\frac{i\pi}{2}\varepsilon\right) K^R(u, n)$.

Finally, for the left K-matrices, we take (as in the previous chapter) [52, 53]

$$K^L(u,p) = K^R(-u-\rho,p)\, M\,, \tag{8.10}$$

which is a solution of left boundary Yang-Baxter equation (7.19), and corresponds to imposing the "same" boundary conditions on the two ends.

Using then the R-matrices and K-matrices mentioned above we can use the same description presented in the subsection 7.1.3 to construct the transfer matrices.

### 8.1.3 Symmetries of the transfer matrix

It has been shown in the previous chapter (for all the cases except $D^{(2)}_{n+1}$) and in [88, 89] that the transfer matrices (7.20) constructed using the K-matrices (7.14) and (8.8) have the QG symmetries in Table 8.2. For

| $\hat{g}$ | QG symmetry | Representation at each site |
|---|---|---|
| $A^{(2)}_{2n-1}$ | $U_q(C_{n-p})\otimes U_q(D_p)$ $(p\neq 1)$ | $(2(n-p),1)\oplus(1,2p)$ |
| $A^{(2)}_{2n}$ | $U_q(B_{n-p})\otimes U_q(C_p)$ | $(2(n-p)+1,1)\oplus(1,2p)$ |
| $B^{(1)}_{n}$ | $U_q(B_{n-p})\otimes U_q(D_p)$ $(p\neq 1)$ | $(2(n-p)+1,1)\oplus(1,2p)$ |
| $C^{(1)}_{n}$ | $U_q(C_{n-p})\otimes U_q(C_p)$ | $(2(n-p),1)\oplus(1,2p)$ |
| $D^{(1)}_{n}$ | $U_q(D_{n-p})\otimes U_q(D_p)$ $(n>1\,,p\neq 1\,,n-1)$ | $(2(n-p),1)\oplus(1,2p)$ |
| $D^{(2)}_{n+1}$ | $U_q(B_{n-p})\otimes U_q(B_p)$ | $(2(n-p)+1,1)\oplus(1,2p+1)$ |

Table 8.2: QG symmetries of the transfer matrix $t(u,p)$, where $p=0,1,\ldots,n$.

.

$0<p<n$, the QG symmetries are given by a tensor product of two factors, to which we refer as the "left" and "right" factors. For $p=0$, the "right" factors are absent; while for $p=n$, the "left" factors are absent. That is,

$$\begin{aligned}
\Big[\Delta_N(H^{(l)}_i(p))\,,t(u,p)\Big] &= \Big[\Delta_N(E^{\pm(l)}_i(p))\,,t(u,p)\Big]=0\,, \qquad i=1\,,\ldots\,,n-p\,,\\
\Big[\Delta_N(H^{(r)}_i(p))\,,t(u,p)\Big] &= \Big[\Delta_N(E^{\pm(r)}_i(p))\,,t(u,p)\Big]=0\,, \qquad i=1\,,\ldots\,,p\,,
\end{aligned} \tag{8.11}$$

where $H^{(l)}_i(p)\,,E^{\pm(l)}_i(p)$ are generators of the "left" algebra $g^{(l)}$; $H^{(r)}_i(p)\,,E^{\pm(r)}_i(p)$ are generators of the "right" algebra $g^{(r)}$; and $\Delta_N$ denotes the $N$-fold coproduct[2].

It can be shown in a similar way that the transfer matrices for the "special" cases (8.5), which are constructed using the K-matrices (8.6)-(8.7), have the QG symmetries in Table 8.3.[3]

| $\hat{g}$ | QG symmetry | Representation at each site |
|---|---|---|
| $A^{(2)}_{2n-1}(p=1)$ | $U_q(C_n)$ | $2n$ |
| $B^{(1)}_{n}$ $(p=1)$ | $U_q(B_n)$ | $2n+1$ |
| $D^{(1)}_{n}$ $(n>1,p=1,n-1)$ | $U_q(D_n)$ | $2n$ |

Table 8.3: QG symmetries of the transfer matrix $t(u,p)$ for the "special" cases (8.5).

The QG symmetries displayed in Tables 8.2 and 8.3 correspond to removing the $p^{th}$ node from the $\hat{g}$ Dynkin diagram, as can be seen in Fig. 8.1.

For the cases $C^{(1)}_n, D^{(1)}_n$ and $D^{(2)}_{n+1}$ (i.e., the last three rows of Table 8.2), the transfer matrices have a $p\leftrightarrow n-p$ duality symmetry

$$\mathcal{U}\,t(u,p)\,\mathcal{U}^{-1} = f(u,p)\,t(u,n-p)\,, \tag{8.12}$$

see [88, 89] for explicit expressions for the quantum-space operator $\mathcal{U}$ and the scalar factor $f(u,p)$. In particular, for $p=\frac{n}{2}$ ($n$ even), the transfer matrix is self-dual

$$\big[\mathcal{U}\,,t(u,\tfrac{n}{2})\big]=0\,, \tag{8.13}$$

[2]The explicit form of $\Delta_N$ for $N=2$ can be found in [88, 89]

[3]The symmetries for the "special" cases with $p=1$ are the same as for $p=0$, while the symmetry for $D^{(1)}_n$ with $p=n-1$ is the same as for $p=n$. (See Table 8.2.) These observations can be readily understood from the Dynkin diagrams, see Fig. 8.1.

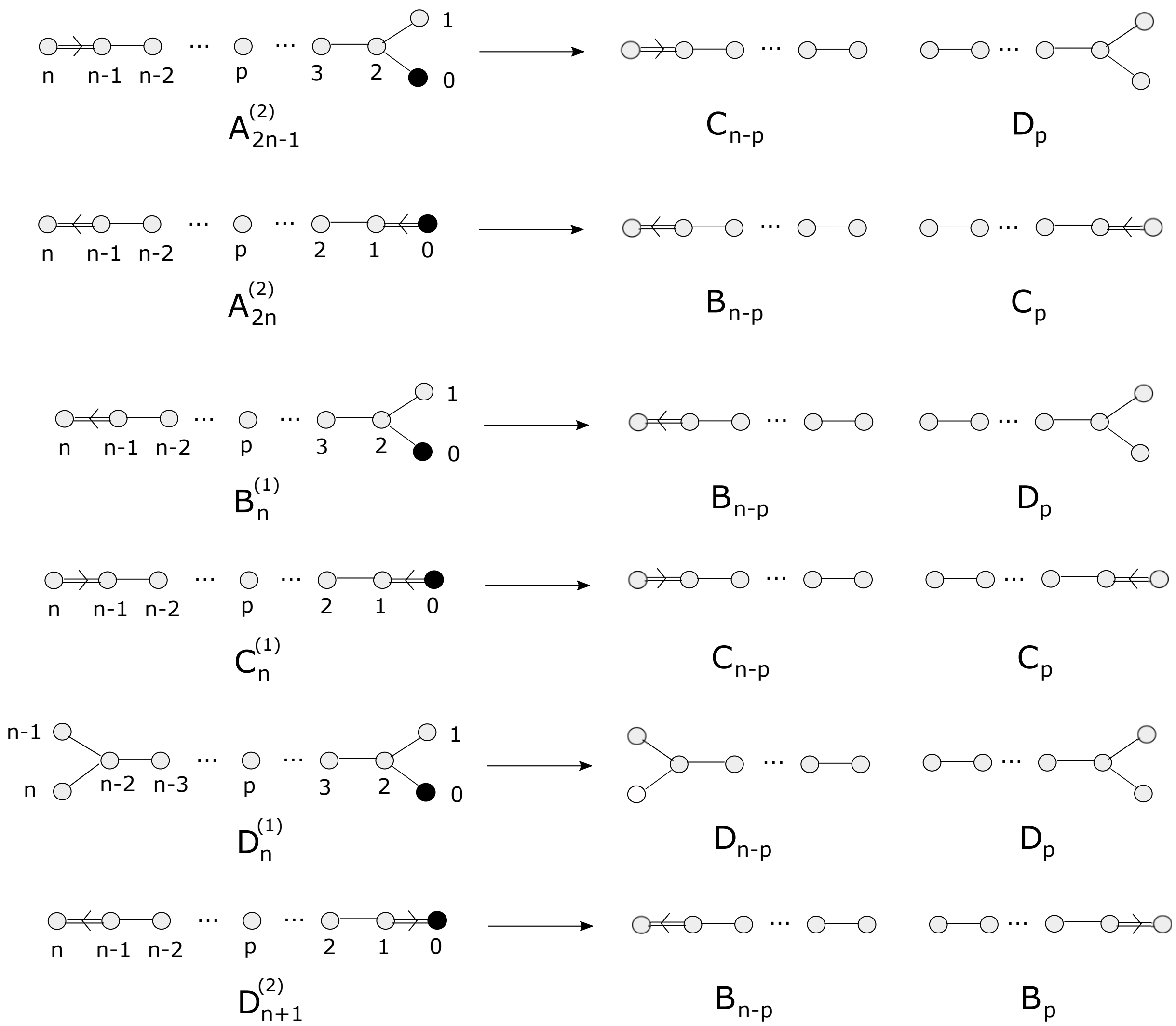


Figure 8.1: Subalgebras of affine Lie algebras corresponding to removing the $p^{th}$ node from the extended Dynkin diagram.

since $f(u, \frac{n}{2}) = 1$. For $p = \frac{n}{2}$ ($n$ even) and $\varepsilon = 1$, there is an additional ("bonus") symmetry, which leads to even higher degeneracies for the transfer-matrix eigenvalues [88, 89].

The cases $A^{(2)}_{2n-1}, B^{(1)}_n$ and $D^{(1)}_n$ (for which the "right" factor in Table 8.2 is $U_q(D_p)$) have a "right" $Z_2$ symmetry

$$\left[\mathcal{Z}^{(r)}\,, t(u,p)\right] = 0\,; \tag{8.14}$$

and the case $D^{(1)}_n$ (for which the "left" factor in Table 8.2 is $U_q(D_{n-p})$) also has a "left" $Z_2$ symmetry

$$\left[\mathcal{Z}^{(l)}\,, t(u,p)\right] = 0\,, \tag{8.15}$$

see [88] for explicit expressions for the quantum-space operators $\mathcal{Z}^{(r)}$ and $\mathcal{Z}^{(l)}$.

## 8.2 Analytical Bethe ansatz

We now proceed to determine the spectrum of the transfer matrix (7.20) for all the cases in Tables 8.2 and 8.3. The results hold for both values $\varepsilon = 0, 1$ except for the case $D^{(2)}_{n+1}$, where we consider only $\varepsilon = 0$. The results for some of these cases have already been known:

- For $p = 0$:
  - $A^{(2)}_{2n}$ [68, 84, 85, 86]
  - $A^{(2)}_{2n-1}$ [69, 85, 87]
  - $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$ [69, 85]
- For $p = n$:
  - $A^{(2)}_{2n}$ [84, 86]
  - $A^{(2)}_{2n-1}$, $D^{(2)}_{n+1}$ [87]
- For $0 < p < n$:
  - $A^{(2)}_{2n}$ [84]

The eigenvalues of the transfer matrix are determined in Secs. 8.2.1, 8.2.2, and the corresponding Bethe equations are obtained in Sec. 8.2.3.

### 8.2.1 Eigenvalues of the transfer matrix

The transfer matrix and Cartan generators can be diagonalized simultaneously

$$\begin{aligned} t(u,p)\,|\Lambda^{(m_1,\dots,m_n)}\rangle &= \Lambda^{(m_1,\dots,m_n)}(u,p)\,|\Lambda^{(m_1,\dots,m_n)}\rangle\,, \\ \Delta_N(H^{(l)}_i(p))\,|\Lambda^{(m_1,\dots,m_n)}\rangle &= h^{(l)}_i\,|\Lambda^{(m_1,\dots,m_n)}\rangle\,, \qquad i = 1,\dots,n-p\,, \\ \Delta_N(H^{(r)}_i(p))\,|\Lambda^{(m_1,\dots,m_n)}\rangle &= h^{(r)}_i\,|\Lambda^{(m_1,\dots,m_n)}\rangle\,, \qquad i = 1,\dots,p\,, \end{aligned} \tag{8.16}$$

as follows from (7.22) and (8.11). We focus here on determining the transfer matrix eigenvalues $\Lambda^{(m_1,\dots,m_n)}(u,p)$; the eigenvalues of the Cartan generators $h^{(l)}_i, h^{(r)}_i$ are determined in Sec. 8.3.1.

We take the analytical Bethe ansatz approach, whereby the eigenvalues of the transfer matrix are obtained by "dressing" the reference-state eigenvalue. The "dressing" is assumed to be "doubled" with respect to the corresponding closed chain. Hence, the main difficulty is to determine the reference-state eigenvalue. For the reference state corresponding to the cases in Table 8.2, we choose [4]

$$|0\,,p\rangle = v_p^{\otimes N}\,, \qquad v_p = \begin{cases} e_1 & \text{for} \quad p = 0 \\ e_p & \text{for} \quad p = 1\,,\dots\,,n \end{cases}\,, \tag{8.17}$$

where $e_i$ are $d$-dimensional elementary basis vectors $(e_i)_j = \delta_{i,j}$. Like the usual reference state $e_1^{\otimes N}$, the state (8.17) is an eigenstate of the transfer matrix with no Bethe roots ($m_1 = \ldots = m_n = 0$)

$$t(u,p)\,|0\,,p\rangle = \Lambda^{(0,\dots,0)}(u,p)\,|0\,,p\rangle\,. \tag{8.18}$$

[4] For the special cases in Table 8.3, we choose (see again footnote 3) the reference state $|0\,,0\rangle$ for $A^{(2)}_{2n-1}$, $B^{(1)}_n$, $D^{(1)}_n$ with $p = 1$; and the reference state $|0\,,n\rangle$ for $D^{(1)}_n$ with $p = n-1$.

However, in addition, this state is a highest weight of the "left" algebra

$$\Delta_N(E_i^{+(l)}(p))\,|0\,,p\rangle = 0\,, \qquad i = 1,\ldots,n-p\,, \tag{8.19}$$

and a lowest weight of the "right" algebra

$$\Delta_N(E_i^{-(r)}(p))\,|0\,,p\rangle = 0\,. \qquad i = 1,\ldots,p\,. \tag{8.20}$$

In view of the crossing invariance (7.23) and the known results for $p=0$ [68, 69] and for $D_{n+1}^{(2)}$ [87], we propose that the eigenvalues of the transfer matrix for general values of $p$ are given by the T-Q equation

$$\begin{aligned}\Lambda^{(m_1,\ldots,m_n)}(u,p) =&\phi(u,p)\Bigg\{A(u)\,z_0(u)\,y_0(u,p)\,c(u)^{2N} + \tilde{A}(u)\,\tilde{z}_0(u)\,\tilde{y}_0(u,p)\,\tilde{c}(u)^{2N} \\ &+ \Big\{\sum_{l=1}^{n-1}\Big[z_l(u)\,y_l(u,p)\,B_l(u) + \tilde{z}_l(u)\,\tilde{y}_l(u,p)\,\tilde{B}_l(u)\Big] + w_1(u)\,y_n(u,p)\,B_n(u) \\ &+ w_2\Big[z_n(u)\,y_n(u,p)\,B_n(u) + \tilde{z}_n(u)\,\tilde{y}_n(u,p)\,\tilde{B}_n(u)\Big]\Big\}\,b(u)^{2N}\Bigg\}\,.\end{aligned} \tag{8.21}$$

The overall factor $\phi(u,p)$ is given by

$$\phi(u,p) = \begin{cases} (-1)^{\xi}\left(\frac{\gamma e^{u}+1}{\gamma+e^{u}}\right)\left(\frac{\gamma e^{-u-\rho}+1}{\gamma+e^{-u-\rho}}\right) & \text{for} \quad A_{2n}^{(2)}\,, C_n^{(1)}\,, A_{2n-1}^{(2)}\,(p\neq 1)\,, \\ & \quad B_n^{(1)}\,(p\neq 1)\,, D_n^{(1)}\,(p\neq 1\,,n-1) \\ (-1)^{\xi} & \text{for} \quad A_{2n-1}^{(2)}\,(p=1)\,, B_n^{(1)}\,(p=1)\,, \\ & \quad D_n^{(1)}\,(p=1\,,n-1) \\ \frac{\cosh(u-(n-2p)\eta)\,\cosh(u-(n+2p)\eta)}{\cosh(u+(n-2p)\eta)\,\cosh(u-(3n-2p)\eta)} & \text{for} \quad D_{n+1}^{(2)} \end{cases} \tag{8.22}$$

where $\gamma$ is defined in (7.15), and the parameters $\xi$ and $\rho$ are given in Table 8.1. The tilde denotes crossing $\tilde{A}(u) = A(-u-\rho)$, etc. The functions $A(u)$ and $B_l(u)$ for $\hat{g} = A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$ (for $n>1$), $B_n^{(1)}$, $C_n^{(1)}$ (for $n>1$), $D_n^{(1)}$ (for $n>2$) are defined as

$$\begin{aligned} A(u) &= \frac{Q^{[1]}(u+2\eta)}{Q^{[1]}(u-2\eta)}\,, \\ B_l(u) &= \frac{Q^{[l]}(u-2(l+2)\eta)}{Q^{[l]}(u-2l\eta)}\,\frac{Q^{[l+1]}(u-2(l-1)\eta)}{Q^{[l+1]}(u-2(l+1)\eta)}\,, \\ &\quad l=1,...,n-3 \text{ for } D_n^{(1)} \\ &\quad l=1,...,n-2 \text{ for } A_{2n-1}^{(2)}\,, C_n^{(1)} \\ &\quad l=1,...,n-1 \text{ for } A_{2n}^{(2)}\,, B_n^{(1)}\,. \end{aligned} \tag{8.23}$$

Moreover, for the values of $l$ not included above:

$$A_{2n-1}^{(2)}: \qquad B_{n-1}(u) = \frac{Q^{[n-1]}(u-2(n+1)\eta)}{Q^{[n-1]}(u-2(n-1)\eta)}\,\frac{Q^{[n]}(u-2(n-2)\eta)}{Q^{[n]}(u-2n\eta)} \times \frac{Q^{[n]}(u-2(n-2)\eta+i\pi)}{Q^{[n]}(u-2n\eta+i\pi)}\,, \tag{8.24}$$

$$A_{2n}^{(2)}: \qquad B_n(u) = \frac{Q^{[n]}(u-2(n+2)\eta)}{Q^{[n]}(u-2n\eta)}\,\frac{Q^{[n]}(u-2(n-1)\eta+i\pi)}{Q^{[n]}(u-2(n+1)\eta+i\pi)}\,, \tag{8.25}$$

$$B_n^{(1)}: \qquad B_n(u) = \frac{Q^{[n]}(u-2(n-2)\eta)}{Q^{[n]}(u-2n\eta)}\,\frac{Q^{[n]}(u-2(n+1)\eta)}{Q^{[n]}(u-2(n-1)\eta)}\,, \tag{8.26}$$

$$C_n^{(1)}: \qquad B_{n-1}(u) = \frac{Q^{[n-1]}(u-2(n+1)\eta)}{Q^{[n-1]}(u-2(n-1)\eta)}\,\frac{Q^{[n]}(u-2(n-3)\eta)}{Q^{[n]}(u-2(n+1)\eta)}\,, \tag{8.27}$$

$$D_n^{(1)}: \qquad B_{n-2}(u) = \frac{Q^{[n-2]}(u-2n\eta)}{Q^{[n-2]}(u-2(n-2)\eta)} \frac{Q^{[n-1]}(u-2(n-3)\eta)}{Q^{[n-1]}(u-2(n-1)\eta)} \frac{Q^{[n]}(u-2(n-3)\eta)}{Q^{[n]}(u-2(n-1)\eta)}\,, \tag{8.28}$$

$$B_{n-1}(u) = \frac{Q^{[n-1]}(u-2(n-3)\eta)}{Q^{[n-1]}(u-2(n-1)\eta)} \frac{Q^{[n]}(u-2(n+1)\eta)}{Q^{[n]}(u-2(n-1)\eta)}\,. \tag{8.29}$$

For the values of $n$ not included above:

$$A_1^{(2)}: \quad A(u) = \frac{Q^{[1]}(u+2\eta)\, Q^{[1]}(u+2\eta+i\pi)}{Q^{[1]}(u-2\eta)\, Q^{[1]}(u-2\eta+i\pi)}\,, \tag{8.30}$$

$$C_1^{(1)}: \quad A(u) = \frac{Q^{[1]}(u+4\eta)}{Q^{[1]}(u-4\eta)}\,, \tag{8.31}$$

$$D_2^{(1)}: \quad A(u) = \frac{Q^{[1]}(u+2\eta)}{Q^{[1]}(u-2\eta)} \frac{Q^{[2]}(u+2\eta)}{Q^{[2]}(u-2\eta)}\,,$$

$$B_1(u) = \frac{Q^{[1]}(u-6\eta)}{Q^{[1]}(u-2\eta)} \frac{Q^{[2]}(u+2\eta)}{Q^{[2]}(u-2\eta)}\,. \tag{8.32}$$

Finally, for $D_{n+1}^{(2)}$:

$$\begin{aligned} A(u) &= \frac{Q^{[1]}(u+\eta)}{Q^{[1]}(u-\eta)} \frac{Q^{[1]}(u+\eta+i\pi)}{Q^{[1]}(u-\eta+i\pi)}, \\ B_l(u) &= \frac{Q^{[l]}(u-(l+2)\eta)}{Q^{[l]}(u-l\eta)} \frac{Q^{[l]}(u-(l+2)\eta+i\pi)}{Q^{[l]}(u-l\eta+i\pi)} \\ &\quad \times \frac{Q^{[l+1]}(u-(l-1)\eta)}{Q^{[l+1]}(u-(l+1)\eta)} \frac{Q^{[l+1]}(u-(l-1)\eta+i\pi)}{Q^{[l+1]}(u-(l+1)\eta+i\pi)}, \qquad l = 1, ..., n-1, \\ B_n(u) &= \frac{Q^{[n]}(u-(n+2)\eta)}{Q^{[n]}(u-n\eta)} \frac{Q^{[n]}(u-(n-2)\eta+i\pi)}{Q^{[n]}(u-n\eta+i\pi)}\,. \end{aligned} \tag{8.33}$$

In the above equations (8.23) - (8.33) for the functions $A(u)$ and $B_l(u)$, the functions $Q^{[l]}(u)$ are given by

$$Q^{[l]}(u) = \prod_{j=1}^{m_l} \sinh\left(\tfrac{1}{2}(u-u_j^{[l]})\right) \sinh\left(\tfrac{1}{2}(u+u_j^{[l]})\right)\,, \qquad Q^{[l]}(-u) = Q^{[l]}(u)\,, \tag{8.34}$$

where the zeros $u_j^{[l]}$ (and their number $m_l$) are still to be determined. Note that these expressions for $A(u)$ and $B_l(u)$ are "doubled" with respect to those in [94] for the corresponding closed chains.

The functions $c(u)$ and $b(u)$ are given by

$$c(u) = \begin{cases} 2\sinh\left(\frac{u}{2}-2\eta\right)\cosh\left(\frac{u}{2}-\kappa\eta\right) & \text{for} \quad A_{2n}^{(2)}\,, A_{2n-1}^{(2)}\,, \\ 2\sinh\left(\frac{u}{2}-2\eta\right)\sinh\left(\frac{u}{2}-\kappa\eta\right) & \text{for} \quad B_n^{(1)}\,, C_n^{(1)}\,, D_n^{(1)}\,, \\ 4\sinh\left(u-2\eta\right)\sinh\left(u-\kappa\eta\right) & \text{for} \quad D_{n+1}^{(2)}\,, \end{cases} \tag{8.35}$$

and

$$b(u) = \begin{cases} 2\sinh\left(\frac{u}{2}\right)\cosh\left(\frac{u}{2}-\kappa\eta\right) & \text{for} \quad A_{2n}^{(2)}\,, A_{2n-1}^{(2)}\,, \\ 2\sinh\left(\frac{u}{2}\right)\sinh\left(\frac{u}{2}-\kappa\eta\right) & \text{for} \quad B_n^{(1)}\,, C_n^{(1)}\,, D_n^{(1)}\,, \\ 4\sinh\left(u\right)\sinh\left(u-\kappa\eta\right) & \text{for} \quad D_{n+1}^{(2)}\,. \end{cases} \tag{8.36}$$

For all $\hat{g}$ except $D_{n+1}^{(2)}$, the functions $z_l(u)$ are given by

$$z_l(u) = \frac{\sinh u \sinh(u-2\kappa\eta)\cosh\left(u-\omega\eta+(2-\delta)\frac{i\pi}{4}\right)}{\sinh(u-2l\eta)\sinh\left(u-2(l+1)\eta\right)\cosh\left(u-\kappa\eta+(2-\delta)\frac{i\pi}{4}\right)}\,, \tag{8.37}$$

where $\omega$ and $\delta$ are given in Table 8.1. For $D_{n+1}^{(2)}$

$$z_l(u) = \begin{cases} \frac{\cosh(u-(n-1)\eta)\,\sinh(2u-4n\eta)\,\sinh(u-(n+1)\eta)\sinh(2u)}{\sinh(u-n\eta)\,\cosh(u-n\eta)\,\sinh(2u-2l\eta)\,\sinh(2u-2(l+1)\eta)} & l = 0, ..., n-1\,, \\ z_{n-1}(u)\frac{\sinh(u-(n-1)\eta)}{\sinh(u-(n+1)\eta)} & l = n\,. \end{cases} \tag{8.38}$$

Finally, the quantities $w_1(u)$ and $w_2$ are defined as

$$w_1(u) = \begin{cases} \frac{\sinh u \sinh(u-2\kappa\eta)}{\sinh(u-(\kappa+1)\eta)\sinh(u-(\kappa-1)\eta)} & \text{for} \quad A^{(2)}_{2n}\,, B^{(1)}_{n}\,, \\ 0 & \text{for} \quad A^{(2)}_{2n-1}\,, C^{(1)}_{n}\,, D^{(1)}_{n}\,, D^{(2)}_{n+1}\,, \end{cases}$$

$$w_2 = \begin{cases} 1 & \text{for} \quad D^{(2)}_{n+1} \\ 0 & \text{otherwise} \end{cases}\,. \tag{8.39}$$

In the expression (8.21) for the transfer-matrix eigenvalue, only the functions $y_l(u,p)$ remain to be specified. For $y_l(u,p) = 1$, the expression (8.21) reduces (apart from the overall factor) to the transfer-matrix eigenvalue for the case $p=0$ for all the cases except $D^{(2)}_{n+1}$ [68, 69]. The functions $y_l(u,p)$ for general values of $p$ are determined in the following section.

### 8.2.2 Determining $y_l(u,p)$

We now proceed to determine the functions $y_0(u,p)\,, \ldots\,, y_n(u,p)$ for general values of $p$. We emphasize that these are the only functions (besides the overall factor $\phi(u,p)$ (8.22), through the quantity $\gamma$ (7.15)) in the expression (8.21) for the transfer-matrix eigenvalue with explicit dependence on $p$.

For the special cases in Table 8.3, the functions $y_l(u,p)$ are simply given by

$$y_l(u,p) = 1\,, \qquad l = 0\,, \ldots\,, n\,, \tag{8.40}$$

i.e., the same as for the case $p=0$. We therefore focus our attention in the remainder of this section on the cases in Table 8.2.

We make the ansatz

$$y_l(u,p) = \begin{cases} F(u) & \text{for} \quad 0 \le l \le p-1 \\ G(u) & \text{for} \quad p \le l \le n \end{cases}\,, \tag{8.41}$$

and

$$\tilde{G}(u) \equiv G(-u-\rho) = G(u)\,, \tag{8.42}$$

which guarantees that the only Bethe equation with an extra factor (in comparison with the case $p=0$) is the equation for the $p^{th}$ Bethe roots $\{u^{[p]}_j\}$, as discussed further in Sec. 8.2.3.

