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Planar reversible vector fields, averaging theory and polycycles in non-smooth vector fields

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Dissertação apresentada como parte dos requisitos para obtenção do título de Mestre em Matemática, junto ao Programa de Pós-Graduação em Matemática, do Instituto de Biociências, Letras e Ciências Exatas da Universidade Estadual Paulista “Júlio de Mesquita Filho”, Câmpus de São José do Rio Preto.

Orientador: Prof. Dr. Claudio Aguinaldo Buzzi
Co-orientador: Prof. Dr. Jaume Llibre

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RESUMO

Neste trabalho veremos os retratos de fase, no disco de Poincaré, das formas normais locais das singularidades simétricas dos campos reversíveis do tipo $(2;0)$ e $(2;1)$ de baixa codimensão; uma aplicação da Teoria da Média no campo da Astrofísica, com o objetivo de estudar as órbitas periódicas de um modelo do universo de Friedmann-Robertson-Walker; e algumas generalizações de resultados conhecidos sobre policiclos em campos de vetores suaves para o caso não suave, focando um melhor entendimento de sua estabilidade e da bifurcação de ciclos limite.

Palavras-chave: Sistemas Dinâmicos, Retratos de fase, Reversibilidade, Teoria da Média, Policiclos.

ABSTRACT

In this work one will see the phase portraits, in the Poincaré disk, of the local normal forms of symmetrical singularities of reversible vector fields of type $(2;0)$ *and* $(2;1)$*; an application of the Averaging Theory at the field of Astrophysics, aiming the study of the periodic orbits in a model of the Friedmann-Robertson-Walker universe; and some generalizations of well established results about the polycycles in smooth vector fields to the non-smooth cases, aiming a better understanding of its stability and the bifurcation of limit cycles.*

Keywords: Dynamical Systems, Phase portraits, Reversibility, Averaging Theory, Polycycles.

LIST OF FIGURES

CONTENTS

1 INTRODUCTION

Let $P,\ Q\colon \mathbb{R}^2 \to \mathbb{R}$ be two C^k, $k \geqslant 1$, functions. A *planar C^k differential system* is a system of the form

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1.1}$$

where the dot in system (1.1) denote the derivative with respect to the independent variable t. The map $X = (P,Q)$ is called a *vector field*. If P and Q are polynomials such that the maximum of the degrees of P and Q is n, then system (1.1) is a *planar polynomial differential system of degree n*, or just a *polynomial system*. If $n = 1$, then system (1.1) is a *planar linear differential system*. This last class of system is already completely understood, see for instance the books [19, 25, 50]. However, if $n \geqslant 2$, then we know very few things. The class of planar polynomial systems with degree $n \geqslant 2$, i.e. the *planar nonlinear polynomial differential systems*, is too wide and thus it is common to study more specific subclasses and to classify them by their topological behavior.

With this in mind, we point out the subclass of the *reversible vector fields*. Given a C^k planar vector field X and a C^k diffeomorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi = \varphi^{-1}$ (i.e. φ is an *involution*) we say that X is a φ-reversible vector field of type $(2;r)$, $r \in \{0,1,2\}$, if

$$D\varphi(x,y)X(x,y) = -X(\varphi(x,y)), \tag{1.2}$$

for all $(x,y) \in \mathbb{R}^2$ and $Fix(\varphi) = \{(x,y) \in \mathbb{R}^2\colon \varphi(x,y) = (x,y)\}$ is a r-dimensional manifold. We observe that $D\varphi(x,y)$ denotes the *Jacobian matrix* of φ applied at the point (x,y). In a simple way, X is φ-reversible if after applying the change of coordinates $(u,v) = \varphi(x,y)$ one obtains $-X$.

Knowing that any planar polynomial vector field X can be extended analytically to the sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3\colon x^2+y^2+z^2 = 1\}$ through the *Poincaré compactification* (see Section 2.2), great advances in the topological classification of the planar nonlinear polynomial differential systems were made by Peixoto [44] and furthermore extended by Sotomayor [54]. From these works we point out the notion of *generic families*. Given a set $\mathfrak{X}$ we say that $\xi \in \mathfrak{X}$ is *generic* if ξ is an element of a collection $\Sigma \subset \mathfrak{X}$ such that:

(a) Σ is large with respect to $\mathfrak{X}$;

(b) its elements are amenable to simple description.

In a more mathematical way, if $\mathfrak{X}$ is endowed with some interesting topology, then condition (a) can be replaced by

(a_1) Σ is open and dense in $\mathfrak{X}$.

In [56] Teixeira applied the notion of generic families in the set of one and two-parameters families of germs of reversible vector fields of type $(2;1)$. Moreover, Buzzi [9] also applied this notion in the set of one-parameter families of germs of reversible vector fields of type $(2;0)$ with a singularity at the origin. Knowing that the unique germ of reversible vector field of type $(2;2)$ is $X = 0$ we conclude that the works of Teixeira and Buzzi together provide a topological classification of all the symmetrical singularities of germs of planar reversible vector fields of low codimension. In other words, given a symmetrical singularity of a planar reversible vector field we know all the low condimension bifurcations that can occur on it. Furthermore, Medrado and Teixeira [35, 36] also gave a classification of the symmetrical singularities of the reversible vector fields of type $(3;2)$ of codimension zero, one and two. For more works about reversibility see Buzzi, Roberto and Teixeira [14], for a time-reversible system, and Pereira and Pessoa [46, 47], for reversible vector fields over the sphere.