The explicit form of $F(u)$ and $G(u)$ are

$$G(u) = \frac{\cosh\left(\frac{u}{2} - \frac{\omega\eta}{2} - (\delta-4\varepsilon)\frac{i\pi}{8}\right)\cosh\left(\frac{u}{2} - \frac{\bar{\omega}\eta}{2} - (\delta-4\varepsilon)\frac{i\pi}{8}\right)}{\cosh\left(\frac{u}{2} - \frac{(\omega-4p)\eta}{2} - (\delta-4\varepsilon)\frac{i\pi}{8}\right)\cosh\left(\frac{u}{2} - \frac{(\bar{\omega}+4p)\eta}{2} - (\delta-4\varepsilon)\frac{i\pi}{8}\right)}\,, \tag{8.43}$$

$$F(u) = \frac{\cosh^2\left(\frac{u}{2} + \frac{(\omega-4p)\eta}{2} + (\delta-4\varepsilon)\frac{i\pi}{8}\right)}{\cosh^2\left(\frac{u}{2} - \frac{\omega\eta}{2} - (\delta-4\varepsilon)\frac{i\pi}{8}\right)}\,G(u)\,, \tag{8.44}$$

for $\hat{g} = A^{(2)}_{2n}$, $A^{(2)}_{2n-1}$, $B^{(1)}_{n}$, $C^{(1)}_{n}$, $D^{(1)}_{n}$. Note that $\omega\,, \bar{\omega}\,, \delta$ are given in Table 8.1. Moreover,

$$G(u) = \frac{\cosh^2(u-n\eta)}{\cosh(u-(n-2p)\eta)\,\cosh(u-(n+2p)\eta)}\,, \tag{8.45}$$

$$F(u) = \frac{\cosh^2\left(u+(n-2p)\eta\right)}{\cosh^2\left(u-n\eta\right)}\,G(u)\,, \tag{8.46}$$

for $\hat{g} = D^{(2)}_{n+1}$. Note that $G(u) = 1$ for $p=0$ in all cases. The rest of this section is dedicated to explaining how the above expressions can be obtained, starting with $F(u)$.

According to (8.41), $y_0(u,p)$ is equal to $F(u)$ for any value of $p$ except $p=0$. We can use this fact to determine $F(u)$ by arranging to kill all the terms in (8.21) except the one with $y_0(u,p)$, which can be accomplished by judiciously introducing inhomogeneities. Indeed, it is well known that arbitrary inhomogeneities $\{\theta_i\}$ can be introduced in the transfer matrix $t(u,p\,;\{\theta_i\})$ while maintaining the commutativity property

$$[t(u,p\,;\{\theta_i\})\,, t(v,p\,;\{\theta_i\})] = 0\,. \tag{8.47}$$

By appropriately choosing the inhomogeneities, all the terms in (8.21) except the first one can be made to vanish. A similar procedure has been used in e.g. [87, 74].

As an example, let us consider the case $A^{(2)}_{2n}$. The only effect on the eigenvalue (8.21) of introducing inhomogeneities $\{\theta_i\}$ in the transfer matrix is to modify the expressions for $c(u)$, $\tilde{c}(u)$ and $b(u)$ (8.35), (8.36) as follows:

$$c(u)^{2N} = \left[2\sinh\left(\frac{u}{2} - 2\eta\right)\cosh\left(\frac{u}{2} - \kappa\eta\right)\right]^{2N}$$
$$\longmapsto \prod_{i=1}^{N}\left[2\sinh\left(\frac{u+\theta_i}{2} - 2\eta\right)\cosh\left(\frac{u+\theta_i}{2} - \kappa\eta\right)\right]\left[2\sinh\left(\frac{u-\theta_i}{2} - 2\eta\right)\cosh\left(\frac{u-\theta_i}{2} - \kappa\eta\right)\right], \quad (8.48)$$

$$\tilde{c}(u)^{2N} = \left[2\sinh\left(\frac{u}{2}\right)\cosh\left(\frac{u}{2} - (\kappa-2)\,\eta\right)\right]^{2N}$$
$$\longmapsto \prod_{i=1}^{N}\left[2\sinh\left(\frac{u+\theta_i}{2}\right)\cosh\left(\frac{u+\theta_i}{2} - (\kappa-2)\,\eta\right)\right]\left[2\sinh\left(\frac{u-\theta_i}{2}\right)\cosh\left(\frac{u-\theta_i}{2} - (\kappa-2)\,\eta\right)\right], \quad (8.49)$$

$$b(u)^{2N} = \left[2\sinh\left(\frac{u}{2}\right)\cosh\left(\frac{u}{2} - \kappa\eta\right)\right]^{2N}$$
$$\longmapsto \prod_{i=1}^{N}\left[2\sinh\left(\frac{u+\theta_i}{2}\right)\cosh\left(\frac{u+\theta_i}{2} - \kappa\,\eta\right)\right]\left[2\sinh\left(\frac{u-\theta_i}{2}\right)\cosh\left(\frac{u-\theta_i}{2} - \kappa\,\eta\right)\right]. \quad (8.50)$$

By choosing $\theta_i = u$, the modified expressions for $\tilde{c}(u)$ and $b(u)$ (but not $c(u)$) evidently become zero; hence, the only term in (8.21) that survives is the first term, which is proportional to $y_0(u,p) = F(u)$. On the other hand, by acting with the transfer matrix $t(u,p\,;\{\theta_i = u\})$ with $N = 1$ and $n = p = 1$ on the reference state (8.17), we explicitly obtain the corresponding eigenvalue. Comparing these two results, keeping in mind that the reference state is the Bethe state with no Bethe roots and therefore $A(u) = 1$, we can solve for $F(u)$. By repeating this procedure for $n = 2$ and $p = 1, 2$, we infer the general result (8.44), which can then be easily checked in a similar way for higher values of $n, p, N$.

In order to determine $G(u)$, we return to the homogeneous case $\theta_i = 0$, so that all the functions $y_0(u,p), \ldots, y_n(u,p)$ again appear in (8.21). Using (8.21), the ansätze (8.41) and (8.42), and the result (8.44) for $F(u)$, we obtain an expression for the reference-state eigenvalue ($A(u) = B_l(u) = 1$) in terms of $G(u)$. We also calculate this eigenvalue explicitly by acting with $t(u,p)$ (with $N = 1$) on the reference state (8.17). By comparing both expressions, we can solve for $G(u)$. We again use the results for small values of $n$ and $p$ to infer the general result (8.43). Having obtained both $F(u)$ and $G(u)$ for general values of $n$ and $p$, the reference-state eigenvalue can be easily checked for higher values of $n, p, N$.

Using the same procedure for the other $\hat{g}$ in Table 8.2, we arrive at the results (8.43) - (8.46). As already noted, for the special cases in Table 8.3, we have $y_l(u,p) = 1$ (8.40).

### 8.2.3 Bethe equations

The expression (8.21) for the transfer-matrix eigenvalues is in terms of the zeros $u^{[l]}_j$ of the functions $Q^{[l]}(u)$, which are still to be determined. In principle, these zeros can be determined by solving corresponding Bethe equations, which we now present. We find that these Bethe equations are the same as for the case $p = 0$ [68, 69], except for the presence of an extra factor $\Phi_{l,p,n}(u)$ (8.76), (8.83) that is different from 1 only if $l = p$. The only dependence on $p$ in the Bethe equations is in this factor.

**For $\hat{g} = A^{(2)}_{2n}\,, A^{(2)}_{2n-1}\,, B^{(1)}_{n}\,, C^{(1)}_{n}\,, D^{(1)}_{n}$**

We determine the Bethe equations from the requirement that the expression (8.21) for the transfer-matrix eigenvalues have vanishing residues at the poles. In this way, we obtain the following Bethe equations for all the cases in Tables 8.2 and 8.3 except for $D^{(2)}_{n+1}$:

$$\left[\frac{\sinh\left(\frac{u^{[1]}_k}{2} + \eta\right)}{\sinh\left(\frac{u^{[1]}_k}{2} - \eta\right)}\right]^{2N} \Phi_{1,p,n}(u^{[1]}_k) = \frac{Q^{[1]}_k\left(u^{[1]}_k + 4\eta\right)}{Q^{[1]}_k\left(u^{[1]}_k - 4\eta\right)}\,\frac{Q^{[2]}\left(u^{[1]}_k - 2\eta\right)}{Q^{[2]}\left(u^{[1]}_k + 2\eta\right)}\,, \quad k = 1, \ldots, m_1\,, \quad (8.51)$$

$$\Phi_{l,p,n}(u_k^{[l]}) = \frac{Q^{[l-1]}\left(u_k^{[l]}-2\eta\right)}{Q^{[l-1]}\left(u_k^{[l]}+2\eta\right)} \frac{Q_k^{[l]}\left(u_k^{[l]}+4\eta\right)}{Q_k^{[l]}\left(u_k^{[l]}-4\eta\right)} \frac{Q^{[l+1]}\left(u_k^{[l]}-2\eta\right)}{Q^{[l+1]}\left(u_k^{[l]}+2\eta\right)}\,, \quad k=1,\ldots,m_l\,, \tag{8.52}$$

$$\begin{aligned} &l=1,...,n-3 \quad \text{for} \quad D_n^{(1)}\ (n>2) \\ &l=1,...,n-2 \quad \text{for} \quad C_n^{(1)}\ (n>1),\ A_{2n-1}^{(2)}\ (n>1) \\ &l=1,...,n-1 \quad \text{for} \quad A_{2n}^{(2)}, B_n^{(1)}\,, \end{aligned}$$

where $Q^{[l]}(u)$ is given by (8.34), and $Q_k^{[l]}(u)$ is defined by

$$Q_k^{[l]}(u) = \prod_{j=1, j\neq k}^{m_l} \sinh\left(\tfrac{1}{2}(u-u_j^{[l]})\right) \sinh\left(\tfrac{1}{2}(u+u_j^{[l]})\right)\,. \tag{8.53}$$

Moreover, for the values of $l$ not included above:

$A_{2n-1}^{(2)}$ :

$$\begin{aligned} \Phi_{n-1,p,n}(u_k^{[n-1]}) &= \frac{Q^{[n-2]}\left(u_k^{[n-1]}-2\eta\right)}{Q^{[n-2]}\left(u_k^{[n-1]}+2\eta\right)} \frac{Q_k^{[n-1]}\left(u_k^{[n-1]}+4\eta\right)}{Q_k^{[n-1]}\left(u_k^{[n-1]}-4\eta\right)} \\ &\times \frac{Q^{[n]}\left(u_k^{[n-1]}-2\eta\right)}{Q^{[n]}\left(u_k^{[n-1]}+2\eta\right)} \frac{Q^{[n]}\left(u_k^{[n-1]}-2\eta+i\pi\right)}{Q^{[n]}\left(u_k^{[n-1]}+2\eta+i\pi\right)}\,, \end{aligned} \tag{8.54}$$

$$\begin{aligned} \Phi_{n,p,n}(u_k^{[n]}) &= \frac{Q^{[n-1]}\left(u_k^{[n]}-2\eta\right)}{Q^{[n-1]}\left(u_k^{[n]}+2\eta\right)} \frac{Q^{[n-1]}\left(u_k^{[n]}-2\eta+i\pi\right)}{Q^{[n-1]}\left(u_k^{[n]}+2\eta+i\pi\right)} \\ &\times \frac{Q_k^{[n]}\left(u_k^{[n]}+4\eta\right)}{Q_k^{[n]}\left(u_k^{[n]}-4\eta\right)} \frac{Q_k^{[n]}\left(u_k^{[n]}+4\eta+i\pi\right)}{Q_k^{[n]}\left(u_k^{[n]}-4\eta+i\pi\right)}\,, \end{aligned} \tag{8.55}$$

$A_{2n}^{(2)}$ :

$$\Phi_{n,p,n}(u_k^{[n]}) = \frac{Q^{[n-1]}\left(u_k^{[n]}-2\eta\right)}{Q^{[n-1]}\left(u_k^{[n]}+2\eta\right)} \frac{Q_k^{[n]}\left(u_k^{[n]}+4\eta\right)}{Q_k^{[n]}\left(u_k^{[n]}-4\eta\right)} \frac{Q_k^{[n]}\left(u_k^{[n]}-2\eta+i\pi\right)}{Q_k^{[n]}\left(u_k^{[n]}+2\eta+i\pi\right)}\,, \tag{8.56}$$

$B_n^{(1)}$ :

$$\Phi_{n,p,n}(u_k^{[n]}) = \frac{Q^{[n-1]}\left(u_k^{[n]}-2\eta\right)}{Q^{[n-1]}\left(u_k^{[n]}+2\eta\right)} \frac{Q_k^{[n]}\left(u_k^{[n]}+2\eta\right)}{Q_k^{[n]}\left(u_k^{[n]}-2\eta\right)}\,, \tag{8.57}$$

$C_n^{(1)}$ :

$$\Phi_{n-1,p,n}(u_k^{[n-1]}) = \frac{Q^{[n-2]}\left(u_k^{[n-1]}-2\eta\right)}{Q^{[n-2]}\left(u_k^{[n-1]}+2\eta\right)} \frac{Q_k^{[n-1]}\left(u_k^{[n-1]}+4\eta\right)}{Q_k^{[n-1]}\left(u_k^{[n-1]}-4\eta\right)} \frac{Q^{[n]}\left(u_k^{[n-1]}-4\eta\right)}{Q^{[n]}\left(u_k^{[n-1]}+4\eta\right)}\,, \tag{8.58}$$

$$\Phi_{n,p,n}(u_k^{[n]}) = \frac{Q^{[n-1]}\left(u_k^{[n]}-4\eta\right)}{Q^{[n-1]}\left(u_k^{[n]}+4\eta\right)} \frac{Q_k^{[n]}\left(u_k^{[n]}+8\eta\right)}{Q_k^{[n]}\left(u_k^{[n]}-8\eta\right)}\,, \tag{8.59}$$

$D_n^{(1)}$ :

$$\begin{aligned} \Phi_{n-2,p,n}(u_k^{[n-2]}) &= \frac{Q^{[n-3]}\left(u_k^{[n-2]}-2\eta\right)}{Q^{[n-3]}\left(u_k^{[n-2]}+2\eta\right)} \frac{Q_k^{[n-2]}\left(u_k^{[n-2]}+4\eta\right)}{Q_k^{[n-2]}\left(u_k^{[n-2]}-4\eta\right)} \\ &\times \frac{Q^{[n-1]}\left(u_k^{[n-2]}-2\eta\right)}{Q^{[n-1]}\left(u_k^{[n-2]}+2\eta\right)} \frac{Q^{[n]}\left(u_k^{[n-2]}-2\eta\right)}{Q^{[n]}\left(u_k^{[n-2]}+2\eta\right)}\,, \end{aligned} \tag{8.60}$$

$$\Phi_{n-1,p,n}(u_k^{[n-1]}) = \frac{Q^{[n-2]}\left(u_k^{[n-1]}-2\eta\right)}{Q^{[n-2]}\left(u_k^{[n-1]}+2\eta\right)} \frac{Q_k^{[n-1]}\left(u_k^{[n-1]}+4\eta\right)}{Q_k^{[n-1]}\left(u_k^{[n-1]}-4\eta\right)}\,, \tag{8.61}$$

$$\Phi_{n,p,n}(u_k^{[n]}) = \frac{Q^{[n-2]}\left(u_k^{[n]}-2\eta\right)}{Q^{[n-2]}\left(u_k^{[n]}+2\eta\right)} \frac{Q_k^{[n]}\left(u_k^{[n]}+4\eta\right)}{Q_k^{[n]}\left(u_k^{[n]}-4\eta\right)}\,. \tag{8.62}$$

The Bethe equations for values of $n$ not included above:

$$A_1^{(2)}: \quad \left[\frac{\sinh(\frac{u_k^{[1]}}{2}+2\eta)}{\sinh(\frac{u_k^{[1]}}{2}-2\eta)}\right]^{2N} \Phi_{1,p,1}(u_k^{[1]}) = \frac{Q_k^{[1]}\left(u_k^{[1]}+4\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-4\eta\right)} \frac{Q_k^{[1]}\left(u_k^{[1]}+4\eta+i\pi\right)}{Q_k^{[1]}\left(u_k^{[1]}-4\eta+i\pi\right)}, \tag{8.63}$$

$$A_3^{(2)}: \quad \left[\frac{\sinh(\frac{u_k^{[1]}}{2}+\eta)}{\sinh(\frac{u_k^{[1]}}{2}-\eta)}\right]^{2N} \Phi_{1,p,2}(u_k^{[1]}) = \frac{Q_k^{[1]}\left(u_k^{[1]}+4\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-4\eta\right)} \frac{Q^{[2]}\left(u_k^{[1]}-2\eta\right)}{Q^{[2]}\left(u_k^{[1]}+2\eta\right)} \frac{Q^{[2]}\left(u_k^{[1]}-2\eta+i\pi\right)}{Q^{[2]}\left(u_k^{[1]}+2\eta+i\pi\right)},$$

$$\Phi_{2,p,2}(u_k^{[2]}) = \frac{Q^{[1]}\left(u_k^{[2]}-2\eta\right)}{Q^{[1]}\left(u_k^{[2]}+2\eta\right)} \frac{Q^{[1]}\left(u_k^{[2]}-2\eta+i\pi\right)}{Q^{[1]}\left(u_k^{[2]}+2\eta+i\pi\right)} \times \frac{Q_k^{[2]}\left(u_k^{[2]}+4\eta\right)}{Q_k^{[2]}\left(u_k^{[2]}-4\eta\right)} \frac{Q_k^{[2]}\left(u_k^{[2]}+4\eta+i\pi\right)}{Q_k^{[2]}\left(u_k^{[2]}-4\eta+i\pi\right)}, \tag{8.64}$$

$$A_2^{(2)}: \quad \left[\frac{\sinh(\frac{u_k^{[1]}}{2}+\eta)}{\sinh(\frac{u_k^{[1]}}{2}-\eta)}\right]^{2N} \Phi_{1,p,1}(u_k^{[1]}) = \frac{Q_k^{[1]}\left(u_k^{[1]}+4\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-4\eta\right)} \frac{Q_k^{[1]}\left(u_k^{[1]}-2\eta+i\pi\right)}{Q_k^{[1]}\left(u_k^{[1]}+2\eta+i\pi\right)}, \tag{8.65}$$

$$B_1^{(1)}: \quad \left[\frac{\sinh(\frac{u_k^{[1]}}{2}+\eta)}{\sinh(\frac{u_k^{[1]}}{2}-\eta)}\right]^{2N} \Phi_{1,p,1}(u_k^{[1]}) = \frac{Q_k^{[1]}\left(u_k^{[1]}+2\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-2\eta\right)}, \tag{8.66}$$

$$C_1^{(1)}: \quad \left[\frac{\sinh(\frac{u_k^{[1]}}{2}+2\eta)}{\sinh(\frac{u_k^{[1]}}{2}-2\eta)}\right]^{2N} \Phi_{1,p,1}(u_k^{[1]}) = \frac{Q_k^{[1]}\left(u_k^{[1]}+8\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-8\eta\right)}, \tag{8.67}$$

$$C_2^{(1)}: \quad \left[\frac{\sinh(\frac{u_k^{[1]}}{2}+\eta)}{\sinh(\frac{u_k^{[1]}}{2}-\eta)}\right]^{2N} \Phi_{1,p,2}(u_k^{[1]}) = \frac{Q_k^{[1]}\left(u_k^{[1]}+4\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-4\eta\right)} \frac{Q^{[2]}\left(u_k^{[1]}-4\eta\right)}{Q^{[2]}\left(u_k^{[1]}+4\eta\right)}, \tag{8.68}$$

$$\Phi_{2,p,2}(u_k^{[2]}) = \frac{Q^{[1]}\left(u_k^{[2]}-4\eta\right)}{Q^{[1]}\left(u_k^{[2]}+4\eta\right)} \frac{Q_k^{[2]}\left(u_k^{[2]}+8\eta\right)}{Q_k^{[2]}\left(u_k^{[2]}-8\eta\right)}, \tag{8.69}$$

$$D_2^{(1)}: \quad \left[\frac{\sinh(\frac{u_k^{[1]}}{2}+\eta)}{\sinh(\frac{u_k^{[1]}}{2}-\eta)}\right]^{2N} \Phi_{1,p,2}(u_k^{[1]}) = \frac{Q_k^{[1]}\left(u_k^{[1]}+4\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-4\eta\right)}, \tag{8.70}$$

$$\left[\frac{\sinh(\frac{u_k^{[2]}}{2}+\eta)}{\sinh(\frac{u_k^{[2]}}{2}-\eta)}\right]^{2N} \Phi_{2,p,2}(u_k^{[2]}) = \frac{Q_k^{[2]}\left(u_k^{[2]}+4\eta\right)}{Q_k^{[2]}\left(u_k^{[2]}-4\eta\right)}, \tag{8.71}$$

$$D_3^{(1)}: \quad \left[\frac{\sinh(\frac{u_k^{[1]}}{2}+\eta)}{\sinh(\frac{u_k^{[1]}}{2}-\eta)}\right]^{2N} \Phi_{1,p,3}(u_k^{[1]}) = \frac{Q_k^{[1]}\left(u_k^{[1]}+4\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-4\eta\right)} \frac{Q^{[2]}\left(u_k^{[1]}-2\eta\right)}{Q^{[2]}\left(u_k^{[1]}+2\eta\right)} \times \frac{Q^{[3]}\left(u_k^{[1]}-2\eta\right)}{Q^{[3]}\left(u_k^{[1]}+2\eta\right)}, \tag{8.72}$$

$$\Phi_{2,p,3}(u_k^{[2]}) = \frac{Q^{[1]}\left(u_k^{[2]}-2\eta\right)}{Q^{[1]}\left(u_k^{[2]}+2\eta\right)} \frac{Q_k^{[2]}\left(u_k^{[2]}+4\eta\right)}{Q_k^{[2]}\left(u_k^{[2]}-4\eta\right)}, \tag{8.73}$$

$$\Phi_{3,p,3}(u_k^{[3]}) = \frac{Q_k^{[1]}\left(u_k^{[3]}-2\eta\right)}{Q_k^{[1]}\left(u_k^{[3]}+2\eta\right)} \frac{Q_k^{[3]}\left(u_k^{[3]}+4\eta\right)}{Q_k^{[3]}\left(u_k^{[3]}-4\eta\right)}. \tag{8.74}$$

The $u_k^{[1]} \leftrightarrow u_k^{[2]}$ symmetry of the Bethe equations (8.70), (8.71) is a reflection of the $U_q(D_2)$ symmetry (see again Table 8.2) and the fact $D_2 = A_1 \otimes A_1$.

The important factor $\Phi_{l,p,n}(u)$ in the Bethe equations for most of the cases in Table 8.2 is given by [5]

$$\Phi_{l,p,n}(u) = \frac{y_l(u+2l\eta,p)}{y_{l-1}(u+2l\eta,p)} = \begin{cases} \frac{G(u+2p\eta)}{F(u+2p\eta)} & \text{for } l=p \\ 1 & \text{for } l\neq p \end{cases}, \tag{8.75}$$

where the second equality follows from (8.41). Using the expressions for $G(u)$ (8.43) and $F(u)$ (8.44), we conclude that $\Phi_{l,p,n}(u)$ is given by

$$\Phi_{l,p,n}(u) = \begin{cases} \left[\frac{\cosh\left(\frac{u}{2} - \delta_{l,p}\left(\frac{(\omega-2p)}{2}\eta + \frac{i\pi}{8}(\delta-4\varepsilon)\right)\right)}{\cosh\left(\frac{u}{2} + \delta_{l,p}\left(\frac{(\omega-2p)}{2}\eta + \frac{i\pi}{8}(\delta-4\varepsilon)\right)\right)}\right]^2 & \text{for } \quad B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n}^{(2)} \\ & \text{and for } \quad A_{2n-1}^{(2)} \text{ with } l<n\,, \\ \left[\frac{\sinh\left(u - \delta_{l,p}\left((\omega-2p)\eta + \frac{i\pi}{4}(\delta-4\varepsilon)\right)\right)}{\sinh\left(u + \delta_{l,p}\left((\omega-2p)\eta + \frac{i\pi}{4}(\delta-4\varepsilon)\right)\right)}\right]^2 & \text{for } A_{2n-1}^{(2)} \text{ with } l=n\,. \end{cases} \tag{8.76}$$

Note that $\Phi_{l,p,n}(u)$ is different from 1 only if $l=p$. That is, the Bethe equations are the same as for the case $p=0$ [68, 69], except for an extra factor in the equation for the $p^{th}$ Bethe roots $\{u_j^{[p]}\}$.

The factor $\Phi_{l,p,n}(u)$ for all the special cases in Table 8.3 is simply given by

$$\Phi_{l,p,n}(u) = 1\,, \tag{8.77}$$

as follows from (8.40).

For $p=n$, the Bethe equations for $A_{2n}^{(2)}$ with $\varepsilon=1$ reduce to those found in [86]; and (again for $p=n$) the Bethe equations for $A_{2n-1}^{(2)}$ with $\varepsilon=0$ reduce to those found in [87]. We have numerically verified the completeness of all the above Bethe ansatz solutions for small values of $n$ and $N$ (for all $p=0,\ldots,n$ and $\varepsilon=0,1$), along the lines in [86, 87].

#### For $\hat{g} = D_{n+1}^{(2)}$

We emphasize that, for $D_{n+1}^{(2)}$, we consider only the case $\varepsilon=0$. We obtain the following Bethe equations:
For $n=1$ with $p=0,1$:

$$\left[\frac{\sinh(u_k^{[1]}+\eta)}{\sinh(u_k^{[1]}-\eta)}\right]^{2N} = \frac{Q_k^{[1]}\left(u_k^{[1]}+2\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-2\eta\right)}\,, \quad k=1,\ldots,m_1\,. \tag{8.78}$$

For $n>1$ with $p=0,\ldots,n$:

$$\begin{aligned}\left[\frac{\sinh(u_k^{[1]}+\eta)}{\sinh(u_k^{[1]}-\eta)}\right]^{2N} \Phi_{1,p,n}(u_k^{[1]}) &= \frac{Q_k^{[1]}\left(u_k^{[1]}+2\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-2\eta\right)} \frac{Q_k^{[1]}\left(u_k^{[1]}+2\eta+i\pi\right)}{Q_k^{[1]}\left(u_k^{[1]}-2\eta+i\pi\right)} \\ &\times \frac{Q^{[2]}\left(u_k^{[1]}-\eta\right)}{Q^{[2]}\left(u_k^{[1]}+\eta\right)} \frac{Q^{[2]}\left(u_k^{[1]}-\eta+i\pi\right)}{Q^{[2]}\left(u_k^{[1]}+\eta+i\pi\right)}\,, \\ & k=1,\ldots,m_1\,, \end{aligned} \tag{8.79}$$

[5]The exceptions are as follows:

$$A_{2n-1}^{(2)}\,, p=n: \quad \Phi_{l,p,n}(u) = \begin{cases} \frac{\tilde{y}_{n-1}(u+2n\eta,p)}{y_{n-1}(u+2n\eta,p)} & \text{for } l=n \\ 1 & \text{for } l\neq n \end{cases},$$

$$C_n^{(1)}\,, p=n: \quad \Phi_{l,p,n}(u) = \begin{cases} \frac{\tilde{y}_{n-1}(u+2(n+1)\eta,p)}{y_{n-1}(u+2(n+1)\eta,p)} & \text{for } l=n \\ 1 & \text{for } l\neq n \end{cases},$$

$$D_n^{(1)}\,, p=n: \quad \Phi_{l,p,n}(u) = \begin{cases} \frac{y_{n-1}(u+2(n-1)\eta,p)}{y_{n-2}(u+2(n-1)\eta,p)} & \text{for } l=n \\ 1 & \text{for } l\neq n \end{cases}.$$

$$\begin{aligned}\Phi_{l,p,n}(u_k^{[l]}) =& \frac{Q^{[l-1]}\left(u_k^{[l]}-\eta\right)}{Q^{[l-1]}\left(u_k^{[l]}+\eta\right)} \frac{Q^{[l-1]}\left(u_k^{[l]}-\eta+i\pi\right)}{Q^{[l-1]}\left(u_k^{[l]}+\eta+i\pi\right)} \\ &\times \frac{Q_k^{[l]}\left(u_k^{[l]}+2\eta\right)}{Q_k^{[l]}\left(u_k^{[l]}-2\eta\right)} \frac{Q_k^{[l]}\left(u_k^{[l]}+2\eta+i\pi\right)}{Q_k^{[l]}\left(u_k^{[l]}-2\eta+i\pi\right)} \\ &\times \frac{Q^{[l+1]}\left(u_k^{[l]}-\eta\right)}{Q^{[l+1]}\left(u_k^{[l]}+\eta\right)} \frac{Q^{[l+1]}\left(u_k^{[l]}-\eta+i\pi\right)}{Q^{[l+1]}\left(u_k^{[l]}+\eta+i\pi\right)}, \\ & k=1,\ldots,m_l\,, \quad l=2,\ldots,n-1\,,\end{aligned} \tag{8.80}$$

$$\begin{aligned}\Phi_{n,p,n}(u_k^{[n]}) =& \frac{Q^{[n-1]}\left(u_k^{[n]}-\eta\right)}{Q^{[n-1]}\left(u_k^{[n]}+\eta\right)} \frac{Q^{[n-1]}\left(u_k^{[n]}-\eta+i\pi\right)}{Q^{[n-1]}\left(u_k^{[n]}+\eta+i\pi\right)} \frac{Q_k^{[n]}\left(u_k^{[n]}+2\eta\right)}{Q_k^{[n]}\left(u_k^{[n]}-2\eta\right)}, \\ & k=1,\ldots,m_n\,.\end{aligned} \tag{8.81}$$

The factor $\Phi_{l,p,n}(u)$ in the above Bethe equations is given by

$$\Phi_{l,p,n}(u) = \frac{y_l(u+l\eta,p)}{y_{l-1}(u+l\eta,p)} = \begin{cases} \frac{G(u+p\eta)}{F(u+p\eta)} & \text{for } l=p \\ 1 & \text{for } l\neq p \end{cases}. \tag{8.82}$$

Using the results for $G(u)$ (8.45) and $F(u)$ (8.46), we obtain

$$\Phi_{l,p,n}(u) = \left[\frac{\cosh\left(u-\delta_{l,p}(n-p)\eta\right)}{\cosh\left(u+\delta_{l,p}(n-p)\eta\right)}\right]^2. \tag{8.83}$$

As for (8.76), this factor $\Phi_{l,p,n}(u)$ is different from 1 only if $l=p$.

For $p=n$, these Bethe equations reduce to the one found in [87]. We have numerically verified the completeness of the above Bethe ansatz solutions for small values of $n$ and $N$ (for all $p=0,\ldots,n$) along the lines in [87].

### Towards a universal formula for the Bethe equations

Let us denote in this subsection the affine Lie algebras $\hat{g}$ in Tables 8.2 and 8.3 by $g^{(t)}$, where $g$ is a (non-affine) Lie algebra with rank $\mathtt{r}$, and $t=1$ (untwisted) or $t=2$ (twisted).[6] The above formulas for the $g^{(t)}$ Bethe equations can be rewritten in a more compact form in terms of representation-theoretic quantities following [94]:[7]

$$\begin{aligned}&\prod_{s=0}^{t-1}\left[\frac{\sinh\left(\frac{u_k^{[l]}}{2}+(\lambda_1,\theta^s\alpha_l)\,\eta+\frac{i\pi s}{2}\right)}{\sinh\left(\frac{u_k^{[l]}}{2}-(\lambda_1,\theta^s\alpha_l)\,\eta+\frac{i\pi s}{2}\right)}\right]^{2N}\Phi_{l,p,n}(u_k^{[l]}) \\ &= \prod_{s=0}^{t-1}\prod_{l'=1}^{n}\prod_{j=1}^{m_{l'}}{}' \frac{\sinh\left[\frac{1}{2}\left(u_k^{[l]}-u_j^{[l']}\right)+(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}{\sinh\left[\frac{1}{2}\left(u_k^{[l]}-u_j^{[l']}\right)-(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]} \frac{\sinh\left[\frac{1}{2}\left(u_k^{[l]}+u_j^{[l']}\right)+(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}{\sinh\left[\frac{1}{2}\left(u_k^{[l]}+u_j^{[l']}\right)-(\alpha_l,\theta^s\alpha_{l'})\,\eta+\frac{i\pi s}{2}\right]}, \\ &\qquad k=1,\ldots,m_l\,, \qquad l=1,\ldots,n\,,\end{aligned} \tag{8.84}$$

where the product over $j$ has the restriction $(j,l')\neq(k,l)$. The simple roots $\alpha_i$ of $g$ are given in the orthogonal basis by

$$\begin{aligned}\alpha_i &= e_i - e_{i+1}, \qquad i=1,\ldots,\mathtt{r}-1\,, \\ \alpha_{\mathtt{r}} &= \begin{cases} e_{\mathtt{r}}-e_{\mathtt{r}+1} & \text{for } A_{\mathtt{r}} \\ e_{\mathtt{r}} & \text{for } B_{\mathtt{r}} \\ 2e_{\mathtt{r}} & \text{for } C_{\mathtt{r}} \\ e_{\mathtt{r}-1}+e_{\mathtt{r}} & \text{for } D_{\mathtt{r}} \end{cases},\end{aligned} \tag{8.85}$$

[6] The notation $g^{(t)}$ introduced here for affine Lie algebras should not be confused with the "left" and "right" algebras $g^{(l)}$ and $g^{(r)}$ introduced in Sec. 8.1.3.

[7] For $D^{(2)}_{n+1}$, a rescaling $\eta\to\frac{\eta}{2}$ in (8.84) is necessary in order to match with the Bethe equations as written in Sec. 8.2.3.

where $e_i$ are $\mathtt{r}$-dimensional elementary basis vectors (except for $A_{\mathtt{r}}$, in which case the dimension is $\mathtt{r}+1$). The notation $(*\,,*)$ denotes the ordinary scalar product, and $\lambda_1$ is the first fundamental weight of $g$, with $(\lambda_1\,,\alpha_i) = \delta_{i,1}$. For the twisted cases $g^{(2)}$, the order-2 automorphisms $\theta$ of $g$ are given by

$$\begin{aligned}
&\theta\alpha_i = \alpha_{2n-i}\,, \quad i=1,\ldots,2n-1 && \text{for} \quad A^{(2)}_{2n-1}\,,\\
&\theta\alpha_i = \alpha_{2n+1-i}\,, \quad i=1,\ldots,2n && \text{for} \quad A^{(2)}_{2n}\,,\\
&\theta\alpha_i = \alpha_i\,, \quad i=1,\ldots,n-1\,, \quad \theta\alpha_n = \alpha_{n+1} && \text{for} \quad D^{(2)}_{n+1}\,.
\end{aligned} \tag{8.86}$$

The factor $\Phi_{l,p,n}(u)$ in (8.84) is understood to be the appropriate one for $g^{(t)}$, see (8.76), (8.77), (8.83). It would be interesting to also have a universal expression for this factor.

## 8.3 Dynkin labels of the Bethe states

In this section we obtain formulas for the Dynkin labels of the Bethe states in terms of the numbers of Bethe roots of each type. Since the Dynkin labels of an irrep determine its dimension, these formulas help determine the degeneracies of the transfer-matrix eigenvalues.

### 8.3.1 Eigenvalues of the Cartan generators

We now argue that the eigenvalues of the Cartan generators for the Bethe states (8.16) are given in terms of the cardinalities of the Bethe roots of each type by

$$\begin{aligned}
h^{(l)}_i &= m_{p+i-1} - m_{p+i} - \xi\,\delta_{i,n-p}m_n - \xi'\,\delta_{i,n-p-1}m_n\,, && i=1,...,n-p\,,\\
h^{(r)}_i &= m_i - m_{i-1} + \xi\,\delta_{i,n}m_n + \xi'\,\delta_{i,n-1}m_n\,, && i=1,...,p\,,
\end{aligned} \tag{8.87}$$

where $\xi$ and $\xi'$ are given in Table 8.1.

The first step is to compute the asymptotic behavior of $\Lambda^{(m_1,\ldots,m_n)}(u,p)$ by computing the expectation value

$$\langle\Lambda^{(m_1,\ldots,m_n)}|t(u,p)|\Lambda^{(m_1,\ldots,m_n)}\rangle \tag{8.88}$$

for $u\to\infty$. The main idea is to perform a gauge transformation to the "unitary" gauge [88, 89], so that the asymptotic limit of the monodromy matrices in $t(u,p)$ become expressed in terms of the QG generators. We assume that the Bethe states $|\Lambda^{(m_1,\cdots,m_n)}\rangle$ are highest-weight states of the "left" algebra

$$\Delta_N(E^{+(l)}_i(p))\,|\Lambda^{(m_1,\ldots,m_n)}\rangle = 0\,, \qquad i=1,\ldots,n-p\,, \tag{8.89}$$

and lowest-weight states of the "right" algebra

$$\Delta_N(E^{-(r)}_i(p))\,|\Lambda^{(m_1,\ldots,m_n)}\rangle = 0\,, \qquad i=1,\ldots,p\,, \tag{8.90}$$

as is the reference state (8.19), (8.20). We eventually obtain

$$\begin{aligned}
\Lambda^{(m_1,\ldots,m_n)}(u,p) \sim \sigma(u)\,e^{-2\kappa\,\eta\,N}\Bigg\{ d - 2n + \sum_{j=1}^{p}\left[\mathfrak{f}^{(r)}e^{4\eta(-j+h^{(r)}_{p+1-j})} + \frac{1}{\mathfrak{f}^{(r)}}e^{-4\eta(-j+h^{(r)}_{p+1-j})}\right]\\
+ \sum_{j=p+1}^{n}\left[\mathfrak{f}^{(l)}e^{-4\eta(n-j+h^{(l)}_{j-p})} + \frac{1}{\mathfrak{f}^{(l)}}e^{4\eta(n-j+h^{(l)}_{j-p})}\right]\Bigg\} \quad \text{for} \quad u\to\infty\,,
\end{aligned} \tag{8.91}$$

where

$$\sigma(u) = \begin{cases} 2^{-2N}e^{2Nu} & \text{for } A^{(2)}_{2n-1}\,, A^{(2)}_{2n}\,, B^{(1)}_n\,, C^{(1)}_n\,, D^{(1)}_n \\ e^{4Nu} & \text{for } D^{(2)}_{n+1} \end{cases}\,, \tag{8.92}$$

and

$$\mathfrak{f}^{(r)} = \begin{cases} -1 & \text{for } A^{(2)}_{2n}\,, C^{(1)}_n \\ e^{4\eta} & \text{for } A^{(2)}_{2n-1}\,, B^{(1)}_n\,, D^{(1)}_n \\ e^{2\eta} & \text{for } D^{(2)}_{n+1} \end{cases}\,, \tag{8.93}$$

$$\mathfrak{f}^{(l)} = \begin{cases} e^{-2\eta} & \text{for } A^{(2)}_{2n}\,, B^{(1)}_n\,, D^{(2)}_{n+1} \\ -e^{-4\eta} & \text{for } A^{(2)}_{2n-1}\,, C^{(1)}_n \\ 1 & \text{for } D^{(1)}_n \end{cases}\,. \tag{8.94}$$

Note that the result (8.91) is in terms of the eigenvalues of the Cartan generators for the Bethe states.

The second step is to compute again the asymptotic behavior of $\Lambda^{(m_1,\ldots,m_n)}(u,p)$, but now using instead the T-Q equation (8.21). We obtain in this way

$$\Lambda^{(m_1,\ldots,m_n)}(u,p) \sim \sigma(u)\, e^{-2\kappa\,\eta\,N}\Bigg\{d-2n+\sum_{l=0}^{n-1}\Big[\mathfrak{g}_l e^{4\eta(l-n+m_{l+1}-m_l+\xi\delta_{l,n-1}m_n+\xi'\delta_{l,n-2}m_n)} + \frac{1}{\mathfrak{g}_l}e^{-4\eta(l-n+m_{l+1}-m_l+\xi\delta_{l,n-1}m_n+\xi'\delta_{l,n-2}m_n)}\Big]\Bigg\} \quad \text{for} \quad u\to\infty\,, \tag{8.95}$$

where

$$\mathfrak{g}_l = \begin{cases}\mathfrak{f}^{(r)}e^{4\eta(n-p)} & l\le p-1\\ \mathfrak{f}^{(l)}e^{4\eta} & l\ge p\end{cases}\,, \tag{8.96}$$

$\sigma(u)$ is given by (8.92), and $\mathfrak{f}^{(r)}, \mathfrak{f}^{(l)}$ are given by (8.94). Moreover, we define $m_0$ as

$$m_0 = N\,. \tag{8.97}$$

Note that the result (8.95) is in terms of the cardinalities of the Bethe roots of each type.

Finally, by comparing (8.91) and (8.95), we obtain the desired result (8.87).

### 8.3.2 Formulas for the Dynkin labels

The "left" Dynkin labels are expressed in terms of the eigenvalues of the "left" Cartan generators by (see, e.g. [86, 87])

$$\begin{aligned} a_i^{(l)} &= h_i^{(l)} - h_{i+1}^{(l)}, \qquad i=1,\ldots,n-p-1\,,\\ a_{n-p}^{(l)} &= \begin{cases} 2h_{n-p}^{(l)} & \text{for } g^{(l)}=B_{n-p} \text{ i.e., for } A_{2n}^{(2)}, B_n^{(1)}, D_{n+1}^{(2)}\\ h_{n-p}^{(l)} & \text{for } g^{(l)}=C_{n-p} \text{ i.e., for } A_{2n-1}^{(2)}, C_n^{(1)}\\ h_{n-p-1}^{(l)}+h_{n-p}^{(l)} & \text{for } g^{(l)}=D_{n-p} \text{ i.e., for } D_n^{(1)}\end{cases}\,. \end{aligned} \tag{8.98}$$

Similarly, the "right" Dynkin labels are expressed in terms of the eigenvalues of the "right" Cartan generators by

$$\begin{aligned} a_i^{(r)} &= -h_i^{(r)} + h_{i+1}^{(r)} \qquad i=1,\ldots,p-1\,,\\ a_p^{(r)} &= \begin{cases} -2h_p^{(r)} & \text{for } g^{(r)}=B_p \text{ i.e., for } D_{n+1}^{(2)}\\ -h_p^{(r)} & \text{for } g^{(r)}=C_p \text{ i.e., for } A_{2n}^{(2)}, C_n^{(1)}\\ -h_{p-1}^{(r)}-h_p^{(r)} & \text{for } g^{(r)}=D_p \text{ i.e., for } A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}\end{cases}\,. \end{aligned} \tag{8.99}$$

We introduce extra minus signs in (8.99) (in comparison with corresponding formulas in (8.98)) since the Bethe states are *lowest* weights of the "right" algebra (8.90). The algebras $g^{(l)}$ and $g^{(r)}$ for the various affine algebras $\hat{g}$ are given in Table 8.2.

Finally, using the results (8.87) for the eigenvalues of the Cartan generators in terms of the cardinalities of the Bethe roots of each type, we obtain formulas for the Dynkin labels in terms of the cardinalities of the Bethe roots. Explicitly, for the "left" Dynkin labels ($p=0,1,\ldots,n-1$):

$$\begin{aligned} a_i^{(l)} &= m_{p+i-1} - 2m_{p+i} + m_{p+i+1}\,,\\ & i=1,\ldots,n-p-1 \quad \text{for} \quad A_{2n}^{(2)}, B_n^{(1)}, D_{n+1}^{(2)}\,,\\ & i=1,\ldots,n-p-2 \quad \text{for} \quad A_{2n-1}^{(2)}, C_n^{(1)}\,,\\ & i=1,\ldots,n-p-3 \quad \text{for} \quad D_n^{(1)}\,. \end{aligned} \tag{8.100}$$

Moreover, for the values of $i$ not included above:

$$A_{2n}^{(2)}, B_n^{(1)}, D_{n+1}^{(2)}: \qquad a_{n-p}^{(l)} = 2m_{n-1} - 2m_n\,, \tag{8.101}$$

$$\begin{aligned} A_{2n-1}^{(2)}, C_n^{(1)}: \qquad a_{n-p-1}^{(l)} &= m_{n-2} - 2m_{n-1} + 2m_n\,,\\ a_{n-p}^{(l)} &= m_{n-1} - 2m_n\,, \end{aligned} \tag{8.102}$$

$$\begin{aligned} D_n^{(1)}: \qquad a_{n-p-2}^{(l)} &= m_{n-3} - 2m_{n-2} + m_{n-1} + m_n\,,\\ a_{n-p-1}^{(l)} &= m_{n-2} - 2m_{n-1}\,,\\ a_{n-p}^{(l)} &= m_{n-2} - 2m_n\,. \end{aligned} \tag{8.103}$$

For the "right" Dynkin labels ($p = 1, \ldots, n$):

$$\begin{aligned} a_i^{(r)} &= m_{i-1} - 2m_i + m_{i+1}\,, \\ & i = 1,...,p-1 \quad \text{for} \quad A_{2n}^{(2)}\,, B_n^{(1)}\,, D_{n+1}^{(2)}\,, \\ & i = 1,...,p-2 \quad \text{for} \quad A_{2n-1}^{(2)}\,, C_n^{(1)} \\ & i = 1,...,p-3 \quad \text{for} \quad D_n^{(1)}\,. \end{aligned} \tag{8.104}$$

Moreover, for the values of $i$ not included above:

$$A_{2n}^{(2)}: \qquad a_p^{(r)} = m_{p-1} - m_p\,, \tag{8.105}$$

$$B_n^{(1)}: \qquad a_p^{(r)} = m_{p-2} - m_p\,, \tag{8.106}$$

$$A_{2n-1}^{(2)}: \qquad \begin{aligned} a_{p-1}^{(r)} &= m_{p-2} - 2m_{p-1} + (1+\delta_{p,n})m_p\,, \\ a_p^{(r)} &= m_{p-2} - (1+\delta_{p,n})m_p\,, \end{aligned} \tag{8.107}$$

$$C_n^{(1)}: \qquad \begin{aligned} a_{p-1}^{(r)} &= m_{p-2} - 2m_{p-1} + (1+\delta_{p,n})m_p\,, \\ a_p^{(r)} &= m_{p-1} - (1+\delta_{p,n})m_p\,, \end{aligned} \tag{8.108}$$

$$D_n^{(1)} \qquad \begin{aligned} a_{p-2}^{(r)} &= m_{p-3} - 2m_{p-2} + m_{p-1} + \delta_{p,n}m_p\,, \\ a_{p-1}^{(r)} &= m_{p-2} - 2m_{p-1} + m_p + (\delta_{p,n-1} - \delta_{p,n})m_n\,, \\ a_p^{(r)} &= m_{p-2} - m_p - (\delta_{p,n-1} + \delta_{p,n})m_p\,, \end{aligned} \tag{8.109}$$

$$D_{n+1}^{(2)}: \qquad a_p^{(r)} = 2m_{p-1} - 2m_p\,. \tag{8.110}$$

We remind the reader that $m_0$ is defined in (8.97).

For the cases of overlap with previous results (namely, $A_{2n}^{(2)}$ with $p = 0, n$ [86]; $A_{2n-1}^{(2)}$ with $p = 0, n$ [87]; and $D_{n+1}^{(2)}$ with $p = n$ [87]), the results match.

### 8.3.3 Examples

We now illustrate the results of Sec. 8.3.2 with two simple examples.

#### $A_{2n}^{(2)}$ with $n = 3$

As a first example, we consider the case $A_{2n}^{(2)}$ with $n = 3$, two sites ($N = 2$), and either $\varepsilon = 0$ or $\varepsilon = 1$. The four possibilities $p = 0, 1, 2, 3$ are summarized in Table 8.4. By solving the Bethe equations (see Sec. 8.2.3) with a generic value of anisotropy $\eta$, we obtain solutions (not shown[8]) with the values of $m_1, m_2, m_3$ displayed in the table. The corresponding Dynkin labels obtained using the formulas from Sec. 8.3.2, are also displayed in the table. Finally, the irreducible representations of the "left" and "right" algebras corresponding to these Dynkin labels (obtained e.g. using LieART [96]) are shown in the final column. By explicit diagonalization of the transfer matrix, we confirm that the degeneracies of the eigenvalues exactly match with the dimensions of the corresponding irreps.

#### $D_n^{(1)}$ with $n = 4$

As a second example, we consider the case $D_n^{(1)}$ with $n = 4$, two sites ($N = 2$), and with $\varepsilon = 0$. The three cases $p = 0, 2, 4$ are summarized in Table 8.5. (We omit the "special" cases $p = 1, 3$, whose results are the same as for $p = 0, 4$, respectively, see Table 8.3.) By solving the Bethe equations (see Sec. 8.2.3) with a generic value of anisotropy $\eta$, we obtain solutions (not shown) with the values of $m_1, m_2, m_3, m_4$ displayed in the table. The corresponding Dynkin labels obtained using the formulas from Sec. 8.3.2, are also displayed in the table. Finally, the irreps of the "left" and "right" algebras corresponding to these Dynkin labels are shown in the final column.

Notice that the values of $m$'s and Dynkin labels for $p = 0$ and $p = 4$ in Table 8.5 are exactly the same, which is due to the $p \leftrightarrow n - p$ duality (8.12).

[8] For the cases $p = 0$ and $p = n$, such solutions can be found in tables in [86].

| | $m_1$ | $m_2$ | $m_3$ | $a_1^{(l)}$ | $a_2^{(l)}$ | $a_3^{(l)}$ | Irreps. |
|---|---|---|---|---|---|---|---|
| $p=0$ $U_q(B_3)$ | 0 | 0 | 0 | 2 | 0 | 0 | **27** |
| | 1 | 0 | 0 | 0 | 1 | 0 | **21** |
| | 2 | 2 | 2 | 0 | 0 | 0 | **1** |
| | $m_1$ | $m_2$ | $m_3$ | $a_1^{(l)}$ | $a_2^{(l)}$ | $a_1^{(r)}$ | Irreps. |
| $p=1$ $U_q(B_2)\otimes U_q(C_1)$ | 0 | 0 | 0 | 0 | 0 | 2 | (**1**,**3**) |
| | 1 | 0 | 0 | 1 | 0 | 1 | 2(**5**,**2**) |
| | 2 | 0 | 0 | 2 | 0 | 0 | (**14**,**1**) |
| | 2 | 1 | 0 | 0 | 2 | 0 | (**10**,**1**) |
| | 2 | 2 | 2 | 0 | 0 | 0 | 2(**1**,**1**) |
| | $m_1$ | $m_2$ | $m_3$ | $a_1^{(l)}$ | $a_1^{(r)}$ | $a_2^{(r)}$ | Irreps. |
| $p=2$ $U_q(B_1)\otimes U_q(C_2)$ | 0 | 0 | 0 | 0 | 2 | 0 | (**1**,**10**) |
| | 1 | 0 | 0 | 0 | 0 | 1 | (**1**,**5**) |
| | 1 | 1 | 0 | 2 | 1 | 0 | 2(**3**,**4**) |
| | 2 | 2 | 0 | 4 | 0 | 0 | (**5**,**1**) |
| | 2 | 2 | 1 | 2 | 0 | 0 | (**3**,**1**) |
| | 2 | 2 | 2 | 0 | 0 | 0 | 2(**1**,**1**) |
| | $m_1$ | $m_2$ | $m_3$ | $a_1^{(r)}$ | $a_2^{(r)}$ | $a_3^{(r)}$ | Irreps. |
| $p=3$ $U_q(C_3)$ | 0 | 0 | 0 | 2 | 0 | 0 | **21** |
| | 1 | 0 | 0 | 0 | 1 | 0 | **14** |
| | 1 | 1 | 1 | 1 | 0 | 0 | 2(**6**) |
| | 2 | 2 | 2 | 0 | 0 | 0 | 2(**1**) |

Table 8.4: Numbers of Bethe roots and Dynkin labels for $A_{2n}^{(2)}$ with $n=3, N=2$.

The degeneracy pattern is particularly interesting for the case $p=2$ in Table 8.5. Indeed, by explicitly diagonalizing the transfer matrix for this case[9], we find the following degeneracies

$$\{1,1,12,16,16,18\}\,. \tag{8.111}$$

That is, one eigenvalue is repeated 18 times; two distinct eigenvalues are each repeated 16 times; etc. What is happening is that the irreps

$$(\mathbf{1},\mathbf{9})\,,(\mathbf{9},\mathbf{1}) \tag{8.112}$$

(see Table 8.5) are degenerate, thereby giving rise to the 18-fold degeneracy, due to the self-duality (8.13). Moreover, the irreps

$$(\mathbf{1},\mathbf{3})\,,(\mathbf{3},\mathbf{1})\,,(\mathbf{1},\bar{\mathbf{3}})\,,(\bar{\mathbf{3}},\mathbf{1}) \tag{8.113}$$

(see again Table 8.5) are all degenerate, thereby giving rise to the 12-fold degeneracy, due to the self-duality (8.13) and $Z_2$ symmetries (8.14), (8.15).

For eigenvalues corresponding to more than one irrep, it is enough to solve the Bethe equations corresponding to just *one* of those irreps, such as the irrep with the minimal values of $m$'s. Hence, for the example (8.112), it is enough to consider the reference state ($m_1=m_2=m_3=m_4=0$). For the example (8.113), it is enough to consider the state with $m_1=1, m_2=m_3=m_4=0$. Note that a non-minimal set $\{m_1,m_2,\ldots,m_n\}$ generally does *not* form a monotonic decreasing sequence, i.e. does *not* satisfy $m_1\ge m_2\ge\ldots\ge m_n$.[10]

For $\varepsilon=1$ (and still $n=4, p=2$), the transfer matrix has an additional "bonus" symmetry [88]. Consequently, the two irreps $(\mathbf{4},\mathbf{4})$ in Table 8.5 become degenerate (giving rise to a 32-fold degeneracy), and the two irreps $(\mathbf{1},\mathbf{1})$ become degenerate (giving rise to a 2-fold degeneracy). Interestingly, these levels have the singular (exceptional) Bethe roots $u^{(1)}=2\eta\,, u^{(2)}=4\eta$; and for the 2-fold degenerate level, these Bethe roots are repeated. This phenomenon is discussed further in Appendix K.

[9]We emphasize that we restrict to generic values of $\eta$.

[10] It can happen that an eigenvalue corresponding to a *single* irrep is described by more than one set of Bethe roots, and therefore by more than one set of $m$'s; and (for some cases with $\frac{n}{2}\le p<n$), the set of $m$'s corresponding to the Dynkin labels for the irrep may not be minimal. For example, for $C_4^{(1)}$ with $p=3$ and $N=2$, the transfer matrix has an eigenvalue with degeneracy 12 and Dynkin labels $(a_1^{(l)},a_1^{(r)},a_2^{(r)},a_3^{(r)})=(1,1,0,0)$, which according to the formulas in section 8.3.2 corresponds to $(m_1,m_2,m_3,m_4)=(1,1,1,0)$. Indeed, one can solve the Bethe equations (8.51), (8.52), (8.58), (8.59) and find such a solution for this eigenvalue. However, this set of $m$'s is not minimal, as one can find another solution of these Bethe equations for this eigenvalue with only $(m_1,m_2,m_3,m_4)=(1,1,0,0)$. Another example is $D_4^{(2)}$ with $p=2$ and $N=2$, for which there is an eigenvalue with degeneracy 3 and Dynkin labels $(a_1^{(l)},a_1^{(r)},a_2^{(r)})=(2,0,0)$, corresponding to $(m_1,m_2,m_3)=(2,2,1)$; but by solving the Bethe equations we can also find it with $(m_1,m_2,m_3)=(2,1,1)$.

| | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $a_1^{(l)}$ | $a_2^{(l)}$ | $a_3^{(l)}$ | $a_4^{(l)}$ | Irreps. | |
|---|---|---|---|---|---|---|---|---|---|---|
| $p=0$ $U_q(D_4)$ | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | **35** | |
| | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | **28** | |
| | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | **1** | |
| | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $a_1^{(l)}$ | $a_2^{(l)}$ | $a_1^{(r)}$ | $a_2^{(r)}$ | Irreps. | |
| $p=2$ $U_q(D_2)\otimes U_q(D_2)$ | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | $(\mathbf{1},\mathbf{9})$ | 18 |
| | 2 | 2 | 0 | 0 | 2 | 2 | 0 | 0 | $(\mathbf{9},\mathbf{1})$ | |
| | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | $2(\mathbf{4},\mathbf{4})$ | |
| | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | $(\mathbf{1},\bar{\mathbf{3}})$ | 12 |
| | 1 | 2 | 1 | 1 | 0 | 0 | 2 | 0 | $(\mathbf{1},\mathbf{3})$ | |
| | 2 | 2 | 0 | 1 | 2 | 0 | 0 | 0 | $(\mathbf{3},\mathbf{1})$ | |
| | 2 | 2 | 1 | 0 | 0 | 2 | 0 | 0 | $(\bar{\mathbf{3}},\mathbf{1})$ | |
| | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | $2(\mathbf{1},\mathbf{1})$ | |
| | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $a_1^{(r)}$ | $a_2^{(r)}$ | $a_3^{(r)}$ | $a_4^{(r)}$ | Irreps. | |
| $p=4$ $U_q(D_4)$ | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | **35** | |
| | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | **28** | |
| | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | **1** | |

Table 8.5: Numbers of Bethe roots and Dynkin labels for $D_n^{(1)}$ with $n=4, N=2$.

## 8.4 Duality and the Bethe ansatz

For the cases $C_n^{(1)}$, $D_n^{(1)}$ and $D_{n+1}^{(2)}$, the $p \leftrightarrow n-p$ duality property of the transfer matrix (8.12) is reflected in the Bethe ansatz solution. For concreteness, we restrict our attention here to the case $C_n^{(1)}$, for which

$$f(u,p) = -\phi(u,p)\,, \tag{8.114}$$

where $\phi(u,p)$ is given by (8.22).