Therefore, based in the works of Buzzi [9] and Teixeira [56], Chapters 3 and 4 of this dissertation concerns with the global phase portrait, in the Poincaré disk, of all the topological normal forms given by [56] and [9]. Since such normal forms are *local*, we observe that their global phase portraits *does not* represent the global phase portrait of all reversible vector fields of type $(2;0)$ and $(2;1)$ of low codimension. However, for a classification of all the quadratic reversible vector fields of type $(2;1)$, see Llibre and Medrado [30]. Furthermore, see Theorem 3.7 for the phase portraits of the normal forms obtained by Buzzi and Theorem 4.3 for the phase portraits of the normal forms obtained by Teixeira. Given a vector field X, our approach works as follows.

(a) First we workout what we call the *local behavior* of the vector field X, i.e. we

1) study all the possible finite singularities;
2) use the Grobman-Hartman Theorem, The Stable Manifold Theorem and the Blow Up technique to obtain the local phase portrait of X at each of the finite singularities;
3) look for singularity bifurcations as the saddle-node, the center-focus and the Hopf bifurcation;
4) use tools as the Poincaré-Hopf Theorem and the Bendixson Criterion to understand when and where a limit cycle can appear;
5) study the equator of the compactification $p(X)$ of X;

(b) Then we look for topological informations which involves more than one singularity as heteroclinic connections and the formation of graphs;

(c) Finally we look for convenient curves in which the flow of X crosses it in a convenient way to shrink the possibilities for the α and ω-limit of the separatrices.

We observe that the work contained in both chapters is a co-work with professors Claudio Buzzi and Jaume Llibre, with the work [13] of Chapter 3 already published and the work [11] of Chapter 4 submitted for publication.

Chapter 5 consist in a application of the *Averaging Theory* in the field of Astrophysics, i.e. the application of the Averaging Theory in the four-dimensional Hamiltonian

$$H = \frac{1}{2}\left(y^2 - x^2 + p_y^2 - p_x^2\right) + \frac{1}{4}\left(ax^4 + 2bx^2y^2 + cy^4\right) - \omega\left(xp_y - yp_x\right), \tag{1.3}$$

which models in a simplified way what is called the *Friedmann-Robertson-Walker universe*. We observe that this is a co-work with professors Claudio Buzzi and Jaume Llibre and it was already published. See [12]. Our goal in this work is the description, in an analytical way, of the periodic orbits around the origin of the four-dimensional vector field derived from the Hamiltonian (1.3). See Theorem 5.1. Furthermore, our approach works as follows.

(a) First we applied the change of coordinates $(x, y, p_x, p_y) = \sqrt{\varepsilon}(X, Y, p_X, p_Y)$, obtaining a four-dimensional vector field which is topologically equivalent to the original system, derived from the Hamiltonian (1.3). We observe that this new system has the necessary ε-parameter;

(b) Then we applied a linear change of coordinates $(X, Y, p_X, p_Y) = (u, v, p_u, p_v)$ such that in these new coordinate the system is in its Jordan coordinates, i.e. its linear part is equal its Jordan Normal form;

(c) Then we applied the polar change of coordinates

$$(u, v, p_u, p_v) = (r\cos\theta, r\sin\theta, \rho\cos(\theta + \phi), \rho\sin(\theta + \phi)),$$

obtained a never-vanishing $\dot{\theta}$;

(d) Since $\dot{\theta}$ never vanishes, we can take θ as the independent variable of the polar system and then shrink the dimension of the vector field from four to three;

(e) In this three-dimensional nonautonomous vector field which has θ as its independent variable, we write the variable ρ as a function of (θ, r, ϕ, h), where h is the fixed-value of the first integral obtained from (1.3) in these new coordinate system. Thus, we obtain a two-dimension vector field which depends on the parameter h;

(f) Hence, we apply the Averaging Theory in this planar nonautonomous vector field obtaining information about its periodic orbits, which translates in to informations about the periodic orbits of the original system.

Chapter 6 concerns with the polycycles of a non smooth vector field (also known as *piecewise* vector fields). More precisely, if Γ is a polycycle of a non-smooth vector field, composed by tangential singularities, hyperbolic saddles and semi-hyperbolic saddles, then in Chapter 6 we prove the following.

(a) The stability of Γ depends on the stability of its singularities. See Theorem 6.4 and Corollary 6.5;

(b) If all the singularities of Γ compress the flow (resp. all the singularities repels the flow) around it, then Γ is stable (resp. unstable). Furthermore, if small enough perturbation of Γ has a limit cycle, then it is unique, hyperbolic and stable (resp. unstable). See Theorem 6.7;

(c) If Γ has n singularities satisfying some conditions, then there exists a perturbation of Γ such that at least n limit cycles bifurcate from it. See Theorem 6.8;

(d) If Γ is composed by a hyperbolic saddle and a quadratic-regular tangential singularity, then the bifurcation diagrams of Γ was completely described in the generic cases. See Theorem 6.9.

Our approach in Chapter 6 relies in the extension of previous results that are already well established in the smooth case, i.e. in polycycles of vector fields of class C^∞ (see [17, 20, 23, 38, 49]), together with the characterization of the flow of non-smooth vector fields near tangential singularities, obtained by Andrade, Gomide and Novaes [2]. We also observe that this work is a undergoing collaboration with professors Claudio Buzzi and Douglas Novaes.

For the sake of self-containedness, Chapter 2 concerns with some preliminaries necessary for Chapters 3, 4, 5 and 6. Finally, at the end we have a conclusion pointing out the highlights of each chapter.

7 CONCLUSION

We would like to point out that we are very impressed on how the systematic chasing for convenient curves such that the flow crosses it in a convenient way proved to be a very fruitful tool in the classification of the phase portraits of the planar reversible vector fields. We also point out the qualitative simplification done in the Friedmann-Robertson-Walker system, shrinking the dimensional from four to two and hence recovering all the theory of planar vector fields. Finally, we point out the fact that polycycles in non-smooth vector fields, at least when generic, behaves in a very similar way from its siblings in the smooth realm.

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