The duality property of the transfer matrix (8.12) implies that corresponding eigenvalues satisfy

$$\Lambda(u,p) = f(u,p)\,\Lambda(u,n-p)\,. \tag{8.115}$$

Let us define the rescaled eigenvalue $\lambda(u,p)$ such that

$$\Lambda(u,p) = \phi(u,p)\,\lambda(u,p)\,. \tag{8.116}$$

In terms of $\lambda(u,p)$, the duality relation (8.115) takes the form

$$\lambda(u,p) = \frac{1}{f(u,p)}\,\lambda(u,n-p)\,, \tag{8.117}$$

as follows from (8.114), (8.116) and $f(u,n-p) = 1/f(u,p)$.

Let us now try to understand how the duality relation (8.117) emerges from the Bethe ansatz solution (8.21), which in terms of $\lambda(u,p)$ (8.116) reads

$$\begin{aligned}\lambda(u,p) =& A(u)\, z_0(u)\, y_0(u,p)\, c(u)^{2N} + \tilde{A}(u)\, \tilde{z}_0(u)\, \tilde{y}_0(u,p)\, \tilde{c}(u)^{2N} \\ &+ \left\{ \sum_{l=1}^{n-1} \Big[ z_l(u)\, y_l(u,p)\, B_l(u) + \tilde{z}_l(u)\, \tilde{y}_l(u,p)\, \tilde{B}_l(u) \Big] \right\} b(u)^{2N}\,.\end{aligned} \tag{8.118}$$

For the self-dual case $p=n/2$, the relation (8.117) is obvious, since $f(u,n/2)=1$. For the case $p=0$, we note the identity

$$\frac{y_l(u,0)}{y_l(u,n)} = \frac{1}{f(u,0)}\,, \qquad l = 0,1,\ldots,n-1\,. \tag{8.119}$$

Since $A(u)$ and $\{B_l(u)\}$ for $p=0$ are the same as for $p=n$ (the Bethe equations for $p=0$ are the same as for $p=n$), it follows from (8.118) and (8.119) that

$$\lambda(u,0) = \frac{1}{f(u,0)}\,\lambda(u,n)\,, \tag{8.120}$$

in agreement with (8.117).

To derive the duality relation (8.117) from the Bethe ansatz solution for $0 < p < n/2$ requires more effort. For simplicity, let us consider as an example the case $n = 3$ with $p = 1$, which is related by duality to $p = 2$. The rescaled eigenvalue is given by (8.118)

$$\begin{aligned}\lambda(u,p) &= z_0(u)\, y_0(u,p)\, \frac{Q^{[1]}(u+2\eta)}{Q^{[1]}(u-2\eta)} \left[2\sinh(\frac{u}{2}-2\eta)\sinh(\frac{u}{2}-8\eta)\right]^{2N} \\ &+ \Big\{ z_1(u)\, y_1(u,p)\, \frac{Q^{[1]}(u-6\eta)}{Q^{[1]}(u-2\eta)} \frac{Q^{[2]}(u)}{Q^{[2]}(u-4\eta)} \\ &+ z_2(u)\, y_2(u,p)\, \frac{Q^{[2]}(u-8\eta)}{Q^{[2]}(u-4\eta)} \frac{Q^{[3]}(u)}{Q^{[3]}(u-8\eta)} \Big\} \left[2\sinh(\frac{u}{2})\sinh(\frac{u}{2}-8\eta)\right]^{2N} + \ldots\,, \end{aligned} \tag{8.121}$$

where the crossed terms (indicated by the ellipsis) have not been explicitly written. Let us define the barred Q-functions

$$\bar{Q}^{[l]}(u) = \prod_{j=1}^{\bar{m}_l} \sinh\left(\tfrac{1}{2}(u-\bar{u}_j^{[l]})\right) \sinh\left(\tfrac{1}{2}(u+\bar{u}_j^{[l]})\right)\,, \qquad \bar{Q}^{[l]}(-u) = \bar{Q}^{[l]}(u)\,, \tag{8.122}$$

(in terms of unbarred ones $Q^{[l]}(u)$) as follows:

$$S(u) - S(-u) = \mathfrak{c}\, \sinh^{2N}(\frac{u}{2})\, \sinh(u)\, Q^{[2]}(u)\,, \tag{8.123}$$

$$S(u) = \chi(u+2\eta)\, Q^{[1]}(u+2\eta)\, \bar{Q}^{[1]}(u-2\eta)\,, \tag{8.124}$$

$$\bar{Q}^{[2]}(u) = Q^{[2]}(u)\,, \tag{8.125}$$

$$\bar{Q}^{[3]}(u) = Q^{[3]}(u)\,, \tag{8.126}$$

where

$$\chi(u) = 1 + \cosh(u)\,, \qquad \mathfrak{c} = 2\sinh(2\eta(1+2m_1-m_2-N))\,, \tag{8.127}$$

and

$$\bar{m}_1 = N - m_1 + m_2\,, \qquad \bar{m}_2 = m_2\,, \qquad \bar{m}_3 = m_3\,. \tag{8.128}$$

(The above results for $\bar{m}_1$ and $\mathfrak{c}$ follow from the asymptotic limit $u \to \infty$ of (8.123).) We show below that, if $Q^{[l]}(u)$ are the Q-functions for $p = 1$, then $\bar{Q}^{[l]}(u)$ are the Q-functions for $p = 2$.

### 8.4.1 Duality of the Bethe equations

We first show that (8.123)-(8.126) map the $p = 1$ Bethe equations:

$$\left[\frac{\sinh\left(\frac{u_k^{[1]}}{2}+\eta\right)}{\sinh\left(\frac{u_k^{[1]}}{2}-\eta\right)}\right]^{2N} \left[\frac{\cosh\left(\frac{u_k^{[1]}}{2}-2\eta\right)}{\cosh\left(\frac{u_k^{[1]}}{2}+2\eta\right)}\right]^{2} = \frac{Q_k^{[1]}\left(u_k^{[1]}+4\eta\right)}{Q_k^{[1]}\left(u_k^{[1]}-4\eta\right)} \frac{Q^{[2]}\left(u_k^{[1]}-2\eta\right)}{Q^{[2]}\left(u_k^{[1]}+2\eta\right)}\,, \tag{8.129}$$

$$1 = \frac{Q^{[1]}\left(u_k^{[2]}-2\eta\right)}{Q^{[1]}\left(u_k^{[2]}+2\eta\right)} \frac{Q_k^{[2]}\left(u_k^{[2]}+4\eta\right)}{Q_k^{[2]}\left(u_k^{[2]}-4\eta\right)} \frac{Q^{[3]}\left(u_k^{[2]}-2\eta\right)}{Q^{[3]}\left(u_k^{[2]}+2\eta\right)}\,, \tag{8.130}$$

$$1 = \frac{Q^{[2]}\left(u_k^{[3]}-4\eta\right)}{Q^{[2]}\left(u_k^{[3]}+4\eta\right)} \frac{Q_k^{[3]}\left(u_k^{[3]}+8\eta\right)}{Q_k^{[3]}\left(u_k^{[3]}-8\eta\right)}\,, \tag{8.131}$$

to the $p = 2$ Bethe equations:

$$\left[\frac{\sinh\left(\frac{\bar{u}_k^{[1]}}{2}+\eta\right)}{\sinh\left(\frac{\bar{u}_k^{[1]}}{2}-\eta\right)}\right]^{2N} = \frac{\bar{Q}_k^{[1]}\left(\bar{u}_k^{[1]}+4\eta\right)}{\bar{Q}_k^{[1]}\left(\bar{u}_k^{[1]}-4\eta\right)}\frac{\bar{Q}^{[2]}\left(\bar{u}_k^{[1]}-2\eta\right)}{\bar{Q}^{[2]}\left(\bar{u}_k^{[1]}+2\eta\right)}, \tag{8.132}$$

$$\left[\frac{\cosh\left(\frac{\bar{u}_k^{[2]}}{2}-\eta\right)}{\cosh\left(\frac{\bar{u}_k^{[2]}}{2}+\eta\right)}\right]^{2} = \frac{\bar{Q}^{[1]}\left(\bar{u}_k^{[2]}-2\eta\right)}{\bar{Q}^{[1]}\left(\bar{u}_k^{[2]}+2\eta\right)}\frac{\bar{Q}_k^{[2]}\left(\bar{u}_k^{[2]}+4\eta\right)}{\bar{Q}_k^{[2]}\left(\bar{u}_k^{[2]}-4\eta\right)}\frac{\bar{Q}^{[3]}\left(\bar{u}_k^{[2]}-2\eta\right)}{\bar{Q}^{[3]}\left(\bar{u}_k^{[2]}+2\eta\right)}, \tag{8.133}$$

$$1 = \frac{\bar{Q}^{[2]}\left(\bar{u}_k^{[3]}-4\eta\right)}{\bar{Q}^{[2]}\left(\bar{u}_k^{[3]}+4\eta\right)}\frac{\bar{Q}_k^{[3]}\left(\bar{u}_k^{[3]}+8\eta\right)}{\bar{Q}_k^{[3]}\left(\bar{u}_k^{[3]}-8\eta\right)}, \tag{8.134}$$

Evidently, it follows from (8.125) and (8.126) that (8.131) implies (8.134).

Setting $u = u_k^{[2]}$ in (8.123), remembering that $Q^{[2]}(u_k^{[2]}) = 0$, we obtain the relation

$$\frac{Q^{[1]}\left(u_k^{[2]}-2\eta\right)}{Q^{[1]}\left(u_k^{[2]}+2\eta\right)} = \frac{\chi\left(u_k^{[2]}+2\eta\right)}{\chi\left(u_k^{[2]}-2\eta\right)}\frac{\bar{Q}^{[1]}\left(u_k^{[2]}-2\eta\right)}{\bar{Q}^{[1]}\left(u_k^{[2]}+2\eta\right)}. \tag{8.135}$$

With the help of this relation, it follows that (8.130) implies (8.133).

Setting $u = \pm u_k^{[1]} + 2\eta$ in (8.123), noting that therefore $Q^{[1]}(u-2\eta) = 0$ and $S(-u) = 0$, we obtain the pair of relations

$$\begin{aligned}
&\chi(u_k^{[1]}+4\eta)\, Q^{[1]}(u_k^{[1]}+4\eta)\, \bar{Q}^{[1]}(u_k^{[1]}) = \mathfrak{c}\, \sinh^{2N}(\frac{u_k^{[1]}}{2}+\eta)\, \sinh(u_k^{[1]}+2\eta)\, Q^{[2]}(u_k^{[1]}+2\eta)\,,\\
&\chi(u_k^{[1]}-4\eta)\, Q^{[1]}(u_k^{[1]}-4\eta)\, \bar{Q}^{[1]}(u_k^{[1]}) = -\mathfrak{c}\, \sinh^{2N}(\frac{u_k^{[1]}}{2}-\eta)\, \sinh(u_k^{[1]}-2\eta)\, Q^{[2]}(u_k^{[1]}-2\eta)\,.
\end{aligned} \tag{8.136}$$

Forming the ratio of these relations, we arrive at the Bethe equation (8.129). Similarly, setting $u = \pm\bar{u}_k^{[1]} - 2\eta$ in (8.123), we obtain the Bethe equation (8.132).

### 8.4.2 Duality of the transfer-matrix eigenvalues

In order to relate the transfer-matrix eigenvalues for $p = 1$ and $p = 2$, we observe from (8.123) that

$$\mathfrak{c} = \frac{S(u)-S(-u)}{\sinh^{2N}(\frac{u}{2})\, \sinh(u)\, Q^{[2]}(u)} = \frac{S(u-4\eta)-S(-u+4\eta)}{\sinh^{2N}(\frac{u}{2}-2\eta)\, \sinh(u-4\eta)\, Q^{[2]}(u-4\eta)}\,, \tag{8.137}$$

where the second equality follows from shifting $u \mapsto u-4\eta$. Making use of (8.124) and (8.125), and rearranging terms, we obtain the relation

$$\begin{aligned}
&\sinh^{2N}(\frac{u}{2}-2\eta)\, \sinh(u-4\eta)\, \chi(u+2\eta)\, \frac{Q^{[1]}(u+2\eta)}{Q^{[1]}(u-2\eta)}\\
&\quad + \sinh^{2N}(\frac{u}{2})\, \sinh(u)\, \chi(u-6\eta)\, \frac{Q^{[1]}(u-6\eta)}{Q^{[1]}(u-2\eta)}\frac{Q^{[2]}(u)}{Q^{[2]}(u-4\eta)}\\
&\quad = \sinh^{2N}(\frac{u}{2}-2\eta)\, \sinh(u-4\eta)\, \chi(u-2\eta)\, \frac{\bar{Q}^{[1]}(u+2\eta)}{\bar{Q}^{[1]}(u-2\eta)}\\
&\quad + \sinh^{2N}(\frac{u}{2})\, \sinh(u)\, \chi(u-2\eta)\, \frac{\bar{Q}^{[1]}(u-6\eta)}{\bar{Q}^{[1]}(u-2\eta)}\frac{\bar{Q}^{[2]}(u)}{\bar{Q}^{[2]}(u-4\eta)}\,.
\end{aligned} \tag{8.138}$$

This relation implies that

$$\begin{aligned}
& z_0(u)\, y_0(u,1)\, \frac{Q^{[1]}(u+2\eta)}{Q^{[1]}(u-2\eta)} \left[2\sinh(\frac{u}{2}-2\eta)\sinh(\frac{u}{2}-8\eta)\right]^{2N} \\
& + z_1(u)\, y_1(u,1)\, \frac{Q^{[1]}(u-6\eta)}{Q^{[1]}(u-2\eta)} \frac{Q^{[2]}(u)}{Q^{[2]}(u-4\eta)} \left[2\sinh(\frac{u}{2})\sinh(\frac{u}{2}-8\eta)\right]^{2N} \\
& = \frac{1}{f(u,1)} \Big\{ z_0(u)\, y_0(u,2)\, \frac{\bar{Q}^{[1]}(u+2\eta)}{\bar{Q}^{[1]}(u-2\eta)} \left[2\sinh(\frac{u}{2}-2\eta)\sinh(\frac{u}{2}-8\eta)\right]^{2N} \\
& + z_1(u)\, y_1(u,2)\, \frac{\bar{Q}^{[1]}(u-6\eta)}{\bar{Q}^{[1]}(u-2\eta)} \frac{\bar{Q}^{[2]}(u)}{\bar{Q}^{[2]}(u-4\eta)} \left[2\sinh(\frac{u}{2})\sinh(\frac{u}{2}-8\eta)\right]^{2N} \Big\} .
\end{aligned} \tag{8.139}$$

Finally, in view of also (8.121), (8.125), (8.126) and the identity

$$\frac{y_2(u,1)}{y_2(u,2)} = \frac{1}{f(u,1)} , \tag{8.140}$$

we conclude that the duality relation (8.117) is indeed satisfied by the Bethe ansatz solution for $n=3\,, p=1$.

### 8.4.3 Duality of the Dynkin labels

It is interesting to see if the formulas in Sec. 8.3.2 for the Dynkin labels are compatible with duality. For the case $n=3, p=1$, where the QG symmetry is $U_q(C_2)\otimes U_q(C_1)$, the Dynkin labels are given by

$$\begin{aligned}
a_1^{(l)} &= m_1 - 2m_2 + 2m_3 , \\
a_2^{(l)} &= m_2 - 2m_3 , \\
a_1^{(r)} &= N - m_1 .
\end{aligned} \tag{8.141}$$

On the other hand, for the dual case $n=3, p=2$, where the QG symmetry is $U_q(C_1)\otimes U_q(C_2)$, the Dynkin labels are given by

$$\begin{aligned}
\bar{a}_1^{(l)} &= \bar{m}_2 - 2\bar{m}_3 , \\
\bar{a}_1^{(r)} &= N - 2\bar{m}_1 + \bar{m}_2 , \\
\bar{a}_2^{(r)} &= \bar{m}_1 - \bar{m}_2 ,
\end{aligned} \tag{8.142}$$

where we again use a bar to denote quantities for the $p=2$ case. If a transfer-matrix eigenvalue ($\Lambda(u,1)$ or equivalently its dual $\Lambda(u,2)$) forms a single irreducible representation of the QG, then we expect that the corresponding Dynkin labels (8.141) and (8.142) should be related by the duality relations[11]

$$\begin{aligned}
\bar{a}_1^{(l)} &= a_1^{(r)} , \\
\bar{a}_i^{(r)} &= a_i^{(l)} , \qquad i=1,2 .
\end{aligned} \tag{8.143}$$

Making use of the relation (8.128) between $\{m_l\}$ and $\{\bar{m}_l\}$, we find that the relations (8.143) are indeed satisfied, provided that the $m$'s satisfy the constraint

$$N = m_1 + m_2 - 2m_3 \qquad \text{or equivalently} \qquad \bar{m}_1 - 2\bar{m}_2 + 2\bar{m}_3 = 0 . \tag{8.144}$$

Some simple examples for $N=2$ are displayed in Table 8.6.

Interestingly, not all transfer-matrix eigenvalues have Bethe roots that satisfy the constraint (8.144). (A simple example is the reference-state eigenvalue, for which $m_1=m_2=m_3=0$.) Such transfer-matrix eigenvalues correspond to *reducible* representations of the QG (i.e., they correspond to a direct sum of two or more irreps). Indeed, it was noted in the example 2 of section 7.5.4 that for $C_n^{(1)}$ with odd $n$ and $p=\frac{n\pm1}{2}$, there are additional degeneracies in the spectrum, which may be due to some yet unknown discrete symmetry.

[11] For general values of $n$ and $p$, we expect the duality relations

$$\begin{aligned}
\bar{a}_i^{(l)} &= a_i^{(r)} , \qquad i=1,\dots,p , \\
\bar{a}_i^{(r)} &= a_i^{(l)} , \qquad i=1,\dots,n-p ,
\end{aligned}$$

where the unbarred and barred quantities correspond to $p$ and $n-p$, respectively.

| | $m_1$ | $m_2$ | $m_3$ | $a_1^{(l)}$ | $a_2^{(l)}$ | $a_1^{(r)}$ | Irreps. |
|---|---|---|---|---|---|---|---|
| $p=1$ $U_q(C_2)\otimes U_q(C_1)$ | 2 | 0 | 0 | 2 | 0 | 0 | $(\mathbf{10},\mathbf{1})$ |
| | 2 | 2 | 1 | 0 | 0 | 0 | $2(\mathbf{1},\mathbf{1})$ |
| | $\bar{m}_1$ | $\bar{m}_2$ | $\bar{m}_3$ | $\bar{a}_1^{(l)}$ | $\bar{a}_1^{(r)}$ | $\bar{a}_2^{(r)}$ | Irreps. |
| $p=2$ $U_q(C_1)\otimes U_q(C_2)$ | 0 | 0 | 0 | 0 | 2 | 0 | $(\mathbf{1},\mathbf{10})$ |
| | 2 | 2 | 1 | 0 | 0 | 0 | $2(\mathbf{1},\mathbf{1})$ |

Table 8.6: Numbers of Bethe roots, which satisfy the constraint (8.144), and the corresponding Dynkin labels for $C_n^{(1)}$ with $n=3, N=2$ and $p=1,2$.

### 8.4.4 Further remarks

We have seen that, for the case $C_n^{(1)}$ with $n=3$, the relations (8.123)-(8.126) implement the duality transformation $p=1 \leftrightarrow p=2$ on the Bethe ansatz solution. Note that the Bethe roots corresponding to transfer-matrix eigenvalues related by this duality satisfy $u_k^{[2]}=\bar{u}_k^{[2]}$ and $u_k^{[3]}=\bar{u}_k^{[3]}$; i.e. only the type-1 Bethe roots $(u_k^{[1]},\bar{u}_k^{[1]})$ are different. We expect that, for $C_n^{(1)}$ with other values of $n$, as well as for $D_n^{(1)}$ and $D_{n+1}^{(2)}$, generalizations of the relations (8.123)-(8.126) can be found to implement the duality transformations $p \leftrightarrow n-p$ on the Bethe ansatz solutions. For supersymmetric (graded) integrable spin chains, a different type of "duality" transformation can be defined, which can be implemented on the corresponding Bethe ansatz solutions by relations somewhat analogous to (8.123)-(8.126), see e.g. [98, 99] and references therein.

# Chapter 9

# Conclusions and further developments

We used quantum group symmetries and some additional discrete symmetries to explain de degeneracies and multiplicities of spin chains constructed from anisotropic R-matrices and diagonal K-matrices depending on a discrete parameter $p$. We have proved that such spin chains have QG symmetry corresponding to removing the node $p$ of the extended Dynkin diagram of $\hat{g}$.

The cases $C_n^{(1)}$ and $D_n^{(1)}$ were proved to have a duality symmetry $p \to n-p$ which makes the spin chains for $p$ and $n-p$ have the same degeneracies. The existence of such symmetry could be expected by looking the middle column of the Table 6.1 or the form of their Dynkin diagrams in the Figure 8.1. In addition to that, when $n$ is even and $p = \frac{n}{2}$ there is a self duality symmetry that makes representations of the type $(1, a)$ and $(a, 1)$ become degenerated. When there is self-duality but also $\epsilon = 1$ ($\gamma_0 = -1$) there is an extra symmetry which we called bonus symmetry making states with representation $2(a, a)$ become degenerated, i.e., have degeneracy $2a^2$.

For $A_{2n-1}^{(2)}$, $B_n^{(1)}$ and $D_n^{(1)}$ we also have a left $Z_2$ symmetry transforming complex representations $(\bar{R}, 1)$ into their conjugates $(R, 1)$. In addition to that $D_n^{(1)}$ has also a right $Z_2$ symmetry transforming $(1, \bar{R})$ into $(1, R)$.

We constructed explicitly the transformations which generate all the symmetries described above and prove that they are responsible for the already mentioned effects. The proofs (even the ones in the appendix J which were done using Mathematica) were performed without to specify a number of sites $N$, and are therefore valid for any finite number of sites. It is worth to mention that in the process to prove those symmetries we found new properties for the $R$-matrix which in our knowledge are also new.

Still for $C_n^{(1)}$ and $D_n^{(1)}$, when $n$ is odd and $p = \frac{n \pm 1}{2}$ there is some unknown symmetry making states with different dimensions become degenerated. In the example presented in section 7.5.5 the representations $(1, 9)$ and one $(6, 4)$ become a 33 and the $(15, 1)$ and the other $(6, 4)$ become a 39. This is happening also for $D_{n+1}^{(2)}$ in [89]. All the other symmetries we worked with related representations with same dimension. What is happening for these cases is different and it would be very interesting to find what is the symmetry which causes this effect.

All the models described in this work (except $D_{n+1}^{(2)}$ whose symmetries were understood in [89]) were constructed using diagonal $K$-matrices depending on a discrete parameter $p$. There are, however some additional diagonal $K$-matrices solutions with a boundary parameter [62] which in principal would generate spin chains with QG symmetry. We leave these cases for future investigations.

In chapter 8 we have done analytical Bethe ansatz for all the cases discussed in the Chapter 7 for both $\epsilon = 0, 1$ and the case with $D_{n+1}^{(2)}$ [1] (for $\epsilon = 0$).

Usually, for a same model, the difference for the Bethe equations of close spin chains and open spin chains appear in two points: the $N$ in the lhs of the Bethe equations becomes $2N$, and the $Q$-functions become doubled. In our case we found an additional effect, we have an overall factor $\Phi_{l,p,n}(u)$ which is different from 1 for $l = p$ and equal to 1 otherwise. We numerically checked completeness for a small number of sites.

We also conjectured a generalization for open chains for the Reshetikhin's formula for the Bethe equations. Such formula depends on the simple roots of the algebras.

In addition to that we constructed formulas relating the Dynkin labels of the Bethe states with the number of Bethe roots of each type. These formulas help to determine the degeneracies of the transfer-matrix eigenvalues.

There is one case, however, which remains to be solved. The $D_{n+1}^{(2)}$ for $\epsilon = 1$ has so far resisted to our attempts to find its Bethe ansatz. We have made some progress for this case with $n = 1$ in [91] where

[1] the investigations for this case started in [87]).

we constructed an ansatz which gives all the levels with odd degeneracy but none of the ones with even degeneracy. In principle, we expected that all the levels would have odd degeneracy but it was found in [89] that some degeneracies are doubled. This does not happen for the other algebras considered and we do not know so far what symmetry is generating this effect. It happens that all the states our ansatz do not find are exactly the ones which have the doubled degeneracy. The difficulty increases with the number of sites, since the proportion of states with even degeneracy grows fast with the number of sites.

# Appendix A

# Zero Curvature for KdV hierarchy

Here we write down the time component of the two dimensional gauge potential generating the first three flows for the KdV hierarchy. Let

$$A_{t_N,KdV} = \mathcal{D'}^{(N)} + \mathcal{D'}^{(N-1)} + \cdots + \mathcal{D'}^{(0)}, \quad \mathcal{D'}^{(j)} \in \mathcal{G'}^{j} \tag{A.1}$$

where we find by solving the zero curvature representation (3.6) in the homogeneous gradation. For $N = 3$ we have

$$\begin{aligned}
\mathcal{D'}^{(3)} &= \zeta^3 h, \\
\mathcal{D'}^{(2)} &= -\zeta^2 E_\alpha + J\zeta^2 E_{-\alpha}, \\
\mathcal{D'}^{(1)} &= \frac{1}{2}\partial_x J\zeta E_{-\alpha} + \frac{1}{2}J\zeta h, \\
\mathcal{D'}^{(0)} &= -\frac{1}{2}JE_\alpha + \left(\frac{1}{4}\partial_x^2 J + \frac{1}{2}J^2\right) E_{-\alpha} + \frac{1}{4}\partial_x J h,
\end{aligned} \tag{A.2}$$

and $N = 5$

$$\begin{aligned}
\mathcal{D'}^{(5)} &= \zeta^5 h, \\
\mathcal{D'}^{(4)} &= -\zeta^4 E_\alpha + J\zeta^4 E_{-\alpha}, \\
\mathcal{D'}^{(3)} &= \frac{1}{2}\partial_x J\zeta^3 E_{-\alpha} + \frac{1}{2}J\zeta^3 h, \\
\mathcal{D'}^{(2)} &= -\frac{1}{2}J\zeta^2 E_\alpha + \left(\frac{1}{4}\partial_x^2 J + \frac{1}{2}J^2\right)\zeta^2 E_{-\alpha}\zeta^2 + \frac{1}{4}\partial_x J\zeta^2 h, \\
\mathcal{D'}^{(1)} &= \left(\frac{1}{8}\partial_x^3 J + \frac{3}{4}J\partial_x J\right)\zeta E_{-\alpha} + \left(\frac{1}{8}\partial_x^2 J + \frac{3}{8}J^2\right)\zeta h \\
\mathcal{D'}^{(0)} &= \left(-\frac{1}{8}\partial_x^2 J - \frac{3}{8}J^2\right) E_\alpha + \left(\frac{1}{16}\partial_x^3 J + \frac{3}{8}J\partial_x J\right) h \\
+ &\quad \left(\frac{1}{16}\partial_x^4 J + \frac{3}{8}(\partial_x J)^2 + \frac{1}{2}J\partial_x^2 J + \frac{3}{8}J^3\right) E_{-\alpha}
\end{aligned} \tag{A.3}$$

and for $N = 7$

$$
\begin{aligned}
\mathcal{D}'^{(7)} &= \zeta^7 h, \\
\mathcal{D}'^{(6)} &= -\zeta^6 E_\alpha + J\zeta^6 E_{-\alpha}, \\
\mathcal{D}'^{(5)} &= \frac{1}{2}\partial_x J\zeta^5 E_{-\alpha} + \frac{1}{2}J\zeta^5 h, \\
\mathcal{D}'^{(4)} &= -\frac{1}{2}J\zeta^4 E_\alpha + \left(\frac{1}{4}\partial_x^2 J + \frac{1}{2}J^2\right)\zeta^4 E_{-\alpha} + \frac{1}{4}\partial_x J\zeta^4 h, \\
\mathcal{D}'^{(3)} &= \left(\frac{1}{8}\partial_x^3 J + \frac{3}{4}J\partial_x J\right)\zeta^3 E_{-\alpha} + \left(\frac{1}{8}\partial_x^2 J + \frac{3}{8}J^2\right)\zeta^3 h \\
\mathcal{D}'^{(2)} &= \left(-\frac{1}{8}\partial_x^2 J - \frac{3}{8}J^2\right)\zeta^2 E_\alpha + \left(\frac{1}{16}\partial_x^3 J + \frac{3}{8}J\partial_x J\right)\zeta^2 h \\
+ &\quad \left(\frac{1}{16}\partial_x^4 J + \frac{3}{8}(\partial_x J)^2 + \frac{1}{2}J\partial_x^2 J + \frac{3}{8}J^3\right)\zeta^2 E_{-\alpha} \\
\mathcal{D}'^{(1)} &= \left(\frac{1}{32}\partial_x^5 J + \frac{5}{8}(\partial_x J)(\partial_x^2 J) + \frac{5}{16}J(\partial_x^3 J) + \frac{15}{16}J^2(\partial_x J)\right)\zeta E_{-\alpha} \\
+ &\quad \left(\frac{1}{32}(\partial_x^4 J) + \frac{5}{32}(\partial_x J)^2 + \frac{5}{16}J(\partial_x^2 J) + \frac{5}{16}J^3\right)\zeta h \\
+ &\quad \mathcal{D}'^{(0)} = \left(-\frac{1}{32}(\partial_x^4 J) - \frac{5}{32}(\partial_x J)^2 - \frac{5}{16}J(\partial_x^2 J) - \frac{5}{16}J^3\right)E_\alpha \\
+ &\quad \left(\frac{1}{64}(\partial_x^6 J) + \frac{5}{16}(\partial_x^2 J)^2 + \frac{15}{32}(\partial_x J)(\partial_x^3 J)\right)E_{-\alpha} + \\
+ &\quad \left(\frac{3}{16}J(\partial_x^4 J) + \frac{35}{32}J(\partial_x J)^2 + \frac{25}{32}J^2(\partial_x^2 J) + \frac{5}{16}J^4\right)E_{-\alpha} + \\
+ &\quad \left(\frac{1}{64}(\partial_x^5 J) + \frac{5}{16}(\partial_x J)(\partial_x^2 J) + \frac{5}{32}J(\partial_x^3 J) + \frac{15}{32}J^2(\partial_x J)\right)h
\end{aligned}
\tag{A.4}
$$

# Appendix B

# Equivalence between mKdV and KdV variables

We now verify the equivalence between mKdV and KdV variables. From (3.70) we find

$$\begin{aligned}\partial_x Q = &-\frac{(1+\epsilon)}{2\sigma}\partial_x\left(\Lambda - p\right)e^{\Lambda-p}(e^{q}+e^{-q}+\eta)\\ &-\frac{(1+\epsilon)}{2\sigma}\partial_x q e^{\Lambda-p}(e^{q}-e^{-q})+\frac{2(1-\epsilon)}{\sigma}\partial_x(p-\Lambda)e^{p-\Lambda}\end{aligned} \tag{B.1}$$

and using (3.72), (3.59) and(3.60) we obtain

$$\partial_x Q = 2\beta_- - \frac{QP}{2} + Q\Omega \tag{B.2}$$

Consider now $\partial_x P = J_1 + J_2 = \partial_x(w_1 + w_2)$. In terms of Miura transformation

$$\partial_x P \;\; = \;\; \epsilon\partial_x(v_1+v_2) - (v_1^2+v_2^2) = \epsilon\partial_x^2 p - \frac{1}{2}(\partial_x p)^2 - \frac{1}{2}(\partial_x q)^2.) \tag{B.3}$$

Acting with $\partial_x$ in (3.72) and using (B.3) and (3.70) we find

$$\begin{aligned}&\frac{Q}{2}\left(\partial_x P + 2\partial_x\Omega\right) = \frac{\partial_x Q}{2}(P-2\Omega) + \frac{1}{2}Q\left[(\partial_x p)^2 - (\partial_x q)^2\right]\\ &- Q\partial_x p\partial_x\Lambda + \frac{(1+\epsilon)}{2\sigma}\partial_x p\partial_x q e^{\Lambda-p}(e^{q}-e^{-q}) + \frac{2\epsilon}{\sigma^2}\partial_x q(e^{q}+e^{-q})\end{aligned} \tag{B.4}$$

where $p = v_1 + v_2, \quad q = v_1 - v_2$. Substituting the equation (3.70) in the equation (3.73) we find

$$\begin{aligned}&\frac{Q}{2}\left[(\partial_x p)^2 - (\partial_x q)^2\right] = \frac{2\partial_x p}{\sigma^2}(e^{q}-e^{-q}) - \frac{2\epsilon}{\sigma^2}\partial_x q(e^{q}+e^{-q})\\ &-\frac{Q}{2}\left(-\Omega^2 + \Omega P - \frac{P^2}{4} + \frac{Q^2}{4}\right) + \beta_+ Q - \beta_-(P-2\Omega)\end{aligned} \tag{B.5}$$

Substituting this result in (B.4) and eliminating the mKdV variables using (3.59) and (3.60) we obtain

$$\partial_x P + 2\partial_x\Omega = -\frac{1}{4}(P^2+Q^2) + \Omega P - \Omega^2 + 2\beta_+ \tag{B.6}$$

Eqns. (B.2) and (B.6) correspond precisely to the Type-II Bäcklund for the KdV hierarchy.

# Appendix C

# Consistency with Equations of Motion

In this appendix we verify that the compatibility of Bäcklund transformations lead us to the equation of motion.

We start with the spatial part which is common to all $N$. For the KdV equation it is given by

$$\partial_x P = \frac{\beta^2}{2} - \frac{1}{2}Q^2. \tag{C.1}$$

In what follows it will be useful calculate its spatial derivatives:

$$\partial_x^2 P = -Q\partial_x Q; \tag{C.2}$$
$$\partial_x^3 P = -(\partial_x Q)^2 - Q(\partial_x^2 Q); \tag{C.3}$$
$$\partial_x^4 P = -3(\partial_x Q)(\partial_x^2 Q) - Q(\partial_x^3 Q); \tag{C.4}$$
$$\partial_x^5 P = -3(\partial_x^2)^2 - 4(\partial_x Q)(\partial_x^3 Q) - Q(\partial_x^4 Q); \tag{C.5}$$
$$\partial_x^6 P = -10(\partial_x^2 Q)(\partial_x^3 Q) - 5(\partial_x Q)(\partial_x^4 Q) - Q(\partial_x^5 Q); \tag{C.6}$$
$$\partial_x^7 P = -10(\partial_x^3 Q)^2 - 15(\partial_x^2 Q)(\partial_x^4 Q) - 6(\partial_x Q)(\partial_x^5 Q) - Q\partial_x^6 Q; \tag{C.7}$$
$$\partial_x^8 P = -35(\partial_x^3 Q)(\partial_x^4 Q) - 21(\partial_x^2 Q)(\partial_x^5 Q) - 7(\partial_x Q)(\partial_x^6 Q) - Q(\partial_x^7 Q). \tag{C.8}$$

**N=3 (KdV)**

The temporal part of the KdV BT is given by

$$4\partial_{t_3} P = -Q(\partial_x^2 Q) + \frac{1}{2}\left[(\partial_x Q)^2 + (\partial_x P)^2\right]. \tag{C.9}$$

In order to verify the consistency of this transformation we act with the spatial derivative to obtain

$$4\partial_x\partial_{t_3} P = -Q\partial_x^3 Q + 3(\partial_x P)(\partial_x^2 P), \tag{C.10}$$

eliminating the term $-Q\partial_x^3 Q$ from equation (C.4) we find

$$4\partial_x\partial_{t_3} P = \partial_x^4 P + 3(\partial_x P)(\partial_x^2 P) + 3(\partial_x Q)(\partial_x^2 Q). \tag{C.11}$$

Substituting

$$\partial_x P = J_1 + J_2, \qquad \partial_x Q = J_1 - J_2 \tag{C.12}$$

(C.11) becomes precisely the sum of two KdV equations.

**N=5**

The temporal part of the BT for N=5 equation is given by

$$\begin{aligned} 16\partial_{t_5} P = & -Q(\partial_x^4 Q) + (\partial_x Q)(\partial_x^3 Q) + 5(\partial_x P)(\partial_x^3 P) + \frac{5}{2}(\partial_x^2 P)^2 - \frac{1}{2}(\partial_x^2 Q)^2 \\ & + \frac{5}{2}(\partial_x P)\left[(\partial_x P)^2 + 3(\partial_x Q)^2\right] \end{aligned} \tag{C.13}$$

Acting $\partial_x$ in the equation (C.13) we obtain

$$16\partial_x\partial_{t_5}P = -\,Q(\partial_x^5 Q) + 10(\partial_x^2 P)(\partial_x^3 P) + 5(\partial_x P)(\partial_x^4 P) + \frac{15}{2}(\partial_x P)^2(\partial_x^2 P) + \frac{15}{2}(\partial_x^2 P)(\partial_x Q)^2 + 15(\partial_x P)(\partial_x Q)(\partial_x^2 Q). \tag{C.14}$$

Then we isolate the term $-Q(\partial_x^5 Q)$ from equation (C.6) to find

$$16\partial_x\partial_{t_5}P = \partial_x^6 P + 10(\partial_x^2 Q)(\partial_x^3 Q) + 5(\partial_x Q)(\partial_x^4 Q) + 10(\partial_x^2 P)(\partial_x^3 P) + 5(\partial_x P)(\partial_x^4 P) + \frac{15}{2}(\partial_x^2 P)\left[(\partial_x P)^2 + (\partial_x P)^2\right] + 15(\partial_x P)(\partial_x Q)(\partial_x^2 Q), \tag{C.15}$$

Substituting (C.12) we obtain the sum of two equations for $N = 5$, i.e., eqn. (3.9).

**N=7**

The temporal BT for the $N = 7$ equation is

$$64\partial_{t_7}P = -\,Q(\partial_x^6 Q) + (\partial_x Q)(\partial_x^5 Q) + 7(\partial_x P)(\partial_x^5 P) - (\partial_x^2 Q)(\partial_x^4 Q) + 14(\partial_x^2 P)(\partial_x^4 P) + \frac{1}{2}(\partial_x^3 Q)^2 + \frac{21}{2}(\partial_x^3 P)^2 + \frac{35}{2}(\partial_x^3 P)(\partial_x P)^2 + \frac{35}{2}(\partial_x^3 P)(\partial_x Q)^2 + 35(\partial_x^3 Q)(\partial_x P)(\partial_x Q) + \frac{35}{2}(\partial_x P)\left[(\partial_x^2 P)^2 + (\partial_x^2 Q)^2\right] + 35(\partial_x^2 P)(\partial_x^2 Q)^2(\partial_x Q) + \frac{35}{8}\left[(\partial_x P)^2 + (\partial_x Q)^2\right] + \frac{105}{4}(\partial_x Q)^2(\partial_x P)^2. \tag{C.16}$$

Likewise we did for other values of $N$, acting $\partial_x$ in the above equation. Then we isolate $-Q(\partial_x^7 Q)$ from equation (C.8) to find

$$64\partial_x\partial_{t_7}P = \partial_x^8 P + (\partial_x P)(\partial_x^6 P) + 7(\partial_x Q)(\partial_x^6 Q) + 21(\partial_x^2 P)(\partial_x^5 P) + 21(\partial_x^2 Q)(\partial_x^5 Q) + 35(\partial_x^3 P)(\partial_x^4 P) + 35(\partial_x^3 Q)(\partial_x^4 Q) + \frac{35}{2}(\partial_x^4 P)\left[(\partial_x P)^2 + (\partial_x Q)^2\right] + 35(\partial_x P)(\partial_x Q)(\partial_x^4 Q) + 70(\partial_x P)\left[(\partial_x^2 P)(\partial_x^3 P) + (\partial_x^2 Q)(\partial_x^3 Q)\right] + 70(\partial_x Q)\left[(\partial_x^2 Q)(\partial_x^3 P) + (\partial_x^2 P)(\partial_x^3 Q)\right] + \frac{35}{2}(\partial_x^2 P)^3 + \frac{105}{2}(\partial_x^2 P)(\partial_x^2 Q)^2 + \frac{35}{2}(\partial_x^2 P)(\partial_x P)^3 + \frac{35}{2}(\partial_x^2 Q)(\partial_x Q)^3 + \frac{105}{2}(\partial_x^2 Q)(\partial_x Q)(\partial_x P)^2 + \frac{105}{2}(\partial_x^2 P)(\partial_x P)(\partial_x Q)^2 \tag{C.17}$$

Substituting (C.12) we obtain the equation of motion for $N = 7$ (3.10).

# Appendix D

# Representation of the $\widehat{sl}(2,1)$ affine Lie superalgebra

In this work we are considering the following representation of the $\widehat{sl}(2,1)$ affine superalgebra,

$$K_1^{(2n+1)} = \begin{pmatrix} 0 & -\lambda^n & 0 \\ -\lambda^{n+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2^{(2n+1)} = \begin{pmatrix} \lambda^{n+\frac{1}{2}} & 0 & 0 \\ 0 & \lambda^{n+\frac{1}{2}} & 0 \\ 0 & 0 & 2\lambda^{n+\frac{1}{2}} \end{pmatrix}, \tag{D.1}$$

$$M_1^{(2n+1)} = \begin{pmatrix} 0 & -\lambda^n & 0 \\ \lambda^{n+1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_2^{(2n)} = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & -\lambda^n & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{D.2}$$

$$F_1^{\left(2n+\frac{3}{2}\right)} = \begin{pmatrix} 0 & 0 & \lambda^{n+\frac{1}{2}} \\ 0 & 0 & -\lambda^{n+1} \\ \lambda^{n+1} & -\lambda^{n+\frac{1}{2}} & 0 \end{pmatrix}, \quad F_2^{\left(2n+\frac{1}{2}\right)} = \begin{pmatrix} 0 & 0 & -\lambda^n \\ 0 & 0 & \lambda^{n+\frac{1}{2}} \\ \lambda^{n+\frac{1}{2}} & -\lambda^n & 0 \end{pmatrix}, \tag{D.3}$$

$$G_1^{\left(2n+\frac{1}{2}\right)} = \begin{pmatrix} 0 & 0 & \lambda^n \\ 0 & 0 & \lambda^{n+\frac{1}{2}} \\ \lambda^{n+\frac{1}{2}} & \lambda^n & 0 \end{pmatrix}, \quad G_2^{\left(2n+\frac{3}{2}\right)} = \begin{pmatrix} 0 & 0 & -\lambda^{n+\frac{1}{2}} \\ 0 & 0 & -\lambda^{n+1} \\ \lambda^{n+1} & \lambda^{n+\frac{1}{2}} & 0 \end{pmatrix}. \tag{D.4}$$

# Appendix E

# $N = 5$ Lax component

The Lax component $A_{t_5}$ takes the following form,

$$A_{t_5} = \left(\begin{array}{cc|c} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ \hline b_{31} & b_{32} & b_{33} \end{array}\right), \tag{E.1}$$

where,

$$\begin{aligned} b_{11} &= \lambda^{5/2} + \lambda^2\,\partial_x^2\phi - \frac{i\lambda^{3/2}}{2}\bar\psi\partial_x\bar\psi + \lambda\left(\frac{1}{2}(\partial_x\phi)^3 - \frac{1}{4}\partial_x^3\phi - \frac{3i}{4}\partial_x\phi\,\bar\psi\partial_x\bar\psi\right) \\ &+\lambda^{1/2}\left(\frac{i}{2}(\partial_x\phi)^2\,\bar\psi\partial_x\bar\psi + \frac{i}{8}\partial_x\bar\psi\partial_x^2\bar\psi - \frac{i}{8}\bar\psi\partial_x^3\bar\psi\right) + \frac{5}{8}(\partial_x\phi)^2\partial_x^3\phi + \frac{5}{8}\partial_x\phi(\partial_x^2\phi)^2 \\ &+\frac{5i}{4}(\partial_x\phi)^3\,\bar\psi\partial_x\bar\psi - \frac{5i}{16}\partial_x\phi\,\bar\psi\partial_x^3\bar\psi - \frac{5i}{16}\partial_x^2\phi\,\bar\psi\partial_x^2\bar\psi - \frac{5i}{16}\partial_x^3\phi\,\bar\psi\partial_x\bar\psi - \frac{3}{8}(\partial_x\phi)^5, \end{aligned} \tag{E.2}$$

$$\begin{aligned} b_{12} &= -\lambda^2 + \frac{\lambda}{2}\left(\partial_x^2\phi + (\partial_x\phi)^2 - i\bar\psi\partial_x\bar\psi\right) - \frac{1}{8}(\partial_x^2\phi)^2 + \frac{1}{8}\partial_x^4\phi + \frac{1}{4}\partial_x\phi\partial_x^3\phi - \frac{3}{4}(\partial_x\phi)^2\partial_x^2\phi \\ &-\frac{3}{8}(\partial_x\phi)^4 + \frac{i}{4}\partial_x\phi\bar\psi\partial_x^2\bar\psi + i(\partial_x\phi)^2\bar\psi\partial_x\bar\psi + \frac{i}{2}\partial_x^2\phi\,\bar\psi\partial_x\bar\psi + \frac{i}{8}\partial_x\bar\psi\partial_x^2\bar\psi - \frac{i}{8}\bar\psi\partial_x^3\bar\psi, \end{aligned} \tag{E.3}$$

$$\begin{aligned} b_{13} &= \lambda^2\sqrt{i}\bar\psi + \frac{\lambda^{3/2}\sqrt{i}}{2}\left(\partial_x\phi\,\bar\psi + \partial_x\bar\psi\right) + \frac{\lambda\sqrt{i}}{4}\left(\partial_x\phi\partial_x\bar\psi - 2(\partial_x\phi)^2\bar\psi - \partial_x^2\phi\bar\psi + \partial_x^2\bar\psi\right) \\ &+\frac{\lambda^{1/2}\sqrt{i}}{8}\left(\partial_x\phi\partial_x^2\bar\psi - 3(\partial_x\phi)^2\partial_x\bar\psi - \partial_x^2\phi\partial_x\bar\psi + \partial_x^3\phi\bar\psi - 3\partial_x\phi\partial_x^2\phi\bar\psi - 3(\partial_x\phi)^3\bar\psi + \partial_x^3\bar\psi\right) \\ &+\frac{\sqrt{i}}{16}\left(\partial_x\phi\,\partial_x^3\bar\psi - \partial_x^2\phi\,\partial_x^2\bar\psi + \partial_x^3\phi\,\partial_x\bar\psi - \partial_x^4\phi\,\bar\psi + \partial_x^4\bar\psi\right) - \frac{\sqrt{i}}{2}\partial_x\phi\partial_x^2\phi\partial_x\bar\psi \\ &+\frac{\sqrt{i}}{4}\left(\partial_x^2\phi\,(\partial_x\phi)^2\,\bar\psi - (\partial_x\phi)^3\partial_x\bar\psi - (\partial_x\phi)^2\,\partial_x^2\bar\psi\right) \\ &+\frac{\sqrt{i}}{8}\left(3(\partial_x\phi)^4\,\bar\psi - 3\partial_x\phi\,\partial_x^3\phi\,\bar\psi - (\partial_x^2\phi)^2\bar\psi\right), \end{aligned} \tag{E.4}$$

$$\begin{aligned} b_{21} &= -\lambda^3 + \frac{\lambda^2}{2}\left(-\partial_x^2\phi + (\partial_x\phi)^2 - i\bar\psi\partial_x\bar\psi\right) + \lambda\left(\frac{1}{4}\partial_x\phi\,\partial_x^3\phi - \frac{1}{8}(\partial_x^2\phi)^2 - \frac{1}{8}\partial_x^4\phi\right. \\ &-\frac{3}{8}(\partial_x\phi)^4 + \frac{3}{4}(\partial_x\phi)^2\partial_x^2\phi + i(\partial_x\phi)^2\bar\psi\partial_x\bar\psi - \frac{i}{4}\partial_x\phi\bar\psi\partial_x^2\bar\psi - \frac{i}{2}\partial_x^2\phi\bar\psi\partial_x\bar\psi \\ &\left.+\frac{i}{8}\partial_x\bar\psi\partial_x^2\bar\psi - \frac{i}{8}\bar\psi\partial_x^3\bar\psi\right), \end{aligned} \tag{E.5}$$

$$
\begin{aligned}
b_{22} \;=\;& \lambda^{5/2} + \lambda^2 \partial_x\phi - \frac{i\lambda^{3/2}}{2}\bar\psi\partial_x\bar\psi + \lambda\left(\frac{1}{4}\partial_x^3\phi - \frac{1}{2}(\partial_x\phi)^3 + \frac{3i}{4}\partial_x\phi\bar\psi\partial_x\bar\psi\right) \\
&+\lambda^{1/2}\left(\frac{i}{2}(\partial_x\phi)^2\bar\psi\partial_x\bar\psi + \frac{i}{8}\partial_x\bar\psi\partial_x^2\bar\psi - \frac{i}{8}\bar\psi\partial_x^3\bar\psi\right) + \frac{1}{16}\partial_x^5\phi - \frac{5}{8}\partial_x\phi(\partial_x^2\phi)^2 \\
&-\frac{5}{8}\partial_x^3\phi(\partial_x\phi)^2 + \frac{3}{8}(\partial_x\phi)^5 + \frac{5i}{16}\partial_x^2\phi\bar\psi\partial_x^2\bar\psi + \frac{5i}{16}\partial_x^3\phi\bar\psi\partial_x\bar\psi + \frac{5i}{16}\partial_x\phi\bar\psi\partial_x^3\bar\psi \\
&-\frac{5i}{4}(\partial_x\phi)^3\bar\psi\partial_x\bar\psi, \qquad\qquad \text{(E.6)}
\end{aligned}
$$

$$
\begin{aligned}
b_{23} \;=\;& \lambda^{5/2}\sqrt{i}\bar\psi + \frac{\lambda^2\sqrt{i}}{2}\left(\partial_x\bar\psi - \partial_x\phi\bar\psi\right) + \frac{\lambda^{3/2}\sqrt{i}}{4}\left(\partial_x^2\phi\bar\psi - 2(\partial_x\phi)^2\bar\psi - \partial_x\phi\partial_x\bar\psi + \partial_x^2\bar\psi\right) \\
&+\frac{\lambda\sqrt{i}}{8}\left(\partial_x^2\phi\partial_x\bar\psi - \partial_x\phi\partial_x^2\bar\psi - \partial_x^3\phi\bar\psi - 3(\partial_x\phi)^2\partial_x\bar\psi + 3(\partial_x\phi)^3\bar\psi - 3\partial_x\phi\partial_x^2\phi\bar\psi + \partial_x^3\bar\psi\right) \\
&+\frac{\lambda^{1/2}\sqrt{i}}{16}\left(\partial_x^4\phi\,\bar\psi - \partial_x^3\phi\,\partial_x\bar\psi - \partial_x\phi\,\partial_x^3\bar\psi + \partial_x^2\phi\,\partial_x^2\bar\psi + \partial_x^4\bar\psi\right) - \frac{\lambda^{1/2}\sqrt{i}}{2}\partial_x\phi\,\partial_x^2\phi\,\partial_x\bar\psi \\
&+\frac{\lambda^{1/2}\sqrt{i}}{4}\Big((\partial_x\phi)^3\partial_x\bar\psi - (\partial_x\phi)^2\partial_x^2\bar\psi - \partial_x^2\phi(\partial_x\phi)^2\bar\psi + \frac{3}{2}(\partial_x\phi)^4\,\bar\psi - 12\partial_x^3\phi\,\partial_x\phi\,\bar\psi \\
&\qquad -4(\partial_x^2\phi)^2\,\bar\psi\Big), \qquad\qquad \text{(E.7)}
\end{aligned}
$$

$$
\begin{aligned}
b_{31} \;=\;& \lambda^{5/2}\sqrt{i}\bar\psi + \frac{\lambda^2\sqrt{i}}{2}(\partial_x\phi\bar\psi - \partial_x\bar\psi) + \frac{\lambda^{3/2}\sqrt{i}}{4}\left(\partial_x^2\phi\bar\psi - \partial_x\phi\partial_x\bar\psi - 2(\partial_x\phi)^2\bar\psi + \partial_x^2\bar\psi\right) \\
&+\frac{\lambda\sqrt{i}}{8}\left(\partial_x\phi\partial_x^2\bar\psi - \partial_x^2\phi\partial_x\bar\psi + \partial_x^3\phi\bar\psi + 3\partial_x\phi\partial_x^2\phi\bar\psi - 3(\partial_x\phi)^3\bar\psi + 3(\partial_x\phi)^2\partial_x\bar\psi - \partial_x^3\bar\psi\right) \\
&+\frac{\lambda^{1/2}\sqrt{i}}{16}\left(\partial_x^4\phi\,\bar\psi - \partial_x^3\phi\,\partial_x\bar\psi - \partial_x\phi\partial_x^3\bar\psi + \partial_x^2\phi\,\partial_x^2\bar\psi + \partial_x^4\bar\psi\right) - \frac{\lambda^{1/2}\sqrt{i}}{2}\partial_x\phi\,\partial_x^2\phi\,\partial_x\bar\psi \\
&+\frac{\lambda^{1/2}\sqrt{i}}{4}\Big((\partial_x\phi)^3\partial_x\bar\psi - (\partial_x\phi)^2\partial_x^2\bar\psi - \partial_x^2\phi(\partial_x\phi)^2\bar\psi - \frac{1}{2}(\partial_x^2\phi)^2\bar\psi - 12\partial_x\phi\partial_x^3\phi\bar\psi \\
&\qquad +12(\partial_x\phi)^4\,\bar\psi\Big), \qquad\qquad \text{(E.8)}
\end{aligned}
$$

$$
\begin{aligned}
b_{32} \;=\;& \lambda^2\sqrt{i}\bar\psi - \frac{\lambda^{3/2}\sqrt{i}}{2}(\partial_x\phi\,\bar\psi + \partial_x\bar\psi) + \frac{\lambda\sqrt{i}}{4}\left(\partial_x\phi\partial_x\bar\psi - \partial_x^2\phi\,\bar\psi - 2(\partial_x\phi)^2\,\bar\psi + \partial_x^2\bar\psi\right) \\
&+\frac{\lambda^{1/2}\sqrt{i}}{8}\left(3\partial_x\phi\partial_x^2\phi\bar\psi + 3(\partial_x\phi)^3\bar\psi - \partial_x^3\phi\bar\psi + \partial_x^2\phi\partial_x\bar\psi - \partial_x\phi\partial_x^2\bar\psi + 3(\partial_x\phi)^2\partial_x\bar\psi - \partial_x^3\bar\psi\right) \\
&+\frac{\sqrt{i}}{16}\left(\partial_x\phi\,\partial_x^3\bar\psi - \partial_x^2\phi\,\partial_x^2\bar\psi - \partial_x^4\phi\,\bar\psi + \partial_x^3\phi\,\partial_x\bar\psi + \partial_x^4\bar\psi\right) - \frac{\sqrt{i}}{2}\partial_x\phi\,\partial_x^2\phi\,\partial_x\bar\psi \\
&-\frac{\sqrt{i}}{4}(\partial_x\phi)^2\,\partial_x^2\bar\psi - \frac{\sqrt{i}}{4}(\partial_x\phi)^3\,\partial_x\bar\psi - \frac{\sqrt{i}}{8}(\partial_x^2\phi)^2\,\bar\psi - \frac{3}{8}\partial_x^3\phi\,\partial_x\phi\,\bar\psi + \frac{\sqrt{i}}{4}(\partial_x\phi)^2\,\partial_x^2\phi\,\bar\psi \\
&+\frac{3}{8}(\partial_x\phi)^4\,\bar\psi, \qquad\qquad \text{(E.9)}
\end{aligned}
$$

$$
b_{33} \;=\; 2\lambda^{5/2} - i\lambda^{3/2}\bar\psi\partial_x\bar\psi + i\lambda^{1/2}\left((\partial_x\phi)^2\,\bar\psi\partial_x\bar\psi + \frac{1}{4}\partial_x\bar\psi\partial_x^2\bar\psi - \frac{1}{4}\bar\psi\partial_x^3\bar\psi\right). \qquad \text{(E.10)}
$$

# Appendix F

# Coefficients of the Bäcklund transformations for $N = 5$ member

The coefficients $c_i$ in Bäcklund equations (4.30) are given by,

$$
\begin{aligned}
c_0 &= -\partial_x^4\phi_+ \cosh\left(\frac{\phi_+}{2}\right) + \left(\partial_x^2\phi_+\right)^2 \sinh\left(\frac{\phi_+}{2}\right) + 3\left(\partial_x^3\phi_+\right)\left(\partial_x\phi_+\right)\sinh\left(\frac{\phi_+}{2}\right) \\
&\quad + \left(\partial_x^2\phi_+\right)\left(\partial_x\phi_+\right)^2 \cosh\left(\frac{\phi_+}{2}\right) - \frac{3}{4}\left(\partial_x\phi_+\right)^4 \sinh\left(\frac{\phi_+}{2}\right), && \text{(F.1)} \\
c_1 &= \partial_x^3\phi_+ \cosh\left(\frac{\phi_+}{2}\right) - \left(\partial_x\phi_+\right)^3 \cosh\left(\frac{\phi_+}{2}\right) + 4\left(\partial_x^2\phi_+\right)\left(\partial_x\phi_+\right)\sinh\left(\frac{\phi_+}{2}\right), && \text{(F.2)} \\
c_2 &= -\left(\partial_x^2\phi_+\right)\cosh\left(\frac{\phi_+}{2}\right) + 2\left(\partial_x\phi_+\right)^2 \sinh\left(\frac{\phi_+}{2}\right), && \text{(F.3)} \\
c_3 &= \partial_x\phi_+ \cosh\left(\frac{\phi_+}{2}\right), && \text{(F.4)} \\
c_4 &= -2\sinh\left(\frac{\phi_+}{2}\right), && \text{(F.5)} \\
c_5 &= 4\left(\partial_x^4\phi_+\right)\cosh\phi_+ - 6\left(\partial_x^2\phi_+\right)\left(\partial_x\phi_+\right)^2\cosh\phi_+ + 2\left(\partial_x^2\phi_+\right)^2\sinh\phi_+ \\
&\quad -4\left(\partial_x^3\phi_+\right)\left(\partial_x\phi_+\right)\sinh\phi_+ + \frac{3}{2}\left(\partial_x\phi_+\right)^4\sinh\phi_+, && \text{(F.6)} \\
c_6 &= 4\left(\partial_x^2\phi_+\right)\cosh\phi_+ - 4\left(\partial_x\phi_+\right)^2\sinh\phi_+, && \text{(F.7)} \\
c_7 &= 2(\partial_x\phi_+)(\cosh\phi_+), && \text{(F.8)} \\
c_8 &= 2\sinh\phi_+, && \text{(F.9)} \\
c_9 &= \left[-20\cosh\left(\frac{\phi_+}{2}\right) + 20\cosh\left(\frac{3\phi_+}{2}\right) + 80\cosh\left(\frac{5\phi_+}{2}\right)\right]\left(\partial_x^2\phi_+\right) \\
&\quad + \left[35\sinh\left(\frac{\phi_+}{2}\right) - \frac{15}{2}\sinh\left(\frac{3\phi_+}{2}\right) + \frac{75}{2}\sinh\left(\frac{5\phi_+}{2}\right)\right]\left(\partial_x\phi_+\right)^2, && \text{(F.10)} \\
c_{10} &= \left[70\cosh\left(\frac{\phi_+}{2}\right) - 25\cosh\left(\frac{3\phi_+}{2}\right) + 35\cosh\left(\frac{5\phi_+}{2}\right)\right]\partial_x\phi_+, && \text{(F.11)} \\
c_{11} &= -20\sinh\left(\frac{\phi_+}{2}\right) + 10\sinh\left(\frac{3\phi_+}{2}\right) + 30\sinh\left(\frac{5\phi_+}{2}\right), && \text{(F.12)} \\
c_{12} &= 40\partial_x^2\phi_+\left(\cosh\phi_+ - \cosh(3\phi_+)\right) - 20(\partial_x\phi_+)^2\left(5\sinh\phi_+ + \sinh(3\phi_+)\right), && \text{(F.13)} \\
c_{13} &= 30\sinh\phi_+ - 10\sinh(3\phi_+), && \text{(F.14)} \\
c_{14} &= -120\sinh\left(\frac{\phi_+}{2}\right) + 80\sinh\left(\frac{3\phi_+}{2}\right) + 240\sinh\left(\frac{5\phi_+}{2}\right) - 60\sinh\left(\frac{7\phi_+}{2}\right) \\
&\quad -100\sinh\left(\frac{9\phi_+}{2}\right), && \text{(F.15)} \\
c_{15} &= 240\sinh\phi_+ - 120\sinh(3\phi_+) + 24\sinh(5\phi_+). && \text{(F.16)}
\end{aligned}
$$

And the coefficients $g_j$ in the Bäcklund equations (4.31) are the following,

$$\begin{aligned} g_0 &= \Big[-\frac{1}{2}(\partial_x^2\phi_+)^2 - \frac{3}{2}(\partial_x^3\phi_+)(\partial_x\phi_+) + \frac{3}{8}(\partial_x\phi_+)^4\Big]\cosh\Big(\frac{\phi_+}{2}\Big) \\ &+\Big[\frac{1}{2}\partial_x^4\phi_+ - \frac{1}{2}(\partial_x^2\phi_+)(\partial_x\phi_+)^2\Big]\sinh\Big(\frac{\phi_+}{2}\Big), \end{aligned} \tag{F.17}$$

$$g_1 = \left[-\frac{1}{2}\partial_x^3\phi_+ + \frac{1}{2}(\partial_x\phi_+)^3\right]\sinh\Big(\frac{\phi_+}{2}\Big) - 2(\partial_x^2\phi_+)(\partial_x\phi_+)\cosh\Big(\frac{\phi_+}{2}\Big), \tag{F.18}$$

$$g_2 = \frac{1}{2}(\partial_x^2\phi_+)\sinh\Big(\frac{\phi_+}{2}\Big) - (\partial_x\phi_+)^2\cosh\Big(\frac{\phi_+}{2}\Big), \tag{F.19}$$

$$g_3 = -\frac{1}{2}\partial_x\phi_+\sinh\Big(\frac{\phi_+}{2}\Big), \tag{F.20}$$

$$g_4 = \cosh\Big(\frac{\phi_+}{2}\Big), \tag{F.21}$$

$$g_5 = [10\sinh\phi_+ - 5\sinh(2\phi_+)]\,\partial_x\phi_+, \tag{F.22}$$

$$\begin{aligned} g_6 &= -\left[\frac{5}{2} + 20\cosh\phi_+ + \frac{35}{2}\cosh(2\phi_+)\right](\partial_x^2\phi_+)(\partial_x\phi_+) \\ &- [10\sinh\phi_+ + 5\sinh(2\phi_+)]\,\partial_x^3\phi_+ - \left[\frac{15}{2}\sinh\phi_+ + \frac{15}{4}\sinh(2\phi_+)\right](\partial_x\phi_+)^3, \end{aligned} \tag{F.23}$$

$$\begin{aligned} g_7 &= -\left[\frac{35}{2}\cosh\left(\frac{\phi_+}{2}\right) + \frac{45}{4}\cosh\left(\frac{3\phi_+}{2}\right) + \frac{45}{4}\cosh\left(\frac{5\phi_+}{2}\right)\right](\partial_x\phi_+)^2 \\ &- \left[15\sinh\left(\frac{3\phi_+}{2}\right) + 15\sinh\left(\frac{5\phi_+}{2}\right)\right]\partial_x^2\phi_+, \end{aligned} \tag{F.24}$$

$$g_8 = \left[-15\sinh\left(\frac{\phi_+}{2}\right) - \frac{45}{2}\sinh\left(\frac{3\phi_+}{2}\right) - \frac{15}{2}\sinh\left(\frac{5\phi_+}{2}\right)\right](\partial_x\phi_+), \tag{F.25}$$

$$g_9 = 10\cosh\left(\frac{\phi_+}{2}\right) - 5\cosh\left(\frac{3\phi_+}{2}\right) - 5\cosh\left(\frac{5\phi_+}{2}\right), \tag{F.26}$$

$$g_{10} = [-120\sinh\phi_+ - 40\sinh(2\phi_+) + 40\sinh(3\phi_+) + 20\sinh(4\phi_+)]\,\partial_x\phi_+, \tag{F.27}$$

$$\begin{aligned} g_{11} &= 60\cosh\left(\frac{\phi_+}{2}\right) - 40\cosh\left(\frac{3\phi_+}{2}\right) - 40\cosh\left(\frac{5\phi_+}{2}\right) + 10\cosh\left(\frac{7\phi_+}{2}\right) \\ &+10\cosh\left(\frac{9\phi_+}{2}\right). \end{aligned} \tag{F.28}$$

# Appendix G

# R-matrices

The R-matrices are given by

$$R(u) = c(u) \sum_{\alpha \neq \alpha'} e_{\alpha\alpha} \otimes e_{\alpha\alpha} + b(u) \sum_{\alpha \neq \beta, \beta'} e_{\alpha\alpha} \otimes e_{\beta\beta} + \left( e(u) \sum_{\alpha < \beta, \alpha \neq \beta'} + \bar{e}(u) \sum_{\alpha > \beta, \alpha \neq \beta'} \right) e_{\alpha\beta} \otimes e_{\beta\alpha} + \sum_{\alpha,\beta} a_{\alpha\beta}(u) e_{\alpha\beta} \otimes e_{\alpha'\beta'} \,, \tag{G.1}$$

where $e_{\alpha\beta}$ are the elementary $d \times d$ matrices, with $d$ given by (7.1). Moreover,

$$\left.\begin{array}{lcl} c(u) & = & 2\sinh(\frac{u}{2} - 2\eta) \\ b(u) & = & 2\sinh(\frac{u}{2}) \\ e(u) & = & -2e^{-\frac{u}{2}}\sinh(2\eta) \end{array}\right\} \times \left\{ \begin{array}{lll} \cosh(\frac{u}{2} - \kappa\eta) & \text{for} & A^{(2)}_{2n}\,, A^{(2)}_{2n-1} \\ \sinh(\frac{u}{2} - \kappa\eta) & \text{for} & B^{(1)}_n\,, C^{(1)}_n\,, D^{(1)}_n \end{array}\right. \,, \tag{G.2}$$

$$\bar{e}(u) = e^{u} e(u) \,,$$

$$a_{\alpha\beta}(u) = \begin{cases} 2\sinh(\frac{u}{2}) \times \begin{cases} \cosh(\frac{u}{2} - (\kappa-2)\eta) & \text{for} \quad A^{(2)}_{2n}\,, A^{(2)}_{2n-1} \\ \sinh(\frac{u}{2} - (\kappa-2)\eta) & \text{for} \quad B^{(1)}_n\,, C^{(1)}_n\,, D^{(1)}_n \end{cases} & \alpha = \beta, \alpha \neq \alpha' \\[2ex] b(u) + \begin{cases} 2\sinh(2\eta)\sinh((2n-1)\eta) & \text{for} \quad B^{(1)}_n \\ -2\sinh(2\eta)\cosh((2n+1)\eta) & \text{for} \quad A^{(2)}_{2n} \end{cases} & \alpha = \beta, \alpha = \alpha' \\[2ex] 2\sinh(2\eta)e^{\mp\frac{u}{2}} \times \begin{cases} \mp\epsilon_\alpha\epsilon_\beta e^{(\pm\kappa + 2(\bar{\alpha}-\bar{\beta}))\eta}\sinh(\frac{u}{2}) & \\ \quad -\delta_{\alpha\beta'}\cosh(\frac{u}{2} - \kappa\eta) & \text{for} \quad A^{(2)}_{2n}\,, A^{(2)}_{2n-1} \\ \epsilon_\alpha\epsilon_\beta e^{(\pm\kappa + 2(\bar{\alpha}-\bar{\beta}))\eta}\sinh(\frac{u}{2}) & \\ \quad -\delta_{\alpha\beta'}\sinh(\frac{u}{2} - \kappa\eta) & \text{for} \quad B^{(1)}_n\,, C^{(1)}_n\,, D^{(1)}_n \end{cases} & \alpha \lessgtr \beta \end{cases} \tag{G.3}$$

where

$$\kappa = \begin{cases} 2n & \text{for } A^{(2)}_{2n-1} \\ 2n+1 & \text{for } A^{(2)}_{2n} \\ 2n-1 & \text{for } B^{(1)}_n \\ 2n+2 & \text{for } C^{(1)}_n \\ 2n-2 & \text{for } D^{(1)}_n \end{cases} \,, \tag{G.4}$$

$$\begin{array}{ll} \epsilon_\alpha = \begin{cases} 1 & \text{for} \quad 1 \leq \alpha \leq n \\ -1 & \text{for} \quad n+1 \leq \alpha \leq 2n \end{cases} & \text{for } A^{(2)}_{2n-1}\,, C^{(1)}_n \\ \epsilon_\alpha = 1 & \text{for } A^{(2)}_{2n}\,, B^{(1)}_n\,, D^{(1)}_n \end{array} \tag{G.5}$$

$$\bar{\alpha} = \begin{cases} \alpha - \frac{1}{2} & 1 \le \alpha \le n \\ \alpha + \frac{1}{2} & n+1 \le \alpha \le 2n \qquad \text{for } A^{(2)}_{2n-1}\,, C^{(1)}_{n} \end{cases}$$

$$\bar{\alpha} = \begin{cases} \alpha + \frac{1}{2} & 1 \le \alpha < \frac{d+1}{2} \\ \alpha & \alpha = \frac{d+1}{2} \\ \alpha - \frac{1}{2} & \frac{d+1}{2} < \alpha \le d \qquad \text{for } A^{(2)}_{2n}\,, B^{(1)}_{n}\,, D^{(1)}_{n} \end{cases} \tag{G.6}$$

$$\alpha' = d + 1 - \alpha\,, \\ \alpha, \beta = 1\,, \ldots\,, d\,. \tag{G.7}$$

All but one of these R-matrices are the same as in [56], up to the change of variables $x = e^u\,, k = e^{2\eta}$ and an overall factor. The one exception is the R-matrix for $A^{(2)}_{2n-1}$, which we obtain from the $C^{(1)}_n$ R-matrix in [56] by replacing $\xi = k^{2n+2}$ by $\xi = -k^{2n}$; i.e. by changing $\xi \mapsto -\xi k^{-2}$. It is the same as the $A^{(2)}_{2n-1}$ R-matrix in the appendix of [58] up to some redefinitions of the anisotropy and spectral parameters, and an overall factor. This $A^{(2)}_{2n-1}$ R-matrix was used in [69, 85, 87].

We use the $D^{(2)}_{n+1}$ R-matrix given by Jimbo [56], except we use the variables $u$ and $\eta$ instead of $x$ and $k$, respectively, which are related as follows:

$$x = e^u\,, \qquad\qquad k = e^{2\eta}\,. \tag{G.8}$$

We also multiply the Jimbo R-matrix by an overall factor $e^{-2u}\, e^{-2(n+1)\eta}$ in order to have nice crossing and unitarity properties. (See also [97, 57].) Hence, this R-matrix is given by

$$R(u) = e^{-2u}\, e^{-2(n+1)\eta} R_J(u) \tag{G.9}$$

with

$$\begin{aligned} R_J(u) = & \left(e^{2u} - e^{4\eta}\right)\left(e^{2u} - e^{4n\eta}\right) \sum_{\alpha \ne n+1, n+2} e_{\alpha\alpha} \otimes e_{\alpha\alpha} + e^{2\eta}\left(e^{2u} - 1\right)\left(e^{2u} - e^{4n\eta}\right) \sum_{\substack{\alpha \ne \beta, \beta' \\ \alpha \text{ or } \beta \ne n+1, n+2}} \\ & \cdot e_{\alpha\alpha} \otimes e_{\beta\beta} - \left(e^{4\eta} - 1\right)\left(e^{2u} - e^{4n\eta}\right)\left(\sum_{\substack{\alpha<\beta, \alpha\ne\beta' \\ \alpha,\beta \ne n+1,n+2}} + e^{2u} \sum_{\substack{\alpha>\beta, \alpha\ne\beta' \\ \alpha,\beta \ne n+1,n+2}}\right) e_{\alpha\beta} \otimes e_{\beta\alpha} \\ & - \frac{1}{2}\left(e^{4\eta} - 1\right)\left(e^{2u} - e^{4n\eta}\right)\left(\left(e^u + 1\right)\left(\sum_{\alpha<n+1, \beta=n+1,n+2} + e^u \sum_{\alpha>n+2, \beta=n+1,n+2}\right)\right. \\ & \cdot \left(e_{\alpha\beta} \otimes e_{\beta\alpha} + e_{\beta'\alpha'} \otimes e_{\alpha'\beta'}\right) + \left(e^u - 1\right)\left(-\sum_{\alpha<n+1, \beta=n+1,n+2} + e^u \sum_{\alpha>n+2, \beta=n+1,n+2}\right) \\ & \left.\cdot \left(e_{\alpha\beta} \otimes e_{\beta'\alpha} + e_{\beta'\alpha'} \otimes e_{\alpha'\beta}\right)\right) + \sum_{\alpha,\beta\ne n+1,n+2} a_{\alpha\beta}(u) e_{\alpha\beta} \otimes e_{\alpha'\beta'} + \frac{1}{2} \sum_{\alpha\ne n+1,n+2, \beta=n+1,n+2} \\ & \cdot \left(b^+_\alpha(u)\left(e_{\alpha\beta} \otimes e_{\alpha'\beta'} + e_{\beta'\alpha'} \otimes e_{\beta\alpha}\right) + b^-_\alpha(u)\left(e_{\alpha\beta} \otimes e_{\alpha'\beta} + e_{\beta\alpha'} \otimes e_{\beta\alpha}\right)\right) \\ & + \sum_{\alpha=n+1,n+2} \left(c^+(u) e_{\alpha\alpha} \otimes e_{\alpha'\alpha'} + c^-(u) e_{\alpha\alpha} \otimes e_{\alpha\alpha}\right. \\ & \left. + d^+(u) e_{\alpha\alpha'} \otimes e_{\alpha'\alpha} + d^-(u) e_{\alpha\alpha'} \otimes e_{\alpha\alpha'}\right)\,, \end{aligned} \tag{G.10}$$

where for $\alpha, \beta \ne n+1, n+2$

$$a_{\alpha\beta}(u) = \begin{cases} (e^{4\eta} e^{2u} - e^{4n\eta})(e^{2u} - 1) & \alpha = \beta \\ (e^{4\eta} - 1)(e^{4n\eta} e^{2\eta(\bar{\alpha} - \bar{\beta})}(e^{2u} - 1) - \delta_{\alpha\beta'}(e^{2u} - e^{4n\eta})) & \alpha < \beta \\ (e^{4\eta} - 1) e^{2u}(e^{2\eta(\bar{\alpha} - \bar{\beta})}(e^{2u} - 1) - \delta_{\alpha\beta'}(e^{2u} - e^{4n\eta})) & \alpha > \beta \end{cases}\,, \tag{G.11}$$

$$b^{\pm}_\alpha(u) = \begin{cases} \pm e^{2\eta(\alpha - 1/2)}(e^{4\eta} - 1)(e^{2u} - 1)(e^u \pm e^{2n\eta}) & \alpha < n+1 \\ e^{2\eta(\alpha - n - 5/2)}(e^{4\eta} - 1)(e^{2u} - 1) e^u (e^u \pm e^{2n\eta}) & \alpha > n+2 \end{cases}\,, \tag{G.12}$$

$$c^{\pm}(u) = \pm\frac{1}{2}(e^{4\eta} - 1)(e^{2n\eta} + 1) e^u (e^u \mp 1)(e^u \pm e^{2n\eta}) + e^{2\eta}(e^{2u} - 1)(e^{2u} - e^{4n\eta})\,, \tag{G.13}$$

$$d^{\pm}(u) = \pm\frac{1}{2}(e^{4\eta}-1)(e^{2n\eta}-1)e^{u}(e^{u}\pm 1)(e^{u}\pm e^{2n\eta})\,, \tag{G.14}$$

and

$$\bar{\alpha} = \begin{cases} \alpha+1 & 1\le\alpha<n+1 \\ n+\frac{3}{2} & \alpha=n+1 \\ n+\frac{3}{2} & \alpha=n+2 \\ \alpha-1 & n+2<\alpha\le 2n+2 \end{cases}\,, \tag{G.15}$$

$$\alpha' = 2n+3-\alpha\,. \tag{G.16}$$

The elementary matrices $e_{\alpha\beta}$ have dimension $(2n+2)\times(2n+2)$ with

$$\alpha,\beta = 1,\ldots,2n+2\,. \tag{G.17}$$

# Appendix H

# $U_q(g^{(l)}) \otimes U_q(g^{(r)})$ and $\tilde{T}^{\pm}(p)$

We show here that the asymptotic gauge-transformed monodromy matrix $\tilde{T}^{\pm}(p)$ (7.37) can be expressed in terms of coproducts of the generators of a QG of the form $U_q(g^{(l)}) \otimes U_q(g^{(r)})$, where $g^{(l)}$ and $g^{(r)}$ are (non-affine) simple Lie algebras of type $B$, $C$ or $D$, with rank $n-p$ and $p$, respectively. Specifically, the pairs of algebras $(g^{(l)}, g^{(r)})$ are given in Table H.1, where $\hat{g}$ is the affine Lie algebra in the list (7.4) that is associated to the R-matrix. The algebras $g^{(l)} \oplus g^{(r)}$ are in fact the subalgebras of $\hat{g}$ obtained by removing the $p^{th}$ node from the (extended) Dynkin diagram of $\hat{g}$, which has $n+1$ nodes. We emphasize that the possible values of $p$ are $0, 1, \ldots, n$; it is understood that the "right" algebra $g^{(r)}$ is absent for $p=0$, while the "left" algebra $g^{(l)}$ is absent for $p=n$.

| $\hat{g}$ | $(g^{(l)}, g^{(r)})$ |
|---|---|
| $A_{2n}^{(2)}$ | $(B_{n-p}, C_p)$ |
| $A_{2n-1}^{(2)}$ | $(C_{n-p}, D_p) \quad (p \neq 1)$ |
| $B_n^{(1)}$ | $(B_{n-p}, D_p) \quad (n>1, p \neq 1)$ |
| $C_n^{(1)}$ | $(C_{n-p}, C_p)$ |
| $D_n^{(1)}$ | $(D_{n-p}, D_p) \quad (n>1, p \neq 1, n-1)$ |

Table H.1: Pairs of Lie algebras $(g^{(l)}, g^{(r)})$ corresponding to the affine Lie algebras $\hat{g}$, where $p = 0, 1, \ldots, n$.

## H.1 Generators

We denote the generators corresponding to the simple roots of $g^{(l)}$ and $g^{(r)}$ by

$$H_i^{(l)}(p)\,, \quad E_i^{\pm\,(l)}(p)\,, \qquad i = 1, \ldots, n-p\,,$$

and

$$H_i^{(r)}(p)\,, \quad E_i^{\pm\,(r)}(p)\,, \qquad i = 1, \ldots, p\,,$$

respectively. (To lighten the notation, we shall refrain from displaying the dependence of these generators on $p$ when there is no ambiguity in so doing.) The "left" generators satisfy the commutation relations

$$\begin{aligned} \left[H_i^{(l)}(p)\,, H_j^{(l)}(p)\right] &= 0\,, \\ \left[H_i^{(l)}(p)\,, E_j^{\pm\,(l)}(p)\right] &= \pm\alpha_i^{(j)} E_j^{\pm\,(l)}(p)\,, \\ \left[E_i^{+\,(l)}(p)\,, E_j^{-\,(l)}(p)\right] &= \delta_{i,j} \sum_{k=1}^{n-p} \alpha_k^{(j)} H_k^{(l)}(p)\,, \end{aligned} \tag{H.1}$$

and the "right" generators similarly satisfy the commutation relations

$$\begin{aligned} \left[H_i^{(r)}(p)\,, H_j^{(r)}(p)\right] &= 0\,, \\ \left[H_i^{(r)}(p)\,, E_j^{\pm\,(r)}(p)\right] &= \pm\alpha_i^{(j)} E_j^{\pm\,(r)}(p)\,, \\ \left[E_i^{+\,(r)}(p)\,, E_j^{-\,(r)}(p)\right] &= \delta_{i,j} \sum_{k=1}^{p} \alpha_k^{(j)} H_k^{(r)}(p)\,. \end{aligned} \tag{H.2}$$

Moreover, the "left" and "right" generators commute with each other

$$\left[H_i^{(l)}(p)\,, E_j^{\pm\,(r)}(p)\right] = \left[E_i^{\pm\,(l)}(p)\,, H_j^{(r)}(p)\right] = \left[E_i^{\pm\,(l)}(p)\,, E_j^{\pm\,(r)}(p)\right] = \left[E_i^{\pm\,(l)}(p)\,, E_j^{\mp\,(r)}(p)\right] = 0\,. \qquad \text{(H.3)}$$

The simple roots $\{\alpha^{(1)}, \ldots, \alpha^{(m)}\}$ (where $m$ is either $n-p$ or $p$) in the orthogonal basis are given by

$$\begin{aligned} \alpha^{(j)} &= e_j - e_{j+1}\,, \qquad j = 1, \ldots, m-1\,, \\ \alpha^{(m)} &= \begin{cases} e_m & \text{for } B_m \\ 2e_m & \text{for } C_m \\ e_{m-1} + e_m & \text{for } D_m \end{cases}\,, \end{aligned} \qquad \text{(H.4)}$$

where $e_j$ are the elementary $m$-dimensional basis vectors $(e_j)_i = \delta_{i,j}$ (i.e., $e_1 = (1,0,0,\ldots,0)\,, e_2 = (0,1,0,\ldots,0)$, etc.).

In terms of the $\hat{g}$ generators[1]

$$\begin{aligned} H_i &= e_{i,i} - e_{d+1-i,d+1-i}\,, \qquad i = 1, \ldots, n\,, \\ E_i^+ &= e_{i,i+1} + e_{d-i,d+1-i}\,, \qquad i = 1, \ldots, n-1\,, \\ E_n^+ &= \begin{cases} e_{n,n+1} + e_{d-n,d+1-n} & \text{if } g^{(l)} = B_{n-p} \text{ i.e., for } A_{2n}^{(2)}\,, B_n^{(1)} \\ \sqrt{2}e_{n,n+1} & \text{if } g^{(l)} = C_{n-p} \text{ i.e., for } A_{2n-1}^{(2)}\,, C_n^{(1)} \\ e_{n-1,n+1} + e_{n,n+2} & \text{if } g^{(l)} = D_{n-p} \text{ i.e., for } D_n^{(1)} \end{cases}\,, \\ E_0^+ &= \begin{cases} \sqrt{2}e_{d,1} & \text{if } g^{(r)} = C_p \text{ i.e., for } A_{2n}^{(2)}\,, C_n^{(1)} \\ e_{d-1,1} + e_{d,2} & \text{if } g^{(r)} = D_p \text{ i.e., for } A_{2n-1}^{(2)}\,, B_n^{(1)}\,, D_n^{(1)} \end{cases}\,, \\ E_i^- &= (E_i^+)^t\,, \qquad i = 0, 1, \ldots, n\,, \end{aligned} \qquad \text{(H.5)}$$

the "left and "right" generators are given by

$$\begin{aligned} H_i^{(l)}(p) &= H_{p+i}\,, \\ E_i^{\pm\,(l)}(p) &= E_{p+i}^{\pm}\,, \qquad i = 1, \ldots, n-p\,, \end{aligned} \qquad \text{(H.6)}$$

and

$$\begin{aligned} H_i^{(r)}(p) &= -H_{p+1-i}\,, \\ E_i^{\pm\,(r)}(p) &= E_{p-i}^{\pm}\,, \qquad i = 1, \ldots, p\,, \end{aligned} \qquad \text{(H.7)}$$

respectively. Indeed, one can check that the commutation relations (H.1) - (H.3) are satisfied. Note that the "broken" generators $E_p^{\pm}$ in (H.5) do not belong to either the "left" (H.6) or "right" (H.7) set of generators; indeed, dropping the $\hat{g}$ generators $E_p^{\pm}$ corresponds to deleting the $p^{th}$ node from the (extended) Dynkin diagram of $\hat{g}$.

## H.2 Coproducts

We now present the coproducts for the quantum groups $U_q(g^{(l)})$ and $U_q(g^{(r)})$.

[1] Note that $e_{ij}$ are the elementary $d \times d$ matrices introduced below (7.3), where $d$ is defined in (7.1). We see from (H.6) that the generators in (H.5) with $i = 1, \ldots, n$ are in fact the generators of $g^{(l)}$ with $p = 0$; and we see from (H.7) that $E_0^{\pm}$ in (H.5) are the $n^{th}$ generators of $g^{(r)}$ with $p = n$.

### H.2.1 "Left" generators

The coproducts for the "left" generators are given by

$$\Delta(H_j^{(l)}) = H_j^{(l)} \otimes \mathbb{I} + \mathbb{I} \otimes H_j^{(l)}\,, \qquad j = 1, \ldots, n-p\,,$$
$$\Delta(E_j^{\pm\,(l)}) = E_j^{\pm\,(l)} \otimes e^{(\eta+i\pi)H_j^{(l)} - \eta H_{j+1}^{(l)}} + e^{-(\eta+i\pi)H_j^{(l)} + \eta H_{j+1}^{(l)}} \otimes E_j^{\pm\,(l)}\,, \qquad j = 1, \ldots, n-p-1\,,$$
$$\Delta(E_{n-p}^{\pm\,(l)}) = \begin{cases} E_{n-p}^{\pm\,(l)} \otimes e^{(\eta+i\pi)H_{n-p}^{(l)}} \\ \qquad + e^{-(\eta+i\pi)H_{n-p}^{(l)}} \otimes E_{n-p}^{\pm\,(l)} & \text{if } g^{(l)} = B_{n-p} \text{ i.e., for } A_{2n}^{(2)}\,, B_n^{(1)} \\ \\ E_{n-p}^{\pm\,(l)} \otimes e^{2\eta H_{n-p}^{(l)}} + e^{-2\eta H_{n-p}^{(l)}} \otimes E_{n-p}^{\pm\,(l)} & \text{if } g^{(l)} = C_{n-p} \text{ i.e., for } A_{2n-1}^{(2)}\,, C_n^{(1)} \\ \\ E_{n-p}^{\pm\,(l)} \otimes e^{\eta H_{n-p-1}^{(l)} + (\eta+i\pi)H_{n-p}^{(l)}} \\ \qquad + e^{-\eta H_{n-p-1}^{(l)} - (\eta+i\pi)H_{n-p}^{(l)}} \otimes E_{n-p}^{\pm\,(l)} & \text{if } g^{(l)} = D_{n-p} \text{ i.e., for } D_n^{(1)} \end{cases} . \tag{H.8}$$

These coproducts satisfy

$$\left[\Delta(H_i^{(l)})\,, \Delta(E_j^{\pm\,(l)})\right] = \pm\alpha_i^{(j)} \Delta(E_j^{\pm\,(l)})\,, \tag{H.9}$$

and

$$\Omega_{ij}^{(l)} \Delta(E_i^{+\,(l)}) \Delta(E_j^{-\,(l)}) - \Delta(E_j^{-\,(l)}) \Delta(E_i^{+\,(l)}) \Omega_{ij}^{(l)} \tag{H.10}$$
$$= \begin{cases} \delta_{i,j} \frac{\sinh\left[2\eta \sum_{k=1}^{n-p} \alpha_k^{(j)} \Delta(H_k^{(l)})\right]}{\sinh(2\eta)} & \text{for } A_{2n}^{(2)}\,, B_n^{(1)}\,, D_n^{(1)} \\ \\ \delta_{i,j}(1+\delta_{i,n-p}) \frac{\sinh\left[2\eta \sum_{k=1}^{n-p} \alpha_k^{(j)} \Delta(H_k^{(l)})\right]}{\sinh(2(1+\delta_{i,n-p})\eta)} & \text{for } A_{2n-1}^{(2)}\,, C_n^{(1)} \end{cases} , \tag{H.11}$$

where $\Omega_{ij}^{(l)}$ is given by

$$\Omega_{ij}^{(l)} = \begin{cases} \begin{cases} e^{i\pi H_{\max(i,j)}^{(l)}} \otimes \mathbb{I} & |i-j| = 1 \\ \mathbb{I} \otimes \mathbb{I} & \text{otherwise} \end{cases} & \text{for } A_{2n}^{(2)}\,, B_n^{(1)} \\ \\ \begin{cases} e^{i\pi H_{\max(i,j)}^{(l)}} \otimes \mathbb{I} & |i-j| = 1 \text{ and } 1 \le \min(i,j) \le n-p-2 \\ \mathbb{I} \otimes \mathbb{I} & \text{otherwise} \end{cases} & \text{for } A_{2n-1}^{(2)}\,, C_n^{(1)}\,, D_n^{(1)} \end{cases} . \tag{H.12}$$

### H.2.2 "Right" generators

The coproducts for the "right" generators are given by

$$\Delta(H_j^{(r)}) = H_j^{(r)} \otimes \mathbb{I} + \mathbb{I} \otimes H_j^{(r)}\,, \qquad j = 1, \ldots, p\,,$$
$$\Delta(E_j^{\pm\,(r)}) = E_j^{\pm\,(r)} \otimes e^{(\eta+i\pi)H_j^{(r)} - \eta H_{j+1}^{(r)}} + e^{-(\eta+i\pi)H_j^{(r)} + \eta H_{j+1}^{(r)}} \otimes E_j^{\pm\,(r)}\,, \qquad j = 1, \ldots, p-1\,,$$
$$\Delta(E_p^{\pm\,(r)}) = \begin{cases} E_p^{\pm\,(r)} \otimes e^{2\eta H_p^{(r)}} + e^{-2\eta H_p^{(r)}} \otimes E_p^{\pm\,(r)} & \text{if } g^{(r)} = C_p \text{ i.e., for } A_{2n}^{(2)}\,, C_n^{(1)} \\ \\ E_p^{\pm\,(r)} \otimes e^{(\eta+i\pi)H_{p-1}^{(r)} + \eta H_p^{(r)}} \\ \qquad + e^{-(\eta+i\pi)H_{p-1}^{(r)} - \eta H_p^{(r)}} \otimes E_p^{\pm\,(r)} & \text{if } g^{(r)} = D_p \text{ i.e., for } A_{2n-1}^{(2)}\,, B_n^{(1)}\,, D_n^{(1)} \end{cases} . \tag{H.13}$$

These coproducts satisfy

$$\left[\Delta(H_i^{(r)})\,, \Delta(E_j^{\pm\,(r)})\right] = \pm\alpha_i^{(j)} \Delta(E_j^{\pm\,(r)})\,, \tag{H.14}$$

and

$$\Omega_{ij}^{(r)} \Delta(E_i^{+\,(r)}) \Delta(E_j^{-\,(r)}) - \Delta(E_j^{-\,(r)}) \Delta(E_i^{+\,(r)}) \Omega_{ij}^{(r)} \tag{H.15}$$
$$= \begin{cases} \delta_{i,j} \frac{\sinh\left[2\eta \sum_{k=1}^{p} \alpha_k^{(j)} \Delta(H_k^{(r)})\right]}{\sinh(2\eta)} & \text{for } A_{2n-1}^{(2)}, B_n^{(1)}\,, D_n^{(1)} \\ \\ \delta_{i,j}(1+\delta_{i,p}) \frac{\sinh\left[2\eta \sum_{k=1}^{p} \alpha_k^{(j)} \Delta(H_k^{(r)})\right]}{\sinh(2(1+\delta_{i,p})\eta)} & \text{for } A_{2n}^{(2)}\,, C_n^{(1)} \end{cases} , \tag{H.16}$$

where $\Omega_{ij}^{(r)}$ is given by

$$\Omega_{ij}^{(r)} = \begin{cases} \begin{cases} e^{i\pi H^{(r)}_{\max(i,j)}} \otimes \mathbb{I} & |i-j| = 1 \text{ and } 1 \leq \min(i,j) \leq p-2 \text{ and } i \neq p \\ \mathbb{I} \otimes \mathbb{I} & \text{otherwise} \end{cases} & \text{for } A_{2n}^{(2)}, C_n^{(1)}, \\ \\ \begin{cases} e^{i\pi H^{(r)}_{\max(i,j)}} \otimes \mathbb{I} & |i-j| = 1 \text{ and } 1 \leq \min(i,j) \leq p-2, \\ e^{i\pi\left(H_i^{(r)} + H_j^{(r)}\right)} \otimes \mathbb{I} & |i-j| = 2 \text{ and } (i = p \text{ or } j = p) \\ \mathbb{I} \otimes \mathbb{I} & \text{otherwise} \end{cases} & \text{for } A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)} \end{cases} . \quad \text{(H.17)}$$

## H.3 $\tilde{T}^{\pm}(p)$

The matrix elements of the asymptotic gauge-transformed monodromy matrix $\tilde{T}^{\pm}(p)$ (7.37) can be expressed in terms of the coproducts of the "left" and "right" generators introduced above. We now exhibit a set of matrix elements $\tilde{T}_{ij}^{+}(p)$ that includes all $\Delta_{(N)}(E_1^{+\,(l)}), \ldots, \Delta_{(N)}(E_{n-p}^{+\,(l)})$ and all $\Delta_{(N)}(E_1^{+\,(r)}), \ldots, \Delta_{(N)}(E_p^{+\,(r)})$.

For $j \neq n$ and for all the considered affine algebras, we find that

$$\tilde{T}_{j+1,j}^{+}(p) = \begin{cases} -\psi\, e^{(\eta+i\pi)\Delta_{(N)}(H_{p-j}^{(r)}) + \eta\Delta_{(N)}(H_{p-j+1}^{(r)})} \Delta_{(N)}(E_{p-j}^{+\,(r)}) & j = 1, ..., p-1 \\ 0 & j = p \\ \psi\, e^{(-\eta+i\pi)\Delta_{(N)}(H_{j-p}^{(l)}) - \eta\Delta_{(N)}(H_{j-p+1}^{(l)})} \Delta_{(N)}(E_{j-p}^{+\,(l)}) & j = p+1, ..., n-1 \end{cases} , \quad \text{(H.18)}$$

where

$$\psi = \frac{e^{-(\kappa N - 1)\eta}}{2^{N-1}} \sinh(2\eta) . \quad \text{(H.19)}$$

The set of matrix elements $\{\tilde{T}_{2,1}^{+}(p), \ldots, \tilde{T}_{n,n-1}^{+}(p)\}$ evidently contains all the generators except $\Delta_{(N)}(E_p^{+\,(r)})$ and $\Delta_{(N)}(E_{n-p}^{+\,(l)})$.

For the $p$-th "right" generator $\Delta_{(N)}(E_p^{+\,(r)})$ we have

$$\tilde{T}_{1,\sigma(n)}^{+}(p) = \begin{cases} 0 & p = 0 \\ -\frac{2}{\sqrt{2}}\psi e^{\eta} \cosh(2\eta) \Delta_{(N)}(E_p^{+\,(r)}) & p = 1, ..., n \text{ for } g^{(r)} = C_p \\ & \quad \text{i.e., for } A_{2n}^{(2)}, C_n^{(1)} \\ \psi e^{(-\eta+i\pi)\Delta_{(N)}(H_{p-1}^{(r)}) + \eta\Delta_{(N)}(H_p^{(r)})} \Delta_{(N)}(E_p^{+\,(r)}) & p = 2, ..., n \text{ for } g^{(r)} = D_p \\ & \quad \text{i.e., for } A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)} \end{cases} \quad \text{(H.20)}$$

where

$$\sigma(n) = \begin{cases} 2n-1 & \text{for } A_{2n-1}^{(2)}, D_n^{(1)} \\ 2n & \text{for } B_n^{(1)}, C_n^{(1)} \\ 2n+1 & \text{for } A_{2n}^{(2)} \end{cases} . \quad \text{(H.21)}$$

For the $(n-p)$-th "left" generator $\Delta_{(N)}(E_{n-p}^{+\,(l)})$ we have (for $p = 0, 1, \ldots, n-1$)

$$\tilde{T}_{n+1,\bar{\sigma}(n)}^{+}(p) = \begin{cases} \psi e^{(-\eta+i\pi)\Delta_{(N)}(H_{n-p}^{(l)})} \Delta_{(N)}(E_{n-p}^{+\,(l)}) & \text{for } g^{(l)} = B_{n-p} \\ & \quad \text{i.e., for } A_{2n}^{(2)}, B_n^{(1)} \\ -\frac{2}{\sqrt{2}}\psi e^{\eta} \cosh(2\eta) \Delta_{(N)}(E_{n-p}^{+\,(l)}) & \text{for } g^{(l)} = C_{n-p} \\ & \quad \text{i.e., for } A_{2n-1}^{(2)}, C_n^{(1)} \\ -\psi e^{-\eta\Delta_{(N)}(H_{n-p-1}^{(l)}) + (\eta+i\pi)\Delta_{(N)}(H_{n-p}^{(l)})} \Delta_{(N)}(E_{n-p}^{+\,(l)}) & p \neq n, n-1 \text{ for } g^{(l)} = D_{n-p} \\ & \quad \text{i.e., for } D_n^{(1)} \end{cases} , \quad \text{(H.22)}$$

where

$$\bar{\sigma}(n) = \begin{cases} n & \text{for } A_{2n}^{(2)}, A_{2n-1}^{(2)}, B_n^{(1)}, C_n^{(1)} \\ n-1 & \text{for } D_n^{(1)} \end{cases} . \quad \text{(H.23)}$$

Similar expressions can be found for $\tilde{T}_{ij}^{-}(p)$ in terms of $\Delta_{(N)}(E_1^{-\,(l)}), \ldots, \Delta_{(N)}(E_{n-p}^{-\,(l)})$ and $\Delta_{(N)}(E_1^{-\,(r)}), \ldots, \Delta_{(N)}(E_p^{-\,(r)})$.

# Appendix I

# The Hamiltonian

The transfer matrix (7.20) contains [52] the Hamiltonian $\mathcal{H}(p) \sim t'(0,p)$. More explicitly, using the regularity properties

$$\begin{aligned} R(0) &= \xi(0)\mathcal{P}\,, \\ K^R(0,p) &= \mathbb{I}\,, \end{aligned} \tag{I.1}$$

one obtains

$$\mathcal{H}(p) = \sum_{k=1}^{N-1} h_{k,k+1} + \frac{1}{2}{K_1^R}'(0,p) + \frac{1}{\mathrm{tr}\,K^L(0,p)}\,\mathrm{tr}_a\,K_a^L(0,p)h_{Na}\,, \tag{I.2}$$

where the two-site Hamiltonian $h_{k,k+1}$ is given by

$$h_{k,k+1} = \frac{1}{\xi(0)}\mathcal{P}_{k,k+1}R'_{k,k+1}(0)\,. \tag{I.3}$$

The Hamiltonian is gauge invariant [53]

$$\mathcal{H}(p) = \sum_{k=1}^{N-1} \tilde{h}_{k,k+1}(p) + \frac{1}{2}{\tilde{K}_1^R}{}'(0,p) + \frac{1}{\mathrm{tr}\,\tilde{K}^L(0,p)}\,\mathrm{tr}_a\,\tilde{K}_a^L(0,p)\tilde{h}_{Na}\,, \tag{I.4}$$

where the gauge-transformed two-site Hamiltonian is given by

$$\begin{aligned} \tilde{h}_{k,k+1}(p) &= \frac{1}{\xi(0)}\mathcal{P}_{k,k+1}\tilde{R}'_{k,k+1}(0,p) \\ &= h_{k,k+1} + B'_{k+1}(0,p) - B'_k(0,p)\,, \end{aligned} \tag{I.5}$$

where we have used the definition (7.24) of the gauge-transformed R-matrix to pass to the second line.

## I.1 Special cases

For the special case with $p = 0$, the K-matrix $K^R(u,0)$ is proportional to the identity matrix (7.17). It follows[1] that only the first term in (I.2) contributes [65]

$$\mathcal{H}(0) = \sum_{k=1}^{N-1} h_{k,k+1}\,. \tag{I.6}$$

Similarly, for the special case with $p = n$ and $d = 2n$, the gauge-transformed K-matrix $\tilde{K}^R(u,n)$ is proportional to the identity matrix, see (7.27). Hence, only the first term in (I.4) contributes

$$\mathcal{H}(n) = \sum_{k=1}^{N-1} \tilde{h}_{k,k+1}(n) \qquad (d = 2n)\,. \tag{I.7}$$

This explains the observation in [87] that the Hamiltonian for this case is given by a sum of two-body terms. A similar result holds for the special case with $p = n$ and $d = 2n + 1$ [86].

[1] Indeed, the second term in (I.2) is evidently proportional to the identity matrix; moreover, using an identity from [65, 86, 87], one can show that the third term in (I.2) is also proportional to the identity matrix.

# Appendix J

# Proofs of four lemmas

We outline here proofs of Lemmas 1, 5, 9 and 13 for any value of $n$. For all of these proofs, it is useful to rewrite the R-matrix (G.1) as follows

$$R(u) = c(u)\, R^{(1)} + b(u)\, R^{(2)} + e(u)\, R^{(3)} + \bar{e}(u)\, R^{(4)} + R^{(5)}(u)\,, \tag{J.1}$$

where

$$R^{(1)} = \sum_{\alpha\neq\alpha'} e_{\alpha\alpha}\otimes e_{\alpha\alpha} = \sum_{\alpha} e_{\alpha\alpha}\otimes e_{\alpha\alpha} - e_{n+1,n+1}\otimes e_{n+1,n+1}\,(1-\delta_{d,2n}), \tag{J.2}$$

$$R^{(2)} = \sum_{\alpha\neq\beta,\beta'} e_{\alpha\alpha}\otimes e_{\beta\beta} = \sum_{\alpha,\beta} e_{\alpha\alpha}\otimes e_{\beta\beta} - \sum_{\beta\neq\beta'} e_{\beta\beta}\otimes e_{\beta\beta} - \sum_{\beta} e_{\beta'\beta'}\otimes e_{\beta\beta}, \tag{J.3}$$

$$R^{(3)} = \sum_{\alpha<\beta,\alpha\neq\beta'} e_{\alpha\beta}\otimes e_{\beta\alpha} = \sum_{\alpha<\beta} e_{\alpha\beta}\otimes e_{\beta\alpha} - \sum_{\beta>\frac{d+1}{2}} e_{\beta'\beta}\otimes e_{\beta\beta'}, \tag{J.4}$$

$$R^{(4)} = \sum_{\alpha>\beta,\alpha\neq\beta'} e_{\alpha\beta}\otimes e_{\beta\alpha} = \sum_{\alpha>\beta} e_{\alpha\beta}\otimes e_{\beta\alpha} - \sum_{\beta<\frac{d+1}{2}} e_{\beta'\beta}\otimes e_{\beta\beta'}, \tag{J.5}$$

$$R^{(5)}(u) = \sum_{\alpha,\beta} a_{\alpha\beta}(u)\, e_{\alpha\beta}\otimes e_{\alpha'\beta'}\,. \tag{J.6}$$

We follow a similar basic strategy for all the proofs: express all the matrices in terms of the elementary matrices $e_{ij}$ and the identity matrix $\mathbb{I}$, perform the matrix products using the identity

$$e_{ij}\, e_{kl} = \delta_{jk}\, e_{il}\,, \tag{J.7}$$

and then effectuate the resulting Kronecker deltas. Since many terms are generated by this procedure, we use the software `Mathematica` to perform the necessary algebra. Since the proofs are too long to present all the details, we explain the main steps, and point out some of the subtleties. We start with the simplest proof (Lemma 13), and then work our way to the most difficult one (Lemma 1).

## J.1 Lemma 13

We wish to prove the relation

$$Z_1^{(l)}\, R_{12}(u)\, Z_1^{(l)} = Z_2^{(l)}\, R_{12}(u)\, Z_2^{(l)} \tag{J.8}$$

for the $D_n^{(1)}$ R-matrix. We begin by rewriting $Z^{(l)}$ (7.111) as

$$Z^{(l)} = \mathbb{I} - e_{n,n} - e_{n+1,n+1} + e_{n,n+1} + e_{n+1,n}\,. \tag{J.9}$$

The relation (J.8) is in fact separately satisfied by each of the terms in the expression (J.1) for the R-matrix, which we now discuss in turn.

### J.1.1 $R^{(1)}$ and $R^{(2)}$

Since we consider here only the $D_n^{(1)}$ R-matrix, here $d = 2n$; therefore, the second term in (J.2) is absent. For $R^{(1)}$ and $R^{(2)}$, the sums in $\alpha$ and $\beta$ do not have any restriction of the type $\alpha < \beta$ or $\alpha > \beta$; hence, it is straightforward to show using (J.7) that

$$Z_1^{(l)}\, R^{(1)}\, Z_1^{(l)} = Z_2^{(l)}\, R^{(1)}\, Z_2^{(l)}\,,$$
$$Z_1^{(l)}\, R^{(2)}\, Z_1^{(l)} = Z_2^{(l)}\, R^{(2)}\, Z_2^{(l)}\,. \tag{J.10}$$

### J.1.2 $R^{(3)}$ and $R^{(4)}$

These terms require much more effort. Let us start by considering the first term in $R^{(3)}$, and calculating

$$Z_1^{(l)} \left( \sum_{\alpha<\beta} e_{\alpha\beta} \otimes e_{\beta\alpha} \right) Z_1^{(l)}. \tag{J.11}$$

Using the relation (J.7) we obtain an expression depending on Kronecker deltas. But we cannot directly effectuate these Kronecker deltas to evaluate the sums because of the condition $\alpha < \beta$. We can put terms such as $\delta_{n,\beta}\,\delta_{n+1,\alpha}$, $\delta_{n,\beta}\,\delta_{n,\alpha}$ and $\delta_{n+1,\beta}\,\delta_{n+1,\alpha}$ to zero, because they do not obey $\alpha < \beta$. After doing this, we remain with expressions such as

$$\sum_{\alpha<\beta} e_{n,\beta} \otimes e_{\beta,\alpha}\, \delta_{n,\alpha}. \tag{J.12}$$

Notice that we cannot simply set $\alpha = n$ in this expression. In order to satisfy the condition $\alpha < \beta$, if $\alpha = n$, then $\beta \in \{n+1, \dots, 2n\}$. Hence, we can rewrite (J.12) as

$$\sum_{\alpha<\beta} e_{n,\beta} \otimes e_{\beta,\alpha}\, \delta_{n,\alpha} = e_{n,n+1} \otimes e_{n+1,n} + \sum_{\beta=n+2}^{2n} e_{n,\beta} \otimes e_{\beta,n}, \tag{J.13}$$

where we separate the term with $\beta = n+1$ from the sum, since this helps to cancel with other terms. For the same reason, we can rewrite

$$\sum_{\alpha<\beta} e_{n,\beta} \otimes e_{\beta,\alpha}\, \delta_{n+1,\alpha} = \sum_{\beta=n+2}^{2n} e_{n,\beta} \otimes e_{\beta,n+1}. \tag{J.14}$$

Using similar logic with all of the terms, we obtain

$$Z_1^{(l)} \left( \sum_{\alpha<\beta} e_{\alpha\beta} \otimes e_{\beta\alpha} \right) Z_1^{(l)} - Z_2^{(l)} \left( \sum_{\alpha<\beta} e_{\alpha\beta} \otimes e_{\beta\alpha} \right) Z_2^{(l)} = e_{1+n,n} \otimes e_{1+n,n} - e_{n,1+n} \otimes e_{n,1+n}. \tag{J.15}$$

We still must consider the contribution of the second term in $R^{(3)}$

$$Z_1^{(l)} \left( - \sum_{\beta>\frac{d+1}{2}} e_{\beta'\beta} \otimes e_{\beta\beta'} \right) Z_1^{(l)}. \tag{J.16}$$

Notice that, since $d = 2n$, the condition $\beta > \frac{d+1}{2}$ is equivalent to $\beta \geq n+1$. Due to this condition, all terms with $\delta_{n,\beta}$ and $\delta_{n+1,2n+1-\beta}$ must vanish. Taking this into account, we obtain

$$Z_1^{(l)} \left( - \sum_{\beta>\frac{d+1}{2}} e_{\beta'\beta} \otimes e_{\beta\beta'} \right) Z_1^{(l)} - Z_2^{(l)} \left( - \sum_{\beta>\frac{d+1}{2}} e_{\beta'\beta} \otimes e_{\beta\beta'} \right) Z_2^{(l)} = -e_{1+n,n} \otimes e_{1+n,n} + e_{n,1+n} \otimes e_{n,1+n}, \tag{J.17}$$

which exactly cancels with (J.15). We conclude that $R^{(3)}$ satisfies

$$Z_1^{(l)}\, R^{(3)}\, Z_1^{(l)} = Z_2^{(l)}\, R^{(3)}\, Z_2^{(l)}\,. \tag{J.18}$$

We prove that $R^{(4)}$ satisfies

$$Z_1^{(l)}\, R^{(4)}\, Z_1^{(l)} = Z_2^{(l)}\, R^{(4)}\, Z_2^{(l)} \tag{J.19}$$

using the same arguments presented for $R^{(3)}$, but considering $\alpha > \beta$ instead of $\alpha < \beta$.

### J.1.3 $R^{(5)}(u)$

For $R^{(5)}(u)$, there are no restrictions on the sums over $\alpha$ and $\beta$; hence, we can directly effectuate all the Kronecker deltas. However, doing this is not enough to show that

$$Z_1^{(l)}\left(\sum_{\alpha,\beta} a_{\alpha\beta}(u)\, e_{\alpha\beta}\otimes e_{\alpha'\beta'}\right) Z_1^{(l)} - Z_2^{(l)}\left(\sum_{\alpha,\beta} a_{\alpha\beta}(u)\, e_{\alpha\beta}\otimes e_{\alpha'\beta'}\right) Z_2^{(l)} = 0\,. \tag{J.20}$$

To this end, it is useful to separate all the terms with $\alpha, \beta \in \{n, n+1\}$ from the sums. For example,

$$\begin{aligned}\sum_{\beta} a_{n,\beta}(u)\, e_{n+1,\beta}\otimes e_{n,\beta'} =& a_{n,n}(u)\, e_{n+1,n}\otimes e_{n,n+1} + a_{n,n+1}(u)\, e_{n+1,n+1}\otimes e_{n,n}+ \\ &+\sum_{\beta=1}^{n-1} a_{n,\beta}(u)\, e_{n+1,\beta}\otimes e_{n,\beta'} + \sum_{\beta=n+2}^{2n} a_{n,\beta}(u)\, e_{n+1,\beta}\otimes e_{n,\beta'}\,.\end{aligned} \tag{J.21}$$

By doing this, we find that all the terms without sums cancel. The remaining terms can also be seen to cancel by using the following properties of the functions $a_{\alpha\beta}(u)$ (G.3) for $D_n^{(1)}$

$$\begin{aligned} a_{n,n} &= a_{n+1,n+1}, \\ a_{n,n+1} &= a_{n+1,n}, \\ a_{n,\beta} &= a_{n+1,\beta} \quad \text{for} \quad 1\le\beta\le n-1 \quad \text{and for} \quad n+2\le\beta\le 2n, \\ a_{\beta,n} &= a_{\beta,n+1} \quad \text{for} \quad 1\le\beta\le n-1 \quad \text{and for} \quad n+2\le\beta\le 2n. \end{aligned} \tag{J.22}$$

We conclude that

$$Z_1^{(l)}\, R^{(5)}(u)\, Z_1^{(l)} = Z_2^{(l)}\, R^{(5)}(u)\, Z_2^{(l)}\,, \tag{J.23}$$

which concludes the proof of (J.8).

## J.2 Lemma 9

We now turn to the proof of the relations

$$\begin{aligned} Z_1^{(r)}\, R_{12}(u)\, Z_1^{(r)} &= Y_2^t(u)\, R_{12}(u)\, Y_2^t(u)\,, \\ Z_2^{(r)}\, R_{12}(u)\, Z_2^{(r)} &= Y_1(u)\, R_{12}(u)\, Y_1(u)\,, \end{aligned} \tag{J.24}$$

for the $A_{2n-1}^{(2)}$, $B_n^{(1)}$ and $D_n^{(1)}$ R-matrices. We begin by rewriting $Z^{(r)}$ and $Y(u)$ (7.100) as follows

$$\begin{aligned} Z^{(r)} &= \mathbb{I} - e_{1,1} - e_{d,d} + e_{1,d} + e_{d,1}\,, \\ Y(u) &= \mathbb{I} - e_{1,1} - e_{d,d} + e^{-u} e_{1,d} + e^{u} e_{d,1}\,. \end{aligned} \tag{J.25}$$

The rest of the proof is very similar to the one for Lemma 13 (J.8). However, whereas in the previous case all the terms are written in such a way that $\alpha, \beta \in \{n, n+1\}$ appear explicitly and not inside the sums, here we should write all the terms in such a way that $\alpha, \beta \in \{1, d\}$ appear explicitly. Another difference is that now not all the terms in the expression (J.1) for the R-matrix separately satisfy the relations (J.24). Indeed, the linear combination $R^{(3)} + e^u\, R^{(4)}$ satisfies these relations, but not $R^{(3)}$ and $R^{(4)}$ separately. Otherwise, all the intermediate strategies are analogous. At the end, we must use the following properties of the functions $a_{\alpha\beta}(u)$ (G.3) for $A_{2n-1}^{(2)}$, $B_n^{(1)}$ and $D_n^{(1)}$

$$\begin{aligned} a_{1,1} &= a_{d,d}, \\ a_{d,1} &= e^{2u}\, a_{1,d}, \\ a_{d,\beta} &= e^{u}\, a_{1,\beta} \quad \text{for} \quad 2\le\beta\le d-1\,, \\ a_{\beta,d} &= e^{-u}\, a_{\beta,1} \quad \text{for} \quad 2\le\beta\le d-1\,. \end{aligned} \tag{J.26}$$

## J.3 Lemma 5

We now present some details about our proof of the duality relation

$$U_2\, R_{12}(u)\, U_2 = W_1(u)\, R_{12}(u)\, W_1(u) \tag{J.27}$$

for the $C_n^{(1)}$ and $D_n^{(1)}$ R-matrices, for which $d = 2n$. In contrast with the previous proofs (J.9), (J.25), the matrices $U$ and $W(u)$ (7.64) cannot be expressed in the form ($\mathbb{I}$ − few terms). We rewrite these matrices instead as

$$U = \sum_{i=1}^{n} (e_{i,n+i} + e_{n+i,i})\,, \tag{J.28}$$

and

$$W(u) = \sum_{i=1}^{n} \left(e^{-\frac{u}{2}}\, e_{i,n+i} + e^{\frac{u}{2}}\, e_{n+i,i}\right). \tag{J.29}$$

We now proceed to analyze separately the contributions of the terms in the expression (J.1) for the R-matrix to the relation (J.27).

### J.3.1 $R^{(1)}$ and $R^{(2)}$

After applying the rule (J.7), we must deal with the ranges of the sums. The ranges for the sums in (J.28) and (J.29) (from 1 to $n$) are different from the ones in (J.2) and (J.3) (from 1 to $2n$). We cannot effectuate the Kronecker deltas to evaluate the sums in $R^{(1)}$ unless we split those sums into two ranges: $1 \leq \alpha \leq n$ and $n+1 \leq \alpha \leq 2n$. In (J.27) we write the $U_2$ on the left hand side of $R^{(1)}$ with a sum in $i$, and the $U_2$ on the right hand side with a sum in $j$. For the range $1 \leq \alpha \leq n$, all the terms with $\delta_{i+n,\alpha}$ and $\delta_{j+n,\alpha}$ are zero, because $\alpha$ is always smaller than $n+i$. For $n+1 \leq \alpha \leq 2n$, all the terms with $\delta_{i,\alpha}$ and $\delta_{j,\alpha}$ are zero, because $\max(i)$ and $\max(j)$ are $n$, while $\alpha$ is always greater or equal to $n+1$. After applying such arguments, we obtain

$$U_2\, R^{(1)}\, U_2 = \sum_{\alpha=1}^{n} e_{\alpha,\alpha} \otimes e_{\alpha+n,\alpha+n} + \sum_{\alpha=n+1}^{2n} e_{\alpha,\alpha} \otimes e_{\alpha-n,\alpha-n}\,. \tag{J.30}$$

By applying analogous arguments for the terms with $W_1(u)$, we find

$$W_1(u)\, R^{(1)}\, W_1(u) = \sum_{\alpha=1}^{n} e_{\alpha+n,\alpha+n} \otimes e_{\alpha,\alpha} + \sum_{\alpha=n+1}^{2n} e_{\alpha-n,\alpha-n} \otimes e_{\alpha,\alpha}\,. \tag{J.31}$$

We conclude that

$$U_2\, R^{(1)}\, U_2 = W_1(u)\, R^{(1)}\, W_1(u)\,, \tag{J.32}$$

since the right-hand-sides of (J.30) and (J.31) become identical upon redefining the $\alpha$'s in the sums. We prove in a similar way that $R^{(2)}$ satisfies

$$U_2\, R^{(2)}\, U_2 = W_1(u)\, R^{(2)}\, W_1(u)\,. \tag{J.33}$$

### J.3.2 $R^{(3)}$ and $R^{(4)}$

The duality relation is not satisfied separately by $R^{(3)}$ and $R^{(4)}$, but is instead satisfied by the linear combination $R^{(3)} + e^u\, R^{(4)}$. That is,

$$U_2 \left(R^{(3)} + e^u\, R^{(4)}\right) U_2 = W_1(u) \left(R^{(3)} + e^u\, R^{(4)}\right) W_1(u)\,. \tag{J.34}$$

In order to manage the cases with $\alpha < \beta$ and $\alpha > \beta$, we split the sums over $\alpha$ and $\beta$ into four ranges:

$$\begin{aligned}
&1 \leq \alpha \leq n \quad \text{and} \quad 1 \leq \beta \leq n\,,\\
&1 \leq \alpha \leq n \quad \text{and} \quad n+1 \leq \beta \leq 2n\,,\\
&n+1 \leq \alpha \leq 2n \quad \text{and} \quad 1 \leq \beta \leq n\,,\\
&n+1 \leq \alpha \leq 2n \quad \text{and} \quad n+1 \leq \beta \leq 2n\,.
\end{aligned} \tag{J.35}$$

For each of these ranges, we put to zero terms that contain Kronecker deltas where $\alpha$ and $\beta$ are outside of the relevant interval. Again, at the end, it is necessary to redefine $\alpha$ and $\beta$ on the sums to see that (J.34) is satisfied.

### J.3.3 $R^{(5)}(u)$

For this term we also split the sums over $\alpha$ and $\beta$ into the four ranges (J.35). All the other strategies are similar to the ones presented above, and we obtain

$$U_2\, R^{(5)}(u)\, U_2 = W_1(u)\, R^{(5)}(u)\, W_1(u)\,. \tag{J.36}$$

## J.4 Lemma 1 for $d = 2n$

In order to prove

$$\left[\tilde{R}^{+}_{12}(p)\,,\tilde{K}^{R}_{2}(u,p)\right] = 0 \tag{J.37}$$

for any value of $n$, we proceed in three steps: finding an explicit expression for the gauge-transformed R-matrix $\tilde{R}_{12}(u,p)$, performing the limit $u \to \infty$ in $e^{-u}\,\tilde{R}_{12}(u,p)$ to obtain $\tilde{R}^{+}_{12}(p)$, and finally evaluating the commutator. We consider here the case $d = 2n$, leaving the case $d = 2n+1$ for the following subsection.

### J.4.1 Finding $\tilde{R}_{12}(u,p)$

In order to obtain an explicit expression for the gauge-transformed R-matrix $\tilde{R}_{12}(u,p)$ (7.24), it is useful to rewrite $B(u)$ (7.26) in terms of elementary matrices

$$B(u) = e^{\frac{u}{2}} \sum_{i=1}^{p} e_{i,i} + \sum_{i=p+1}^{n} e_{i,i} + \sum_{i=n+1}^{2n-p} e_{i,i} + e^{-\frac{u}{2}} \sum_{i=2n-p+1}^{2n} e_{i,i}\,, \tag{J.38}$$

for $1 \leq p \leq n-1$.[1]

We now point out some useful simplifications for the contributions from each of the terms in the expression (J.1) for the R-matrix.

Since $B(u)$ is a diagonal matrix,

$$B_1(u)\, R^{(1)}\, B_1(-u) = R^{(1)}\,, \tag{J.39}$$

$$B_1(u)\, R^{(2)}\, B_1(-u) = R^{(2)}\,. \tag{J.40}$$

Let us now consider the first term in $B_1(u)\, R^{(3)}\, B_1(-u)$, where $\alpha < \beta$. After applying the rule (J.7), we obtain terms such as

$$\sum_{\alpha<\beta} \sum_{i=2n-p+1}^{2n} \sum_{j=n+1}^{2n-p} e_{i,j} \otimes e_{\beta,\alpha}\, \delta_{i,\alpha}\, \delta_{j,\beta}\,, \tag{J.41}$$

for example. Several terms like this appear, but they are all equal to zero, because the $\delta$'s force $\alpha = i$ and $\beta = j$; but $i > j$ in this sum, which contradicts the condition $\alpha < \beta$. For the second term in $B_1(u)\, R^{(3)}\, B_1(-u)$, several terms are zero because the Kronecker deltas force $\beta$ to have values that are not greater than $\frac{d+1}{2}$. Similar arguments can be used for $B_1(u)\, R^{(4)}\, B_1(-u)$.

For $B_1(u)\, R^{(5)}(u)\, B_1(-u)$, after applying the rule (J.7), we can directly use the $\delta$'s to evaluate the sums, because there are no restrictions on the $\alpha$'s and $\beta$'s. The functions $a_{i,j}(u)$ have different expressions depending on whether $i = j$, $i < j$ or $i > j$. For later convenience, we separately calculate the contributions from each of these three cases. For example, consider the term

$$\sum_{i=2n-p+1}^{2n} \sum_{j=1}^{p} a_{i,j}(u)\, e_{i,j} \otimes e_{i',j'}\,. \tag{J.42}$$

This term contributes only to $i > j$, due to the ranges in the sums and the fact $2n - p + 1 > p$.

We refrain from displaying the final result for $\tilde{R}_{12}(u,p)$, which is quite lengthy even after the simplifications noted above.

[1] For $p = 0$ and $p = n$, $\tilde{K}^{R}(u,p) \propto \mathbb{I}$, so (J.37) is trivially satisfied.

### J.4.2 Performing the large-$u$ limit

We now proceed to perform the limit $u \to \infty$ in $e^{-u}\,\tilde{R}_{12}(u,p)$. To this end, we need the following results

$$
\begin{aligned}
&\lim_{u\to\infty} e^{-u} e(u) = 0 = \lim_{u\to\infty} e^{-\frac{u}{2}} e(u) = \lim_{u\to\infty} e^{-\frac{3u}{2}} \bar{e}(u) = \lim_{u\to\infty} e^{-2u} \bar{e}(u),\\
&\lim_{u\to\infty} e^{-u} a^{(3)}_{\alpha\beta}(u) = 0 = \lim_{u\to\infty} e^{-\frac{u}{2}} a^{(3)}_{\alpha\beta}(u) = \lim_{u\to\infty} e^{-2u} a^{(4)}_{\alpha\beta}(u) = \lim_{u\to\infty} e^{-\frac{3u}{2}} a^{(4)}_{\alpha\beta}(u),\\
&\mathfrak{b} \equiv \lim_{u\to\infty} e^{-u} b(u) = \frac{1}{2} e^{-\kappa\eta},\\
&\mathfrak{c} \equiv \lim_{u\to\infty} e^{-u} c(u) = \frac{1}{2} e^{-(\kappa+2)\eta},\\
&\mathfrak{e} \equiv \lim_{u\to\infty} e(u) = -e^{-\kappa\eta}\sinh(2\eta) = \lim_{u\to\infty} e^{-u}\bar{e}(u),\\
&\mathfrak{a}^{(1)} \equiv \lim_{u\to\infty} e^{-u} a^{(1)}_{\alpha\beta}(u) = \frac{1}{2} e^{-(\kappa-2)\eta},\\
&\mathfrak{a}^{(2)} \equiv \lim_{u\to\infty} e^{-u} a^{(2)}_{\alpha\beta}(u) = \frac{1}{2} e^{-\kappa\eta}\\
&\mathfrak{a}^{(3)}_{\alpha,\beta} \equiv \lim_{u\to\infty} a^{(3)}_{\alpha\beta}(u) = e^{-\kappa\eta}\sinh(2\eta)\left(\delta_1^2 e^{2(\kappa+\bar{\alpha}-\bar{\beta})\eta}\epsilon_\alpha\epsilon_\beta - \delta_{\alpha,\beta'}\right),\\
&\mathfrak{a}^{(4)}_{\alpha,\beta} \equiv \lim_{u\to\infty} e^{-u} a^{(4)}_{\alpha\beta} = e^{-\kappa\eta}\sinh(2\eta)\left(e^{2(\bar{\alpha}-\bar{\beta})\eta}\epsilon_\alpha\epsilon_\beta - \delta_{\alpha,\beta'}\right),
\end{aligned}
\tag{J.43}
$$

where

$$
a_{\alpha,\beta}(u) = \begin{cases} a^{(1)}_{\alpha,\beta}(u) \text{ for } \alpha=\beta,\ \alpha\neq\alpha' \\ a^{(2)}_{\alpha,\beta}(u) \text{ for } \alpha=\beta,\ \alpha=\alpha' \\ a^{(3)}_{\alpha,\beta}(u) \text{ for } \alpha<\beta \\ a^{(4)}_{\alpha,\beta}(u) \text{ for } \alpha>\beta \end{cases},
\tag{J.44}
$$

and the definition of $a^{(i)}_{\alpha,\beta}(u)$ can be read off directly from (G.3).

With the help of these results, we find that $\tilde{R}^{+}_{12}(p)$ (7.36) is given, for $d=2n$ and $1\le p\le n-1$, by

$$
\begin{aligned}
\tilde{R}^{+}_{12}(p) = {} & \mathfrak{c}\sum_{\alpha} e_{\alpha,\alpha}\otimes e_{\alpha,\alpha} + \mathfrak{b}\sum_{\alpha\neq\beta,\beta'} e_{\alpha,\alpha}\otimes e_{\beta,\beta} - \mathfrak{e}\left(\sum_{\beta=p+1}^{n} + \sum_{\beta=2n-p+1}^{2n}\right) e_{\beta',\beta}\otimes e_{\beta,\beta'}\\
& + \mathfrak{e}\left(\sum_{\substack{\alpha,\beta=1\\ \alpha>\beta}}^{p} + \sum_{\substack{\alpha,\beta=p+1\\ \alpha>\beta}}^{n} + \sum_{\substack{\alpha,\beta=n+1\\ \alpha>\beta}}^{2n-p} + \sum_{\substack{\alpha,\beta=2n-p+1\\ \alpha>\beta}}^{2n} + \sum_{\alpha=1}^{p}\sum_{\beta=2n-p+1}^{2n} + \sum_{\alpha=n+1}^{2n-p}\sum_{\beta=p+1}^{n}\right) e_{\alpha,\beta}\otimes e_{\beta,\alpha}\\
& + \mathfrak{a}^{(1)}\sum_{\alpha} e_{\alpha,\alpha}\otimes e_{\alpha',\alpha'} + \sum_{\alpha=1}^{p}\sum_{\beta=2n+1-p}^{2n} \mathfrak{a}^{(3)}_{\alpha,\beta}\, e_{\alpha,\beta}\otimes e_{\alpha',\beta'}\\
& + \left(\sum_{\substack{\alpha,\beta=1\\ \alpha>\beta}}^{p} + \sum_{\substack{\alpha,\beta=p+1\\ \alpha>\beta}}^{n} + \sum_{\substack{\alpha,\beta=n+1\\ \alpha>\beta}}^{2n-p} + \sum_{\substack{\alpha,\beta=2n-p+1\\ \alpha>\beta}}^{2n} + \sum_{\alpha=n+1}^{2n-p}\sum_{\beta=p+1}^{n}\right)\mathfrak{a}^{(4)}_{\alpha,\beta}\, e_{\alpha,\beta}\otimes e_{\alpha',\beta'}\,.
\end{aligned}
\tag{J.45}
$$

### J.4.3 Evaluating the commutator

In order to evaluate the commutator (J.37), we rewrite $\tilde{K}^R(u,p)$ (7.27) in terms of elementary matrices, and obtain

$$
\tilde{K}^R_2(u,p) = \mathbb{I}\otimes\left[\sum_{i=1}^{p} + \left(\frac{\gamma e^u+1}{\gamma+e^u}\right)\sum_{i=p+1}^{n} + \left(\frac{\gamma e^u+1}{\gamma+e^u}\right)\sum_{i=n+1}^{2n-p} + \sum_{i=2n-p+1}^{2n}\right] e_{i,i}\,.
\tag{J.46}
$$

It is then just a matter of applying the same ideas presented above, and putting to zero all the terms that do not belong to the relevant range. In this way, one can see that each of the terms in (J.45) commutes with (J.46).

## J.5 Lemma 1 for $d = 2n + 1$

The cases where $d = 2n+1$ can be analogously proved. However, it is more suitable to separate the "middle" terms in $B(u)$ and $\tilde{K}_2^R(u,p)$, i.e. we set

$$B(u) = e^{\frac{u}{2}} \sum_{i=1}^{p} e_{i,i} + \sum_{i=p+1}^{n} e_{i,i} + e_{n+1,n+1} + \sum_{i=n+2}^{2n-p+1} e_{i,i} + e^{-\frac{u}{2}} \sum_{i=2n-p+2}^{2n+1} e_{i,i} \tag{J.47}$$

and

$$\tilde{K}_2^R(u,p) = \mathbb{I} \otimes e_{n+1,n+1} + \mathbb{I} \otimes \left[ \sum_{i=1}^{p} + \left( \frac{\gamma e^u + 1}{\gamma + e^u} \right) \sum_{i=p+1}^{n} + \left( \frac{\gamma e^u + 1}{\gamma + e^u} \right) \sum_{i=n+2}^{2n-p+1} + \sum_{i=2n-p+2}^{2n+1} \right] e_{i,i} \,, \tag{J.48}$$

for $1 \leq p \leq n-1$. For this case, $\tilde{R}_{12}^{+}(p)$ is given by

$$\begin{aligned}
\tilde{R}_{12}^{+}(p) = {} & \mathfrak{c} \sum_{\alpha \neq \alpha'} e_{\alpha,\alpha} \otimes e_{\alpha,\alpha} + \mathfrak{b} \sum_{\alpha \neq \beta,\beta'} e_{\alpha,\alpha} \otimes e_{\beta,\beta} - \mathfrak{e} \left( \sum_{\beta=2n-p+2}^{2n+1} + \sum_{\beta=p+1}^{n} \right) e_{\beta',\beta} \otimes e_{\beta,\beta'} \\
& + \mathfrak{e} \left( \sum_{\alpha=1}^{p} \sum_{\beta=2n-p+2}^{2n+1} + \sum_{\substack{\alpha,\beta=1 \\ \alpha>\beta}}^{p} + \sum_{\substack{\alpha,\beta=p+1 \\ \alpha>\beta}}^{n} + \sum_{\substack{\alpha,\beta=n+2 \\ \alpha>\beta}}^{2n-p+1} + \sum_{\substack{\alpha,\beta=2n-p+2 \\ \alpha>\beta}}^{2n+1} + \sum_{\alpha=n+2}^{2n-p+1} \sum_{\beta=p+1}^{n} \right) e_{\alpha,\beta} \otimes e_{\beta,\alpha} \\
& + \mathfrak{a}^{(1)} \left( \sum_{\alpha=1}^{n} + \sum_{\alpha=n+2}^{2n+1} \right) e_{\alpha,\alpha} \otimes e_{\alpha',\alpha'} + \sum_{\alpha=1}^{p} \sum_{\beta=2n+2-p}^{2n+1} \mathfrak{a}^{(3)}_{\alpha,\beta} \, e_{\alpha,\beta} \otimes e_{\alpha',\beta'} \\
& + \left( \sum_{\substack{\alpha,\beta=1 \\ \alpha>\beta}}^{p} + \sum_{\substack{\alpha,\beta=p+1 \\ \alpha>\beta}}^{n} + \sum_{\substack{\alpha,\beta=n+2 \\ \alpha>\beta}}^{2n-p+1} + \sum_{\substack{\alpha,\beta=2n-p+2 \\ \alpha>\beta}}^{2n+1} + \sum_{\alpha=n+2}^{2n-p+1} \sum_{\beta=p+1}^{n} \right) \mathfrak{a}^{(4)}_{\alpha,\beta} \, e_{\alpha,\beta} \otimes e_{\alpha',\beta'} \\
& + \mathfrak{a}^{(2)} e_{n+1,n+1} \otimes e_{n+1,n+1} - \mathfrak{e} \, e^{2(n+1)\eta} \sum_{\beta=p+1}^{n} e^{-2\bar{\beta}\eta} e_{n+1,\beta} \otimes e_{n+1,\beta'} \\
& - \mathfrak{e} \, e^{-2(n+1)\eta} \sum_{\beta=n+2}^{2n-p+1} e^{2\bar{\beta}\eta} e_{\beta,n+1} \otimes e_{\beta',n+1} \\
& + \mathfrak{e} \left( \sum_{\alpha=p+1}^{n} e_{n+1,\alpha} \otimes e_{\alpha,n+1} + \sum_{\alpha=n+2}^{2n-p+1} e_{\alpha,n+1} \otimes e_{n+1,\alpha} \right).
\end{aligned} \tag{J.49}$$

For $p = n$, it is suitable to write $B(u)$ and $\tilde{K}_2^R(u,p)$ as

$$B(u) = e^{\frac{u}{2}} \sum_{i=1}^{n} e_{i,i} + e_{n+1,n+1} + \sum_{i=n+2}^{2n+1} e_{i,i} \,, \tag{J.50}$$

and

$$\tilde{K}_2^R(u,p) = \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes e_{n+1,n+1} + \left( \frac{\gamma e^u + 1}{\gamma + e^u} \right) \mathbb{I} \otimes e_{n+1,n+1} \,. \tag{J.51}$$

For this case, $\tilde{R}_{12}^{+}(p)$ is given by

$$\begin{aligned}
\tilde{R}_{12}^{+}(p) = {} & \mathfrak{c} \sum_{\alpha \neq \alpha'} e_{\alpha,\alpha} \otimes e_{\alpha,\alpha} + \mathfrak{b} \sum_{\alpha \neq \beta,\beta'} e_{\alpha,\alpha} \otimes e_{\beta,\beta} - \mathfrak{e} \sum_{\beta=n+2}^{2n+1} e_{\beta',\beta} \otimes e_{\beta,\beta'} \\
& + \mathfrak{e} \left( \sum_{\alpha=1}^{n} \sum_{\beta=n+2}^{2n+1} + \sum_{\substack{\alpha,\beta=1 \\ \alpha>\beta}}^{n} + \sum_{\substack{\alpha,\beta=n+2 \\ \alpha>\beta}}^{2n+1} \right) e_{\alpha,\beta} \otimes e_{\beta,\alpha} \\
& + \mathfrak{a}^{(1)} \left( \sum_{\alpha=1}^{n} + \sum_{\alpha=n+2}^{2n+1} \right) e_{\alpha,\alpha} \otimes e_{\alpha',\alpha'} + \mathfrak{a}^{(2)} \, e_{n+1,n+1} \otimes e_{n+1,n+1}
\end{aligned}$$

$$+\sum_{\alpha=1}^{n}\sum_{\beta=n+2}^{2n+1}\mathfrak{a}^{(3)}_{\alpha,\beta}\,e_{\alpha,\beta}\otimes e_{\alpha',\beta'}+\left(\sum_{\substack{\alpha,\beta=1\\ \alpha>\beta}}^{n}+\sum_{\substack{\alpha,\beta=n+2\\ \alpha>\beta}}^{2n+1}\right)\mathfrak{a}^{(4)}_{\alpha,\beta}\,e_{\alpha,\beta}\otimes e_{\alpha',\beta'}\,. \tag{J.52}$$

For $p=0$, $\tilde{K}^R(u,p)\propto\mathbb{I}$, so (J.37) is trivially satisfied.

# Appendix K

# Bonus symmetry and singular solutions

For the cases $C_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}$ with $p = \frac{n}{2}$ ($n$ even) and $\varepsilon = 1$, the transfer matrix has a "bonus" symmetry (i.e., a symmetry in addition to self-duality), leading to higher degeneracies in comparison with $\varepsilon = 0$ [88, 89]. We observe here that the solutions of the Bethe equations corresponding to such degenerate levels are singular (exceptional).

As an example, we consider the case $C_n^{(1)}$ with $n = 2, p = 1$. From the $U_q(C_1) \otimes U_q(C_1)$ symmetry of the transfer matrix, we expect (for generic values of $\eta$) the following Hilbert space decompositions

$$N = 2 : \quad [(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})]^{\otimes 2} = 2(\mathbf{1}, \mathbf{1}) \oplus 2(\mathbf{2}, \mathbf{2}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \,, \tag{K.1}$$

$$N = 3 : \quad [(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})]^{\otimes 3} = 5(\mathbf{2}, \mathbf{1}) \oplus 5(\mathbf{1}, \mathbf{2}) \oplus 3(\mathbf{3}, \mathbf{2}) \oplus 3(\mathbf{2}, \mathbf{3}) \oplus (\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{4}) \,. \tag{K.2}$$

However, by diagonalizing the transfer matrix directly, we observe the following degeneracy patterns

$$N = 2 : \quad \{1, 1, 4, 4, 6\} \quad \text{when } \varepsilon = 0 \,, \tag{K.3}$$

$$\{2, 8, 6\} \quad \text{when } \varepsilon = 1 \,, \tag{K.4}$$

$$N = 3 : \quad \{4, 4, 4, 4, 4, 8, 12, 12, 12\} \quad \text{when } \varepsilon = 0 \,, \tag{K.5}$$

$$\{4, 8, 8, 8, 12, 24\} \quad \text{when } \varepsilon = 1 \,. \tag{K.6}$$

Let us first consider the case $N = 2$. Comparing the decomposition (K.1) with the degeneracies for $\varepsilon = 0$ (K.3), we see that they do not completely match: the $(\mathbf{3}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{3})$ are degenerate (thereby giving rise to the 6-fold degeneracy) due to the self-duality (8.13). However, the degeneracies for $\varepsilon = 1$ (K.4) are even higher: the two $(\mathbf{2}, \mathbf{2})$ are degenerate (thereby giving rise to the 8-fold degeneracy) and the two $(\mathbf{1}, \mathbf{1})$ are degenerate (thereby giving rise to the 2-fold degeneracy) due to the "bonus" symmetry.

The key new point is that, among the Bethe roots corresponding to the levels with 8-fold degeneracy and 2-fold degeneracy, is the exact Bethe root $u^{[1]} = 2\eta$ (which is repeated for the 2-fold degenerate level), for which the Bethe equations have a zero or pole.

The bonus symmetry is also present for $N = 3$, see (K.2), (K.5), (K.6). The levels that are degenerate due to the bonus symmetry (namely, the level with 24-fold degeneracy, and two levels with 8-fold degeneracy) again contain the singular solution $u^{[1]} = 2\eta$, which is repeated for the 8-fold degenerate levels.

For all the examples that we have checked (another example is noted in Sec. 8.3.3), singular solutions occur if and only if the states are affected by the bonus symmetry. However, a general understanding of this phenomenon is still lacking.

# Appendix L

# Bethe ansatz solutions for some additional cases

In the main part of this work, we do *not* consider the K-matrices (7.14) for the cases $A^{(2)}_{2n-1}$ and $B^{(1)}_n$ with $p = 1$, and $D^{(1)}_n$ with $p = 1\,, n-1$, as emphasized in (8.5). These K-matrices are excluded because the corresponding transfer matrices do *not* have QG symmetry corresponding to removing one node from the Dynkin diagram. (This is the reason why we consider instead the K-matrices (8.6) and (8.7) for these cases.) Nevertheless, the transfer matrices for these cases are integrable, and we have also determined their spectra. We briefly note here the Bethe ansatz solutions for these cases.

For these cases (i.e., for the transfer matrices constructed using the K-matrices (7.14) for $A^{(2)}_{2n-1}$ and $B^{(1)}_n$ with $p = 1$, and for $D^{(1)}_n$ with $p = 1\,, n-1$), the transfer matrix eigenvalues are in fact given by (8.21), where the functions $y_l(u, p)$ are given by (8.41), (8.43), (8.44). Hence, the Bethe equations for $A^{(2)}_{2n-1}$, $B^{(1)}_n$ and $D^{(1)}_n$ with $p = 1$ are again those in Sec. 8.2.3, with the functions $\Phi_{l,p,n}$ given by (8.76).

For $D^{(1)}_n$ $(n > 3)$ with $p = n-1$, the Bethe equations for $l \leq n-2$ are the ones given in (8.51),(8.52),(8.60); but the Bethe equations for $l = n-1, n$ are given by

$$\left[\frac{\cosh\left(\frac{u_k^{[n-1]}}{2} + \eta + \frac{i\pi\varepsilon}{2}\right)}{\cosh\left(\frac{u_k^{[n-1]}}{2} - \eta + \frac{i\pi\varepsilon}{2}\right)}\right]^2 = \frac{Q^{[n-2]}\left(u_k^{[n-1]} - 2\eta\right)}{Q^{[n-2]}\left(u_k^{[n-1]} + 2\eta\right)} \frac{Q_k^{[n-1]}\left(u_k^{[n-1]} + 4\eta\right)}{Q_k^{[n-1]}\left(u_k^{[n-1]} - 4\eta\right)}\,, \tag{L.1}$$

$$\left[\frac{\cosh\left(\frac{u_k^{[n]}}{2} + \eta + \frac{i\pi\varepsilon}{2}\right)}{\cosh\left(\frac{u_k^{[n]}}{2} - \eta + \frac{i\pi\varepsilon}{2}\right)}\right]^2 = \frac{Q^{[n-2]}\left(u_k^{[n]} - 2\eta\right)}{Q^{[n-2]}\left(u_k^{[n]} + 2\eta\right)} \frac{Q_k^{[n]}\left(u_k^{[n]} + 4\eta\right)}{Q_k^{[n]}\left(u_k^{[n]} - 4\eta\right)}\,, \tag{L.2}$$

instead of by (8.61) and (8.62). In contrast with the QG-invariant case, the LHS of (L.2) has a nontrivial $(\neq 1)$ factor, even though $l = n \neq p$.

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